# A Simple Proof of Silver's Theorem

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**Abstract**

By using combinatorial properties of stationary sets, we give a simple proof of some generalization of Silver's Theorem i.e. if $\kappa$ is an uncountable regular cardinal such that $\aleph_\kappa$ is a singular strong limit cardinal, then the following hold.

(1). If $\{\alpha < \kappa : \aleph_\alpha^{<\kappa} \leq \aleph_{\alpha \cdot 2}\}$ is stationary, then $2^{\aleph_\kappa} \leq \aleph_{\kappa \cdot 2}$.

(2). If $\{\alpha < \kappa : \aleph_\alpha^{<\kappa} \leq \aleph_{\alpha+\gamma}\}$, where $0 < \gamma < \kappa$, is stationary, then $2^{\aleph_\kappa} \leq \aleph_{\kappa+\gamma}$.

**Keywords:** Silver's theorem, Singular cardinal, Stationary set

## 1. Introduction

In 1974, Silver showed that "if $\kappa$ is a singular cardinal of uncountable cofinality and GCH holds below $\kappa$, then it also holds for $\kappa$". The theorem is known as Silver's Theorem. It is one of the most surprising theorems dealing with singular cardinals since before it was discovered, most set theorists believed that it was possible for a singular cardinal to be the least counterexample of GCH (Jech, 1995, p. 408-424). The original proof of the theorem (Silver, 1974, p. 265-268) used some sophisticated techniques such as forcing and the method of ultrapowers (Jech, 2010, p. 94-121). In order to understand the proof one should have a strong background in set theory.

In the following years, Jensen, Baumgartner, and Prikry independently found direct proofs of Silver's Theorem which used only the knowledge of infinite combinatorics (Baumgartner and Prikry, 1976, p. 17-21, Baumgartner and Prikry, 1977, p. 108-113). These new proofs are significantly simpler than the original one.

In this paper, we give a simple proof of the following theorem which generalizes Silver's Theorem.

**The main theorem.** Let $\kappa$ be an uncountable regular cardinal. If $\aleph_\kappa$ is a singular strong limit cardinal, then the following hold.

(1). If $\{\alpha < \kappa : \aleph_\alpha^{<\kappa} \leq \aleph_{\alpha \cdot 2}\}$ is stationary, then $2^{\aleph_\kappa} \leq \aleph_{\kappa \cdot 2}$.

(2). If $\{\alpha < \kappa : \aleph_\alpha^{<\kappa} \leq \aleph_{\alpha+\gamma}\}$, where $0 < \gamma < \kappa$, is stationary, then $2^{\aleph_\kappa} \leq \aleph_{\kappa+\gamma}$.

Galvin and Hajnal showed that the above theorem can be derived from one of their main theorems in (Galvin and Hajnal, 1975, p. 491-498) (see Corollaries 16 and 18). By modifying the techniques in (Baumgartner and Prikry, 1976, p. 17-21) and (Baumgartner and Prikry, 1977, p. 108-113), we give a direct proof of the theorem.

## 2. Preliminaries

We assume the reader has some background in ordinals and cardinals. We shall give below a brief exposition of some notions used in this paper. Full details can be found in (Jech, 2006).

All basic concepts in set theory used in this paper are defined in the usual ways. We use $A, B, C, \ldots$ for sets; $\alpha, \beta, \gamma$ as well as $\xi$ and $\theta$ for ordinals; $\kappa$ and $\lambda$ for cardinals. $\mathcal{P}(A)$, $|A|$, and ${}^AB$ denote the power set of $A$, the cardinality of $A$, and the set of all functions from $A$ into $B$, respectively. The class of ordinals is denoted by **ON** and ${}^A$**ON** denotes the class of all functions from $A$ into **ON**. If $|A| = \lambda$ and $|B| = \kappa$, $\kappa^\lambda = |{}^AB|$. Let ${}^{<\alpha}A = \bigcup\{{}^\gamma A : \gamma < \alpha\}$ and $\kappa^{<\lambda} = |{}^{<\alpha}A|$ if $|A| = \kappa$ and $|\alpha| = \lambda$. We write $f{\restriction}_C$ for the restriction of a function $f$ to $C$ and $T_{|C}$ for $\{A \cap C \mid A \in T\}$.

If $U \subseteq \alpha$, we call $U$ *unbounded* in $\alpha$ if and only if for every $\beta < \alpha$ there exists $\gamma \in U$ such that $\gamma \geq \beta$. The *cofinality* of $\alpha$, $\mathrm{cf}(\alpha)$, is the least ordinal $\beta$ such that there is a map $f : \beta \to \alpha$ whose range is unbounded in $\alpha$. A cardinal $\kappa$ is *regular* if $\mathrm{cf}(\kappa) = \kappa$, otherwise $\kappa$ is *singular*. For a fixed regular uncountable cardinal $\kappa$, $C \subseteq \kappa$ is *closed in* $\kappa$ if and only if $supA \in C$ whenever $A$ is a nonempty subset of $C$ such that $supA < \kappa$ and $S \subseteq \kappa$ is *stationary in* $\kappa$ if and only if $S$ intersects every closed and unbounded subset of $\kappa$. $\kappa$ is a *strong limit cardinal* if $2^\lambda < \kappa$ for all $\lambda < \kappa$.

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Throughout this paper, let $\kappa$ be an uncountable regular cardinal such that $\aleph_\kappa$ is singular. The definitions of unbounded, closed, and stationary sets are understood to be defined in $\kappa$. It follows straightforwardly from the above definitions that every stationary set is unbounded and if $S$ is stationary and $C$ is closed and unbounded, then $S \cap C$ is stationary.

Some other basic facts concerning the above notions which will be used later are the following.

**Lemma 1** *If $\bigcup\{S_\alpha : \alpha < \lambda\}$, where $\lambda < \kappa$, is stationary, then at least one of the sets $S_\alpha$ is stationary.*

**Lemma 2** *(Fodor) If $S$ is stationary and $f$ is a function such that $\forall \alpha \in S$, $f(\alpha) < \alpha$, then there is a stationary set $S' \subseteq S$ such that $f$ is constant on $S'$.*

The proofs of the above lemmas can be found in (Jech, 2006).

**Lemma 3** *Suppose $S$ is stationary and $f$ is a function such that for all $\alpha \in S, f(\alpha) < \omega_\alpha$. Then there is an ordinal $\gamma < \kappa$ and a stationary set $S' \subseteq S$ such that for all $\alpha \in S'$, $f(\alpha) < \omega_\gamma$.*

Proof. First, define a function $g$ such that $g(\alpha)$ = the least ordinal $\gamma$ such that $f(\alpha) < \omega_\gamma < \omega_\alpha$ for all limit ordinals $\alpha \in S$. By applying Lemma 2 to $g$, this lemma follows.

By modifying the proof of Lemma 3 only a little, we obtain the following lemma.

**Lemma 4** *Suppose $S$ is stationary and $f$ is a function such that for all $\alpha \in S, f(\alpha) < \omega_{\alpha\cdot 2}$. Then there is an ordinal $\gamma < \kappa$ and a stationary set $S' \subseteq S$ such that for all $\alpha \in S'$, $f(\alpha) < \omega_{\alpha+\gamma}$.*

## 3. The main theorem

**Lemma 5** *Let $X = \bigcup\{F(Y) : Y \in Z\}$ where $|Z| \leq \aleph_\kappa$ and $F$ is a function. If $|X| \geq \aleph_{\kappa+\gamma}$, where $0 < \gamma < \kappa$, then $\forall\beta < \gamma \exists Y \in Z, |F(Y)| \geq \aleph_{\kappa+\beta+1}$.*

Proof. This can be easily seen by contraposition.

**Lemma 6** *Let $X \subseteq \mathcal{P}(\omega_\kappa)$ be such that $|X| > \aleph_{\kappa+\gamma}$, where $0 < \gamma < \kappa$, and $R$ be a relation on $X$ such that $A_1RA_2$ or $A_2RA_1$ for all distinct $A_1, A_2 \in X$. Then there exists $B \in X$ such that $|\{A \in X : ARB\}| \geq \aleph_{\kappa+\gamma}$.*

Proof. Let $Y$ be a subset of $X$ of cardinality $\aleph_{\kappa+\gamma}$. If there is $B \in Y$ which satisfies the lemma, then we are done. Suppose $\forall C \in Y, |\{A \in X : ARC\}| < \aleph_{\kappa+\gamma}$. Let $Z = \bigcup\{\{A \in X : ARC\} : C \in Y\}$. Then $|Z| \leq \aleph_{\kappa+\gamma}$. Since $|X| > \aleph_{\kappa+\gamma}$, $Z \neq X$ and so $\exists B \in X, B \notin Z$. By the definition of $Z$, $BRA$ is false for all $A \in Y$. Hence $ARB$ for all $A \in Y - \{B\}$ and then $|\{A \in X : ARB\}| \geq |Y| = \aleph_{\kappa+\gamma}$.

**Lemma 7** *Let $\gamma < \kappa$, $K$ be a stationary set, and $X \subseteq \mathcal{P}(\omega_\kappa)$. For each $A \in X$, fix $f_A \in {}^K\mathbf{ON}$. Assume that $\{\alpha \in K : f_{A_1}(\alpha) = f_{A_2}(\alpha)\}$ is bounded for all distinct $A_1, A_2 \in X$ and $\aleph_\kappa$ is strong limit. Then if $|X| > \aleph_{\kappa+\gamma+1}$ and $\forall A \in X \forall \alpha \in K, f_A(\alpha) < \omega_{\alpha+\gamma+1}$, there exist $B \in X$, a stationary set $K' \subseteq K$, $Y \subseteq X$ with $|Y| > \aleph_{\kappa+\gamma}$, and a one-to-one function $h_\alpha$ for each $\alpha \in K'$ such that $\forall A \in Y, f_A(\alpha) < f_B(\alpha)$ and $h_\alpha$ mapping $f_B(\alpha)$ into $\omega_{\alpha+\gamma}$ for all $\alpha \in K'$.*

Proof. Assume $|X| > \aleph_{\kappa+\gamma+1}$ and $\forall A \in X \forall \alpha \in K, f_A(\alpha) < \omega_{\alpha+\gamma+1}$. Define a relation $R$ on $X$ by letting $A_1RA_2$ if and only if $\{\alpha \in K : f_{A_1}(\alpha) < f_{A_2}(\alpha)\}$ is stationary. Since $K = \{\alpha \in K : f_{A_1}(\alpha) < f_{A_2}(\alpha)\} \cup \{\alpha \in K : f_{A_1}(\alpha) = f_{A_2}(\alpha)\} \cup \{\alpha \in K : f_{A_1}(\alpha) > f_{A_2}(\alpha)\}$ for any $A_1, A_2 \in X$, by Lemma 1 and the assumption that $\{\alpha \in K : f_{A_1}(\alpha) = f_{A_2}(\alpha)\}$ is bounded for all distinct $A_1, A_2 \in X$, we have $A_1RA_2$ or $A_2RA_1$ for all distinct $A_1, A_2 \in X$. By Lemma 6, there exists $B \in X$ such that $|\{A \in X : ARB\}| \geq \aleph_{\kappa+\gamma+1}$. Since $\aleph_\kappa$ is a singular strong limit cardinal where $\kappa$ is regular, $2^\kappa < \aleph_\kappa$. Hence there are at most $2^\kappa < \aleph_\kappa$ stationary sets. Since $\{A \in X : ARB\} = \bigcup\{\{A \in X : ARB \text{ and } K' \subseteq S_A\} : K' \text{ is stationary}\}$, where $S_A = \{\alpha \in K : f_A(\alpha) < f_B(\alpha)\}$, by Lemma 5, there exists a stationary set $K' \subseteq K$ such that $|\{A \in X : ARB \text{ and } K' \subseteq S_A\}| \geq \aleph_{\kappa+\gamma+1}$.

By the above assumption, for each $\alpha \in K'$, $|f_B(\alpha)| \leq \aleph_{\alpha+\gamma}$, so there exists a one-to-one function $h_\alpha$ mapping $f_B(\alpha)$ into $\omega_{\alpha+\gamma}$.

**Lemma 8** *Let $0 < \gamma < \kappa$, $K$ be a stationary set, and $X \subseteq \mathcal{P}(\omega_\kappa)$. For each $A \in X$, fix $f_A \in {}^K\mathbf{ON}$. Assume that $\{\alpha \in K : f_{A_1}(\alpha) = f_{A_2}(\alpha)\}$ is bounded for all distinct $A_1, A_2 \in X$ and $\aleph_\kappa$ is strong limit. Then if $|X| > \aleph_{\kappa+\gamma}$ and $\forall A \in X \forall \alpha \in K, f_A(\alpha) < \omega_{\alpha+\gamma}$, there exist $B \in X$, a stationary set $K' \subseteq K$, $Y \subseteq X$ with $|Y| > \aleph_\kappa$, and one-to-one functions $g_\alpha$ and $h_\alpha$ for each $\alpha \in K'$ such that $\forall A \in Y, g_\alpha(f_A(\alpha)) < g_\alpha(f_B(\alpha))$ and $h_\alpha$ mapping $g_\alpha(f_B(\alpha))$ into $\omega_\alpha$ for all $\alpha \in K'$.*

Proof. We will prove by transfinite induction on $\gamma$.
$\gamma=\beta+1$: By Lemma 7, there exist $B_1 \in X$, a stationary set $K_1 \subseteq K$, $X_1 \subseteq X$ with $|X_1| > \aleph_{\kappa+\beta}$, and a one-to-one function $\hat{g}_\alpha$ for each $\alpha \in K_1$ such that $\forall A \in X_1, f_A(\alpha) < f_{B_1}(\alpha)$ and $\hat{g}_\alpha$ mapping $f_{B_1}(\alpha)$ into $\omega_{\alpha+\beta}$ for all $\alpha \in K_1$. If $\beta = 0$, we are done. Assume $\beta > 0$. For each $A \in X_1$, define $F_A(\alpha) = \hat{g}_\alpha(f_A(\alpha))$ for all $\alpha \in K_1$. Hence $\forall A \in X_1 \forall \alpha \in K_1, F_A(\alpha) < \omega_{\alpha+\beta}$. Since each $\hat{g}_\alpha$ is one-to-one, $\{\alpha \in K_1 : F_{A_1}(\alpha) = F_{A_2}(\alpha)\} \subseteq \{\alpha \in K : f_{A_1}(\alpha) = f_{A_2}(\alpha)\}$ and hence $\{\alpha \in K_1 : F_{A_1}(\alpha) = F_{A_2}(\alpha)\}$ is bounded for all distinct $A_1, A_2 \in X_1$. By the induction hypothesis, there exist $B \in X_1 \subseteq X$, a stationary set $K' \subseteq K_1 \subseteq K$, $Y \subseteq X_1 \subseteq X$ with $|Y| > \aleph_\kappa$, and one-to-one functions $g_\alpha$ and $h_\alpha$ for each $\alpha \in K'$ such that $\forall A \in Y, g_\alpha(F_A(\alpha)) < g_\alpha(F_B(\alpha))$ and $h_\alpha$ mapping $g_\alpha(F_B(\alpha))$ into $\omega_\alpha$ for all $\alpha \in K'$. Since $g_\alpha$ and $\hat{g}_\alpha$ are one-to-one, so is $g_\alpha \circ \hat{g}_\alpha$. Since $\forall A \in Y \forall \alpha \in K', g_\alpha(F_A(\alpha)) = g_\alpha(\hat{g}_\alpha(f_A(\alpha))) = (g_\alpha \circ \hat{g}_\alpha)(f_A(\alpha))$, we are done.

$\gamma$ is a limit ordinal: For each $A \in X$, define $\phi_A(\alpha) = min\{\beta < \gamma : f_A(\alpha) < \omega_{\alpha+\beta+1}\}$ for all $\alpha \in K$ and let $\|\phi_A\|$ be the least ordinal $\beta$ such that $S_A^\beta = \{\alpha \in K : \phi_A(\alpha) = \beta\}$ is stationary. Note that such $\beta$ exists by Lemma 1 since $K = \bigcup\{S_A^\beta : \beta < \gamma\}$ for all $A \in X$. For each $\beta < \gamma$, let $Z_\beta = \{A \in X : \|\phi_A\| = \beta\}$. Clearly, $\bigcup\{Z_\beta : \beta < \gamma\} = X$ and $\forall \beta < \gamma \forall A \in Z_\beta, S_A^\beta$ is stationary. Since $|X| \geq \aleph_{\kappa+\gamma+1}$, $\exists \beta < \gamma, \left|Z_\beta\right| \geq \aleph_{\kappa+\gamma+1}$. For each $S \subseteq K$, let $H(S) = \{A \in Z_\beta : S_A^\beta = S\}$. Hence $\bigcup\{H(S) : S \text{ is stationary}\} = Z_\beta$. By Lemma 5, there is a stationary set $S$ such that $|H(S)| \geq \aleph_{\kappa+\gamma+1} > \aleph_{\kappa+\beta+1}$. Since $\forall A \in H(S) \forall \alpha \in S, \phi_A(\alpha) = \beta$, we have $\forall A \in H(S) \forall \alpha \in S, f_A(\alpha) < \omega_{\alpha+\beta+1}$. The rest of the proof follows by the induction hypothesis.

**Lemma 9** *Let $T = \{A \subseteq \omega_\kappa : |A| = \kappa$ and $\forall \alpha < \kappa(|A \cap \omega_\alpha| < \kappa)\}$. Assume $\aleph_\kappa$ is a strong limit cardinal. Then the following hold.*

*1. If $S = \{\alpha < \kappa : \left|T_{|\omega_\alpha}\right| \leq \aleph_{\alpha\cdot 2}\}$ is stationary, then $|T| \leq \aleph_{\kappa\cdot 2}$.*

*2. If $S = \{\alpha < \kappa : \left|T_{|\omega_\alpha}\right| \leq \aleph_{\alpha+\gamma}\}$, where $0 < \gamma < \kappa$, is stationary, then $|T| \leq \aleph_{\kappa+\gamma}$.*

Proof. We shall prove 1 first.

For every $\xi \in S$ and $A \in T$, let $f_\xi$ be a one-to-one map of $T_{|\omega_\xi}$ into $\omega_{\xi\cdot 2}$ and $F_A(\xi) = f_\xi(A \cap \omega_\xi)$.

Let $A_1, A_2 \subseteq \omega_\kappa$ be such that $A_1 \neq A_2$. Then $\exists \alpha(A_1 \cap \omega_\alpha \neq A_2 \cap \omega_\alpha)$. To show that $\{\gamma \in S : F_{A_1}(\gamma) = F_{A_2}(\gamma)\}$ is bounded, let $\beta \in S$ be such that $F_{A_1}(\beta) = F_{A_2}(\beta)$ i.e. $f_\beta(A_1 \cap \omega_\beta) = f_\beta(A_2 \cap \omega_\beta)$. Since $f_\beta$ is one-to-one, $A_1 \cap \omega_\beta = A_2 \cap \omega_\beta$. Since $A_1 \cap \omega_\alpha \neq A_2 \cap \omega_\alpha, \beta < \alpha$ and so $\{\gamma \in S : F_{A_1}(\gamma) = F_{A_2}(\gamma)\}$ is bounded.

Since $\forall \xi \in S, F_A(\xi) < \omega_{\xi\cdot 2}$ for every $A \in T$, by Lemma 4, for each $A \in T$, there exist $\gamma < \kappa$ and a stationary set $S' \subseteq S$ such that $\forall \xi \in S', F_A(\xi) < \omega_{\xi+\gamma}$.

Assume $|T| > \aleph_{\kappa\cdot 2}$. Let $T' \subseteq T$ be such that $|T'| = \aleph_{(\kappa\cdot 2)+1}$. Clearly, $T' = \bigcup_{(\gamma,S')\in\kappa\times\mathcal{P}(\kappa)} \{A \in T' : S'$ is a stationary subset of $S$ and $\forall \xi \in S', F_A(\xi) < \omega_{\xi+\gamma}\}$. Since there are at most $\kappa \cdot 2^\kappa = 2^\kappa < \aleph_\kappa < \aleph_{(\kappa\cdot 2)+1}$ pairs of $(\gamma, S')$ and $\aleph_{(\kappa\cdot 2)+1}$ is regular, there exist $\gamma > 0$, a stationary set $S_1 \subseteq S$, and $T_1 \subseteq T' \subseteq T$ such that $|T_1| = \aleph_{(\kappa\cdot 2)+1}$ and $\forall A \in T_1 \forall \xi \in S_1, F_A(\xi) < \omega_{\xi+\gamma}$.

By Lemma 7, there exist $B \in T_1$, a stationary set $K \subseteq S_1$, $Y \subseteq T_1$ with $|Y| > \aleph_\kappa$, and one-to-one functions $g_\alpha$ and $h_\alpha$ for each $\alpha \in K$ such that $\forall A \in Y, g_\alpha(F_A(\alpha)) < g_\alpha(F_B(\alpha))$ and $h_\alpha$ mapping $g_\alpha(F_B(\alpha))$ into $\omega_\alpha$ for all $\alpha \in K$.

For each $A \in Y$, let $S_A = \{\alpha \in K : g_\alpha(F_A(\alpha)) < g_\alpha(F_B(\alpha))\}$ and define $G_A(\alpha) = h_\alpha(g_\alpha(F_A(\alpha)))$ for all $\alpha \in K$. Since $G_A(\alpha) < \omega_\alpha$ for all $\alpha \in S_A$ and $K \subseteq S_A$ and so $S_A$ is stationary for all $A \in Y$, by Lemma 3, there exist $\theta_A < \kappa$ and a stationary set $U_A \subseteq S_A$ such that $\forall \alpha \in U_A, G_A(\alpha) < \omega_{\theta_A}$ for all $A \in Y$. The total number of such pairs $(U_A, \theta_A)$ is at most $|\mathcal{P}(S_A)| \cdot \kappa = 2^\kappa \cdot \kappa = 2^\kappa < \aleph_\kappa$. Since $Y = \bigcup_{(U,\theta)\in\mathcal{P}(\kappa)\times\kappa} \{A \in Y : U_A = U$ and $\theta_A = \theta\}$, by Lemma 5, there exists $(U, \theta)$ such that $|\{A \in Y : U_A = U$, and $\theta_A = \theta\}| > \aleph_\kappa$. Since $U \subseteq \kappa$ and $\theta < \kappa, \left|{}^U\omega_\theta\right| = \aleph_\theta^\kappa \leq max(\aleph_\theta^{\aleph_\theta}, \kappa^\kappa) = max(2^{\aleph_\theta}, 2^\kappa) < \aleph_\kappa < |\{A \in Y : U_A = U$ and $\theta_A = \theta\}|$. Then $|\{G_A \restriction_U : A \in Y, U_A = U$, and $\theta_A = \theta\}| \leq \aleph_\theta^\kappa < |\{A \in Y : U_A = U$ and $\theta_A = \theta\}|$. Hence there exist $A_1, A_2 \in Y$ such that $A_1 \neq A_2, U_{A_1} = U_{A_2} = U, \theta_{A_1} = \theta_{A_2} = \theta$, and $G_{A_1}(\alpha) = G_{A_2}(\alpha)$ for all $\alpha \in U$. Since $g_\alpha$ and $h_\alpha$ are one-to-one, $F_{A_1}(\alpha) = F_{A_2}(\alpha)$ for all $\alpha \in U$. Thus $\{\alpha : F_{A_1}(\alpha) = F_{A_2}(\alpha)\} \supseteq U$. We have shown that $\{\alpha < \kappa : F_{A_1}(\alpha) = F_{A_2}(\alpha)\}$ is bounded while $U$ is unbounded, a contradiction.

The proof of 2 is easier. We first define $F_A$ as above. There is no need to use Lemma 4 since we already have the fact that $\forall A \in T \forall \xi \in S, F_A(\xi) < \omega_{\xi+\gamma}$ from the assumption. After assuming $|T| > \aleph_{\kappa+\gamma}$ and $T' \subseteq T$ is such that $|T'| = \aleph_{\kappa+\gamma+1}$, we can apply Lemma 8 immediately and the rest of the proof is similar.

**Theorem 10** *Suppose $\aleph_\kappa$ is a strong limit cardinal.*

*1. If $\{\alpha < \kappa : \aleph_\alpha^{<\kappa} \leq \aleph_{\alpha\cdot 2}\}$ is stationary, then $2^{\aleph_\kappa} \leq \aleph_{\kappa\cdot 2}$.*

*2. If $\{\alpha < \kappa : \aleph_\alpha^{<\kappa} \leq \aleph_{\alpha+\gamma}\}$, where $0 < \gamma < \kappa$, is stationary, then $2^{\aleph_\kappa} \leq \aleph_{\kappa+\gamma}$.*

Proof. Let $T$ as in Lemma 9 be fixed. Define $F : {}^\kappa(\omega_\kappa - \{0\}) \to T$ by $F(g) = \{\omega_\alpha + \sum_{\xi\leq\alpha} g(\xi) : \alpha < \kappa\}$ for every

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$g \in {}^{\kappa}(\omega_\kappa - \{0\})$. Note that for each $g \in {}^{\kappa}(\omega_\kappa - \{0\})$, $F(g) \in T$ since $|F(g) \cap \omega_\alpha| \leq \left|\{\omega_\xi : \xi < \alpha\}\right| = |\alpha| < \kappa$ for all $\alpha < \kappa$.

To show that $F$ is one-to-one, let $g_1, g_2 \in {}^{\kappa}(\omega_\kappa - \{0\})$ be such that $g_1 \neq g_2$. Let $\gamma$ be the least ordinal such that $g_1(\gamma) \neq g_2(\gamma)$. Without lost of generality suppose that $g_1(\gamma) < g_2(\gamma)$. Then $\omega_\alpha + \sum_{\xi\leq\alpha} g_2(\xi) = \omega_\alpha + \sum_{\xi\leq\alpha} g_1(\xi) < \omega_\gamma + \sum_{\xi\leq\gamma} g_1(\xi) < \omega_\gamma + \sum_{\xi\leq\gamma} g_2(\xi) \leq \omega_\beta + \sum_{\xi\leq\beta} g_2(\xi)$ for all $\alpha < \gamma$ and $\beta \geq \gamma$. Thus $\omega_\gamma + \sum_{\xi\leq\gamma} g_1(\xi) \notin F(g_2)$ and so $F(g_1) \neq F(g_2)$. Hence $|T| \geq \aleph_\kappa^\kappa$. Since $\aleph_\kappa$ is a strong limit cardinal where $\kappa$ is regular, $2^{\aleph_\kappa} = \aleph_\kappa^{cf(\aleph_\kappa)} = \aleph_\kappa^\kappa$. Hence $|T| \geq 2^{\aleph_\kappa}$. Since $T \subseteq \mathcal{P}(\omega_\kappa)$, $|T| = 2^{\aleph_\kappa}$.

Since for each $\alpha < \kappa$, $T_{|\omega_\alpha} \subseteq \{A \subseteq \omega_\alpha : |A| < \kappa\}$, $\left|T_{|\omega_\alpha}\right| \leq \aleph_\alpha^{<\kappa}$ for all $\alpha < \kappa$. By the assumption of 1, $\{\alpha < \kappa : \aleph_\alpha^{<\kappa} \leq \aleph_{\alpha\cdot 2}\}$ is stationary and so is $\{\alpha < \kappa : \left|T_{|\omega_\alpha}\right| \leq \aleph_{\alpha\cdot 2}\}$. By Lemma 9, $2^{\aleph_\kappa} = |T| \leq \aleph_{\kappa\cdot 2}$.

The proof of 2 is similar.

**Corollary 11**

*1. If* $\{\alpha < \kappa : 2^{\aleph_\alpha} \leq \aleph_{\alpha\cdot 2}\}$ *is stationary, then* $2^{\aleph_\kappa} \leq \aleph_{\kappa\cdot 2}$.

*2. If* $\{\alpha < \kappa : 2^{\aleph_\alpha} \leq \aleph_{\alpha+\gamma}\}$*, where* $0 < \gamma < \kappa$*, is stationary, then* $2^{\aleph_\kappa} \leq \aleph_{\kappa+\gamma}$.

Proof. For 1, assume $\{\alpha < \kappa : 2^{\aleph_\alpha} \leq \aleph_{\alpha\cdot 2}\}$ is stationary. By the above theorem, it remains to show that $\{\alpha < \kappa : {\aleph_\alpha}^{<\kappa} \leq \aleph_{\alpha\cdot 2}\}$ is stationary and $\aleph_\kappa$ is a strong limit cardinal.

Since ${\aleph_\alpha}^{<\kappa} \leq {\aleph_\alpha}^{\kappa} \leq {\aleph_\alpha}^{\aleph_\alpha} = 2^{\aleph_\alpha}$ for all $\alpha$ such that $\kappa < \aleph_\alpha$, $\{\alpha < \kappa : 2^{\aleph_\alpha} \leq \aleph_{\alpha\cdot 2}\} \cap \{\alpha < \kappa : \kappa < \aleph_\alpha\} \subseteq \{\alpha < \kappa : {\aleph_\alpha}^{<\kappa} \leq \aleph_{\alpha\cdot 2}\}$. Since $\{\alpha < \kappa : 2^{\aleph_\alpha} \leq \aleph_{\alpha\cdot 2}\}$ is stationary and $\{\alpha < \kappa : \kappa < \aleph_\alpha\}$ is closed and unbounded, $\{\alpha < \kappa : 2^{\aleph_\alpha} \leq \aleph_{\alpha\cdot 2}\} \cap \{\alpha < \kappa : \kappa < \aleph_\alpha\}$ is stationary and so is $\{\alpha < \kappa : {\aleph_\alpha}^{<\kappa} \leq \aleph_{\alpha\cdot 2}\}$.

To show that $\aleph_\kappa$ is a strong limit cardinal, let $\alpha < \kappa$. Since $\{\xi < \kappa : 2^{\aleph_\xi} \leq \aleph_{\xi\cdot 2}\}$ is unbounded, there exists $\alpha \leq \beta < \kappa$ such that $2^{\aleph_\alpha} \leq 2^{\aleph_\beta} \leq \aleph_{\beta\cdot 2} < \aleph_\kappa$.

The proof of 2 is similar.

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