Introduction to Multifractals

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Abstract

This is an easy read introduction to multifractals. We start with a thorough study of the Binomial measure from a multifractal point of view, introducing the main multifractal tools. We then continue by showing how to generate more general multiplicative measures and close by presenting an extensive set of examples on which we elaborate how to 'read' a multifractal spectrum.

1 Introduction

Multifractal analysis is concerned with describing the local singular behavior of measures or functions in a geometrical and statistical fashion. It was first introduced in the context of turbulence [M2, M3, FP, Gr1, GP1, HP, HJKPS, M5], and then studied as a mathematical tool in increasingly general settings [EM, KP, CM, BMP, R1, F2, AP, J, LV, HW, O, S2, RSch] Multifractals are applied in many contexts such as DLA patterns investigation [ME], earth quake distribution analysis [HI], signal processing [LM], and internet data traffic modelling [RL]. Here, we present only some very basic results in a simple setting, accessible also to graduate students with only little knowledge of probability theory. Intending to motivate the reader for further study, proofs and heavy notation are avoided. As this monograph is continuously 'under construction' the author welcomes comments towards an improvement of the presentation and appologizes for parts which are still in a preliminary status.

With multifractals, an new approach to dealing with (preferably irregular, erratic) data – or with geometrical and/or probabilistic objects – and a new set of models for such is at hand. The main reason for this novelty lies in the fact that multiplicative iterative schemes are essentially different from additive ones. One could object that a simple 'exponentiating' respectively 'taking logarithms' leads from one to the other. While this is true at every step of the iteration, it becomes false when going to the limit. In addition, controling the correlation structure of the 'exponentiated' process is combersome and not always doable. We hope that this will become clear as we go along in our presentation.

2 A paradigm: The Binomial Measure

Purpose and techniques of multifractal analysis are best explained in the most simple situation: the binomial measure on the unit interval.

2.1 Definition

The binomial measure is a probability measure μ which is defined conveniently via a recursive construction. Start by splitting $I := [0,1]$ into two subintervals I_0 and I_1 of equal length and assign the masses m_0 and $m_1 = 1 - m_0$ to them. With the two subintervals one proceeds in the same manner and so forth: at stage two, e.g. the four subintervals I_{00}, I_{01}, I_{10}, and I_{11} have masses $m_0 m_0$, $m_0 m_1$, $m_1 m_0$, and $m_1 m_1$ respectively. At stage n, the total mass 1 is distributed among the 2^n dyadic intervals of order n such that $I_{\varepsilon_1 \ldots \varepsilon_n}$ has mass $m_{\varepsilon_1} \cdot \ldots \cdot m_{\varepsilon_n}$. This defines a sequence of measures μ_n, all piecewise uniform. Since $\mu_k(I_{\varepsilon_1 \ldots \varepsilon_n}) = \mu_n(I_{\varepsilon_1 \ldots \varepsilon_n})$ for all $k \geq n$ we may define the limit measure μ by $\mu(I_{\varepsilon_1 \ldots \varepsilon_n}) = m_{\varepsilon_1} \cdot \ldots \cdot m_{\varepsilon_n}$, in other words, μ_n converges weakly towards μ. By construction, the restrictions of μ to the intervals I_0 and I_1 have the same structure as μ itself. In fact, they are reduced copies of μ where the reductions in space and mass are by $1/2$ and m_i, respectively. We conclude that μ is *self-similar* in a very strict way (see Fig. 1): for all intervals $[a,b]$

$$\mu([a,b]) = m_0 \mu([2a, 2b]) + m_1 \mu([2a-1, 2b-1]). \tag{1}$$

It is instructing to prove this formula by pluging in the special intervals $[a,b] = I_{\varepsilon_1 \ldots \varepsilon_n}$, starting with $I_0 = [0,1/2]$, $I_1 = [1/2,1]$, $I_{00} = [0,1/4]$ etc. Note in particular that $\mu([2a,2b]) = 0$ for $[a,b] \subset [1/2,1]$. An alternative proof uses that $\mu_{n+1}([a,b]) = m_0 \mu_n([2a,2b])$ for every $[a,b] \subset [0,1/2]$ and $\mu_{n+1}([a,b]) = m_1 \mu_n([2a-1,2b-1])$ for $[a,b] \subset [1/2,1]$.

Another way of defining μ is the following. Let $x = .\sigma_1 \sigma_2 \ldots$ be the dyadic representation of a point in $[0,1]$. Here, we don't have to care about points with multiple expansions since our results concern 'almost all points x'. Imagine that the digits σ_k are picked randomly such that $P[\sigma_k = i] = m_i$ independently of k. Then, μ is the law—or probability distribution—of the corresponding x on $[0,1]$.

This measure μ has no density, unless $m_0 = m_1 = 1/2$, which can be seen in two ways. First, for a typical point t the densities, say $\phi_n(t)$, of the approximating measures μ_n tend either to infinity or to 0. Second, $M(x) = \mu([0,x])$ has zero derivative almost everywhere, whence $M(x) \neq 0 = \int_0^x M'(t)dt$. We are about to prove these two facts using the LLN, but it is also well illustrated with Fig. 1. Nevertheless, any coarse graining (or sampling) of μ on the dyadic intervals $I_{\varepsilon_1 \ldots \varepsilon_n}$, is equal to μ_n which has the density ϕ_n. It is, therefore, essential to understand the limit behavior of such an approximation μ_n.

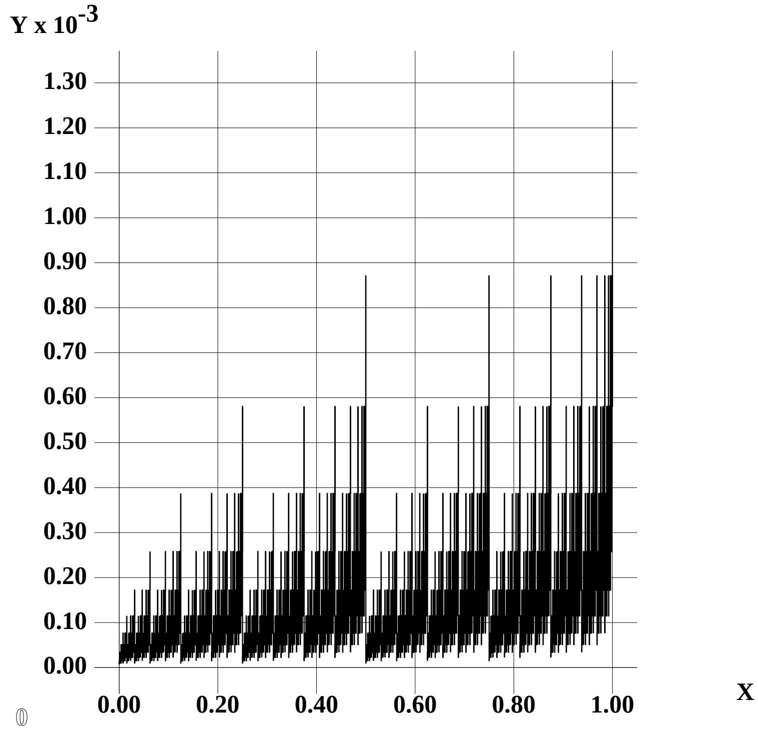


Figure 1: The Binomial measure with $m_0 = .4$ as obtained after 13 iterations.

2.2 Singular behavior

The absence of a density for μ is responsible for its erratic, or 'fractal' appearance (see Fig. 1). It is the aim of multifractal analysis to characterize this erratic behavior. The multiplicative construction of μ makes clear that the mass of a sequence of intervals $\mu(I_{\varepsilon_1\ldots\varepsilon_n})$ will decay roughly exponentially fast as the $I_{\varepsilon_1\ldots\varepsilon_n}$ shrinks down to a point x, say approximately as $2^{-n\alpha(x)}$. Such an exponential rate $\alpha(x)$ could be though of as a generalization of the local degree of differentiability of M at x. Indeed, $M(x') - M(x) \simeq |x' - x|^\alpha$ which is called *Hölder continuity* of order α at x. Therefore, set

$$\alpha_n(x) := \frac{\log \mu(I_{\varepsilon_1\ldots\varepsilon_n})}{\log |I_{\varepsilon_1\ldots\varepsilon_n}|} = -\frac{1}{n} \log_2 \mu(I_{\varepsilon_1\ldots\varepsilon_n}),$$

and

$$\alpha(x) := \lim_{n\to\infty} \alpha_n(x)$$

whenever this limit exists.

A simple calculation reveals that $\alpha(x)$ does indeed exist for 'many' points x and that it takes quite different values depending on the dyadic expansion $x = \sum_k \varepsilon_k 2^{-k}$. Denoting the number of ones among the first n binary digits of x by $l_n(x)$ we find immediately

$\mu(I_{\varepsilon_1\ldots\varepsilon_n}) = m_0^{n-l_n(x)} m_1^{l_n(x)}$and

$$\alpha(x) = -\lim_{n\to\infty} \frac{n - l_n(x)}{n} \log_2 m_0 + \frac{l_n(x)}{n} \log_2 m_1$$

So, $\alpha(x)$ can take all values between $\log_2 m_0$ and $\log_2 m_1$ and some of these values will be assumed 'more likely' than others. Before elaborating further on this issue we would like to mention informally that the points x where $\alpha(x)$ assumes a given value α will typically form highly interwoven fractal sets, whence the term *multifractal*. Thereby, the term fractal is not so much referring to this 'fractured' appearance as rather to the fact that the aforementioned sets have a dimension which is not integer. A precise statement of this fact, however, has to wait for later (or can be looked up in [R1]).

Let us now become more precise about 'more frequent' and 'more rare' values $\alpha(x)$ of the binomial measure. The first, trivial observation is that $\alpha(x)$ is entirely determined by the limiting frequency of digits of x, $\chi(x) := \lim_n l_n(x)/n$. It is clear how to produce points x where $\chi(x)$ assumes some given value between 0 and 1. E.g. for $x = .001001001...$ we have $\chi(x) = 1/3$. As one lowers the value of $\chi(x)$ one has to plug in more and more zeros and the 'choice' of x becomes more and more restricted. This fact is what we would like to make more precise.

In order to gain information about how 'likely' $\chi(x)$ assumes a certain value we use theorems from probability theory. Picking x randomly with uniform distribution U, we can consider its binary digits ε_n to be random variables. The Law of Large Numbers implies that $(1/n)l_n(x) = (1/n)(\varepsilon_1 + \ldots + \varepsilon_n)$ converges to $1/2$, whence $\chi(x) = 1/2$ and $\alpha(x) = -(1/2)\log_2 m_0 m_1$ for U-almost all x. Equivalently, we could argue directly by considering m_{σ_i} as random variables and conclude with the LLN that for U-almost all x

$$\alpha_n(x) = \frac{\log \mu(I_{\varepsilon_1\ldots\varepsilon_n})}{\log |I_{\varepsilon_1\ldots\varepsilon_n}|} = -\frac{1}{n}\sum_{k=1}^{n} \log_2 m_{\sigma_k} \to \mathbb{E}_U[-\log_2 m_{\sigma_i}] = -\frac{1}{2}\log_2 m_0 m_1. \qquad (2)$$

For convenience we set $\alpha_0 := \mathbb{E}_U[-\log_2 m_{\sigma_i}] = -(1/2)\log_2 m_0 m_1$.

Usually, one is happy with an 'almost sure' result such as (2). Here, we would like to ask the question what one could say about points x with $\alpha(x)$ different from (2). Again, we may put the LLN to use towards the question, how often $l_n(x)/n$ converges to a number different from $1/2$. Therefore, let us pick x randomly *with distribution* μ, i.e. the binary digits of such a point equal 1 with probability m_1. Then, the LLN gives $\chi(x) = m_1$, or, equivalently

$$\alpha_n(x) = \frac{\log \mu(I_{\varepsilon_1\ldots\varepsilon_n})}{\log |I_{\varepsilon_1\ldots\varepsilon_n}|} \to \mathbb{E}_\mu[-\log_2 m_{\sigma_i}] = -m_0\log_2(m_0) - m_1\log_2(m_1). \qquad (3)$$

We conclude that the 'almost sure' result (2) depends heavily on the distribution according to which points are picked randomly. This is, after all, an obvious, but nevertheless notable remark.

Let us stop for a moment and comment on the differentiability of M. From $-\frac{1}{2}\log_2 m_0 m_1 > 1$ we conclude that

$$M'(x) \;=\lim_{n\to\infty}\frac{\mu(I_{\varepsilon_1\ldots\varepsilon_n})}{2^{-n}}=0$$

almost surely (in the uniform sense), as mentioned above. Alternatively, using the obvious formula $\phi_n(x)=\mu(I_{\varepsilon_1\ldots\varepsilon_n})/2^{-n}$ for the density of μ_n (as always $x=\sum_i \varepsilon_i 2^{-i}$) we find that $\phi_n(x)\simeq 2^{n(1-\alpha(x))}$. Thus, the densities tend to zero or infinity, depending on the value of $\alpha(x)$. As we have just seen, it will actually be zero for (uniformly) almost all x, and ∞ for μ-almost all x.

2.3 Large Deviation Principles

Before we can come to a general treatment of the possible values of $\alpha(x)$ we need to gain some intuition.

The theory of Large Deviation Principles (LDP) deals with a problem intimately related to our quest. In order to make the connection let us note that (2) implies the following: if we pick an interval $I_{\varepsilon_1\ldots\varepsilon_n}$ for each n randomly in an uniform way then the probability of finding $(1/n)\log_2\mu(I_{\varepsilon_1\ldots\varepsilon_n})$ outside $[\alpha_0-\varepsilon,\alpha_0+\varepsilon]$ must tend to zero for any fixed ε. Theorems on LDP state more sharply that this decay must be exponentially fast.

A first explanation for this exponential decay comes from the so-called Cramer-Chernoff bound. It reads as follows. Consider a sequence of i.i.d. random variables W_n and set

$$V_n \;:=\frac{1}{n}(W_1+\ldots+W_n).$$

While the LLN states that V_n converges to $\mathbb{E}W$ almost surely, the theorem of **Cramer-Chernoff** says the following:

Theorem 1 *If* $\mathbb{E}W<a$ *and* $P[W>a]\neq 0$*, then*

$$\frac{1}{n}\log P\left[V_n\geq a\right]\to\inf_{q>0}(\log\mathbb{E}[\exp(qW)]-qa).$$

For a proof at least of the upper bound note that for all $q\geq 0$

$$P\left[V_n\geq a\right]=P\left[e^{qnV_n}\geq e^{nqa}\right]\leq\frac{\mathbb{E}e^{q(W_1+\ldots+W_n)}}{e^{nqa}}=\left(\mathbb{E}[e^{qW}]e^{-qa}\right)^n$$

using Tschebischev's inequality and independence.

This result is generalized by the well-known theorem on LDP by **Gärtner-Ellis** [Ell, Thm 2]. We present its contents in a simplified version fit to suit our purpose. The familiarity with theorem 1 is nevertheless still apparent.

Theorem 2 *Let Y_n be an arbitrary sequence of random variables on a sequence of probability spaces, and let $a_n \to \infty$. Assume that the following limit exists*

$$-\frac{1}{a_n} \log \mathbb{E}_n[\exp\langle q, Y_n\rangle] \to c(q) \tag{4}$$

and assume furthermore that $c(q)$ is finite, concave and differentiable. Then,

$$\frac{1}{a_n} \log P_n[-a_n^{-1} Y_n \in A] \to \inf_q (qa - c(q)) \qquad (n \to \infty \text{ and } A \to \{a\}). \tag{5}$$

The term 'large deviations' stems from the fact that (5) deals with probabilities of the renormalized $-a_n^{-1}Y_n$ deviating from its expected value for large n.

Let us illustrate how this result relates to the occurance of coarse Hölder exponents $\alpha_n(x)$ for measures in general and for the binomial in particular. To this end, let P_n be the uniform distribution on the set of all dyadic intervals $I_k^{(n)} \subset [0,1]$ of order n, i.e. $P_n[I_{K(\omega)}^{(n)} = I_k^{(n)}] = P_n[K = k] = 1/2^n$ for all $k = 1, \dots, 2^n$. Denote expectation w.r.t. P_n by $\mathbb{E}_n$ and consider the sequence of random variables $Y_n = \log \mu(I_K^{(n)})$. Also, let us choose $a_n = n \log 2$.

In this setting we have $-a_n^{-1} Y_n = \alpha_n(x)$ where $x = \sum_k \varepsilon_k 2^{-k}$ as before. In order to apply theorem 2 one has to calculate the asymptotic behavior of the moment generating functions:

$$c(q) := \lim_{n\to\infty} \frac{-1}{n} \log_2 \mathbb{E}_n[\exp(qY_n)] = \lim_{n\to\infty} \frac{-1}{n} \log_2 2^{-n} \sum_{k=1}^{2^n} \mu(I_k^{(n)})^q \tag{6}$$

The basic assumption of the LDP theorem 2 is that this limit c exists and is a differentiable, concave function. Theorem 2 implies then that the following limit does exist and takes the stated value:

$$\begin{aligned} &\frac{1}{n} \log_2 \#\{k = 0, \dots 2^n - 1 \,:\, \alpha - \varepsilon \le (1/n) \log_2 \mu(I_k^{(n)}) \le \alpha + \varepsilon\} \\ &\quad = \quad 1 + \frac{1}{n} \log_2 P_n\Big[\frac{-1}{n \log 2} Y_n \in [\alpha - \varepsilon, \alpha + \varepsilon]\Big] \\ &\quad \to \quad 1 + c^*(\alpha) \qquad (n \to \infty, \varepsilon \to 0). \end{aligned} \tag{7}$$

Hereby, c^* denotes the Legendre transform, i.e. $c^*(\alpha) = \inf_q (q\alpha - c(q))$. For a typical shape of c^* see Fig. 3, which actually shows $c^*(\alpha) + 1$. It is natural to consider $1 + c^*$ rather than c^* itself since (7) is easily interpretated as a limit of renormalized histograms.

For the binomial measure in particular, the binomial formula gives

$$\sum_{k=1}^{2^n} \mu(I_k^{(n)})^q = \left(m_0^q + m_1^q\right)^n,$$

and

$$c(q) = 1 - \log_2 \left(m_0^q + m_1^q \right). \tag{8}$$

Thus, theorem 2 can be applied for the binomial measure.

In order to derive some simple formulas for Legendre transforms let us assume for a moment that c is strictly concave and twice continuously differentiable for all q. (Such is the case for the binomial.) First, using calculus to compute the infimum in the definition of c^* reveals that $c^*(\alpha) = q\alpha - c(q)$ at $\alpha = c'(q)$. Note that c' is strictly increasing, positive and will in general not cover the entire positive α-asis. Now, by taking implicit derivatives we find that $c^*(\alpha)$ is differentiable with derivative $(c^*)'(\alpha) = q$ at $\alpha = c'(q)$, and that c^* is strictly concave by monotony of $c'(q)$. Let us derive, finally, a formula for the Legendre transform of c^*, i.e. for $c^{**}(t) := \inf_\alpha(\alpha t - c^*(\alpha))$: as above $c^{**}(t) = \alpha t - c^*(\alpha)$ at $t = (c^*)'(\alpha)$. Thus, if $\alpha = c'(q)$ then $t = q$. Using also that $c^*(\alpha) = q\alpha - c(q)$ at $\alpha = c'(q)$ one finds $c^{**}(q) = c(q)$. In other words, for such functions c, the Legendre transform is its own inverse.

We end this short introduction of LDP-s with an intuitive argument explaining why the Legendre transform shows up in theorem 2. To this end, we will assume that the limits in (6) as well as in (7) exist and denote them by $c(q)$ and $1 + g(\alpha)$ for the time being. We would like to establish that $g = c^*$. By assumption, approximately $2^{n(g(\alpha)+1)}$ of the dyadic intervals $I_k^{(n)}$ carry mass $\mu(I_k^{(n)}) \simeq 2^{-n\alpha}$. Rearranging terms we find:

$$\begin{aligned}
2^{-nc(q)} &\simeq 2^{-n} \sum_{k=1}^{2^n} \mu(I_k^{(n)})^q = 2^{-n} \sum_{\alpha} \sum_{\mu(I_k^{(n)}) \simeq 2^{-n\alpha}} \mu(I_k^{(n)})^q \\
&= 2^{-n} \sum_{\alpha} 2^{n(g(\alpha)+1)} 2^{-nq\alpha} = \sum_{\alpha} 2^{-n(q\alpha - g(\alpha))} \\
&\simeq 2^{-n \inf_\alpha (q\alpha - g(\alpha))}.
\end{aligned}$$

In the last step we used that the bulk contribution to this sum will come from the term with the largest exponent. Thus, we argued that $g^* = c$ which is somewhat weaker than the desired $g = c^*$. But it is sufficient when c has enough regularity as described above.

2.4 Practical issues

In practice, on estimates c from a log-log plot of the sample moment $\sum_{k=1}^{2^n} \mu(I_k^{(n)})^q$ (8) against scale 2^{-n} and interprets the LDP result (7) as a convergence of normalized, logarithmic histograms (Fig. 2). Indeed, since the distribution P_n used here is uniform, it reduces to counting. The histogram will be in terms of

$$\frac{-1}{n \log 2} Y_n = \frac{\log \mu(I_K^{(n)})}{\log |I_K^{(n)}|} =: \alpha(I_K^{(n)}),$$

which have been called the *coarse Hölder exponents* of μ as these values provide information on the degree of Hölder continuity of M at x: $\alpha_n(x) = \alpha(I_{\varepsilon_1 \ldots \varepsilon_n})$. The LDP results states, thus, that the number of $\alpha(I_k^{(n)})$ at given resolution n 'close' to some given α grows like $2^{n(c^*(\alpha)+1)}$ (which is exponentially slower than the total number 2^n). This approach to the multiplicative structure of the measure μ and to its singularities has been called *coarse graining* and will be introduced in the next section with all rigor.

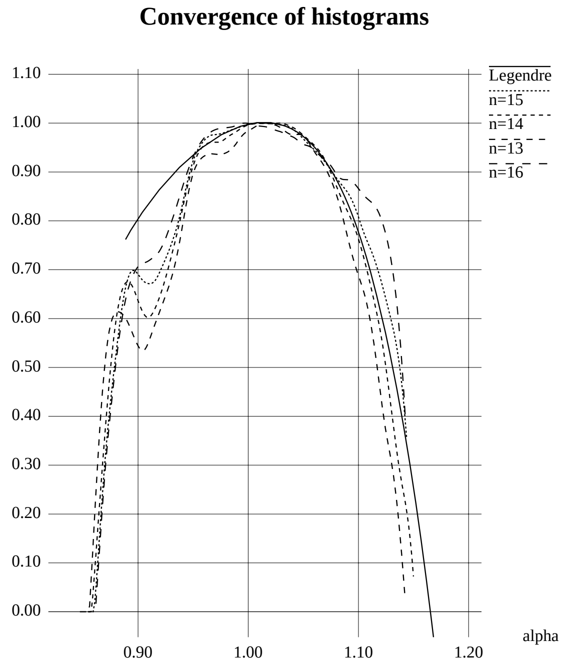


Figure 2: Convergence of the normalized histograms f_G^ε of coarse Hölder exponents for some real world data: normalizing the total time of a trace recording Internet packets arriving at a gateway to unit time the number of packets arriving in a time slot over the total number of packets in the trace are interpreted as a probability distribution or measure. The estimates f_G^ε are as in Fig. 10 and the f_L is added as the solid line. The convergence of f_G^ε to f_L is convincing, but a 'bump' around $\alpha = .9$ cannot go unnoticed. It indicates that the traffic is composed of two different components as explained in Fig. 4, arising from the use of two different communication speeds.

2.5 Locating the singularities $\alpha(x)$

To conclude this section and to give the promised insight into the 'appearance' of the $\alpha(x)$ let us take a more careful look into the Large Deviation result. The LLN, as we have seen, tells us that the peak of the histograms (7) will be close to α_0 (2). To obtain information about other parts of the histograms we need to have a way of choosing intervals (or points x) where the 'unusual' happens, i.e. where $\alpha(I_k^{(n)})$ is 'far' from α_0.

The technical term is a 'change of probability' meaning that the intervals $I_k^{(n)}$ are chosen randomly according to a law μ_q which insures the convergence of $\alpha(I_k^{(n)})$ towards some value a_q provided that points are picked randomly with distribution μ_q. This distribution μ_q is defined in the same way as μ but with probabilities $\overline{m}_0 := m_0^q 2^\beta$ and $\overline{m}_1 := m_1^q 2^\beta$ where β is such that $\overline{m}_0 + \overline{m}_1 = 1$. Hence,

$$\beta = \beta(q) \ = -\log_2 \left(m_0^q + m_1^q\right) = c(q) - 1.$$

Note that

$$\beta^*(\alpha) = \inf_q (q\alpha - \beta(q)) = c^*(\alpha) + 1.$$

Choosing the digits σ_k of the dyadic expansion of a point x such that $P[\sigma_k = i] = m_i^q 2^\beta$ amounts to picking x randomly with law μ_q. Let us apply now the LLN to this situation to get

$$\alpha_n(x) = \frac{\log \mu(I_{\varepsilon_1 \dots \varepsilon_n})}{\log |I_{\varepsilon_1 \dots \varepsilon_n}|} \to \mathbb{E}_{\mu_q}[-\log_2(m_{\sigma_i})] = -\sum_{i=0}^{1} \overline{m}_i \log_2 m_i = \beta'(q).$$

In other words, for the points picked randomly with distribution μ_q the α_n converge (almost surely) to $a_q := \beta'(q)$. Whence, these points lie all in

$$K_{a_q} := \{x \ : \ \alpha(x) := \lim_n \alpha_n(x) = a_q\},$$

and μ_q helps us 'concentrate' on the part of the histograms close to a_q.

To find out about 'where' this distribution μ_q concentrates let us note that for the same points x in K_α we find that

$$\frac{\log \mu_q(I_{\varepsilon_1 \dots \varepsilon_n})}{\log |I_{\varepsilon_1 \dots \varepsilon_n}|} = -\frac{1}{n} \sum_{k=1}^{n} \log_2 \overline{m}_{\sigma_k} = -\frac{1}{n} \sum_{k=1}^{n} \log_2(m_{\sigma_k}^q \cdot 2^\beta) \to q a_q - \beta(q) = \beta^*(a_q). \tag{9}$$

This result is helpful in two ways. First, a very rough but useful estimation (which can be made precise [RM3, p 137]) shows, how many intervals have $\alpha(I_k^{(n)}) \simeq a_q$. These intervals are the ones contributing the bulk probability to μ_q. Using (9),

$$1 \simeq \sum_{\alpha(I_k^{(n)}) \simeq a_q} \mu_q(I_k^{(n)}) \simeq \#\{k \ : \ \alpha(I_k^{(n)}) \simeq a_q\} \cdot 2^{-n\beta^*(a_q)},$$

whence this number is approximately $2^{n\beta^*(a_q)} = 2^{n(c^*(a_q)+1)}$. But this is the content of the LDP and provides, thus, a second proof.

Second, (9) allows us to determine the Hausdorff dimension [F] of K_α. For those familiar with this notion, let us fix q and let $\alpha = \beta'(q)$. Then (9) means that μ_q is equivalent to the $\beta^*(\alpha)$-dimensional Hausdorff measure [F] restricted to K_α. Since K_α has full μ_q-measure, $\beta^*(\alpha)$ is a lower bound on the dimension of K_α. From the coarse graining approach it follows easily that this bound is in fact exact [R1].

In summary, we verified that in this simple situation three approaches coincide: one through a 'partition function' c or β, one through 'coarse graining' and one using the concept of 'dimensions'. In a notion which we are about to introduce this reads as

$$f_{\mathrm{L}}(\alpha) = f_{\mathrm{G}}(\alpha) = f_{\mathrm{H}}(\alpha). \tag{10}$$

3 Multifractal spectra and the multifractal formalism

We introduce now rigorously what has been motivated in the preceding section.

Much effort has been made in order to obtain rigorous mathematical extensions of the aforementioned result (10) to more general cases [KP, CM, O, F2, AP, LV, R1]. The general setting is as follows.

Assume that a distribution of points in d-space is given in form of a measure μ: the probability for a point to fall in a set E is $\mu(E)$. If this distribution is singular one cannot describe it by means of a density and multifractal analysis proves useful in characterizing the complicated geometrical properties of μ. The basic idea is to classify the singularities of μ by strength. This strength is measured as a singularity exponent $\alpha(x)$, called Hölder exponent. Usually, points of equal strength lie on interwoven fractal sets K_α:

$$K_\alpha := \{x \in \mathbb{R}^d \,:\, \alpha(x) := \lim_{B \to \{x\}} \frac{\log \mu(B)}{\log |B|} = \alpha\}, \tag{11}$$

which explains the name 'multifractal'. Here, $B \to \{x\}$ means that B is a ball containing x, and that its diameter $|B|$ tends to zero. The geometry of the singular distribution μ can then be characterized by giving the 'size' of the sets K_α, more precisely their Hausdorff dimension [F]:

$$f_{\mathrm{H}}(\alpha) := \dim(K_\alpha).$$

This definition is most useful in purely mathematical settings. It is not required, though, for the understanding of this paper. For the interested reader we refer to [F, AP, R1, LV] for further details.

In applications, one assumes that μ has bounded support[1] and considers a *coarse grained* version f_{G}, also called *large deviation spectrum*:

$$f_{\mathrm{G}}(\alpha) := \lim_{\varepsilon \to 0} \limsup_{\delta \to 0} \frac{\log N_\delta(\alpha, \varepsilon)}{\log 1/\delta}$$

with the convention $\log 0 := -\infty$. Here, N_δ denotes the number of cubes C of size δ with *coarse Hölder exponent* $\alpha(C)$ 'roughly equal to α'. More precisely, denote by G_δ the set of all cubes of the form $C = [l_1\delta, (l_1+1)\delta) \times \ldots \times [l_d\delta, (l_d+1)\delta)$ with integer $l_1, \ldots, l_d$ and with $\mu(C) \neq 0$. Then, we set

$$C^* = [(l_1-1)\delta, (l_1+2)\delta) \times \ldots \times [(l_d-1)\delta, (l_d+2)\delta),$$

$$\alpha(C) := \frac{\log \mu(C^*)}{\log \delta}, \tag{12}$$

[1] The complement of the support is the union of the intervals with no measure, resp. over which a function is constant.

and

$$N_\delta(\alpha,\varepsilon) = \#\{C \in G_\delta \,:\, \alpha(C) \in (\alpha-\varepsilon, \alpha+\varepsilon]\}.$$

As is pointed out in [R1, PR] using C^* instead of C in (12) greatly improves the theoretical properties as well as the numerical behavior of f_{G}. The reason for this is that C provides a poor approximation of a ball centered in a point of the distribution μ, especially in points close to the border of the support of μ. Since singular measures are typically supported on fractals, these problems are present on all scales leading to wrong results. A further advantage of using C^* is the fact that the spectrum $f_{\mathrm{G}}(\alpha)$ does not change when replacing the continuous limit $\delta \to 0$ by the discrete limit $\delta_n = c2^{-n}$ $(n \to \infty)$. Finally, we should point out, that this choice δ_n provokes no 'border effects' for measures supported on an *interval* of length c. In this special case C^* may again be replaced by C without changing the outcome. These properties have been used throughout our numerical analysis. For the ease of notation we will write $N_n^\varepsilon(\alpha) := N_{\delta_n}(\alpha,\varepsilon)$.

Though tempting it is *wrong* to interpret f_{G} as the box dimension (see [F, R1]) of K_α. This function f_{G} is better explained in statistical terms: Note first that the number N_δ of cubes in G_δ behaves roughly as $N_\delta \simeq \delta^{-D_0}$ where D_0 denotes the box dimension of the support of μ. It follows that $f_{\mathrm{G}}(\alpha) \le D_0$ for all α. Now suppose that one picks a cube C out of G_δ randomly and determines its coarse Hölder exponent $\alpha(C) := \log\mu(C^*)/\log\delta$. Then, the probability of finding $\alpha(C) \simeq \alpha$ behaves roughly like

$$N_\delta(\alpha,\varepsilon)/N_\delta = P_\delta[\alpha(C) \simeq \alpha] \simeq \delta^{D_0 - f_G(\alpha)}. \tag{13}$$

This is the statistical interpretation of f_{G}. Note in particular that in the limit $\delta \to 0$ the only Hölder exponent which is observed with non-vanishing probability is α_0, where $f_{\mathrm{G}}(\alpha_0) = D_0$.

What shapes of the spectra f_{H} and f_{G} can we expect?[2] Could they be trivial functions ($f(\alpha) = -\infty$)? Certainly not, since there is always at least α_0 with $f_{\mathrm{G}}(\alpha_0) = D_0$. Let give a simple argument for 'self-similar measures' with $D_0 = 1$ using the Law of Large Numbers (LLN). A general and rigorous proof is obtained by combining theorem 4 below with the fact that $\tau(1) = 0 \ne -\infty$.

Write

$$\alpha(x) = \lim_{n\to\infty} -\frac{1}{n}\log_2\mu(C_n^*(x)) = \lim_{n\to\infty} -\frac{1}{n}\sum_{k=1}^{n}\log_2\frac{\mu(C_k^*(x))}{\mu(C_{k-1}^*(x))}$$

where $C_n(x)$ is the unique cube in $G_{1/2^n}$ containing x. Then, the assumption of self-similarity means that the random variables $\log_2\mu(C_k^*(x))/\mu(C_{k-1}^*(x))$ are i.i.d. (compare Section 2 and [R1, AP]). Denote the common expectation by α_0. The LLN implies that almost surely $\alpha(x) = \alpha_0$ when picking points x randomly with 'uniform' distribution, i.e. when picking C randomly in G_δ. This establishes the claim.

[2]Some answers of a general kind can be found in [LV].

In special cases such as the binomial measure with $m_0 = m_1 = 1/2$ (uniform distribution) α_0 is the only Hölder exponent. More precisely, $\alpha(x) = \alpha_0 = 1$ for all $x \in [0,1]$, $f_{\mathrm{H}}(\alpha_0) = f_{\mathrm{G}}(\alpha_0) = 1$ and $f_{\mathrm{H}}(\alpha) = f_{\mathrm{G}}(\alpha) = -\infty$ for $\alpha \neq \alpha_0$ in this case. Such measures with only one Hölder exponent are called uniform or *monofractal*.

In general, other Hölder exponents occur. For the binomial, e.g. we find $\alpha(0) = -\log_2(m_0)$ $\alpha(1) = -\log_2(m_1)$ etc. Also, the coarse graining will show non-trivial spectra, i.e. on every finite level of approximation G_δ one will have a whole histogram of *coarse Hölder exponents* $\alpha(I_k^{(n)})$. For $\alpha \neq \alpha_0$, however, the probability of finding $\alpha(I_k^{(n)}) \simeq \alpha$ will decrease exponentially fast to 0 as $\delta \to 0$ (13). A rigorous proof of this fact is most easily obtained — at least under certain conditions — by applying the Principle of Large Deviations (LDP) of Gärtner-Ellis (see [Ell]). Translated into our setting the LDP states, in simple terms, that

$$P_n[-\frac{1}{n}\log_2 \mu(C_n^*(x)) \simeq \alpha] \simeq 2^{nc^*(\alpha)}$$

with some scaling function c^*. Note that $c^*(\alpha) < 0$ unless $\alpha = \alpha_0$. A rigorous formulation is the following:

Theorem 3 ([Ell, R1]) *Assume that the 'moment generating function'*

$$c(q) := \lim_{n\to\infty} \frac{-1}{n} \log_2 \mathbb{E}[\exp(q \log \mu(C_n^*(x)))]$$

exists and is convex and differentiable for all $q \in \mathbb{R}$. Then,

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n\log 2} P_n\left[\left|-\frac{1}{n}\log_2 \mu(C_n^*(x)) - \alpha\right| \leq \varepsilon\right] = c^*(\alpha)$$

where $c^*(\alpha) = \inf_q(q\alpha - c(q))$ is the Legendre transform of c.

So, it is natural to introduce the *partition function* $\tau(q)$

$$\tau(q) := \lim_{\delta\to 0} \frac{\log S_\delta(q)}{\log \delta} \qquad \text{with} \qquad S_\delta(q) := \sum_{C\in G_\delta} \mu(C^*)^q.$$

As a matter of fact, $\tau(q)$ stands at the beginning of multifractal analysis and has since played an central role [M, M5, FP, Gr1, HP, HJKPS, JKP, R1, LV].

It is notable that $S_\delta(0)$ simply counts the number of cubes with non-vanishing measure. Thus, $-\tau(0)$ is actually the box-dimension of the support of μ, i.e.

$$D_0 = -\tau(0).$$

It follows then from the definitions that $c(q) = \tau(q) - \tau(0) = \tau(q) + D_0$. For the binomial measure defined in Section 2 one finds with (8)

$$\tau(q) = -\log_2(m_0^q + m_1^q).$$

For other examples see Section 4 below.

Provided that Ellis' theorem applies, i.e. assuming that $\tau(q)$ exists and is differentiable, it follows that (13) holds with $c^* = f_{\mathrm{G}}(\alpha) - D_0$, i.e.

$$f_{\mathrm{G}}(\alpha) = \tau^*(\alpha). \tag{14}$$

This has been termed the *multifractal formalism*. The similarity to the well-known thermo-dynamical formalism [V, R3] is immediate (see also [R3]).

Since $\tau(q)$ is obtained by averaging, it depends more regularly on the data than $f_{\mathrm{G}}(\alpha)$ and is easier to compute. It is important to note, though, that it contains in general less information than f_{G}. Let us make this point more precise.

The partition function is always convex since $S_\delta(q)$ is convex for all δ. But it is not necessarily differentiable in every q and the multifractal formalism may not hold for all α. For some simple and convincing counterexamples see Fig. 4 and [R1, MR, RM2, RM3, LV]. It is natural, thus, to introduce the *Legendre spectrum*

$$f_{\mathrm{L}}(\alpha) := \tau^*(\alpha).$$

This spectrum is sometimes referred to as obtained by the *method of moments.*

While (14) may be wrong for certain α, the opposite relation holds for all q as is shown in [R3] an d[R]:

Theorem 4

$$\tau(q) = f_{\mathrm{G}}^*(q) = \inf_{\alpha \in \mathbb{R}} \Big(q\alpha - f_{\mathrm{G}}(\alpha) \Big)$$

As a first consequence, D_0 is indeed the maximal value of f_{G} in general. Secondly, $f_{\mathrm{L}} = \tau^* = {f_{\mathrm{G}}}^{**}$ is the concave hull of f_{G}. Thus,

$$f_{\mathrm{G}}(\alpha) \leq f_{\mathrm{L}}(\alpha).$$

Thirdly, it follows that even a not everywhere differentiable $\tau(q)$ determines $f_{\mathrm{G}}(\alpha)$ at least in its concave points. To be more precise let $\alpha^+ := \tau'(q+)$ for $q > 0$ and $\alpha^- := \tau'(q-)$ for $q < 0$ denote the one-sided derivatives of $\tau(q)$ which must exist since $\tau(q)$ is convex. Then [R3, R],

$$\begin{aligned} f_{\mathrm{G}}(\alpha^+) &= q\alpha^+ - \tau(q) = \tau^*(\alpha^+) \qquad && (q > 0) \\ f_{\mathrm{G}}(\alpha^-) &= q\alpha^- - \tau(q) = \tau^*(\alpha^-) \qquad && (q < 0). \end{aligned} \tag{15}$$

An alternative way of displaying the scaling of moments is through the so-called *generalized dimensions*

$$D_q := \frac{\tau(q)}{q-1}.$$

Besides D_0, a notable value of D_q is $\alpha_1 = \tau'(1) = D_1$. It has been termed information dimension [Gr2, GP1, GP2, OWY]: With respect to the given distribution μ we have $\alpha(x) = \alpha_1 = D_1$ almost surely. For a binomial measure α_1 is given by (3).

We conclude the section by noting that in all generality we have

$$f_{\mathrm{H}}(\alpha) \leq f_{\mathrm{G}}(\alpha) \leq fla.$$

This has been shown in [RM3] and in [R3]. If equality holds for a particular measure μ then the *multifractal formalism* is said to hold for μ.

4 How to read a spectrum

Before listing some general rules on how to derive underlying properties of a measure from the shape of its multifractal spectrum we give an intuitive interpretation of the actual values $(\alpha, f(\alpha))$ seen in a spectrum.

4.1 Values

The multifractal spectra provide a global description of the singularities of the observed measure μ. The parameter α quantifies the degree of regularity in a point x: loosely speaking, the measure of an interval $[x, x + \Delta x]$ — in applications usually the number of events occurring in this interval — behaves as $(\Delta x)^\alpha$ (11,12).

For a uniform distribution one finds $\alpha(x) = 1$ for all x. More generally, for any $a > 0$ the distribution with density x^{a-1} on $[0, 1]$ has $\alpha(0) = a$ and $\alpha(x) = 1$ for all $x \in (0, 1]$. Values $\alpha(x) < 1$ indicate, thus, a burst of events around x 'on all levels' (bursts of bursts), while $\alpha(x) > 1$ is found in regions where events occur sparsely.

The spectrum $f_{\mathrm{G}}(\alpha)$ captures how 'frequently' a value $\alpha(x) = \alpha$ is found: the number of dyadic intervals C of size Δx with $\alpha(C) = \alpha$ behaves as $(\Delta x)^{-f_{\mathrm{G}}(\alpha)}$. For the 'almost sure' value α_0 one has $f_{\mathrm{G}}(\alpha_0) = D_0$ which is necessarily the maximal value of f_{G}. A more precise characterization is given by (13).

The spectrum $f_{\mathrm{H}}(\alpha)$ gives the size of the 'set of Hölder exponent α' K_α in terms of its dimension.

4.2 Shape

In order to give an idea of how to extract information from the shape of a spectrum f we proceed by giving examples for which the spectra are known explicitly. The plots of Fig. 3 to 9 are obtained by first solving implicit equations for $\tau(q)$ similar as the one for the Binomial measure of Section 2:

$$m_0^q 2^\tau + m_1^q 2^\tau = 1.$$

Applying then the multifractal formalism (which has beed shown to hold for these examples in [CM, AP, R1]) one finds the multifractal spectrum of the measure (or parts of it, see below) by taking Legendre transform. The resulting graphs can be plotted by varying the parameter q:

$$\alpha(q) = \tau'(q) \qquad\qquad \tau^*(\alpha) = q\tau'(q) - \tau(q).$$

The most simple shape of a multifractal spectrum is found for those measures for which the multifractal formalism applies and for which τ is a twice continuously differentiable function. Such is the case for the binomial measure (see Fig. 3), as well as more general self-similar measures, that is, measures with multiplicative rescaling structure (see below). Obvously, the spectrum $f = f_L$ is then a smooth, concave function, with an overall shape like the symbol $\cap$ (see Fig. 3). These are properties which most spectra encountered in the real world do not share.

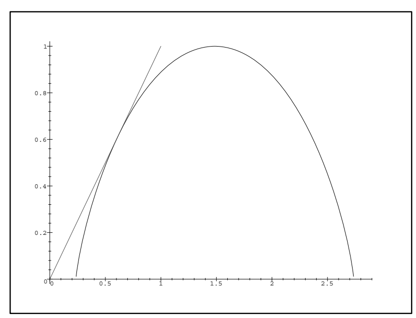


Figure 3: The most simple shape of a multifractal spectrum is found for $f_L = \tau^*$ with smooth τ as here for the binomial measure with $m_0 = .15$, $m_1 = .85$, and $r_0 = r_1 = 1/2$, and $r_0 = r_1 = 1/2$. However, all spectra of measures touch the bisector (dashed) and reach the maximum $D_0 = -\tau(0)$. Here, μ is supported on the unit interval which has dimension $D_0 = 1$. The extremal Hölder exponents are $\alpha_{\min} = \log(.85)/\log(.5) \simeq .234$ and $\alpha_{\max} = \log(.15)/\log(.5) \simeq 2.737$.

Here are some features which are common to *all* spectra of *measures*:

- The spectrum of a *measure* touches the internal bisector of the axis.
- Moreover, for any measure $f(\alpha) \leq \alpha$ for all α.
- The spectrum touches the horizontal line through $(0, D_0)$ where $D_0 = -\tau(0)$.
- Moreover, $f(\alpha) \leq D_0$ for all α.

To prove the first two points we begin by noting that $\tau(1) = 0$ because $1 \leq S_\delta(1) \leq 3$. Now, in the cases where $f = \tau^*$ the claim follows immediately by taking Legendre transform. But, even in the general case, i.e. when $f = \tau^*$ is not guaranteed, we have always $\tau^* = f$ by theorem 4. Now, if there was an α such that $f(\alpha) > \alpha$ then $\tau(1) \geq f(\alpha) - \alpha > 0$, a contradiction. Similarly, if there was $\varepsilon > 0$ such that $f(\alpha) \leq \alpha - \varepsilon$ for all α then $\tau(1) \leq \varepsilon < 0$, another contradiction. This proves the two claims[3]. The other claims follow in a similar fashion.

Whenever a spectrum f fails to show a **concave $\cap$-shape** we have evidence that μ is not purely multiplicative, or self-similar. A search of models with similar features as the observed spectra may reveal telling details on the structure of the distribution μ. A most prominent example is found with the left sided spectra of DLA [M4, MEH, RM1] where the shape of the spectrum hints to an infinite rather than finite set of multiplicative rescaling laws present with DLA.

In the sequel we provide various examples with atypical spectra and explain the particular appearances.

First, let us note a few generalizations of the binomial. For all of them, the multifractal formalism is valid with smooth τ as is shown in [AP, F2] and [R3].

- Arbitrary contraction ratios r_i instead of $1/2$. More precisely, the ratio between the length of the subintervals $I_{i_1\ldots i_n k}$ and the length of their 'mother' $I_{i_1\ldots i_n}$ will be r_k. The invariance of μ reads then as

$$\mu([a,b]) = m_0\mu([a/r_0, b/r_0]) + m_1\mu([a/r_1 + 1 - 1/r_1, b/r_1 + 1 - 1/r_1])$$

  (compare (1)) and the formula for $\tau(q)$ generalizes to

$$m_0^q {r_0}^{-\tau} + m_1^q {r_1}^{-\tau} = 1.$$

  As a consequence, $\tau(0) < 1$ if the support is fractal, i.e. if $r_0 + r_1 < 1$ (compare Fig. 3).

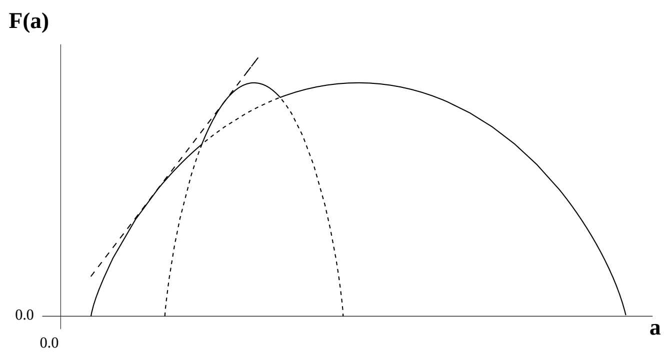
- Arbitrary number n of subintervals instead of 2 :

$$\sum_{i=0}^{n-1} m_i^q {r_i}^{-\tau} = 1.$$

- Random contraction ratios r_i and weigths m_i :

$$\mathbb{E}\sum_i m_i^q {r_i}^{-\tau} = 1.$$

[3] In the first two points we have emphasized the assumption that the multifractal spectrum f be computed from a *measure*. More generally, one could define Hölder exponents and spectra for functions by replacing $\mu(B)$ in the definition of $\alpha(x)$ (see (12)) by the maximal increment of the function over the interval (cube) B and other quantities (compare [R3]). Then, the zero of τ is typically not in 1, but say in $q = 1/H$, and the tangent to f which goes through the origin has slope $1/H$.



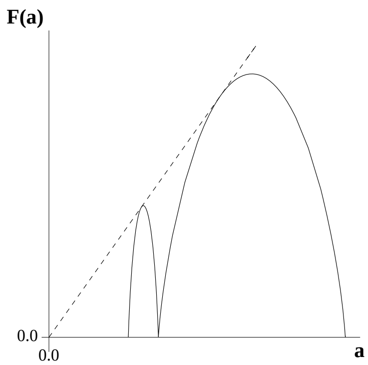


Figure 4: The spectrum of the sum $\mu = \mu_1 + \mu_2$ of two measures which live on disjoint supports is simply the maximum of the individual spectra. This will in general result in a non-concave spectrum as shown in two cases here. The dashed parts show the internal bisector of the axes and the spectra of the binomial measures μ_1 and μ_2 where they do not coincide with the spectrum of μ.

Second, a few examples with non-concave spectra. Note, that in particular the multifractal formalism can not hold here. The easiest way of breaking the concavity-property is by considering sums of binomial measures $\mu = \mu_1 + \mu_2$. If the supports of μ_1 and μ_2 are disjoint we have

$$f(\alpha) = \max(f_1(\alpha), f_2(\alpha)) \qquad \tau(q) = \min(\tau_1(q), \tau_2(q)).$$

which is valid for both, f_{G} and f_{H}. It does not and cannot hold, however, for f_{L} simply because the maximum of concave functions is not concave (see Fig. 4). With this remark, it is now clear that the multifractal formalism must break down (here: $f_{\mathrm{H}} = f_{\mathrm{G}} < f_L$) for sums $\mu_1 + \mu_2$ unless their spectra are identical.

The failure of the multifractal formalism and the non-concave shape of f_{G} is in this example a direct consequence of a sort of **phase transition**: At the α-value where the irregularity of the spectrum f_{G} occurs, the major contributor to the set of singularities K_α changes from μ_1 to μ_2.

Similar phenomena of **phase transitions** have been observed with the considerably richer class of self-affine measures which are invariant similar as in (1) but with general affine maps of the plane replacing $x/2$ and $x/2 + 1/2$ [R]. Here, $\tau(q)$ is often found to be differentiable, thus at least $f_{\mathrm{G}}(\alpha) = f_{\mathrm{L}}(\alpha)$. The phase transition occurs here as the main contributor changes between the two eigendirections, i.e. as it changes from the horizontal rectangles to the vertical rectangels and vice versa. See Fig. 5 through 8.

Thus, we take departures from concavity of the spectrum as evidence of the absence of a ‘universal’ multiplicative law, thus, in the most simple first assumption as evidence for the presence of several measures following such laws.

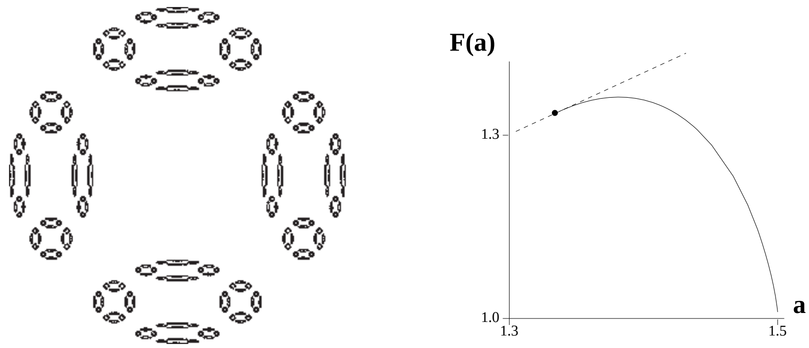


Figure 5: An image of a self-affine measure with a 'circular' appearance. The phase transition observed here is mild: $\tau(q)$ is once but not twice differentiable. The sudden end in the trajectory of f corresponds to $\tau(q)$ becoming linear.

Less violent departures from concavity are linear parts in the spectrum. In the cases observed so far, this comes along with $f_{\mathrm{H}} < f_{\mathrm{G}}$ and may be produced either by a high order zero of a limiting density or a hierarchy of atoms. In the first case the measure is in fact nonsingular, a fact that can be encountered in situations as obvious as in Fig. 9 and as unexpected as in Fig. 10 [RM4].

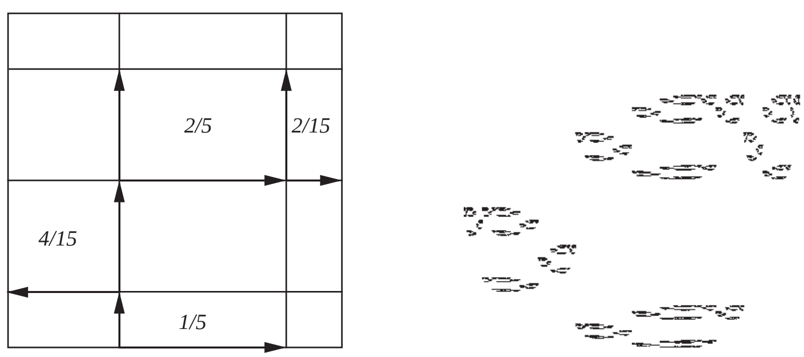


Figure 6: On the right the image of a self-affine multifractal composed of 30'000 points obtained by a random algorithm. The affine maps and the probabilities involved in its construction are indicated on the left.

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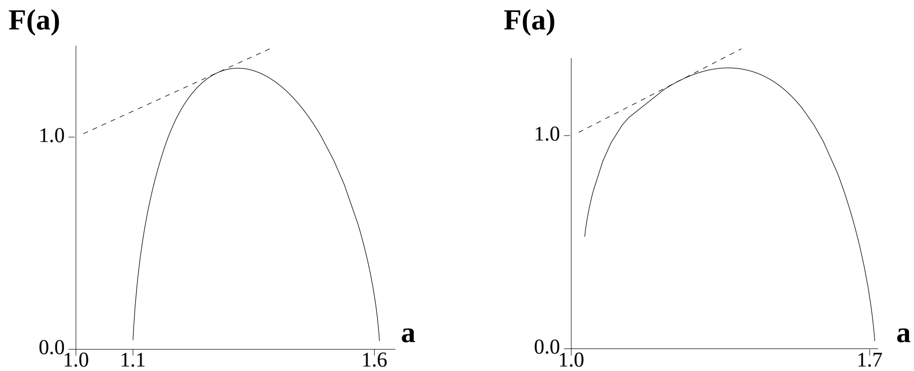

Figure 7: On the left the spectrum of the self-affine measure given in Figure 6 above. Its asymmetry reflects the fact that contraction ratios are asymmetrical with respect to the weights $p_1 = 1/5$, $p_2 = 4/15$, $p_3 = 2/5$ and $p_4 = 2/15$. On the right the spectrum when changing the weights to $p_1 = p_2 = p_3 = p_4 = 1/4$. With this choice, which is of course asymmetrical as well, we find a mild phase transition: the structure function $\tau(q)$ is still smooth. This results in a concavity of the spectrum which is disturbed but not destroyed.

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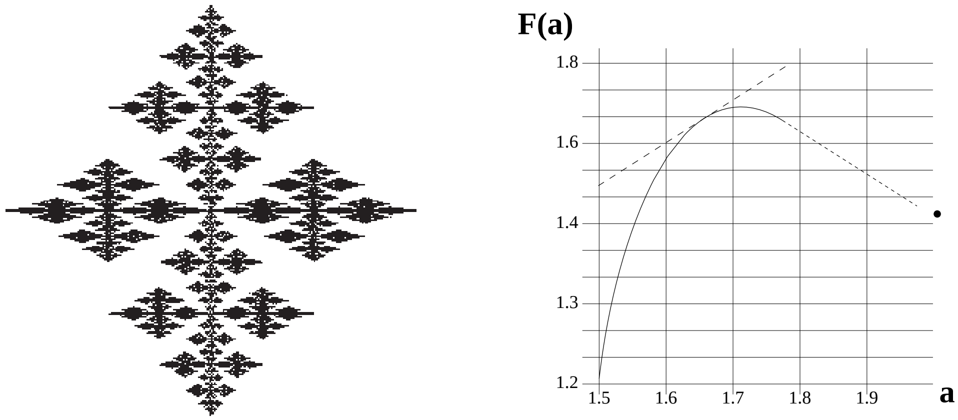

Figure 8: An image of a self-affine measure which resembles a 'rosette'. The phase transition is at its extreme: $\tau(q)$ is not differentiable which results in the linear part or gap of the spectrum. This is combined with a trivial continuation, i.e. $\tau(q)$ becomes linear and the rest of the spectrum reduces to a point.

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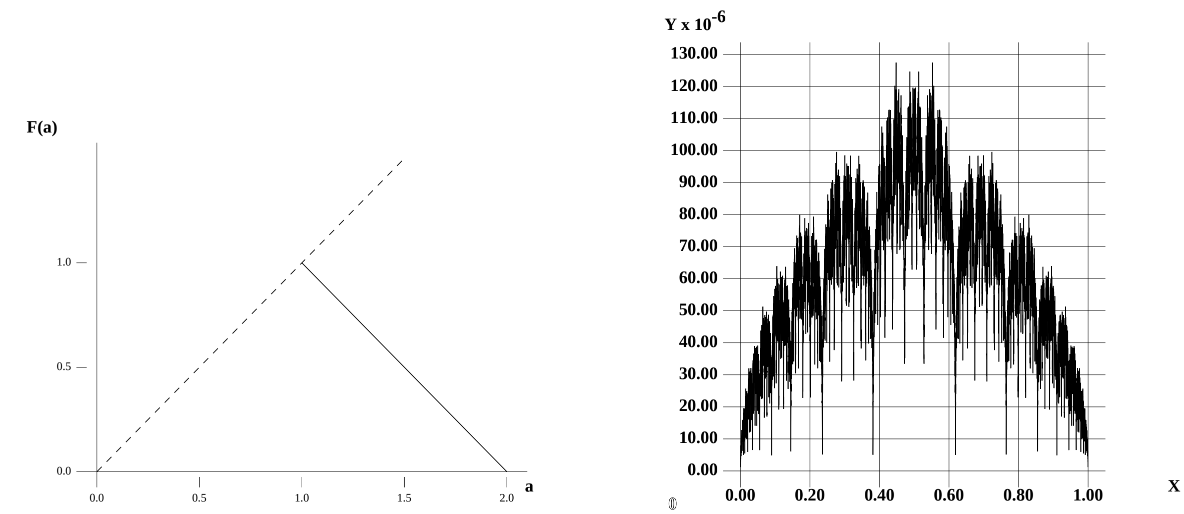

Figure 9: On the left the spectrum f_{G} of the measure with density $\simeq t^h$, i.e. $\mu_h([0,t] = t^{h+1})$, for $h = 1$. Note that f_{H} reduces to two points $(1,1)$ and $(h+1,0)$. On the right the Bernoulli convolution ν_r with $r = .617$ [RM4] which has spectra of the same shape as μ_h with $h = -\log(2r)/\log r$. Though regular it has a 'wild' appearance. Indeed, one of his close 'brothers' (ν_r with $r = (\sqrt{5}-1)/2 = .618$, the golden mean) is singular with a nontrivial spectrum.

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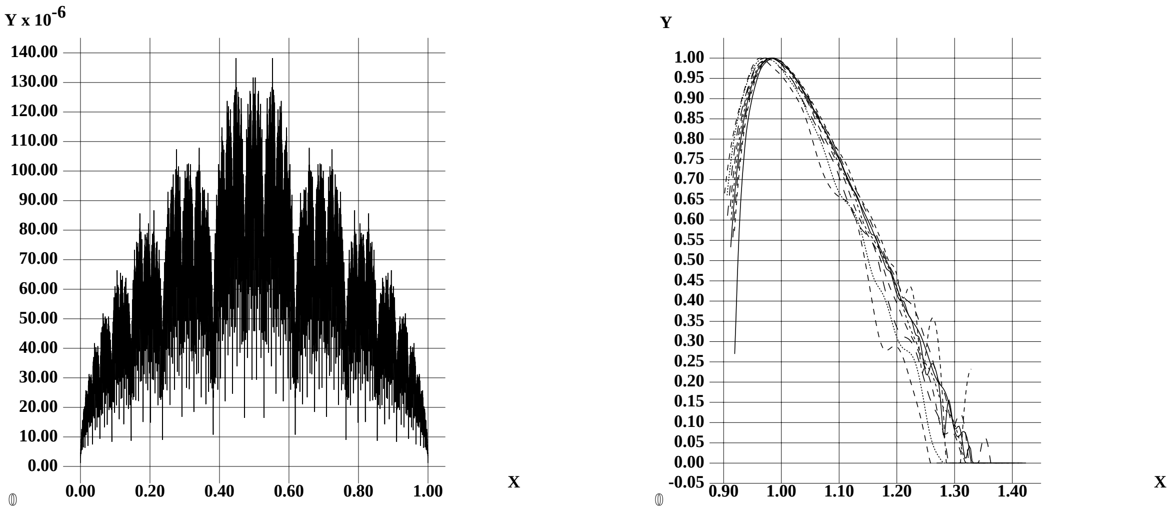

Figure 10: The Bernoulli convolution ν_r [RM4] with $r = (\sqrt{5}-1)/2 = .618$ has a nontrivial spectrum as is demonstrated numerically with the collapsing estimations $f_{\mathrm{G}}{}^{\varepsilon}$ at various scales $\varepsilon = 2^{-16}$, $\varepsilon = 2^{-15} \ldots \varepsilon = 2^{-8}$ on the right. In particular, the spectrum is strictly concave which agrees with the fact that the multifractal formalism holds [LN].

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