Technical Notes and Correspondence

Instability of Feedback Systems

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Abstract—**A generalization is given of a result due to Takeda and Bergen [1].**

In this note we give a generalization of a "passivity-type" of instability result due to Takeda and Bergen [1]. Basically, we derive the same result as in [1], but subject to fewer assumptions.

Takeda and Bergen consider a feedback system described by

$$e_1 = u_1 - y_2 \tag{1}$$

$$e_2 = u_2 + y_1 \tag{2}$$

$$y_1 = G_1 e_1 \tag{3}$$

$$y_2 = G_2 e_2 \tag{4}$$

where e_1, e_2, y_1, y_2, u_1, u_2 belong to the extended functional space $L_{2e} = L_{2e}[0,\infty)$ (see [2] for definition of terms), and G_1, G_2, map L_{2e} into itself. To be specific, it is assumed that for each $u_1, u_2 \in L_2$ there exist $e_1, e_2, y_1, y_2 \in L_{2e}$ such that (1)–(4) hold. The system (1)–(4) is said to be *stable* if whenever $u_1, u_2 \in L_2$ (the unextended space), any corresponding $e_1, e_2, y_1, y_2 \in L_{2e}$ such that (1)–(4) hold, actually belong to L_2. The system (1)–(4) is said to be *unstable* otherwise.

The objective is to derive conditions on G_1 and G_2 which insure the instability of the system (1)–(4). For this purpose, the following assumptions are made in [1]:

Assumption 1: There is a family of constants (α_T, $T \in [0,\infty)$) such that

$$\|(G_1 x)_T\| \leqslant \alpha_T \|x_T\|, \qquad T \in [0,\infty), \quad x \in L_{2e} \tag{5}$$

where $(\cdot)_T$ denotes the truncation [2] of a function to the interval $[0,T]$.

Assumption 2: Define

$$M_1 = \{x \in L_2 : G_1 x \in L_2\}. \tag{6}$$

Then there is a finite constant γ such that

$$\|G_1 x\| \leqslant \gamma \|x\|, \qquad \forall x \in M. \tag{7}$$

Assumption 3: G_1 is linear.

Assumption 4: M_1 is a proper subset of L_2.

Assumptions 1–4 are technical assumptions which imply that the operator G_1 is unstable in a particular way. Specifically [1], [2], if Assumptions 1–4 hold, then M_1 is a proper closed subspace of L_2 so that $M_1^\perp$ (the orthogonal complement of M_1) contains some nonzero elements.

In proving a "passivity-type" instability result, Takeda and Bergen make the following additional assumptions:

Assumption 5: There exists a constant ϵ such that

$$\langle x, G_x \rangle \geqslant \epsilon \|x\|^2, \qquad \forall x \in M_1 \tag{8}$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product on L_2.

Assumption 6: G_2 maps L_2 into itself.

Assumption 7: G_2 satisfies

$$G_2 x = 0 \Rightarrow x = 0. \tag{9}$$

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Assumption 8: There exists a constant δ such that

$$\langle x, G_2 x \rangle \geqslant \delta \|G_2 x\|^2. \tag{10}$$

Assumption 9: ϵ and δ together satisfy

$$\epsilon + \delta > 0. \tag{11}$$

In [1], it is shown that if Assumptions 1–9 hold, then the system (1)–(4) is unstable, and that, in particular, $y_1 \notin L_2$ whenever $u_1 = 0$ and $u_2 \in M_1^\perp/\{0\}$.

We now come to the objective of this note. In [1] a large number of assumptions are necessitated because the ultimate conclusion is that $y_1 \notin L_2$ for a certain class of inputs. However, instability only requires that *either* y_1 *or* y_2 does not belong to L_2 for a certain class of inputs. In Theorem 1 below, this fact is exploited to eliminate some assumptions. A further reduction in the assumptions is obtained through a new method of proof.

Theorem 1: Suppose Assumptions 1–5 and Assumptions 8 and 9 hold. Suppose also that Assumption 10 is true.

Assumption 10:

$$G_2 x = 0, \qquad x \in M_1^\perp, \Rightarrow x = 0. \tag{12}$$

Then either $y_1 \notin L_2$ or $y_2 \notin L_2$, whenever $u_1 = 0$ and $u_2 \in M_1^\perp/\{0\}$.

Proof: Let $u_1 = 0$, $u_2 \in M_1^\perp/\{0\}$, and assume by way of contradiction that $y_1 \in L_2$, $y_2 \in L_2$. Then it follows that $e_1 \in L_2$, $e_2 \in L_2$. By Assumption 2, this implies that $e_1 \in M_1$. Next, pick $\alpha > 0$ sufficiently small so that

$$\epsilon - \alpha + \delta > 0. \tag{13}$$

This is possible in view of Assumption 9. Now, from Assumptions 5 and 2 , we have that for all $x \in M_1$,

$$\langle x, G_1 x \rangle \geqslant \epsilon \|x\|^2 \geqslant (\epsilon - \alpha)\|x\|^2 + \alpha \|x\|^2 \geqslant (\epsilon - \alpha)\|x\|^2 + (\alpha/\gamma^2)\|Gx\|^2. \tag{14}$$

From the system equations, we have

$$\langle y_1, e_1 \rangle + \langle y_2, e_2 \rangle = \langle e_2 - u_2, e_1 \rangle + \langle u_1 - e_1, e_2 \rangle = \langle -u_2, e_1 \rangle + \langle u_1, e_2 \rangle = 0 \tag{15}$$

because $u_1 = 0$, $u_2 \in M_1^\perp$, and $e_1 \in M^\perp$. On the other hand, by (10) and (14), we have

$$\langle y_1, e_1 \rangle + \langle y_2, e_2 \rangle \geqslant (\epsilon - \alpha)\|e_1\|^2 + (\alpha/\gamma^2)\|y_1\|^2 + \delta \|y_2\|^2. \tag{16}$$

Replacing e_1 by $-y_2$, and using (15) gives

$$0 \geqslant (\epsilon - \alpha + \delta)\|y_2\|^2 + (\alpha/\gamma^2)\|y_1\|^2 \tag{17}$$

from which it follows that

$$y_1 = 0, \quad y_2 = 0. \tag{18}$$

Hence, we have

$$y_2 = 0, \quad e_2 = y_1 + u_2 \in M_1^\perp$$

which, by Assumption 10, implies that $e_2 = 0$. However, this is a contradiction since $u_2 \neq 0$. This shows that the original assumption is wrong, whence either $y_1 \notin L_2$ or $y_2 \notin L_2$. Q.E.D.

Comparing Theorem 1 with the earlier result in [1], we note two differences: 1) Assumption 6 is eliminated altogether and 2) Assumption 7 is replaced by a weaker requirement, Assumption 10. Of these, the first difference arises because we are only concluding here that either y_1 or y_2 does not belong to L_2, while in [1] it is possible to conclude that $y_1 \notin L_2$. The second difference arises because we exploit the "conditional finite gain" assumption on G_1 to recast Assumption 5 in the form of (14). Checking back over the proof of Theorem 1, it is clear that if we add Assumption 6 to the hypotheses of Theorem 1, then we too can conclude that $y_1 \notin L_2$. In this case, we have a slight improvement over the result in [1], since [1, Assumption 7] is replaced by the present Assumption 10.

REFERENCES

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Direct Solution Method for $A_1E + EA_2 = -D$

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***Abstract*—A direct method—a method without truncation or convergence errors—for the solution of $A_1E+EA_2=-D$, where $A_1 \in \Re^{n_1\times n_1}$ and $A_2 \in \Re^{n_2 \times n_2}$, is described. The only assumption on the matrices A_1 and A_2 is that the spectra of A_1 and $-A_2$ be disjoint. The method requires storage for order $2n_1^2+3n_1n_2+2n_2^2$ variables and requires order $n_1(n_1^2+n_1n_2+n_2^2)n_2$ multiplications and divisions.**

I. INTRODUCTION

Consider the matrix differential equation

$$\dot{X} = A_1X + XA_2 + D \tag{1a}$$

with the initial condition

$$X(0) = C \tag{1b}$$

where $A_1 \in \Re^{n_1\times n_1}$ and $A_2 \in \Re^{n_2\times n_2}$. Assume the spectra of A_1 and $-A_2$ are disjoint; that is, assume $\mathcal{S}[A_1] \cap \mathcal{S}[-A_2] = \emptyset$. Then the solution of (1) is

$$X(t) = e^{A_1t}(C-E)e^{A_2t} + E \tag{2}$$

where E is the solution, which exists by the above assumption [1, p. 231], of the matrix algebraic equation

$$A_1E + EA_2 = -D. \tag{3}$$

In a recent paper, Davison [2] described an algorithm by which to obtain numerical values of the solution (2) for $t = nh$ $(n = 0,1,2,\cdots)$. He invoked the stronger assumption $\mathcal{S}[A_1] \subset \mathcal{L}$ and $\mathcal{S}[A_2] \subset \mathcal{L}$, where $\mathcal{L} = \{s: s \in \mathcal{C}$ and $\operatorname{Re}[s] < 0\}$. By this assumption, the solution of (3) can be expressed as [1, p. 175]

$$E = \int_0^\infty e^{A_1t} D e^{A_2t}\, dt. \tag{4}$$

Now, in his algorithm he adopts a Padé (2,2) approximation to the matrix exponential e^{A_ih} $(i = 1,2)$ and employs a forward Euler numerical integration method to evaluate (4).[1] The latter requires an iteration to convergence—the converge error bound being specified and thus known—as the interval of integration in (4) is infinite. The Padé approximation is invoked in two places, in the evaluation of (2) for $X(nh)$, when E is known, and in the evaluation of (4) for E. As to the evaluation of (4), the truncation error due to both the Padé approximation and the forward Euler method and the iteration in using the forward Euler method can be eliminated by adopting a direct solution method for (3). Furthermore, a direct solution method need not as Davison implies engender a need for greater computer storage space or central processor time.

Lacoss and Shakal [3] have shown one solution method which is reasonably efficient computationally. However, it is not, as they imply, a direct solution method for E. In addition to $\mathcal{S}[A_1] \cap \mathcal{S}[-A_2] = \emptyset$, they assume A_i $(i=1,2)$ is similar—with known similarity transformation matrix—to a diagonal matrix. (They suggest how to proceed when A_i is not similar to a diagonal matrix.) Thus, their method requires evaluation of the eigenvalues and eigenvectors of A_i. As this, in general, requires an iterative process [4, p. 485], their method fails in this respect to be a direct method.

The method, a direct method, herein proposed invokes no assumption other than that first given for (3) to possess a solution. Note: The method does require evaluation of an annihilating polynomial of A_1 or of A_2. This, though, can be accomplished without iteration [4].

II. DIRECT SOLUTION METHOD

Let $a_i(\lambda) = \lambda^{m_i} + a_{i1}\lambda^{m_i-1} + \cdots + a_{i\,m-1}\lambda + a_{i0}$ $(i=1,2)$ denote a monic annihilating polynomial of A_i. Note: The characteristic polynomial $c_i(\lambda)$ and the minimal polynomial $m_i(\lambda)$ are annihilating polynomials of A_i. Next, let

$$M_0 = 0 \tag{5a}$$

$$M_1 = -D \tag{5b}$$

and

$$M_k = A_1M_{k-1} - M_{k-1}A_2 + A_1M_{k-2}A_2, \qquad (k=2,3,\cdots). \tag{5c}$$

Then the solution of (3) can be expressed as

$$E = -\left[\sum_{k=1}^{m_2} a_{1\,m_1-k} M_k\right]\left[a_1(-A_2)\right]^{-1} \tag{6a}$$

or

$$E = \left[a_2(-A_1)\right]^{-1}\left[\sum_{k=1}^{m_2} a_{2\,m_2-k}(-1)^k M_k\right]. \tag{6b}$$

This method with $a_i(\lambda) = c_i(\lambda)$—hence, also $m_i = n_i$—is due to Jameson [5]. The proof for this somewhat more general form is the same as that given by Jameson, because he invoked only the annihilating polynomial property of $c_i(\lambda)$ in his proof.

III. FURTHER DISCUSSION

The number of multiplications (and divisions) required to evaluate E by (6a)—set $i=1$—and by (6b)—set $i=2$—is

$$m_in_j^3 + (m_i^2 - m_i + n_i)n_j^2 + (m_i^3 - m_i + 1)n_j - 1 \tag{7}$$

where $j = 2$ (alternatively 1) when $i = 1$ (alternatively 2). If LU decomposition rather than direct inversion is used, the number of multiplications is somewhat less, namely,

$$\left(m_i - \tfrac{2}{3}\right)n_j^3 + \left(m_i^2 - m_i + n_i\right)n_j^2 + \left(m_i^3 - m_i + \tfrac{2}{3}\right)n_j. \tag{8}$$

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[1] There is an order inconsistency here in that the Padé approximation is of order 4 (in h) and the forward Euler method is of order 1.