Optimal and Hierarchical Controls in Dynamic Stochastic Manufacturing Systems: A Survey

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Abstract

Most manufacturing systems are large and complex and operate in an uncertain environment. One approach to managing such systems is that of hierarchical decomposition. This paper reviews the research devoted to proving that a hierarchy based on the frequencies of occurrence of different types of events in the systems results in decisions that are asymptotically optimal as the rates of some events become large compared to those of others. The paper also reviews the research on stochastic optimal control problems associated with manufacturing systems, their dynamic programming equations, existence of solutions of these equations, and verification theorems of optimality for the systems. Manufacturing systems that are addressed include single machine systems, dynamic flowshops, and dynamic jobshops producing multiple products. These systems may also incorporate random production capacity and demands, and decisions such as production rates, capacity expansion, and promotional campaigns. Related computational results and areas of applications are also presented.

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1 Introduction

Most manufacturing firms are large, complex systems characterized by several decision subsystems, such as finance, personnel, marketing, and operations. They may have a number of plants and warehouses, and they produce a large number of different products using a wide variety of machines and equipment. Moreover, these systems are subject to discrete events such as construction of new facilities, purchase of new equipment and scrappage of old, machine setups, failures, and repairs, and new product introductions. These events could be deterministic or stochastic. Management must recognize and react to these events. Because of the large size of these systems and the presence of these events, obtaining exact optimal policies to run these systems is nearly impossible both theoretically and computationally.

One way to cope with these complexities is to develop methods of hierarchical decision making for these systems. The idea is to reduce the overall complex problem into manageable approximate problems or subproblems, to solve these problems, and to construct a solution for the original problem from the solutions of these simpler problems.

There are several different (and not mutually exclusive) ways by which to reduce the complexity. These include decomposing the problem into problems of smaller subsystems with a proper coordinating mechanism; aggregating products and subsequently disaggregating them; replacing random processes with their averages and possibly other moments; modeling uncertainties in the production planning problem via diffusion processes; and so on. Development of such approaches for large, complex systems was identified as a particular fruitful research area by the *Committee on the Next Decade in Operations Research* (1988), as well as by the *Panel on Future Directions in Control Theory* chaired by Fleming (1988). A great deal of research has been conducted in the areas of Operations Research, Operations Management, Systems Theory, and Control Theory. For their importance in practice, see the surveys of the literature by Libosvar (1988), Rogers et al. (1991), Bitran and Tirupati (1993), and Cheng (1999), a bibliography complied by Bukh (1992), and books by Stadtler (1988) and Switalski (1989). Some other references on hierarchical systems are Simon (1962), Mesarovic et al. (1970), Smith and Sage (1973), Singh (1982), Saksena et al. (1984), and Auger (1989). It should be noted, however, that most of them concern deterministic systems.

Each approach mentioned above is suited to certain types of models and assumptions. The approach we shall first discuss is that of modeling uncertainties in the production planning problem via diffusion processes. The idea is initiated by Sethi and Thompson (1981a, b) and Bensoussan et al. (1984). Because controlled diffusion problems can often be solved (see Ghosh et al. (1993, 1997), Harrison and Taksar (1983), and Harrison et al. (1983)), one uses the controlled diffusion models to approximate stochastic manufacturing systems. Kushner and Ramachandran (1989) begin with a sequence of systems whose limit is a controlled diffusion process. It should be noted that the traffic intensities of the systems in sequence converge to the critical intensity of one. They show that the sequence of value functions associated with the given sequence converges to the value function of the limiting problem. This enables them to construct a sequence of asymptotic optimal policies defined to be those for which the difference between the associated cost and the value function converges to zero as the traffic intensity approaches its critical value. The most important application of this approach concerns the scheduling of networks of queues. If a network of queues is operating under heavy traffic, that is, when the rate of customers entering some of the stations in the network is very close to the rate of service at those stations, the problem of scheduling the network can be approximated by a dynamic control problem involving diffusion processes. The optimal policies that are obtained for the dynamic control problem involving diffusion approximation are interpreted in terms of the original problem. A justification of this procedure based on simulation is provided in Harrison and Wein (1989, 1990), Wein (1990), and Kumar and Kumar (1994), for example; see also the survey on fluid models and strong approximations by Chen and Mandelbaum (1994). Furthermore, Krichagina et al. (1993) and Krichagina et al. (1994) apply this approach to the problem of controlling the production rate of a single product using a single unreliable machine in order to minimize the total discounted inventory/backlog costs. They imbed the given system into a sequence of systems in heavy traffic. Their purpose is to obtain asymptotic optimal policies for the sequence of systems that can be expressed only in terms of the parameters of the original system.

It should be noted that these approaches do not provide us with an estimate of how much the policies constructed for the given original system deviate from the optimal solution, especially when the optimal solution is not known, which is most often the case. As we shall see later, the hierarchical approach under consideration in this survey enables one to provide just such an estimate in many cases.

The next approach we shall discuss is that of aggregation-disaggregation. Bitran et al. (1986) formulate a model of a manufacturing system in which uncertainties arise from demand estimates and forecast revisions. They consider first a two-level product hierarchical structure, which is characterized by families and items. Hence, the production planning decisions consist of determining the sequence of the product families and the production lot sizes for items within each family, with the objective of minimizing the total cost. Then, they consider demand forecasts and forecast revisions during the planning horizon. The authors assume that the mean demand for each family is invariant and that the planners can estimate the improvement in the accuracy of forecasts, which is measured by the standard deviation of forecast errors. Bitran et al. (1986) view the problem as a two-stage hierarchical production planning problem. The aggregate problem is formulated as a deterministic mixed integer program that provides a lower bound on the optimal solution. The solution to this problem determines the set of product families to be produced in each period. The second-level problem is interpreted as the disaggregate stage where lot sizes are determined for the individual product to be scheduled in each period. Only a heuristic justification has been provided for the approach described. Some other references in the area are Bitran and Hax (1977), Hax and Candea (1984), Gelders and Van Wassenhove (1981), Ari and Axsåter (1988), and Nagi (1991).

Lasserre and Mercé (1990) assume that the aggregate demand forecast is deterministic, while the detailed level forecast is nondeterministic within known bounds. Their aim is to obtain an aggregate plan for which there exists a feasible dynamic disaggregation policy. Such an aggregate plan is called a *robust* plan, and they obtain necessary and sufficient conditions for robustness; see also Gfrerer and Zäpfel (1994).

Finally we consider the approach of *replacing random processes with their averages and possibly other moments*, see Sethi and Zhang (1994a, 1998) and Sethi et al. (2000e). The idea of the approach is to derive a limiting control problem which is simpler to solve than the given original problem. The limiting problem is obtained by replacing the stochastic machine capacity process by the average total capacity of machines and by appropriately modifying the objective function. The solution of this problem provides us with longer-term decisions. Furthermore, given these decisions, there are a number of ways by which we can construct short-term production decisions. By combining these decisions, we create an approximate solution of the original, more complex problem.

The specific points to be addressed in this review are results on the asymptotic optimality of the constructed solution and the extent of the deviation of its cost from the optimal cost for the original problem. The significance of these results for the decision-making hierarchy is that management at the highest level of the hierarchy can ignore the day-to-day fluctuation in machine capacities, or more generally, the details of shop floor events, in carrying out long-term planning decisions. The lower operational level management can then derive approximate optimal policies for running the actual (stochastic) manufacturing system.

While the approach could be extended for applications in other areas, the purpose here is to review models of a variety of representative manufacturing systems in which some of the exogenous processes, deterministic or stochastic, are changing much faster than the remaining ones, and to apply the methodology of hierarchical decision making to them. We are defining a fast changing process as a process that is changing so rapidly that from any initial condition, it reaches its stationary distribution in a time period during which there are few, if any, fluctuations in the other processes.

In what follows we review applications of the approach to stochastic manufacturing problems, where the objective function is to minimize a total discounted cost, a long-run average cost, or a risk-sensitive criterion. We also summarize results on dynamic programming equations, existence of their solutions, and verification theorems of optimality for single/parallel machine systems, dynamic flowshops, and dynamic jobshops producing multiple products. Sections 2 and 3 are devoted to discounted cost models. In Section 2, we review the existence of solutions to the dynamic programming equations associated with stochastic manufacturing systems with the discounted cost criterion. The verification theorems of optimality and the characterization of optimal controls are also given. Section 3 discusses the results on open-loop and/or feedback hierarchical controls that have been developed and shown to be asymptotically optimal for the systems. The computational issues are also included in this section. Sections 4 and 5 are devoted to average cost models. In Section 4, we review the existence of solutions to the ergodic equations corresponding to stochastic manufacturing systems with the long-run average cost criterion and the corresponding verification theorems and the characterization of optimal controls. Section 5 surveys hierarchical controls for single machine systems, flowshops, and jobshops with the long-run average cost criterion or the risk-sensitive long-run average cost criterion. Markov decision processes with weak and strong interactions are also included. Important insights have been gained from the research reviewed here, see Sethi (1997). Some of these insights are given where appropriate. Section 6 concludes the paper.

2 Optimal Control with the Discounted Cost Criterion

The class of convex production planning models is an important paradigm in the operations management/operations research literature. The earliest formulation of a convex production planning problem in a discrete-time framework dates back to Modigliani and Hohn (1955). They were interested in obtaining a production plan over a finite horizon in order to satisfy a deterministic demand and minimize the total discounted convex costs of production and inventory holding. Since then the model has been further studied and extended in both continuous-time and discrete-time frameworks by a number of researchers, including Johnson (1957), Arrow et al.(1958), Veinott (1964), Adiri and Ben-Israel (1966), Sprzeuzkouski (1967), Lieber (1973), and Hartl and Sethi (1984). A rigorous formulation of the problem along with a comprehensive discussion of the relevant literature appears in Bensoussan et al.(1983).

Extensions of the convex production planning problem to incorporate stochastic demand have been analyzed mostly in the discrete-time framework. A rigorous analysis of the stochastic problem has been carried out in Bensoussan et al. (1983). Continuous-time versions of the model that incorporate additive white noise terms in the dynamics of the inventory process were analyzed by Sethi and Thompson (1981a) and Bensoussan et al. (1984).

Earlier works that relate most closely to problems under consideration here include Kimemia and Gershwin (1983), Akella and Kumar (1986), Fleming et al. (1987), Sethi et al. (1992a), and Lehoczky et al. (1991). These works incorporate piecewise deterministic processes (PDP) either in the dynamics or in the constraints of the model. Fleming et al. (1987) consider the demand to be a finite state Markov process. In the models of Kimemia and Gershwin (1983), Akella and Kumar (1986), Sethi et al. (1992a) and Lehoczky et al. (1991), the production capacity rather than the demand for production is modeled as a stochastic process. In particular, the process of machine breakdown and repair is modeled as a birth-death process, thus making the production capacity over time a finite state Markov process. Feng and Yan (2000) incorporate a Markovian demand in a discrete state version of the model of Akella and Kumar (1986).

Here we will discuss the optimality of single/parallel machine systems, N-machine flowshops, and general jobshops.

2.1 Single or parallel machine systems

Akella and Kumar (1986) deal with a single machine (with two states: up and down), single product problem. They obtained an explicit solution for the threshold inventory level, in terms of which the optimal policy is as follows: Whenever the machine is up, produce at the maximum possible rate if the inventory level is less than the threshold, produce on demand if the inventory level is exactly equal to the threshold, and not produce at all if the inventory level exceeds the threshold. When their problem is generalized to convex costs and more than two machine states, it is no longer possible to obtain an explicit solution. Using the viscosity solution technique, Sethi et al. (1992a) investigate this general problem. They study the elementary properties of the value function. They show that the value function is a convex function, and that it is strictly convex provided the inventory cost is strictly convex. Moreover, it is shown to be a viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation and to have upper and lower bounds each with polynomial growth. Following the idea of Thompson and Sethi (1980), they define what are known as the turnpike sets in terms of the corresponding value function. They prove that the turnpike sets are attractors for the optimal trajectories and provide sufficient conditions under which the optimal trajectories enter the convex closure in finite time. Also, they give conditions to ensure that the turnpike sets are non-empty.

To more precisely state their results, we need to specify the model of a single/parallel machine manufacturing system. Let $\boldsymbol{x}(t)$, $\boldsymbol{u}(t)$, $\boldsymbol{z}$, and $m(t)$ denote, respectively, the surplus (inventory/shortage) level, the production rate, the demand rate, and the machine capacity level at time $t \in [0, \infty)$. We assume shortages to be backlogged. Here and throughout the paper, vectors will be denoted by bold-faced letters. We assume that $\boldsymbol{x}(t) \in R^n$, $\boldsymbol{u}(t) \in R^n_+$, (i.e., $\boldsymbol{u}(t) \geq 0$), and $\boldsymbol{z}$ is a constant positive vector in R^n_+. Furthermore, we assume that $m(\cdot)$ is a Markov process with a finite space $\mathcal{M} = \{0, 1, ..., p\}$. We can now write the dynamics of the system as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(t) - \boldsymbol{z}, \quad \boldsymbol{x}(0) = \boldsymbol{x}. \tag{2.1}$$

Definition 2.1. A control process (production rate) $\boldsymbol{u}(\cdot) = \{\boldsymbol{u}(t) : t \geq 0\}$ is called *admissible* with respect to the initial capacity m if (i) $\boldsymbol{u}(\cdot)$ is history-dependent or, more precisely, adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$ with $\mathcal{F}_t = \sigma\{m(s) : 0 \leq s \leq t\}$, the σ-field generated by $m(t)$; (ii) $0 \leq \langle \boldsymbol{r}, \boldsymbol{u}(t) \rangle \leq m(t)$ for all $t \geq 0$ for some positive vector $\boldsymbol{r}$, where $\langle \cdot, \cdot \rangle$ between $\boldsymbol{r}$ and $\boldsymbol{u}(t)$ represents inner product of vectors $\boldsymbol{r}$ and $\boldsymbol{u}(t)$.

Let $\mathcal{A}(m)$ denote the set of all admissible control processes with the initial condition $m(0) = m$.

Definition 2.2. A real-valued function $\boldsymbol{u}(\boldsymbol{x}, m)$ on $R^n \times \mathcal{M}$ is called an *admissible feedback control*, or simply a *feedback control*, if (i) for any given initial $\boldsymbol{x}$, the equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(\boldsymbol{x}(t), m(t)) - \boldsymbol{z}, \quad \boldsymbol{x}(0) = \boldsymbol{x}$, has a unique solution; (ii) $\boldsymbol{u}(\cdot) = \{\boldsymbol{u}(t) = \boldsymbol{u}(\boldsymbol{x}(t), m(t)) : t \geq 0\} \in \mathcal{A}(m)$.

Let $h(\boldsymbol{x})$ and $c(\boldsymbol{u})$ denote the surplus cost and the production cost functions, respectively. For every $\boldsymbol{u}(\cdot) \in \mathcal{A}(m)$, $\boldsymbol{x}(0) = \boldsymbol{x}$, and $m(0) = m$, define the cost criterion

$$J(\boldsymbol{x}, m, \boldsymbol{u}(\cdot)) = E \int_0^\infty e^{-\rho t} [h(\boldsymbol{x}(t)) + c(\boldsymbol{u}(t))] dt, \tag{2.2}$$

where $\rho > 0$ is the given discount rate. The problem is to choose an admissible control $\boldsymbol{u}(\cdot)$ that minimizes $J(\boldsymbol{x}, m, \boldsymbol{u}(\cdot))$. We define the value function as

$$v(\boldsymbol{x}, m) = \inf_{\boldsymbol{u}(\cdot) \in \mathcal{A}(m)} J(\boldsymbol{x}, m, \boldsymbol{u}(\cdot)). \tag{2.3}$$

We make the following assumptions on the cost functions $h(\boldsymbol{x})$ and $c(\boldsymbol{u})$.

(A.2.1) $h(\boldsymbol{x})$ is nonnegative and convex with $h(0) = 0$. There are positive constants C_{21}, C_{22}, C_{23}, $\kappa_{21} \geq 0$, and $\kappa_{22} \geq 0$ such that $C_{21}|\boldsymbol{x}|^{\kappa_{21}} - C_{22} \leq h(\boldsymbol{x}) \leq C_{23}(1 + |\boldsymbol{x}|^{\kappa_{22}})$.

(A.2.2) $c(\boldsymbol{u})$ is nonnegative, $c(0) = 0$, and $c(\boldsymbol{u})$ is twice differentiable. Moreover, $c(\boldsymbol{u})$ is either strictly convex or linear.

(A.2.3) $m(\cdot)$ is a finite state Markov chain with generator Q, where $Q = (q_{ij})$, $i, j \in \mathcal{M}$ is a $(p+1) \times (p+1)$ matrix such that $q_{ij} \geq 0$ for $i \neq j$ and $q_{ii} = -\sum_{i \neq j} q_{ij}$. That is, for any function $f(\cdot)$ on $\mathcal{M}$,

$$Qf(\cdot)(m) = \sum_{\ell \neq m} q_{m\ell}[f(\ell) - f(m)].$$

With these three assumptions we can state the following theorem concerning the properties of the value function $v(\cdot, \cdot)$, proved in Fleming et al. (1987).

Theorem 2.1. (i) *For each m, $v(\cdot, m)$ is convex on R^n, and $v(\cdot, m)$ is strictly convex if $h(\cdot)$ is so*; (ii) *There exist positive constants C_{24}, C_{25}, and C_{26} such that for each m, $C_{24}|\boldsymbol{x}|^{\kappa_{21}} - C_{25} \leq v(\boldsymbol{x}, m) \leq C_{26}(1 + |\boldsymbol{x}|^{\kappa_{22}})$.*

We next consider the HJB equation associated with the problem. Let $F(m, \boldsymbol{w}) = \inf\{\langle \boldsymbol{u} - \boldsymbol{z}, \boldsymbol{w}\rangle : 0 \le \langle \boldsymbol{u}, \boldsymbol{r}\rangle \le m\}$, where $\boldsymbol{r}$ is given in Definition 2.1. Then, the HJB equation is written formally as follows:

$$\rho v(\boldsymbol{x}, m) = F(m, v'_{\boldsymbol{x}}(\boldsymbol{x}, m)) + h(\boldsymbol{x}) + Qv(\boldsymbol{x}, \cdot)(m), \tag{2.4}$$

for $\boldsymbol{x} \in R^n$, $m \in \mathcal{M}$, where $v'_{\boldsymbol{x}}(\boldsymbol{x}, m)$ is the partial derivative (gradient) of $v(\cdot, \cdot)$ with respect to $\boldsymbol{x}$.

In general, the value function $v(\boldsymbol{x}, m)$ may not be differentiable. In order to make sense of the HJB equation (2.4), we consider its *viscosity solution*, see Fleming and Soner (1992). To define a viscosity solution, we first introduce the superdifferential and subdifferential of a given function $f(\boldsymbol{x})$ on R^n.

Definition 2.3. The *superdifferential* $D^+ f(\boldsymbol{x})$ and the *subdifferential* $D^- f(\boldsymbol{x})$ of any function $f(\boldsymbol{x})$ on R^n are defined, respectively, as follows:

$$D^+ f(\boldsymbol{x}) = \left\{ s \in R^n : \limsup_{|\boldsymbol{r}| \to 0} \frac{f(\boldsymbol{x} + \boldsymbol{r}) - f(\boldsymbol{x}) - \langle \boldsymbol{r}, s\rangle}{|\boldsymbol{r}|} \le 0 \right\},$$
$$D^- f(\boldsymbol{x}) = \left\{ s \in R^n : \liminf_{|\boldsymbol{r}| \to 0} \frac{f(\boldsymbol{x} + \boldsymbol{r}) - f(\boldsymbol{x}) - \langle \boldsymbol{r}, s\rangle}{|\boldsymbol{r}|} \ge 0 \right\}.$$

Definition 2.4. We say that $v(\boldsymbol{x}, m)$ is a *viscosity solution* of equation (2.4) if the following holds: (i) $v(\boldsymbol{x}, m)$ is continuous in $\boldsymbol{x}$ and there exist $C_{27} > 0$ and $\kappa_{23} > 0$ such that $|v(\boldsymbol{x}, m)| \le C_{27}(1 + |\boldsymbol{x}|^{\kappa_{23}})$; (ii) for all $\boldsymbol{n} \in D^+ v(\boldsymbol{x}, m)$, $\rho v(\boldsymbol{x}, m) - F(m, v'_{\boldsymbol{x}}(\boldsymbol{x}, m)) + h(\boldsymbol{x}) + Qv(\boldsymbol{x}, \cdot)(m) \le 0$; and (iii) for all $\boldsymbol{n} \in D^- v(\boldsymbol{x}, m)$, $\rho v(\boldsymbol{x}, m) - F(m, v'_{\boldsymbol{x}}(\boldsymbol{x}, m)) + h(\boldsymbol{x}) + Qv(\boldsymbol{x}, \cdot)(m) \ge 0$.

Lehoczky et al. (1991) prove the following theorem.

Theorem 2.2. *The value function $v(\boldsymbol{x}, m)$ defined in* (2.3) *is the unique viscosity solution to the HJB equation* (2.4).

Remark 2.1. If there is a continuously differentiable function that satisfies the HJB equation (2.4), then it is a viscosity solution, and therefore, it is the value function. Furthermore, we have the following result.

Theorem 2.3. *The value function $v(\cdot, m)$ is continuously differentiable and satisfies the HJB equation* (2.4).

For its proof, see Theorem 3.1 in Sethi and Zhang (1994a). Next, we give a verification theorem.

Theorem 2.4. (Verification Theorem) *Suppose that there is a continuously differentiable function $\hat{v}(\boldsymbol{x}, m)$ that satisfies the HJB equation* (2.4). *If there exists $\boldsymbol{u}^*(\cdot) \in \mathcal{A}(m)$, for which the* *corresponding* $\boldsymbol{x}^*(t)$ *satisfies* (2.1) *with* $\boldsymbol{x}^*(0) = \boldsymbol{x}$, $\boldsymbol{w}^*(t) = \hat{v}'_{\boldsymbol{x}}(\boldsymbol{x}^*(t), m(t))$, *and* $F(m(t), \boldsymbol{w}^*(t)) = \langle \boldsymbol{u}^*(t) - \boldsymbol{z}, \boldsymbol{w}^*(t)\rangle + c(\boldsymbol{u}^*(t))$, *almost everywhere in* t *with probability one, then* $\hat{v}(\boldsymbol{x}, m) = v(\boldsymbol{x}, m)$ *and* $\boldsymbol{u}^*(t)$ *is optimal, i.e.,* $\hat{v}(\boldsymbol{x}, m) = v(\boldsymbol{x}, m) = J(\boldsymbol{x}, m, \boldsymbol{u}^*(\cdot))$.

For its proof, see Lemma H.3 of Sethi and Zhang (1994a). We now give an application of the verification theorem. With Assumption (A.2.2), we can use the verification theorem to derive an optimal feedback control for $n = 1$. From Theorem 2.4, an optimal feedback control $u^*(x, m)$ must minimize $(u - z)v'_x(x, m) + c(u)$. Thus,

$$u^*(x,m) = \begin{cases} 0 & \text{if } \; v'_x(x,m) \geq 0 \\ (\dot{c})^{-1}(-v'_x(x,m)) & \text{if } \; -\dot{c}(m) \leq v'_x(x,m) < 0 \\ m & \text{if } \; v'_x(x,m) < -\dot{c}(m), \end{cases}$$

when the second derivative of $c(u)$ is strictly positive, and

$$u^*(x,m) = \begin{cases} 0 & \text{if } \; v'_x(x,m) > -c \\ \min\{z, m\} & \text{if } \; v'_x(x,m) = -c \\ m & \text{if } \; v'_x(x,m) < -c, \end{cases}$$

when $c(u) = cu$ for some constant $c \geq 0$.

Recall that $v(\cdot, m)$ is a convex function. Thus, $u^*(x, m)$ is increasing in x. From a result on differential equations (see Hartman (1982)), $\dot{x}(t) = u^*(x(t), m(t)) - z$, $x(0) = x$, has a unique solution $x^*(t)$ for each sample path of the capacity process. Hence, the control given above is the *optimal feedback control*. From this application, we can see that the points satisfying $v'_x(x, m) = -\dot{c}(z)$ are critical in describing the optimal feedback control. So we give the following definition.

Definition 2.5. The *turnpike set* $\mathcal{G}(m, z)$ is defined by $\mathcal{G}(m, z) = \{x \in R : v'_x(x, m) = -\dot{c}(z)\}$.

Next we will discuss the monotonicity of the turnpike set. To do this, define $i_0 \in \mathcal{M}$ to be such that $i_0 < z < i_0 + 1$. Observe that for $m \leq i_0$, $\dot{x}(t) \leq m - z \leq i_0 - z < 0$. Therefore, $x(t)$ goes to $-\infty$ monotonically as $t \to \infty$, if the capacity state m is absorbing. Hence, only those $m \in \mathcal{M}$, for which $m \geq i_0 + 1$, are of special interest to us.

In view of Theorem 2.1, if $h(\cdot)$ is strictly convex, then each turnpike set reduces to a singleton, i.e., there exists an x_m such that $\mathcal{G}(m, z) = \{x_m\}$, $m \in \mathcal{M}$. If the production cost is linear, i.e., $c(u) = cu$ for some constant c, then x_m is the threshold inventory level with capacity m. Specifically, if $x > x_m$, $u^*(x, m) = 0$, and if $x < x_m$, $u^*(x, m) = m$ (full *available* capacity).

Let us make the following observation. If the capacity $m > z$, then the optimal trajectory will move toward the turnpike set x_m. Suppose the inventory level is x_m for some m and the capacity increases to $m_1 > m$; it then becomes costly to keep the inventory at level x_m, since a lower inventory level may be more desirable given the higher current capacity. Thus, we expect $x_{m_1} \leq x_m$. Sethi et al. (1992a) show that this intuitive observation is true. We state their result as the following theorem.

Theorem 2.5. *Assume $h(\cdot)$ to be differentiable and strictly convex. Then $x_{i_0} \geq x_{i_0+1} \geq \cdots \geq x_m \geq c_z$, where $c_z = (\dot{h})^{-1}(-\rho\dot{c}(z))$.*

2.2 Dynamic flowshops

We consider a dynamic flowshop that produces a single finished product using N machines in tandem that are subject to breakdown and repair. In comparison to the single/parallel machine systems, the flowshop problem with internal buffers and the resulting state constraints is much more complicated. Certain boundary conditions need to be taken into account for the associated HJB equation, see Soner (1986). Optimal control policy can no longer be described simply in terms of some hedging points. Lou et al. (1994) show that the optimal control policy for a two-machine flowshop with linear costs of production can be given in terms of two switching manifolds. However, the switching manifolds are not easy to obtain. One way to compute them is to approximate them by continuous piecewise-linear functions as done by Van Ryzin et al. (1993), in the absence of production costs. To rigorously deal with the general flowshop problem under consideration, the HJB equation in terms of the directional derivatives (HJBDD) at inner and boundary points are introduced by Presman et al. (1993, 1995). They show that the value function corresponding to the dynamic flowshop problem is a solution of the HJBDD equation. They also establish a verification theorem. Presman et al. (1997b) extend these results to dynamic flowshops with limited buffers.

Because dynamic flowshops are special cases of dynamic jobshops reviewed in detail in the next section, we will not discuss them in detail here separately.

2.3 Dynamic jobshops

Consider a manufacturing system producing a variety of products in demand using machines in a general network configuration, which generalizes both the parallel and the tandem machine models. Each product follows a process plan—possibly from a number of alternative process plans—that specifies the sequence of machines it must visit and the operations performed by them. A process plan may call for multiple visits to a given machine, as is the case in semiconductor manufacturing; Lou and Kager (1989), Srivatsan et al. (1994), Uzsoy et al. (1996), and Yan et al. (1994, 1996). Often the machines are unreliable. Over time they break down and must be repaired. A manufacturing system so described will be termed a *dynamic jobshop*. Now we give the mathematically description of a jobshop suggested by Presman et al. (1997a), as a revision of the description by Sethi and Zhou (1994). First we give some definitions.

Definition 2.6. A *manufacturing digraph* is a graph (Δ, Π), where Δ is a set of $N_b + 2$ (≥ 3), *vertices*, and Π is a set of ordered pairs called *arcs*, satisfying the following properties: (i) there is only one source, labeled 0, and only one sink, labeled $N_b + 1$, in the digraph; (ii) no vertex in the graph is isolated; and (iii) the digraph does not contain any cycle.

Remark 2.2. Condition (ii) is not an essential restriction. Inclusion of isolated vertices is merely a nuisance. This is because an isolated vertex is like a warehouse that can only ship out parts of a particular type to meet their demand. Since no machine (or production) is involved, its inclusion or exclusion does not affect the optimization problem under consideration. Condition (iii) is imposed to rule out the following two trivial situations: (a) a part of type i in buffer i gets processed on a machine without any transformation and returns to buffer i, and (b) a part of type i is processed and converted back into a part of type j, $j \neq i$, and is then processed further on a number of machines to be converted back into a part of type i. Moreover, if we had included any cycle in our manufacturing system, the flow of parts that leave buffer i only to return to buffer i would be zero in any optimal solution. It is unnecessary, therefore, to complicate the problem by including cycles.

Definition 2.7. In a manufacturing digraph, the source is called the supply node and the sink represents the customers. Vertices immediately preceding the sink are called *external buffers*, and all others are called *internal buffers*.

In order to obtain the system dynamics from a given manufacturing digraph, a systematic procedure is required to label the state and control variables. For this purpose, note that our manufacturing digraph (Δ, Π) contains a total of $N_b + 2$ vertices including the source, the sink, d internal buffers, and $N_b - d$ external buffers for some integer d and N_b, $0 \leq d \leq N_b - 1$, $N_b \geq 1$. The proof of the following theorem is similar to Theorem 2.2 in Sethi and Zhou (1994).

Theorem 2.6. *We can label all the vertices from 0 to N_b+1 in a way so that the label numbers of the vertices along every path are in a strictly increasing order, that is, the source is labeled 0, the sink is labeled N_b+1, and the external buffers are labeled $d+1, d+2, ..., N_b$.*

Definition 2.8. For each arc (i,j), $j \neq N_b+1$, in a manufacturing digraph, the rate at which parts in buffer i are converted to parts in buffer j is labeled as *control* u_{ij}. Moreover, the control u_{ij} associated with the arc (i,j) is called an *output* of i and an *input* to j. In particular, outputs of the source are called *primary controls* of the digraph. For each arc (i, N_b+1), $i = d+1, ..., N_b$, the demand for products in buffer i is denoted by z_i.

In what follows, we shall also set

$$\begin{aligned} u_{i,N_b+1} &= z_i,\ i = d+1, ..., N_b, \\ u_{i,j} &= 0,\ \text{for } (i,j) \notin \Pi,\ 0 \leq i \leq N_b \text{ and } 1 \leq j \leq N_b+1, \end{aligned}$$

for a unified notation suggested in Presman et al. (1997a). While z_i and $u_{i,j}$ for $(i,j) \notin \Pi$ are not controls, we shall for convenience refer to $u_{i,j}$, $0 \leq i \leq N_b$, $0 \leq j \leq N_b+1$, as controls. In this way, we can consider the controls as an $(N_b+1) \times (N_b+1)$ matrix $\boldsymbol{u} = (u_{ij})$ of the following form:

$$\begin{array}{ccccccccccccc}
u_{0,1} & u_{0,2} & ... & u_{0,i} & u_{0,i+1} & ... & u_{0,d} & u_{0,d+1} & ... & u_{0,N_b-1} & u_{0,N_b} & 0 \\
0 & u_{1,2} & ... & u_{1,i} & u_{1,i+1} & ... & u_{1,d} & u_{1,d+1} & ... & u_{1,N_b-1} & u_{1,N_b} & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & ... & u_{i-1,i} & u_{i-1,i+1} & ... & u_{i-1,d} & u_{i-1,d+1} & ... & u_{i-1,N_b-1} & u_{i-1,N_b} & \\
0 & 0 & ... & 0 & u_{i,i+1} & ... & u_{i,d} & u_{i,d+1} & ... & u_{i,N_b-1} & u_{i,N_b} & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & ... & 0 & 0 & ... & u_{d-1,d} & u_{d-1,d+1} & ... & u_{d-1,N_b-1} & u_{d-1,N_b} & 0 \\
0 & 0 & ... & 0 & 0 & ... & 0 & u_{d,d+1} & ... & u_{d,N_b-1} & u_{d,N_b} & 0 \\
0 & 0 & ... & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & u_{d+1,N_b+1} \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & ... & 0 & 0 & ... & 0 & 0 & ... & 0 & 0 & u_{N_b,N_b+1}
\end{array}$$

The set of all such controls is written as $\mathcal{U}$, i.e., $\mathcal{U} = \{\boldsymbol{u} = (u_{ij}) : 0 \leq i \leq N_b, 1 \leq j \leq N_b + 1, u_{ij} = 0 \text{ for } (i,j) \notin \Pi\}$. Before writing the dynamics and the state constraints corresponding to the manufacturing digraph (Δ, Π) containing N_b+2 vertices consisting of a source, a sink, d internal buffers, and $N_b - d$ external buffers associated with the $N_b - d$ distinct final products to be manufactured (or characterizing a jobshop), we give the description of the control constraints. We label all the vertices according to Theorem 2.8. For simplicity in the sequel, we shall call the buffer whose label is i as buffer i, $i = 1, 2, ..., N_b$. The control constraints depend on the placement of the machines, and the different placements on the same digraph will give rise to different jobshops. In other words, a jobshop corresponds to a unique digraph, whereas a digraph may correspond to many different jobshops. Therefore, to uniquely characterize a jobshop using graph theory, we need to introduce the concept of a placement of machines, or simply a placement. Let $N_b \leq \pi - N_b + d$, where π denotes the total number of arcs in Π.

Definition 2.9. In a manufacturing digraph (Δ, Π), a set $\mathcal{K} = \{K_1, K_2, ..., K_N\}$ is called a *placement of machines* $1, 2, ..., N$, if $\mathcal{K}$ is a partition of $\hat{\Pi} = \{(i,j) \in \Pi : j \neq N_b + 1\}$, namely, $\emptyset \neq K_n \subset \hat{\Pi}$, $K_n \cap K_\ell = \emptyset$ for $n \neq \ell$, and $\cup_{k=1}^{N} K_k = \hat{\Pi}$.

A dynamic jobshop can be uniquely specified by a triple $(\Delta, \Pi, \mathcal{K})$, which denotes a manufacturing system that corresponds to a manufacturing digraph (Δ, Π) along with a placement of machines $\mathcal{K} = (K_1, K_2, ..., K_N)$. Consider a jobshop $(\Delta, \Pi, \mathcal{K})$, let $u_{ij}(t)$ be the control at time t associated with arc (i,j), $(i,j) \in \Pi$. Suppose we are given a stochastic process $\boldsymbol{m}(t) = (m_1(t), ..., m_N(t))$ on the probability space $(\Omega, \mathcal{F}, P)$ with $m_n(t)$ representing the capacity of the nth machine at time t, $n = 1, ..., N$. The controls $u_{ij}(t)$ with $(i,j) \in K_n$, $n = 1, ..., N$, $t \geq 0$, should satisfy the following constraints: $0 \leq \sum_{(i,j) \in K_n} u_{ij}(t) \leq m_n(t)$ for all $t \geq 0$, $n = 1, ..., N$, where we have assumed that the required machine capacity p_{ij} (for unit production rate of type j from part type i) equals 1, for convenience in exposition. The analysis can be readily extended to the case when the required machine capacity for the unit production rate of part j from part i is any given positive constant.

We denote the surplus at time t in buffer i by $x_i(t)$, $i \in \Delta \setminus \{0, N_b + 1\}$. Note that if $x_i(t) > 0$, $i = 1, ..., N_b$, we have an inventory in buffer i, and if $x_i(t) < 0$, $i = d+1, ..., N_b$, we have a shortage of finished product i. The dynamics of the system are, therefore,

$$\begin{cases} \dot{x}_i(t) = \left(\sum_{\ell=0}^{i-1} u_{\ell i}(t) - \sum_{\ell=i+1}^{N_b} u_{i\ell}(t) \right), \ 1 \leq i \leq d, \\ \dot{x}_i(t) = \left(\sum_{\ell=0}^{d} u_{\ell i}(t) - z_i \right), \ d+1 \leq i \leq N_b, \end{cases} \tag{2.5}$$

for some integer d and $\boldsymbol{x}(0) := (x_1(0), ..., x_{N_b}(0)) = (x_1, ..., x_{N_b}) = \boldsymbol{x}$. Since internal buffers provide inputs to machines, a fundamental physical fact about them is that they must not have shortages.

In other words, we must have

$$\begin{cases} x_i(t) \geq 0, \ t \geq 0, \ i = 1, ..., d, \\ -\infty < x_i < +\infty, \ t \geq 0, \ i = d+1, ..., N_b. \end{cases} \tag{2.6}$$

Let $\boldsymbol{u}_\ell(t) = (u_{\ell,\ell+1}(t), ..., u_{\ell,N_b}(t))'$, $\ell = 0, ..., d$, and $\boldsymbol{u}_{d+1}(t) = (z_{d+1}, ..., z_{N_b})'$. The relation (2.5) can be written in the following vector form:

$$\dot{\boldsymbol{x}}(t) = (\dot{x}_1(t), ..., \dot{x}_{N_b}(t))' = D\boldsymbol{u}(t), \tag{2.7}$$

where $D : R^J \to R^{N_b}$ is the corresponding linear operator with $J = (N_b - d) + \sum_{\ell=0}^{d}(N_b - \ell)$, and $\boldsymbol{u}(t) = (\boldsymbol{u}_0(t), ..., \boldsymbol{u}_{d+1}(t))'$. Let $\mathcal{S} = R_+^d \times R^{N_b - d}$. Furthermore, let $\mathcal{S}^b$ be the boundary of $\mathcal{S}$, and the interior $\mathcal{S}^o = \mathcal{S} \setminus \mathcal{S}^b$.

We are now in the position to formulate our stochastic optimal control problem for the jobshop defined by (2.5)-(2.7). For $\boldsymbol{m} = (m_1, ..., m_N)$, let

$$\begin{aligned} U(\boldsymbol{m}) \quad = \quad & \{\boldsymbol{u} = (u_{ij}) : \boldsymbol{u} \in \mathcal{U}, 0 \leq \sum_{(i,j)\in K_n} u_{ij} \leq m_n, 1 \leq n \leq N, \\ & u_{i,N_b+1} = z_i, d+1 \leq i \leq N_b\}, \end{aligned}$$

and for $\boldsymbol{x} \in \mathcal{S}$ and $\boldsymbol{m}$,

$$U(\boldsymbol{x}, \boldsymbol{m}) = \Big\{\boldsymbol{u} : \boldsymbol{u} \in U(\boldsymbol{m}) \text{ and } x_n = 0 \Rightarrow \sum_{i=0}^{n-1} u_{in} - \sum_{i=n+1}^{N_b} u_{ni} \geq 0, n = 1, ..., d\Big\}.$$

Definition 2.10. We say that a control $\boldsymbol{u}(\cdot) \in \mathcal{U}$ is admissible with respect to the initial state vector $\boldsymbol{x} = (x_1, \cdots, x_{N_b}) \in S$ and $\boldsymbol{m} \in \mathcal{M}$, if (i) $\boldsymbol{u}(\cdot)$ is an $\mathcal{F}_t$-adapted measurable process with $\mathcal{F}_t = \sigma\{\boldsymbol{m}(s) : 0 \leq s \leq t\}$; (ii) $\boldsymbol{u}(t) \in U(\boldsymbol{m}(t))$ for all $t \geq 0$; and (iii) the corresponding state process $\boldsymbol{x}(t) = (x_1(t), \cdots, x_{N_b}(t)) \in \mathcal{S}$ for all $t \geq 0$.

Remark 2.3. The condition (iii) is equivalent to $\boldsymbol{u}(t) \in U(\boldsymbol{x}(t), \boldsymbol{m}(t))$, $t \geq 0$.

Let $\mathcal{A}(\boldsymbol{x}, \boldsymbol{m})$ denote the set of all admissible control with respect to the initial buffer level $\boldsymbol{x} \in \mathcal{S}$ and the initial machine capacity $\boldsymbol{m}$. The problem is to find an admissible control $\boldsymbol{u}(\cdot)$ that minimizes the cost

$$J(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}(\cdot)) = E\int_0^\infty e^{-\rho t} H(\boldsymbol{x}(t), \boldsymbol{u}(t))dt, \tag{2.8}$$

where $H(\cdot, \cdot)$ defines the cost of surplus and production, $\boldsymbol{x}$ is the initial state, and $\boldsymbol{m}$ is the initial value of $\boldsymbol{m}(t)$. The value function is then defined as

$$v(\boldsymbol{x}, \boldsymbol{m}) = \inf_{\boldsymbol{u}(\cdot)\in\mathcal{A}(\boldsymbol{x},\boldsymbol{m})} J(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}(\cdot)). \tag{2.9}$$

We impose the following assumptions on the random process $\boldsymbol{m}(t) = (m_1(t), ..., m_N(t))$ and the cost function $H(\cdot,\cdot)$ throughout this section.

(A.2.4) $H(\cdot,\cdot)$ is nonnegative and convex. For all $\boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathcal{S}$ and $\boldsymbol{u}, \hat{\boldsymbol{u}}$, there exist constants C_{28} and $\kappa_{25} \geq 0$ such that $|H(\boldsymbol{x},\boldsymbol{u}) - H(\hat{\boldsymbol{x}},\hat{\boldsymbol{u}})| \leq C_{28}(1 + |\boldsymbol{x}|^{\kappa_{25}} + |\hat{\boldsymbol{x}}|^{\kappa_{25}})(|\boldsymbol{x} - \hat{\boldsymbol{x}}| + |\boldsymbol{u} - \hat{\boldsymbol{u}}|)$.

(A.2.5) Let $\mathcal{M} = \{\boldsymbol{m}^1, \ldots, \boldsymbol{m}^p\}$ for some given integer $p \geq 1$. The capacity process $\boldsymbol{m}(t) \in \mathcal{M}$, $t \geq 0$, is a finite state Markov chain with generator $Q = (q_{k\hat{k}})$ such that $q_{k\hat{k}} \geq 0$ if $k \neq \hat{k}$ and $q_{kk} = -\sum_{\hat{k} \neq k} q_{k\hat{k}}$. Moreover, Q is irreducible.

Presman et al. (1997a) prove the following theorem.

Theorem 2.7. *The optimal control $\boldsymbol{u}^*(\cdot) \in \mathcal{A}(\boldsymbol{x},\boldsymbol{m})$ exists, and can be represented as a feedback control. That is, there exists a function $\boldsymbol{u}^*(\cdot,\cdot)$ such that for any $\boldsymbol{x}$ we have $\boldsymbol{u}^*(t) = \boldsymbol{u}^*(\boldsymbol{x}^*(t), \boldsymbol{m}(t))$, $t \geq 0$, where $\boldsymbol{x}^*(\cdot)$ is the optimal state process – the solution of* (2.7) *for $\boldsymbol{u}(t) = \boldsymbol{u}^*(\boldsymbol{x}(t), \boldsymbol{m}(t))$ with $\boldsymbol{x}(0) = \boldsymbol{x}$. Moreover, if $H(\boldsymbol{x},\boldsymbol{u})$ is strictly convex in $\boldsymbol{u}$, then the optimal feedback control $\boldsymbol{u}^*(\cdot,\cdot)$ is unique.*

Now we consider the Lipschitz property of the value function. It should be noted that unlike in the case without state constraints, the Lipschitz property in our case does not follow directly. The reason for this is that in the presence of state constraints, a control which is admissible with respect to $\boldsymbol{x}(0) = \boldsymbol{x} \in \mathcal{S}$ is not necessarily admissible for $\boldsymbol{x}(0) = \boldsymbol{x}'$ when $\boldsymbol{x}' \neq \boldsymbol{x}$.

Theorem 2.8. *The value function is convex, and satisfies the condition $|v(\boldsymbol{x},\boldsymbol{m}) - v(\hat{\boldsymbol{x}},\boldsymbol{m})| \leq C_{29}(1 + |\boldsymbol{x}|^{\kappa_{25}} + |\hat{\boldsymbol{x}}|^{\kappa_{25}})|\boldsymbol{x} - \hat{\boldsymbol{x}}|$ for some positive constant C_{29} and all $\boldsymbol{x},\ \hat{\boldsymbol{x}} \in \mathcal{S}$.*

Because the problem of the jobshop involves state constraints, we can write the HJBDD equation for the problem as follows:

$$\rho v(\boldsymbol{x},\boldsymbol{m}) = \inf_{\boldsymbol{u} \in U(\boldsymbol{x},\boldsymbol{m})} \{\partial v(\boldsymbol{x},\boldsymbol{m})/\partial D\boldsymbol{u} + H(\boldsymbol{x},\boldsymbol{u})\} + Qv(\boldsymbol{x},\boldsymbol{m}). \tag{2.10}$$

Theorem 2.9. (Verification Theorem) (i) *The value function $v(\boldsymbol{x},\boldsymbol{m})$ satisfies equation* (2.10) *for all $\boldsymbol{x} \in \mathcal{S}$.*

(ii) *If some continuous convex function $\hat{v}(\boldsymbol{x},\boldsymbol{m})$ satisfies* (2.10) *and the growth condition given in Theorem* 2.8 *with $\hat{\boldsymbol{x}} = 0$, then $\hat{v}(\boldsymbol{x},\boldsymbol{m}) \leq v(\boldsymbol{x},\boldsymbol{m})$. Moreover, if there exists a feedback control $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{m})$ providing the infimum in* (2.10) *for $\hat{v}(\boldsymbol{x},\boldsymbol{m})$, then $\hat{v}(\boldsymbol{x},\boldsymbol{m}) = v(\boldsymbol{x},\boldsymbol{m})$, and $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{m})$ is an optimal feedback control.*

(iii) *Assume that* $H(\boldsymbol{x}, \boldsymbol{u})$ *is strictly convex in* $\boldsymbol{u}$ *for each fixed* $\boldsymbol{x}$. *Let* $\boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{m})$ *denote the minimizer function of the right-hand side of* (2.10). *Then,* $\dot{\boldsymbol{x}}(t) = D\boldsymbol{u}^*(\boldsymbol{x}(t), \boldsymbol{m}(t))$, $\boldsymbol{x}(0) = \boldsymbol{x}$, *has a solution* $\boldsymbol{x}^*(t)$, *and* $\boldsymbol{u}^*(t) = \boldsymbol{u}^*(\boldsymbol{x}^*(t), \boldsymbol{m}(t))$ *is the optimal control.*

Remark 2.4. The HJBDD (2.10) coincides at inner points of $\mathcal{S}$ with the usual dynamic programming equation for convex PDP problems. Here PDP is the abbreviation of piecewise deterministic processes introduced by Vermes (1985) and Davis (1993). The HJBDD gives at boundary points of $\mathcal{S}$, a boundary condition in the following sense. Let the restriction of $v(\boldsymbol{x}, \boldsymbol{m})$ on some l-dimensional face, $0 < l < J$, of the boundary of $\mathcal{S}$ be differentiable at an inner point $\boldsymbol{x}_0$ of this face. Note that this restriction is convex and is differentiable almost everywhere on this face. Then there is a vector $\tilde{\nabla} v(\boldsymbol{x}_0, \boldsymbol{m})$ such that $v'_{\boldsymbol{m}}(\boldsymbol{x}_0, \boldsymbol{m}) = \langle \tilde{\nabla} v(\boldsymbol{x}_0, \boldsymbol{m}), \boldsymbol{p} \rangle$ for any admissible direction at $\boldsymbol{x}_0$. It follows from the continuity of the value function that

$$\min_{\boldsymbol{u} \in U(\boldsymbol{x}_0, \boldsymbol{m})} \left\{ \langle \tilde{\nabla} v(\boldsymbol{x}_0, \boldsymbol{m}), D\boldsymbol{u} \rangle + H(\boldsymbol{x}_0, \boldsymbol{u}) \right\} = \min_{\boldsymbol{u} \in U(\boldsymbol{m})} \left\{ \langle \tilde{\nabla} v(\boldsymbol{x}_0, \boldsymbol{m}), D\boldsymbol{u} \rangle + H(\boldsymbol{x}_0, \boldsymbol{u}) \right\}.$$

This boundary condition on $v(\cdot, \cdot)$ can be interpreted as follows. First, the optimal control policy on the boundary has the same intuitive explanation as in the interior. The important difference is that we now have to worry about the feasibility of the policy. What the boundary condition accomplishes is to shape the value function on the boundary of $\mathcal{S}$ in such a way that the unconstrained optimal policy is also feasible.

According to (2.10), optimal feedback control policies are obtained in terms of the directional derivatives of the value function. Note now that the uniqueness of the optimal control follows directly from the strict convexity of function $H(\cdot, \cdot)$ in $\boldsymbol{u}$ and the fact that any convex combination of admissible controls for any given $\boldsymbol{x}$ is also admissible. For proving the remaining statements of Theorems 2.8 and 2.9, see Presman et al. (1997a).

Remark 2.5. Presman et al. (1997a, b) show that Theorems 2.7-2.9 also hold when the systems are subject to lower and upper bound constraints on work-in-process.

3 Hierarchical Controls with Discounted Cost Criterion

In this section the problems of hierarchical production planning with the discounted cost is discussed. We present asymptotic results for hierarchical production planning in manufacturing systems with machines subject to breakdown and repair. The idea is to reduce the original problem into simpler problems and to describe a procedure to construct controls, derived from the solution to the simpler problems, for the original systems. The simpler problems turn out to be the limiting problems obtained by averaging the given stochastic machine capacities and modifying the objective function in a reasonable way to account for the convexity of the cost function. Therefore, by showing that the associated value function for the original systems converge to the value functions of the limit systems, we can construct controls for the original systems from the optimal control of the limit systems. The controls so constructed are asymptotically optimal as the fluctuation rate of the machine capacities goes to infinity. Furthermore, error estimates of the asymptotic optimality are provided in terms of their corresponding cost functions.

Here we will discuss hierarchical controls in single/parallel machine systems, flowshops, jobshops, and production-investment and production-marketing systems. Finally, some computational results are given.

3.1 Single or parallel machine systems

Sethi and Zhang (1994b) and Sethi et al. (1994b) consider a stochastic manufacturing system with surplus $\boldsymbol{x}^\varepsilon(t) \in R^n$ and production rate $\boldsymbol{u}^\varepsilon(t) \in R^n_+$ satisfying $\dot{\boldsymbol{x}}^\varepsilon(t) = \boldsymbol{u}^\varepsilon(t) - \boldsymbol{z},\ \boldsymbol{x}(0) = \boldsymbol{x}$, where $\boldsymbol{z} \in R^n_+$ is the constant demand rate and $\boldsymbol{x}$ is the initial surplus $\boldsymbol{x}^\varepsilon(0)$.

Let $m(\varepsilon, t) \in \mathcal{M} = \{0, 1, 2, \cdots, p\}$ denote the machine capacity process of our manufacturing system, where ε is a small parameter to be specified later. Then the production rate $\boldsymbol{u}^\varepsilon(t) \geq 0$ must satisfy $\langle \boldsymbol{r}, \boldsymbol{u}^\varepsilon(t)\rangle \leq m(\varepsilon, t)$ for some positive vector $\boldsymbol{r}$. We consider the cost $J^\varepsilon(\boldsymbol{x}, m, \boldsymbol{u}^\varepsilon(\cdot))$ with $m(\varepsilon, 0) = m$ and $\boldsymbol{x}^\varepsilon(0) = \boldsymbol{x}$ defined by

$$J^\varepsilon(\boldsymbol{x}, m, \boldsymbol{u}^\varepsilon(\cdot)) = E\int_0^\infty e^{-\rho t}[h(\boldsymbol{x}^\varepsilon(t)) + c(\boldsymbol{u}^\varepsilon(t))]dt, \tag{3.1}$$

where $\rho > 0$ is the discount rate, $h(\cdot)$ is the cost of surplus, and $c(\cdot)$ is the cost of production. The problem is to find a control $\boldsymbol{u}^\varepsilon(\cdot) \geq 0$ with $\langle \boldsymbol{r}, \boldsymbol{u}^\varepsilon(t)\rangle \leq m(\varepsilon, t)$, that minimizes $J^\varepsilon(\boldsymbol{x}, m, \boldsymbol{u}^\varepsilon(\cdot))$. We make the following assumptions on the machine capacity process and the cost function on production rate and the surplus.

(A.3.1) $c(\boldsymbol{u})$ and $h(\boldsymbol{x})$ are convex. For all $\boldsymbol{x}, \hat{\boldsymbol{x}}$, there exist constants C_{31} and κ_{31} such that $0 \leq h(\boldsymbol{x}) \leq C_{31}(1 + |\boldsymbol{x}|^{\kappa_{31}+1})$ and $|h(\boldsymbol{x}) - h(\hat{\boldsymbol{x}})| \leq C_{31}(1 + |\boldsymbol{x}|^{\kappa_{31}} + |\hat{\boldsymbol{x}}|^{\kappa_{31}})|\boldsymbol{x} - \hat{\boldsymbol{x}}|$.

(A.3.2) Let $Q^\varepsilon = Q^{(1)} + \varepsilon^{-1}Q^{(2)}$, where $\varepsilon > 0$ and $Q^{(\ell)}$ is an $(p+1) \times (p+1)$ matrix such that

$Q^{(\ell)} = (q_{ij}^{(\ell)})$ with $q_{ij}^{(\ell)} \geq 0$ if $i \neq j$ and $q_{ii}^{(\ell)} = -\sum_{j \neq i} q_{ij}^{(\ell)}$, for $\ell = 1, 2$. The capacity process $0 \leq m(\varepsilon, t) \in \mathcal{M}$ is a finite state Markov process governed by $Q^{(\varepsilon)}$, i.e., $L\psi(\cdot)(i) = Q^{\varepsilon}\psi(\cdot)(i)$, for any function ψ on $\mathcal{M}$.

(A.3.3) The $Q^{(2)}$ is weakly irreducible, i.e., the equations $\nu Q^{(2)} = 0$ and $\sum_{j=0}^{p} \nu_j = 1$ have a unique solution $\nu = (\nu_0, \nu_1, \cdots, \nu_p) > 0$. We call ν to be the *equilibrium distribution* of $Q^{(2)}$.

Remark 3.1. Jiang and Sethi (1991) and Khasminskii et al. (1997) consider a model in which the irreducibility assumption in (A.3.4) can be relaxed to incorporate machine state processes with a generator that consists of several irreducible submatrices. In these models, some jumps are associated with a fast process, while others are associated with a slow process; see Section 5.4.

Definition 3.1. We say that a control $\boldsymbol{u}^{\varepsilon}(\cdot) = \{\boldsymbol{u}^{\varepsilon}(t) : \ t \geq 0\}$ is *admissible* if (i) $\boldsymbol{u}^{\varepsilon}(t) \geq 0$ is a measurable process adapted to $\mathcal{F}_t = \sigma\{m(\varepsilon, s), 0 \leq s \leq t\}$; (ii) $\langle \boldsymbol{r}, \boldsymbol{u}^{\varepsilon}(t) \rangle \leq m(\varepsilon, t)$ for all $t \geq 0$.

We use $\mathcal{A}^{\varepsilon}(m)$ to denote the set of all admissible controls with the initial condition $m(\varepsilon, 0) = k$. Then our control problem can be written as follows:

$$\mathcal{P}^{\varepsilon} : \begin{cases} \text{minimize} & J^{\varepsilon}(\boldsymbol{x}, m, \boldsymbol{u}^{\varepsilon}(\cdot)) = E \int_0^{\infty} e^{-\rho t}[h(\boldsymbol{x}^{\varepsilon}(t)) + c(\boldsymbol{u}^{\varepsilon}(t))]dt, \\ \text{subject to} & \dot{\boldsymbol{x}}^{\varepsilon}(t) = \boldsymbol{u}^{\varepsilon}(t) - \boldsymbol{z}, \boldsymbol{x}^{\varepsilon}(0) = \boldsymbol{x}, \ \boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{A}^{\varepsilon}(m), \\ \text{value function} & v^{\varepsilon}(\boldsymbol{x}, m) = \inf_{\boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{A}^{\varepsilon}(m)} J^{\varepsilon}(\boldsymbol{x}, m, \boldsymbol{u}^{\varepsilon}(\cdot)). \end{cases} \tag{3.2}$$

Similar to Theorem 2.1, one can show that the value function $v^{\varepsilon}(\boldsymbol{x}, m)$ is convex in $\boldsymbol{x}$ for each m. The value function $v^{\varepsilon}(\cdot, \cdot)$ satisfies the dynamic programming equation

$$\rho v^{\varepsilon}(\boldsymbol{x}, m) = \min_{\boldsymbol{u} \geq 0, \boldsymbol{r} \cdot \boldsymbol{u} \leq m} [\langle (\boldsymbol{u} - \boldsymbol{z}), \partial v^{\varepsilon}(\boldsymbol{x}, m)/\partial \boldsymbol{x} \rangle + h(\boldsymbol{x}) + c(\boldsymbol{u})] + L v^{\varepsilon}(\boldsymbol{x}, \cdot)(m), m \in \mathcal{M}, \tag{3.3}$$

in the sense of viscosity solutions. Sethi et al. (1994b) consider a control problem in which the stochastic machine capacity process is averaged out. Let $\mathcal{A}^0$ denote the control space

$$\mathcal{A}^0 = \{U(t) = (\boldsymbol{u}^0(t), \boldsymbol{u}^1(t), \cdots, \boldsymbol{u}^p(t)) : \ \boldsymbol{u}^i(t) \geq 0, \ \langle \boldsymbol{r}, \boldsymbol{u}^i(t) \rangle \leq i, \ 0 \leq i \leq p\}.$$

Then we define the control problem $\mathcal{P}^0$ as follows:

$$\mathcal{P}^0 : \begin{cases} \text{minimize} & J^0(\boldsymbol{x}, U(\cdot)) = E \int_0^{\infty} e^{-\rho t}[h(\boldsymbol{x}(t)) + \sum_{i=0}^{p} \nu_i c(\boldsymbol{u}^i(t))]dt, \\ \text{subject to} & \dot{\boldsymbol{x}}(t) = \sum_{i=0}^{p} \nu_i \boldsymbol{u}^i(t) - \boldsymbol{z}, \ \boldsymbol{x}(0) = \boldsymbol{x}, \ U(\cdot) \in \mathcal{A}^0, \\ \text{value function} & v(\boldsymbol{x}) = \inf_{U(\cdot) \in \mathcal{A}^0} J^0(\boldsymbol{x}, \boldsymbol{u}(\cdot)). \end{cases} \tag{3.4}$$

Sethi and Zhang (1994b) and Sethi et al. (1994b) construct a solution of $\mathcal{P}^{\varepsilon}$ from a solution of $\mathcal{P}^0$, and show it to be asymptotically optimal as stated below.

Theorem 3.1. (i) *There exists a constant C_{32} such that* $|v^\varepsilon(\boldsymbol{x},m)-v(\boldsymbol{x})| \le C_{32}(1+|\boldsymbol{x}|^{\kappa_{31}})\sqrt{\varepsilon}$.

(ii) *Let $U(\cdot) \in \mathcal{A}^0$ denote an ε-optimal control. Then,* $\boldsymbol{u}^\varepsilon(t) = \sum_{i=0}^{p} I_{\{m(\varepsilon,t)=i\}}\boldsymbol{u}^i(t)$ is asymptotically optimal, i.e.,

$$|J^\varepsilon(\boldsymbol{x},m,\boldsymbol{u}^\varepsilon(\cdot)) - v^\varepsilon(\boldsymbol{x},m)| \le C_{32}(1+|\boldsymbol{x}|^{\kappa_{31}})\sqrt{\varepsilon}. \tag{3.5}$$

(iii) *Assume in addition that $c(\boldsymbol{u})$ is twice differentiable with* $(\frac{\partial^2 c(\boldsymbol{u})}{\partial u_i \partial u_j}) \ge c_0 I_{n\times n}$, the function *$h(\cdot)$ is differentiable, and constants C_{33} and $\kappa_{32} > 0$ exist such that*

$$|h(\boldsymbol{x}+\boldsymbol{y}) - h(\boldsymbol{x}) - \langle h'_{\boldsymbol{x}}(\boldsymbol{x}),\boldsymbol{y}\rangle| \le C_{33}(1+|\boldsymbol{x}|^{\kappa_{32}})|\boldsymbol{y}|^2.$$

Then, there exists a locally Lipschitz optimal feedback control $U^*(\boldsymbol{x})$ for $\mathcal{P}^0$. Let

$$\boldsymbol{u}^*(\boldsymbol{x},m(\varepsilon,t))) = \sum_{i=0}^{p} I_{\{m(\varepsilon,t)=i\}}\boldsymbol{u}^{i*}(\boldsymbol{x}). \tag{3.6}$$

Then, $\boldsymbol{u}^\varepsilon(t) = \boldsymbol{u}^*(\boldsymbol{x}(t),m(\varepsilon,t))$ is an asymptotically optimal feedback control for $\mathcal{P}^\varepsilon$ with the convergence rate of $\sqrt{\varepsilon}$, i.e., (3.5) *holds.*

Insight 3.1. (Based on Theorem 3.1(i) and (ii).) If the capacity transition rate is sufficiently fast in relation to the discount rate, then the value function is essentially independent of the initial capacity state. This is because the transients die out and the capacity process settles into its stationary distribution long before the discount factor $e^{-\rho t}$ has decreased substantially from its initial value of 1.

Remark 3.2. Part (ii) of the theorem states that from an ε-optimal open-loop control of the limiting problem, we can construct an $\sqrt{\varepsilon}$-optimal open-loop control for the original problem. With further restrictions on the cost function, Part (iii) of the theorem states that from the ε-optimal feedback control of the limiting problem, we can construct an $\sqrt{\varepsilon}$-optimal feedback control for the original problem.

Remark 3.3. It is important to point out that the hierarchical feedback control (3.6) can be shown to be a threshold-type control if the production cost $c(\boldsymbol{u})$ is linear. Of course, the value of the threshold depends on the state of the machines. For single product problems with constant demand, this means that production takes place at the maximum rate if the inventory is below the threshold, no production takes place above it, and production rate equals the demand rate once the threshold is attained. This is also the form of the optimal policy for these problems as shown, e.g., in Kimemia and Gershwin (1983), Akella and Kumar (1986), and Sethi et al. (1992a).

The threshold level for any given machine capacity state in these cases is also known as a hedging point in that state following Kimemia and Gershwin (1983). In these simple problems, asymptotic optimality is maintained as long as the threshold, say, $\theta(\varepsilon)$, goes to 0 as $\varepsilon \to 0$. Thus, there is a possibility of obtaining better policies than (3.6) that are asymptotically optimal. In fact, one can even minimize over the class of threshold policies for the parallel-machines problems discussed in this section.

Soner (1993) and Sethi and Zhang (1994c) consider $\mathcal{P}^\varepsilon$ in which $Q = \frac{1}{\varepsilon}Q(\boldsymbol{u})$ depends on the control variable $\boldsymbol{u}$. They show that under certain assumptions, the value function v^ε converges to the value function of a limiting problem. Moreover, the limiting problem can be expressed in the same form as $\mathcal{P}^0$ except that the equilibrium distribution $\nu_i, i = 0, 1, 2, \cdots, p$, are now control-dependent. Thus, ν_i in Assumption (A.3.3) is now replaced by $\nu_i(\boldsymbol{u}(t))$ for each i; see also (3.22). Then an asymptotically optimal control for $\mathcal{P}^\varepsilon$ can be obtained as in (3.6) from the optimal control of the limiting problem. As yet, no convergence rate has been obtained in this case.

An example of $Q(\boldsymbol{u})$ in a one-machine case with two (up and down) states is

$$Q(\boldsymbol{u}) = \begin{pmatrix} -\mu & \mu \\ \lambda(\boldsymbol{u}) & -\lambda(\boldsymbol{u}) \end{pmatrix}.$$

Thus, the breakdown rate $\lambda(\boldsymbol{u})$ of the machine depends on the rate of production $\boldsymbol{u}$, while the repair rate μ is independent of the production rate. These are reasonable assumptions in practice.

3.2 Dynamic flowshops

For manufacturing systems with N machines in tandem and with unlimited capacities of the internal buffers, Sethi et al. (1992c) obtain a limiting problem. Then they use a near-optimal control of the limiting problem to construct an open-loop control for the original problem, which is asymptotically optimal as the transition rates between the machine states go to infinity. The case of a limited capacity internal buffer is treated in Sethi et al. (1992d, 1993, 1997c). Recently, based on the Lipschitz continuity of the value function given by Presman et al. (1997b), Sethi et al. (2000d) construct a hierarchical control for the N-machine flowshop with limited buffers.

Since many of the flowshop results have been generalized to the more general case of jobshops discussed in the next section, we shall not provide a separate review of the flowshop results. However, results derived specifically for flowshops will be given at the end of the next section as special cases of the jobshop.

3.3 Dynamic jobshops

Sethi and Zhou (1994) consider hierarchical production planning in a general manufacturing system given in Section 2.3. For the jobshop $(\Delta, \Pi, \mathcal{K})$, let $u_{ij}^{\varepsilon}(t)$ be the control at time t associated with arc (i, j), $(i, j) \in \Pi$. Suppose we are given a stochastic process $\boldsymbol{m}(\varepsilon, t) = (m_1(\varepsilon, t), ..., m_N(\varepsilon, t))$ on the probability space $(\Omega, \mathcal{F}, P)$ with $m_n(\varepsilon, t)$ representing the capacity of the nth machine at time t, $n = 1, ..., N$, where $\varepsilon > 0$ is a small parameter to be precisely specified later. The controls $u_{ij}^{\varepsilon}(t)$ with $(i, j) \in K_n$, $n = 1, ..., N$, $t \geq 0$, should satisfy the following constraints:

$$0 \leq \sum_{(i,j)\in K_n} u_{ij}^{\varepsilon}(t) \leq m_n(\varepsilon, t) \text{ for all } t \geq 0,\ n = 1, ..., N, \tag{3.7}$$

where we have assumed that the required machine capacity p_{ij} (for unit production rate of type j from part type i) equals 1, for convenience in exposition. The analysis in this paper can be readily extended to the case when the required machine capacity for the unit production rate of part j from part i is any given positive constant.

We denote the level at time t in buffer i by $x_i^{\varepsilon}(t)$, $i \in \Delta \setminus \{0, N_b + 1\}$. Note that if $x_i^{\varepsilon}(t) > 0$, $i = 1, ..., N_b$, we have an inventory in buffer i, and if $x_i^{\varepsilon}(t) < 0$, $i = d+1, ..., N_b$, we have a shortage of finished product i. The dynamics of the system are, therefore,

$$\left\{ \begin{array}{l} \dot{x}_i^{\varepsilon}(t) = \left(\sum_{\ell=0}^{i-1} u_{\ell i}^{\varepsilon}(t) - \sum_{\ell=i+1}^{N_b} u_{i\ell}^{\varepsilon}(t) \right),\ 1 \leq i \leq d, \\ x_i^{\varepsilon}(t) = \left(\sum_{\ell=0}^{d} u_{\ell i}^{\varepsilon}(t) - z_i \right),\ d+1 \leq i \leq N_b, \end{array} \right. \tag{3.8}$$

with $\boldsymbol{x}^{\varepsilon}(0) := (x_1^{\varepsilon}(0), ..., x_{N_b}^{\varepsilon}(0)) = (x_1, ..., x_{N_b}) = \boldsymbol{x}$. Let $\boldsymbol{u}_{\ell}^{\varepsilon}(t) = (u_{\ell,\ell+1}^{\varepsilon}(t), ..., u_{\ell,N_b}^{\varepsilon}(t))'$, $\ell = 0, ..., d$, and $\boldsymbol{u}_{d+1}^{\varepsilon}(t) = (z_{d+1}, ..., z_{N_b})'$. Similar to Section 2.3, we rewrite (3.8) in the vector form as $\dot{\boldsymbol{x}}^{\varepsilon}(t) = (\dot{x}_1^{\varepsilon}(t), ..., \dot{x}_{N_b}^{\varepsilon}(t))' = D\boldsymbol{u}^{\varepsilon}(t)$.

Definition 3.2. We say that a control $\boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{U}$ is admissible with respect to the initial state vector $\boldsymbol{x} = (x_1, \cdots, x_{N_b}) \in S$ and $\boldsymbol{m} \in \mathcal{M}$, if (i) $\boldsymbol{u}^{\varepsilon}(\cdot)$ is an $\mathcal{F}_t^{\varepsilon}$-adapted measurable process with $\mathcal{F}_t^{\varepsilon} = \sigma\{\boldsymbol{m}(\varepsilon, s) : 0 \leq s \leq t\}$; (ii) $\boldsymbol{u}^{\varepsilon}(t) \in U(\boldsymbol{m}(\varepsilon, t))$ for all $t \geq 0$; and (iii) the corresponding state process $\boldsymbol{x}^{\varepsilon}(t) = (x_1^{\varepsilon}(t), \cdots, x_{N_b}^{\varepsilon}(t)) \in \mathcal{S}$ for all $t \geq 0$.

Let $\mathcal{A}^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m})$ denote the set of all admissible control with respect to $\boldsymbol{x} \in \mathcal{S}$ and the machine capacity vector $\boldsymbol{m}$. The problem is to find an admissible control $\boldsymbol{u}^{\varepsilon}(\cdot)$ that minimize the cost criterion

$$J^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}^{\varepsilon}(\cdot)) = E \int_0^{\infty} e^{-\rho t}[h(\boldsymbol{x}^{\varepsilon}(t)) + c(\boldsymbol{u}^{\varepsilon}(t))]dt, \tag{3.9}$$

where $h(\cdot)$ defines the surplus cost, $c(\cdot)$ is the production cost, $\boldsymbol{x}$ is the initial state, and $\boldsymbol{m}$ is the initial value of $\boldsymbol{m}(\varepsilon, t)$. The value function is then defined as

$$v^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m}) = \inf_{\boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{A}^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m})} J^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}^{\varepsilon}(\cdot)). \tag{3.10}$$

We impose the following assumptions on the capacity process $\boldsymbol{m}(\varepsilon, t) = (m_1(\varepsilon, t), ..., m_N(\varepsilon, t))$ and the cost functions $h(\cdot)$ and $c(\cdot)$ throughout this section.

(A.3.4) Let $\mathcal{M} = \{\boldsymbol{m}^1, ..., \boldsymbol{m}^p\}$ for some given integer $p \geq 1$, where $\boldsymbol{m}^j = (m_1^j, ..., m_N^j)$, with $m_k^j, k = 1, ..., N$, denoting the capacity of the kth machine, $j = 1, ..., p$. The capacity process $\boldsymbol{m}^{\varepsilon}(t) \in \mathcal{M}$ is a finite state Markov chain with the infinitesimal generator $Q = Q^{(1)} + \varepsilon^{-1} Q^{(2)}$, where $Q^{(1)} = (q_{ij}^{(1)})$ and $Q^{(2)} = (q_{ij}^{(2)})$ are matrices such that $q_{ij}^{(\ell)} \geq 0$ if $j \neq i$, and $q_{ii}^{(\ell)} = -\sum_{j \neq i} q_{ij}^{(\ell)}$ for $\ell = 1, 2$. Moreover, $Q^{(2)}$ is irreducible and, without any loss of generality, it is taken to be the one that satisfies $\min_{ij}\{|q_{ij}^{(2)}| : q_{ij}^{(2)} \neq 0\} = 1$.

(A.3.5) Assume that $Q^{(2)}$ is weakly irreducible. Let $\nu = (\nu_1, ..., \nu_p)$ denote the equilibrium distribution of $Q^{(2)}$, that is, ν is the only nonnegative solution to the equations $\nu Q^{(2)} = 0$ and $\sum_{i=1}^p \nu_i = 1$.

(A.3.6) $h(\cdot)$ and $c(\cdot)$ are convex functions. For all $\boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathcal{S}$ and $\boldsymbol{u}, \hat{\boldsymbol{u}}$, there exist constants C_{34} and $\kappa_{32} \geq 0$ such that $0 \leq h(\boldsymbol{x}) \leq C_{34}(1 + |\boldsymbol{x}|^{\kappa_{32}})$, $|h(\boldsymbol{x}) - h(\hat{\boldsymbol{x}})| \leq C_{34}(1 + |\boldsymbol{x}|^{\kappa_{32}} + |\hat{\boldsymbol{x}}|^{\kappa_{32}})|\boldsymbol{x} - \hat{\boldsymbol{x}}|$, and $|c(\boldsymbol{u}) - c(\hat{\boldsymbol{u}})| \leq C_{34}|\boldsymbol{u} - \hat{\boldsymbol{u}}|$.

We use $\mathcal{P}^{\varepsilon}$ to denote our control problem:

$$\mathcal{P}^{\varepsilon}: \begin{cases} \min & J^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}^{\varepsilon}(\cdot)) = E \int_0^{\infty} e^{-\rho t}[h(\boldsymbol{x}^{\varepsilon}(t)) + c\boldsymbol{u}^{\varepsilon}(t))]dt, \\ \text{s.t.} & \dot{\boldsymbol{x}}^{\varepsilon}(t) = D\boldsymbol{u}^{\varepsilon}(t),\ \boldsymbol{x}^{\varepsilon}(0) = \boldsymbol{x},\ \boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{A}^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m}), \\ \text{value fn} & v^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m}) = \inf_{\boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{A}^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m})} J^{\varepsilon}(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}^{\varepsilon}(\cdot)). \end{cases} \tag{3.11}$$

In order to obtain the limiting problem, we consider the deterministic controls defined below.

Definition 3.3. For $\boldsymbol{x} \in \mathcal{S}$, let $\mathcal{A}^0(\boldsymbol{x})$ denote the set of the following measurable controls $U(\cdot) = ((\boldsymbol{u}_0^{1,0}(\cdot), ..., \boldsymbol{u}_{d+1}^{1,0}(\cdot)), ..., (\boldsymbol{u}_0^{p,0}(\cdot), ..., \boldsymbol{u}_{d+1}^{p,0}(\cdot)))$ with $\sum_{(i,j) \in K_n} u_{ij}^{\ell,0}(t) \leq m_n^{\ell}$, $\ell = 1, ..., p$, $n = 1, ..., N$, and the corresponding solution $\boldsymbol{x}(\cdot)$ of the system

$$\begin{cases} \dot{x}_j(t) = \sum_{\ell=0}^{j-1} \sum_{i=1}^p \nu_i u_{\ell j}^{i,0}(t) - \sum_{\ell=j+1}^{N_b} \sum_{i=1}^p \nu_i u_{j\ell}^{i,0}(t),\ x_j(0) = x_j,\ 1 \leq j \leq d, \\ \dot{x}_j(t) = \sum_{\ell=0}^{d} \sum_{i=1}^p \nu_i u_{\ell j}^{i,0}(t) - z_j,\ x_j(0) = x_j,\ d+1 \leq j \leq N_b, \end{cases} \tag{3.12}$$

satisfies $\boldsymbol{x}(t) \in \mathcal{S}$ for all $t \geq 0$.

The objective of the limiting problem is to choose a control $U(\cdot) \in \mathcal{A}^0$ that minimizes

$$J^0(\boldsymbol{x}, U(\cdot)) = \int_0^\infty e^{-\rho t} \left[h(\boldsymbol{x}(t)) + \sum_{\ell=1}^{p} \nu_\ell c(\boldsymbol{u}^{\ell,0}(t)) \right] dt.$$

We write (3.12) in the vector form $\dot{\boldsymbol{x}}(t) = D \sum_{\ell=1}^{p} \nu_\ell \boldsymbol{u}^{\ell,0}(t),\ \boldsymbol{x}(0) = \boldsymbol{x}$. We use $\mathcal{P}^0$ to denote the limiting problem and derive it as follows:

$$\mathcal{P}^0 : \begin{cases} \text{min} & J^0(\boldsymbol{x}, U(\cdot)) = \int_0^\infty e^{-\rho t}[h(\boldsymbol{x}(t)) + \sum_{\ell=1}^{p} \nu_\ell c(\boldsymbol{u}^{\ell,0}(t))]dt, \\ \text{s.t.} & \dot{\boldsymbol{x}}(t) = D \sum_{\ell=1}^{p} \nu_\ell \boldsymbol{u}^{\ell,0}(t),\ \boldsymbol{x}(0) = \boldsymbol{x},\ U(\cdot) \in \mathcal{A}^0(\boldsymbol{x}), \\ \text{value fn.} & v(\boldsymbol{x}) = \inf_{U(\cdot) \in \mathcal{A}^0(\boldsymbol{x})} J^0(\boldsymbol{x}, U(\cdot)). \end{cases}$$

Based on the Lipschitz continuity of the value function given in Section 2.3, Sethi and Zhou (1994) prove the following theorem which says that the problem $\mathcal{P}^0$ is indeed a limiting problem in the sense that the value function $v^\varepsilon(\boldsymbol{x}, \boldsymbol{m})$ of $\mathcal{P}^\varepsilon$ converges to the value function $v(\boldsymbol{x})$ of $\mathcal{P}^0$. Furthermore, the theorem also gives the corresponding convergence rate.

Theorem 3.2. *For each $\delta \in (0, \frac{1}{2})$, there exists a positive constant C_{35} such that for all $\boldsymbol{x} \in \mathcal{S}$ and sufficiently small ε, we have $|v^\varepsilon(\boldsymbol{x}, \boldsymbol{m}) - v(\boldsymbol{x})| \leq C_{35}(1 + |\boldsymbol{x}|^{\kappa_{32}})\varepsilon^{\frac{1}{2}-\delta}$.*

Insight 3.2. Comparison of Theorems 3.1 and 3.2 reveals that there is a slight loss in the order of the error bound when going from a single/parallel machine system to a jobshop on account of the state constraints inherent in a jobshop. The presence of the state constraints results in a capacity loss phenomenon, because a machine, even while in the working order, cannot produce if the output buffer provides an input to a failed machine.

Sethi et al. (2002) also show that Theorem 3.2 is true for a general jobshop system with limited buffers. Similar to Theorem 3.1, for a given $\boldsymbol{x} \in S$, they describe the procedure of constructing an asymptotic optimal control $\boldsymbol{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{m})$ of the original problem $\mathcal{P}^\varepsilon$ beginning with any near-optimal control $U(\cdot) \in \mathcal{A}^0(\boldsymbol{x})$ of the limiting problem $\mathcal{P}^0$. We illustrate their procedure for the special case of a flowshop with two-machine and single product, that is, (3.8) with $m = 1$, $N_b = 2$ and $u_{02}^\varepsilon(t) \equiv 0$. First we focus on the open-loop control. Let us fix an initial state $\boldsymbol{x} \in S$. Let $U(\cdot) = (\boldsymbol{u}^{1,0}(\cdot), \cdots, \boldsymbol{u}^{p,0}(\cdot)) \in \mathcal{A}^0$, where $\boldsymbol{u}^{j,0}(t) = (u_{01}^{j,0}(t), u_{12}^{j,0}(t))$ is an $\varepsilon^{\frac{1}{2}-\delta}$-*optimal control* for $\mathcal{P}^0$, i.e., $|J^0(\boldsymbol{x}, U(\cdot)) - v(\boldsymbol{x})| \leq \varepsilon^{\frac{1}{2}-\delta}$. Because the work-in-process level must be nonnegative, unlike in the case of parallel machine systems, the control $\sum_{j=1}^{p} I_{\{\boldsymbol{m}(\varepsilon,t)=\boldsymbol{m}^j\}} \boldsymbol{u}^{j,0}(t)$ may not be admissible. Thus, we need to modify it. Let us define a time $t^* \leq \infty$ as follows: $t^* = \inf\{t : \int_0^t [\sum_{j=1}^{p} (m_1^j -$

$\nu_j u_{01}^{j,0}(s) + \nu_j u_{12}^{j,0}(s))]ds \geq \varepsilon^{\frac{1}{2}-\delta}\}$. We define another control process $\tilde{U}(t) = (\tilde{\boldsymbol{u}}^{1,0}(\cdot), \cdots, \tilde{\boldsymbol{u}}^{p,0}(\cdot))$ as follows: for $j = 1, \cdots, p$,

$$\tilde{\boldsymbol{u}}^{j,0}(t) = (\tilde{u}_{01}^{j,0}(t), \tilde{u}_{12}^{j,0}(t)) = \begin{cases} (m_1^j, 0) & \text{if } t < t^*, \\ (u_{01}^{j,0}(t), u_{12}^{j,0}(t)) & \text{if } t \geq t^*. \end{cases} \tag{3.13}$$

It is easy to check that $\tilde{U}(\cdot) \in \mathcal{A}^0(\boldsymbol{x})$. Let

$$\boldsymbol{w}^\varepsilon(t) = \sum_{j=1}^{p} \nu_j \tilde{\boldsymbol{u}}^j(t) I_{\{\boldsymbol{m}(\varepsilon,t)=\boldsymbol{m}^j\}}, \tag{3.14}$$

and let $\boldsymbol{y}^\varepsilon(t) = (y_1^\varepsilon(t), y_2^\varepsilon(t))$ be the corresponding trajectory defined as

$$\begin{cases} y_1^\varepsilon(t) = x_1 + \int_0^t (w_1^\varepsilon(s) - w_2^\varepsilon(s))ds, \\ y_2^\varepsilon(t) = x_2 + \int_0^t (w_2^\varepsilon(s) - z_2)ds. \end{cases}$$

Note that $E|\boldsymbol{y}^\varepsilon(t) - \tilde{\boldsymbol{x}}(t)|^2 \leq C(1+t^2)\varepsilon$. However, $\boldsymbol{y}^\varepsilon(t)$ may not be in S for some $t \geq 0$. To obtain an admissible control for $\mathcal{P}^\varepsilon$, we need to modify $\boldsymbol{y}^\varepsilon(t)$ so that the state trajectory stays in S. This is done as follows. Let $\boldsymbol{u}^\varepsilon(t) = (u_1^\varepsilon(t), u_2^\varepsilon(t)) := \boldsymbol{y}^\varepsilon(t) I_{\{y_1^\varepsilon(t)\geq 0\}}$. Then, for the control $\boldsymbol{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{m})$ constructed (3.13)-(3.14) above, it is shown in Sethi et al. (1993) that $|J^\varepsilon(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}^\varepsilon(\cdot)) - v^\varepsilon(\boldsymbol{x}, \boldsymbol{m})| = O(\varepsilon^{\frac{1}{2}-\delta})$. Moreover, the case of more than two machines is treated in Sethi et al. (1992c).

Next, we give explicitly an asymptotically optimal feedback control for a two-machine flowshop. The problem is addressed by Sethi and Zhou (1996a, b), who consider $\mathcal{P}^\varepsilon$ in (3.11) with

$$c(\boldsymbol{u}) = 0 \text{ and } h(\boldsymbol{x}) = c_1 x_1 + c_2^+ x_2^+ + c_2^- x_2^-, \tag{3.15}$$

where c_1, c_2^+ and c_2^- are given nonnegative cost coefficients and $x_2^+ = \max\{x_2, 0\}$ and $x_2^- = -\min\{x_2, 0\}$. In order to illustrate their results, we choose a simple situation in which each of the two machines has a capacity m when up and 0 when down, and has a breakdown rate $\lambda > 0$ and the repair rate $\mu > 0$. Furthermore, we shall assume that the average capacity $\bar{m} = m\mu/(\lambda+\mu)$ of each machine is strictly larger than the demand z_2.

The optimal control of the corresponding limiting (deterministic) problem $\mathcal{P}^0$ is:

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{cases} (0,0) & \text{if } x_1 \geq 0, x_2 > 0, \\ (0, z_2) & \text{if } x_1 > 0, x_2 = 0, \\ (0, \bar{m}) & \text{if } x_1 > 0, x_2 < 0, \\ (\bar{m}, \bar{m}) & \text{if } x_1 = 0, x_2 < 0, \\ (z_2, z_2) & \text{if } x_1 = 0, x_2 = 0. \end{cases} \tag{3.16}$$

Insight 3.3. An optimal control is to get to $(0,0)$ in the cheapest possible way, and then stay there.

From (3.16), Sethi and Zhou (1996a, b) construct the following asymptotically optimal feedback control:

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x},\boldsymbol{m}) = \begin{cases} (0,0), & x_1 \geq 0, x_2 > \theta_2(\varepsilon), \\ (0,\min\{k_2,z_2\}), & x_1 > \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon), \\ (0,k_2), & x_1 > \theta_1(\varepsilon), x_2 < \theta_2(\varepsilon), \\ (\min\{k_1,k_2\},k_2), & x_1 = \theta_1(\varepsilon), x_2 < \theta_2(\varepsilon), \\ (\min\{k_1,k_2,z_2\},\min\{k_2,z_2\}), & x_1 = \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon), \\ (k_1,k_2), & 0 < x_1 < \theta_1(\varepsilon), x_2 < \theta_2(\varepsilon), \\ (k_1,\min\{k_2,z_2\}), & 0 < x_1 < \theta_1(\varepsilon), x_2 = \theta_2(\varepsilon), \\ (k_1,\min\{k_1,k_2,z_2\}), & x_1 = 0, x_2 = \theta_2(\varepsilon), \\ (k_1,\min\{k_1,k_2\}), & x_1 = 0, x_2 < \theta_2(\varepsilon), \end{cases} \tag{3.17}$$

where $\boldsymbol{m} = (k_1,k_2)$ with $k_1 \in \{0,m\}$ and $k_2 \in \{0,m\}$, and $(\theta_1(\varepsilon),\theta_2(\varepsilon)) \to (0,0)$ as $\varepsilon \to 0$; see Figure 1.

Note that the optimal control (3.16) of $\mathcal{P}^0$ uses the obvious bang-bang and singular controls to go to $(0,0)$ and then stay there. In the same spirit, the control in (3.17) uses bang-bang and singular controls to approach $(\theta_1(\varepsilon),\theta_2(\varepsilon))$. For a detailed heuristic explanation of asymptotic optimality, see Samaratunga et al. (1997) and Sethi (1997); for a rigorous proof, see Sethi and Zhou (1996a, b).

Remark 3.5. The policy in Figure 1 cannot be termed a threshold-type policy, since there is no maximum tendency to go to $x_1 = \theta_1(\varepsilon)$, when the inventory level $x_1(t)$ is below $\theta_1(\varepsilon)$ and $x_2(t) > \theta_2(\varepsilon)$. In fact, Sethi and Zhou (1996a, b) show that a threshold-type policy, known also as a Kanban policy, is not even asymptotically optimal when $c_1 > c_2^+$. Also, it is known that the optimal feedback policy for two-machine flowshops involve switching manifolds that are much more complicated than the manifolds $x_1 = \theta_1$ and $x_2 = \theta_2$ required to specify a threshold-type policy. This implies that in the discounted flowshop problems, one cannot find an optimal feedback policy within the class of threshold-type policies. While θ_1 and θ_2 could still be called hedging points, there is no notion of optimal hedging points insofar as they are used to specify a feedback policy. See Samaratunga et al. (1997) for a further discussion on this point.

3.4 Computational results

Connolly et al. (1992), Van Ryzin et al. (1993), Violette (1993), and Violette and Gershwin (1991) have carried out a good deal of computational work in connection with manufacturing systems without state constraints. Such systems include single or parallel machine systems described in Sections 3.1, 3.2, and 3.3 as well as no-wait flowshops (or flowshops without internal buffers) treated in Kimemia and Gershwin (1983). Darakananda (1989) developed a simulation software called *Hiercsim* based on the control algorithms of Gershwin et al. (1985) and Gershwin (1989). It should be noted that controls constructed in these algorithms have been shown under some conditions to be asymptotically optimal by Sethi and Zhang (1994b) and Sethi et al. (1994b).

One of the main weaknesses of the early version of Hiercsim for the purpose of this review is its inability to deal with internal storage, see also Violette and Gershwin (1991). Bai (1991) and Bai and Gershwin (1990) developed a hierarchical scheme based on partitioning machines in the original flowshop or jobshop into a number of virtual machines each devoted to single part type production. Violette (1993) developed a modified version of Hiercsim to incorporate the method of Bai and Gershwin (1990). Violette and Gershwin (1991) perform a simulation study indicating that the modified method is efficient and effective. We shall not review it further, since the procedure based on partitioning of machines is unlikely to be asymptotically optimal.

Sethi and Zhou (1996b) have constructed asymptotically optimal hierarchical controls $\boldsymbol{u}^\varepsilon(\boldsymbol{x}, \boldsymbol{m})$, given in (3.17) with switching manifolds depicted in Figure 1, for the two-machine flowshop defined by (3.8) with $d = 1$, $N_b = 1$, and $u_{02}^\varepsilon(t) \equiv 0$, and (3.15). Samaratunga et al. (1997) have compared the performance of these hierarchical controls (HC) to that of optimal control (OC) and of two other existing heuristic methods known as Kanban Control (KC) and Two-Boundary Control (TBC). Like HC, KC is a two parameter policy defined as follows:

$$\boldsymbol{u}_{KC}^\varepsilon(\boldsymbol{x}, \boldsymbol{m}) = \begin{cases} (m_1, 0) & \text{if } \ 0 \leq x_1 < \theta_1(\varepsilon), x_2 > \theta_2(\varepsilon), \\ \boldsymbol{u}^\varepsilon(\boldsymbol{x}, \boldsymbol{m}) & \text{otherwise.} \end{cases} \tag{3.18}$$

Note that KC is a threshold-type policy. TBC is a three-parameter policy developed by Lou and Van Ryzin (1989). Because it is much more complicated than HC or KC and because its performance is not significantly different from HC as can be seen in Samaratunga et al. (1997), we shall not discuss it any further in this survey. In what follows, we provide the computational results obtained in Samaratunga et al. (1997) for the problem (3.11) and (3.15) with $\lambda = 1, \mu =$

$5, m = 2, c_1^+ = 0.1, c_2^+ = 0.2,$ and $c_2^- = 1.0$. Then we discuss the results.

In Table 1, different initial states are selected and the best parameter values are computed for these different initial states for HC and KC; note from Remark 3.6 that in general there are no parameter values that are best for all possible initial states. In the last row, the initial state (2.70, 1.59) is such that the best hedging point for HC and KC are (2.70,1.59). Table 2 uses the parameter values obtained in Table 1 in the row with the initial state (0,0). Samaratunga et al. (1997) analyze these computational results and provide the following comparison of OC and KC.

HC vs. OC: In Tables 1 and 2, the cost of HC is quite close to the optimal cost, if the initial state is sufficiently removed from point (0,0). Moreover, the further the initial (x_1, x_2) is from point (0,0), the better the approximation HC provides to OC. This is because the hedging points are close to point (0,0), and hierarchical and optimal controls agree at points in the state space that are further from (0,0) or further from hedging points. In these cases, transients contribute a great deal to the total cost and transients of HC and OC agree in regions far away from (0,0).

HC vs. KC: Let us now compare HC and KC in detail. Of course, if the initial state is in a shortage situation ($x_2 \leq 0$), then HC and KC must have identical costs. This can be easily seen in Table 1 or Table 2 when initial (x_1, x_2) = (0, -5), (0, -10), (0, -20), (5, -5), (10, -10) and (20, -20).

On the other hand, if the initial surplus is positive, cost of HC is either the same as or slightly smaller than the cost of KC, as should be expected. This is because, KC being a threshold-type policy, the system approaches $\theta_1(\varepsilon)$ even when there is large positive surplus, implying higher inventory costs. In Tables 1 and 2, we can see this in rows with initial (x_1, x_2) = (0, 5), (0, 10), (0, 20), and (20, 20). Moreover, by the same argument, the values of $\theta_1(\varepsilon)$ for KC must not be larger than those for HC in Table 1. Indeed, in cases with large positive surplus, the value of $\theta_1(\varepsilon)$ for KC must be smaller than that for HC. Furthermore, in these cases with positive surplus, the cost differences in Table 2 must be larger than those in Table 1, since Table 2 uses hedging point parameters that are best for initial (x_1, x_2) = (0,0). These parameters are the same for HC and KC. Thus, the system with an initial surplus has higher inventories in the internal buffer with KC than with HC.

Note also that if the surplus is very large, then KC in order to achieve lower inventory costs sets $\theta_1(\varepsilon) = 0$, with the consequence that its cost is the same as that for HC. For example, this happens when the initial (x_1, x_2) = (0,50) in Table 1. As should be expected, the difference in cost for initial $(x_1, x_2) = (0{,}50)$ in Table 2 is quite large compared to the corresponding difference in Table 1.

3.5 Production-investment models

Sethi et al. (1992b) incorporate an additional capacity expansion decision in the model discussed in Section 3.1. They consider a stochastic manufacturing system with the surplus $\boldsymbol{x}^\varepsilon(t) \in R^n$ and production rate $\boldsymbol{u}^\varepsilon(t) \in R^n$ that satisfy $\dot{\boldsymbol{x}}^\varepsilon(t) = \boldsymbol{u}^\varepsilon(t) - \boldsymbol{z},\ \boldsymbol{x}^\varepsilon(0) = \boldsymbol{x}$, where $\boldsymbol{z} \in R^n$ denotes the constant demand rate and $\boldsymbol{x}$ is the initial surplus level. They assume $\boldsymbol{u}^\varepsilon(t) \geq 0$ and $\langle \boldsymbol{r}, \boldsymbol{u}^\varepsilon(t) \rangle \leq m(\varepsilon, t)$ for some $\boldsymbol{r} \geq 0$, where $m(\varepsilon, t)$ is the machine capacity process described by (3.20). The specification of $m(\varepsilon, t)$ involves the instantaneous purchase of some given additional capacity at some time τ, $0 \leq \tau \leq \infty$, at a cost of K, where $\tau = \infty$ means not to purchase it at all; see Sethi et al. (1994a) for an alternate model in which the investment in the additional capacity is continuous. For the model under consideration, the control variable is a pair $(\tau, \boldsymbol{u}(\cdot))$ of a Markov time $\tau \geq 0$ and a production process $\boldsymbol{u}(\cdot)$ over time. The cost criterion J^ε is given by

$$J^\varepsilon(\boldsymbol{x}, m, \tau, \boldsymbol{u}^\varepsilon(\cdot)) = E\left[\int_0^\infty e^{-\rho t} H(\boldsymbol{x}^\varepsilon(t), \boldsymbol{u}^\varepsilon(t))dt + K e^{-\rho\tau}\right], \tag{3.19}$$

where $m(\varepsilon, 0) = m$ is the initial capacity and $\rho > 0$ is the discount rate. The problem is to find an admissible control $(\tau, \boldsymbol{u}^\varepsilon(\cdot))$ that minimizes $J^\varepsilon(\boldsymbol{x}, m, \tau, \boldsymbol{u}^\varepsilon(\cdot))$.

Define $m_1(\varepsilon, \cdot)$ and $m_2(\varepsilon, \cdot)$ as two Markov processes with state spaces $\mathcal{M}_1 = \{0, 1, \cdots, p_1\}$ and $\mathcal{M}_2 = \{0, 1, \cdots, p_1 + p_2\}$, respectively. Here, $m_1(\varepsilon, \cdot) \geq 0$ denotes the existing production capacity process and $m_2(\varepsilon, \cdot) \geq 0$ denotes the capacity process of the system if it were to be supplemented by the additional new capacity at time 0. Let $\mathcal{F}_1(t) = \sigma\{m_1(\varepsilon, s) : 0 \leq s \leq t\}$ and $\mathcal{F}(t) = \sigma\{m(\varepsilon, t) : 0 \leq s \leq t\}$.

Define the capacity process $m(\varepsilon, t)$ as follows: For each $\mathcal{F}_1(t)$-Markov time $\tau \geq 0$,

$$m(\varepsilon, t) = \begin{cases} m_1(\varepsilon, t) & \text{if } t < \tau, \\ m_2(\varepsilon, t - \tau) & \text{if } t \geq \tau, \end{cases} \quad \text{and } m(\varepsilon, \tau) = m_2(\varepsilon, 0) := m_1(\varepsilon, \tau) + p_2. \tag{3.20}$$

Here p_2 denotes the maximum additional capacity resulting from the investment in the new capacity. We make the following assumptions on the cost function $H(\cdot, \cdot)$ and the process $m(\varepsilon, t)$.

(A.3.7) $G(\boldsymbol{x}, \boldsymbol{u})$ is a nonnegative jointly convex function that is strictly convex in either $\boldsymbol{x}$ or $\boldsymbol{u}$ or both. For all $\boldsymbol{x}, \hat{\boldsymbol{x}} \in R^n$ and $\boldsymbol{u}, \hat{\boldsymbol{u}} \in R^n_+$, there exist constant C_{35} and κ_{33} such that $|H(\boldsymbol{x}, \boldsymbol{u}) - H(\hat{\boldsymbol{x}}, \hat{\boldsymbol{u}})| \leq C_{35}[(1 + |\boldsymbol{x}|^{\kappa_{33}} + |\hat{\boldsymbol{x}}|^{\kappa_{33}})|\boldsymbol{x} - \hat{\boldsymbol{x}}| + |\boldsymbol{u} - \hat{\boldsymbol{u}}|]$.

(A.3.8) $m_1(\varepsilon,t)\in\mathcal{M}_1$ and $m_2(\varepsilon,t)\in\mathcal{M}_2$ are Markov processes with generators $\varepsilon^{-1}Q_1$ and $\varepsilon^{-1}Q_2$, respectively, where $Q_1=(q_{ij}^{(1)})$ and $Q_2=(q_{ij}^{(2)})$ are matrices such that $q_{ij}^{(\ell)}\geq 0$ if $i\neq j$ and $q_{ii}^{(\ell)}=-\sum_{i\neq j}q_{ij}^{(\ell)}$ for $\ell=1,2$. Moreover, Q_1 and Q_2 are both irreducible.

Definition 3.4. We say that a control $(\tau,\boldsymbol{u}^\varepsilon(\cdot))$ is *admissible* if (i) τ is an $\mathcal{F}_1(t)$-Markov time; (ii) $\boldsymbol{u}^\varepsilon(t)$ is $\mathcal{F}(t)$-adapted and $\langle\boldsymbol{r},\boldsymbol{u}^\varepsilon(t)\rangle\leq m(\varepsilon,t)$ for $t\geq 0$.

We use $\mathcal{A}^\varepsilon(\boldsymbol{x},m)$ to denote the set of all admissible controls $(\tau,\boldsymbol{u}^\varepsilon(\cdot))$. Then the problem is:

$$\mathcal{P}^\varepsilon:\begin{cases}\min_{(\tau,\boldsymbol{u}^\varepsilon(\cdot))\in\mathcal{A}^\varepsilon} & J^\varepsilon(\boldsymbol{x},m,\tau,\boldsymbol{u}^\varepsilon(\cdot)),\\ \quad\text{subject to} & \dot{\boldsymbol{x}}^\varepsilon(t)=\boldsymbol{u}^\varepsilon(t)-\boldsymbol{z},\ \boldsymbol{x}^\varepsilon(0)=\boldsymbol{x}.\end{cases}$$

We use $v^\varepsilon(\boldsymbol{x},m)$ to denote the value function of the problem and define an auxiliary value function $v_a^\varepsilon(\boldsymbol{x},\hat{m})$ to be K plus the optimal cost with the capacity process $m_2(\varepsilon,t)$ with the initial capacity $\hat{m}\in\mathcal{M}_2$ and no future capital expansion possibilities. Then the dynamic programming equations are as follows:

$$\begin{aligned}&\min\{\min_{\boldsymbol{u}\geq 0,\langle\boldsymbol{r},\boldsymbol{u}\rangle\leq m}[\langle(\boldsymbol{u}-\boldsymbol{z}),\partial v^\varepsilon(\boldsymbol{x},m)/\partial\boldsymbol{x}\rangle+G(\boldsymbol{x},\boldsymbol{u})]+\varepsilon^{-1}Q_1v^\varepsilon(\boldsymbol{x},\cdot)(m)\\ &\qquad -\rho v^\varepsilon(\boldsymbol{x},m),v_a^\varepsilon(\boldsymbol{x},m+p_2)-v^\varepsilon(\boldsymbol{x},m)\}=0,\quad m\in\mathcal{M}_1,\\ &\min_{\boldsymbol{u}\geq 0,\langle\boldsymbol{r},\boldsymbol{u}\rangle\leq m}[\langle(\boldsymbol{u}-\boldsymbol{z}),\partial v_a^\varepsilon(\boldsymbol{x},m)/\partial\boldsymbol{x}\rangle+G(\boldsymbol{x},\boldsymbol{u})]+\varepsilon^{-1}Q_2v_a^\varepsilon(\boldsymbol{x},\cdot)(m)\\ &\quad -\rho(v_a^\varepsilon(\boldsymbol{x},m)-K)=0,\quad m\in\mathcal{M}_2.\end{aligned}$$

Let $\nu^{(1)}=(\nu_0^{(1)},\nu_1^{(1)},\cdots,\nu_{p_1}^{(1)})$ and $\nu^{(2)}=(\nu_0^{(2)},\nu_1^{(2)},\cdots,\nu_{p_1+p_2}^{(2)})$ denote the equilibrium distributions of Q_1 and Q_2, respectively. We now proceed to develop the limiting problem. We first define the control sets for the limiting problem. Let $U_1=\{(\boldsymbol{u}^0,\cdots,\boldsymbol{u}^{p_1}):\boldsymbol{u}^i\geq 0,\langle\boldsymbol{r},\boldsymbol{u}^i\rangle\leq i\}$ and $U_2=\{(\boldsymbol{u}^0,\cdots,\boldsymbol{u}^{p_1+p_2}):\boldsymbol{u}^i\geq 0,\langle\boldsymbol{r},\boldsymbol{u}^i\rangle\leq i\}$. Then $U_1\subset R^{n\times(p_1+1)}$ and $U_2\subset R^{n\times(p_1+p_2+1)}$.

Definition 3.5. We use $\mathcal{A}^0(\boldsymbol{x})$ to denote the set of the following controls (*admissible controls* for the limiting problem): (i) A deterministic time σ; (ii) A deterministic $U(t)$ such that for $t<\sigma$, $U(t)=(\boldsymbol{u}^0(t),\cdots,\boldsymbol{u}^{p_1}(t))\in U_1$ and for $t\geq\sigma$, $U(t)=(\boldsymbol{u}^0(t),\cdots,\boldsymbol{u}^{p_1+p_2}(t))\in U_2$.

Let

$$\begin{aligned}J^0(\boldsymbol{x},\sigma,U(\cdot)) &= \int_0^\sigma e^{-\rho t}\sum_{i=0}^{p_1}\nu_i^{(1)}G(\boldsymbol{x}(t),\boldsymbol{u}^i(t))dt\\ &\qquad+\int_\sigma^\infty e^{-\rho t}\sum_{i=0}^{p_1+p_2}\boldsymbol{u}_i^{(2)}G(\boldsymbol{x}(t),\boldsymbol{u}^i(t))dt+e^{-\rho\sigma}K,\end{aligned}$$

$$\boldsymbol{u}(t) \quad = \quad \begin{cases} \sum_{i=0}^{p_1} \nu_i^{(1)} \boldsymbol{u}^i(t) & \text{if } \ t < \sigma, \\ \sum_{i=0}^{p_1+p_2} \nu_i \boldsymbol{u}^i(t) & \text{if } \ t \geq \sigma. \end{cases}$$

We can now define the following limiting optimal control problem:

$$\mathcal{P}^0 : \begin{cases} \min\limits_{(\sigma, U(\cdot)) \in \mathcal{A}^0} & J^0(\boldsymbol{x}, \sigma, U(\cdot)) \\ \text{subject to} & \dot{\boldsymbol{x}}(t) = \bar{\boldsymbol{u}}(t) - \boldsymbol{z}, \ \boldsymbol{x}(0) = \boldsymbol{x}. \end{cases} \tag{3.21}$$

Let $v(\boldsymbol{x})$ denote the value functions for $\mathcal{P}^0$, and $v_a(\boldsymbol{x}))$ denote $\min_{(0,U(\cdot))\in\mathcal{A}^0} J^0(\boldsymbol{x}, 0, U(\cdot))$. Let $(\tau, U(\cdot)) \in \mathcal{A}^0$ denote any admissible control for the limiting problem $\mathcal{P}^0$, where

$$U(t) = \begin{cases} (\boldsymbol{u}^0(t), \cdots, \boldsymbol{u}^{p_1}(t)) \in U_1 & \text{if } \ t < \sigma, \\ (\boldsymbol{u}^0(t), \cdots, \boldsymbol{u}^{p_1+p_2}(t)) \in U_2 & \text{if } \ t \geq \sigma. \end{cases}$$

We take

$$\boldsymbol{u}^\varepsilon(t) = \begin{cases} \sum_{i=0}^{p_1} \boldsymbol{u}^i(t) I_{\{m_1(\varepsilon,t)=i\}} & \text{if } \ t < \tau, \\ \sum_{i=0}^{p_1+p_2} \boldsymbol{u}^i(t) I_{\{m_2(\varepsilon,t)=i\}} & \text{if } \ t \geq \tau. \end{cases}$$

Then the control $(\tau, \boldsymbol{u}^\varepsilon(\cdot))$ is admissible for $\mathcal{P}^\varepsilon$. The following result is proved in Sethi et al. (1992b).

Theorem 3.3. (i) *There exists a constant* C_{36} *such that* $|v^\varepsilon(\boldsymbol{x}, m) - v(\boldsymbol{x})| + |v_a^\varepsilon(\boldsymbol{x}, m) - v_a(\boldsymbol{x})| \leq C_{36}(1 + |\boldsymbol{x}|^{\kappa_{33}})\sqrt{\varepsilon}$.

(ii) *Let* $(\tau, U(\cdot)) \in \mathcal{A}^0$ *be an* ε*-optimal control for the limiting problem* $\mathcal{P}^0$ *and let* $(\tau, \boldsymbol{u}^\varepsilon(\cdot)) \in \mathcal{A}^\varepsilon$ *be the control constructed above. Then,* $(\tau, \boldsymbol{u}^\varepsilon(\cdot))$ *is asymptotically optimal with error bound* $\sqrt{\varepsilon}$*, i.e.,* $|J^\varepsilon(\boldsymbol{x}, m, \tau, \boldsymbol{u}^\varepsilon(\cdot)) - v^\varepsilon(\boldsymbol{x}, m)| \leq C_{36}(1 + |\boldsymbol{x}|^{\kappa_{33}})\sqrt{\varepsilon}$.

3.6 Other multilevel models

Sethi and Zhang (1992b, 1995a) extend the model in Section 3.1 to incorporate promotional or advertising decisions that influence the product demands. Zhou and Sethi (1994) demonstrate how workforce and production decisions can be decomposed hierarchically in a stochastic version of the classical HMMS model (see Holt et al. (1960)). Manufacturing systems involving preventive maintenance are studied by Boukas and Haurie (1990), Boukas (1991), Boukas et al. (1993), and Boukas et al. (1994). The maintenance activity involves lubrication, routine adjustments, etc., which reduce the machine failure rates. The objective in these systems is to choose the rate of maintenance and the rate of production in order to minimize the total discounted cost of surplus, production, and maintenance.

In this section, we shall only discuss the model developed in Sethi and Zhang (1995a), who consider the case when both capacity and demand are finite state Markov processes constructed from generators that depend on the production and promotional decisions, respectively. In order to specify their marketing-production problem, let $m(\varepsilon, t) \in \mathcal{M}$ as in Section 3.1 and $\boldsymbol{z}(\delta, t) \in \{\boldsymbol{z}^0, \boldsymbol{z}^1, \cdots, \boldsymbol{z}^{\hat{p}}\}$ for a given δ, denote the capacity process and the demand process, respectively.

Definition 3.6. We say that a control $(\boldsymbol{u}^\varepsilon(\cdot), w(\cdot)) = \{(\boldsymbol{u}^\varepsilon(t), w^\delta(t)) : \ t \geq 0\}$ is *admissible*, if (i) $(\boldsymbol{u}^\varepsilon(\cdot), w^\delta(\cdot))$ is right-continuous having left-hand limit (RCLL); (ii) $(\boldsymbol{u}^\varepsilon(\cdot), w^\delta(\cdot))$ is $\sigma\{(m(\varepsilon, s), \boldsymbol{z}(\delta, s)) : \ 0 \leq s \leq t\}$-adapted, and satisfies $\boldsymbol{u}^\varepsilon(t) \geq 0$, $\langle \boldsymbol{r}, \boldsymbol{u}^\varepsilon(t)\rangle \leq m(\varepsilon, t)$, and $0 \leq w^\delta(t) \leq 1$ for all $t \geq 0$.

We use $\mathcal{A}^{\varepsilon,\delta}(\boldsymbol{x}, m, \boldsymbol{z})$ to denote the set of all admissible controls. Then the control problem can be written as follows:

$$\mathcal{P}^{\varepsilon,\delta}: \begin{cases} \text{maximize} & J^{\varepsilon,\delta}(\boldsymbol{x}, m, \boldsymbol{z}, \boldsymbol{u}^\varepsilon(\cdot), w^\delta(\cdot)) \\ & = E\int_0^\infty e^{-\rho t} G(\boldsymbol{x}^{\varepsilon,\delta}(t), \boldsymbol{z}(\delta, t), \boldsymbol{u}^\varepsilon(t), w^\delta(t))dt \\ \text{subject to} & \begin{cases} \dot{\boldsymbol{x}}^{\varepsilon,\delta}(t) = \boldsymbol{u}^\varepsilon(t) - \boldsymbol{z}(\delta, t), & \boldsymbol{x}^{\varepsilon,\delta}(0) = \boldsymbol{x} \\ m(\varepsilon, t) \sim \varepsilon^{-1} Q(\boldsymbol{u}^\varepsilon(t)), & m(\varepsilon, 0) = m \\ \boldsymbol{z}(\delta, t) \sim \delta^{-1}\hat{Q}(w(t)), & \boldsymbol{z}(\delta, 0) = \boldsymbol{z} \\ (\boldsymbol{u}(\cdot), w^\delta(\cdot)) \in \mathcal{A}^{\varepsilon,\delta}(\boldsymbol{x}.m, \boldsymbol{z}). & \end{cases} \\ \text{value function} & v^{\varepsilon,\delta}(\boldsymbol{x}, m, \boldsymbol{z}) = \inf\limits_{(\boldsymbol{u}^\varepsilon(\cdot), w^\delta(\cdot)) \in \mathcal{A}^{\varepsilon,\delta}(\boldsymbol{x}, m, \boldsymbol{z})} J^{\varepsilon,\delta}(\boldsymbol{x}, m, \boldsymbol{z}, \boldsymbol{u}^\varepsilon(\cdot), w^\delta(\cdot)) \end{cases} \tag{3.22}$$

where by $m(\varepsilon, t) \sim \varepsilon^{-1} Q(\boldsymbol{u}^\varepsilon(t))$, we mean that the Markov process $m(\varepsilon, t)$ has the generator $\varepsilon^{-1} Q(\boldsymbol{u}^\varepsilon(t))$. We use $\mathcal{A}^{0,\delta}$ to denote the admissible control space

$$\begin{aligned} \mathcal{A}^{0,\delta} = \quad & \{(U(t), w(t)) = (\boldsymbol{u}^0(t), \boldsymbol{u}^1(t), \cdots, \boldsymbol{u}^p(t), w(t)) : \boldsymbol{u}^i(t) \geq 0, \langle \boldsymbol{r}, \boldsymbol{u}^i(t)\rangle \leq i, 0 \leq w(t) \leq 1, \\ & (U(t), w(t)) \text{ is } \sigma\{\boldsymbol{z}(\delta, s) : \ 0 \leq s \leq t\} \text{ adapted and RCLL}\}, \end{aligned}$$

for the problem

$$
\mathcal{P}^{0,\delta}: \begin{cases} \text{maximize} & J^{0,\delta}(\boldsymbol{x},\boldsymbol{z},U(\cdot),w(\cdot)) \\ & = \displaystyle\int_0^\infty e^{-\rho t}\sum_{i=0}^p \nu_i(U(t))G(\boldsymbol{x}^\delta(t),\boldsymbol{z}(\delta,t),\boldsymbol{u}^i(t),w(t))dt \\ \text{subject to} & \begin{cases} \dot{\boldsymbol{x}}^\delta(t) = \displaystyle\sum_{i=0}^p \nu_i(U(t))\boldsymbol{u}^i(t) - \boldsymbol{z}(\delta,t), & \boldsymbol{x}^\delta(0)=\boldsymbol{x} \\ \boldsymbol{z}(\delta,t) \sim \dfrac{1}{\delta}\hat{Q}(w(t)), & \boldsymbol{z}(\delta,0)=\boldsymbol{z} \\ (U(\cdot),w(\cdot)) \in \mathcal{A}^{0,\delta}. \end{cases} \\ \text{value function} & v^{0,\delta}(\boldsymbol{x},\boldsymbol{z}) = \displaystyle\inf_{(U(\cdot),w(\cdot))\in\mathcal{A}^{0,\delta}} J^{0,\delta}(\boldsymbol{x},\boldsymbol{z},U(\cdot),w(\cdot)). \end{cases} \tag{3.23}
$$

Let $(U(\cdot),w(\cdot)) \in \mathcal{A}^{0,\delta}$ denote an optimal open-loop control. We construct $\boldsymbol{u}^{\varepsilon,\delta}(t) = \sum_{i=0}^p \boldsymbol{u}^i(t) I_{\{m(\varepsilon,t)=i\}}$ and $w^{\varepsilon,\delta}(t) = w(t)$. Then $(\boldsymbol{u}^{\varepsilon,\delta}(t), w^{\varepsilon,\delta}(t)) \in \mathcal{A}^{\varepsilon,\delta}$, and it is asymptotically optimal, i.e.,

$$\lim_{\delta\to 0} |J^{\varepsilon,\delta}(\boldsymbol{x},m,\boldsymbol{z},\boldsymbol{u}^{\varepsilon,\delta}(\cdot),w^{\varepsilon,\delta}(\cdot)) - v^{\varepsilon,\delta}(\boldsymbol{x},m,\boldsymbol{z})| = 0.$$

Similarly, let $(U(\boldsymbol{x},\boldsymbol{z}), w(\boldsymbol{x},\boldsymbol{z})) \in \mathcal{A}^{\varepsilon,\delta}$ denote an optimal feedback control for $\mathcal{P}^{0,\delta}$. Suppose that $(U(\boldsymbol{x},\boldsymbol{z}), w(\boldsymbol{x},\boldsymbol{z}))$ is locally Lipschitz for each $\boldsymbol{z}$. Let $\boldsymbol{u}^{\varepsilon,\delta}(t) = \sum_{i=0}^p \boldsymbol{u}^i(\boldsymbol{x}^{\varepsilon,\delta}(t), m(\varepsilon,t), \boldsymbol{z}(\delta,t)) I_{\{m(\varepsilon,t)=i\}}$ and $w^{\varepsilon,\delta}(t) = w(\boldsymbol{x}^{\varepsilon,\delta}(t),\boldsymbol{z}(\delta,t))$. Then the feedback control $(\boldsymbol{u}^{\varepsilon,\delta}(\cdot), w^{\varepsilon,\delta}(\cdot))$ is asymptotically optimal for $\mathcal{P}^{\varepsilon,\delta}$, i.e.,

$$\lim_{\delta\to 0} |J^{\varepsilon,\delta}(\boldsymbol{x},m,\boldsymbol{z},\boldsymbol{u}^{\varepsilon,\delta}(\cdot),w^{\varepsilon,\delta}(\cdot)) - v^{\varepsilon,\delta}(\boldsymbol{x},m,\boldsymbol{z})| = 0.$$

We have described only the hierarchy that arises from a large δ and a small ε. In this case, promotional decisions are obtained under the assumption that the available production capacity is equal to the average capacity. Subsequently, production decisions taking into account the stochastic nature of the capacity can be constructed. Other possible hierarchies result when both δ and ε are small or when ε is large and δ is small.

3.7 Single or parallel machine systems with risk-sensitive discounted cost criterion

Consider the single/parallel machine system producing multiple products described in Subsection 3.1, but with the risk-sensitive discounted cost criterion defined by

$$J^{\varepsilon,\sqrt{\varepsilon}}(\boldsymbol{u}^\varepsilon(\cdot)) = \sqrt{\varepsilon}\log E\left[\exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^\infty e^{-\rho t}[h(\boldsymbol{x}^\varepsilon(t)) + c(\boldsymbol{u}^\varepsilon(t))]dt\right\}\right]. \tag{3.24}$$

A motivation for choosing this risk-sensitive criterion is to incorporate the decision maker's attitude toward risk. In (3.24), $\sqrt{\varepsilon}$ is called the risk-sensitivity coefficient. A positive coefficient indicates a risk-averse behavior, whereas $\sqrt{\varepsilon} = 0$ signifies risk neutrality. The problem is then as follows:

$$\mathcal{P}^{\varepsilon,\sqrt{\varepsilon}} : \begin{cases} \text{minimize} & J^{\varepsilon,\sqrt{\varepsilon}}(\boldsymbol{u}^\varepsilon(\cdot)), \\ \text{subject to} & \dot{\boldsymbol{x}}^\varepsilon(t) = \boldsymbol{u}^\varepsilon(t) - \boldsymbol{z},\ \boldsymbol{x}^\varepsilon(0) = \boldsymbol{x},\ \boldsymbol{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(m), \\ \text{value function} & v^{\varepsilon,\sqrt{\varepsilon}}(\boldsymbol{x}, m) = \inf_{\boldsymbol{u}^\varepsilon(\cdot)\in\mathcal{A}^\varepsilon(m)} J^{\varepsilon,\sqrt{\varepsilon}}(\boldsymbol{u}^\varepsilon(\cdot)). \end{cases}$$

For this problem, Zhang (1995) establishes results similar to Theorem 3.1.

In the next two sections we focus on the long-run average cost criterion, which presents a stark contrast to the discounted-cost criterion. The discounted-cost criterion considers near-term costs to be more important than costs occurring in the long term. The long-run cost criterion, on the other hand, ignores the near-term costs and considers only the distant future costs. In Section 4, we consider the theory of optimal control of stochastic manufacturing problems with the long-run average cost criterion. We use the vanishing discount approach for this purpose. Section 5 is concerned with hierarchical controls for long-run average cost problems, for risk-sensitive average cost problems, and for Markov decision processes with weak and strong interactions.

4 Optimal Control with Long-Run Average Cost Criteria

Beginning with Bielecki and Kumar (1988), there has been a considerable interest in studying the problem of convex production planning in stochastic manufacturing systems with the objective of minimizing long-run average cost. Bielecki and Kumar (1988) dealt with a single machine (with two states: up and down), single product problem with linear holding and backlog costs. Because of the simple structure of their problem, they were able to obtain an explicit solution, and thus verify the optimality of the resulting policy. They showed that the so-called hedging point policy is optimal in their simple model. It is a policy to produce at full capacity if the surplus is smaller than a threshold level, produce nothing if the surplus is higher than the threshold level, and produce as much as possible, but no more than the demand rate when the surplus is at the threshold level.

Sharifnia (1988) dealt with an extension of the Bielecki-Kumar model with more than two machine states. Liberopoulous and Caramanis (1995) showed that Sharifnia's method for evaluating hedging point policies applies even when the transition rates of the machine states depend on the production rate. Liberopoulous and Hu (1995) obtained monotonicity of the threshold levels corresponding to different machine states. Srivatsan and Dallery (1998) generalized the Bielecki-Kumar problem to allow for two products. They limited their focus to only the class of hedging point policies and attempted to partially characterize an optimal solution within that class. Bai and Gershwin (1990) and Bai (1991) use heuristic argument to obtain suboptimal controls in two-machine flowshops. In addition to nonnegative constraints on the inventory levels in the internal buffer, they also consider inventory level in this buffer to be bounded above by the size of the buffer. Moreover, Srivatsan et al. (1994) apply their results to semiconductor manufacturing (jobshop). All of these papers, however, are heuristic in nature, since they do not rigorously prove the optimality of the policies for their extensions of the Bielecki-Kumar model.

Presman et al. (1998) extend the hedging point policy to the problem of two products by using the potential function related to the dynamic programming equation of the two-product problem. Using a verification theorem for the two-product problem, they prove the optimality of the hedging point policy. Furthermore, Sethi and Zhang (1999) prove the optimality of the hedging point policy for a general n-product problem under a special class of cost functions. For its deterministic version, Sethi et al. (1996) provide explicit optimal control policies and the value function.

The difficulty in proving the optimality in general rather than for these special cases lies in the fact that when the problem is generalized to include convex cost and multiple machine capacity levels, explicit solutions are no longer possible. One then needs to develop appropriate dynamic programming equations, existence of their solutions, and verification theorems for optimality. In this section, we will review some works related to this.

4.1 Single or parallel machine systems

We consider an n-product manufacturing system given in Section 2.1. For any $\boldsymbol{u}(\cdot) \in \mathcal{A}(m)$, define

$$\bar{J}(\boldsymbol{x}, m, \boldsymbol{u}(\cdot)) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T [h(\boldsymbol{x}(t)) + c(\boldsymbol{u}(t))]dt, \tag{4.1}$$

where $\boldsymbol{x}(\cdot)$ is the surplus process corresponding to the production process $\boldsymbol{u}(\cdot)$ with $\boldsymbol{x}(0) = \boldsymbol{x}$, and $h(\cdot)$, $c(\cdot)$ and $m(\cdot)$ are as given in Section 2.1. Our goal is to choose $\boldsymbol{u}(\cdot) \in \mathcal{A}(m)$ so as to minimize the average cost $\bar{J}(\boldsymbol{x}, m, \boldsymbol{u}(\cdot))$. In addition to Assumption (A.2.2) on the production cost $c(\cdot)$, we assume the production capacity process $m(\cdot)$ and the surplus cost function $h(\cdot)$ to satisfy the following:

(A.4.1) $h(\cdot)$ is nonnegative and convex with $h(0) = 0$. There are positive constants C_{41}, C_{42}, and κ_{41} such that $h(\boldsymbol{x}) \geq C_{41}|\boldsymbol{x}|^{\kappa_{41}} - C_{42}$. Moreover, there are constants C_{43} and κ_{42} such that $|h(\boldsymbol{x}) - h(\hat{\boldsymbol{x}})| \leq C_{43}(1 + |\boldsymbol{x}|^{k_{42}-1} + |\hat{\boldsymbol{x}})|^{\kappa_{42}-1})|\boldsymbol{x} - \hat{\boldsymbol{x}}|$.

(A.4.2) $m(t)$ is a finite state Markov chain with generator Q, where $Q = (q_{ij})$, $i, j \in \mathcal{M}$ is a $(p+1) \times (p+1)$ matrix such that $q_{ij} \geq 0$ for $i \neq j$ and $q_{ii} = -\sum_{i \neq j} q_{ij}$. We assume that Q is weakly irreducible. Let $\nu = (\nu_0, \nu_1, ..., \nu_p)$ be the equilibrium distribution vector of $m(t)$.

(A.4.3) The average capacity $\bar{m} = \sum_{j=0}^{p} j\nu_j > \sum_{i=1}^{n} z_i$.

Definition 4.1. A control $\boldsymbol{u}(\cdot) \in \mathcal{A}(m)$ is called *stable* if $\lim_{T\to\infty} E|\boldsymbol{x}(T)|^{\kappa_{42}+1}/T = 0$, where $\boldsymbol{x}(\cdot)$ is the surplus process corresponding to the control $\boldsymbol{u}(\cdot)$ with $(\boldsymbol{x}(0), m(0)) = (\boldsymbol{x}, m)$ and κ_{42} is defined in Assumption (A.4.1). Let $\mathcal{B}(m) \subset \mathcal{A}(m)$ denote the class of stable controls.

It can be shown that there exists a constant λ^*, independent of the initial condition $(\boldsymbol{x}(0), m(0)) = (\boldsymbol{x}, m)$, and a stable Markov control policy $\boldsymbol{u}^*(\cdot) \in \mathcal{A}(m)$ such that $\boldsymbol{u}^*(\cdot)$ is optimal, i.e., it minimizes the average cost defined by (4.1) over all $\boldsymbol{u}(\cdot) \in \mathcal{A}(m)$, and furthermore,

$$\lim_{T\to\infty} \frac{1}{T} E \int_0^T [h(\boldsymbol{x}^*(t)) + c(\boldsymbol{u}^*(t))]dt = \lambda^*, \tag{4.2}$$

where $\boldsymbol{x}^*(\cdot)$ is the surplus process corresponding to $\boldsymbol{u}^*(\cdot)$ with $(\boldsymbol{x}(0), m(0)) = (\boldsymbol{x}, m)$. Moreover, for any other (stable) control $\boldsymbol{u}(\cdot) \in \mathcal{B}(m)$,

$$\liminf_{T\to\infty} \frac{1}{T} E \int_0^T [h(\boldsymbol{x}(t)) + c(\boldsymbol{u}(t))]dt \geq \lambda^*. \tag{4.3}$$

Since we use the vanishing discount approach to treat the problem, we provide the required results for the discounted problem. First, we introduce a corresponding control problem with the cost discounted at a rate $\rho > 0$. For $\boldsymbol{u}(\cdot) \in \mathcal{A}(m)$, we define the expected discounted cost as

$$J^\rho(\boldsymbol{x}, m, \boldsymbol{u}(\cdot)) = E \int_0^\infty e^{-\rho t}[h(\boldsymbol{x}(t)) + c(\boldsymbol{u}(t))]\, dt.$$

The value function of the discounted problem is defined as

$$V^\rho(\boldsymbol{x}, m) = \inf_{\boldsymbol{u}(\cdot)\in\mathcal{A}(m)} J^\rho(\boldsymbol{x}, m, \boldsymbol{u}(\cdot)). \tag{4.4}$$

In order to study the long-run average cost control problem using the *vanishing discount approach*, we must first obtain some properties of the value function $V^\rho(\boldsymbol{x}, m)$. Sethi et al. (1997a) prove the following properties.

Theorem 4.1. (i) *There exists a constant* $\rho_0 > 0$ *such that* $\{\rho V^\rho(0,0) : 0 < \rho \leq \rho_0\}$ *is bounded.*

(ii) *The function* $W^\rho(\boldsymbol{x}, m) = V^\rho(\boldsymbol{x}, m) - V^\rho(0,0)$ *is convex in* $\boldsymbol{x}$. *It is locally uniformly bounded, i.e., there exists a constant* $C_{44} > 0$ *such that* $|V^\rho(\boldsymbol{x}, m) - V^\rho(0,0)| \leq C_{44}(1 + |\boldsymbol{x}|^{\kappa_{42}+1})$ *for all* $(\boldsymbol{x}, m) \in R \times \mathcal{M}$, $\rho \geq 0$.

(iii) $W^\rho(\boldsymbol{x},m)$ *is locally uniformly Lipschitz continuous in* $\boldsymbol{x}$, *with respect to* $\rho>0$, *i.e., for any* $X>0$, *there exists a constant* $C_{45}>0$, *independent of* ρ, *such that* $|W^\rho(\boldsymbol{x},m)-W^\rho(\hat{\boldsymbol{x}},m)|\le C_{45}|\boldsymbol{x}-\hat{\boldsymbol{x}}|$ *for all* $m\in\mathcal{M}$ *and all* $|\boldsymbol{x}|,|\hat{\boldsymbol{x}}|\le X$.

The HJB equation associated with the long-run average cost optimal control problem as formulated above takes the following form

$$\lambda=\inf_{\boldsymbol{u}\in\mathcal{A}(m)}\{\langle W_{\boldsymbol{x}}(\boldsymbol{x},m),\boldsymbol{u}-\boldsymbol{z}\rangle+c(\boldsymbol{u})\}+h(\boldsymbol{x})+QW(\boldsymbol{x},\cdot)(m), \tag{4.5}$$

where λ is a constant, $W(\cdot,m)$ is a real-valued function, known as the potential function or the relative value function defined on $R^n\times\mathcal{M}$. Without requiring that $W(\cdot,m)$ is C^1, it is convenient to write the HJBDD equation for our problem as follows:

$$\lambda=\inf_{\boldsymbol{u}\in\mathcal{A}(m)}\{\partial W(\boldsymbol{x},m)/\partial(\boldsymbol{u}-\boldsymbol{z})+c(\boldsymbol{u})\}+h(\boldsymbol{x})+QW(\boldsymbol{x},\cdot)(m). \tag{4.6}$$

Let $\mathcal{G}$ denote the family of real-valued functions $W(\cdot,\cdot)$ defined on $R\times\mathcal{M}$ such that $W(\cdot,m)$ is convex and $W(\cdot,m)$ has polynomial growth, i.e., there are constants κ_{43} and $C_{46}>0$ such that $|W(\boldsymbol{x},m)|\le C_{46}(1+|\boldsymbol{x}|^{\kappa_{43}+1}),\forall\boldsymbol{x}\in R$.

A solution to the HJB or HJBDD equation is a pair $(\lambda,W(\cdot,\cdot)$ with λ a constant and $W(\cdot,\cdot)\in\mathcal{G}$. The function $W(\cdot,\cdot)$ is called the *potential function* for the control problem, if λ is the minimum long-run average cost. The following result directly follows from Theorem 4.1.

Theorem 4.2. *For* $(\boldsymbol{x},m)\in R^n\times\mathcal{M}$, $\rho V^\rho(\boldsymbol{x},m)\to\lambda$ and $W^\rho(\boldsymbol{x},m)\to W^0(\boldsymbol{x},m)$ on a subsequence of $\rho\to 0$. *Furthermore,* $(\lambda,W^0(\cdot,\cdot))$ *is a viscosity solution to the HJB equation* (4.5).

Using results from convex analysis, Sethi et al. (1997a) prove the following theorem.

Theorem 4.3. $(\lambda,W^0(\cdot,\cdot))$ *defined in Theorem* 4.3 *is a solution to the HJBDD equation* (4.6).

Remark 4.1. When there is no cost of production, i.e., $c(\boldsymbol{u})\equiv 0$, Veatch and Caramanis (1999) introduce the following differential cost function

$$\hat{W}(\boldsymbol{x},m)=\lim_{T\to\infty}\left[E\int_0^T h(\boldsymbol{x}^*(t))dt-T\lambda^*\right],$$

where $m=m(0)$, λ^* is the optimal value, and $\boldsymbol{x}^*(t)$ is the surplus process corresponding to the optimal production process $\boldsymbol{u}^*(\cdot)$ with $\boldsymbol{x}=\boldsymbol{x}^*(0)$. The differential cost function is used in the algorithms to compute a reasonable control policy using infinitesimal perturbation analysis or direct computation of average cost; see Caramanis and Liberopoulos (1992), and Liberopoulos and Caramanis (1995). They prove that the differential cost function $\hat{W}(\boldsymbol{x},m)$ is convex and differentiable in $\boldsymbol{x}$. If $n = 1$, $h(x_1) = |x_1|$ and $\mathcal{M} = \{0, 1\}$, we know from Bielecki and Kumar (1988) that

$$\hat{W}(x, m) = W^0(x, m). \tag{4.7}$$

This means that the differential cost function is the same as the potential function given by Theorem 4.2. However, so far (4.7) has not been established in general. Now we state the following verification theorem proved by Sethi et al. (1998a).

Theorem 4.4. *Let* $(\lambda, W(\cdot, \cdot))$ *be a solution to the HJBDD equation* (4.6)*. Then the following holds.* (i) *If there is a control* $\boldsymbol{u}^*(\cdot) \in \mathcal{A}(m)$ *such that*

$$\inf_{\boldsymbol{u} \in \mathcal{A}(m(t))} \left\{ \frac{\partial W(\boldsymbol{x}^*(t), m(t))}{\partial(\boldsymbol{u} - \boldsymbol{z})} + c(\boldsymbol{u}) \right\} = \frac{\partial W(\boldsymbol{x}^*(t), m(t))}{\partial(\boldsymbol{u}^*(t) - \boldsymbol{z})} + c(\boldsymbol{u}^*(t)) \tag{4.8}$$

for a.e. $t \geq 0$ *with probability one, where* $\boldsymbol{x}^*(\cdot)$ *is the surplus process corresponding to the control* $\boldsymbol{u}^*(\cdot)$, *and* $\lim_{T \to \infty} W(\boldsymbol{x}^*(T), m(T))/T = 0$, *then* $\lambda = J(\boldsymbol{x}, m, \boldsymbol{u}^*(\cdot))$.

(ii) *For any* $\boldsymbol{u}(\cdot) \in \mathcal{A}(m)$, *we have* $\lambda \leq J(\boldsymbol{x}, m, \boldsymbol{u}(\cdot))$.

(iii) *Furthermore, for any (stable) control policy* $\boldsymbol{u}(\cdot) \in \mathcal{B}(m)$, *we have*

$$\liminf_{T \to \infty} (1/T) E \int_0^T [h(\boldsymbol{x}(t)) + c(\boldsymbol{u}(t))] dt \geq \lambda.$$

In the remainder of this section, let us consider the single product case, i.e., $n = 1$. For this case, Sethi et al. (1997a) prove the following result.

Theorem 4.5. *For* λ *and* $W^0(x, m)$ *given in Theorem* 4.2, *we have that* $W^0(x, m)$ *is continuously differentiable in* x and $(\lambda, W^0(\cdot, \cdot))$ *is a classical solution to the HJB equation* (4.5).

Let us define a control policy $\hat{u}(\cdot, \cdot)$ via the potential function $W^0(\cdot, \cdot)$ as follows:

$$\hat{u}(x, m) = \begin{cases} 0 & \text{if } \partial W^0(x, m)/\partial x > -\dot{c}(0), \\ (\dot{c})^{-1}(-\partial W^0(x, m)/\partial x) & \text{if } -\dot{c}(m) \leq \partial W^0(x, m)/\partial x \leq \dot{c}(0), \\ m & \text{if } \partial W^0(x, m)/\partial x < -\dot{c}(m), \end{cases} \tag{4.9}$$

if the function $c(\cdot)$ is strictly convex, or

$$\hat{u}(x, m) = \begin{cases} 0 & \text{if } \partial W^0(x, m)/\partial x > -c, \\ k \wedge z & \text{if } \partial W^0(x, m)/\partial x = -c, \\ m & \text{if } \partial W^0(x, m)/\partial x < -c, \end{cases} \tag{4.10}$$

if $c(u) = cu$. Therefore, the control policy $\hat{u}(\cdot, \cdot)$ satisfies (i) of Theorem 4.4. From the convexity of the potential function $W^0(\cdot, m)$, there are $x_m,\ y_m,\ -\infty < y_m < x_m < \infty$, such that $(x_m, \infty) = \{x\ :\ \partial W^0(x,m)/\partial x > -\dot{c}(0)\}$ and $(-\infty, y_m) = \{x\ :\ \partial W^0(x,m)/\partial x < -\dot{c}(m)\}$. The control policy $\hat{u}(\cdot, \cdot)$ can be written as

$$\hat{u}(x,m) = \begin{cases} 0 & \text{if } x > x_m, \\ (\dot{c})^{-1}(-\partial W^0(x,m)/\partial x) & \text{if } y_m \leq x \leq x_m, \\ m & \text{if } x < y_m. \end{cases}$$

Then we have the following result.

Theorem 4.6. *The control policy $\hat{u}(\cdot, \cdot)$ defined in* (4.9) *and* (4.10), *as the case may be, is optimal.*

By Theorem 4.4, to get Theorem 4.6, we need only to show that $\lim_{t\to\infty} W^0(\hat{x}(t), m(t))/t = 0$. But this is implied by Theorem 4.5 and the fact that $\hat{u}(\cdot, \cdot)$ is a stable control.

Remark 4.2. When $c(u) = 0$, i.e., there is no production cost in the model, the optimal control policy can be chosen to be the so-called *hedging point policy*, which has the following form: there are real numbers $x_k,\ k = 1, ..., m$, such that

$$\hat{u}(x,k) = \begin{cases} 0 & \text{if } x > x_k, \\ k \wedge z & \text{if } x = x_k, \\ k & \text{if } x < x_k. \end{cases}$$

In particular, if $h(x) = c_1 x^+ + c_2 x^-$ with $x^+ = \max\{0, x\}$ and $x^- = \max\{0, -x\}$, we obtain the special case of Bielecki and Kumar (1988). This will be reviewed next. When $c(u) \neq 0$, just as in Section 2.1 for the case with the discounted cost criterion, we can also get some properties related to the turnpike set; see Sethi et al. (2001).

The Bielecki-Kumar Case: Bielecki and Kumar (1988) treated the special case in which $h(x) = c_1 x^+ + c_2 x^-$, $c(u) = 0$, and the production capacity $m(\cdot)$ is a two-state birth-death Markov process. Thus, the binary variable $m(\cdot)$ takes the value one when the machine is up and zero when it is down. Let $1/q_1$ and $1/q_0$ represent the mean time between failures and the mean repair time, respectively. Bielecki and Kumar obtain the following explicit solution:

$$\hat{u}(x,k) = \begin{cases} 0 & \text{if } x > x^*, \\ k \wedge z & \text{if } x = x^*, \\ k & \text{if } x < x^*, \end{cases}$$

where

$$\hat{x}^* = \begin{cases} 0 & \text{if } \frac{q_1(c_1+c_2)}{c_1(1-z)(q_0+q_1)} \text{ and } \frac{1-z}{q_1} > \frac{z}{q_0}, \\ \frac{1}{(q_0/z)-(q_1/(1-z))} \log\left[\frac{q_1(c_1+c_2)}{c_1(1-z)(q_0+q_1)}\right] & \text{otherwise.} \end{cases}$$

Remark 4.3. When the system equation is governed by the stochastic differential equation $dx(t) = b(x(t), \alpha(t), u(t))dt + g(x(t), \alpha(t))d\xi(t)$, where $b(\cdot, \cdot, \cdot)$, $g(\cdot, \cdot)$ are suitable functions and $\xi(t)$ is a standard Brownian motion, Ghosh et al. (1993, 1997) and Basak et al. (1997) have studied the corresponding HJB equation and established the existence of their solutions and the existence of an optimal control under certain conditions. In particular, Basak et al. (1997) allow the matrix $g(\cdot, \cdot)$ to be of any rank between 1 and n.

Remark 4.4. For $n = 2$ and $c(\boldsymbol{u}) = 0$, Srivatsan and Dallery (1998) limit their focus to only the class of hedging point policies and attempt to partially characterize an optimal solution within this class.

Remark 4.5. Abbad et al. (1992) and Filar et al. (1999) consider the perturbed stochastic hybrid system whose continuous part is described by the following stochastic differential equation $dx(t) = \varepsilon^{-1} f(x(t), u(t))dt + \varepsilon^{-1/2} A d\xi(t)$, where $f(\cdot, \cdot)$ is continuous in both arguments, A is an $n \times n$ matrix, and $\xi(t)$ is a Brownian motion. The perturbation parameter ε is assumed to be small. They prove that when ε tends to zero, the optimal solution of the perturbed hybrid system can be approximated by a structured linear program.

Remark 4.6. Duncan el al. (2001) extend the model of Sethi et al. (1997a) to allow for a Markovian demand. Feng and Xiao (2002) incorporate a Markovian demand in a discrete-state version of the model of Bielecki and Kumar (1988).

4.2 Dynamic flowshops

For a dynamic flowshop with the long-run average cost criterion, Presman et al. (2000a) establish a verification theorem similar to Theorem 4.4 in terms of the corresponding HJBDD equations. Based on the verification theorem, they characterize the optimal solution. Furthermore, Presman et al. (2000c) extend these results to the case of a two-machine flowshop with a limited buffer. All these results are special cases of results on dynamic jobshops reviewed in the next section.

4.3 Dynamic jobshops

We consider the dynamic jobshop given by (2.5)-(2.7) in Section 2.3, but here our problem is to find an admissible control $\boldsymbol{u}(\cdot)$ that minimizes the long-run average cost

$$\bar{J}(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}(\cdot)) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T H(\boldsymbol{x}(t), \boldsymbol{u}(t)) dt, \tag{4.11}$$

where $H(\cdot, \cdot)$ defines the cost of surplus and production, $\boldsymbol{x}$ is the initial state, and $\boldsymbol{m}$ is the initial value of $\boldsymbol{m}(t)$. In addition to Assumptions (A.2.4) and (A.2.5) in Section 2.4, we assume the following:

(A.4.4) Let $(\nu_1, ..., \nu_p)$ be the stationary distribution of $\boldsymbol{m}(t)$. Let $p_n = \sum_{j=1}^p m_n^j \nu_j$ and $n(i,j) = \arg\{(i,j) \in K_n\}$ for $(i,j) \in \Pi$. Here p_n represents the average capacity of the machine n, and $n(i,j)$ is the number of machines placed on the arc (i,j). Let $\{p_{ij} > 0 : (i,j) \in K_n\} (n = 1, ..., N)$ be such that $\sum_{(i,j)\in K_n} p_{ij} \le 1$, $\sum_{\ell=0}^d p_{\ell i} p_{n(\ell,i)} > z_i$, $i = d+1, ..., N_b$, and $\sum_{\ell=0}^{i-1} p_{\ell i} p_{n(\ell,i)} > \sum_{\ell=i+1}^{N_b} p_{i\ell} p_{n(i,\ell)}$, $i = 1, ..., d$.

Let $\lambda(\boldsymbol{x}, \boldsymbol{m})$ denote the minimal expected cost, i.e., $\lambda(\boldsymbol{x}, \boldsymbol{m}) = \inf_{\boldsymbol{u}(\cdot)\in\mathcal{A}(\boldsymbol{x},\boldsymbol{m})} \bar{J}(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}(\cdot))$. In order to get the HJB equation for our problem, we introduce some notation. Let $\mathcal{G}$ denote the family of real-valued functions $f(\cdot, \cdot)$ defined on $\mathcal{S} \times \mathcal{M}$ such that $f(\cdot, \boldsymbol{m})$ is convex for any $\boldsymbol{m} \in \mathcal{M}$. Let $\hat{C}(\boldsymbol{x})$ be such that for any $\boldsymbol{m} \in \mathcal{M}$ and any $\boldsymbol{x}, \hat{\boldsymbol{x}} \in S$, $f(\boldsymbol{x}, \boldsymbol{m}) - f(\hat{\boldsymbol{x}}, \boldsymbol{m})| \le \hat{C}(\boldsymbol{x})|\boldsymbol{x} - \hat{\boldsymbol{x}}|$.

Consider the equation

$$\lambda = \inf_{\boldsymbol{u}\in U(\boldsymbol{x},\boldsymbol{m})} \{\partial f(\boldsymbol{x}, \boldsymbol{m})/\partial D\boldsymbol{u} + G(\boldsymbol{x}, \boldsymbol{u})\} + Qf(\boldsymbol{x}, \cdot)(\boldsymbol{m}), \tag{4.12}$$

where λ is a constant, $f(\cdot, \cdot) \in \mathcal{G}$. We have the following verification theorem due to Presman et al. (2000b).

Theorem 4.7. *Assume* (i) $(\lambda, f(\cdot, \cdot))$ *with* $f(\cdot, \cdot) \in \mathcal{G}$ *satisfies* (4.12); (ii) *there exists* $\boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{m})$ *for which*

$$\inf_{\boldsymbol{u}\in U(\boldsymbol{x},\boldsymbol{m})} \left\{ \frac{\partial f(\boldsymbol{x}, \boldsymbol{m})}{\partial D\boldsymbol{u}} + H(\boldsymbol{x}, \boldsymbol{u}) \right\} = \frac{\partial f(\boldsymbol{x}, \boldsymbol{m})}{\partial D\boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{m})} + H(\boldsymbol{x}, \boldsymbol{u}^*(\boldsymbol{x}, \boldsymbol{m})), \tag{4.13}$$

and the equation $\dot{\boldsymbol{x}}(t) = D\boldsymbol{u}^*(\boldsymbol{x}(t), \boldsymbol{m}(t))$, *has for any initial condition* $(\boldsymbol{x}^*(0), \boldsymbol{m}(0)) = (\boldsymbol{x}^0, \boldsymbol{m}^0)$, *a solution* $\boldsymbol{x}^*(t)$ *such that* $\lim_{T\to\infty} Ef(\boldsymbol{x}^*(T), \boldsymbol{m}(T))/T = 0$. *Then* $\boldsymbol{u}^*(t) = \boldsymbol{u}^*(\boldsymbol{x}^*(t), \boldsymbol{m}(t))$ *is an optimal control. Furthermore,* $\lambda(\boldsymbol{x}^0, \boldsymbol{m}^0)$ *does not depend on* $\boldsymbol{x}^0$ and $\boldsymbol{m}^0$, *and it coincides with* λ.

Moreover, for any $T > 0$,

$$
\begin{aligned}
f(\boldsymbol{x}^0, \boldsymbol{m}^0) &= \inf_{\boldsymbol{u}(\cdot) \in \mathcal{A}(\boldsymbol{x}^0, \boldsymbol{m}^0)} E\left[\int_0^T (H(\boldsymbol{x}(t), \boldsymbol{u}(t)) - \lambda)\, dt + f(\boldsymbol{x}(T), \boldsymbol{m}(T))\right] \\
&= E\left[\int_0^T (H(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t)) - \lambda)\, dt + f(\boldsymbol{x}^*(T), \boldsymbol{m}(T))\right]. \qquad (4.14)
\end{aligned}
$$

Next we try to construct a pair $(\lambda, W(\cdot,\cdot))$ which satisfies (4.12). To get this pair, we use the vanishing discount approach. Consider a corresponding control problem with the cost discounted at a rate $\rho > 0$. For $\boldsymbol{u}(\cdot) \in \mathcal{A}(\boldsymbol{x}, \boldsymbol{m})$, we define the expected discounted cost as

$$J^\rho(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}(\cdot)) = E\int_0^\infty e^{-\rho t} G(\boldsymbol{x}(t), \boldsymbol{u}(t)) dt.$$

Define the value function of the discounted cost problem as

$$V^\rho(\boldsymbol{x}, \boldsymbol{m}) = \inf_{\boldsymbol{u}(\cdot) \in \mathcal{A}(\boldsymbol{x}, \boldsymbol{m})} J^\rho(\boldsymbol{x}, \boldsymbol{m}, \boldsymbol{u}(\cdot)).$$

Theorem 4.8. *There exists a sequence $\{\rho_k : k \geq 1\}$ with $\rho_k \to 0$ as $k \to \infty$ such that for $(\boldsymbol{x}, \boldsymbol{m}) \in \mathcal{S} \times \mathcal{M}$, $\lim_{k\to\infty} \rho_k V^{\rho_k}(\boldsymbol{x}, \boldsymbol{m}) = \lambda$, and $\lim_{k\to\infty}[V^{\rho_k}(\boldsymbol{x}, \boldsymbol{m}) - V^{\rho_k}(0, \boldsymbol{m}^0)] = W^0(\boldsymbol{x}, \boldsymbol{m})$, where, $W^0(\boldsymbol{x}, \boldsymbol{m}) \in \mathcal{G}$.*

Theorem 4.9. *In our problem, $\lambda(\boldsymbol{x}, \boldsymbol{m})$ does not depend on $(\boldsymbol{x}, \boldsymbol{m})$, and the pair $(\lambda, W^0(\cdot,\cdot))$ defined in Theorem 4.8 is a solution to (4.12).*

For the proof of Theorems 4.8 and 4.9, see Presman et al. (2000b).

Remark 4.7. Assumption (A.4.5) is not needed in the discounted case. But it is necessary for the finiteness of the long-run average cost in the case when $h(\cdot,\cdot)$ tends to $+\infty$ as $x_{N_b} \to -\infty$.

5 Hierarchical Controls with the Long-Run Average Cost Criterion

In this section, the results on hierarchical controls with the long-run average cost criterion are reviewed. Hierarchical controls for stochastic manufacturing systems including single/parallel machine system, the flowshops, and the general jobshops are discussed. For each model, the corresponding limiting problem is given, and the optimal value of the original problem is shown to converge to the optimal value of the limiting problem. Also constructed is an asymptotic optimal control for the original problem by using a near-optimal control of the limiting problem. The rate of convergence and error bounds of the constructed control are provided.

5.1 Single or parallel machine systems

Let us consider a manufacturing system whose system dynamics satisfy the differential equation

$$\dot{\boldsymbol{x}}^{\varepsilon}(t) = -\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}^{\varepsilon}(t) + \boldsymbol{u}^{\varepsilon}(t) - \boldsymbol{z}, \quad \boldsymbol{x}^{\varepsilon}(0) = \boldsymbol{x} \in R^n, \tag{5.1}$$

where $\boldsymbol{a} = (a_1, ..., a_n)$ with $a_i > 0$. The attrition rate a_i represents the deterioration rate of the inventory of the finished product type i when $x_i^{\varepsilon}(t) > 0$, and it represents a rate of cancellation of backlogged orders when $x_i^{\varepsilon}(t) < 0$. We assume symmetric deterioration and cancellation rates for product i only for convenience in exposition. It is easy to extend our results when $a_i^+ > 0$ denotes the deterioration rate and $a_i^- > 0$ denotes the order cancellation rate.

Let $m(\varepsilon, t) \in \mathcal{M} = \{0, 1, ..., m\}$, $t \geq 0$, denote a Markov process generated by $Q^{(1)} + (1/\varepsilon)Q^{(2)}$, where $\varepsilon > 0$ is a small parameter and $Q^{(\ell)} = (q_{ij}^{(\ell)})$, $i, j \in \mathcal{M}$, is an $(m+1) \times (m+1)$ matrix such that $q_{ij}^{(\ell)} \geq 0$ for $i \neq j$ and $q_{ii}^{(\ell)} = -\sum_{j \neq i} q_{ij}^{(\ell)}$ for $\ell = 1, 2$. We let $m(\varepsilon, t)$ represent the machine capacity state at time t.

Definition 5.1. A production control process $\boldsymbol{u}^{\varepsilon}(\cdot) = \{\boldsymbol{u}^{\varepsilon}(t) : t \geq 0\}$ is admissible, if (i) $\boldsymbol{u}^{\varepsilon}(t)$ is a measurable process adapted to $\mathcal{F}_t^{(\varepsilon)} \equiv \sigma(m(\varepsilon, s), 0 \leq s \leq t)$; (ii) $u_k^{\varepsilon}(t) \geq 0$, $\sum_{k=1}^n u_k^{\varepsilon}(t) \leq m(\varepsilon, t)$ for all $t \geq 0$.

We denote by $\mathcal{A}^{\boldsymbol{a},\varepsilon}(m)$ to be the set of all admissible controls with the initial condition $m(\varepsilon, 0) = m$.

Definition 5.2. A function $\boldsymbol{u}(\boldsymbol{x}, m)$ defined on $R^n \times \mathcal{M}$ is called an admissible feedback control or simply a feedback control, if (i) for any given initial surplus and production capacity, the equation $\dot{\boldsymbol{x}}^{\varepsilon}(t) = -\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}^{\varepsilon}(t) + \boldsymbol{u}(\boldsymbol{x}^{\varepsilon}(t), m(\varepsilon, t)) - \boldsymbol{z}$ has a unique solution; (ii) the control defined by $\boldsymbol{u}^{\varepsilon}(\cdot) = \{\boldsymbol{u}^{\varepsilon}(t) = \boldsymbol{u}(\boldsymbol{x}^{\varepsilon}(t), m(\varepsilon, t)), t \geq 0\} \in \mathcal{A}^{\boldsymbol{a},\varepsilon}(m)$.

With a slight abuse of notation, we simply call $\boldsymbol{u}(\boldsymbol{x}, m)$ a feedback control when no ambiguity arises. For any $\boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{A}^{\boldsymbol{a},\varepsilon}(m)$, define the expected long-run average cost

$$\bar{J}^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, m, \boldsymbol{u}^{\varepsilon}(\cdot)) = \limsup_{T \to \infty} \frac{1}{T} E \int_0^T [h(\boldsymbol{x}^{\varepsilon}(t)) + c(\boldsymbol{u}^{\varepsilon}(t))] dt, \tag{5.2}$$

where $\boldsymbol{x}^{\varepsilon}(\cdot)$ is the surplus process corresponding to the production process $\boldsymbol{u}^{\varepsilon}(\cdot)$ in $\mathcal{A}^{\boldsymbol{a},\varepsilon}(m)$ with $\boldsymbol{x}^{\varepsilon}(0) = \boldsymbol{x}$, and $h(\cdot)$ and $c(\cdot)$ are given as in Section 2.1. The problem is to obtain $\boldsymbol{u}^{\varepsilon}(\cdot) \in \mathcal{A}^{\boldsymbol{a},\varepsilon}(m)$

that minimizes $\bar{J}^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, m, \boldsymbol{u}^\varepsilon(\cdot))$. We formally summarize our control problem as follows:

$$\bar{\mathcal{P}}^{\boldsymbol{a},\varepsilon}: \begin{cases} \min \ \bar{J}^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, m, \boldsymbol{u}^\varepsilon(\cdot)) = \limsup_{T\to\infty} \dfrac{1}{T} E \displaystyle\int_0^T [h(\boldsymbol{x}^\varepsilon(t)) + c(\boldsymbol{u}^\varepsilon(t))]dt, \\ \text{s.t. } \dot{\boldsymbol{x}}(t) = -\text{diag}(\boldsymbol{a})\boldsymbol{x}^\varepsilon(t) + \boldsymbol{u}^\varepsilon(t) - \boldsymbol{z}, \ \boldsymbol{x}^\varepsilon(0) = \boldsymbol{x}, \ \boldsymbol{u}^\varepsilon(\cdot) \in \mathcal{A}^{\boldsymbol{a},\varepsilon}(m), \\ \text{minimum average cost } \lambda^\varepsilon = \displaystyle\inf_{\boldsymbol{u}^\varepsilon(\cdot)\in\mathcal{A}^{\boldsymbol{a},\varepsilon}(m)} \bar{J}^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, m, \boldsymbol{u}^\varepsilon(\cdot)). \end{cases}$$

Here we assume that the production cost function $c(\cdot)$ and the surplus cost function $h(\cdot)$ satisfy (A.3.1), and the machine capacity process $m(\varepsilon, t)$ satisfies (A.3.2) and (A.3.3). Furthermore, similar to Assumption (A.4.3), we also assume

(A.5.1) $\sum_{j=0}^p j\nu_j \geq \sum_{i=1}^n z_i$.

As in Fleming and Zhang (1998), the positive attrition rate $\boldsymbol{a}$ implies a uniform bound for $\boldsymbol{x}^\varepsilon(t)$. In view of the fact that the control $\boldsymbol{u}^\varepsilon(\cdot)$ is bounded between 0 and m, this implies that any solution $\boldsymbol{x}^\varepsilon(\cdot)$ to (5.1) must satisfy

$$\begin{aligned} |x_i^\varepsilon(t)| &= \left| x_i e^{-a_i t} + e^{-a_i t} \int_0^t e^{a_i s}(u_i^\varepsilon(s) - z_i) ds \right| \\ &\leq |x_i| e^{-a_i t} + \frac{m + z_i}{a_i}, \quad i = 1, ..., n. \end{aligned} \tag{5.3}$$

Thus, under the positive deterioration/cancellation rate, the surplus process $\boldsymbol{x}(t)$ remains bounded. The average cost optimality equation associated with the average-cost optimal control problem in $\mathcal{P}^{\boldsymbol{a},\varepsilon}$, as shown in Sethi et al. (1997), takes the form

$$\begin{aligned} \bar{\lambda}^\varepsilon &= \inf_{u_i \geq 0, \sum_{i=1}^n u_i \leq k} \left\{ \frac{\partial W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, m)}{\partial(-\text{diag}(\boldsymbol{a})\boldsymbol{x} + \boldsymbol{u} - \boldsymbol{z})} + c(\boldsymbol{u}) \right\} \\ &\quad + h(\boldsymbol{x}) + \left(Q^{(1)} + \frac{1}{\varepsilon} Q^{(2)} \right) W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, \cdot)(m), \end{aligned} \tag{5.4}$$

where $W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, m)$ is the potential function of the problem $\mathcal{P}^{\boldsymbol{a},\varepsilon}$. The analysis begins with a proof of the boundedness of λ^ε. Sethi et al. (1997b) prove the following result.

Theorem 5.1. *The minimum average expected cost* λ^ε of $\mathcal{P}^{\boldsymbol{a},\varepsilon}$ *is bounded in* ε, *i.e., there exists a constant* $C_{51} > 0$ *such that* $0 \leq \lambda^\varepsilon \leq C_{51}$ for all $\varepsilon > 0$.

In order to construct open-loop and feedback hierarchical controls for the system, one derives the limiting control problem as $\varepsilon \to 0$. As in Sethi et al. (1994b), consider the enlarged control space

$$\begin{aligned} \mathcal{A}^{\boldsymbol{a}} = \{ U(\cdot) = (\boldsymbol{u}^0(\cdot), \boldsymbol{u}^1(\cdot), ..., \boldsymbol{u}^p(\cdot)) : u_i^k(t) \geq 0, \forall i \text{ and } \sum_{i=1}^p u_i^k(t) \leq k, \ t \geq 0, \\ U(\cdot) \ \text{ is a deterministic process} \}. \end{aligned}$$

Then, define the limiting control problem $\bar{\mathcal{P}}^{\boldsymbol{a}}$ as follows:

$$\bar{\mathcal{P}}^{\boldsymbol{a}}: \begin{cases} \bar{J}^{\boldsymbol{a}}(\boldsymbol{x}, U(\cdot)) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T [h(\boldsymbol{x}(s)) + \sum_{k=0}^p \nu_k c(\boldsymbol{u}^k(s))]ds, \\ \text{s.t. } \dot{\boldsymbol{x}}(t) = -\text{diag}(\boldsymbol{a})\boldsymbol{x}(t) + \sum_{k=0}^p \nu_k \boldsymbol{u}^k(t) - \boldsymbol{z},\ \boldsymbol{x}(0) = \boldsymbol{x},\ U(\cdot) \in \mathcal{A}^{\boldsymbol{a}}, \\ \text{minimum average cost } \lambda = \inf_{U(\cdot)\in\mathcal{A}^{\boldsymbol{a}}} \bar{J}^{\boldsymbol{a}}(\boldsymbol{x}, U(\cdot)). \end{cases}$$

The average-cost optimality equation associated with the limiting control problem $\bar{\mathcal{P}}^{\boldsymbol{a}}$ is

$$\bar{\lambda} = \inf_{u_i^k \geq 0, \sum_{i=1}^n u_i^k \leq k, k\in\mathcal{M}} \left\{ \frac{\partial W^{\boldsymbol{a}}(\boldsymbol{x})}{\partial(-\text{diag}(\boldsymbol{a})\boldsymbol{x} + \sum_{k=0}^p \nu_k \boldsymbol{u}^k - \boldsymbol{z})} + \sum_{k=0}^p \nu_k c(\boldsymbol{u}^k) \right\} + h(\boldsymbol{x}), \tag{5.5}$$

where $W^{\boldsymbol{a}}(\boldsymbol{x})$ is a potential function for $\mathcal{P}^{\boldsymbol{a}}$. From Sethi et al. (1997b), we know that there exist $\bar{\lambda}$ and $\bar{W}^{\boldsymbol{a}}(\boldsymbol{x})$ such that (5.5) holds. Moreover, $\bar{W}^{\boldsymbol{a}}(\boldsymbol{x})$ is the limit of $\bar{W}^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, k)$ as $\varepsilon \to 0$. Armed with Theorem 5.1, one can derive the convergence of the minimum average expected cost λ^ε as ε goes to zero, and establish the convergence rate. The following two theorems are proved in Sethi et al. (1997b).

Theorem 5.2. *There exists a constant* C_{52} *such that for all* $\varepsilon > 0$, $|\lambda^\varepsilon - \lambda| \leq C_{52}\varepsilon^{\frac{1}{2}}$. *This implies in particular that* $\lim_{\varepsilon\to 0} \lambda^\varepsilon = \lambda$.

Theorem 5.3. (Open-loop control) *Let* $\bar{U}(\cdot) = (\bar{\boldsymbol{u}}^0(\cdot), \bar{\boldsymbol{u}}^1(\cdot), ..., \bar{\boldsymbol{u}}^p(\cdot)) \in \mathcal{A}^{\boldsymbol{a}}$ *be an optimal control for* $\bar{\mathcal{P}}^{\boldsymbol{a}}$, *and let* $\boldsymbol{u}^\varepsilon(t) = \sum_{i=0}^p I_{\{m(\varepsilon,t)=i\}} \bar{\boldsymbol{u}}^i(t)$. *Then* $\boldsymbol{u}^\varepsilon(\cdot) \in \mathcal{A}^{\boldsymbol{a},\varepsilon}(m(\varepsilon, 0))$, *and* $\boldsymbol{u}^\varepsilon(\cdot)$ *is asymptotically optimal for* $\bar{\mathcal{P}}^{\boldsymbol{a},\varepsilon}$, *i.e.,* $|\lambda^\varepsilon - \bar{J}^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, m(\varepsilon, 0), \boldsymbol{u}^\varepsilon(\cdot))| \leq C_{53}\varepsilon^{\frac{1}{2}}$ *for some positive constant* C_{53}.

Remark 5.1. A similar averaging approach is introduced in Altman and Gaitsgory (1993, 1997), Nguyen and Gaitsgory (1997), Shi et al. (1998), and Nguyen (1999). They consider a class of nonlinear hybrid systems in which the parameters of the dynamics of the system may jump at discrete moments of time, according to a controlled Markov chain with finite state and action spaces. They assume that the unit of the length of intervals between the jumps is small. They prove that the optimal solution of the hybrid systems governed by the controlled Markov chain can be approximated by the solution of a limiting deterministic optimal control problem.

Remark 5.2. Without deterioration and cancellation rates ($\boldsymbol{a} = 0$), the system dynamics equation is given by $\dot{\boldsymbol{x}}^\varepsilon(t) = \boldsymbol{u}^\varepsilon(t) - \boldsymbol{z}$. With the same cost function given by (5.2), we can still obtain the convergence of the minimum average cost, see Sethi and Zhang (1998a). Furthermore, Sethi et al. (2000a) obtain the optimal feedback control for the limiting control problem $\bar{\mathcal{P}}^{\boldsymbol{a}}$ with $n = 2$, $c(\boldsymbol{u}) = 0$, and $h(\boldsymbol{x}) = c_1^+ x_1^+ + c_1^- x_1^- + c_2^+ x_2^+ + c_2^- x_2^-$ (see (3.15)). Consequently, an asymptotically optimal feedback control for the original control problem $\bar{\mathcal{P}}^{\boldsymbol{a},\varepsilon}$ can be constructed.

5.2 Dynamic flowshops

For dynamic flowshops, Sethi et al. (2000b) prove the convergence of the minimum average expected cost in the hierarchical framework. Furthermore, they use the technique of *partial pathwise lifting* and *pathwise shrinking* to construct an asymptotically optimal control. These will be reviewed in the next section for the more general case of dynamic jobshops.

5.3 Dynamic jobshops

We consider the jobshop given in Section 3.3. The dynamics of the system are given by

$$\begin{cases} \dot{x}_i^\varepsilon(t) = -a_i x_i^\varepsilon(t) + (\sum_{\ell=0}^{i-1} u_{\ell i}^\varepsilon(t) - \sum_{\ell=i+1}^{N_b} u_{i\ell}^\varepsilon(t)),\ 1 \le i \le d, \\ x_i^\varepsilon(t) = -a_i x_i^\varepsilon(t) + (\sum_{\ell=0}^{d} u_{\ell i}^\varepsilon(t) - z_i),\ d+1 \le i \le N_b, \end{cases}$$

with $\boldsymbol{x}^\varepsilon(0) := (x_1^\varepsilon(0), ..., x_{N_b}^\varepsilon(0)) = (x_1, ..., x_{N_b}) = \boldsymbol{x}$. We write this in the vector form as

$$\begin{aligned} \dot{\boldsymbol{x}}^\varepsilon(t) &= (\dot{x}_1^\varepsilon(t), ..., \dot{x}_{N_b}(t))' \\ &= -\mathrm{diag}(\boldsymbol{a})\boldsymbol{x}^\varepsilon(t) + D(\boldsymbol{u}_0^\varepsilon(t), ..., \boldsymbol{u}_{d+1}^\varepsilon(t))', \end{aligned} \tag{5.6}$$

where $\boldsymbol{a} = (a_1, ..., a_{N_b})'$, $\boldsymbol{u}_\ell^\varepsilon(t) = (u_{\ell,\ell+1}^\varepsilon(t), ..., u_{\ell,N_b}^\varepsilon(t))',\ \ \ell = 0, ..., d$, and $\boldsymbol{u}_{d+1}^\varepsilon(t) = (z_{d+1}, ..., z_{N_b})',\ \ \ell = d+1, ..., N_b$. Our problem is to find an admissible control $\boldsymbol{u}(\varepsilon, \cdot)$ that minimizes the average cost

$$\bar{J}^\varepsilon(\boldsymbol{x}, \boldsymbol{m}) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T [h(\boldsymbol{x}^\varepsilon(t)) + c(\boldsymbol{u}^\varepsilon(t))]dt, \tag{5.7}$$

where $h(\cdot)$ defines the cost of inventory/shortage, $c(\cdot)$ is the production cost, $\boldsymbol{x}$ is the initial state, and $\boldsymbol{m}$ is the initial value of $\boldsymbol{m}(\varepsilon, t) = (m_1(\varepsilon, t), ..., m_N(\varepsilon, t))$.

In addition to Assumptions (A.3.4)-(A.3.6) in Section 3.3 on the cost functions $h(\cdot)$ and $c(\cdot)$ and the machine capacity process $\boldsymbol{m}(\varepsilon, t)$, we assume that $\boldsymbol{m}(\varepsilon, t)$ satisfies the following:

(A.5.2) Let $p_n = \sum_{j=1}^p m_n^j \nu_j$, and $n(i,j) = \arg\{(i,j) \in K_n\}$ for $(i,j) \in \Pi$, that is, p_n is the average capacity of the machine n, and $n(i,j)$ is the number of the machine located on the arc (i,j). Let $\{p_{ij} > 0 : (i,j) \in K_n\}(n = 1, ..., N)$ be such that $\sum_{(i,j)\in K_n} p_{ij} \le 1$, $\sum_{\ell=0}^m p_{\ell i} p_{n(\ell,i)} > z_i,\ \ i = d+1, ..., N_b$, and $\sum_{\ell=0}^{i-1} p_{\ell i} p_{n(\ell,i)} > \sum_{\ell=i+1}^{N_b} p_{i\ell} p_{n(i,\ell)},\ \ i = 1, ..., d$.

We use $\mathcal{A}^\varepsilon(\boldsymbol{x}, \boldsymbol{m})$ to denote the set of all admissible controls with respect to $\boldsymbol{x} \in \mathcal{S}$ and $\boldsymbol{m}(\varepsilon, 0) = \boldsymbol{m}$. Let $\lambda^\varepsilon(\boldsymbol{x}, \boldsymbol{m})$ denote the minimal expected cost, i.e.,

$$\lambda^\varepsilon(\boldsymbol{x}, \boldsymbol{m}) = \inf_{\boldsymbol{u}(\cdot)\in\mathcal{A}^\varepsilon(\boldsymbol{x},\boldsymbol{m})} \bar{J}^\varepsilon(\boldsymbol{x}, \boldsymbol{m}). \tag{5.8}$$

In the case of the long-run average cost criterion used here, we know, by Theorem 2.4 in Presman et al. (2000b), that under Assumption (A.5.2), $\lambda^\varepsilon(\boldsymbol{x},\boldsymbol{m})$ is independent of the initial condition $(\boldsymbol{x},\boldsymbol{m})$. Thus we will use λ^ε instead of $\lambda^\varepsilon(\boldsymbol{x},\boldsymbol{m})$. We use $\mathcal{P}^\varepsilon$ to denote our control problem, i.e.,

$$\mathcal{P}^\varepsilon : \begin{cases} \text{minimize } J^\varepsilon(\boldsymbol{x},\boldsymbol{m}^0) = \limsup_{T\to\infty} \frac{1}{T} E \int_0^T h(\boldsymbol{x}^\varepsilon(t)) + u(\boldsymbol{u}^\varepsilon(t))dt, \\ \text{subject to } \begin{cases} \dot{\boldsymbol{x}}^\varepsilon(t) = -\text{diag}(\boldsymbol{a})\boldsymbol{x}^\varepsilon(t) + D\boldsymbol{u}^\varepsilon(t),\ \boldsymbol{x}^\varepsilon(0)=\boldsymbol{x}, \\ \boldsymbol{u}^\varepsilon(\cdot) \in \mathcal{A}^\varepsilon(\boldsymbol{x},\boldsymbol{m}) \end{cases} \\ \text{value function } \lambda^\varepsilon = \inf_{\boldsymbol{u}^\varepsilon(\cdot)\in\mathcal{A}^\varepsilon(\boldsymbol{x},\boldsymbol{m}^0)} J^\varepsilon(\boldsymbol{x},\boldsymbol{m}). \end{cases} \tag{5.9}$$

As in Section 5.1, the positive attrition rate $\boldsymbol{a}$ implies a uniform bound for $\boldsymbol{x}^\varepsilon(t)$. Next we examine elementary properties of the potential function and obtain the limiting control problem as $\varepsilon \to 0$. The HJBDD equation, as shown in Sethi et al. (1998b, 2000c), takes the form

$$\begin{aligned} \lambda^\varepsilon &= \inf_{\boldsymbol{u}\in U(\boldsymbol{x},\boldsymbol{m}^j)} \left\{ \frac{\partial W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x},\boldsymbol{m}^j)}{\partial(-\text{diag}(\boldsymbol{a})\boldsymbol{x}+D\boldsymbol{u})} + c(\boldsymbol{u}) \right\} + h(\boldsymbol{x}) \\ &\quad + \left(Q^{(1)} + \frac{1}{\varepsilon}Q^{(2)}\right) W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x},\cdot)(\boldsymbol{m}^j), \end{aligned} \tag{5.10}$$

where $W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x},\boldsymbol{m}^j)$ is the potential function of the problem $\mathcal{P}^\varepsilon$. Moreover, following Presman et al. (2000b), we can show that there exists a potential function $W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x},\boldsymbol{m})$ such that the pair $(\lambda^\varepsilon, W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x},\boldsymbol{m}))$ is a solution of (5.10), where λ^ε is the minimum average expected cost for $\mathcal{P}^\varepsilon$. First, we can get the boundedness of λ^ε.

Theorem 5.4. *There exists a constant* $M_1 > 0$ *such that* $0 \le \lambda^\varepsilon \le M_1$ *for all* $\varepsilon > 0$.

For its proof, see Sethi et al. (2000c). Now we derive the limiting control problem as $\varepsilon \to 0$.

As in Sethi and Zhou (1994), for $\boldsymbol{x} \in \mathcal{S}$, let $\mathcal{A}^0(\boldsymbol{x})$ denote the set of measurable controls

$$\begin{aligned} U(\cdot) &= (\boldsymbol{u}^1(\cdot),...,\boldsymbol{u}^p(\cdot)) \\ &= ((\boldsymbol{u}_0^{1,0}(\cdot),...,\boldsymbol{u}_d^{1,0})(\cdot),...,(\boldsymbol{u}_0^{p,0}(\cdot),...,\boldsymbol{u}_d^{p,0})(\cdot)), \end{aligned}$$

with $\boldsymbol{u}_k^{j,0}(\cdot) = (u_{k,k+1}^{j,0}(\cdot),...,u_{k,N_b}^{j,0}(\cdot))$, such that $0 \le \sum_{(i,j)\in K_n} u_{ij}^{j,0}(t) \le m_n^j$ for all $t \ge 0$, $j = 1,...,p$, and $n = 1,...,N$, and the corresponding solutions $\boldsymbol{x}(\cdot)$ of the system

$$\begin{cases} \dot{x}_k(t) = -a_k x_k(t) + \left(\sum_{j=1}^p \gamma_j \sum_{\ell=0}^{k-1} u_{\ell k}^j(t) - \sum_{j=1}^p \gamma_j \sum_{\ell=k+1}^N u_{k\ell}^j(t)\right), \\ k = 1,...,d, \text{ and} \\ \dot{x}_k(t) = -a_k x_k(t) + \left(\sum_{j=1}^p \gamma_j \sum_{\ell=1}^d u_{\ell k}^j(t) - d_k\right),\ k = d+1,...,N, \end{cases}$$

with $(x_1(0),...,x_N(0)) = (x_1,...,x_N)$ satisfy $\boldsymbol{x}(t) \in \mathcal{S}$ for all $t \ge 0$.

The objective is to choose a control $U(\cdot) \in \mathcal{A}^0(\boldsymbol{x})$ that minimizes

$$\bar{J}(U(\cdot)) = \limsup_{T\to\infty} \frac{1}{T} \int_0^T [h(\boldsymbol{x}(s)) + \sum_{j=0}^{p} \gamma_j c(\boldsymbol{u}^j(s))] ds.$$

We use $\mathcal{P}^0$ to denote the above problem, and regard this as our limiting problem. Then we define the limiting control problem $\mathcal{P}^0$ as follows:

$$\mathcal{P}^0 : \begin{cases} \bar{J}(U(\cdot)) = \limsup_{T\to\infty} \frac{1}{T} \int_0^T \left[h(\boldsymbol{x}(s)) + \sum_{j=0}^{p} \gamma_j c(\boldsymbol{u}^j(s))\right] ds, \\ \begin{cases} \dot{x}_k(t) = -a_k x_k(t) + \left(\sum_{j=1}^{p} \gamma_j \sum_{\ell=0}^{k-1} u^j_{\ell k}(t) - \sum_{j=1}^{p} \gamma_j \sum_{\ell=k+1}^{N} u^j_{k\ell}(t)\right), \\ \quad \text{with } x_k(0) = x_k, \ k = 1, ..., d, \ \text{and} \\ \dot{x}_k(t) = -a_k x_k(t) + \left(\sum_{j=1}^{p} \gamma_j \sum_{\ell=0}^{d} u^j_{\ell m}(t) - d_k\right), \\ \quad \text{with } x_k(0) = x_k \ \ k = d+1, ..., N, \\ U(\cdot) \in \mathcal{A}^0(\boldsymbol{x}), \end{cases} \\ \text{minimum average cost } \lambda = \inf_{U(\cdot)\in\mathcal{A}^0} \bar{J}(U(\cdot)). \end{cases}$$

The average cost optimality equation associated with the limiting control problem $\mathcal{P}^0$ is

$$\lambda = \inf_{U^0 \in \mathcal{A}^0} \left\{ \frac{\partial W^{\boldsymbol{a}}(\boldsymbol{x})}{\partial(-\mathrm{diag}(\boldsymbol{a})\boldsymbol{x} + DU^0)} + \sum_{j=0}^{p} \gamma_j c(\boldsymbol{u}^j) \right\} + h(\boldsymbol{x}), \tag{5.11}$$

where $W^{\boldsymbol{a}}(\boldsymbol{x})$ is a potential function for $\mathcal{P}^0$ and $U^0 = \sum_{j=1}^{p} \gamma_j \boldsymbol{u}^j$. From Presman et al. (2000b), we know that there exist $\bar{\lambda}$ and $W^{\boldsymbol{a}}(\boldsymbol{x})$ such that (5.11) holds. Moreover, $W^{\boldsymbol{a}}(\boldsymbol{x})$ is the limit of $W^{\boldsymbol{a},\varepsilon}(\boldsymbol{x}, \boldsymbol{m})$ as $\varepsilon \to 0$. The following convergence result for the minimum average expected cost λ^ε, as ε goes to zero, is established in Sethi et al. (2000c).

Theorem 5.5. *For any $\delta \in [0, \frac{1}{2})$ there exists a constant $\hat{C}_{56} > 0$ such that for all sufficiently small $\varepsilon > 0$, $|\lambda^\varepsilon - \lambda| \le \hat{C}_{56}\varepsilon^\delta$. This implies in particular that $\lim_{\varepsilon\to 0} \lambda^\varepsilon = \lambda$.*

5.4 Markov decision processes with weak and strong interactions

Markovian decision processes (MDP) have received much attention in recent years because of their capability in dealing with a large class of practical problems under uncertainty. The formulation of many practical problems, such as queueing and machine replacement, fits well in the framework of Markov decision processes; see Derman (1970). In this section we present results that provide a justification for hierarchical controls of a class of Markov decision problems. We focus on the problem of a finite state continuous-time Markov decision process that has both weak and strong interactions. More specifically, the state of the process can be divided into several groups such that transitions among the states within each group occur much more frequently than the transitions among the states belonging to different groups. By replacing the states in each group by the corresponding average distribution, we can derive a limiting problem which is simpler to solve. Given an optimal solution to limiting problem, we can construct a solution for the original problem which is asymptotically optimal. Proofs of results in this section can be found in Zhang (1996).

Let us consider a Markov decision process $x(\cdot) = \{x(t) : \ t \geq 0\}$ and a control process $u(\cdot) = \{u(t) = u(x(t)) : \ t \geq 0\}$ such that $u(t) \in U$, $t \geq 0$, where U is a set with finite elements. Let $Q^\varepsilon(u(t)) = (q^\varepsilon_{ij}(u(t)))$, $t \geq 0$, denote the generator of $x(\cdot)$ such that $Q^\varepsilon(u) = \dfrac{1}{\varepsilon}A(u) + B(u)$, where $A(u) = \text{diag}\{A^1(u), \ldots, A^r(u)\}$, $A^k(u) = (a^k_{ij}(u))_{p_k \times p_k}$ with $a^k_{ij}(u) \geq 0$ for $j \neq i$ and $\sum_j a^k_{ij}(u) = 0$, $B(u) = (b_{ij}(u))_{p \times p}$ with $p = p_1 + \cdots + p_r$, $b_{ij}(u) \geq 0$ for $j \neq i$ and $\sum_j b_{ij}(u) = 0$, and ε is a small parameter. For each $k = 1, \ldots, r$, let $\mathcal{M}_k = \{s_{k1}, \ldots, s_{kp_k}\}$, where p_k is the dimension of $A^k(u)$. The state space of $x(\cdot)$ is given by $\mathcal{M} = \mathcal{M}_1 \times \ldots \times \mathcal{M}_r$, i.e.,

$$\mathcal{M} = \{s_{11}, \ldots, s_{1p_1}, s_{21}, \ldots, s_{2p_2}, \ldots, s_{r1}, \ldots, s_{rp_r}\}.$$

The matrix $A(u)$ dictates the fast transition of the process $x(\cdot)$ within each group $\mathcal{M}_k$, $k = 1, \ldots, r$, and the matrix $B(u)$ together with the average distribution of the Markov chain generated by $A^k(u)$ dictates the slow transition of $x(\cdot)$ between different groups. When ε is small, the process $x(\cdot)$ has a strong interaction within any group $\mathcal{M}_k$, $k = 1, \ldots, r$, and has a weak interaction among these groups. Let $u = u(i)$ denote a function such that $u(i) \in \mathcal{U}$ for all $i \in \mathcal{M}$. We call u an admissible control and use Υ to denote all such functions. For each $k = 1, \ldots, r$, let

$$\Gamma_k = \{U^k := (u^{k1}, \ldots, u^{kp_k}) : \ \text{such that } u^{kj} \in U, \ j = 1, \ldots, p_k\}.$$

The control set for the limiting problem is defined as $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$, i.e.,

$$\begin{aligned} \Gamma \quad &= \quad \{U = (U^1, \ldots, U^r) = (u^{11}, \ldots, u^{1p_1}, \ldots, u^{r1}, \ldots, u^{rp_r}) \\ &\qquad : \ \text{such that } U^k \in \Gamma_k, \ k = 1, \ldots, r\}. \end{aligned}$$

We define matrices $\bar{A}^k$, $\bar{B}$, and $\bar{Q}^\varepsilon$ as follows. For each $U = (U^1, \ldots, U^r) \in \Gamma$, let $\bar{A}^k(U^k) =$

$(a_{ij}^k(s_{ki}, u^{ki}))$, $k = 1, \ldots, r$, $\bar{B}(U) = (\bar{b}_{ij}(U))$ where

$$\bar{b}_{ij}(U) = \begin{cases} b_{ij}(u^{1i}) & \text{if } 1 \le i \le p_1, \\ b_{ij}(u^{2(i-p_1)}) & \text{if } p_1 + 1 \le i \le p_1 + p_2, \\ \cdots & \cdots \\ b_{ij}(u^{r(i-p+p_r)}) & \text{if } p - p_r + 1 \le i \le p. \end{cases}$$

with $p = p_1 + \cdots + p_r$, and

$$\bar{Q}^\varepsilon(U) = \frac{1}{\varepsilon}\text{diag}\left\{\bar{A}^1(U^1), \ldots, \bar{A}^r(U^r)\right\} + \bar{B}(U). \tag{5.12}$$

For each $U^k \in \Gamma_k$, let $\nu^k(U^k)$ denote the average distribution of the Markov chain generated by $\bar{A}^k(U^k)$ for $k = 1, \ldots, r$. Define $\nu(U) = \text{diag}\{\nu^1(U^1), \ldots, \nu^r(U^r)\}$ and $I_e = \text{diag}\{e_{p_1}, \ldots, e_{p_r}\}$ with $e_{p_k} = (1, \ldots, 1)'$ being p_k dimensional column vector. Using $\nu(U)$ and I_e, we define another matrix $\bar{Q}(U)$ as a function of $U \in \Gamma$:

$$\bar{Q}(U) = \nu(U)\bar{B}(U)I_e. \tag{5.13}$$

Note that the kth row of $\bar{Q}(U)$ depends only on U^k. Thus, $\bar{Q}(U) = (\bar{q}_{ij}(U^i))_{r\times r}$. We write, with an abuse of notation, $\bar{Q}(U^k)f(\cdot)(k) = \sum_{k' \ne k} \bar{q}_{kk'}(U^k)(f(k') - f(k))$ instead of $\bar{Q}(U)f(\cdot)(k)$, for a function f defined on $\{1, \ldots, r\}$. We assume the following:

(A.5.3) For each $U \in \Gamma$, U is a finite set, $\bar{A}^k(U^k)$ is irreducible, $k = 1, \ldots, r$, and for each $\varepsilon > 0$, $\bar{Q}^\varepsilon(U)$ is irreducible, where $\bar{Q}^\varepsilon(U)$ is defined in (5.12).

We consider the following problem:

$$\bar{\mathcal{P}}^\varepsilon : \begin{cases} \text{minimize} & J^\varepsilon(u) = \limsup\limits_{T\to\infty} \dfrac{1}{T} E \displaystyle\int_0^T G(x(t), u(x(t)))dt, \\ \text{subject to} & x(t) \sim Q^\varepsilon(u(t)),\ x(0) = i,\ u \in \Upsilon, \\ \text{value function} & \lambda^\varepsilon = \inf\limits_{u \in \mathcal{A}^\varepsilon} J^\varepsilon(u). \end{cases} \tag{5.14}$$

Note that the average cost function is independent of the initial value $x(0) = i$, and so is the value function. The dynamic programming (DP) equation for $\bar{\mathcal{P}}^\varepsilon$ is

$$\lambda^\varepsilon = \min_{u \in U} \left\{G(i, u) + Q^\varepsilon(u)W^\varepsilon(\cdot)(i)\right\}, \tag{5.15}$$

where $W^\varepsilon(\cdot)$ is a function defined on $\mathcal{M}$.

Theorem 5.6. (i) *For each fixed* $\varepsilon > 0$*, there exists a pair* $(\lambda^\varepsilon, W^\varepsilon(\cdot))$ *that satisfies the DP equation* (5.15).

(ii) *The DP equation* (5.15) *has a unique solution in the sense that for any other solution* $(\tilde{\lambda}^\varepsilon, \tilde{W}^\varepsilon(\cdot))$ *to* (5.15), $\tilde{\lambda}^\varepsilon = \lambda^\varepsilon$ *and* $\tilde{W}^\varepsilon(\cdot) = W^\varepsilon(\cdot) + K$ *for some constant* K.

(iii) *Let* $u_{*\varepsilon} = u_{*\varepsilon}(i) \in U$ *denote a minimizer of the right-hand side of* (5.15). Then $u_{*\varepsilon}(i) \in \mathcal{A}^\varepsilon$ *is optimal, i.e.,* $J^\varepsilon(u_{*\varepsilon}) = \lambda^\varepsilon$.

Motivated by Sethi and Zhang (1994a), we next define the limiting problem. For $U^k \in \Gamma_k$, define $\bar{G}(k, U^k) = \sum_{j=1}^{m_k} \nu_j^k(U^k) G(s_{kj}, u^{kj}),\ k = 1, \ldots, r,$ where $\nu^k(U^k) = (\nu_1^k(U^k), \ldots, \nu_{m_k}^k(U^k))$ is the average distribution of the Markov chain generated by $A^k(U^k)$. Let $\mathcal{A}^0$ denote a class of functions $U = U(k) \in \Gamma_k,\ k = 1, \ldots, r$. For convenience, we will call $U = (U(1), \ldots, U(r)) \in \mathcal{A}^0$, an admissible control for the limiting problem $\bar{\mathcal{P}}^0$ defined below:

$$\bar{\mathcal{P}}^0 : \begin{cases} \text{minimize} & J^0(U) = \limsup_{T \to \infty} \dfrac{1}{T} E \displaystyle\int_0^T \bar{G}(\bar{x}(t), U(\bar{x}(t))) dt, \\ \text{subject to} & \bar{x}(t) \sim \bar{Q}(U(\bar{x}(t))),\ \bar{x}(0) = k,\ U \in \mathcal{A}^0, \\ \text{value function} & \lambda^0 = \inf_{U \in \mathcal{A}^0} J^0(U). \end{cases} \tag{5.16}$$

It can be shown that $\bar{Q}(U)$ is irreducible for each $U \in \Gamma$. The DP equation for $\bar{\mathcal{P}}^0$ is

$$\lambda^0 = \min_{U^k \in \Gamma_k} \left\{ \bar{G}(k, U^k) + \bar{Q}(U^k) W^0(\cdot)(k) \right\}, \tag{5.17}$$

for some function $W^0(\cdot)$.

Theorem 5.7. (i) *There exists a pair* $(\lambda^0, W^0(\cdot))$ *that satisfies the DP equation* (5.17).

(ii) *The DP equation* (5.17) *has a unique solution in the sense that for any other solution* $(\tilde{\lambda}^0, \tilde{W}^0(\cdot))$ *to* (5.17), $\tilde{\lambda}^0 = \lambda^0$ *and* $\tilde{W}^0(\cdot) = W^0(\cdot) + K$ *for some constant* K.

(iii) *Let* $U_* \in \Gamma$ *denote a minimizer of the right-hand side of* (5.17). *Then* $U_* \in \mathcal{A}^0$ *is optimal, i.e.,* $J^0(U_*) = \lambda^0$.

Remark 5.3. Note that the number of the DP equations for $\bar{\mathcal{P}}^\varepsilon$ is equal to $p = p_1 + \cdots + p_r$, while the number of those for $\bar{\mathcal{P}}^0$ is only r. Since for each $k = 1, \ldots, r$, $p_k \geq 2$, it follows that $p - r \geq r$. The difference between p and r could be very large for either large r or a large p_k for some k. Since the computational effort involved in solving the DP equations depends in part on the number of the equations to be solved (see Hillier and Lieberman (1989)), the effort in solving the DP equations for $\bar{\mathcal{P}}^0$ is substantially less than that of solving $\bar{\mathcal{P}}^\varepsilon$ for $(p - r)$ large.

Insight 5.1. In the presence of states with weak and strong interactions, a near-optimal control of the original problem can be constructed from the solution of an approximate problem with reduced number of states, each of which is formed by aggregating the original states within a group using the stationary distribution of these states given the reduced state corresponding to the group.

We construct a control u_ε as follows:

$$u_\varepsilon = u_\varepsilon(x) = \sum_{k=1}^{r}\sum_{j=1}^{p_k} I_{\{x=s_{kj}\}} u_*^{kj}, \tag{5.18}$$

where I_A is the indicator of a set A.

Theorem 5.8. *Let $U_* \in \mathcal{A}^0$ denote an optimal control for $\bar{\mathcal{P}}^0$ and let $u_\varepsilon \in \mathcal{A}^\varepsilon$ be the corresponding control constructed as in* (5.18). *Then, u_ε is asymptotically optimal and with error bound ε, i.e., $J^\varepsilon(u_\varepsilon) - \lambda^\varepsilon = O(\varepsilon)$.*

Remark 5.4. Filar et al. (1999) introduce diffusions in these models; see Remark 4.5. In their model, all jumps are associated with a slow process, while the continuous part including the diffusions is associated with a small parameter ε (see Remark 4.5) representing a fast process. As ε tends to zero, the problem is shown to reduce to a structured linear program. Its optimal solution can then be used as an approximation to the optimal solution of the original system associated with a small ε.

5.5 Single or parallel machine systems with risk-sensitive average cost criterion

Consider a single product, single/parallel machine manufacturing system described in Subsection 5.1 with the objective of minimizing the risk-sensitive average cost criterion over an infinite horizon. Then, the objective is to choose $u^\varepsilon(\cdot) \in \mathcal{A}^{a,\varepsilon}(m)$ so as to minimize

$$J^\varepsilon(u^\varepsilon(\cdot)) = \limsup_{T\to\infty} \frac{\varepsilon}{T} \log \left[E \exp \left(\frac{1}{\varepsilon} \int_0^T [h(x^\varepsilon(t)) + c(u^\varepsilon(t))]\, dt \right) \right], \tag{5.19}$$

where $x^\varepsilon(\cdot)$ is the surplus process corresponding to the production process $u^\varepsilon(\cdot)$. Let $\lambda^\varepsilon = \inf_{u^\varepsilon(\cdot)\in\mathcal{A}^{a,\varepsilon}(m)} J^\varepsilon(u^\varepsilon(\cdot))$. Fleming and Zhang (1998) establish the verification theorem in terms of a viscosity solution, and give an asymptotically optimal control.

6 Extensions and Concluding Remarks

In this paper, we have reviewed various models of manufacturing systems consisting of flexible machines for which a theory of optimal control has been developed and for which hierarchical controls that are asymptotically optimal have been constructed. We have examined systems with random production capacity and/or demand under various configurations including jobshops and multiple hierarchical decision levels. We have considered different performance criteria such as discounted costs, long-run average costs, risk-sensitive controls with discounted costs, and risk-sensitive controls with average costs.

While asymptotic optimal controls have been constructed for these systems under fairly general conditions, many problems remain open. We shall now describe some of them.

All of the results on the hierarchical controls in this survey assume the costs of inventory/shortage and of production to be separable. Lehoczky et al. (1991) assume a nonseparable cost and prove that the value function of the original problem converges to the value function of the limiting problem. Controls are constructed and are only conjectured to be asymptotically optimal.

With regards to systems with state constraints such as flowshops and jobshops discussed in Sections 3 and 5, respectively, only asymptotic optimal open-loop controls are constructed in general. Because of the absence of the Lipschitz property of the constructed feedback controls, their asymptotic optimality is much harder to establish. It has only been done in Sethi and Zhou (1996a, b) and Fong and Zhou (1996, 1997) for two-machine flowshops with a specific cost structure given in (3.15). Generalization of their results to jobshops and to general cost functions represents a challenging research problem.

When the Markov processes involved depend on control variables, as they do in Soner (1993), Sethi and Zhang (1994c, 1995a), and Yin and Zhang (1997), no error bounds are available for constructed asymptotic optimal controls. Estimation of these errors and extensions of the results to Markov processes depending as well on the state variables remain open problems.

In the case of the long-run average cost criterion, with the exception of single/parallel machine systems in Section 5.1, we have constructed open-loop hierarchical controls only in presence of an attrition rate. Extensions of these results to systems without an attrition term remain open. No hierarchical feedback controls that are optimal have been constructed. In case of optimal control of these ergodic systems, verification theorems have been obtained, but the existence of optimal controls is proved only in the single product case of Section 4.1. In other cases, even the existence of optimal controls remains an open issue.

An important class of manufacturing systems consists of systems that have machines which are not completely flexible, and thus involve setup costs and/or setup times, when switching from production of one product to that of another. Such systems have been considered by Gershwin (1986), Gershwin et al. (1988), Sharifnia et al. (1991), Caramanis et al. (1991), Connolly et al. (1992), Hu and Caramanis (1992), and Srivatsan and Gershwin (1990). They have examined various possible heuristic policies and have carried out numerical computations and simulations. They have not studied their policies from the viewpoint of asymptotic optimality. Sethi and Zhang (1995b) have made some progress in this direction.

Based on the theoretical work on hierarchical control of stochastic manufacturing systems, Srivatsan (1993) and Srivatsan et al. (1994) have developed a hierarchical framework and describe its experimental implementation in a semiconductor research laboratory at MIT. It is expected that such research would lead to the development of real-time decision making algorithms suitable for use in actual flexible manufacturing facilities; see also Caramanis and Sharifnia (1991).

Finally, while error bounds have been obtained only in some cases, they do not provide information on how small ε, the rate of slow and fast rates, have to be for asymptotic hierarchical controls to be acceptable in practice. This issue can only be investigated computationally, and that was done by Samaratunga et al. (1997) only for a two-machine flowshop. From a practical viewpoint, it is important to perform such investigations for dynamic jobshops.

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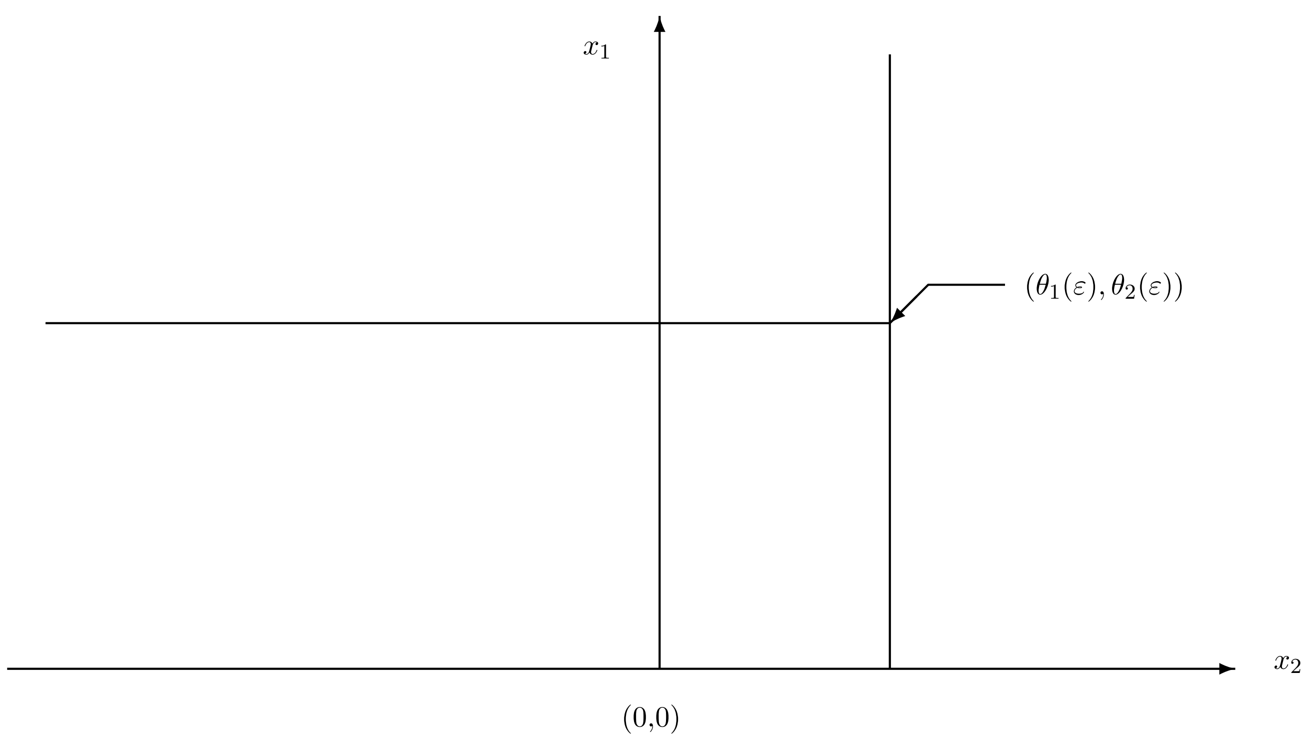

Figure 1 Switching Manifold for Asymptotically Optimal Feedback Control

Initial State (x_1, x_2)	Control Policy				
	HC		KC		OC
	Cost	Parameters	Cost	Parameters	Cost
(0,50)	771.45	(0.00,1.00)	771.45	(0.00,1.00)	770.31
(0,20)	252.72	(3.51,1.52)	253.53	(0.00,3.00)	231.38
(0,10)	150.77	(3.00,2.00)	151.85	(0.00,3.22)	101.13
(0,5)	132.08	(2.34,2.06)	132.16	(2.29,1.81)	69.11
(0,0)	132.76	(2.75,1.58)	132.76	(2.75,1.58)	66.56
(0,-5)	288.17	(3.75,1.50)	288.17	(3.75,1.50)	239.45
(0,-10)	617.27	(4.25,1.25)	617.27	(4.25,1.25)	590.67
(0,-20)	1469.54	(1.00,0.00)	1469.54	(1.00,0.00)	1466.54
(20,20)	414.78	(1.00,1.00)	414.98	(0.50,2.50)	406.96
(10,10)	194.74	(2.33,2.36)	194.74	(2.33,2.35)	165.71
(5,5)	136.80	(2.79,1.64)	136.80	(2.49,1.79)	84.49
(5,-5)	267.87	(4.98,1.22)	267.87	(4.98,1.22)	214.46
(10,-10)	586.04	(6.41,0.72)	586.04	(6.41,0.72)	539.86
(20,-20)	1420.34	(1.00,0.00)	1420.34	(1.00,0.00)	1411.65
(2.70,1.59)	129.46	(2.70,1.59)	129.46	(2.70,1.59)	65.39

Note: Simulation Relative Error $\leq \pm 2\%$, Confidence Level = 95%. Comparison is carried out for the same machine failure breakdown sample paths for all policies. OC is obtained from a Markov decision process formulation of the problem.

Table 1. Comparison of Control Policies with Best Threshold Values for Various Initial States.

Initial Inventory (x_1, x_2)	Control Policy		
	HC Cost	KC Cost	OC Cost
(0,50)	771.45	794.96	770.31
(0,20)	252.78	269.12	231.38
(0,10)	150.94	156.79	101.13
(0,5)	132.31	132.31	69.11
(0,0)	132.76	132.76	66.56
(0,-5)	288.34	288.34	239.45
(0,-10)	617.85	617.85	590.67
(0,-20)	1471.18	1471.18	1466.54
(20,20)	415.03	415.03	406.96
(10,10)	194.83	194.83	165.71
(5,5)	136.82	136.82	84.49
(5,-5)	270.75	270.75	214.46
(10,-10)	583.85	583.85	539.86
(20,-20)	1426.58	1426.58	1411.65

Note: Simulation Relative Error $\leq \pm$ 2%, Confidence Level = 95%. Comparison is carried out for the same machine failure breakdown sample paths. Therefore, the relative comparison is free of statistical uncertainty. Thresholds values used for HC as well as KC are (2.75,1.58) obtained from the (0,0) initial inventory row of Table 1.

Table 2. Comparison of Control Policies with Threshold Values (2.75,1.58) for HC and KC.