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ONE-WAY INTERVALS OF CIRCLE MAPS

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Abstract. An interval in the circle S^1 is *one-way* with respect to a map $f : S^1 \to S^1$ if under repeated applications of f all points of the interval move in the same direction. The main result is that every locally one-way interval is either one-way or is the union of two overlapping one-way subintervals. An example is given which illustrates that the latter case can occur; however, it is proved that the latter case cannot occur if the interval is covered by the image of the map. As a corollary, it is shown that if f has periodic points, then every interval which contains no periodic points is either one-way or is the union of two overlapping one-way subintervals.

1. Introduction

We orient the unit circle S^1 counterclockwise, which allows us to speak of the *positive* and *negative directions* in S^1. If $n \geq 3$ and $x_1, x_2, \ldots, x_n \in S^1$, we write $x_1 < x_2 < \cdots < x_n$ if $x_1, x_2, \ldots, x_n$ are distinct points and, if moving away from x_1 in S^1 in the positive direction, one encounters the points $x_2, x_3, \ldots, x_n$ in that order before one encounters x_1 again. If in the expression $x_1 < x_2 < \cdots < x_n$, one or more of the $<$'s are replaced by $\leq$'s, then let this expression have the obvious meaning.

Let a, b be distinct points of S^1. The preceding notation allows us to define $(a, b) = \{x \in S^1 : a < x < b\}$, $[a, b] = \{x \in S^1 : a \leq x \leq b\}$, $(a, b] = \{x \in S^1 : a < x \leq b\}$ and $[a, b) = \{x \in S^1 : a \leq x < b\}$. We call (a, b) an *open interval* and $[a, b]$ a *closed interval*.

Let $f : S^1 \to S^1$ be a map, and let J be a connected open proper subset of S^1. J is *free* (with respect to f) if no iterate of a point of J returns to J (i.e., for every $x \in J$ and $n \geq 1$, $f^n(x) \notin J$). J is *positive* (with respect to f) if J is not free and whenever $x \in J$ and $f^n(x) \in J$ for some $n \geq 1$, then $f^n(x) \neq x$ and $(x, f^n(x)) \subset J$. J is *negative* (with respect to f) if J is not free, and whenever $x \in J$ and $f^n(x) \in J$ for some $n \geq 1$, then $f^n(x) \neq x$ and $(f^n(x), x) \subset J$. J is *one-way* (with respect to f) if it is either free, positive, or negative. J is *locally one-way* (with respect to f) if every point of J lies in a one-way open subinterval of J.

The dynamic behavior of a map on a one-way interval is relatively uncomplicated, because all sequences of iterates $\{f_n(x)\}_{n=1}^{\infty}$ intersect the interval in monotone subsequences moving in the same direction. The properties of one-way intervals for maps of the real line are studied in [1] where it is proved (in Lemma 9 on page 75 of Chapter 4) that intervals containing no periodic points are one-way. This result

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fails for maps of the circle, as we show in an example. The notion of a one-way interval for a map of the circle is introduced in [2] where dynamic properties of circle maps are explored. In the present paper, we continue the study of one-way intervals for maps of the circle. Our principal results are:

Example. There is a map $f : S^1 \to S^1$ and points $a_1 < a_2 < \cdots < a_5$ in S^1 such that the connected open subset $S^1 - \{a_1\}$ is locally one-way (and, thus, contains no periodic points) but is not one-way. Moreover, (a_1, a_4) and (a_3, a_1) are free, and (a_1, a_5) is negative and (a_2, a_1) is positive.

Theorem. *If $f : S^1 \to S^1$ is a map and (a, d) is a locally one-way open interval in S^1, then either (a, d) is one-way or there are points b and c in (a, d) such that $a < b < c < d$ and (a, c) is negative and (b, d) is positive. Furthermore, if $(a, d) \subset f(S^1)$, then (a, d) is one-way.*

Corollary 1. *If $f : S^1 \to S^1$ is an onto map, then every locally one-way open interval is one-way.*

Corollary 2. *If $f : S^1 \to S^1$ is a map with a non-empty set P of periodic points, then at most one component of $S^1 - \mathrm{cl}(P)$ is not one-way. Moreover, if (a, d) is a component of $S^1 - \mathrm{cl}(P)$ which is not one-way, then there are points b and c in (a, d) such that $a < b < c < d$ and (a, c) is negative and (b, d) is positive.*

Corollary 3. *If $f : S^1 \to S^1$ is an onto map with a non-empty set P of periodic points, then every component of $S^1 - \mathrm{cl}(P)$ is one-way.*

The hypothesis that the map has periodic points in Corollaries 2 and 3 cannot be omitted. For consider an irrational rotation of S^1. It has no periodic points. So every subinterval of S^1 is free of periodic points. However, no subinterval of S^1 is one-way.

Some of the results in this paper are from the second author's Ph.D. thesis at the University of Wisconsin-Milwaukee. Others are from a paper submitted by the first author to the Westinghouse Science Competition when she was a senior at Nicolet High School in Glendale, Wisconsin.

The remainder of the paper is divided into three sections. Section 2 establishes some lemmas used in the proof of the Theorem and its corollaries. Section 3 contains the proofs of the Theorem and corollaries. Section 4 presents the Example.

2. Preliminary lemmas

Lemma 1. *If $f : S^1 \to S^1$ is a map and (a, b) is a positive open interval in S^1, then for every $x \in (a, b)$, there is a $y \in (a, b)$ and an $n \geq 1$ such that $f^n(y) \in (x, b)$.*

Proof. Let $e : \mathbb{R} \to S^1$ be the exponential covering map $e(t) = e^{2\pi i t}$. We can assume there is a $z \in (a, b)$ such that $a < z < f^n(z) \leq x < b$. Let $a' < z' < x' < b' < a' + 1$ be points of $\mathbb{R}$ such that e maps a', z', x', and b' to a, z, x, and b respectively. Let $g : \mathbb{R} \to \mathbb{R}$ be a map which covers f^n (i.e., $e \circ g = f^n \circ e$) such that $z' < g(z') \leq x'$. Since $f|(a, b)$ has no fixed points, then $t < g(t)$ for $a' < t < b'$. So $(x', b') \subset g((z', b'))$. Hence, $(x, b) \subset f^n((z, b))$. □

The following result is Lemma 3.2 of [2].

Lemma 2. *If $f : S^1 \to S^1$ is a map, and J is an open interval in S^1 which contains no periodic points and is not one-way, then $\bigcup_{n=0}^{\infty} f^n(J) = S^1$.* □

Lemma 3. *If $f : S^1 \to S^1$ is a map, (a,b) is a positive open interval in S^1, $a < x < y < b$ and $n \geq 1$, then there is an $i \geq 1$ such that $f^{in}(x) \notin (a,y]$.*

Proof. If not, the monotone increasing sequence $\{f^{in}(x)\}_{i\geq 0}$ converges to a point $z \in (a,y]$. It then follows that $f^n(z) = z$, contradicting the positiveness of (a,b). □

Lemma 4. *If $f : S^1 \to S^1$ is a map, (a,b) is a positive open interval in S^1, and $c \in (a,b)$, then there is an $x \in (c,b)$ such that $f^n([c,x]) \cap [c,x] = \varnothing$ for every $n \geq 1$.*

Remark. The following proof is an adaptation to the circle of part of the proof of Proposition 6 on pages 73–74 of [1]. A more complete adaptation of this proposition to the circle appears in [3] as Proposition 2.1.

Proof. Assume that for each $x \in (c,b)$, there is an $n \geq 1$ such that $f^n([c,x]) \cap [c,x] \neq \varnothing$. We will derive a contradiction.

Since (a,b) is positive, there is an $x_0 \in (c,b)$ such that no iterate of c lies in $(a,x_0]$.

Claim 1: *For all $j,k \geq 1$, $f^j(c) \notin \operatorname{int}(f^k([c,x_0]))$.* Assume $f^j(c) \in \operatorname{int}(f^k([c,x_0]))$ for some $j,k \geq 1$. $f^k([c,x_0])$ is a closed interval because $c \notin f^k([c,x_0])$. Hence, there are points z and z' in S^1 such that $c < z < z' < x_0$ and $f^j(c) \in \operatorname{int}(f^k([z,z']))$. The continuity of f provides a $y \in (c,z)$ such that $f^i([c,y]) \cap [c,x_0] = \varnothing$ for $1 \leq i \leq j$ and $f^j([c,y]) \subset \operatorname{int}(f^k([z,z']))$. By hypothesis, there is an $n \geq 1$ such that $f^n([c,y]) \cap [c,y] \neq \varnothing$. Then $n > j$, and there is a $w \in [c,y]$ such that $f^n(w) \in [c,y]$. Since $f^j(w) \in f^k([z,z'])$, then $f^j(w) = f^k(x)$ for some $x \in [z,z']$. Therefore, $f^{n-j+k}(x) = f^n(w) \in [c,y]$. Since $a < c < y < z < z' < b$, $x \in [z,z']$ and $f^{n-j+k}(x) \in [c,y]$, we have contradicted the positiveness of (a,b). This establishes Claim 1.

Set $A = \{n \geq 1 : f^n([c,x_0]) \cap [c,x_0] \neq \varnothing\}$.

Claim 2: *For each $n \in A$, $f^n([c,x_0]) = [y_n,d]$ where $c < y_n \leq x_0 < d$; $c < y_k < y_j$ for $j,k \in A$ and $j < k$; and $\{y_n\}_{n\in A}$ converges to c.* For each $n \geq 1$, since $f^n(c) \notin \operatorname{int}(f^n([c,x_0]))$, then $f^n(c)$ is one of the endpoints of the closed interval $f^n([c,x_0])$. Let y_n denote the other endpoint. For $n \in A$, since $c \notin f^n([c,x_0])$, $f^n(c) \notin [c,x_0]$ and $f^n([c,x_0]) \cap [c,x_0] \neq \varnothing$, then necessarily $c < y_n \leq x_0 < f^n(c)$. We claim that $f^j(c) = f^k(c)$ for all $j,k \in A$. For if there are $j,k \in A$ such that $x_0 < f^j(c) < f^k(c)$, then $f^j(c) \in (y_k, f^k(c)) = \operatorname{int}(f^k([c,x_0]))$, contradicting Claim 1. Hence, there is a point $d \in S^1$ such that $f^n(c) = d$ for every $n \in A$. Therefore, $c < y_n \leq x_0 < d$ and $f^n([c,x_0]) = [y_n,d]$ for each $n \in A$.

Let $j,k \in A$ such that $j < k$. We assert that $c < y_k < y_j$. For suppose $c < y_j \leq y_k$. Then $f^k([c,x_0]) \subset f^j([c,x_0])$. It follows that the infinite union $\bigcup_{n=1}^{\infty} f^n([c,x_0])$ is equal to the finite union $\bigcup_{n=1}^{k-1} f^n([c,x_0])$. Since the finite union is a closed set not containing c, there is an $x \in (c,x_0)$ such that $[c,x]$ is disjoint from $\bigcup_{n=1}^{\infty} f^n([c,x_0])$. However, by hypothesis, there is an $n \geq 1$, such that $[c,x] \cap f^n([c,x]) \neq \varnothing$. Since $f^n([c,x]) \subset f^n([c,x_0])$, we have reached a contradiction. Our assertion follows.

If $x \in (c,x_0)$, then $f^n([c,x]) \cap [c,x] \neq \varnothing$ for some $n \geq 1$. Since $[c,x] \subset [c,x_0]$, it follows that $n \in A$ and $[y_n,d] \cap [c,x] \neq \varnothing$. Consequently, $y_n \in [c,x]$. This proves $\{y_n\}_{n\in A}$ converges to c, and completes Claim 2.

Set $m = \min\{k - j : j,k \in A \text{ and } j < k\}$.

Claim 3: $f^m((c,d]) = (c,d]$. Choose $i \in A$ such that $i + m \in A$, and set $S = \bigcup_{i\leq n\in A} f^n([c,x_0])$. Then $S = \bigcup_{i\leq n\in A} [y_n,d]$. Since $c < y_n < d$ for $n \in A$, and since $\{y_n\}_{n\in A}$ converges to c, then $S = (c,d]$.

We assert that $\{n \in A : n \geq i\} = \{i + pm : p \geq 0\}$. First let $p \geq 0$. Since $i, i+m \in A$, then Claim 2 implies $f^{i+m}([c, x_0]) = [y_{i+m}, d] \supset [y_i, d] = f^i([c, x_0])$. Repeated application of f^m yields $f^{i+pm}([c, x_0]) \supset f^i([c, x_0])$. Since $f^i([c, x_0])$ intersects $[c, x_0]$, so does $f^{i+pm}([c, x_0])$. Hence, $i + pm \in A$. On the other hand, if $n \in A$ and $n \geq i$, then there is a $p \geq 0$ such that $i + pm \leq n < i + (p+1)m$. Then $n = i + pm$ follows from the definition of m. This proves the assertion. Consequently, $S = \bigcup_{p=0}^{\infty} f^{i+pm}([c, x_0])$. Thus, $f^m(S) = \bigcup_{p=1}^{\infty} f^{i+pm}([c, x_0])$. Since $f^i([c, x_0]) \subset f^{i+m}([c, x_0])$, it follows that $f^m(S) = S$, proving Claim 3.

Since $f^m((c, d]) = (c, d]$, then $f^m(c) = c$, contradicting the positiveness of (a, b). □

Let P denote the set of periodic points of a map $f : S^1 \to S^1$. Since a one-way interval contains no periodic points, then every point of S^1 which has a one-way neighborhood lies in $S^1 - \mathrm{cl}(P)$. Conversely:

Lemma 5. *If $f : S^1 \to S^1$ is a map with a non-empty set P of periodic points, and if a point x of S^1 has no one-way neighborhood, then $x \in \mathrm{cl}(P)$.*

Proof. Assume $x \notin \mathrm{cl}(P)$. We will derive a contradiction. Lemma 2 implies that for each open interval neighborhood J of x which is disjoint from $\mathrm{cl}(P)$, $\bigcup_{n=1}^{\infty} f^n(J)$ covers $S^1 - J$. It follows that $f(S^1) \supset S^1 - \{x\}$. Since $f(S^1)$ is a closed subset of S^1, we conclude that f is onto.

We now refer the reader to the third paragraph of the proof of Theorem A of [2]. That paragraph, with some cosmetic changes, completes the proof of the present lemma. □

3. Proof of the Theorem and corollaries

The Theorem will be derived from the following three propositions. In all three propositions, $f : S^1 \to S^1$ is a map and $a < b < c < d$ are points of S^1.

Proposition 1. *If (a, c) is positive and (b, d) is one-way, then (a, d) is positive. Also if (b, d) is negative and (a, c) is one-way, then (a, d) is negative.*

Proof. Assume (a, c) is positive, (b, d) is one-way, and (a, d) is not one-way. We will derive a contradiction.

Lemma 1 provides an $x \in (a, c)$ and an $m \geq 1$ such that $f^m(x) \in (b, c)$. Let $a' \in (a, x)$ such that (a', d) is not one-way. Then Lemma 2 provides a $y \in (a', d)$ and an $n \geq 1$ such that $f^n(y) = a'$. It follows that $a < f^n(y) < x < f^m(x) < y < d$. See Figure 1.

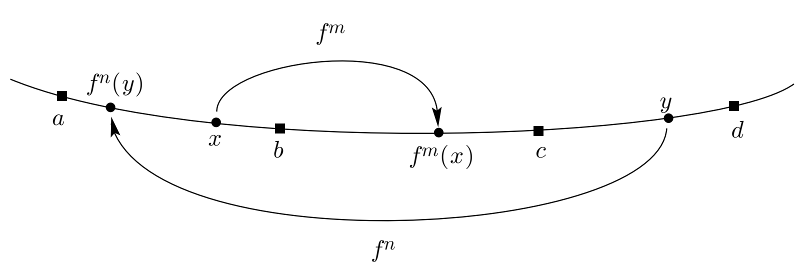


Figure 1

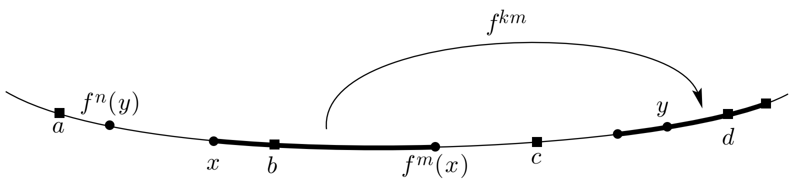


FIGURE 2

We claim that there is an $i \geq 1$ such that $x < f^m(x) \leq f^{im}(x) \leq y < f^{(i+1)m}(x)$. If (b,d) is negative or free, then $f^{2m}(x) \notin (a,d)$; and the claim follows if we set $i = 1$. On the other hand, if (b,d) is positive, then Lemma 3 provides an $i \geq 1$ such that $f^{(i-1)m}(f^m(x)) \subset (b,y]$ and $f^{im}(f^m(x)) \notin (b,y]$; and the claim follows.

Since $x \notin f^{im}([x, f^m(x)])$, then $f^{im}([x, f^m(x)]) \supset [f^{im}(x), f^{(i+1)m}(x)]$. Hence, there is a $z \in [x, f^m(x)]$ such that $f^{im}(z) = y$. See Figure 2. Therefore, $f^{im+n}(z) = f^n(y)$. So $a < f^{im+n}(z) < x \leq z < c$. This contradicts the positiveness of (a,c).

The situation in which (b,d) is negative and (a,c) is one-way can be transformed into the preceding situation dealt with simply by reversing the orientation on S^1. □

Proposition 2. *If (a,c) is free or negative, (b,d) is free or positive, and (a,d) is not one-way, then there are points b' in $(a,b]$ and c' in $[c,d)$ such that (a,c') is negative and (b',d) is positive.*

Proof. Claim 1: *If (a,c) is free and (b,d) is positive or free, then there is a point $c' \in [c,d)$ such that (a,c') is negative.* The union of all the one-way open subintervals of (a,d) with left endpoint a is a one-way open interval (a,c') where $c \leq c' < d$. If (a,c') is negative, we are done. If (a,c') is positive, then (a,d) is one-way by Proposition 1; so (a,c') cannot be positive. Assume (a,c') is free. There is a point $x \in (c',d]$ such that $f^k([c',x]) \cap [c',x] = \varnothing$ for each $k \geq 1$. (This follows from Lemma 4 in the case that (b,d) is positive and from the freeness of (b,d) otherwise.) Since (a,x) is not one-way, there is a $y \in (a,x)$ and an $m \geq 1$ such that $a < f^m(y) < y < x$. Since (a,c') is free, then $y \in (c',x)$. Since (b,d) is positive, then $f^m(y) \in (a,b]$. Since (a,y) is not one-way, then Lemma 2 provides a point $z \in (a,y)$ and an $n \geq 1$ such that $f^n(z) = y$. Then $z \notin [c',x]$. Hence, $z \in (a,c')$ and $f^{m+n}(z) = f^m(y) \in (a,c')$, contradicting the freeness of (a,c'). We conclude that (a,c') must be negative.

By reversing the orientation in Claim 1, we obtain:

Claim 2: *If (a,c) is negative or free and (b,d) is free, then there is a point $b' \in (a,b]$ such that (b',d) is positive.*

Clearly, an application of Claim 1, or of Claim 2, or of Claim 1 followed by Claim 2 yields a proof of Proposition 2. □

Proposition 3. *If (a,c) and (b,d) are one-way and $(b,c) \subset f(S^1)$, then (a,d) is one-way.*

Proof. By Proposition 1, we need only consider the situation in which (a,c) is negative or free, and (b,d) is positive or free. Assume (a,d) is not one-way. We will derive a contradiction.

Let $x \in (b,c)$. Since $x \notin f^n((a,x])$ for every $n \geq 1$, and $x \notin f^n([x,d])$ for every $n \geq 1$, then $x \notin f^n((a,d))$ for every $n \geq 1$. By hypothesis, $f(y) = x$ for some

$y \in S^1$. Lemma 2 provides a $z \in (a,d)$ and an $m \geq 0$ such that $f^m(z) = y$. Hence, $x = f(y) = f^{m+1}(z) \in f^{m+1}((a,d))$. We have reached a contradiction. □

Proof of the Theorem. Let $f : S^1 \to S^1$ be a map and let (a,d) be a locally one-way interval in S^1. Then (a,d) contains a one-way open interval (b',c'). We enlarge (b',c') to a maximal one-way open subinterval (b,c) of (a,d) by the following process. First take the union of all the one-way open subintervals of (a,d) with right endpoint c' to obtain a one-way open subinterval (b,c') of (a,d). Then take the union of all the one-way open subintervals of (a,d) with left endpoint b to obtain a one-way open subinterval (b,c) of (a,d).

Case 1. (b,c) *is free.* We prove $b = a$ and $c = d$. For suppose $b \neq a$. Since (a,d) is locally one-way, there is a one-way open subinterval (x,x') of (a,d) such that $x < b < x' < c$. Since (b,c) is maximal, then Proposition 1 implies (x,x') can't be positive, and Proposition 2 implies (x,x') can't be free or negative, a contradiction. $c = d$ is proved similarly. Hence, (a,d) is one-way.

Case 2. (b,c) *is positive.* We first prove that $c = d$. For if $c \neq d$, then there is a one-way open subinterval (y,y') of (a,d) such that $b < y < c < y'$. Then Proposition 1 implies that (b,y') is positive, contradicting the maximality of (b,c).

If $b = a$, then (a,d) is one-way and we are done. So assume $b \neq a$. Then there is a one-way open subinterval (x',x) of (a,d) such that $x' < b < x < d$. Since (b,d) is maximal, then Proposition 1 implies (x',x) must be free or negative. Then Proposition 2 allows us to assume (x',x) is negative. The union of all the one-way open subintervals of (a,d) with right endpoint x is a negative open subinterval (x'',x) of (a,d) which is the maximal one-way open subinterval of (a,d) with right endpoint x. We claim that $x'' = a$. For if $x'' \neq a$, then there is a one-way open subinterval (z,z') of (a,d) such that $z < x'' < z' < x$. Then Proposition 1 implies that (z,x) is negative, contradicting the maximality of (x'',x). Thus, $a < b < x < d$ where (a,x) is negative and (b,d) is positive.

Case 3: (b,c) *is negative.* We can transform Case 3 to Case 2 by simply reversing the orientation on S^1.

We have now proved the first conclusion of the Theorem: either (a,d) is one-way or there are points b and c in (a,d) such that $a < b < c < d$ and (a,c) is negative and (b,d) is positive.

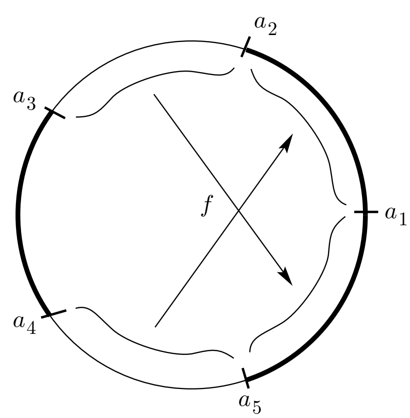
To complete the proof of the Theorem suppose $(a,d) \subset f(S^1)$ and (a,d) is not one-way. Then there are points b and c in (a,d) such that $a < b < c < d$ and (a,c) and (b,d) are one-way. But then Proposition 3 implies (a,d) is one-way, a contradiction. □

Corollary 1 is an obvious consequence of the Theorem.

Proof of Corollaries 2 *and* 3. Let $f : S^1 \to S^1$ be a map with a non-empty set P of periodic points. Lemma 5 implies that each component of $S^1 - \mathrm{cl}(P)$ is a locally one-way open interval. If f is onto, then by Corollary 1 each component of $S^1 - \mathrm{cl}(P)$ is one-way. So assume f is not onto. Since $P \subset f(S^1)$, then $\mathrm{cl}(P) \subset f(S^1)$. Let $x \in S^1 - f(S^1)$, and let (a,d) be the component of $S^1 - \mathrm{cl}(P)$ which contains x. Then $a, d \in \mathrm{cl}(P) \subset f(S^1)$. Since $f(S^1)$ is connected and contains a and d but not x, then $[d,a] \subset f(S^1)$. Hence, every component of $S^1 - \mathrm{cl}(P)$ except (a,d) is contained in $f(S^1)$. Thus, every component of $S^1 - \mathrm{cl}(P)$ with the possible exception of (a,d) is one-way. Furthermore, by the Theorem, either (a,d) is one-way or there are points b and c in (a, d) such that $a < b < c < d$ and (a, c) is negative and (b, d) is positive. □

4. The example

Let $a_1 < a_2 < a_3 < a_4 < a_5$ be points of S^1. Let $f : S^1 \to S^1$ be a map such that $f([a_5, a_2] \cup [a_3, a_4]) = \{a_1\}$, $f((a_2, a_3)) = [a_5, a_1)$ and $f((a_4, a_5)) = (a_1, a_2]$. See Figure 3. It is easily verified that (a_1, a_4) and (a_3, a_1) are free, (a_1, a_5) is negative and (a_2, a_1) is positive. Thus $S^1 - \{a_1\}$ is locally one-way but not one-way. Moreover, $S^1 - \{a_1\}$ is covered by two overlapping free intervals, by overlapping negative and free intervals, by overlapping free and positive intervals, and by overlapping negative and positive intervals. Since $S^1 - \{a_1\}$ is not covered by $f(S^1)$ and is not one-way, then these four types of overlapping interval pairs (free-free, negative-free, free-positive, and negative-positive) are the only types of overlapping interval pairs that are allowed by the proof of the Theorem. Moreover, the phenomenon described in Proposition 2 is illustrated here: the free-free, negative-free and free-positive overlapping interval pairs enlarge to a negative-positive overlapping interval pair.



thickened arc = $f^1(\{a_1\})$

Figure 3

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