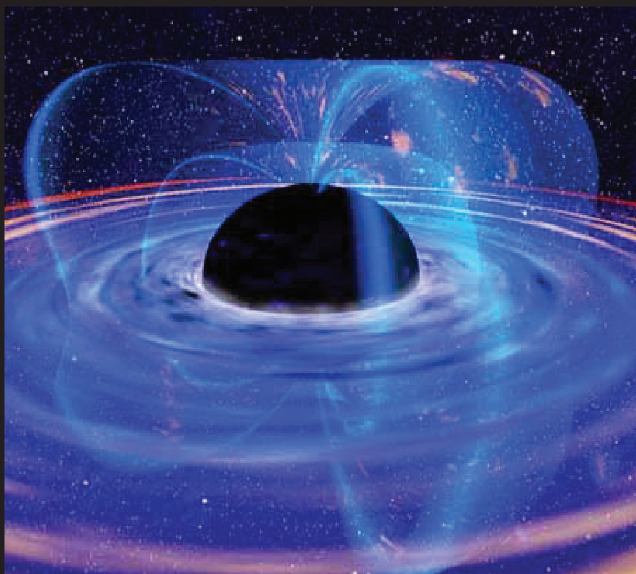
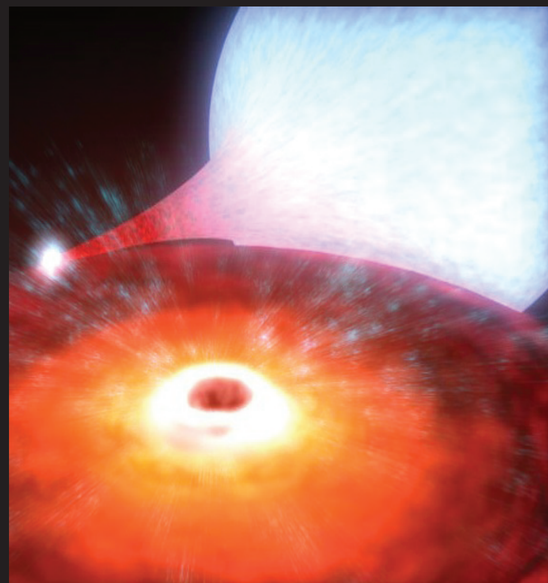
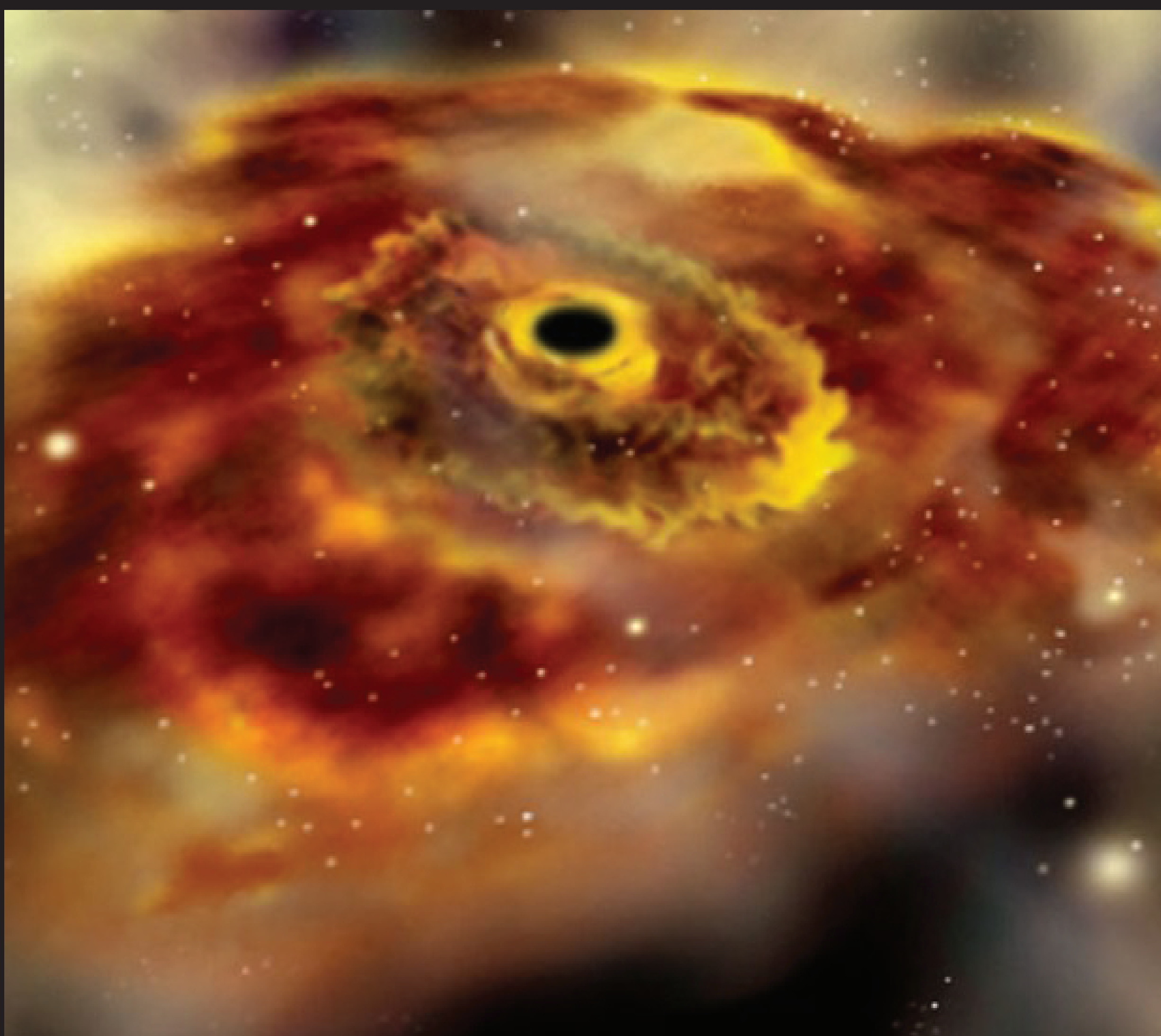
National Aeronautics and Space Administration







Black Hole Math

This collection of activities is based on a weekly series of space science problems distributed to thousands of teachers during the 2004-2008 school years. They were intended as supplementary problems for students looking for additional challenges in the math and physical science curriculum in grades 10 through 12. The problems are designed to be ‘one-pagers’ consisting of a Student Page, and Teacher’s Answer Key. This compact form was deemed very popular by participating teachers.

The topic for this collection is Black Holes, which is a very popular, and mysterious subject among students hearing about astronomy. Students have endless questions about these exciting and exotic objects as many of you may realize! Amazingly enough, many aspects of black holes can be understood by using simple algebra and pre-algebra mathematical skills. This booklet fills the gap by presenting black hole concepts in their simplest mathematical form.

General Approach:

The activities are organized according to progressive difficulty in mathematics. Students need to be familiar with scientific notation, and it is assumed that they can perform simple algebraic computations involving exponentiation, square-roots, and have some facility with calculators. The assumed level is that of Grade 10-12 Algebra II, although some problems can be worked by Algebra I students. Some of the issues of energy, force, space and time may be appropriate for students taking high school Physics.

For more weekly classroom activities about astronomy and space visit the NASA website,

http://spacemath.gsfc.nasa.gov

Add your email address to our mailing list by contacting Dr. Sten Odenwald at

Sten.F.Odenwald@nasa.gov

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Mathematics Topic Matrix

Topic	Problem Numbers																													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Inquiry																															
Technology, rulers	X	X																													
Numbers, patterns, percentages																					X										
Averages	X	X																										X			
Time, distance,speed						X	X	X																				X			
Areas and volumes																															
Scale drawings				X	X		X	X	X	X	X	X			X		X														
Geometry																													X	X	X
Scientific Notation										X			X	X					X	X	X	X	X	X	X	X	X				
Unit Conversions																															
Fractions																															
Graph or Table Analysis	X	X	X																										X	X	X
Solving for X						X							X																		
Evaluating Fns													X	X	X	X	X	X	X	X		X	X	X	X	X	X		X	X	X
Modeling																															
Probability																															
Rates/Slopes			X																												
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Conics																															
Piecewise Fns																															
Trigonometry																															
Integration																															
Differentiation																															
Limits																															

Mathematics Topic Matrix (cont'd)

Topic	Problem Numbers																													
	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53									
Inquiry																															
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Averages																															
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Geometry	X	X	X	X	X	X	X	X	X	X	X				X	X		X	X	X		X									
Scientific Notation																					X										
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Fractions																															
Graph or Table Analysis	X	X	X	X	X	X	X	X	X	X	X				X	X		X	X	X		X									
Solving for X																															
Evaluating Fns		X	X	X	X	X	X	X	X	X	X	X	X	X			X				X	X									
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How to use this book

Teachers continue to look for ways to make math meaningful by providing students with problems and examples demonstrating its applications in everyday life. Space Mathematics offers math applications through one of the strongest motivators-Space. Technology makes it possible for students to experience the value of math, instead of just reading about it. Technology is essential to mathematics and science for such purposes as “access to outer space and other remote locations, sample collection and treatment, measurement, data collection and storage, computation, and communication of information.” 3A/M2 authentic assessment tools and examples. The NCTM standards include the statement that "Similarity also can be related to such real-world contexts as photographs, models, projections of pictures" which can be an excellent application for Black Hole data.

Black Hole Math is designed to be used as a supplement for teaching mathematical topics. The problems can be used to enhance understanding of the mathematical concept, or as a good assessment of student mastery.

An integrated classroom technique provides a challenge in math and science classrooms, through a more intricate method for using Black Hole Math. Read the scenario that follows:

Ms. Green decided to pose a Mystery Math Activity for her students. She challenged each student with math problem from the Black Hole Space Math book. She wrote the problems on the board for students to solve upon entering the classroom; she omitted the words Black Hole from each problem. Students had to solve the problem correctly in order to make a guess to solve the “Mystery.” If the student got the correct answer they received a free math homework pass for that night. Since the problems are a good math review prior to the end of the year final exam, all students had to do all of the problems, even if they guessed the correct answer.

Black Hole Math can be used as a classroom challenge activity, assessment tool, enrichment activity or in a more dynamic method as is explained in the above scenario. It is completely up to the teacher, their preference and allotted time. What it does provide, regardless of how it is used in the classroom, is the need to be proficient in math. It is needed especially in our world of advancing technology and physical science

Alignment with Standards

AAAS: Project:2061 Benchmarks

(3-5) - Quantities and shapes can be used to describe objects and events in the world around us. 2C/E1 --- Mathematics is the study of quantity and shape and is useful for describing events and solving practical problems. 2A/E1 **(6-8)** Mathematicians often represent things with abstract ideas, such as numbers or perfectly straight lines, and then work with those ideas alone. The "things" from which they abstract can be ideas themselves; for example, a proposition about "all equal-sided triangles" or "all odd numbers". 2C/M1 **(9-12)** - Mathematical modeling aids in technological design by simulating how a proposed system might behave. 2B/H1 ---- Mathematics provides a precise language to describe objects and events and the relationships among them. In addition, mathematics provides tools for solving problems, analyzing data, and making logical arguments. 2B/H3 ----- Much of the work of mathematicians involves a modeling cycle, consisting of three steps: (1) using abstractions to represent things or ideas, (2) manipulating the abstractions according to some logical rules, and (3) checking how well the results match the original things or ideas. The actual thinking need not follow this order. 2C/H2

NCTM: Principles and Standards for School Mathematics

Grades 6–8 :

- work flexibly with fractions, decimals, and percents to solve problems;
- understand and use ratios and proportions to represent quantitative relationships;
- develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation; .
- understand the meaning and effects of arithmetic operations with fractions, decimals, and integers;
- develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios.
- represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules;
- model and solve contextualized problems using various representations, such as graphs, tables, and equations.
- use graphs to analyze the nature of changes in quantities in linear relationships.
- understand both metric and customary systems of measurement;
- understand relationships among units and convert from one unit to another within the same system.

Grades 9–12 :

- judge the reasonableness of numerical computations and their results.
- generalize patterns using explicitly defined and recursively defined functions;
- analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions;
- draw reasonable conclusions about a situation being modeled.

<u>Teacher Comments</u>

"Your problems are great fillers as well as sources of interesting questions. I have even given one or two of your problems on a test! You certainly have made the problems a valuable resource!" (Chugiak High School, Alaska)

"I love your problems, and thanks so much for offering them! I have used them for two years, and not only do I love the images, but the content and level of questioning is so appropriate for my high school students, they love it too. I have shared them with our math and science teachers, and they have told me that their students like how they apply what is being taught in their classes to real problems that professionals work on." (Wade Hampton High School ,SC)

"I recently found the Space Math problems website and I must tell you it is wonderful! I teach 8th grade science and this is a blessed resource for me. We do a lot of math and I love how you have taken real information and created reinforcing problems with them. I have shared the website with many of my middle and high school colleagues and we are all so excited. The skills summary allows any of us to skim the listing and know exactly what would work for our classes and what will not. I cannot thank you enough. I know that the science teachers I work with and I love the graphing and conversion questions. The "Are U Nuts" conversion worksheet was wonderful! One student told me that it took doing that activity (using the unusual units) for her to finally understand the conversion process completely. Thank you!" (Saint Mary's Hall MS, Texas)

"I know I'm not your usual clientele with the Space Math problems but I actually use them in a number of my physics classes. I get ideas for real-world problems from these in intro physics classes and in my astrophysics classes. I may take what you have and add calculus or whatever other complications happen, and then they see something other than "Consider a particle of mass 'm' and speed 'v' that…" (Associate Professor of Physics)

A Short Introduction to Black Holes

The basic idea of a black hole is simply an object whose gravity is so strong that light cannot escape from it. It is black because it does not reflect light, nor does its surface emit any light.

Before Princeton Physicist John Wheeler coined the term black hole in the mid-1960s, no one outside of the theoretical physics community really paid this idea much attention.

In 1798, the French mathematician Pierre Laplace first imagined such a body using Newton's Laws of Physics (the three laws plus the Law of Universal Gravitation). His idea was very simple and intuitive. We know that rockets have to reach an escape velocity in order to break free of Earth's gravity. For Earth, this velocity is 11.2 km/sec (40,320 km/hr or 25,000 miles/hr). Now let's add enough mass to Earth so that its escape velocity climbs to 25 km/sec…2000 km/sec…200,000 km/sec, and finally the speed of light: 300,000 km/sec. Because no material particle can travel faster than light, once a body is so massive and small that its escape velocity equals light-speed, it becomes dark. This is what Laplace had in mind when he thought about "black stars." This idea was one of those idle speculations at the boundary of mathematics and science at the time, and nothing more was done with the idea for over 100 years.

Once Albert Einstein had completed developing his Theory of General Relativity in 1915, the behavior of matter and light in the presence of intense gravitational fields was revisited. This time, Newton's basic ideas had to be extended to include situations in which time and space could be greatly distorted. There was an intense effort by mathematicians and physicists to investigate all of the logical consequences of Einstein's new theory of gravity and space. It took less than a year before one of the simplest kinds of bodies was thoroughly investigated through complex mathematical calculations.

The German mathematician Karl Schwarzschild investigated what would happen if all the matter in a body were concentrated at a mathematical point. In Newtonian physics, we call this the center of mass of the body. Schwarzschild chose a particularly simple body: one that was a perfect sphere and not rotating at all. Mathematicians such as Roy Kerr, Hans Reissner, and Gunnar Nordstrom would later work out the mathematical details for other kinds of black holes.

Schwarzschild black holes are actually very simple. Mathematicians even call them elegant because their mathematics is so compact, exact, and beautiful. They have a geometric feature called an "event horizon" (Problem 1) that mathematically distinguishes the inside of the black hole from the outside. These two regions have very different geometric properties for the way that space and time behave. The world outside the event horizon is where we live and contains our universe, but inside the event horizon, space and time behave in very different ways entirely (Problem 9). Once inside, matter and light cannot get back out into the rest of the universe. This horizon has nothing to do, however, with the Newtonian idea of an escape velocity.

By the way, these statements sound very qualitative and vague to students, but the mathematics that goes into making these statements is both complex and exact. With this in mind, there are four basic kinds of black hole solutions to Einstein's equations:

Schwarzschild: These are spherical and do not rotate. They are defined only by their total mass.

Reissner-Nordstrom: These possess mass and charge but do not rotate.

Kerr: These rotate and are flattened at the poles, and only described by their mass and amount of spin (angular momentum).

Kerr-Nordstrom: These possess mass and charge, and they rotate.

There are also other types of black holes that come up when quantum mechanics is applied to understanding gravity or when cosmologists explore the early history of the universe. Among these are

Planck-Mass: These have a mass of 0.00000001 kilograms and a size that is 100 billion billion times smaller than a proton.

Primordial: These can have a mass greater than 10 trillion kilograms and were formed soon after the big bang and can still exist today. Smaller black holes have long-since vanished through evaporation in the time since the big bang.

A Common Misconception

Black holes cannot suck matter into them except under certain conditions. If the sun turned into a black hole, Earth and even Mercury would continue to orbit the new sun and not fall in. There are two common cases in the universe in which matter can be dragged into a black hole. Case 1: If a body orbits close to the event horizon in an elliptical orbit, it emits gravitational radiation, and its orbit will eventually decay in millions of years. Case 2: A disk of gas can form around a black hole, and through friction, matter will slowly slide into the black hole over time.

How Black Holes are Formed

Black holes can come in any size, from microscopic to supermassive. In today's universe, massive stars detonate as supernovae and this can create stellar-mass black holes (1 solar mass = 1.9×10^{30} kg). When enough of these are present in a small volume of space, like the core of a globular cluster, black holes can absorb each other and in principle, can grow to several hundred times the mass of the sun. If there is enough matter (i.e., gas, dust, and stars) for a black hole to “eat,” it can grow even larger. There is a black hole in the star-rich core of the Milky Way that has a mass equal to nearly 3 million suns. The cores of more massive and distant galaxies have supermassive black holes containing the equivalent of 100 million to as much as 10 billion suns. Astronomers are not entirely sure how these supermassive black holes evolved so quickly to their present masses given that the universe is only 14 billion years old.

Currently, there are no known ways to create black holes with masses less than about 0.1 times the sun's mass, and through a speculative process called Hawking Radiation, black holes less than 1 trillion kg in mass would have evaporated by now if they had formed during the Big Bang.

A Short List of Known Black Holes

Stellar-Mass

Name	Constellation	Distance (Light years)	Mass (in solar units)
Cygnus X-1	Cygnus	7,000	16
SS 433	Aquila	16,000	11
Nova Mon 1975	Monocerous	2,700	11
Nova Persi 1992	Perseus	6,500	5
IL Lupi	Lupus	13,000	9
Nova Oph 1977	Ophiuchus	33,000	7
V4641 Sgr	Sagittarius	32,000	7
Nova Vul 1988	Vulpecula	6,500	8
V404 Cygni	Cygnus	8,000	12

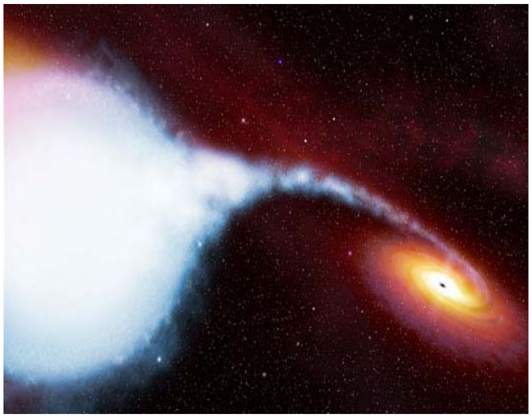
Note: The mass is the sum of the companion star and the black hole masses. '16' means 16 times the mass of the sun.

Galactic - Mass

Name	Constellation	Distance (Light years)	Mass (in solar units)
NGC-205	Andromeda	2,300,000	90,000
Messier-33	Triangulum	2,600,000	50,000
Milky Way SgrA*	Sagittarius	27,000	3,000,000
Messier-31	Andromeda	2,300,000	45,000,000
NGC-1023	Canes Venatici	37,000,000	44,000,000
Messier-81	Ursa Major	13,000,000	68,000,000
NGC-3608	Leo	75,000,000	190,000,000
NGC-4261	Virgo	100,000,000	520,000,000
Messier-87	Virgo	52,000,000	3,000,000,000

Note: The first three are called 'Intermediate-mass' black holes. The remainder are called 'supermassive'.

The Nearest Stellar Black Holes



An artist's concept of Cygnus X-1 shows hot gas from the giant blue star flowing toward the black hole, forming a bright accretion disk. [ESA/Hubble & ESA Information Centre (Kornmesser & Christensen)]

How close is the nearest black hole to our own sun? Because our Milky Way galaxy is a very flat disk of stars, we can use Cartesian coordinates to map out where the nearest black holes are!

Black holes are created when very massive stars explode as supernova. Fortunately, this does not happen very often in our corner of the Milky Way, so black holes are actually very far apart !

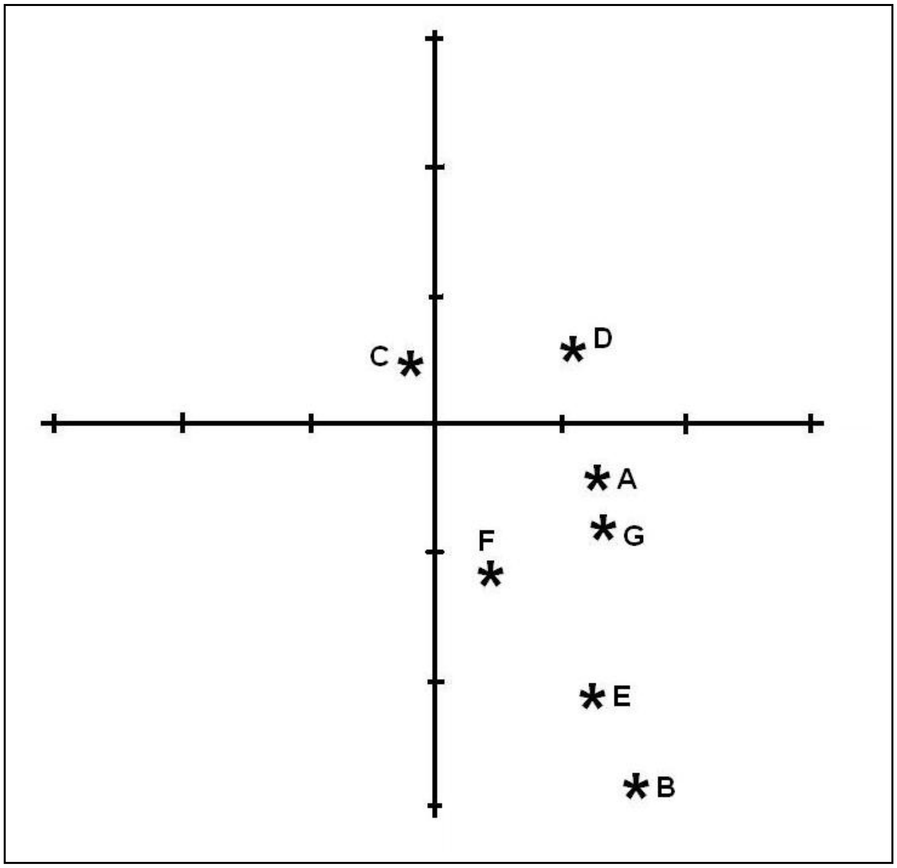
The table below gives the coordinates of the seven nearest black holes to our sun and solar system. The mass of each black hole is given in terms of solar mass units so that '16' means that the mass of the black hole is 16 times that of our sun. All distances (X, Y and D) are given in light years, where 1 light year = 9.6 trillion kilometers.

Name	Mass	X	Y	D
A) Cygnus X-1	16	6600	-2400	
B) SS-433	11	8000	-14000	
C) Nova Monocerotes 1975	11	-1400	2400	
D) Nova Persi 1992	5	5600	3300	
E) IL Lupi	9	6500	-11000	
F) Nova Vulpeculi 1988	8	2200	-6100	
G) V404 Cygni	12	6900	-4000	

Problem 1 – Create a Cartesian coordinate grid with coordinate intervals of 5, 10, 15 representing distances of 5000, 10000, 15000, light years, with the sun at the Origin. On this grid, plot the location of each black hole shown in the table above.

Problem 2 – Using a ruler, measure the distance between the sun and each black hole, convert this to its true distance rounded to the nearest thousands of light years, and enter the result in the last column of the table.

Problem 3 – What is the mean, median and mode distance between stellar black holes in the neighborhood of our sun rounded to the nearest thousands of light years?



Problem 1 – Create a Cartesian coordinate grid with coordinate intervals of 5, 10, 15 representing distances of 5000, 10000, 15000, light years, with the sun at the Origin. On this grid, plot the location of each black hole shown in the table above.

Answer: **See diagram above**.

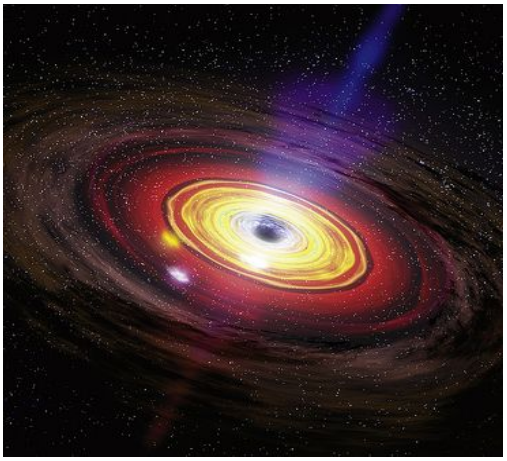
Problem 2 – Using a ruler, measure the distance between the sun and each black hole, convert this to its true distance rounded to the nearest thousands of light years, and enter the result in the last column of the table.

Answer: In order from top to bottom in the table: **7000, 16000, 3000, 7000, 14000, 7000, 8000.** Note: Students may also use the Pythagorean Theorem.

Problem 3 –What is the mean, median and mode distance between stellar black holes in the neighborhood of our sun to the nearest thousands of light years?

Answer: Mean = (7000+16000 +3000 +7000+ 14000+7000+ 8000)/ 7

Mean = 9,000 light years.
Median= 7,000 light years.
Mode=7,000 light years


Artist image of a supermassive black hole. Courtesy (NASA/Hubble/Dana Berry)

Within the dense cores of most galaxies, lurk black holes that have grown over the eons into supermassive objects containing millions of times the mass of a stellar black hole. Some rare galaxies have two or three of these black holes, but far more have only one.

Black holes may never lose mass. They steadily gain mass over the millennia by consuming interstellar gas, and even entire stars that are unfortunate enough to become trapped by their colossal gravity.

The table below gives the distances and locations of the ten closest supermassive black holes to the Milky Way galaxy. The mass of each supermassive black hole is given in terms of solar mass units so that '90,000' means that the mass of the supermassive black hole is 90,000 times that of our sun. All distances (X, Y) are given in millions of light years, where 1 light year = 9.6 trillion kilometers.

Name	Mass	X	Y	D
NGC-205	90,000	1.1	2.0	
Messier-33	50,000	0.5	2.6	
Sagittarius A*	3 million	0	0	
Messier-31	45 million	1.2	2.0	
NGC-1023	44 million	-35.0	12.7	
Messier-81	68 million	6.5	-11.3	
NGC-3608	190 million	-70.5	-25.7	
NGC-4261	520 million	-64.3	-76.7	
Messier-87	3 billion	-33.4	-39.8	

Problem 1 – Use the 2-point distance formula to determine the distance, in millions of light years, between the Milky Way (0,0) and each of the nearby supermassive black holes. Enter your answer in the 'D' column.

Problem 2 – Which supermassive black hole is the closest to Messier-87?

Problem 3 – From the location of NGC-3608 as the new origin (0,0) what would be the new coordinates of the other supermassive black holes?

Name	Mass	X	Y	D
NGC-205	90,000	1.1	2.0	**2**
Messier-33	50,000	0.5	2.6	**3**
Sagittarius A*	3 million	0	0	**0**
Messier-31	45 million	1.2	2.0	**2**
NGC-1023	44 million	-35.0	12.7	**37**
Messier-81	68 million	6.5	-11.3	**13**
NGC-3608	190 million	-70.5	-25.7	**75**
NGC-4261	520 million	-64.3	-76.7	**100**
Messier-87	3 billion	-33.4	-39.8	**52**

Problem 1 – Use the 2-point distance formula to determine the distance, to the nearest million light years, between the Milky Way (0,0) and each of the nearby supermassive black holes. Enter your answer in the 'D' column.

Problem 2 – Which supermassive black hole is the closest to Messier-87?

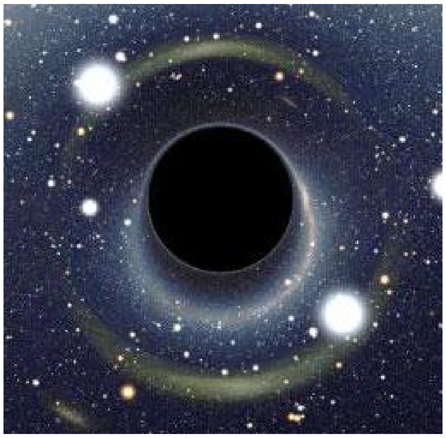
Answer: **NGC-1023** is located $D^2 = (-33.4 - (-35.0))^2 + (-39.8 - (12.7)^2$ so
D = 52 million light years;

Problem 3 – From the location of NGC-3608 as the new origin (0,0) what would be the new coordinates of the other supermassive black holes?

Answer: Subtract the coordinate (-70.5, -25.7) from the other coordinates to get:

Name	Mass	X	Y
NGC-205	90,000	**71.6**	**27.7**
Messier-33	50,000	**71.0**	**28.3**
Sagittarius A*	3 million	**70.5**	**25.7**
Messier-31	45 million	**71.7**	**27.7**
NGC-1023	44 million	**35.5**	**38.4**
Messier-81	68 million	**77.0**	**14.4**
NGC-3608	190 million	**0**	**0**
NGC-4261	520 million	**6.4**	**-51.0**
Messier-87	3 billion	**37.1**	**-14.1**

Exploring the Size and Mass of a Black Hole



Black holes are so incredibly dense that enormous amounts of matter can be compressed into their very small volumes. No known physical event can make black holes smaller than the mass of a small star. But because black holes are a product of gravity, at least theoretically, there is no limit to how big or how small they can be.

The table below gives the predicted radius of black holes containing various amounts of matter. None of these black holes have been observed, but their sizes have been determined from their stated masses. The masses are all given in terms of the mass of our Earth, 5.7×10^{24} kilograms so that '2.0' means a black hole with twice the mass of our Earth.

Mass	Radius
2.0	16.8 cm
3.2	26.9 cm
5.0	42.0 cm
7.5	63.0 cm
8.7	73.1 cm
11.0	96.6 cm

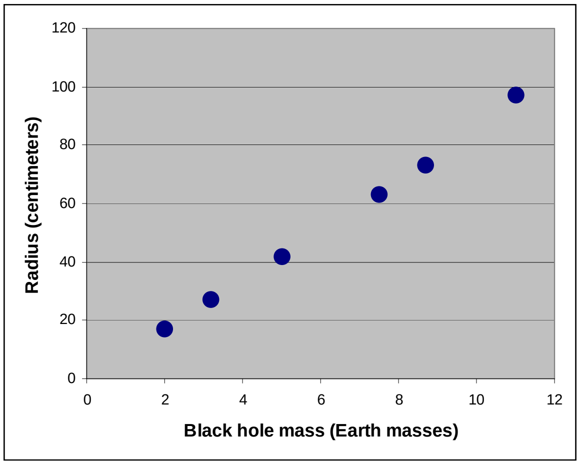
Problem 1 – Graph the data in the table.

Problem 2 – From the graph, use any method to calculate the slope, S, of the data. What are the physical units for the value of this slope?

Problem 3 – From the table, calculate the slope, S, of the data.

Problem 4 – Write a linear equation of the form R(M) = R0 + S M that expresses the black hole Mass-Radius Law.

Problem 5 – To the nearest tenth of a meter, what would you predict as the radius of a black hole with the mass of the planet Jupiter, if the mass of Jupiter is 318 times the mass of Earth?



Problem 1 – Graph the data in the table. (see above)

Problem 2 – From the graph, use any method to calculate the slope of the data. What are the physical units for the value of this slope?

Answer: Select any two points: (2,16.8) and (5,42.0). Draw a line between the points and the slope is just the change in the y values (42-16.8) divided by the change in the x values (5-2) so S = 25.2/3 = **8.4. The units are centimeters per Earth mass**.

Problem 3 – From the table, calculate the slope of the data.

Answer: From the table, if M = 2.0, R = 16.8cm, and if M=5.0, R = 42.0 cm. The slope is just S = (42.0-16.8)/(5.0-2.0) = 25.2/3.0 = **8.4 cm/Earth mass**.

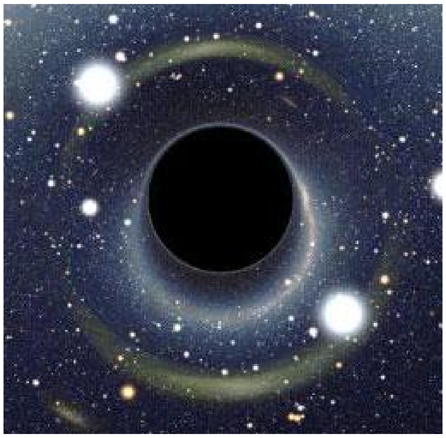
Problem 4 – Write a linear equation of the form y = b + S x that expresses the black hole Mass-Radius Law.

Answer: We know that the slope, S = 8.4. If we substitute the coordinates for one of the points into this equation (2.16.8) we get 16.8 = b + 8.4 (2). Then solving for the y intercept we get b= 0.0, so the formula reads **R(M) = 8.4 M**

Problem 5 – To the nearest tenth of a meter, what would you predict as the radius of a black hole with the mass of the planet Jupiter, if the mass of Jupiter is 318 times the mass of Earth?

Answer: For M = 318 Earths, R(318) = 8.4 x 318 = 2671 centimeters or **2.7 meters**.

The Moon as a Black Hole!



Suppose that a group of hostile aliens passed through our solar system and decided to convert our moon into a black hole!

A body with the mass of our moon (about 7 million trillion tons!) would be compressed into a black hole with a diameter of only 0.2 millimeters!

Problem 1 – In the space below, draw a black disk 0.2 millimeters in diameter to represent the size of Black Hole Moon.

Problem 2 - The Earth as a black hole would have a radius of 8.7 millimeters. In the space below, draw a circle the size of Black Hole Earth.

Problem 3 - If the distance to the moon is 356,000 kilometers, how far from our Black Hole Earth would the new Black Hole Moon be located if its diameter were only 0.2 millimeters?

Problem 1 - Moon black hole shown as the following dot .

Problem 2 - Earth black hole shown above. It has a diameter of 17 millimeters, or about the same diameter as a dime.

Problem 3 - Answer: It would still be 356,000 kilometers because this is NOT a scaled drawing of the black holes sizes, but an illustration of their actual sizes, so the distance between the black disks above would be 356,000 kilometers!

Exploring Black Holes

All bodies produce gravity. The more mass a body has, the more gravity it creates.

It is also true that the smaller you make a body by compressing it, the more intense its gravity is at its surface.

Suppose you made a body that had such an intense gravity that even light could not escape from it.

That body would be called a **black hole**, because anything falling into it, even light, could never escape from it again.

Black holes can come in all imaginable sizes. Suppose that some aliens could turn the planets and moons in our solar system into black holes. How big would they be?

On a black piece of paper, use a ruler and a compass to make circles that are as large as the black holes mentioned in each of the following problems.

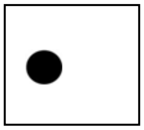
Cut these circles out, and make a black hole mobile of the smaller bodies in the solar system!

Problem 1 - Mercury is a black hole with a radius of 0.5 millimeters.
Problem 2 - Venus is a black hole with a radius of 7 millimeters
Problem 3 - Earth is a black hole with a radius of 9 millimeters
Problem 4 - The moon is a black hole with a radius of 0.1 millimeters
Problem 5 - Mars is a black hole with a radius of 1.0 millimeter
Problem 6 - Pluto is a black hole with a radius of 0.02 millimeters

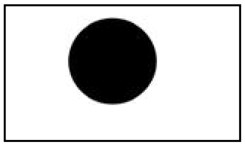
The giant planets will need black circles that are much bigger!

Problem 7 - Jupiter is a black hole with a radius of 280 centimeters
Problem 8 - Saturn is a black hole with a radius of 83 centimeters
Problem 9 - Uranus is a black hole with a radius of 13 centimeter
Problem 10 - Neptune is a black hole with a radius of 15 centimeter

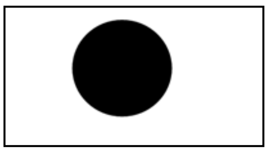
Answer Key (approximate sizes)



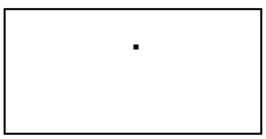
Mercury
Diameter = 1 mm



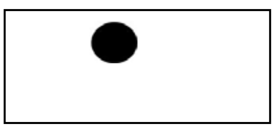
Venus
Diameter= 14 mm



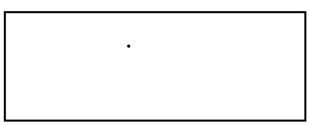
Earth
Diameter= 18 mm



Moon
Diameter= 0.2 mm



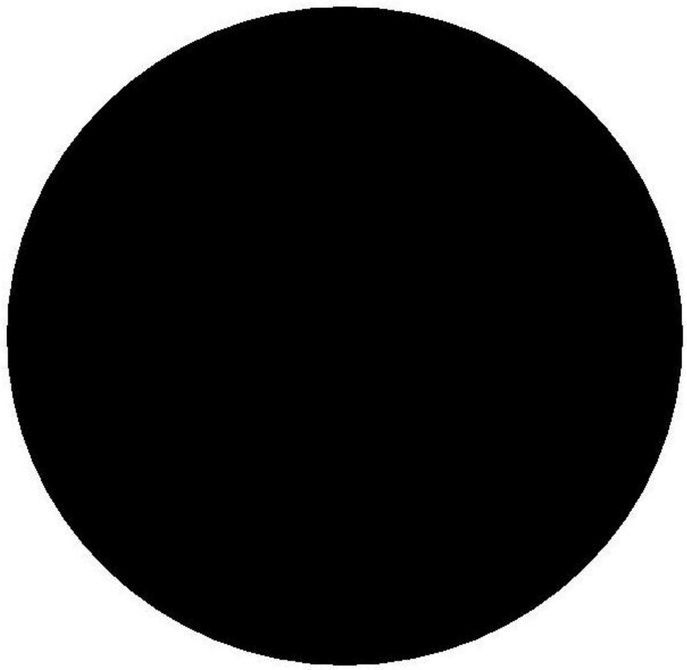
Mars
Diameter= 2 mm



Pluto
Diameter= 0.04 mm

Exploring a Full-Sized Black Hole

This black ball shown below is the exact size of a black hole with a diameter of 9.0 centimeters. Such a black hole would have a mass of **5 times** the mass of our Earth. All of this mass would be INSIDE the ball below.



Although it looks pretty harmless, if this black hole were at arms-length, you would already be dead. In fact, if you were closer to it than the distance from New York to San Francisco, 1 150-pound person would weigh 3 tons and would be crushed by their own weight!

Suppose that you could survive being crushed to death as you got closer to the black hole shown above. To stay in an orbit around the black hole so that you did not fall in, you have to be traveling at a specific speed V, in kilometers per second, that depends on your distance R, in meters from the center of the black hole, is given below:

$$V = \frac{44,700}{\sqrt{R}}$$

Problem 1 - If you were orbiting at the distance of the Space Shuttle (R=6,800 km) from the center of this black hole, what would your orbital speed be in A) kilometers/sec? B) kilometers/hour? C) miles per hour (1 mile = 1.6 km).

Problem 2 - If a small satellite were orbiting 20 centimeters away from the center of the black hole shown above, how fast would it be traveling in A) km/second? B) percentage of the speed of light? (The speed of light = 300,000 km/sec).

Problem 3) If the orbit is a circle, how long: A) would the Space Shuttle in Problem 1 take to go once around in its orbit? B) would it take the satellite in Problem 2 to go once around in its orbit?

Problem 1 - If you were orbiting at the distance of the Space Shuttle (R=6,800 km) from the center of this black hole, what would your orbital speed be in A) kilometers/sec? B) kilometers/hour? C) miles per hour (1 mile = 1.6 km).

Answer; A) The formula says that for R = 6,800,000 meters, **V = 17 km/sec**.
B) 1 hour = 3600 seconds, so V = 17 km/sec x (3600 sec/1 hour) = **61,200 km/hour.**
C) V = 61,200 km/sec x (1 mile / 1.6 km) = **38,250 miles/hr**

Problem 2 - If a small satellite were orbiting 20 centimeters away from the center of the black hole shown above, how fast would it be traveling in A) km/second? B) percentage of the speed of light? (The speed of light = 300,000 km/sec).

Answer; A) R = 0.2 meters, so from the formula V = **100,000 km/sec**
B) Speed = 100% x (100000/300000) so speed = **33% the speed of light**.

Problem 3) If the orbit is a circle, how long: A) would the Space Shuttle in Problem 1 take to go once around in its orbit? B) would it take the satellite in Problem 2 to go once around in its orbit?

A) Orbit circumference, $C = 2\pi R$ so for R = 6,800 km, C = 40,000 kilometers. The Shuttle speed is V=17 km/sec, so the time is T = C/V or **2,353 seconds. This equals about 39 minutes.**

B) A) Orbit circumference, $C = 2\pi R$ so for R = 0.2 meters, C = 1.25 meters. The satellite speed is V=100,000 km/sec. Converting this to meters we get 100,000,000 meters/sec, so the time is T = C/V or **0.000000013 seconds. This is 13 billionths of a second!**

A Scale-Model Black Hole - Orbit Speeds

Black holes can come in all sizes, so let's build one that fits on your desk top, and explore some of its interesting properties!

Get a basket ball (diameter 25 cm) and paint it black. Note: An actual black hole of this size would have a mass equal to about 29 times the Earth!

Most of the real weird things about black holes are hidden in the 'numbers' that define their properties.

Orbit Speed – At a distance of 40 cm from the center of the black hole, a satellite would orbit at a speed of 174,000 km/sec.

Problem 1 - If the speed of light is 300,000 km/sec, what is the orbit speed at 25.1 cm from the center of this 'toy' black hole in terms of a percentage of the light-speed?

Problem 2 - How far above the surface (called the **event horizon**) is the satellite at that distance?

Problem 3 – Suppose you wanted to move the satellite from an orbit distance of 40 centimeters to 29 centimeters in order to study the event horizon more closely. By how much would you have to change the satellites speed?

Distance (cm)	Orbit Speed (km/s)
40	174,000
35	186,000
30	200,000
29	204,000
28	208,000
27	212,000
26	216,000
25.5	218,000
25.1	220,000

Problem 1 - If the speed of light is 300,000 km/sec, what is the orbit speed at 25.1 cm from the center of this 'toy' black hole in terms of a percentage of the light-speed?

Answer: The tables says that at 25.1 cm the speed will be 220,000 km/s so in terms of the speed of light this is 100% x (220,000/300,000) = **73% the speed of light!**

Problem 2 - How far above the surface (called the **event horizon**) is the satellite at that distance?

Answer: The radius of the black hole is 25 centimeters, so the distance of the satellite above the event horizon is 25.1 cm – 25 cm = 0.1 centimeters or **1 millimeters**!

Problem 3 – Suppose you wanted to move the satellite from an orbit distance of 40 centimeters to 29 centimeters in order to study the event horizon more closely. By how much would you have to change the satellites speed?

Answer: The satellites speed would have to increase from 174,000 km/sec to 204,000 km/sec or **30,000 km/sec**.

A Scale-Model Black Hole - Orbit Periods

Black holes can come in all sizes, so let's build one that fits on your desk top, and explore some of its interesting properties!

Get a basket ball (diameter 25 cm) and paint it black. Note: An actual black hole of this size would have a mass equal to about 29 times the Earth!

Most of the real weird things about black holes are hidden in the 'numbers' that define their properties.

Time Distortion – The typical speed of an object that would be orbiting our 'toy' black hole is about 200,000 km/s or 70% the speed of light! That means a satellite would only take a few billionths of a second to make one orbit. As a comparison, for a normal Earth satellite, it takes about 90 minutes! The table below gives two 'time' columns. For example, a clock carried by the satellite at a distance of 30 centimeters from the center of the black hole would record that it took 9 nanoseconds (or 0.000000009 seconds) to make one complete orbit. Because of the distortion of gravity, a distant observer would see the satellite take 23 nanoseconds to make one complete orbit! As viewed by the distant observer, time is actually passing more slowly on the satellite.

Problem 1 – Suppose an instrument to study the black hole spends 1 year orbiting at a distance of 30 cm from the center of the black hole. How much time would have passed as viewed by a distant observer on Earth?

Problem 2 – A meteor falls into the black hole and produces a brilliant flash of light that lasts 6 nanoseconds just before it passes across the event horizon at 25 centimeters. How long will this flash last as seen from Earth?

Distance (cm)	Satellite Period (nanosec)	Observed Period (nanosec)
40	14	21
35	12	22
30	9	23
29	8	24
28	8	26
27	8	29
26	8	39
25.5	7	53
25.1	6	114

Problem 1 – Suppose an instrument to study the black hole spends 1 year orbiting at a distance of 30 cm from the center of the black hole. How much time would have passed as viewed by a distant observer on Earth?

Answer: At 30 cm, the satellite clock takes 9 nanoseconds to orbit once, while the distant observer sees 23 nanoseconds pass. The amount of time dilation is just the ratio of these two times or 23 ns/9 ns = 2.5 times. That means that if one second passes on the satellite, the Observer will see 2.5 seconds pass; if 1 hour passes on the satellite, this will be seen as 2.5 hours by the Observer and so on. **If 1 year passes on the satellite, the distant observer will see 2.5 years pass.**

Problem 2 – A meteor falls into the black hole and produces a brilliant flash of light that lasts 6 nanoseconds just before it passes across the event horizon at 25 centimeters. How long will this flash last as seen from Earth?

Answer: From the table, if the meteor is close to the event horizon, a 6 nanosecond event will be seen to last **114 nanoseconds** as viewed by a distant observer.

A Scale-Model Black Hole - Doppler Shifts

Black holes can come in all sizes, so let's build one that fits on your desk top, and explore some of its interesting properties!

Get a basket ball (diameter 25 cm) and paint it black. Note: An actual black hole of this size would have a mass equal to about 29 times the Earth!

Most of the real weird things about black holes are hidden in the 'numbers' that define their properties.

Wavelength Stretching – A light wave is defined by its wavelength, and because it travels at the speed of light, this wave takes a certain amount of time to pass-by. But if the passage of time is distorted near a black hole, this means that wavelength of light is also distorted. The time between crests of the wave will pass-by in a different time interval. The table on the left shows what happens to a light wave that starts out with a wavelength of 100 nanometers at each of the distances from our toy black hole. (Note: Electromagnetic radiation with a wavelength of 100 nm are called X-rays). The table to the right gives the wavelengths for common types of light energy. For example, as seen by the distant observer on Earth, x-rays emitted from 26 centimeters have there wavelengths stretched from 100 nm to 510 nanometers!

Problem 1 – An astronomer wants to study a burst of X-ray light emitted by a meteor falling into the black hole at a distance of 25.5 cm from the center of the black hole. At what wavelength on Earth will the burst be observed? What is the name for this radiation detected at Earth?

Problem 2 – If the meteor is emitting the x-rays the whole time that it falls from a distance of 40 cm to 25 cm, what will the astronomer on Earth observe about the radiation that she detects?

Distance (cm)	Wavelength (Nm)
40	163
35	187
30	245
29	269
28	306
27	367
26	510
25.5	714
25.1	1584

Problem 1 – An astronomer wants to study a burst of X-ray light emitted by a meteor falling into the black hole at a distance of 25.5 cm from the center of the black hole. At what wavelength on Earth will the burst be observed? What is the name for this radiation detected at Earth?

Answer: **The x-ray light will be shifted to a wavelength of about 714 nanometers. This radiation is called infrared radiation.**

Problem 2 – If the meteor is emitting the x-rays the whole time that it falls from a distance of 40 cm to 25 cm, what will the astronomer on Earth observe about the radiation that she detects?

Answer: **It will look like the meteor is first emitting x-ray light, then as it gets closer to the black hole, the light will shift into ultraviolet, visible and then infrared.**

A Scale-Model Black Hole - Gravity

Black holes can come in all sizes, so let's build one that fits on your desk top, and explore some of its interesting properties!

Get a basket ball (diameter 25 cm) and paint it black. Note: An actual black hole of this size would have a mass equal to about 29 times the Earth!

Most of the real weird things about black holes are hidden in the 'numbers' that define their properties.

Gravity – The amount of gravity generated by a black hole can be tremendous because of all the matter that gets stuffed into such a small region of space! The table below gives the acceleration of gravity at the different distances from the toy black hole in terms of multiples of one Earth Gravity at Earth's surface. For example, if the gravity field had a strength of 10 Earth gravities, you would weigh 10 times your normal '1-G' weight. A 150-pound (667.5 Newtons) person would feel as though they weighed 1,500 pounds (6675 Newtons)!

Problem 1 – A 100-pound student stands 40 centimeters away from the toy black hole. How much will the student weigh at that distance?

Problem 2 – According to the old Bohr Model for atoms, an electron orbits the single proton inside a hydrogen atom. It experiences an acceleration equal to about 9.6×10^{19} Gs. How much less is the acceleration due to the gravity of the toy black hole at 25.1 centimeters?

Problem 3 – If the acceleration is proportional to the inverse-square of the distance from the black hole, at what distance, in kilometers, will the acceleration be exactly 1 G?

Distance (cm)	Earth Gravitys
40	1.2×10^{15}
35	1.0×10^{16}
30	1.4×10^{16}
29	1.5×10^{16}
28	1.6×10^{16}
27	1.7×10^{16}
26	1.8×10^{16}
25.5	1.9×10^{16}
25.1	2.0×10^{16}

Problem 1 – A 100-pound student stands 40 centimeters away from the toy black hole. How much will the student weigh at that distance?

Answer: The table says that at 40 centimeters, the acceleration of gravity will be 1.2×10^{15} Gs, so the student will weigh (100 pounds/ 1 G) x 1.2×10^{15} Gs = **1.2×10^{17} pounds…that's 120 thousand trillion pounds!!**

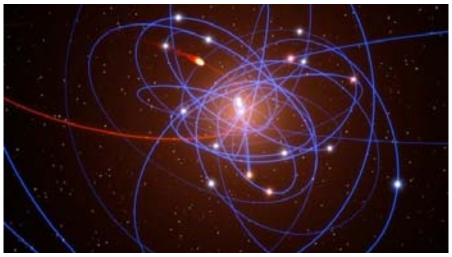
Problem 2 – According to the old Bohr Model for atoms, an electron orbits the single proton inside a hydrogen atom. It experiences an acceleration equal to about 9.6×10^{19} Gs. How much less is the acceleration due to the gravity of the toy black hole at 25.1 centimeters?

Answer: 9.6×10^{19} Gs / 2.0×10^{16} Gs = **4800 times less.**

Problem 3 – If the acceleration id proportional to the inverse-square of the distance from the black hole, at what distance, in kilometers, will the acceleration be exactly 1 G?

Answer: This inverse-square relationship says that if I double my distance from the black hole, the acceleration will decrease by a factor of 4 times so that its strength is now ¼ of its closer value. If the acceleration is 1.0×10^{16} Gs at a distance of 35 centimeters from the center of the toy black hole, we need to diminish the acceleration by a factor of $1/1.0 \times 10^{16}$ to get to 1 G. We will have to increase the distance by $(1.0 \times 10^{16})^{1/2} = 1.0 \times 10^{8}$ times its original distance of 35 cm. This means that at D = 3.5×10^{9} centimeters the acceleration will be 1 G. Since there are 100,000 centimeters in 1 kilometer, this is equal to **35,000 kilometers!**

Exploring the Environment of a Black Hole



A simulated view of the gas stars orbiting the black hole at the center of the Milky Way are shown in blue. (Courtesy of ESO)

Although the physical environment around a black hole can be complicated with gas flowing in, energy being released, and even stars being shreaded apart, mathematically they are very simple.

We have seen that the radius of a black hole is a simple function of the black hole's mass.

For masses, M, given in terms of the mass of Earth, the radius of a black hole is given by $R_h = 8.4M$ in centimeters. For black holes of stellar mass, M, we have $R_h = 2.8M$ in kilometers.

The other important regions surrounding a black hole scale with the horizon radius, R_h, of the black hole.

Problem 1 – What is the event horizon radius, R_h, for a black hole with a mass of 5.0 times our sun's mass?

Problem 2 – If a ray of light passes a black hole, it can be captured into orbit if it gets closer than $R = 1.5\text{x}R_h$. What is the 'Photon Capture' radius for a black hole with a mass of 5 solar masses?

Problem 3 – If a particle of matter gets closer than $R = 3.0\ R_h$ from a black hole, it will not be able to remain in a stable orbit no matter how it moves. It will eventually fall into the black hole. What is the Radius of the Last Stable Particle Orbit, for a black hole with a mass of 5.0 solar masses?

Problem 4 – A black hole has a mass of 10 times the mass of our Earth, with a horizon radius of 84 millimeters. Draw an exact model of this black hole, and shade-in the main regions surrounding the black hole.

Problem 5 – An astronomer detects an asteroid orbiting a black hole with a mass of 8 times our sun, at a distance of 150 kilometers. Is it in a stable orbit, or will it be dragged into the black hole?

Problem 1 – What is the event horizon radius, Rh, for a black hole with a mass of 5.0 times our sun's mass?

Answer: R = 2.8 x 5 = **14 kilometers**.

Problem 2 – If a ray of light passes a black hole, it can be captured into orbit if it gets closer than R = 1.5xRh. What is the 'Photon Capture' radius for a black hole with a mass of 5 solar masses?

Answer: R = 1.5 x 14 km = **21 kilometers**.

Problem 3 – If a particle of matter gets closer than R = 3.0 Rh from a black hole, it will not be able to remain in a stable orbit no matter how it moves. It will eventually fall into the black hole. What is the Radius of the Last Stable Particle Orbit, for a black hole with a mass of 5.0 solar masses?

Answer: R = 3.0 x 14 km = **42 km.**

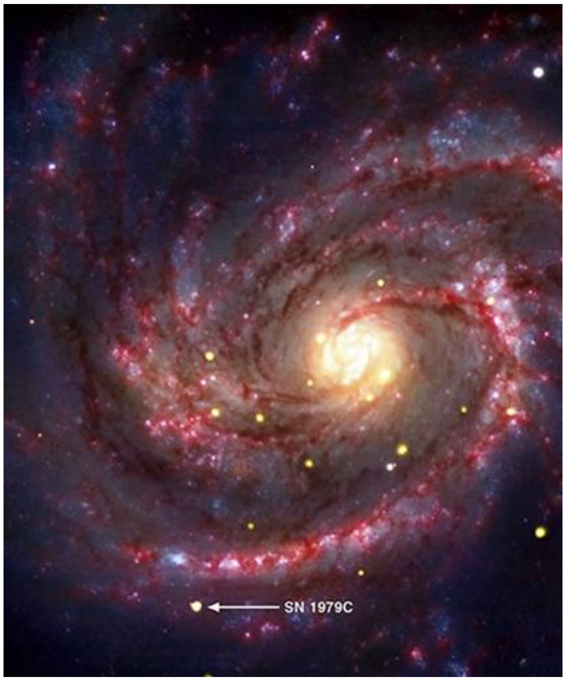
Problem 4 – A black hole has a mass of 3.6 times the mass of our Earth, with a horizon radius of 30 millimeters. Draw an exact model of this black hole, and shade-in the main regions surrounding the black hole.

Answer: R_h = 3.0 cm.
Region 1 = 0 to 3.0 cm = Inside the black hole.
Region 2: R_h to 1.5xR_h; 3.0 cm to 4.5 cm = region of last stable photon orbits.
Region 3: 1.5 R_h to 3.0 R_h; 4.5 cm to 9.0 cm = region of last stable particle orbits.
Region 4: 3.0R_h to +infinity: 9.0cm to +infinity = External region.

Problem 5 – An astronomer detects an asteroid orbiting a black hole with a mass of 8 times our sun, at a distance of 150 kilometers. Is it in a stable orbit, or will it be dragged into the black hole?

Answer: If M = 8.0 solar masses, R_h = 2.8 x 8 = 22.4 kilometers. As a particle of matter, we would check which side of the last stable orbit radius it was on. R = 3.0xR_h, so R = 67.2 kilometers. Our asteroid is outside this boundary at 150 km, **so it can be in a stable orbit.**

The SN1979C Black Hole



The Chandra X-Ray Observatory recently confirmed the discovery of an infant black hole in the nearby galaxy Messiar-100. The product of the supernova of a star with a mass of 20 times our sun, the resulting black hole may only involve about 8 times our sun's mass.

For black holes that do not rotate, called Schwarschild Black Holes, there are several different sizes for such black holes that all scale with the mass of the black hole. When referring to the size of a black hole, astronomers usually mention its mass, which is well defined, rather than its diameter, which depends on the specific kinds of physical processes involved.

The Schwarschild Radius – This is the distance from the center of the black hole at which an incoming person, or light signal, can enter the black hole interior, but cannot emerge back out into the universe. It is also called the Event Horizon. It is a perfectly spherical surface with a radius of Rs = 3.0 M kilometers, where M is the mass of the black hole in multiples of the sun's mass (1 M = 2.0 x 10^{30} kg). For the SN1979C black hole, with an estimated mass of 8 M, what is its Schwarschild Radius, Rs, in kilometers?

Last Photon Orbit – At this distance outside the Event Horizon, an incoming photon of light can enter into an exactly circular orbit, where it will stay until it is disturbed, at which time it will fall into the Event Horizon and never get back out. If Rp =1.5 Rs, what is the radius of the photon orbit for black hole SN1979C in kilometers?

Last Stable Particle Orbit – Inside this distance, a material particle cannot be in a stable circular orbit, but is relentlessly dragged to the Event Horizon and disappears. This occurs at a distance from the black hole center of Rl = 3.0 Rs. How close can a hydrogen atom, an asteroid or a planet remain in a stable circular orbit around the SN1979C black hole?

Problem – An asteroid is spotted at a distance of 700 km from a black hole with a mass of 120 solar masses. Can it escape or remain where it is?

NASA Press release 'Youngest Nearby Black Hole' November 15, 2010

"Data from Chandra, as well as NASA's Swift, the European Space Agency's XMM-Newton and the German ROSAT observatory revealed a bright source of X-rays that has remained steady for the 12 years from 1995 to 2007 over which it has been observed. This behavior and the X-ray spectrum, or distribution of X-rays with energy, support the idea that the object in SN 1979C is a black hole being fed either by material falling back into the black hole after the supernova, or from a binary companion.

The scientists think that SN 1979C formed when a star about 20 times more massive than the Sun collapsed. It was a particular type of supernova where the exploded star had ejected some, but not all of its outer, hydrogen-rich envelope before the explosion, so it is unlikely to have been associated with a gamma-ray burst (GRB). Supernovas have sometimes been associated with GRBs, but only where the exploded star had completely lost its hydrogen envelope. Since most black holes should form when the core of a star collapses and a gamma-ray burst is not produced, this may be the first time that the common way of making a black hole has been observed.

The very young age of about 30 years for the black hole is the observed value, that is the age of the remnant as it appears in the image. Astronomers quote ages in this way because of the observational nature of their field, where their knowledge of the Universe is based almost entirely on the electromagnetic radiation received by telescopes."
(**http://www.nasa.gov/mission_pages/chandra/multimedia/photoH-10-299.html)**

The Schwarschild Radius – This is the distance from the center of the black hole at which an incoming person, or light signal, can enter the black hole interior, but cannot emerge back out into the universe. It is also called the Event Horizon. It is a perfectly spherical surface with a radius of Rs = 3.0 M kilometers, where M is the mass of the black hole in multiples of the sun's mass (1 M = 2.0 x 10^{30} kg). For the SN1979C black hole, with an estimated mass of 8 M, what is its Schwarschild Radius, Rs, in kilometers?

Answer: M = 8, so Rs = 3.0 x 8 = **24 kilometers**.

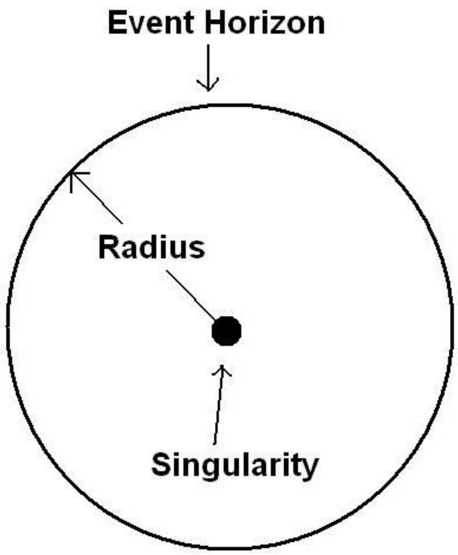
Last Photon Orbit – At this distance outside the Event Horizon, an incoming photon of light can enter into an exactly circular orbit, where it will stay until it is disturbed, at which time it will fall into the Event Horizon and never get back out. If Rp =1.5 Rs, what is the radius of the photon orbit for black hole SN1979C in kilometers?

Answer: Rs = 24 kilometers so Rp = 1.5 x 24 km = **36 kilometers**.

Last Stable Particle Orbit – Inside this distance, a material particle cannot be in a stable circular orbit, but is relentlessly dragged to the Event Horizon and disappears. This occurs at a distance from the black hole center of Rl = 3.0 Rs. How close can a hydrogen atom, an asteroid or a planet remain in a stable circular orbit around the SN1979C black hole?

Answer: R = 3.0 x 24 km = **72 kilometers**.

Problem – An asteroid is spotted at a distance of 700 km from a black hole with a mass of 120 solar masses. Can it escape or remain where it is? Answer: Rs = 3.0 x 120 = 360 kilometers. Rp = 1.5 x 360 km = 540 km; Rl = 3.0 x 360 km = 1080 km. **Since the asteroid is at 700 km, it is inside the distance where it can remain in a stable orbit, so it is about to fall through the black hole's event horizon located some 700-360 = 340 kilometers inside its current position.**



Black holes are objects that have such intense gravitational fields, they do not allow light to escape from them. They also make it impossible for anything that falls into them to escape, because to do so, they would have to travel at speeds faster than light. No forms of matter or energy can travel faster than the speed of light, so that is why black holes are so unusual!

There are three parts to a simple black hole:

Event Horizon - Also called the Schwarzschild radius, that's the part that we see from the outside. It looks like a black, spherical surface with a very sharp edge in space.

Interior Space - This is a complicated region where space and time can get horribly mangled, compressed, stretched, and otherwise a very bad place to travel through.

Singularity - That's the place that matter goes when it falls through the event horizon. It's located at the center of the black hole, and it has an enormous density. You will be crushed into quarks long before you get there!

Black holes can, in theory, come in any imaginable size. The size of a black hole depends on the amount of mass it contains. It's a very simple formula, especially if the black hole is not rotating. These 'non-rotating' black holes are called Schwarzschild Black Holes.

Equation 1) $$R = \frac{2GM}{c^2}$$

Equation 2) $$R = 1.48x10^{-27} M$$

Problem 1 - The two formulas above give the Schwarzschild radius, R, of a black hole in terms of its mass, M. From Equation 1, verify Equation 2, which gives R in meters and M in kilograms, using c= 3 x 10^8 m/sec for the speed of light, and G = $6.67x10^{-11}$ Newtons m^2/kg^2 for the gravitational constant.

Problem 2 - Calculate the Schwarzschild radius, in meters, for Earth where

M = 5.7 x 10^{24} kilograms.

Problem 3 - Calculate the Schwarzschild radius, in kilometers, for the sun, where

M = 1.9 x 10^{30} kilograms.

Problem 4 - Calculate the Schwarzschild radius, in kilometers, for the entire Milky Way, with a mass of 250 billion suns.

Problem 5 - Calculate the Schwarzschild radius, in meters, for a black hole with the mass of an average human being with M = 60 kilograms.

Answer Key

Problem 1 - The two formulas above give the Schwarzschild radius, R, of a black hole in terms of its mass, M. From Equation 1, verify Equation 2, which gives R in meters and M in kilograms, using c= 3 x 10^8 m/sec for the speed of light, and G = 6.67x10^{-11} Newtons m^2/kg^2 for the gravitational constant.

Answer: Radius = 2 x (6.67 x 10^{-11}) / (3 x 10^8)^2 M meters

= **1.48 x 10^{-27} M meters**

where M is the mass of the black hole in kilograms.

Problem 2 - Calculate the Schwarzschild radius, in meters, for Earth where M = 5.7 x 10^{24} kilograms.

Answer: R = 1.48 x 10^{-27} (5.7 x 10^{24}) meters

R = 0.0084 meters !

Problem 3 - Calculate the Schwarzschild radius, in kilometers, for the sun, where M = 1.9 x 10^{30} kilograms.

Answer: R = 1.48 x 10^{-27} (1.9 x 10^{30}) meters

R **= 2.8 kilometers**

Problem 4 - Calculate the Schwarzschild radius, in kilometers, for the entire Milky Way, with a mass of 250 billion suns.

Answer: If a black hole with the mass of the sun has a radius of 2.8 kilometers, a black hole with 250 billion times the sun's mass will be 250 billion times larger, or

R = (2.8 km / sun) x 250 billion suns = **700 billion kilometers.**

Note: The entire solar system has a radius of about 4.5 billion kilometers!

Problem 5 - Calculate the Schwarzschild radius, in meters, for a black hole with a mass of an average human being with M = 60 kilograms.

Answer: R = 1.48 x 10^{-27} (60) meters

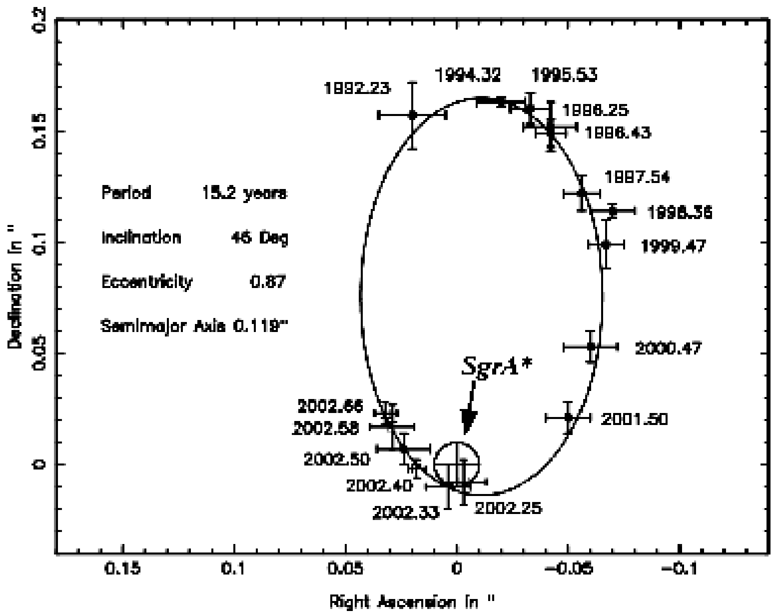
R = 8.9 x 10^{-26} meters.

Note: A proton is only about 10^{-16} meters in diameter.

The Milky Way Black Hole

At the center of our Milky Way Galaxy lies a black hole, called Sagittarius A*, with over 2.6 million times the mass of the Sun. Once a controversial claim, this astounding conclusion is now virtually inescapable and based on observations of stars orbiting very near the galactic center.

The Chandra image to the left shows the x-ray light from a region of space a few light years across. The black hole is invisible, but it is near the center of this image. The gas near the center produces x-ray light as it is heated. Many of the 'stars' in the field probably have much smaller black holes near them that are producing the x-ray light from the gas they are consuming.



Astronomers patiently followed the orbit of a particular star, designated S2. Their results convincingly show that S2 is moving under the influence of the enormous gravity of an unseen object, which must be extremely compact and contain huge amounts of matter yet emits no light -- a supermassive black hole. The drawing above shows the orbit shape.

Problem 1 - Kepler's Third Law can be used to determine the mass of a body by measuring the orbital period, T, and orbit radius, R, of a satellite. If R is given in units of the Astronomical Unit (AU) and T is in years, the relationship becomes $R^3 / T^2 = M$, where M is the mass of the body in multiples of the sun's mass. In these units, for Earth, R = 1.0 AU, and T = 1 year, so M = 1.0 solar masses. In 2006, the Hubble Space Telescope, found that the star Polaris has a companion, Polaris Ab, whose distance from Polaris is 18.5 AU and has a period of 30 years. What is the mass of Polaris?

Problem 2 - The star S2 orbits the supermassive black hole Sagittarius A*. Its period is 15.2 years, and its orbit distance is about 840 AU. What is the estimated mass of the black hole at the center of the Milky Way?

Answer Key:

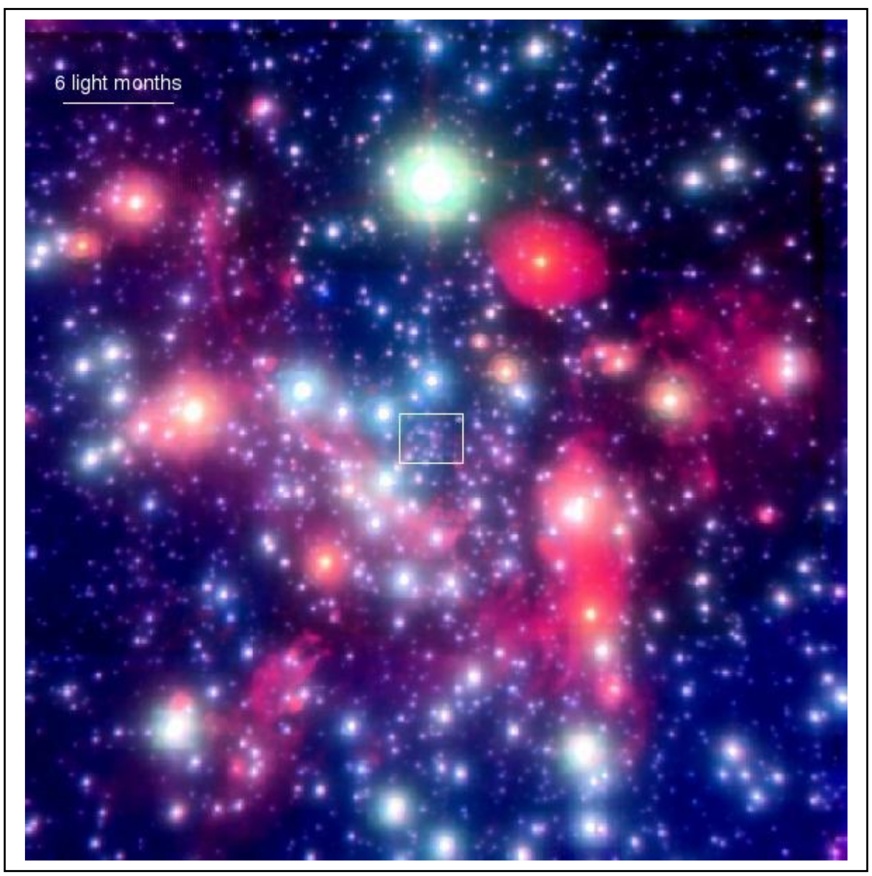
Problem 1 - Kepler's Third Law can be used to determine the mass of a body by measuring the orbital period, T, and orbit radius, R, of a satellite. If R is given in units of the Astronomical Unit (AU) and T is in years, the relationship becomes $R^3 / T^2 = M$, where M is the mass of the body in multiples of the sun's mass. In these units, for Earth, R = 1.0 AU, and T = 1 year, so M = 1.0 solar masses. In 2006, the Hubble Space Telescope, found that the star Polaris has a companion, Polaris Ab, whose distance from Polaris is 18.5 AU and has a period of 30 years. What is the mass of Polaris?

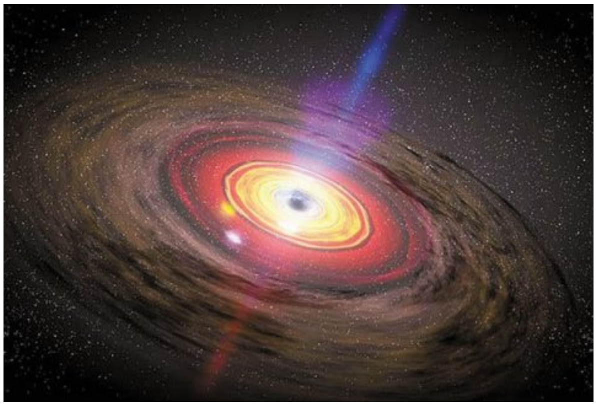
Answer: $M = (18.5)^3 / (30)^2 =$ **7.0 solar masses.**

Problem 2 - The star S2 orbits the supermassive black hole Sagittarius A*. Its period is 15.2 years, and its orbit distance is about 840 AU. What is the estimated mass of the black hole at the center of the Milky Way?

Answer: $M = (840)^3 / (15.2)^2 =$ **2.6×10^6 solar masses.**

The infrared image below shows the central few light years of the Milky Way. The box contains the location of the supermassive black hole and Sagittarius A*. (Courtesy ESA - NAOS)

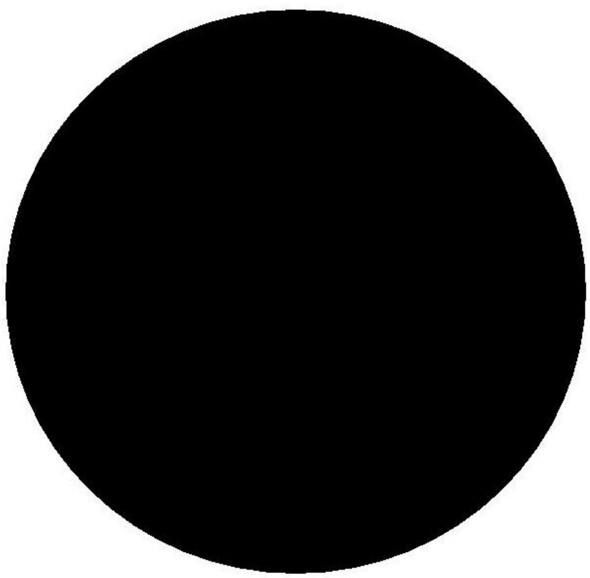




Artist's impression of gas falling into a black hole Image credit: NASA / Dana Berry, SkyWorks Digital

When gas flows into a black hole, it gets very hot and emits light. The gas is heated because the atoms collide with each other as they fall into the black hole. Far away from the black hole, the atoms do not travel very fast so the gas is cool. But close to the black hole, the atoms can be moving at millions of kilometers/hour and the gas can be thousands of degrees hot!

The circle below represents the spherical shape of a black hole with a mass of about 5 times our Earth. Its diameter is 9 centimeters.



The formula that gives the gas temperature, T in Kelvins, at a distance of R in meters from the center of the black hole, is given by:

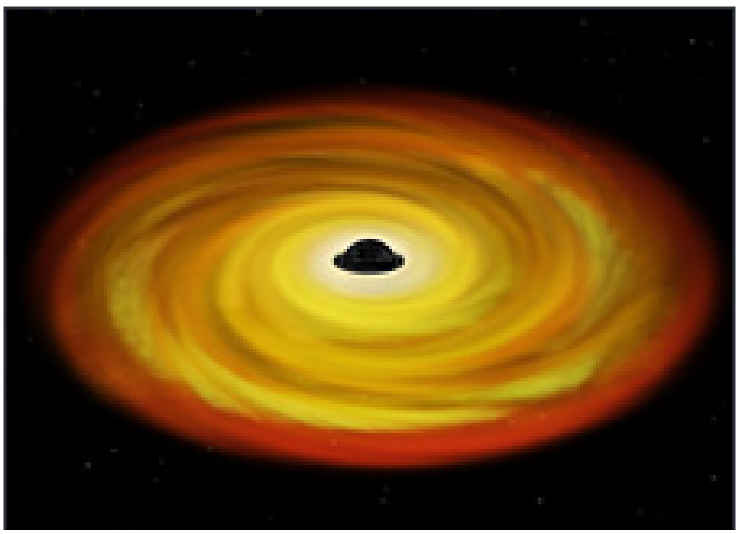
$$T = \frac{35,000}{R^{\frac{3}{4}}}$$

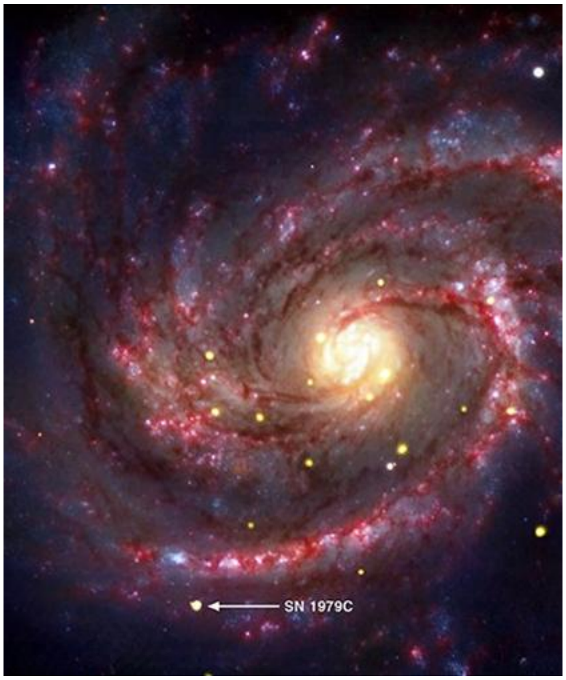
Problem 1 - Sketch a life-sized illustration of the gas surrounding the above black hole and give the temperature at a distance of 1 meter, 50 centimeters and 5 centimeters from the center of the black hole.

Problem 1 - Sketch a life-sized illustration of the gas surrounding the above black hole and give the temperature at a distance of 1 meter, 50 centimeters and 5 centimeters from the center of the black hole.

Answer: At 1 meter, **T = 35,000 K,** which is 6 times the surface temperature of our sun.
At 50 cm or 0.5 meters, **T = 59,000 K**.
At 5 centimeters or 0.05 meters, **T = 331,000 K**.

Students may color many different versions, but they should all show that the most distant gas is cooler (35,000 K) than the gas near the black hole (331,000 K) which could be temperature coded using some plausible scheme like the 'yellow to white' scheme below.





The Chandra X-Ray Observatory recently found evidence for an infant black hole in the nearby galaxy Messiar-100. The black hole is thought to have been produced when a star with a mass of about 20 times that of the sun exploded and left behind a black hole with a mass about 8 times the sun's mass.

The satellite observatory has detected x-rays from the gasses in the orbiting accretion disk that are falling into this young black hole. Infalling gas can be heated to over 100,000,000 K as atoms collide at higher and higher speed during the infall process. The temperature of this x-ray emitting gas is related to its distance from the black hole.

At a distance of R kilometers from a black hole with a mass of M times the sun, suppose that the two equations below relate the temperature of the gas, T, and the wavelength, L, at which the in-flowing gas emits most of its light:

Equation 1 - $T = 100,000,000\left(\frac{M}{R^3}\right)^{1/4}$ Kelvin

Equation 2 - $L = \frac{3,600,000}{T}$ nanometers

where M is in solar mass units, and R is in kilometers.

Problem 1 - Combining these equations using the method of substitution, what is the new formula L(R,M), for the wavelength, L, emitted by the gas as a function of its distance from the black hole center, R, and the mass of the black hole, M?

Problem 2 – X-rays are detected from the vicinity of the SN 1979C black hole at a wavelength of 0.53 nanometers (2,300 electronVolts). If the mass of the black hole is 8 times the sun, at what distance from the center of the black hole is the gas being detected?

Problem 3 – The Event Horizon of a black hole that is not rotating (called a Schwarschild black hole) is located at a distance of Rs = 3.0 M from the center of the black hole, where M is the mass of the black hole in units of our sun, and Rs is in units of kilometers. What is Rs for the SN 1979C black hole, and where is the x-ray emitting gas in relation to the Event Horizon?

NASA Press release 'Youngest Nearby Black Hole' November 15, 2010

"Data from Chandra, as well as NASA's Swift, the European Space Agency's XMM-Newton and the German ROSAT observatory revealed a bright source of X-rays that has remained steady for the 12 years from 1995 to 2007 over which it has been observed. This behavior and the X-ray spectrum, or distribution of X-rays with energy, support the idea that the object in SN 1979C is a black hole being fed either by material falling back into the black hole after the supernova, or from a binary companion.

The scientists think that SN 1979C formed when a star about 20 times more massive than the Sun collapsed. It was a particular type of supernova where the exploded star had ejected some, but not all of its outer, hydrogen-rich envelope before the explosion, so it is unlikely to have been associated with a gamma-ray burst (GRB). Supernovas have sometimes been associated with GRBs, but only where the exploded star had completely lost its hydrogen envelope. Since most black holes should form when the core of a star collapses and a gamma-ray burst is not produced, this may be the first time that the common way of making a black hole has been observed.

The very young age of about 30 years for the black hole is the observed value, that is the age of the remnant as it appears in the image. Astronomers quote ages in this way because of the observational nature of their field, where their knowledge of the Universe is based almost entirely on the electromagnetic radiation received by telescopes."
(**http://www.nasa.gov/mission_pages/chandra/multimedia/photoH-10-299.html)**

Problem 1 - Answer: Substitute Equation 1 into Equation 2 to eliminate T,

$$L(R,M) = \frac{3,600,000}{100,000,000}\left(\frac{R^3}{M}\right)^{1/4}$$

so $L(R,M) = 0.036\left(\frac{R^3}{M}\right)^{1/4}$ nanometers.

Problem 2 – Answer: $0.53 = 0.036\ (8^{-1/4})\ R^{3/4}$ so solve for R to get

$R = (24.8)^{4/3}$

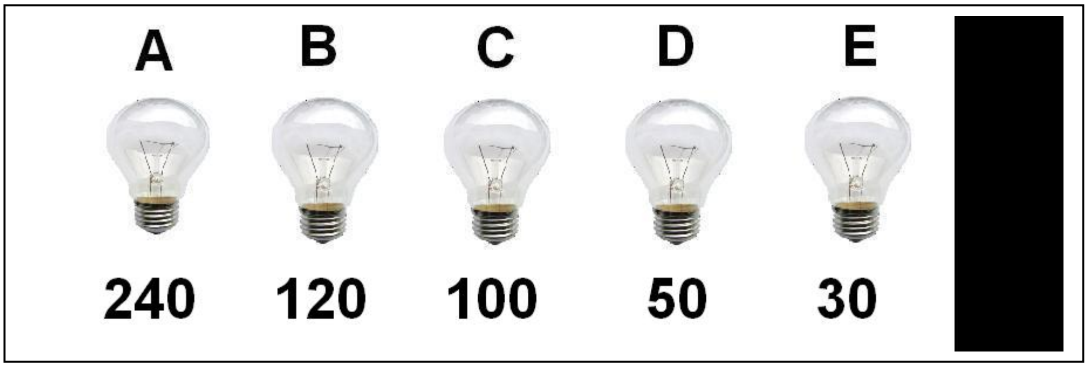
and so R = **72 km**.

Problem 3 – Answer: The Event Horizon is at Rs = 3.0 x 8 = 24 kilometers. **The x-ray emitting gas is located at R = 72 km, just outside the Event Horizon at a distance of about 48 kilometers.**

A Black Hole - Up Close!

The sketch below shows the edge of a black hole on the right hand-side. The distance in centimeters from the edge of the black hole, called the **event horizon**, increases from right to left to a maximum distance of 240 centimeters from the event horizon (Bulb A). In this figure, the radius of the black hole is about 1 meter. This corresponds to a black hole with a mass equal to 120 times the mass of our Earth.

Although all of the light bulbs would be destroyed this close to an actual event horizon, we will pretend, for simplicity, that they can survive.



Someone far away from a black hole will see things very differently than someone close to a black hole. Because of the intense gravitational forces, ordinary light emitted close to a black hole will have its wavelength stretched as viewed by someone far away. The closer the light source, the more wavelength -stretching will be seen by the distant observer.

Suppose all of these light bulbs in the above figure are emitting light at a wavelength of 450 nanometers (nm), and are shining brightly with a color near yellow like our sun. A distant observer will see this light stretched to longer wavelengths. The wavelength they will observe, W, in nanometers depends on the distance of the light source, R to the center of the black hole in centimeters according to the formula:

$$W = \frac{450}{\sqrt{1 - \frac{100}{R}}}$$

For example, the event horizon for this black hole is at R=100 centimeters. If the light bulb is 50 centimeters to the left of the horizon, R = 150 centimeters, and so W = 780 nanometers. The middle of the Visible Band is at about 500 nm, so instead of yellow light, you would see this light bulb emitting very deep red color!

Problem 1 - Suppose the bulbs were located at the distances from the event horizon shown in the figure above. What would be the wavelengths you would observe for Bulbs B, C, D and E?

Problem 2 - The human eye can only detect light at a wavelength shorter than about 650 nm. Which of the light bulbs would appear to be invisible to you and 'black'?

Problem 3 - How close to the event horizon would the light bulb have to be in order for you to only detect it as an invisible heat source emitting at a wavelength of just 14,000 nm?

Problem 1 - Suppose the bulbs were located at the distances indicated above. What would be the wavelengths you would observe for Bulbs B, C, D and E?

Bulb A	240 cm	R = 340 cm	**W = 535 nm**
Bulb B	120 cm	R = 220 cm	**W = 609 nm**
Bulb C	100 cm	R = 200 cm	**W = 636 nm**
Bulb D	50 cm	R = 180 cm	**W = 780 nm**
Bulb E	30 cm	R = 130 cm	**W = 937 nm**

Note: In the visible spectrum
Bulb A = yellow
Bulb B = orange
Bulb C = red
Bulb D = deep crimson or dull red and nearly invisible
Bulb E = infrared and invisible to the eye.

Problem 2 - The human eye can only detect light at a wavelength shorter than about 650 nm. Which of the light bulbs would appear to be invisible to you and 'black'?

Answer: From your location far from the black hole, you see that the Bulbs D and E are not visible to your eyes. **Note: With the proper light detectors you could still see them shining at these longer wavelengths!**

Problem 3 - How close to the event horizon would the light bulb have to be in order for you to only detect it as an invisible heat source emitting at a wavelength of just 14,000 nm?

Answer: We need to solve for R the equation:

$$14,000 = \frac{450}{\sqrt{1-\frac{100}{R}}}$$ from this we get $$1-\frac{100}{R} = \left(\frac{450}{14,000}\right)^2$$

$$1-0.001 = \frac{100}{R}$$ so R = 100.1 centimeters.

This means that the bulb is located **0.1 centimeters or 1 millimeter** just outside the event horizon!

Note: A black hole with a radius of 100 centimeters would have a mass of about 120 times that of Earth, or a little bit more than the planet Saturn.

$$T = t\sqrt{1 - \frac{2GM}{Rc^2}}$$

T = the time measured by someone located on a planet (seconds)

t = the time measured by someone located in space (seconds)

M = the mass of the planet (kg)

R = the distance to the far-away observer from the planet (m)

And the natural constants are:
$G = 6.67 \times 10^{-11}$ Nt m^2/kg^2
$C = 3 \times 10^{8}$ m/sec

The modern theory of gravity, called the Theory of General Relativity, developed by Albert Einstein in 1915 leads to some very unusual predictions, which have all been verified by experiments.

One of the strangest ones is that two people will experience the passage of time very differently if one is standing on the surface of a planet, and the other one is in space. This is because the rate of time passing depends on the strength of the gravitational field that the observer is in.

For example, at the surface of a very dense neutron star, R = 20 km and $M = 1.9 \times 10^{30}$ kg, so

$$T = t\ (1- 0.15)^{1/2} = 0.92\ t$$

This means that for every hour that goes by on the surface of the neutron star (T = 60 minutes), someone in space will see t = 60 / 0.92 = 65 minutes pass from a vantage point in space.

The following problems require accuracy to the 11th decimal place. Most hand calculators only provide 9 digits. Students may use the 'calculator' accessory provided on all PCs and Macs.

Problem 1 - The GPS satellites orbit Earth at a distance of R = 26,560 km. If the mass of Earth is 5.9×10^{24} kg, use the formula to determine the time dilation factor.

Problem 2 - What is the time dilation factor at Earth's surface?

Problem 3 - What is the ratio of the dilation in space to the dilation at earth's surface?

Problem 4 - At the speed of light (3×10^{8} m/sec) how long does it take a radio signal from the GPS satellite to travel 26,560 km to a hand-held GPS receiver?

Problem 5 - The excess time delay between a receiver at Earth's surface, and the GPS satellite is defined by the ratio computed in Problem 3, multiplied by the total travel time in Problem 4. What is the time delay for the GPS-Earth system?

Problem 6 - From your answer to Problem 5, how much extra time does the radio signal take compared to your answer to Problem 4?

Problem 7 - At the speed of light, how far will the radio signal travel during the extra amount of time?

Answer Key:

Problem 1 - The GPS satellites orbit Earth at a distance of R = 26,560 km. If the mass of Earth is 5.9 x 10^{24} kg, use the formula to determine the time dilation factor. Be very careful with the small numbers in the 9th, 10th and 11th decimal places!

Answer: $(1 - 0.0084/2.65\times10^{7}\ \mathrm{m})^{1/2}$ =
$(1 - 3.1 \times 10^{-10})^{1/2}$ =
$(0.99999999969)^{1/2}$ = **0.99999999984**

Problem 2 - What is the time dilation factor at Earth's surface?
$(1 - 0.0084/6.38\times10^{6}\ \mathrm{m})^{1/2}$ =
$(1 - 1.3 \times 10^{-9})^{1/2}$ =
$(0.9999999987)^{1/2}$ = **0.99999999934**

Problem 3 - What is the ratio of the dilation in space to the dilation at Earth's surface?
Answer - 0.99999999984 / 0.99999999934 = **1.00000000050**

Problem 4 - How long does it take a radio signal from the GPS satellite to travel 26,560 km to a hand-held GPS receiver?

Answer - Distance = 26,560 km x (1000 m / km) = 2.65 x 10^{7} meters.
Time = Distance / speed of light
= 2.65 x 10^{7} m / 3 x 10^{8} m/sec = **0.088 seconds.**

Problem 5 - The excess time delay between a receiver at Earth's surface, and the GPS satellite is defined by the ratio computed in Problem 3, multiplied by the total travel time in Problem 4. What is the time delay for the GPS-Earth system?

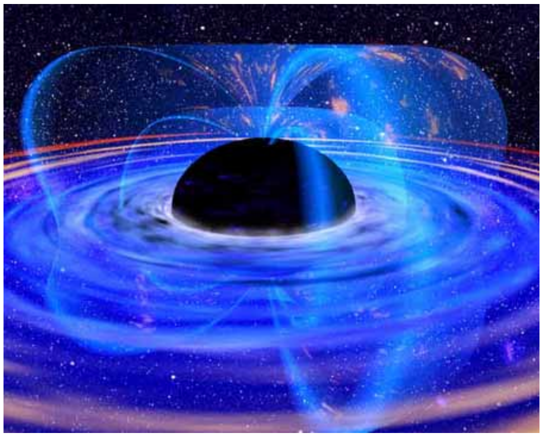
Answer - 0.088 seconds * 1.00000000050 **= 0.088000000044 seconds.**

Problem 6 - From your answer to Problem 5, how much extra time does the radio signal take compared to your answer to Problem 4?
Answer - 0.088000000044 - 0.088 seconds = **0.000000000044 seconds.**

Problem 7 - At the speed of light, how far will the radio signal travel during the extra amount of time?
Answer = 3 x 10^{8} m/sec x 4.4 x 10^{-11} sec = **0.17 meters.**

This shows that Einstein's Theory of General Relativity is required to allow the GPS satellite system to make precise measurements of the locations of objects on Erarth's surface.

Time Dilation Near a Black Hole



Artists illustration of a black hole with an orbiting disk of gas and dust. Friction in the disk causes matter to steadily flow inwards until it reaches the black hole event horizon. Magnetic forces in the disk cause matter to flow in complex jets and plumes. Time dilation causes delays in events taking place near the black hole compared to what distant observers will record.

Time dilation near a black hole is a lot more extreme than what the GPS satellite network experiences in orbit around Earth.

$$T = t\sqrt{1 - \frac{2GM}{Rc^2}}$$

T = time measured by someone located on a planet (sec)

t = time measured by someone located in space (sec)

M = mass of the planet (kg)

R = distance to the far-away observer from the planet (m)

Problem 1 - In the time dilation formula above, evaluate the quantity 2 G M /c^2 for a black hole with a mass of one solar mass (1.9 x 10^{30} kg), and convert the answer to kilometers to two significant figures.

Problem 2 - Re-write the formula in a more tidy form using your answer to Problem 1.

Problem 3 - In the far future, a scientific outpost has been placed in orbit around this solar-mass black hole at a distance of 10 km. What will the time dilation factor be at this location?

Problem 4 - A series of clock ticks were sent out by the satellite once each hour. What will be the time interval in seconds between the clock ticks by the time they reach a distant observer?

Problem 5 - If one tick arrived at 1:00 PM at the distant observer, when will the next clock tick arrive?

Problem 6 - A radio signal was sent by the black hole outpost to a distant observer. At the frequency of the signal, when transmitted from the outpost, the individual wavelengths take 0.000001 seconds to complete one cycle. From your answer to Problem 3, how much longer will they take by the time they arrive at the distant observer?

Answer Key:

Problem 1 - In the time dilation formula above, evaluate the quantity 2 G M /c^2 for a black hole with a mass of one solar mass (1.9 x 10^{30} kg), and convert the answer to kilometers to two significant figures.

Answer: 2 x 6.67 x 10^{-11} x 1.9 x 10^{30} / (3.00 x 10^{8})^2 = 2,816 meters or **2.8 km.**

Problem 2 - Re-write the formula in a more tidy form using your answer to Problem 1.

Answer :

$$T = t\sqrt{1 - \frac{2.8}{R}}$$ where R will now be in units of kilometers.

Problem 3 - In the far future, a scientific outpost has been placed in orbit around this solar-mass black hole at a distance of 10 km. What will the time dilation factor be at this location?

Answer: (1 - 2.8/10)$^{1/2}$ = (0.72)$^{1/2}$ = **0.85**

Problem 4 - A series of clock ticks were sent out by the satellite once each hour .What will be the time interval between the clock ticks by the time they reach a distant observer?

Answer: Time interval = 3600 / 0.85 = **4,200 seconds.**

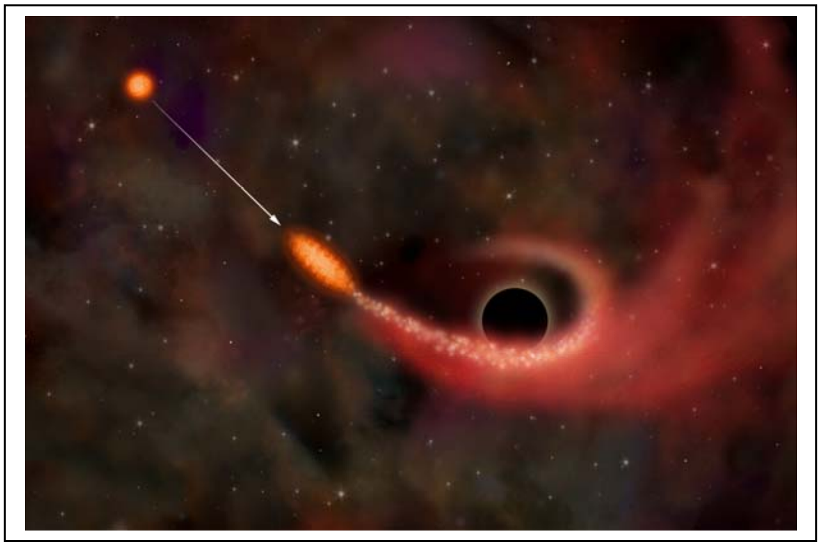
Note: The raw answer would be 4235 seconds, but to 2 significant figures it is 4,200

Problem 5 - If one tick arrived at 1:00 PM at the distant observer, when will the next clock tick arrive?

Answer: 1:00 PM + 4200 seconds = 1:00 PM + 1 Hour + (4200-3600) = 2:00 PM + 600 seconds = **2:10:00 PM**

Problem 6 - A radio signal was sent by the black hole outpost to a distant observer. At the frequency of the signal, when transmitted from the outpost, the individual wavelengths take 0.000001 seconds to complete one cycle. From your answer to Problem 3, how much longer will they take by the time they arrive at the distant observer?

Answer: 0.000001 seconds / 0.85 = **0.0000012 seconds.**



Thanks to two orbiting X-ray observatories, astronomers have the first strong evidence of a supermassive black hole ripping apart a star and consuming a portion of it. The event, captured by NASA's Chandra and ESA's XMM-Newton X-ray Observatories, had long been predicted by theory, but never confirmed until now. Giant black holes in just the right mass range would pull on the front of a closely passing star much more strongly than on the back. Such a strong tidal force would stretch out a star and likely cause some of the star's gasses to fall into the black hole. The infalling gas has been predicted to emit just the same blast of X-rays that have recently been seen in the center of galaxy RX J1242-11 located 700 million light years from the Milky Way, in the constellation Virgo. (NASA news report at http://chandra.harvard.edu/photo/2004/rxj1242/

Problem 1 - The size of the event horizon of a black hole (called the Schwarschild radius) is given by the formula R = 2.8 M, where R is the radius in km, and M is that mass of the black hole in units of the sun's mass. A supermassive black hole can have a mass of 100 million times the sun. What is its Schwarschild radius in: A) kilometers, B) multiples of the Earth orbit radius called an Astronomical Unit (1 AU = 149 million km), C) compared to the orbit of Mars (1.5 AU)

Problem 2 - Black holes are one of the most efficient phenomena in converting matter into energy. As matter falls inward in an orbiting disk of gas, friction heats the gas up, and the energy released can be as much as 7% of the rest mass energy of the infalling matter. The quasar 3C273 has a power output of 3.8×10^{38} Joules/second. If $E = mc^2$ is the formula that converts mass (in kg) into energy (in Joules) and c = the speed of light, 3×10^8 m/sec, how many grams per year does this quasar luminosity imply if 1 year = 3.1×10^7 seconds?

Problem 3 - If the mass of the Sun is 1.9×10^{30} kg, how many suns per year have to be consumed by the 3C273 supermassive black hole at the black hole conversion efficiency of 7%? (Note: 7% efficiency means that for every 100 kg involved, 7 kg are converted into pure energy by $E=mc^2$)

Answer Key:

Problem 1 - The size of the event horizon of a black hole (called the Schwarschild radius) is given by the formula R = 2.8 M, where R is the radius in kilometers, and M is that mass of the black hole in units of the sun's mass. A supermassive black hole can have a mass of 100 million times the sun. What is its Schwarschild radius in: A) kilometers, B) multiples of the Earth orbit radius called an Astronomical Unit (1 AU = 149 million km), C) compared to the orbit of Mars (1.5 AU)

Answer: A) R = **280 million km** B) 280 million / 149 million = **1.9 AU**. C) 1.9/1.5 = **1.3 times the orbit of Mars**. The event horizon would be just beyond the orbit of Mars!

Problem 2 - Black holes are one of the most efficient phenomena in converting matter into energy. As matter falls inward in an orbiting disk of gas, friction heats the gas up, and the energy released can be as much as 7% of the rest mass energy of the infalling matter. The quasar 3C273 has a power output of 3.8×10^{38} Joules/second. If $E = mc^2$ is the formula that converts mass (in kg) into energy (in Joules) and c = the speed of light, 3×10^{8} m/sec, how many kilograms per year does this quasar luminosity imply if 1 year = 3.1×10^{7} seconds?

Answer: 3.8×10^{38} Joules/second x (3.1×10^{7} seconds/year) / $(3 \times 10^{8})^{2}$

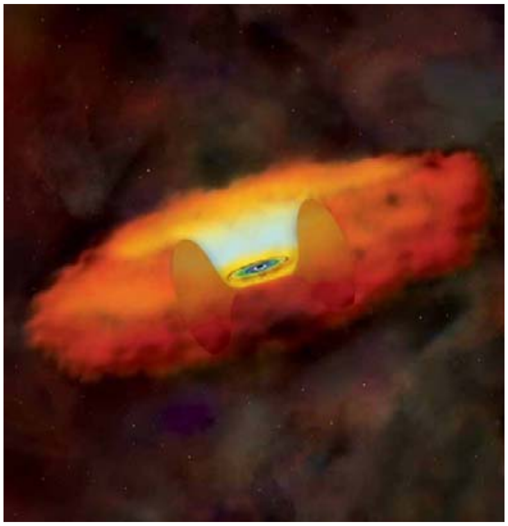
= **1.3×10^{29} kilograms/year**

Problem 3 - If the mass of the Sun is 1.9×10^{30} kg, how many suns per year have to be consumed by the 3C273 supermassive black hole at the black hole conversion efficiency of 7%?

Answer: 7% efficiency means that for every 100 kilograms involved, 7 kilograms are converted into pure energy (by $E=mc^2$). So,

0.07 suns per year / (7/100) = **1.0 suns per year** for 7% efficiency.

Black Hole Power



Black holes are sometimes surrounded by a disk of orbiting matter. This disk is very hot. As matter finally falls into the black hole from the inner edge of that disk, it releases about 7% of its rest-mass energy in the form of light. Some of this energy was already lost as the matter passed through, and heated up, the gases in the surrounding disk. But the over-all energy from the infalling matter is about 7% of its rest-mass in all forms (heat+ light).

The power produced by a black hole is phenomenal, with far more energy per kilogram being created than by ordinary nuclear fusion, which powers the sun.

A black hole accretion disk (M. Weiss NASA/Chandra)

Problem 1 - The event horizon of a black hole has a radius of R = 2.8 M kilometers, where M is the mass of the black hole in multiples of the sun's mass. Assume the event horizon is a spherical surface, so its surface area is $S = 4 \pi R^2$. What is the surface area of A) a stellar black hole with a mass of 10 solar masses? B) a supermassive black hole with a mass of 100 million suns?

Problem 2 - What is the volume of a spherical shell with the surface area of the black holes in Problem 1, with a thickness of one meter?

Problem 3 - If the density of gas near the horizon is 10^{16} atoms/meter^3 of hydrogen, how much matter is in each black hole shell, if the mass of a hydrogen atom is 1.6×10^{-27} kg?

Problem 4 - If $E = m c^2$ is the rest mass energy, E, in Joules, for a particle with a mass of m in kg, what is the rest mass energy equal to the masses in Problem 3 if $c = 3 \times 10^8$ m/sec is the speed of light and only 7% of the mass produced energy?

Problem 5 - Suppose the material was traveling at 1/2 the speed of light as it crossed the event horizon, how much time does it take to travel one meter if $c = 3 \times 10^8$ m/sec is the speed of light?

Problem 6 - The power produced is equal to the energy in Problem 4, divided by the time in Problem 5. What is the percentage of power produced by each black hole compared to the sun's power of 3.8×10^{26} Joules/sec?

Answer Key:

Problem 1 - The event horizon of a black hole has a radius of R = 2.8 M kilometers, where M is the mass of the black hole in multiples of the sun's mass. Assume the event horizon is a spherical surface, so its surface area is $S = 4 \pi R^2$. What is the surface area of: A) a stellar black hole with a mass of 10 solar masses? B) a supermassive black hole with a mass of 100 million suns?

Answer; A) The radius, R, is 2.8 x 10 = 28 km. The surface area $S = 4 \times 3.14 \times (2.8 \times 10^4)^2 =$ **9.8×10^9 m^2,** B) R=2.8×10^{11} m so $S = 4 \times 3.14 \times (2.8 \times 10^{11})^2 =$ **9.8×10^{23} m^2,**

Problem 2 - What is the volume of a spherical shell with the surface area of the black holes in Problem 1, with a thickness of one centimeter?

Answer: Stellar black hole, V = S x 1 meter = **9.8×10^9 m^3**;
Supermassive black hole, **V = 9.8×10^{23} m^3.**

Problem 3 - If the density of gas near the horizon is 10^{16} atoms/meter^3 of hydrogen, how much matter is in each black hole shell, if the mass of a hydrogen atom is 1.6×10^{-27} kilograms?

Answer - Stellar: M = (9.8×10^9 m^3) x (1.0×10^{16} atoms/m^3) x (1.6×10^{-27} kg/atom = **0.16 kg.**
Supermassive: M = (9.8×10^{23} cm^3) x (1.0×10^{16} atoms/cm^3) x (1.6×10^{-27} grams/atom = **1.6×10^{13} kilograms.**

Problem 4 - If $E = m c^2$ is the rest mass energy, E, in Joules, for a particle with a mass of m in kg, what is the rest mass energy equal to the masses in Problem 3 if $c = 3 \times 10^8$ m/sec is the speed of light and only 7% of the mass produced energy?

Answer: Stellar: $E = 0.07 \times (0.16) \times (3 \times 10^8)^2$ = **1.0×10^{15} Joules**
Supermassive $E = 0.07 \times (1.6 \times 10^{13}) \times (3 \times 10^8)^2$ = **1.0×10^{29} Joules**

Problem 5 - Suppose the material was traveling at 1/2 the speed of light as it crossed the event horizon, how much time does it take to travel one meter if $c = 3 \times 10^8$ m/sec is the speed of light?

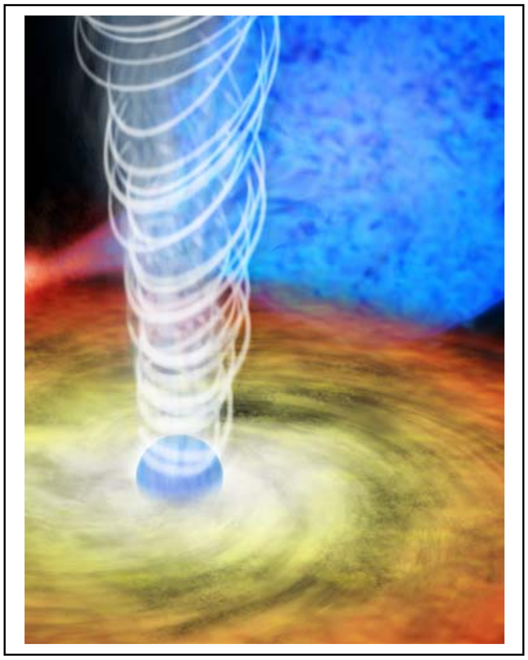
Answer; 1 m / ($0.5 \times 3 \times 10^8$ m/sec) = **6.7×10^{-9} seconds.**

Problem 6 - The power produced is equal to the energy in Problem 4, divided by the time in Problem 5. What is the percentage of power produced by each black hole compared to the sun's power of 3.8×10^{26} Joules/sec?

Answer Stellar: 1.0×10^{15} Joules / 6.7×10^{-9} seconds = **1.5×10^{23} Joules/sec**
Percent = 100% x (1.5×10^{23} / 3.8×10^{26}) = **0.04 %**

Supermassive: 1.0×10^{29} Joules / 6.7×10^{-9} seconds = **1.5×10^{37} Joules/sec**
= (1.5×10^{37} / 3.8×10^{26}) = **39 billion times the sun's power!**

Black Holes and Accretion Disk Temperatures



An artist's impression of a black hole orbiting a companion star, and gravitationally attracting gas from the star into an orbiting accretion disk. Through friction, the gas becomes hotter as it approaches the black hole, turning from red to yellow to white. Most of the gas is swallowed by the black hole, but some is magnetically launched in jets away from the black hole at almost the speed of light. (Credit: M. Weiss, NASA/Chandra)

The farther a particle falls towards a black hole, the faster it travels, and the more kinetic energy it has. Kinetic energy is mathematically defined as K.E. = 1/2 m V^2 where m is the mass of the particle and V is its speed.

Suppose all this energy is converted into heat energy by friction as the particle falls, and that this added energy causes nearby gases to heat up. How hot will the gas get? The equivalent amount of thermal energy, T.E., carried by a single particle is

$$T.E. = \frac{3}{2}kT$$

where Boltzman's Constant k = 1.38 x 10^{-23} Joules/deg. If we set K.E = T.E we get

$$T = \frac{mV^2}{3k}$$

If all the particles in a gas carried this same kinetic energy, then we would say the gas has a temperature of T in kelvins. We also know that the potential energy of the particle is given by

$$P.E. = \frac{GMm}{R}$$

So if we set P.E = T.E we also get the temperature

$$T = \frac{2GMm}{3kR}$$

Problem 1 - The formula T = 2 G M m/(3kR) gives the approximate temperature of hydrogen gas (m = 1.6 x 10^{-27} kg) in an accretion disk around a black hole. To two significant figures, what is the temperature for the material at the distance of Earth's orbit for a solar-mass black hole? (R = 1.47 x 10^{11} m, M = 1.9 x 10^{30} kg, for the constant of gravity G = 6.67 x 10^{-11} Nt m^2/kg^2)?

Problem 2 - How hot would the disk be at the distance of Neptune (R = 4.4 x 10^{12} meters)?

Problem 3 - X-rays are the most common forms of energy produced at temperatures above 100,000 K. Visible light is produced at temperatures above 2,000 K. Infrared radiation is commonly produced for temperatures below 500 K. What would you expect to see if you studied the accretion disk around a solar-mass sized black hole?

Answer Key:

Problem 1 - The formula T = 2/3 G M m/kR gives the approximate temperature of hydrogen gas (m = 1.6 x 10^{-27} kg) in an accretion disk around a black hole. To two significant figures, what is the temperature for a solar-mass black hole disk near the orbit of Earth? (R = 1.47 x 10^{11} m, M = 1.9 x 10^{30} kg, for G = 6.67 x 10^{-11} Nt m^2/kg^2)?

Answer: T = 2/3 x 6.67 x 10^{-11} x 1.9 x 10^{30} x 1.6 x 10^{-27} / (1.38 x 10^{-23} x 1.47 x 10^{11})
= 65,000 K. to 2 significant figures

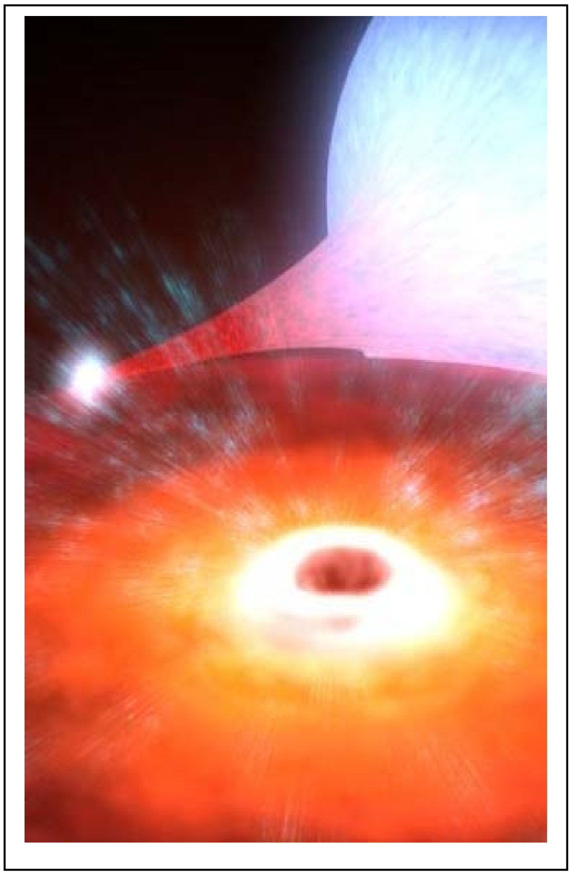
Problem 2 - How hot would the disk be at the distance of Neptune (R = 4.4 x 10^{12} meters)?

Answer: T = 2/3 x 6.67 x 10^{-11} x 1.9 x 10^{30} x 1.6 x 10^{-27} / (1.38 x 10^{-23} x 4.4 x 10^{12})
= **2,200 K.**

Problem 3 - X-rays are the most common forms of energy produced at temperatures above 100,000 K. Visible light is produced at temperatures above 2,000 K. What would you expect to see if you studied the accretion disk around a solar-mass sized black hole?

Answer: **The inner disk region would be an intense source of x-rays and visible light, because the gas is mostly at temperatures above 65,000 K. In the outer disk, the gas is much cooler and emits mostly visible or infrared light.**

Falling Into a Black Hole



An object that falls into a black hole will cross the event horizon, and speed up as it gets closer. This is like a ball traveling faster and faster as it is dropped from a tall building. Suppose the particle fell from infinity. How fast would it be traveling? We can answer this question by considering the concepts of kinetic energy (K.E.) and gravitational potential energy (P.E.):

$$K.E = \frac{1}{2}mV^2 \quad \text{and} \quad P.E. = \frac{GMm}{R}$$

The kinetic energy that the particle with mass, m, will gain as it falls, will depend on the total potential energy it has lost in traveling from infinity to a distance R. By setting the two equations equal to each other, we can relate the kinetic energy a particle gains as it falls to its current distance of R from the center of mass. The quantity, M, is the mass of the gravitating body the particle is falling towards. G is the constant of gravitation which equals 6.67×10^{-11} Nt m^2/kg^2

$$\frac{1}{2}mV^2 = \frac{GMm}{R}$$

We can then solve for the speed, V, in terms of R

$$V = \sqrt{\frac{2GM}{R}}$$

Problem 1 - Suppose a body falls to Earth and strikes the ground. How fast will it be traveling when it hits if M = 5.9×10^{24} kg and R = 6,378 km? Explain why this is the same as Earth's escape velocity?

Problem 2 - NASA's ROSSI satellite was used in 2004 to determine the mass and radius of a neutron star in the binary star system named EXO 0748-676, located about 30,000 light-years away in the southern sky constellation Volans, or the Flying Fish. The neutron star was deduced to have a mass of 1.8 times the sun, and a radius of 11.5 km. A) How fast, in km/sec, will a particle strike the surface of the neutron star if the mass of the sun is 1.9×10^{30} kg? B) In terms of percentage, what will be the speed compared to the speed of light: 300,000 km/sec?

Problem 3 - The star HD226868 is a binary star with an unseen companion. It is also the most powerful source of X-rays in the sky second to the sun - it's called Cygnus X-1. Astronomers have determined the mass of this companion to be 8.7 times the sun. As a black hole, its event horizon radius would be R = 2.8 x 8.7 = 24 km. A) How fast, in km/sec, would a body be traveling as it passed through the event horizon? B) In percentage compared to the speed of light?

Answer Key:

Problem 1 - Suppose a body falls to Earth and strikes the ground. How fast will it be traveling when it hits? Explain why this is the same as Earth's escape velocity?

Answer: R = 6,378 km and M = 5.9×10^{24} kg, so

$V = (2 \times 6.67 \times 10^{-11} \times 5.9 \times 10^{24} / 6.4 \times 10^{6})^{1/2}$
$= 1.1 \times 10^{4}$ m/sec or **11 kilometers/second**.

The particle fell from infinity, so this means that, if you gave a body a speed of 11 km/sec at Earth's surface, it would be able to travel to infinity and escape from Earth.

Problem 2 - NASA's ROSSI satellite was used in 2004 to determine the mass and radius of a neutron star in the binary star system named EXO 0748-676, located about 30,000 light-years away in the southern sky constellation Volans, or the Flying Fish. The neutron star was deduced to have a mass of 1.8 times the sun, and a radius of 11.5 km. A) How fast, in km/sec, will a particle strike the surface of the neutron star if the mass of the sun is 1.9×10^{30} kg? B) In terms of percentage, what will be the speed compared to the speed of light: 300,000 km/sec?

Answer: A) Mass = $1.8 \times 1.9 \times 10^{30}$ kg
$= 3.4 \times 10^{30}$ kg
$V = (2 \times 6.67 \times 10^{-11} \times 3.4 \times 10^{30} / 1.15 \times 10^{4})^{1/2}$
$= 1.98 \times 10^{8}$ m/sec
= **198,000 km/sec.**

B) 198,000/300,000 = **66 % of the speed of light!**

Problem 3 - The star HD226868 is a binary star with an unseen companion. It is also the most powerful source of X-rays in the sky second to the sun - it's called Cygnus X-1. Astronomers have determined the mass of this companion to be 8.7 times the sun. As a black hole, its Event Horizon radius would be R = 2.8 km x 8.7 = 24 km. A) How fast, in km/sec, would a body be traveling as it passed through the event horizon? B) In percentage compared to the speed of light?

Answer: A) Mass = $8.7 \times 1.9 \times 10^{30}$ kg
$= 1.7 \times 10^{31}$ kg.
$V = (2 \times 6.67 \times 10^{-11} \times 1.7 \times 10^{31} / 2.4 \times 10^{4})^{1/2}$
$= 2.98 \times 10^{8}$ m/sec
= **298,000 km/sec.**

B) 298,000/300,000 = **99 % of the speed of light!**

Black Holes and Tidal Forces

$$a = \frac{2GMd}{R^3}$$

A tidal force is a difference in the strength of gravity between two points. The gravitational field of the Moon produces a tidal force across the diameter of Earth, which causes Earth to deform. It also raises tides of several meters in the solid Earth, and larger tides in the liquid oceans.

If the tidal force is stronger than a body's cohesiveness, the body will be disrupted. The minimum distance that a satellite comes to a planet before it is shattered this way is called its Roche Distance. The artistic image to the left shows what tidal disruption could be like for an unlucky moon.

A human falling into a black hole will also experience tidal forces. In most cases these will be lethal! The difference in acceleration between the head and feet could be many thousands of Earth gravities. A person would literally be pulled apart, and his atoms drawn into a narrow string of matter! Some physicists have termed this process spaghettification!

Problem 1 - The equation lets us calculate the tidal acceleration, **a**, across a body with a length of **d**. The acceleration of gravity on Earth's surface is 9.8 m/sec^2. The tidal acceleration between your head and feet is given by the above formula. For M = the mass of Earth (5.9 x 10^{24} kg), R = the radius of Earth (6.4 x 10^6 m) and the constant of gravity whose value is G = 6.67 x 10^{-11} Nt m^2/kg^2 calculate the tidal acceleration, a, if d = 2 meters.

Problem 2 - What is the tidal acceleration across the full diameter of Earth?

Problem 3 - A stellar-mass black hole has the mass of the sun (1.9 x 10^{30} kg), and a radius of 2.8 km. What would be the tidal acceleration across a human at a distance of 100 km?

Problem 4 - A supermassive black hole has 100 million times the mass of the sun (1.9 x 10^{38} kg), and a radius of 280 million km. What would be the tidal acceleration near the event horizon of the supermassive black hole?

Problem 5 - Which black hole could a human enter without being spaghettified?

Answer Key:

Problem 1 - The equation lets us calculate the tidal acceleration, **a**, across a body with a length of **d**. The acceleration of gravity on Earth's surface is 9.8 m/sec^{2}. The tidal acceleration between your head and feet is given by the above formula. For M = the mass of Earth (5.9 x 10^{24} kg), R = the radius of Earth (6.4 x 10^{6} m) and the constant of gravity whose value is G = 6.67 x 10^{-11} Nt m^2/kg^2 calculate the tidal acceleration, a, if d = 2 meters.

Answer: a = 2 x (6.67 x 10^{-11}) x **(**5.9 x 10^{24}**)** x 2 / (6.4 x $10^{6})^{3}$
= 0.000003 x (2)
= **0.000006 m/sec^{2}**

Problem 2 - What is the tidal acceleration across the full diameter of Earth?
Answer: d = 1.28 x 10^{7} m, so a = 0.000003 x 1.28 x 10^{7} = **38 m/sec^{2}**

Problem 3 - A stellar-mass black hole has the mass of the sun (1.9 x 10^{30} kg), and a radius of 2.8 kilometers. What would be the tidal acceleration across a human at a distance of 100 km?

Answer: a = 2 x (6.67 x 10^{-11}) x **(**1.9 x 10^{30}**)** x 2 / (1.0 x $10^{5})^{3}$
= **507,000 m/sec^{2}**

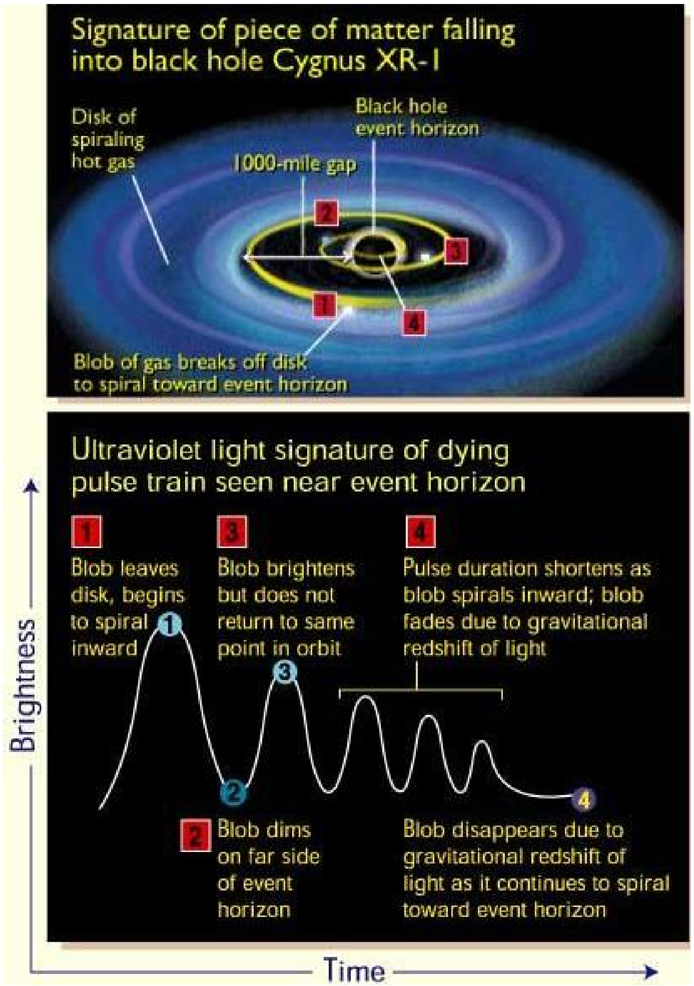
This is equal to 507,000/9.8 = 52,000 times the acceleration of gravity at Earth's surface.

Problem 4 - A supermassive black hole has 100 million times the mass of the sun (1.9 x 10^{38} kg), and an event horizon radius of 280 million km. What would be the tidal acceleration near the event horizon of the supermassive black hole?

Answer: a = 2 x (6.67 x 10^{-11}) x **(**1.9 x 10^{38}**)** x 2 / (2.8 x $10^{11})^{3}$
= 2.3 x10^{-6} m/sec^{2}

Problem 5 - Which black hole could a human enter without being spaghettified?
Answer: The supermassive black hole, because the tidal force is far less than what a human normally experiences on the surface of Earth. That raises the question that as a space traveler, you could find yourself trapped by a supermassive black hole unless you knew exactly what its size was before hand. You would have no physical sensation of having crossed over the black hole's event horizon before it was too late.

Black Hole…Fade-out!



As seen from a distance, not only does the passage of time slow down for someone falling into a black hole, but the image fades to black!

This happens because, during the time that the object reaches the event horizon and passes beyond, a finite number of light particles (photons) will be emitted. Once these have been detected to make an image, there are no more left because the object is on the other side of the event horizon and the photons cannot escape. A star, collapsing to a black hole, will be going very fast as it collapses, then appear to slow down as time dilates. Meanwhile, the image will become redder and redder, until it literally fades to black!

Photographs taken by the Hubble Space Telescope of the black-hole candidate called Cygnus XR-1 detected two instances where a hot gas blob appeared to be slipping past the event horizon for the black hole. Because of the gravitational stretching of light, the fragment disappeared from Hubble's view before it ever actually reached the event horizon. The pulsation of the blob, an effect caused by the black hole's intense gravity, also shortened as it fell closer to the event horizon. Without an event horizon, the blob of gas would have brightened as it crashed onto the surface of the accreting body. See The Astrophysical Journal, 502:L149–L152, 1998 August 1. (Diagram courtesy Ann Field: STScI)

$$L = L_0 \, e^{\frac{-2T}{3\sqrt{3}\ \ 2M}}$$

Problem 1 - The exponential formula above predicts the decay of the light from matter falling in to a black hole. T is the time in seconds measured by distant observer, and M is the mass of the black hole in units of solar masses. How long does it take for the light to fall to half its initial luminosity (i.e. power in units of watts) given by L_0 for a M =1.0 solar mass, stellar black hole?

Problem 2 - How long will your answer be, in years, for a supermassive black hole with M = 100 million times the mass of the Sun?

Problem 3 - The supermassive black hole in Problem 2 'swallows' a star. If the initial luminosity, L_0, of the star is 2.5 times the Sun's, to two significant figures,how long will it take before the brightness of the star fades to 0.0025 Suns, and can no longer be detected from Earth?

Answer Key:

Problem 1 - The exponential formula predicts the decay of the light from matter falling in to a black hole. T is the time in seconds measured by distant observer, and M is the mass of the black hole in units of solar masses. How long does it take for the light to fall to half its initial luminosity (i.e. power in units of watts) given by L_0 for a M =1.0 solar mass, stellar black hole??

Answer : Set L = 1/2 L_0, and M = 1.0, then solve for T. The formula is $0.5 = e^{(-0.19\,T)}$ Take the natural logarithm of both sides to get -0.69 = -0.19 T so **T = 3.6 sec.**

Problem 2 - How long will your answer be, in years, for a supermassive black hole with M = 100 million times the mass of the sun?

Answer: The formula will be $0.5 = e^{-1.9x10^{-9}T}$

So taking the natural log of both sides, $-0.69 = -1.9 \times 10^{-9}$ T, and so T = 3.6×10^{8} sec. If there are 3.1×10^{7} seconds in 1 year, **T = 11.5 years.**

Problem 3 - The supermassive black hole in Problem 2 'swallows' a star. If the initial luminosity, L_0, of the star is 2.5 times the Sun's, to two significant figures, how many years will it take before the brightness of the star fades to 0.0025 Suns, and can no longer be detected from Earth?

Answer: $$0.0025L_{sun} = 2.5L_{sun}e^{-1.9x10^{-9}T}$$

$\mathrm{Ln}(0.001) = -1.9\times10^{-9}$ T

$-6.9 = -1.9\times10^{-9}$ T

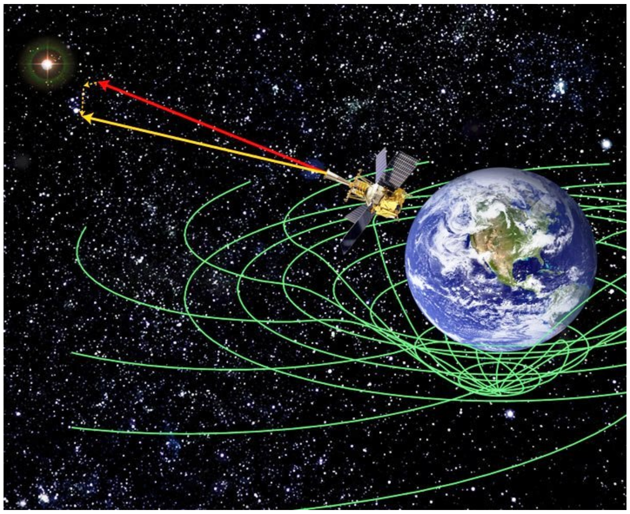
$T = 6.9/1.9\times10^{-9}$

$T = 3.6 \times 10^{9}$ sec

$T = 3.6 \times 10^{9}$ sec x $(1.0 \text{ year} / 3.1 \times 10^{7} \text{ sec})$

T = 120 years to 2 significant figures

Note: Students need to use natural-log not log base-10.



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Physicists at Stanford University have recently completed their analysis of data from the Gravity Probe-B (GP-B) satellite, launched in 2004, and have confirmed two predictions of Albert Einstein's relativistic theory of gravity called General Relativity.

The pointing direction of a high-precision gyroscope was measured for over 50 weeks as it orbited Earth. If Newton's theory of gravity were correct, the pointing direction should stay absolutely the same. If Einstein's theory was correct, it should point in a slightly different direction.

The effect is called 'frame dragging' and was first predicted in 1918 by Austrian physicists Josef Lense (1890-1985) and Hans Thirring (1888-1976) using Einstein's mathematical theory of gravity published in 1915. The rate at which the pointing angle will change is given by the formula for Ω, in degrees/sec, shown below:

$$\Omega = \frac{GJ}{2c^2a^3\left(1-e^2\right)^{3/2}}\left(\frac{360}{2\pi}\right)$$

c is the speed of light: $c = 300{,}000{,}000$ m/s

J is the angular momentum of Earth: $J = 5.861 \times 10^{33}\ m^2\ kg\ sec^{-1}$

G is the Newtonian Gravitational constant $G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$

a is the semi-major axis of the satellite orbit

e is the eccentricity of the satellite orbit

Problem 1- The GP-B satellite orbits at a distance from Earth's center of a = 7020 km, in a circular orbit for which e=0. To two significant figures, what is the value for Omega in A) degrees per second? B) arcseconds per year? (Note 1 degree = 3600 arcseconds and 1 year = 3.1×10^7 seconds)

Problem 2 - The GP-B spacecraft took observations for 50 weeks. About what would be the accumulated angular shift by the end of this time to two significant figures?

Problem 1 - The GP-B satellite orbits at a distance from Earth's center of a = 7020 km, in a circular orbit for which e=0. To two significant figures, what is the value for Omega in A) degrees per second? B) arcseconds per year? (Note 1 degree = 3600 arcseconds and 1 year = 3.1×10^7 seconds)

$$\Omega = \frac{GJ}{2c^2a^3\left(1-e^2\right)^{3/2}}\left(\frac{360}{2\pi}\right)$$ in degrees/sec

$$\Omega = \frac{(6.67x10^{-11})(5.861x10^{33})(360)}{4(3.141)\left(3.0x10^{8}\right)^2\left(7.02x10^{6}\right)^3\left(1-0^2\right)^{3/2}}$$ **3.66 x 10^{-13}** degrees/sec

Answer

A) Ω = **3.6 x 10^{-13} degrees/sec**

B) Ω = **3.6 x 10^{-13}** degrees/sec x (3600 arcsec/1 degree) x (3.1×10^7 sec/1 year)

= **0.04 arcseconds/year**

Problem 2 - The GP-B spacecraft took observations for 50 weeks. About what would be the accumulated angular shift by the end of this time to two significant figures?

Answer: 50 weeks is 50/52 = 0.96 years, so the total shift is just

$\Theta = \Omega$ x 0.96

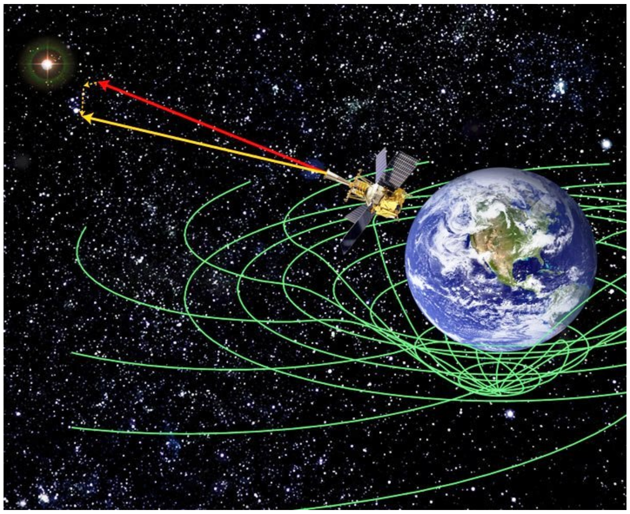
= 0.04 arcseconds/year x (0.96 years)

= 0.038 arcseconds.

Note: This calculation is an approximation to the actual models used to represent the complex spacecraft motion and Earth's gravitational field. Because a more detailed model for Earth and the satellite's motion was used by the GP-B science team, the actual shift detected by the Gravity Probe-B satellite was 0.041 arcseconds, in agreement to within 1% with refined calculations from Einstein's theory.

The equation used in this problem, which predicts the rate of advance of the right ascension of the ascending node of the spacecraft's orbit due to the Lens-Thirring Effect, was obtained from the article:

"*Gravitation, Relativity and Precise Experimentation*" by C.W. Everitt, Proceedings of the First Marcel Grossmann Meeting on General Relativity, pp. 545-615, North Holland, 1977 (p. 567, Equation 22). See the archive of scientific papers at the GP-B website **http://einstein.stanford.edu/content/sci_papers/index.html**



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A prediction of Albert Einstein's relativistic theory of gravity says that the pointing direction of a spinning gyroscope orbiting a massive body should slowly change over time. For Earth, this amount equals degrees/year, and this was recently confirmed by NASA's Gravity Probe-B satellite in 2011.

Einstein's theory predicts much larger shifts if the satellite orbits close to our sun, or to a dense body such as a neutron star.

The effect is called 'frame dragging' and was first predicted in 1918 by Austrian physicists Josef Lense (1890-1985) and Hans Thirring (1888-1976) using Einstein's mathematical theory of gravity published in 1915. The rate, in degrees per second, at which the gyroscope pointing angle will change is given by the formula for Ω, in degrees/sec, shown below:

$$\Omega = \frac{Rac}{r^3 + a^2 r + Ra^2}\left(\frac{360}{2\pi}\right) \quad \text{where} \quad R = \frac{2GM}{c^2} \quad \text{and} \quad a = \frac{2\,\mathrm{R}\,s^2}{5c}\left(\frac{2\pi}{T}\right)$$

and where c is the speed of light (300,000,000 m/s), Rs is the radius of the massive body in meters, M is its mass in kilograms, T is the satellite orbit period in seconds, and G is the Newtonian Gravitational constant 6.67 x 10^{-11} m^3 kg^{-1} s^{-2}. For the GP-B satellite orbiting near Earth at an altitude of 700 km, the measured value for Ω is about 1.2 x 10^{-5} degrees/year.

Problem 1 - In the future, physicists might like to verify this effect near the sun by placing a satellite in a circular orbit at a distance of 10 million kilometers (r = 10^{10} meters). If the radius of the sun is Rs = 6.96x10^8 meters, and its rotation period is T = 24.5 days, and the mass of the sun is M = 2.0 x 10^{30} kg. To two significant figures, what is the value for the Lens-Thirring rate, Ω, in degrees/year? (Note: 1 degree = 3600 arcseconds)

Problem 2 - A neutron star is the compressed nuclear core of a massive star after it has become a supernova. Suppose the mass of a neutron star is equal to our sun, its radius is 12 kilometers, a gyroscope orbits the neutron star at a distance from its center of r = 6,000 kilometers, and its orbit period is T = 8 seconds. To two significant figures, what is Ω for such a dense, compact system in degrees/year?

Problem 1 - In the future, physicists would like to verify this effect near the sun by placing a satellite in a circular orbit at a distance of 10 million kilometers (r = 10^{10} meters). The radius of the sun is Rs = 6.96×10^8 meters, and its rotation period is T = 24.5 days, and the mass of the sun is M = 2.0×10^{30} kg. To two significant figures, what is the value for the Lens-Thirring rate, Ω, in degrees/year?

$$R = \frac{2(6.67x10^{-11})(2.0x10^{30})}{(300,000,000)^2} \quad = \text{ 2,964 m} \qquad a = \frac{2(6.96x10^8)^2}{5(300,000,000)}\left(\frac{2(3.141)}{24.5(24)3600)}\right) = \text{ 1,883 m}$$

then

$$\Omega = \frac{(2964)(1883)(3x10^8)}{\left(10^{10}\right)^3 + 1883^2(10^{10}) + (2964)(1883)^2}\left(\frac{360}{2(3.14)}\right) = 9.60\text{x}10^{-14} \text{ degrees/sec}$$

Ω = 9.6×10^{-14} deg/sec x (365d/1yr) x (24h /1day) x (3600 s / 1 hr) = **3.0×10^{-7} deg/yr**

Note, for GP-B the effect near Earth was 1.2×10^{-5} degrees/year because GP-B was orbiting closer to the mass of Earth than our hypothetical satellite around the sun.

Problem 2 - A neutron star is the compressed nuclear core of a massive star after it has become a supernova. Suppose the mass of a neutron star is equal to our sun, its radius is 12 kilometers, a gyroscope orbits the neutron star at a distance from its center of r = 6,000 kilometers, and its orbit period is T = 8 seconds. To two significant figures, what is Ω for such a dense, compact system in degrees/year?

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then

$$\Omega = \frac{(2964)(0.15)(3x10^8)}{\left(6.0x10^6\right)^3 + \left(0.15\right)^2\left(6.0x10^6\right) + (4150)\left(0.15\right)^2}\left(\frac{360}{2(3.141)}\right)$$

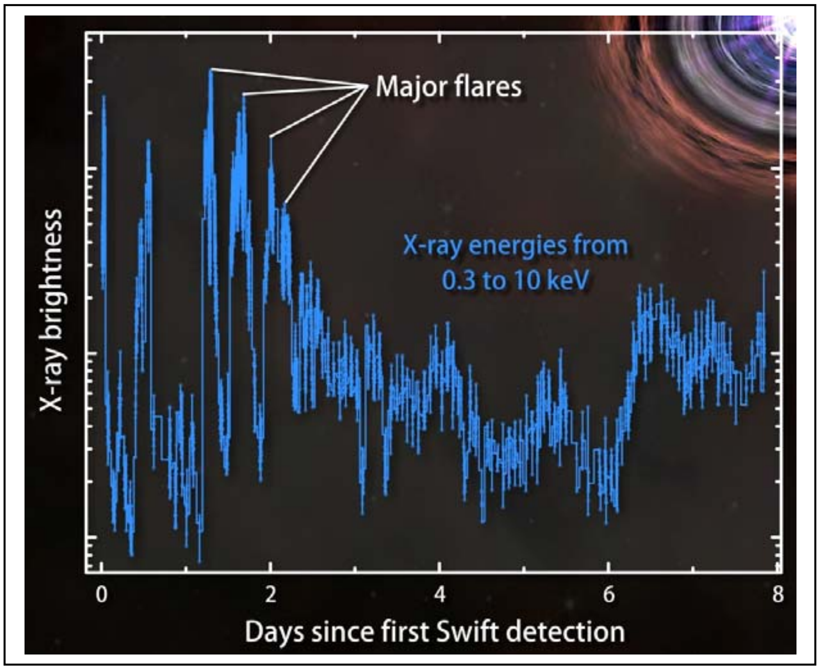
$$\Omega = \frac{(1.33x10^{11})}{\left(2.16x10^{20}\right) + \left(1.35x10^5\right) + (93.4)}\left(\frac{360}{(6.242)}\right) = 3.65 \text{ x } 10^{-8} \text{ degrees/sec}$$

Ω = 3.65×10^{-8} deg/sec x (365d/1yr) x (24h /1day) x (3600 s / 1 hr) = **1.1 deg/yr**

Note this is nearly 100,000 times the corresponding Lens-Thirring rate near Earth.

For a detailed discussion of the derivation of the formula for Ω in the equatorial plane of a spinning body, see Wikipedia:

http://en.wikipedia.org/wiki/Frame-dragging



On March 28, 2011 Swift's Burst Alert Telescope discovered the source in the constellation Draco when it erupted with the first in a series of powerful X-ray blasts. The satellite determined a position for the explosion, now cataloged as gamma-ray burst (GRB) 110328A, and informed astronomers worldwide. As dozens of telescopes turned to study the spot, astronomers quickly noticed that a small, distant galaxy appeared very near the Swift position. A deep image taken by Hubble on April 4 pinpoints the source of the explosion at the center of this galaxy, which lies 3.8 billion light-years away.

That same day, astronomers used NASA's Chandra X-ray Observatory to make a four-hour-long exposure of the puzzling source. The image, which locates the object 10 times more precisely than Swift can, shows that it lies at the center of the galaxy Hubble imaged.

The duration of the x-ray bursts tells astronomers approximately how large the emitting region is, and since the source is a black hole, it gives the approximate diameter of the black hole. The radius of a black hole is related to its mass by the simple formula R = 3 M ,where M is the mass of the black hole in units of the sun's mass, and R is the radius of the Event Horizon in kilometers.

Problem 1 - What is the average duration of the three flare events seen in the X-ray plot above?

Problem 2 - Light travels at a speed of 300,000 km/s. How many kilometers across is the x-ray emitting region based on the average time of the three x-ray flares?

Problem 3 - The size of the x-ray emitting region from Problem 2 is a crude estimate for the diameter of the black hole. For reasons having to do with relativity, a better black hole size estimate will be 100 times smaller than your answer for Problem 2. From this better-estimate, about what is the mass of the black hole GRB110328A in solar masses?

Problem 1 - What is the average duration of the three flare events seen in the X-ray plot above?

Answer: There were about 3 flares in one day, so the average flare duration is about **8 hours**.

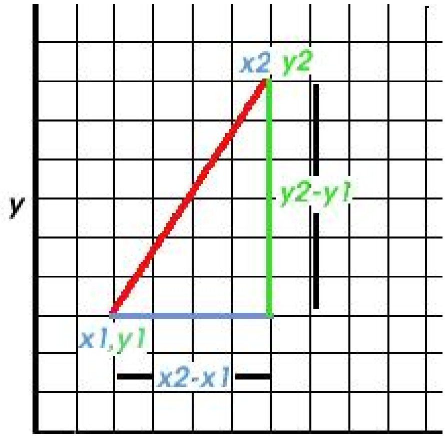
Problem 2 - Light travels at a speed of 300,000 km/s. How many kilometers across is the x-ray emitting region based on the average time of the three x-ray flares?

Answer: Distance = speed x time, so D = 300,000 km/s x 8 hours x (3600 sec/1 hour) = **8.6 billion kilometers.**

Problem 3 - The size of the x-ray emitting region from Problem 2 is a crude estimate for the diameter of the black hole. For reasons having to do with relativity, a better black hole size estimate will be 100 times smaller than your answer for Problem 2. From this better-estimate, about what is mass of the black hole in solar masses?

Answer: If 8.6 billion kilometers is the width of the emitting region, then the radius of the region is about 4.3 billion kilometers, and the estimated radius of the black hole is about 100 times smaller than this or 43 million kilometers. Since the radius of a black hole is R = 3 x M, the mass of the black hole is 43 million = 3 x M, or M = **14 million solar masses**.

Note: Astrophysicists have studied and modeled these kinds of events for decades, and it is generally agreed that gamma-ray bursts are probably caused by beams of particles and radiation leaving the vicinity of the black hole. Because of this, the estimated light-travel size of the emitting region from the changes in the gamma ray or x-ray brightness will greatly over-estimate the actual size of the emitting region. The 'factor of 100' is added to this calculation to account for this 'beaming' effect. Actual astrophysical models of these regions that take into account relativity physics are still in progress and will eventually lead to much better estimates for the black hole size and mass. Also, the relationship between black hole radius and mass that we used only works for black holes that do not rotate, called 'Schwarschild Black Holes'. In actuality, we expect most black holes to be rotation, at speeds that are perhaps even near the speed of light, and these will be significantly larger in size. These are called Kerr Black Holes.



Suppose we had two points P_1 and P_2 on the Cartesian plane at locations $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. By constructing a right triangle with these points defining the hypotenuse of this triangle, it is easy to see that the Pythagorean Theorem would lead to the '2-point' distance formula:

$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Problem 1 – Erica's parents said that she could not bicycle more than 5 miles from home without being accompanied.

Her friend Barbara lives at (+3 ¾ miles, -4 ¼ miles)
Her friend Susan lives at (+2/5 miles, -4 2/5 miles)

If Erica lives at (+1/3 miles, -2/3 miles), how far do her friends live from Erica's house, and which friend can she visit without being accompanied?

Problem 2 – An electrical engineer is designing a wiring harness for cables in a satellite, and wants to use the least amount of gold wire. The points on the satellite base plate that he needs to connect are located at the following locations: A(7,9), B(4,10), C(5,2) and D(2,7) where all units are in centimeters.

What is the total length of the gold wire, to the nearest tenth of a centimeter, connecting B to A to D to C?

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Answer: Barbara : $D^2 = (15/4 - 1/3)^2 + (-17/4-(-2/3))^2$
$D^2 = (41/12)^2 + (-49/12)^2$
$D^2 = 1581/144 + 2401/144$
$D^2 = 3982/144$
D = 5.26 miles

Susan: $D^2 = (2/5-1/3)^2 + (-22/5 - (-2/3))^2$
$D^2 = (6/15 - 5/15)^2 + (-66/15 + 10/15)^2$
$D^2 = (1/15)^2 + (-56/15)^2$
$D^2 = 3137/225$
D = 3.73 miles.

So she can visit Susan.

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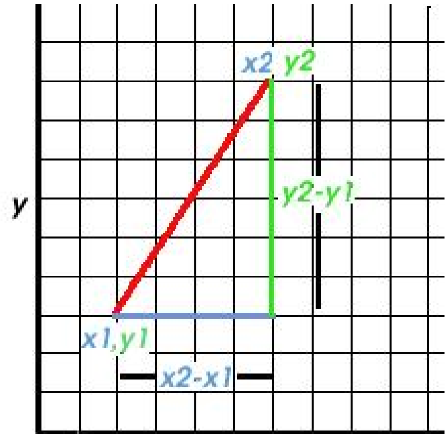
What is the total length of the gold wire, to the nearest tenth of a centimeter, connecting B to A to D to C?

Answer: B to A : $D^2 = (7-4)^2 + (9-10)^2$, so D = 3.16 cm
A to D: $D^2 = (2-7)^2 + (7-9)^2$ so D = 5.38 cm
D to C: $D^2 = (5-2)^2 + (2-7)^2$ so D = 5.83 cm

Total length = 3.16 + 5.38 + 5.83 = 14.37 cm, rounded to the nearest tenth this becomes **14.4 cm of gold wire**.



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$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Problem 1 – Assuming that 3-dimensional space can be created by extending the Cartesian plane by adding a third coordinate axis, Z, what is the Distance Formula for 3-dimensional Cartesian space when each point is defined by the coordinate triad (x,y,z)?

Problem 2 – Suppose that our sun is located at P_1(3, 4, 5) and the nearby star Sirius is located at P_2(8, 10, 7) where the units are in light years. What is the distance between these two stars to the nearest light year?

Problem 2 – A beam of light, traveling at 300,000 km/sec is sent in a round trip between spacecraft located Earth (0,0), Mars (220, 59), Neptune (-3200, -3200) and back to Earth. If the coordinate units are in millions of kilometers, what is

A) The total round-trip distance (Earth, Mars, Neptune, Earth) in billions of kilometers?

B) The round trip time in hours ?

Problem 1 – Assuming that 3-dimensional space can be created by extending the Cartesian plane by adding a third coordinate axis, Z, what is the Distance Formula for 3-dimensional Cartesian space when each point is defined by the coordinate triad (x,y,z)?

Answer: $$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Problem 2 – Suppose that our sun is located at P1(3, 4, 5) and the nearby star Sirius is located at P2(8, 10, 7) where the units are in light years. What is the distance between these two stars to the nearest light year?

$$D^2 = (2-3)^2 + (10-4)^2 + (7-5)^2$$

Problem 4 – A beam of light, traveling at 300,000 km/sec is sent in a round trip between spacecraft located Earth (0,0), Mars (220, 59), Neptune (-3200, -3200) and back to Earth. If the coordinate units are in millions of kilometers, what is A) the total round-trip distance in billions of kilometers? B) The round trip time in hours?

Answer: A) Earth to Mars: $D^2 = (220-0)^2 + (59-0)^2$ so D = 228
Mars to Neptune: $D^2 = (-3200 - (220))^2 + (-3200 - 59)^2$ so D = 4724
Neptune to Earth: $D^2 = (-3200)^2 + (-3200)^2$ so D = 4525
Total round-trip = 228 + 4724 + 4525 = 9477 million kilometers.
This is also **9.477 billion kilometers**.

B) Speed = 300,000 km/s so Time = Distance/speed and so
Time = 9477000000/300000 = 31590 seconds. Since 1 hr = 60x60 = 3600 seconds, this round trip time is just **8.8 hours**.

The Distance Formula in Dilated Space

Many things in Nature cannot be represented on a 'rectangular' Cartesian coordinate grid. One or more of the three possible directions may be dilated so that the resulting grid work is distorted. Even though all points can be labeled with unique coordinates, (x,y,z), the standard Pythagorean Theorem distance formula no longer works. The intervals between the points are not all equal in physical measure (e.g. meters).

For example, although the 1-unit distance separating (2,2) and (1,2) might represent exactly 1 meter, the 1-unit distance between (8,15) and (8,16) may represent 1.5 meters or even 50!

Problem 1 – Suppose we change the y-axis scale by a simple dilation so that 1 unit on the y-axis is equal to 2 units on the x-axis. If a point is located at (3,2) how far is it located from the origin in terms of the x-axis unit scale?

Problem 2 - A carpenter wants to stretch a square, rubber membrane on the top of a rectangular roof before he installs the roofing tiles. The square membrane has the dimensions of 5 meters x 5 meters. The roof has dimensions of 5 meters x 8 meters, and the circular opening is located at the position (3 meters, 7 meters).

A) What is the new Pythagorean Theorem for the stretched membrane in terms of the measurements for x and y?

B) What are the coordinates of the hole translated onto the coordinate grid of the square membrane before it was stretched?

Problem 1 – Suppose we change the y-axis scale by a simple dilation so that 1 unit on the y-axis is equal to 2 units on the x-axis. If a point is located at (3,2) how far is it located from the origin in terms of the x-axis unit scale?

Answer: It is 3 units from the origin along the x-axis and 2 units from the Origin along the y-axis, so by the Pythagorean Theorem we have

$D = (3^2 + (2x2)^2)^{1/2} = (25)^{1/2} =$ **5 units**.

Problem 2 - A carpenter wants to stretch a square, rubber membrane on the top of a rectangular roof before he installs the roofing tiles. The square membrane has the dimensions of 5 meters x 5 meters. The roof has dimensions of 5 meters x 8 meters, and the circular opening is located at the position (3 meters, 7 meters).

A) What is the new Pythagorean Theorem for the stretched membrane in terms of the measurements for x and y?

B) What are the coordinates of the hole translated onto the coordinate grid of the square membrane before it was stretched?

Answer: A) If (x,y) = coordinates on unstretched membrane and (X,Y) are the coordinates on the stretched membrane, we have X = x and Y = 8/5 y, then

$$D^2 = X^2 + (5/8\ Y)^2 \quad \text{and so} \quad \mathbf{D^2 = X^2 + (25/64)\ Y^2}.$$

B) The X axis remains unstretched. The y axis is stretched from 5-meters to 8-meters so the dilation factor is 8/5. So, if the y value of the opening is 7-meters on the stretched membrane, it will be 7 meters x 5/8 = 35/8 or 4 3/8 meters on the unstretched membrane.

The coordinates on the unstretched membrane is **(5, 4 3/8)**

Working with Spacetime – Points and Events

Three-dimensional space is our basic frame of reference for thinking about the world around us, but we all know that time is an important ingredient too. In fact time is so important that physicists consider time and space to be a single mathematical object called, simply, **spacetime**.

Unlike 3-dimensional space, spacetime exists in 4 dimensions, with time being the fourth coordinate.

The diagram above shows time increasing from t=0 to t=5 minutes along the vertical axis, and we have reduced the number of space dimensions from three to one along the x-axis to show the motion of a traveler. The two houses are located at x=0 and x=5 miles apart, and the traveler moves along the green line at a steady travel speed (constant slope). The diagram also shows the two vertical lines for each house, which indicate that the houses remained at x=0 and x=5 during the time the traveler was in motion. This diagram is called a 'spacetime' or 'Minkowski Diagram', and it shows the histories of the two houses and the traveler, which are called **worldlines**.

In this 4-dimensional geometry, all points are called **events**, and all events have four coordinates (x, y, z, t).

Draw a 2-D spacetime diagram with time increasing upwards along the vertical axis and 1-dimension of space increasing to the right along the horizontal axis. Label the x-axis with tick marks at 1 kilometer intervals. Label the t-axis with marks every hour. In the following problems, all coordinates have the form (x,t) where x is the space position and t is the time)

Problem 1 – You are located at (0,0) and a store is located at (5,0). Where on the spacetime diagram will the store be after 5 hours?

Problem 2 - Starting from your position (0,0) you go for a walk to the store and arrive after 5 hours. Draw the worldline for your journey to the store.

Problem 3 – After staying at the store for an hour, you return to where you started at x=0 at 8 hours after you left. Draw the worldline diagram for this complete journey. Where is the store on this diagram by the time you return to x=0?

Problem 1 – You are located at (0,0) and a store is located at (5,0). Where on the spacetime diagram will the store be after 5 hours?

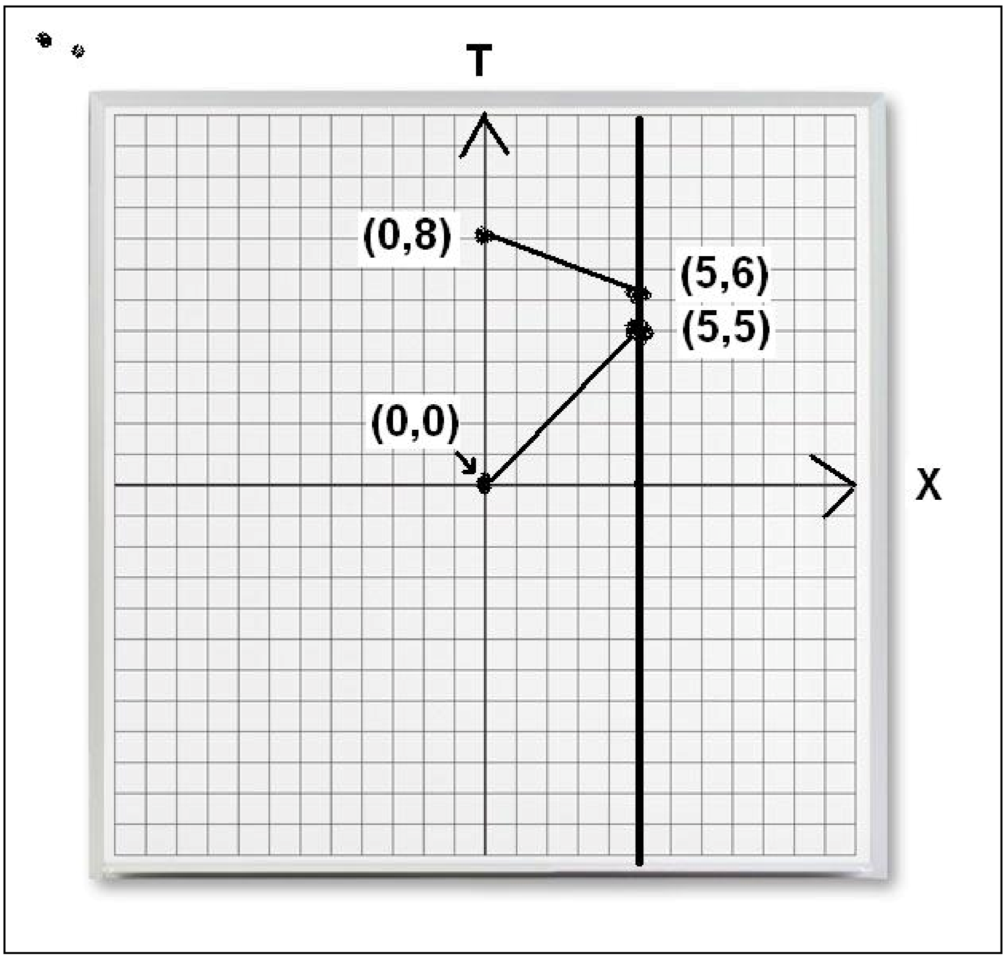
Answer: at coordinates (5,5)

Problem 2 - Starting from your position (0,0) you go for a walk to the store and arrive after 5 hours. Draw the worldline for your journey to the store.

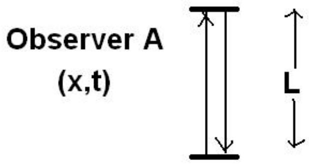
Answer: See diagram below

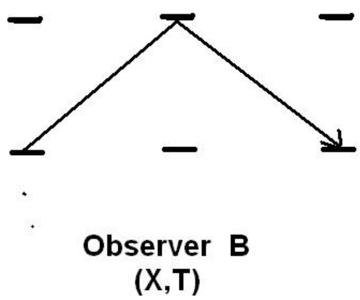
Problem 3 – After staying at the store for an hour, you return to where you started at x=0 at 8 hours after you left. Draw the worldline diagram for this complete journey. Where is the store on this diagram by the time you return to x=0?

Answer: See diagram below



Working with Time – Two Observers





A frame of reference is a coordinate system that follows a specific person or object. Every human benign moves through life anchored to their own personal coordinate system.

If two people are sitting on a couch and measuring things in the room around them, they will measure the same things both in terms of where things are in space, and what the clock on the wall says. If one of these two people start to move, they will no longer agree to the precise values of these measurements!

Problem 1 – Observer A, uses coordinates labeled by (x, t) and has his own meter stick and clock in his hands. In the upper figure, he measures the time it takes a beam of light to travel L meters to a mirror and then back to him. If the speed of light is given by the quantity, c, what is the formula that gives the total time it takes the light to travel to the mirror?

Problem 2 - Observer B uses the coordinates (X,T) which are measured by a meter stick and clock he is carrying. He watches Observer A traveling at a speed of V, and sees the light beam travel along the path shown in the bottom figure. What is the length of the path, l, for the light beam traveling to the mirror that Observer B will see in his coordinate frame as he watches the light clock pass by? (Hint: Use the Pythagorean Theorem).

Problem 3 – If the length of the light clock, L, is the same as measured by Observer A and B, what is the relationship between the time, t, measured on Observer As wristwatch, and time, T, measured by Observer Bs wristwatch?

Problem 4 – If the distance that the light beam travels to the mirror as seen by Observer B is c x T = l, re-write your formula in Problem 3 to show how T and t are related to each other.

Problem 5 – Because Observer A is riding with the light clock, his time his called the <u>proper time</u>. If the proper time measured by Observer A is 1 second, how much time will elapse per 'tick' as observed by Observer B, if V = 90% the speed of light?

Problem 1 – Observer A, uses coordinates labeled by (x, t) and has his own meter stick and clock in his hands. In the lower figure, he measures the time it takes a beam of light to travel L meters to a mirror and then back to him. If the speed of light is given by the quantity, c, what is the formula that gives the total time it takes the light to travel to the mirror?

Answer: $\mathbf{t = L/c}$

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Answer: $\mathbf{l^2 = L^2 + (VT)^2}$

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Answer: $\mathbf{l^2 = (ct)^2 + (VT)^2}$

Problem 4 – If the distance that the light beam travels to the mirror as seen by Observer B is c x T = l, re-write your formula in Problem 3 to show how T and t are related to each other.

Answer: $(cT)^2 = (ct)^2 + (VT)^2$

$$T^2 = t^2 + (V/c)^2\, T^2$$

$$t^2 = T^2 - (V/c)^2\, T^2$$

$$t^2 = T^2\, (1-(V/c)^2)$$

$$\mathbf{t = T\, (1-(V/c)^2)\,)^{1/2}}$$

Problem 5 – Because Observer A is riding with the light clock, his time his called the proper time. If the proper time measured by Observer A is 1 second, how much time will elapse per 'tick' as observed by Observer B, if V = 90% the speed of light?

Answer: t = 1 second and V = 0.9 so T = 1 sec/$(1-(0.9)^2)^{1/2}$ and so T = **2.3 seconds!**

Three-dimensional space is our basic frame of reference for thinking about the world around us, but we all know that time is an important ingredient too. In fact time is so important that physicists consider time and space to be a single mathematical object called, simply, **spacetime**.

Unlike 3-dimensional space, spacetime exists in 4 dimensions, with time being the fourth coordinate.

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In this 4-dimensional geometry, all points are called **events**, and all events have four coordinates (t, x, y, z). Although the Pythagorean Theorem still works to determine the space distance between two events (x1,y1,z1) and (x2,y2,z2), a different Pythagorean Theorem has to be used to give the full 4-dimensional spacetime distance, S, between two events. An approximate form for this new distance formula is given by

Problem 1 – What is the 3-dimensional distance, D, between the two events E_1(1,3,2,5) and E_2(5,4,5,2)?

Problem 2 – What is the 4-dimensional distance between these two events?

Problem 3 – A ray of light travels from (0,3,2,5) to ($19^{1/2}$,4,5,2). What is: A) the 3-dimensional distance that it travels? B) the 4-dimensional distance that it travels?

Problem 4 – Matter can only travel on worldlines for which S > 0. Light rays can only travel on worldlines for which S = 0. Two events are connected by a worldline ‘history’. If the event coordinates (t,x,y,z) are (1,4,6,9) and (5,3,8,10) is this the worldline of a beam of light, or a particle of matter?

Problem 1 – What is the 3-dimensional distance, D, between the two events $E_1(1,3,2,5)$ and $E_2(5,4,5,2)$?

Answer: $D = ((4-3)^2 + (5-2)^2 + (2-5)^2)^{1/2} = \mathbf{(19)^{1/2}}$

Problem 2 – What is the 4-dimensional distance between these two events?

Answer: $S = ((4-3)^2 + (5-2)^2 + (2-5)^2 – (5-1)^2)^{1/2} = (19 – 16)^{1/2} = \mathbf{(3)^{1/2}}$

Problem 3 – A ray of light travels from (0,3,2,5) to $(19^{1/2},4,5,2)$. What is A) the 3-dimensional distance that it travels? B) the 4-dimensional distance that it travels?

Answer: $D = ((4-3)^2 + (5-2)^2 + (2-5)^2)^{1/2} = \mathbf{(19)^{1/2}}$

$S = ((4-3)^2 + (5-2)^2 + (2-5)^2 – ((19)^{1/2} - 0)^2)^{1/2} = (19 – 19)^{1/2} = \mathbf{0}$

Problem 4 – Matter can only travel on worldlines for which S > 0. Light rays can only travel on worldlines for which S = 0. Two events are connected by a worldline 'history'. If the event coordinates (t,x,y,z) are (1,4,6,9) and (5,3,8,15) is this the worldline of a beam of light, or a particle of matter?

Answer: $S = ((3-4)^2 + (8-6)^2 + (15-9)^2 – (5-1)^2)^{1/2} = (41-16)^{1/2} = \mathbf{+5}$

Since S > 0, this worldline can not be that of a beam of light (S=0) but it can be the history of a particle of matter that has traveled from Event 1 to Event 2 after an elapsed time of (5-1) = +4 units.

The Spacetime Interval

The four-dimensional distance between two events tells you if they can be connected by the ends of a meter stick, ($S^2 > 0$), a beam of light ($S^2=0$) or the ticks of a clock ($S^2 <0$) by some other observer. The interval this given a specific name depending on the magnitude of S^2:

$S^2 > 0$ then it is a space-like interval
$S^2 < 0$ then it is a time-like interval
$S^2 = 0$ then it is a null interval.

where the spacetime interval is given by

$$S^2 = (x_2 - x_1)^2 - (t_2 - t_1)^2$$

Problem 1 – Draw a 2-D spacetime diagram with time, t, on the vertical axis and the space coordinate, x, on the horizontal axis.

Problem 2 – Graph the following spacetime events given by the (x,t) coordinates: A(2,5), B(4,2), C(5,4), D(6,7), E(4,-3) and F(8,5).

Problem 3 – Which events are located in the future of event C?

Problem 4 – Which events are located to the right of event C?

Problem 5 – For the following spacetime segments, determine which ones have a spacetime interval S^2 that is space-like, time-like or null: AC, BD, EA, and DF.

Problem 1 – Draw a 2-D spacetime diagram with time, t, on the vertical axis and the space coordinate, x, on the horizontal axis.

Problem 2 – Graph the following spacetime events given by the (x,t) coordinates: A(2,5), B(4,2), C(5,4), D(6,7), E(4,-3) and F(8,5).

Problem 3 – Which events are located in the future of event C?

Answer: Event C is located at (5,4). We are looking for all events that have time coordinates, t for which t > 4. These are the events A, D and F.

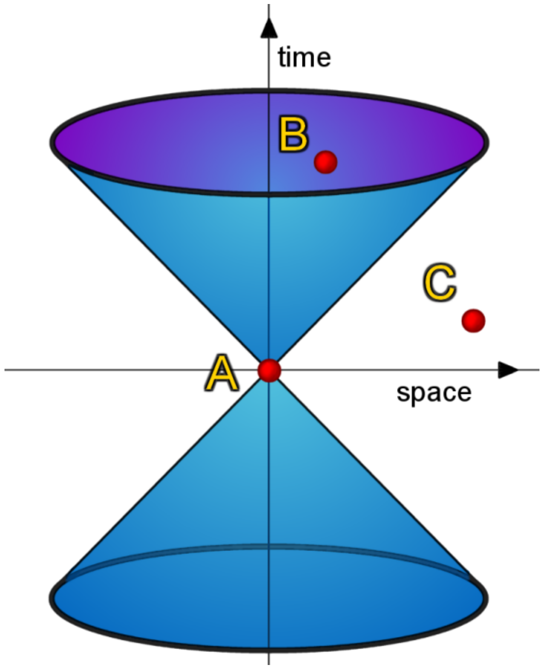
Problem 4 – Which events are located to the right of event C?

Answer: We are looking for all events that have space coordinates, x for which x > 5. These are the events D and F.

Problem 5 – For the following spacetime segments, determine which ones have a spacetime interval S2 that is space-like, time-like or null: AC, BD, EA, and DF. A(2,5), B(4,2), C(5,4), D(6,7), E(4,-3) and F(8,5)

Answer:

AC:	$S^2 = (5-2)^2 - (4-5)^2$,	$S^2 = 9 - 1$	space-like
BD:	$S^2 = (6-4)^2 - (7-2)^2$,	$S^2 = 4 - 25$	time-like
EA:	$S^2 = (2-4)^2 - (6-(-3))^2$	$S^2 = 4 - 81$	time-like
DF:	$S^2 = (8-6)^2 - (5-7)^2$	$S^2 = 4 - 4$	null.



The four-dimensional distance between two events tells you if they can be connected by the ends of a meter stick, ($S^2 > 0$), a beam of light ($S^2=0$) or the ticks of a clock ($S^2 <0$) by some other observer. The interval this given a specific name depending on the magnitude of S^2:

$S^2 > 0$ then it is a space-like interval
$S^2 < 0$ then it is a time-like interval
$S^2 = 0$ then it is a null interval.

where the spacetime interval is given by

$$S^2 = (x_2 - x_1)^2 - (t_2 - t_1)^2$$

Suppose you are located at event A in the diagram above. In order to travel to Event C you will have to move faster than the speed of light because the spacetime interval S^2 is space-like. If you sent a beam of light out into space from Event A, it would reach all the events exactly on the surface of the 45-degree cone, because for those events the value of S^2 is exactly 1.0.

All of the points inside this cone on the top-half of the figure are events you could actually reach if you traveled fast enough. These events are located inside the future light cone of Event A.

Similarly, the sides of the cone in the lower half-plane represent light rays from Events that were emitted in the past and are just now arriving at Event A. The Events inside this cone are events that you could have started from, and traveled at various speeds to arrive at Event A. We say that these Events are located in the past light cone of Event A.

Problem 1 – In this diagram, the slope of the lines connecting Event A with events in the future light cone are given by $R = (t_2-t_1)/(x_2-x_1)$. Prove that 1/R is the same as the average speed, v, of the traveler from Event (t_1,x_1) to (t_2,x_2).

Problem 2 – On the scale of this spacetime diagram, all light rays arriving at, or emitted from Event A have $S^2 = 0.0$. Prove that light rays traveling at the speed of light (300,000 km/s) are represented by 45-degrees lines.

Problem 3 – From Event A, draw the locations of the following three events:

A) The Traveler at Event A standing still.
B) The Traveler at Event A moving at 10% the speed of light.
C) The Traveler at Event A moving at 50% the speed of light.

Problem 1 – In this diagram, the slope of the lines connecting Event A with events in the future light cone are given by $R = (t_2-t_1)/(x_2-x_1)$. Prove that 1/R is the same as the average speed, v, of the traveler from Event (t_1,x_1) to (t_2,x_2).

Answer: The average speed is just the change in distance divided by the change in time, so $v = (x_2-x_1)/(t_2-t_1)$. So R = 1/v and then v = 1/R.

Problem 2 – On the scale of this spacetime diagram, all light rays arriving at, or emitted from Event A have $S^2 = 0.0$. Prove that light rays traveling at the speed of light (300,000 km/s) are represented by 45-degrees lines.

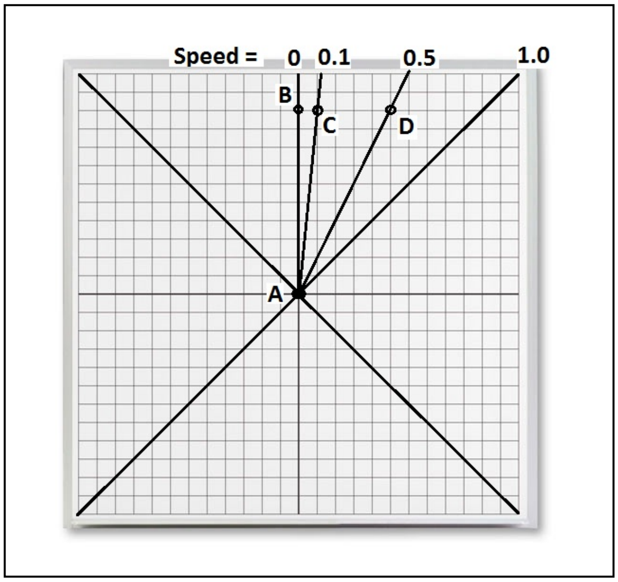
Answer: Because $S^2 = 0.0$ for light rays, we have that $0 = (x_2-x_1)^2 - (t_2-t_1)^2$
And so $(x_2-x_1) = (t_2-t_1)$ and so R = 1.0 and v = 1.0. The slope of the line is 1.0, so using this as the hypotenuse of a right triangle, the sides (x_2-x_1) and (t_2-t_1) are equal and so the triangle is a 45-45-90 right triangle.

Problem 3 – From Event A, draw the locations of the following three events:
A) The Traveler at Event A standing still.
B) The Traveler at Event A moving at 10% the speed of light.
C) The Traveler at Event A moving at 50% the speed of light.

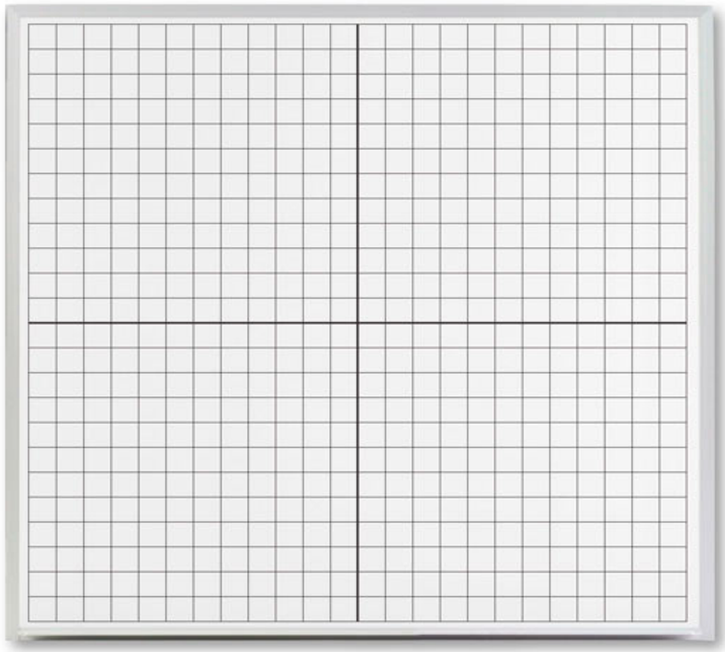
Answer: A) A vertical line along the time axis from Event A.

B) s = 0.10 so R = 10 and so 1 unit along the x axis equals 10 units along the time axis.

C) s = 0.50 so R = 2 and so 1 unit along the x axis equals 2 units along the time axis.



Spacetime Diagrams - I



Spacetime Basics: The coordinate grid above represents time (vertical) extending from -12 units to +12 units, and 1-dimension of space (horizontal) extending from -12 units to +12 units. The Origin (0,0) represents 'Now', and the horizontal axis shows all of the space points existing at Now, and represented by the coordinates (0,+1), (0,+2), (0,+3) etc. The Invariant distance between any two Events (t_1,x_1) and (t_2,x_2) is given by $S^2 = (x_2-x_1)^2 - (t_2-t_1)^2$

Problem 1 – Draw the past lightcone and the future lightcone for the event A located at (0,0).

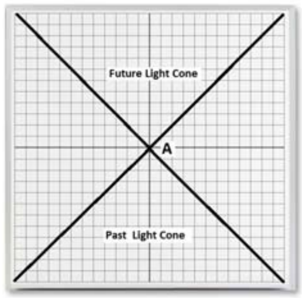
Problem 2 – For events at B (-2, -6) and C (+9,+8) draw the past and future light cones for these two events.

Problem 3 – Can B be on the worldline of A?

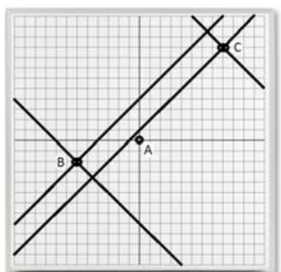
Problem 4 – Can A be on the worldline of C?

Problem 5 – Draw the line representing a light beam sent from B to A. When does the light signal arrive along the vertical worldline of A?

Problem 1 – Draw the past lightcone and the future lightcone for the event A located at (0,0).



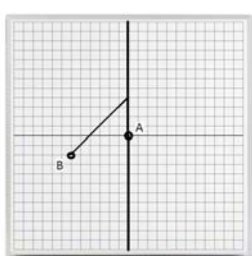
Problem 2 – For events at B (-2, -6) and C (+9,+8) draw the past and future light cones for these two events.



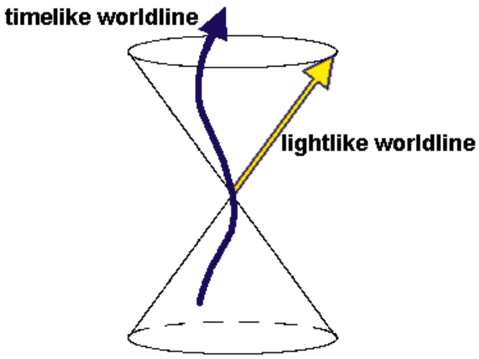
Problem 3 – Can B be on the worldline of A? Answer: No, because Event A is outside the future light cone of B so A and B are not connected by a time-like distance.
$S^2 = (-6-0)^2 - (-2-0)^2$ so S^2 is space-like.

Problem 4 – Can A be on the worldline of C? Answer: A is inside the past light cone of C, so A and C can be connected by an interval of time along a worldline.

Problem 5 – Draw the line representing a light beam sent from B to A. When does the light signal arrive along the vertical worldline of A?



Answer: Light rays travel on worldliness for which $S^2=0$, which are 45-degree lines on the grid. A light ray sent from B will travel future-ward of B, and directed to the right in the direction of A. The light signal will arrive at event (+4,0) on the future worldline of A as shown.



Imagine two Events in spacetime, (x_1,y_1,z_1,t_1) and (x_2,y_2,z_2,t_2) so that the interval $S^2 = 0$. This means that the two events could be part of a worldline representing a ray of light.

If $S^2 < 0$, the interval between the two events is time-like, which means that the two Events could be on the worldline of the same object, which was traveling through space slower than the speed of light.

Every Event in spacetime is located at the vertex of a pair of cones, shown in the diagram above. When the spacetime interval is given by

$$S^2 = (x_2 - x_1)^2 - (t_2 - t_1)^2$$

the cones have sides tilted at exactly 45^o in a spacetime diagram for flat space.

If an Event is on the worldline of a matter particle, all of the past Events for that particle, and all the future Events for that particle will be inside the Past or Future Lightcones of the Events along the worldline.

Problem 1 – A Traveler has arrived on her worldline at the Event (3,2,5,5). Can the Traveler move in a slow rocket ship so that her worldline also includes the Event (3,2,6,10)?

Problem 2 – The Traveler uses a faster rocket ship starting from the same worldline Event (3,2,5,5) and wants her worldline to include Event (3,2,6,7). Is this possible?

Problem 3 – The Traveler's worldline now includes Event (3,2,6,7) and she wants to let an Observer on another worldline know about her arrival at this Event. If the Observer is located at space coordinates (3,2,9) at what time coordinate, T, would the Observer receive a light signal from the Traveler?

Problem 1 – A Traveler has arrived on her worldline at the Event (3,2,5,5). Can the Traveler move in a slow rocket ship so that her worldline also includes the Event (3,2,6,10)?

Answer: $S^2 = (3-3)^2 + (2-2)^2 + (6-5)^2 - (10-5)^2$ so $S^2 = 1-25$ so S^2 is timelike. Event (3,2,6,10) could be on the worldline of the Traveler.

Problem 2 – The Traveler uses a faster rocket ship starting from the same worldline Event (3,2,5,5) and wants her worldline to include Event (3,2,6,7). Is this possible?

Answer: $S^2 = (3-3)^2 + (2-2)^2 + (6-5)^2 - (7-5)^2$ so $S^2 = 1 - 4$ and so S^2 is timelike. Note that the Traveler arrives at the same location in space (3,2,6) but requires less time (2 units of time rather than 5 units) because the rocket is faster.

Problem 3 – The Traveler's worldline now includes Event (3,2,6,7) and she wants to let an Observer on another worldline know about her arrival at this Event. If the Observer is located at space coordinates (3,2,9) at what time coordinate, T, would the Observer receive a light signal from the Traveler?

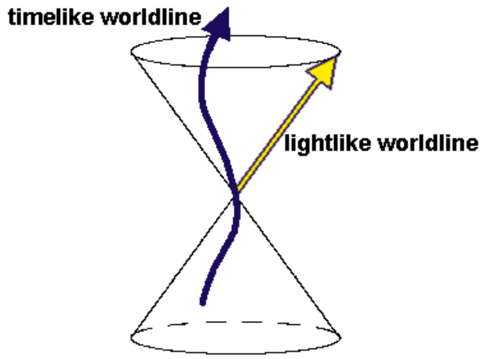
Answer: We know that for light signals, $S^2 = 0$. The calculation of the interval connecting the Traveler and the Observer is just

$$0 = (3-3)^2 + (2-2)^2 + (9-6)^2 - (T-7)^2$$

So $(9-6)^2 = (T-7)^2$
$3 = T-7$

And so **T = 10**.

The Observer will receive the light signal at his worldline coordinate (3,2,9,10)



As a person or other material object moves forward in time, all of the spacetime events in her history, called a worldline, must be separated by a timelike interval. This is true even if the person or particle are in motion.

If, for any time interval the value of S^2 =0, and the interval is null, that means that the object traveled at the speed of light. If $S^2 > 0$, and the interval is space-like, that means that the object traveled faster than the speed of light between the two time intervals.

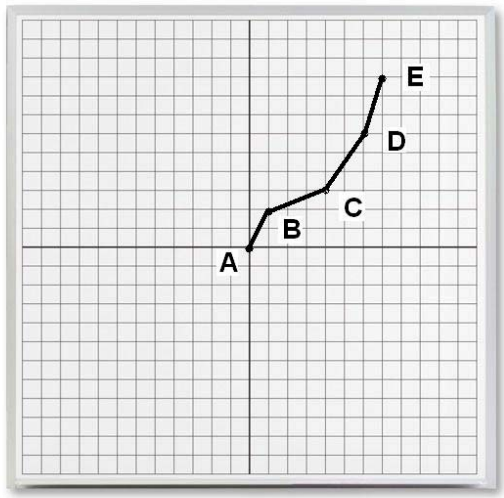
where as usual, the spacetime interval is given by

$$S^2 = (x_2 - x_1)^2 - (t_2 - t_1)^2$$

Problem 1 – Draw a spacetime diagram for the following events along a possible worldline, where the coordinates are given as (x,t): A(0,0), B(1,2), C(4,3), D(6,6), and E(7,9). Connect the points by a line in the sequence given.

Problem 2 – Using the segment test for S^2, is this a possible worldline for a material object?

Problem 1 – Draw a spacetime diagram for the following events along a possible worldline, where the coordinates are given as (x,t): A(0,0), B(1,2), C(4,3), D(6,6), and E(7,9). Connect the points by a line in the sequence given.

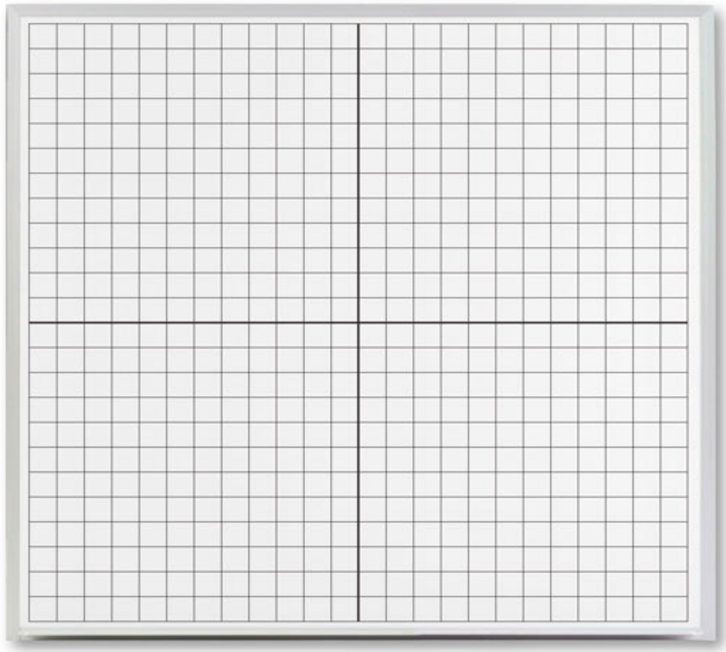


Problem 2 – Using the segment test for S^2, is this a possible worldline for a material object?

Answer:

AB:	$S^2 = 1 - 4$	time-like
BC:	$S^2 = 9 - 1$	space-like
CD:	$S^2 = 4 - 9$	time-like
DE:	$S^2 = 1 - 9$	time-like.

Not all of the intervals are time-like, so this cannot be the worldline for a material object. Between events B and C the interval is space-like which means these events, B and C, can only be connected by a body that moves faster than light.



Spacetime Basics: The coordinate grid above represents time (vertical) extending from -12 units to +12 units, and 1-dimension of space (horizontal) extending from -12 units to +12 units. The Origin (0,0) represents 'Now', and the horizontal axis shows all of the space points existing at t = Now, and represented by the coordinates (0,+1), (0,+2), (0,+3) etc. The Invariant distance between any two Events (t_1,x_1) and (t_2,x_2) is given by $S^2 = (x_2-x_1)^2 - (t_2-t_1)^2$

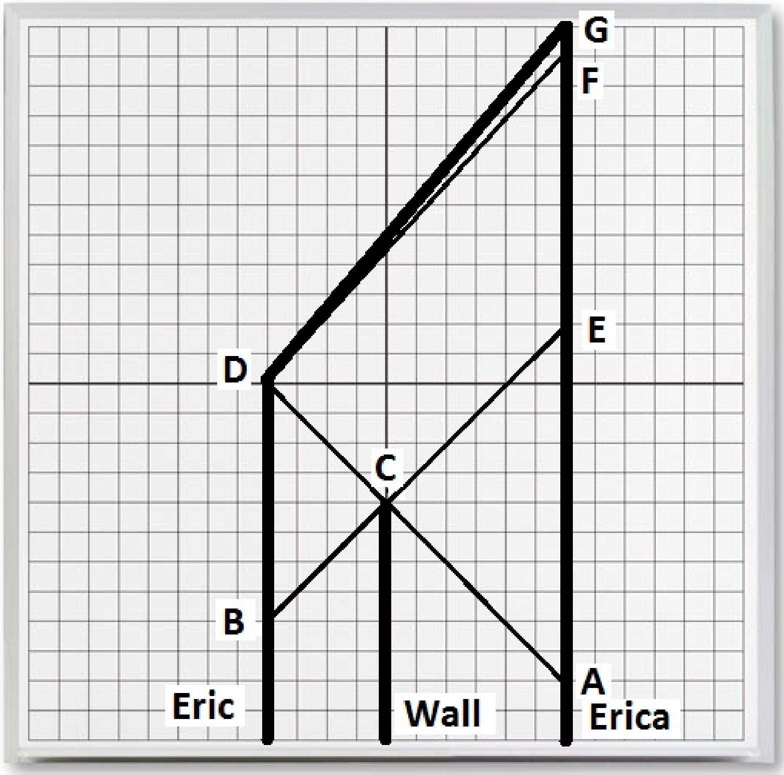
Draw the worldline diagram that represents the following story:

Eric and Erica are in their spacecraft, and there is an asteroid wall located between them. Since a time of -12 units, Eric has been located at x=-4 and Erica has been located at x = +6, with the asteroid wall located at x=0. After some detailed collaborations and measurements, they agree that the most spectacular thing to do is to vaporize the asteroid by using their photon cannons. They agree to synchronize the firing of their two cannons at Events A and so that the energy from both photon streams arrives precisely at the Event C (-4,0) so that the asteroid is obliterated. As soon as the flash of light from the explosion arrives at Eric's spacecraft, Event D, he sends a light signal to Erica which arrives at Event F, and immediately fires up his rocket engines and travels at high speed to visit Erica. He arrives at the Event G (+12,+6).

Problem 1 – How soon after the obliteration of the asteroid does Eric begin his journey?

Problem 2 – How soon after the arrival of the signal at Event F does Eric arrive at Event G?

Problem 3 – How soon after Erica sees the detonation at Event E does she get the message from Eric that he has started his journey?

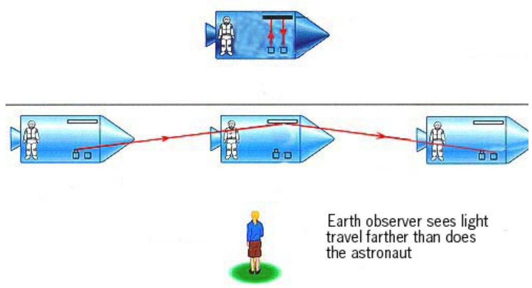


Problem 1 – How soon after the obliteration of the asteroid does Eric begin his journey? Answer: Event C occurred at t= -4, and Event D occurred at t = 0, so Eric left **4 time units later.**

Problem 2 – How soon after the arrival of the signal at Event F does Eric arrive at Event G? Answer: Event F occurred at t=+11, and Event G at t = +12, so Eric arrived **+1 time units later.**

Problem 3 – How soon after Erica sees the detonation at Event E does she get the message from Eric that he has started his journey? Answer: Event E happened at t = +2, and Eric's message arrives at Event F at t= +11, so he arrives +9 time units after Erica sees the asteroid destroyed.

Note: If Eric were certain that the asteroid was evaporated without getting a confirmation light flash at Event D, he could have left at Event B. Students can figure out how soon after Event E he would have arrived in this case.



One of the most important ideas in relativity, which makes all of the calculations possible, is that certain quantities will have identical values no matter which observers observe them. These quantities are called **invariants**. One if these is the speed of light, the other of these invariants is the 4-dimensional distance, S^2, between two Events.

Recall that the 4-dimensional distance between two events with spacetime coordinates (t_1,x_1,y_1,z_1) and (t_2, x_2,y_2,z_2) is given by

$$S^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - (t_2-t_1)^2.$$

By using the invariant called the speed of light, c, we can actually re-write the time part of S^2 so that it has the units of a distance by multiplying it by c^2. Then we have, in standard form, the invariant spacetime distance S^2 defind by:

$$S^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2.$$

For convenience we can write this in 2-dimensions as

$$S^2 = (x_2-x_1)^2 - c^2(t_2-t_1)^2.$$

Problem 1 - Observer A using coordinate system (t,x) is holding a clock that measures the time interval $t_2-t_1=t$. It is located at the origin of its coordinate system so that $x_2-x_1=0$. What is the formula for S^2?

Problem 2 – Observer B using his own coordinate system (T,X) and watches Observer A moving at a speed of V, so that X = VT. What is the S^2 he measures?

Problem 3 – Both Observers will measure the same value for S^2. From Problems 1 and 2, solve for the time T observed by Observer B in terms of the time t measured by Observer A.

Problem 4 – Observer A measures the time interval on his clock as t = 10 seconds. If V = 90% the speed of light, what time interval, T, does Observer B see passing on Observer As clock?

Problem 1 - Observer A using coordinate system (t,x) is holding a clock that measures the time interval t2-t1=t. It is located at the origin of its coordinate system so that $x_2 - x_1 = 0$. What is the formula for S^2?

Answer: $\mathbf{S^2 = -c^2t^2}$

Problem 2 – Observer B using his own coordinate system (T,X) and watches Observer A moving at a speed of V, so that X = VT. What is the S^2 he measures?

Answer: $\mathbf{S^2 = (VT)^2 - c^2T^2}$

Problem 3 – Both Observers will measure the same value for S^2. From Problems 1 and 2, solve for the time T observed by Observer B in terms of the time t measured by Observer A.

Answer: $-c^2t^2 = V^2T^2 - c^2T^2$

$$t^2 = T^2 - (V^2/c^2)T^2$$

$$t^2 = T^2 (1 - (V^2/c^2))$$

$$\mathbf{T = \frac{t}{[1 - (V^2/c^2)]^{1/2}}}$$

Problem 4 – Observer A measures the time interval on his clock as t = 10 seconds. If V = 90% the speed of light, what time interval, T, does Observer B see passing on Observer As clock?

Answer: $T = \frac{10}{[1 - (0.9^2)]^{1/2}}$ so T = 2.29 x 10 minutes or **22.9 minutes**.

Note: The time measured on the clock carried by Observer A is called the <u>Proper Time</u> . This clock is not in the same reference frame as Observer B, so Observer B will measure a different time on his clock. By using the property of the invariance of S^2, we can relate the 'Proper' coordinates of Observer A to what Observer B will measure on his clock.

Compare this derivation with the one in Problem 33.

Time Distortion Near a Black Hole

Near a black hole, it isn't just space that is distorted, but time also behaves differently! It is hard for us to imagine that time-itself can be altered, or even what that can mean.

Although we can never travel backwards in time from the present moment, what we experience as the present moment depends on the frame of reference you are in, and the particular clocks you are using. This is what physicists mean by 'relativity', and the major discovery made by Albert Einstein in 1905.

In relativity theory, the meter sticks and clocks carried by one Observer do not measure the same things as the meter sticks and clocks carried by another Observer, if they are moving relative to each other, or if they are in different gravitational fields.

Suppose that we chose the variables (T,X) to represent the coordinates centered on Observer A, and (t,x) the time and space variables for Observer B. If the two observers are sitting in the same room and not moving, T=t and X=x, and they will not disagree about what they are measuring. But if Observer B is standing on Earth's surface, and Observer A is far away in space, the difference in gravity causes their measurements to disagree. Observer A looking at the clock carried by Observer B on Earth, will see Observer B's clock running slow.

According to Einstein's Theory of General Relativity, gravity causes time to be distorted near a black hole in a simple way that is determined by the formula

$$T^2 = \frac{t^2}{1-\left(\frac{r_s}{r}\right)}$$

where rs is the black hole's Schwarschild radius.

Problem 1 – What happens to T and t when both observers are very far away from the black hole?

Problem 2 – Suppose Observer B carrying the clock that measures time t, is located at a distance of 5.6 km from a 1 solar mass the black hole (rs = 2.8 km). What will Observer A, who is far away from the black hole, see as the passage of time on his clock T, if t=10 seconds pass on Observer Bs clock?

Problem 3 – How close to the surface of the black hole does Observer B have to be so that Observer A sees 10 minutes pass for every 10 seconds on Observer Bs clock if rs = 2.8 km. Give your answer in meters.

Problem 1 – What happens to T and t when both observers are very far away from the black hole?

Answer: In the limit as R becomes very large compared to the radius of the black hole, rs, T becomes more and more similar to t until 'at R=infinity' both clocks always read the same times.

Problem 2 – Suppose Observer B carrying the clock that measures time t, is located at a distance of 5.6 km from a 1 solar mass the black hole (rs = 2.8 km). What will Observer A, who is far away from the black hole, see as the passage of time on his clock T, if t=10 seconds pass on Observer Bs clock?

Answer: From the formula $T^2 = t^2 / (1-rs/R)$ we have

$$T^2 = t^2 / (1-2.8/5.6) \text{ so}$$

$$T^2 = t^2 / (1/2)$$

$$T = 1.41\ t$$

Then for t = 10 seconds on Observer Bs clock, the distant observer will see T = 14.1 seconds pass on his clock! Note: The event is happening in Observer Bs frame of reference, which we would call the Proper Time. It is being observed at a great distance by Observer A on another clock.

Problem 3 – How close to the surface of the black hole does Observer B have to be so that Observer A sees 10 minutes pass for every 10 seconds on Observer Bs clock if rs = 2.8 km. Give your answer in meters.

Answer: 10 minutes = 600 seconds, so we need to solve:

$$(600)^2 = (10)^2 / (1-rs/R)$$

$$1 – rs/R = 102/6002$$

$$1-rs/R = 0.000277$$

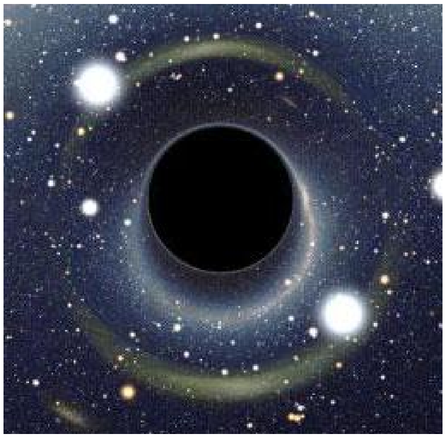
$$rs/R = 0.999723$$

$$R = 1.000277 \text{ rs.}$$

The horizon is located at R = 1.000000 rs, so we have to subtract this from Observer Bs distance.

Since rs = 2.8 km, R = (1.000277 – 1.000000)(280000) meters = **77 meters**.

Exploring Gravity Near a Black Hole



We have seen how two different Observers will measure different distance and time intervals if they are either in motion relative to each other (Special relativity) or if they are in different gravitational fields (General Relativity). We have to allow for these differences in any experiment, but luckily there is a precise mathematical relationship between the different Observers, so we can always figure out exactly what is happening to them.

When, from a great distance, we watch events happening near a black hole we expect to see the time recorded on a clock near a black hole 'slow down'. We have also learned that at the Event Horizon distance of a black hole, rs, something very weird happens to time and space. As viewed from a great distance by Observer B, the clock near the black hole carried by Observer A will slow to a stop as it seems to arrive at the Event Horizon. But what is Observer A near the black hole really experiencing?

In terms of Observer A's coordinates, (r,t), the force of gravity near a black hole is given by the familiar Newtonian equation:

$$F = \frac{GMm}{r^2} \qquad \text{where} \quad G = 6.67 \times 10^{-11} \text{ Newton meter}^2/\text{kg}^2$$

Suppose an Observer A has a mass m = 70 kilograms, and the black hole has a mass M = 1 billion solar masses = 1.9×10^{39} kg. As you complete the problems below, consider that on the surface of Earth, a 70 kg human feels a gravitational force equal to 686 Newtons.

Problem 1 - What is the force of gravity acting on Observer A at a distance just outside the Event Horizon at R = 5 billion kilometers?

Problem 2 - What is the force of gravity acting on Observer A at the event horizon at R = 2.8 billion km?

Problem 3 - What is the force of gravity acting on Observer A just inside the event horizon at R = 1.5 billion km?

Problem 4 - What is the force of gravity acting on Observer A at the center of the black hole called the Singularity at R=0?

Suppose an Observer A has a mass m = 70 kilograms, and the black hole has a mass M = 1 billion solar mass = 1.9×10^{39} kg

Problem 1 - What is the force of gravity acting on Observer A at a distance just outside the Event Horizon at R = 5 billion kilometers?

Answer: $F = (6.67\times10^{-11})(1.9\times10^{39})(70)/(5\times10^{14})^2$ = **35.5 Newtons**

Problem 2 - What is the force of gravity acting on Observer A at the event horizon at R = 2.8 km?

Answer: $F = (6.67\times10^{-11})(1.9\times10^{39})(70)/(2.8\times10^{9})^2$ = **113 Newto**ns

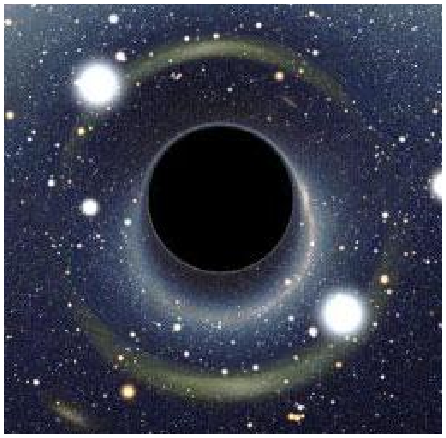
Problem 3 - What is the force of gravity acting on Observer A just inside the event horizon at R = 1.5 km?

Answer: $F = (6.67\times10^{-11})(1.9\times10^{39})(70)/(1.5\times10^{9})^2$ = **394 Newtons**

Problem 4 - What is the force of gravity acting on Observer A at the center of the black hole called the Singularity at R=0?

Answer: **As R goes to zero, F becomes infinite**.

Falling Into a Black Hole and Travel Time



We have seen how two different Observers will measure different distance and time intervals if they are either in motion relative to each other (Special relativity) or if they are in different gravitational fields (General Relativity). We have to allow for these differences in any experiment, but luckily there is a precise mathematical relationship between the different Observers, so we can always figure out exactly what is happening to them.

When, from a great distance, we watch events happening near a black hole we expect to see the time recorded on a clock near a black hole 'slow down'. As we have seen from a few examples, this is not because gravity is making the clock mechanically or electrically run slower. It is because the nature of time itself has altered and become more complex.

We have also learned that at the Event Horizon distance of a black hole, rs, something very weird happens to time and space. As viewed from a great distance by Observer B, the clock near the black hole will slow to a stop as it seems to arrive at the Event Horizon. But what is Observer A near the black hole really experiencing?

Although Observer B will see that it takes Observer A an infinite amount of time to reach the event horizon, the time, t, recorded on Observer As clock to fall to a distance of R from a black hole is given by the formula

$$T = \left(\frac{4M}{3c}\right)\left(\frac{r}{r_s}\right)^{\frac{3}{2}}$$

where c is the speed of light (3×10^{5} km/s), and rs is the event horizon radius in km.

Problem 1 - For a black hole with a mass of M = 1 sun, how long does it take Observer A to fall from a distance of 1000 km to the event horizon if rs = 2.8 km?

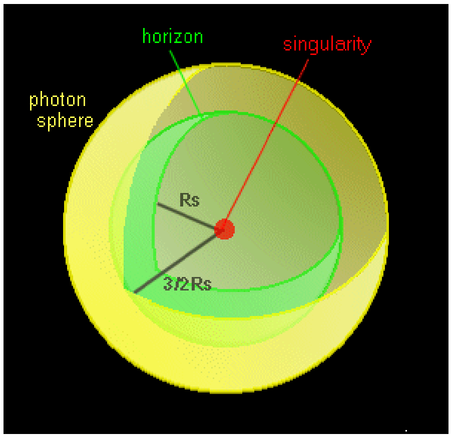
Problem 2 - For a supermassive black hole with a mass of M = 1 billion suns, how many hours does it take Observer A to fall from a distance of 5 billion km to the event horizon if rs = 2.8 billion km?

Problem 1 - For a black hole with a mass of M = 1 sun, how long does it take Observer A to fall from a distance of 1000 km to the event horizon if rs = 2.8 km?

Answer: $(2/3)\ (2.8\ \text{km}/300000)(1000/2.8)^{3/4}$ = **0.0005 seconds**

Problem 2 - For a supermassive black hole with a mass of M = 1 billion suns, how many hours does it take Observer A to fall from a distance of 5 billion km to the event horizon if rs = 2.8 billion km?

Answer: $(2/3)\ (2.8\ \text{billion km}/300000)(5/2.8)^{3/4}$ = 9611 seconds or **2.7 hours!**



As you approach a black hole, you will notice many strange optical illusions as you get close to the event horizon, and assuming you can survive the intense gravitational 'tidal' forces that are trying to rip you apart.

A simple formula lets you calculate how long your trip will take to get from the event horizon to the Singularity at the center.

T = 0.000015 M seconds

where M is the mass of the black hole in units of our sun's mass.

Once you reach the event horizon, there is no longer any hope for you. You cannot slam-on your rockets and try to break free. You can't even go into an orbit and just hang-around until someone can rescue you. There is no amount of rocket energy or force that will let you leave, because to do so you will have to travel faster than the speed of light! All you can do is watch your clock, which will tell you about the next most important event in your life; Your death! Your next destination is the Singularity.

Problem 1 – You just passed across the event horizon of a 1 billion solar mass black hole. How long do you have to live?

Problem 2 – This happens to be a Schwarschild black hole defined by the 4-dimensional distance formula shown below. What happens to this formula when r < rs?

$$S^2 = \left(1 - \frac{rs}{r}\right)(r_2 - r_1)^2 - \frac{(t_2 - t_1)^2}{1 - \frac{rs}{r}}$$

Problem 3 – Outside the black hole, you calculate that the interval between two Events gives $S^2 > 0$ and is space-like. Inside the black hole, using your same clock and meter stick, you find two Events that are separated by the same amount of time and space. What happens to this interval, S^2, inside the black hole when r < rs?

Problem 1 – You just passed across the event horizon of a 1 billion solar mass black hole. How long do you have to live?

Answer: T = 0.000015 (1 billion) = 15,000 seconds or **4.2 hours**.

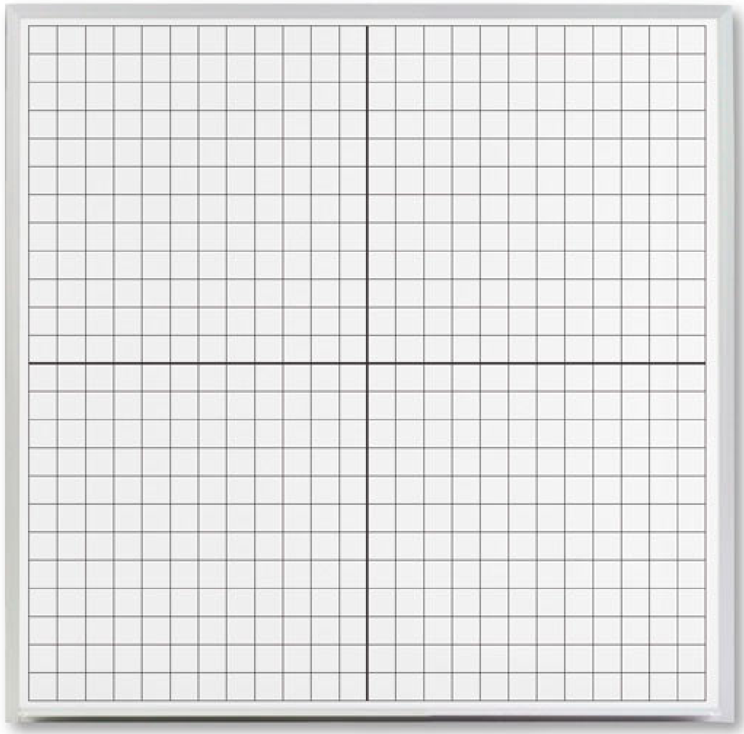
Problem 2 –What happens to this formula when r < rs?

Answer: The first term contributed by spacial differences becomes negative and the second term, which come from time differences, becomes positive.

Problem 3 – Outside the black hole, you calculate that the interval between two Events gives $S^2 > 0$ and is space-like. Inside the black hole, using your same clock and meter stick, you find two Events that are separated by the same amount of time and space. What happens to this interval, S^2, inside the black hole when r < rs?

Answer: Because the signs of the two terms that contribute to S^2 have reversed ,what you originally measured as a space-like distance outside the event horizon at r=rs, now becomes a time-like distance inside the black hole!

A Star Collapsing into a Black Hole



When a massive star explodes as a supernova, its core region implodes and collapses to higher and higher densities. Eventually the entire core mass falls inside its own Event Horizon, and a black hole is formed.

The above grid represents a spacetime grid where time increases vertically, and the radial distance to the center of the star increases along the horizontal axis. Draw the spacetime diagram for the collapsing star, and the formation of its black hole, by following the directions below. All coordinates are given as (t,r) below.

Step 1 – Draw the worldline for the geometric center of the star between events (-12,0) and (+12,0).

Step 2 – Draw the events representing two diametrically opposite points on the surface of the star at A (-12,-10) and B(-12,+10).

Step 3 - Draw the events representing the surface of the star at C(-6,-5) and D(-6,+5).

Step 4 - Draw the events representing the surface of the star arriving at its black hole radius at E(-2,-2) and F(-2,+2).

Step 5 – Draw the creation of the Singularity at event G(0,0)

Step 6 – Draw the future location of the event horizon and Singularity at events H (+12,-2) I (+12,0) and J(+12,+2).

Step 7 – Connect the events with the appropriate worldliness that pass through them.

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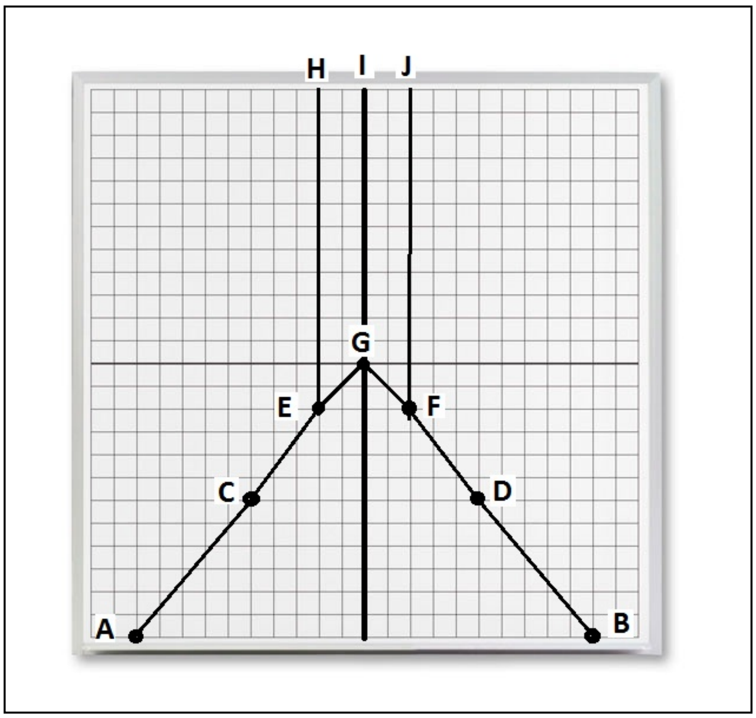
Step 5 – Draw the creation of the Singularity at event G(0,0)

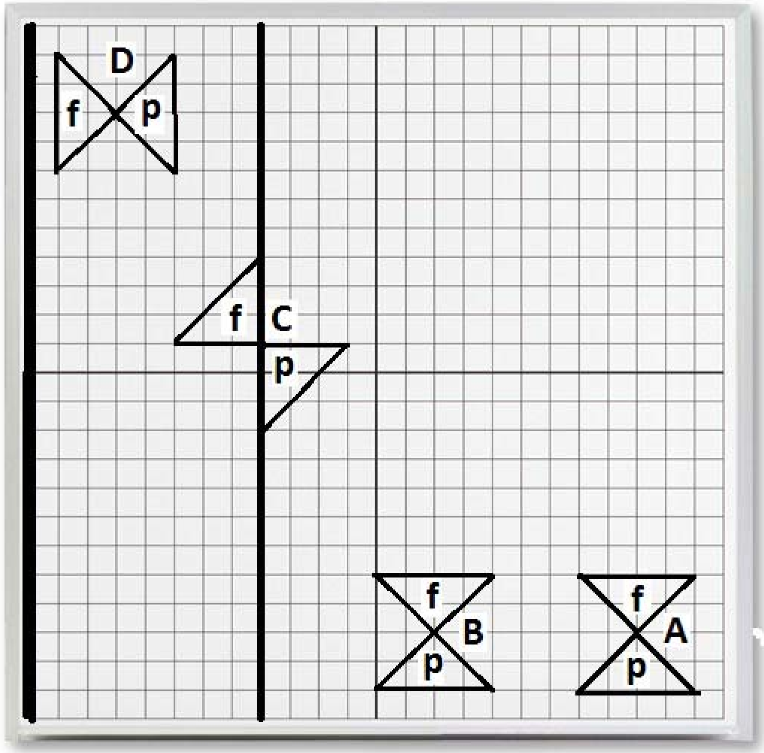
Step 6 – Draw the future location of the event horizon and Singularity at events (+12,-2) (+12,0) and (+12,+2).

Step 7 – Connect the events with the appropriate worldliness that pass through them.



Note: During the collapse of the star, the speed of the surface outside the horizon approaches the speed of light but does not exceed it. This diagram is not meant to be a perfect representation of the collapse due to the curvature of spcetime, however students can verify that the intervals between AC, CE and BD, DF are all time-like, so they lead to consistent worldlines of material particles (surface of star). Also, the event horizon does not exist until t=-2 when the stars mass has actually fallen inside this radius.

Light Cones Inside and Outside a Black Hole



The path of particles and light rays is determined by the geometry of spacetime near the black hole, and this leads to several 'strange' things. The strangest of these is the rotation of the light cone.

The above grid represents a spacetime grid where time increases vertically from 0 to +24, and the radial distance to the center of the star increases to the right along the horizontal axis from 0 to +24. The Singularity is on the far-left edge at r=0, and the event horizon is located along the worldline at r=+8. The light cones for four events, A, B, C and D are also shown. The letter 'p' indicates the past light cone and 'f' is the future light cone for each event.

Problem 1 – If Event A is on the worldline of an astronaut who is far, far, far away from the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Problem 2 – If Event B is on the worldline of an astronaut very close to the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Problem 3 - If Event C is on the worldline of an astronaut who has just arrived at the event horizon. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Problem 4 - If Event D is on the worldline of an astronaut inside the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the Singularity?

Problem 1 – If Event A is on the worldline of an astronaut who is far, far, far away from the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: At a great distance from the black hole, the astronaut can completely avoid falling into the black hole because there are vastly more worldliness available to him, and directions in space to travel.

Problem 2 – If Event B is on the worldline of an astronaut very close to the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: The astronaut still has many choices but now more than half of all possible worldlines that go through this event will intersect with the black hole.

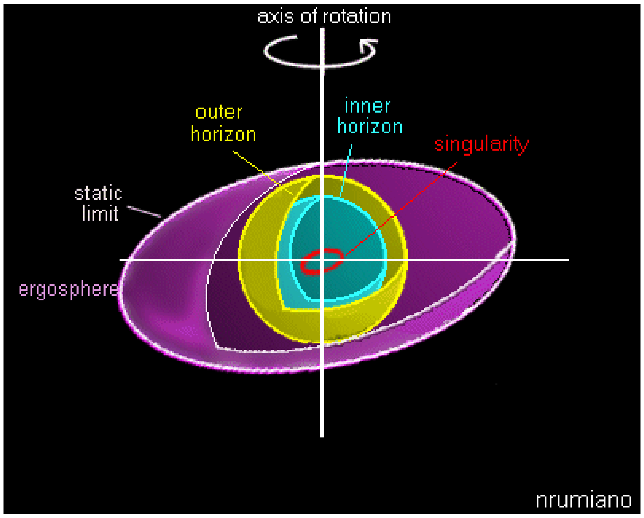
Problem 3 - If Event C is on the worldline of an astronaut who has just arrived at the event horizon. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: At the event horizon, there are no worldliness available to him in his future light cone that do not end inside the black hole. At this distance, the astronaut cannot even go into orbit because he is well inside the Last Stable Orbit for matter. At the horizon, not even photons have a stable orbit.

Problem 4 - If Event D is on the worldline of an astronaut inside the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the Singularity?

Answer: Every worldline that is future ward of this event ends on the Singularity. His past light cone faces the horizon, but without time travel as an option, he cannot retrace events in his past light cone in order to exit the black hole once inside.

Black Holes that Rotate



From planets to stars to galaxies, most objects in the universe rotate, so it is not surprising that rotating black holes will, be common in the universe. The mathematics that describes rotating 'Kerr' black holes was first worked out by the New-Zealand mathematician Roy Kerr in 1963 using Einstein's Theory of General Relativity. This was the same theory that Karl Schwarschild used in 1916 to work out the mathematics of non-rotating 'Schwarschild' black holes.

Like all things that spin, Kerr black holes will have a rotation axis, and will be slightly flattened along this axis, and bulge out in the equatorial plane. This is different from Schwarschild black holes which do not rotate and are perfect spheres.

The inside of a Kerr black hole is more complicated because, instead of having one event horizon, it has two. Also, instead of the Singularity being a mathematical point at r=0, it is deformed through rotation into a 1-dimensional ring in the equatorial plane!

For a Schwarschild black hole, there was a Last Stable Particle Orbit, called the Static Limit, and the Kerr black hole also has this, but with a difference. Because the Kerr black hole rotates, if a particle travels between the static limit and the outer event horizon, although it cannot enter into a stable orbit, it will pick up energy. It can then escape from the black hole carrying off some of the rotational energy from the Kerr black hole.

Between the outer horizon and the inner horizon, unlike for a Schwarschild black hole, a traveler can enter this region, but can also travel on worldliness that avoid the Singularity, so that you can exit this black hole!

Between the inner event horizon and the Singularity is another region of spacetime that you can safely traverse. Unless you approach the ring singularity in the equatorial plane, you can still avoid it, and even continue your journey back outside the black hole!

In the problems below, it is convenient to use c = 1 and G = 1 so that instead of $r = 2GM/c^2$ for the horizon radius of a Schwarschild black hole we just have r = 2M.

Problem 1 – The equation for the static limit is given by $r = M + (M^2 - a^2\cos^2\theta)^{1/2}$ Graph the shape of the static limit for a 1 solar mass black hole (M=1) with the maximum possible amount of spin (a=M). How big is a Kerr black hole compared to a Schwarschild black hole of the same mass.

Problem 2 - The outer and inner horizons are given by $r_o = M + (M^2 - a^2)^{1/2}$ and $r_i = M - (M^2-a^2)^{1/2}$. Graph these horizons for the same black hole as in Problem 1.

Problem 1 – If Event A is on the worldline of an astronaut who is far, far, far away from the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: At a great distance from the black hole, the astronaut can completely avoid falling into the black hole because there are vastly more worldliness available to him, and directions in space to travel.

Problem 2 – If Event B is on the worldline of an astronaut very close to the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: The astronaut still has many choices but now more than half of all possible worldlines that go through this event will intersect with the black hole.

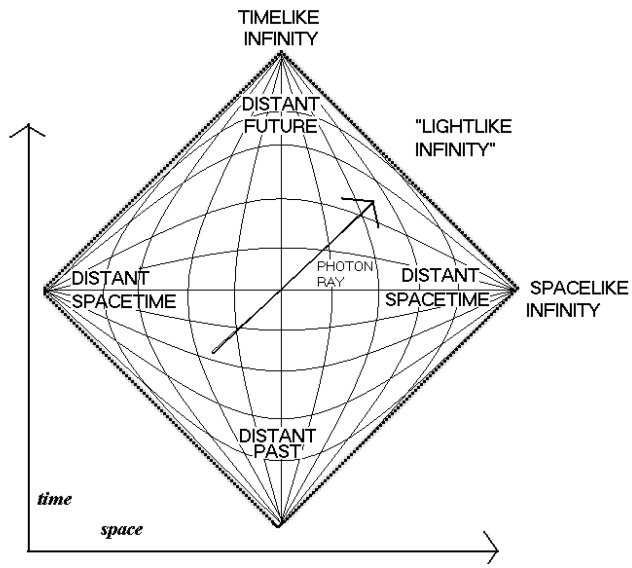
Problem 3 - If Event C is on the worldline of an astronaut who has just arrived at the event horizon. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: At the event horizon, there are no worldliness available to him in his future light cone that do not end inside the black hole. At this distance, the astronaut cannot even go into orbit because he is well inside the Last Stable Orbit for matter. At the horizon, not even photons have a stable orbit.

Problem 4 - If Event D is on the worldline of an astronaut inside the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the Singularity?

Answer: Every worldline that is future ward of this event ends on the Singularity. His past light cone faces the horizon, but without time travel as an option, he cannot retrace events in his past light cone in order to exit the black hole once inside.

Exploring Penrose Diagrams



When we drew our 2-dimensional spacetime diagrams, we could only draw a small portion of the true extent of spacetime, which is infinite in space and infinite in time. This is a problem for drawing black holes because black hole event horizons only form after a very long time as viewed by someone far outside the horizon.

Physicist Roger Penrose developed a new kind of diagram for spacetime that lets you see all locations and times, even the ones at infinity!

A Penrose diagram is a particular way to draw spacetime that lets you include a number of important features such as time and space are infinite, and the light cones for all events are represented by 45-degree lines on the diagram. This is called a conformal mapping because, although it may geometrically distort some features off the geometry, it leaves other desirable features intact. In this case, we want to keep all light cones as 45-degree cones across the entire diagram.

If the center point of the hyperbolic grid is ‘Now’ at the origin X=0, note how the space and time intervals become smaller and smaller as you move to the corners of this diagram. These corners represent ‘asymptotic infinity’ plotted as single point. If you move left or right you encounter the two points that represent the negative (-x) and positive (+x) space-like infinities. If you move vertically along the time axis, you encounter the past (-t), time-like infinity and the future (+t) time-like infinity.

Problem 1 – Suppose you were located at the positive, space-like infinity, and you sent a light signal into the future. What feature on the Penrose diagram represents this light signal, and where does it arrive?

Problem 2 – Suppose that at the past, time-like infinity you sent a light signal into the future. What feature in the Penrose diagram represents this light signal, and where does it end up?

Problem 3 – Suppose that you selected any point within the Penrose diagram and drew a future light-cone. Are there any locations on the Penrose diagram where the future light cone would not include the future time-like infinity? What does this mean, physically?

Problem 4 - Suppose that you selected any point within the Penrose diagram and drew a future light-cone. Are there any locations on the Penrose diagram where the future light

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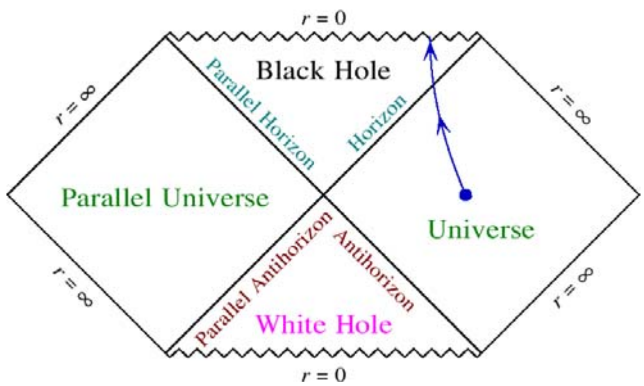
Problem 4 - Suppose that you selected any point within the Penrose diagram and drew a future light-cone. Are there any locations on the Penrose diagram where the future light cone would not include at least one of the space-like infinities? What does this mean, physically?

Penrose Diagram of a Schwarschild Black Hole



Unlike ordinary spacetime, which we can draw with an ordinary Penrose diagram, the spacetime near and inside a black hole is more complicated, and includes new regions of space and time, which behave differently than the ones outside the black hole.

For a Schwarschild black hole, we have the outside universe far from the event horizon, then we have the spacetime inside the event horizon, which includes the Singularity at r=0.

When we drew the previous spacetime diagram for a Schwarschild black hole, it included the fact that, upon entering the event horizon, the light cones rotate by 90 degrees. As a result, the Singularity would be encountered for every worldline in the future light cone of the traveler. The Penrose diagram shows this in a simple way, and lets us keep the orientation of the future light cone of the traveler, always pointed upward in the diagram!

Note that, because the sawtooth line representing the Singularity is drawn parallel to the space-like (horizontal) axis, it is called a space-like singularity. These have the unpleasant property that all worldliness entering the event horizon will terminate on the singularity so they cannot be avoided. There is no 'corner' of space that you could go to to escape your Destiny. The Penrose diagram shows this because the future light cone of the Traveler, drawn at any point on his worldline inside the event horizon faces the Singularity. All of his possible worldliness within his future light cone, traveling at speeds slower than light, all terminate on the Singularity.

Because of mathematical symmetry, the complete Penrose spacetime diagram uncovers a mysterious 'parallel universe', and also predicts a space-like white hole Singularity. The white hole event horizon (antihorizon) is one where particles inside the white hole may <u>only exit</u>, which mirrors the 'one way entry' event horizon of the black hole.

Problem 1 – Given that Travelers may only move within their future light cones, is there any way for a Traveler in the right-hand universe to visit the left-hand universe?

Problem 2 – Can you come up with worldliness for a Traveler from each universe, to enter the same black hole and shake hands just before they die at the Singularity?

Problem 3 – What must happen in order for the two travelers to meet at exactly the right moment inside the black hole?

Problem 4 – Is there any way that you could prove the existence of the Other Universe, and relay this information to your colleagues?

Problem 1 – If Event A is on the worldline of an astronaut who is far, far, far away from the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: At a great distance from the black hole, the astronaut can completely avoid falling into the black hole because there are vastly more worldliness available to him, and directions in space to travel.

Problem 2 – If Event B is on the worldline of an astronaut very close to the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

Answer: The astronaut still has many choices but now more than half of all possible worldlines that go through this event will intersect with the black hole.

Problem 3 - If Event C is on the worldline of an astronaut who has just arrived at the event horizon. What can you say about the future worldline of this astronaut and whether she can avoid falling into the black hole?

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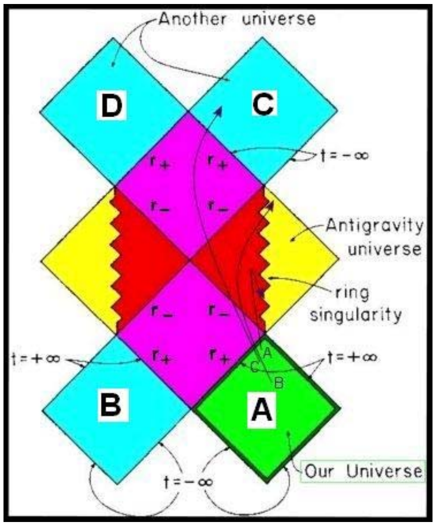
Problem 4 - If Event D is on the worldline of an astronaut inside the black hole. What can you say about the future worldline of this astronaut and whether she can avoid falling into the Singularity?

Answer: Every worldline that is future ward of this event ends on the Singularity. His past light cone faces the horizon, but without time travel as an option, he cannot retrace events in his past light cone in order to exit the black hole once inside.

Penrose Diagram of a Kerr Black Hole



Because a rotating, Kerr-type black hole has four different spacetime regions and a ring Singularity, the Penrose diagram for this spacetime is more complicated:

1 - External spacetime far from black hole, and outside the Outer Event Horizon (r+) colored green and blue.

2 - Spacetime region between the Outer Event Horizon (r+) and the Inner Event Horizon (r-) colored purple.

3 - Spacetime inside the Inner Event Horizon (r-) containing the ring Singularity colored red

4 - Anti-gravity universe on the other side of the ring singularity colored yellow.

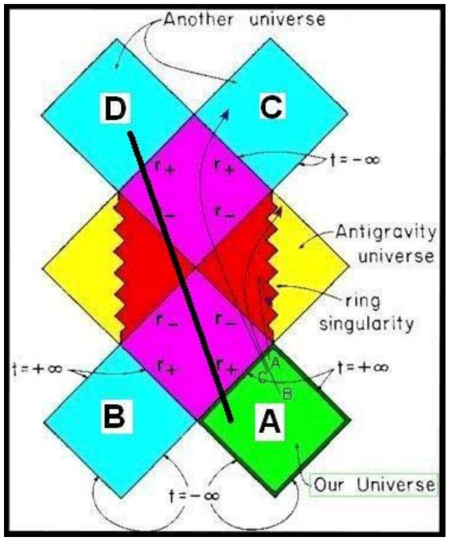
As for all Penrose diagrams, light rays travel on 45-degree lines and light cones point up (future) or down (past) and have an opening angle of 90 degrees. Unlike a Schwarschild black hole, a Traveler can completely avoid the ring singularity if he enters the rotating black hole along its rotation axis. In front of him would be the circular Ring of Death, and he can even head directly for the center of this ring, but will exit the black hole in a bizarre 'anti-gravity' universe very different from his own. In this universe, the mathematics of General Relativity predict that gravity will be repulsive not attractive!

Problem 1 - There are four universes connected by this Kerr black hole: A, B, C and D. By drawing the proper worldliness and light cones, what path might take you from Universe A to Universe D?

Problem 2 - What possible world line might take you from Universe A to Universe B?

Diagram Credit (http://jila.colorado.edu/~ajsh/insidebh/index.html)

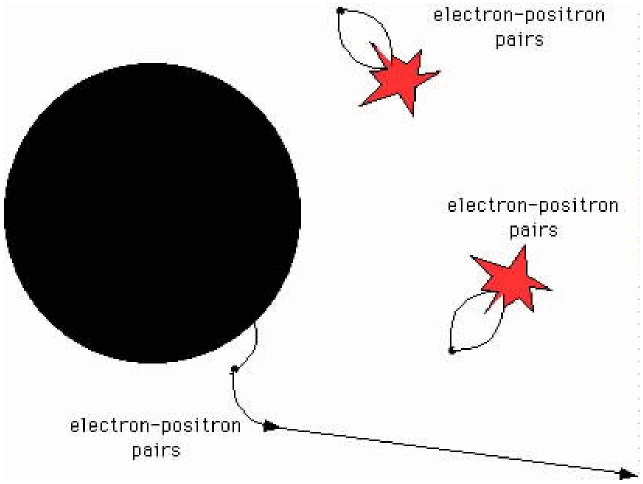
Problem 1 - There are four universes connected by this Kerr black hole: A, B, C and D. By drawing the proper worldliness and light cones, what path might take you from Universe A to Universe D?



Draw a line so that it always lies inside its light cone. No segments along the world line can have a slope that exceeds 1.0 (45 degrees), which is the speed of light.

Problem 2 - What possible world line might take you from Universe A to Universe B?

Answer: **There is no worldline connecting these universes because the traveler would have to leave his future light cone and move backwards in time.**



In 1973, Stephen Hawking deduced that rotating black hole can evaporate and lose mass, thanks to the quantum mechanical properties of 'empty' space.

Pairs of electrons and anti-electrons are constantly appearing and disappearing in space. If this happens near the event horizon, one particle escapes, while the other carries 'negative mass' into the black hole. This causes the black hole to lose mass.

The formula for the evaporation time of a black hole with a mass of M in kilograms is given by

$$t = \frac{10256\pi^2 G^2 M^3}{hc^4}$$

The formula for the temperature of a black hole with a mass of M in kilograms is given by

$$T = \frac{hc^3}{16\pi^2 GMk}$$

where $\pi = 3.141$, and Boltzman Constant k = 1.3806503 × 10^{-23} m^2 kg s^{-2} K^{-1}. Planck's Constant h = 6.628x10^{-34} Joules sec, and the Newtonian Constant of Gravity G = 6.67x10^{-11} Newton $Meter^2/kg^2$.

Problem 1 – By substituting the values for the physical constants into the two formulae above, show that t(seconds) = 8.39 x 10^{-17} M^3 and T (Kelvins) = 1.23x10^{23} / M where M is the mass of the black hole in kilograms.

Problem 2 - The universe has an age of 13.7 billion years. What is the mass of a black hole that, by now, should have completely evaporated if it had formed at the Big Bang? (Assume 3.1 x 10^7 seconds = 1 year)

Problem 1 – By substituting the values for the physical constants into the two formulae above, show that t = 8.39 x 10^{-17} M^3 and T = 1.23x10^{23} / M where M is the mass of the black hole in kilograms.

Answer:

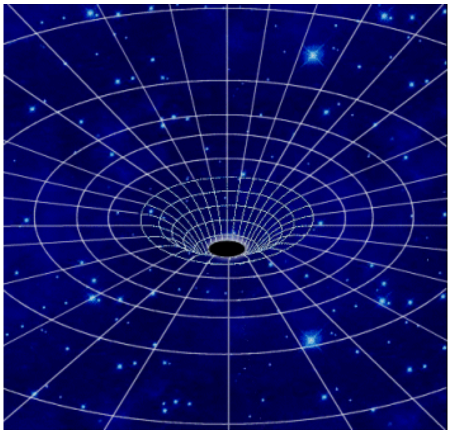
$$t = \frac{10256\pi^2 G^2 M^3}{hc^4} = \frac{10256(3.141)^2(6.67x10^{-11})^2 M^3}{(6.626x10^{-34})(3.0x10^8)^4} = 8.39 \times 10^{-17} M^3$$

$$T = \frac{hc^3}{16\pi^2 GMk} = \frac{(6.626x10^{-34})(3.0x10^8)^3}{16(3.141)^2(6.67x10^{-11})(1.38x10^{-23})M} = 1.23x10^{23} / M$$

Problem 2 - The universe has an age of 13.7 billion years. What is the mass of a black hole that, by now, should have completely evaporated if it had formed at the Big Bang? (Assume 3.1 x 10^7 seconds = 1 year)

Answer: t(universe) = 13.7 billion years x (3.1 x 10^7 sec/year) = 4.25 x 10^{17} seconds, then 4.25 x 10^{17} seconds = 8.39 x 10^{-17} M^3 and solving for M we get **M = 1.7 x 10^{11} kilograms**. This is 170 million tons, or about the mass of a very small asteroid!

Working with Spacetime Near a Black Hole



Near a black hole, Einstein's Theory of General Relativity predicts that space will be come greatly distorted. Some people like to call this the 'warping' of space!

Einstein's theory was studied by Martin Schwarschild in 1916 and he found a precise mathematical formula that describes this space warpage in the case of a black hole that is not rotating.

Remember that we discussed how the Pythagorean Distance Formula has to be changed when the space between two points is dilated along one of more directions. We obtained the formula

where g_1, g_2 and g_3 were constants. This is only a special case of a more general kind of distortion where the amount of dilation changes from point to point! This this case, instead of g_1, g_2 and g_3 being constants like 1.4, 25.213 or 1045623.18952, they could each be functions of the location of each point like $g_1(x,y,z)$, $g_2(x,y,z)$ and $g_3(x,y,z)$. Imagine how complicated life would be if you tried to figure out your car's travel distance in the city if the distances between houses and stores changed in this way!! Sadly, this is exactly the case when you are exploring the neighborhood of a black hole. Luckily the functions g1, g2 and g3 are very simple, because the distortion in space near a 'Schwarschild' black hole only depends on your 1-dimensional' distance from the center of the black hole, symbolized by the variable R! For two points located at $R = R_1$ and $R = R_2$ we have the distance formulas:

Ordinary 'Flat' Space: $D^2 = a\,(R_2-R_1)^2$ where a = 1.000

Black Hole space: $D^2 = (R_2-R_1)^2 / (1-Rs/R)$ where Rs = Event Horizon

Problem 1 – For a black hole with a mass of 1 sun, Rs = 2.8 kilometers. What is the distance formula for the space outside the event horizon of this black hole?

Problem 2 – Suppose that R is given in multiples of the horizon radius so that R = 3 means R = 3 x 2.8 = 8.4 kilometers. Compare the distance formula predictions for Ordinary and Black Hole space. For a black hole with Rs = 10 km, and $R = (R_1 + R_2)/2$, what is the distance, D, between two points located at A) R_1=100000, R_2=200000 B) R_2=25 and R_1=15, and C) R_2=11 and R_1 = 10?

Problem 1 – For a black hole with a mass of 1 sun, Rs = 2.8 kilometers. What is the distance formula for the space outside the event horizon of this black hole?

Answer: $D^2 = (R_2-R_1)^2 / (1-10/R)$

Problem 2 – Suppose that R is given in multiples of the horizon radius so that R = 3 means R = 3 x 2.8 = 8.4 kilometers. Compare the distance formula predictions for Ordinary and Black Hole space. For a black hole with Rs = 10 km, and $R = (R_1 + R_2)/2$, what is the distance, D, between two points located at A) R_1=100000, R_2=200000 B) R_2=25 and R_1=15, and C) R_2=11 and R_1 = 10?

Answer: A) **Ordinary: D = 100000**

Black Hole: R = (2000+1000)/2 = 150000, (or 1,500,000 kilometers) then

$D^2 = (200000-100000)^2 / (1-10/150000)$

$D^2 = (100000)^2 / (1)$

D = 100000

B) Ordinary: D = (25-15) = **10.0**

Black hole: R = (25+15)/2 = 20, (which equals 200 kilometers) then

$D^2 = (25-15)^2 / (1-10/200)$

$D^2 = (10)^2 / (19/20)$

$D^2 = 20 (100)/19$

$D^2 = 105.$

D = 10.26

C) Ordinary: D = (11 – 10) = **1.0**.

Black Hole: R = (11+10)/2 = 10.5, (which equals 105 kilometers) then

$D^2 = (11-10)^2 / (1-10/10.5)$

$D^2 = (1)^2 / (0.047)$

$D^2 = 21.0$

D = 4.6

So, comparing 1.0 to 4.6, we see that when you are just outside the event horizon (R close to Rs=10 kilometers), the separation between nearby points becomes <u>much larger</u> than what you would have predicted using the Pythagorean Theorem in 'flat' space!

Additional Mathematical Resources About Black Holes

It is a challenge to find mathematical resources about black holes that are not too advanced, but that still give the student and teacher some idea of how to think about them more quantitatively. Ironically, if you GOOGLE 'black hole math' you will quickly discover that this very math guide is among the Top-7 options! Clearly there is a big need for this kind of resource that can be used by the K-12 community. Here are just a few others that might be helpful:

Death Spiral Around a Black Hole - Hubble Discovery
http://hubblesite.org/newscenter/archive/releases/2001/03

Chandra Observatory Detects Event Horizon -
http://chandra.harvard.edu/photo/2001/blackholes/

Ask the Astronomer: 87 FAQs About Black Holes:
http://www.astronomycafe.net/qadir/abholes.html

Imagine the Universe: Black Hole FAQs
http://imagine.gsfc.nasa.gov/docs/ask_astro/black_holes.html

New Evidence for Black Holes from NASA
http://science.nasa.gov/headlines/y2001/ast12jan_1.htm

A trip into a black hole
http://antwrp.gsfc.nasa.gov/htmltest/rjn_bht.html

A note from the Author,

Since they first came into public view in the early 1970s, black holes have been a constant source of curiosity and mystery for millions of adults and children. No astronomer has had the experience of visiting a classroom, and NOT being asked questions about these weird objects with which we share our universe.

Beyond answering that they are objects with such intense gravity that even light cannot escape them, we tend to be at a loss for what to say next. The mathematics of Einstein's General Theory of Relativity are extremely complex even for advanced undergraduates in mathematics, so we tend to resort to colorful phrases and actual hand-waving to describe them to eager students.

Actually, there are many important aspects of black holes that can be readily understood by using pre-algebra (scientific notation), Geometry (concepts of space and coordinates, Pythagorean Theorem), Algebra I (working with simple formulae), and Algebra II (working with asymptotic behavior).

This book is a compilation of some of my favorite elementary problems in black hole physics. They will introduce the student to the important concept of the event horizon, time dilation, and how energy is extracted from a black hole to create many kinds of astronomical phenomena. Some of these problems may even inspire a student to tackle a Science Fair or Math Fair problem!

Black holes are indeed something of a mystery, but many of their most well-kept secrets can be understood with just a little mathematics. I hope the problems in this book will inspire students to learn more about them!

Sten Odenwald

Frequently Asked Questions about Black Holes

The FAQs below have been assembled by Dr. Odenwald at his award-winning *Ask the Astronomer* website resource at 'The Astronomy Café' (http://www.astronomycafe.net), which has been in operation since 1995. Visitors submitted hundreds of questions about black holes to Dr. Odenwald. The answers were posted at his *Ask the Astronomer* website, and selected popular questions were re-printed in his two books <u>The Astronomy Café</u> (W.H. Freeman, 1998) and <u>Back to the Astronomy Café</u> (Westview Publishing, 2003), which are available at Amazon.com. Here are a few of the most popular questions asked by visitors:

<u>Is there really a black hole at the center of the Milky Way?</u>

Astronomers have suspected this for over a decade, and there have been many reports in the news media of 'proof' that such a thing existed. Now we have the most direct proof imaginable. The motion of the stars and gas near a region called SgrA* (pronounced 'sadge A star') in the constellation Sagittarius is faster than what you can account for if you just added up the mass of the gas and stars you see and worked out their 'gravitational speed'. This kind of motion has now been directly detected in a star called 'S2' located in orbit around the black hole. An international team of astronomers led by Rainer Schodel at the Max Planck Institute for Extraterrestrial Physics observed this star over the course of 10 years as it completed 2/3 of an orbit around a region centered on SgrA*. The star S2 approaches the central black hole to within three times the distance between the Sun and Pluto - while traveling at no less than 11 million miles per hour. The fast-moving star takes about 15 years to complete a single orbit. They used the Adaptive Optics Instrument on the 8.2-meter Very Large Telescope in Chile and captured a sequence of high-resolution images of this star as it orbited the black hole. The black hole, however, was not visible. Star velocities and variable X-ray emissions from the center of the Milky Way had indicated a compact source of radio waves, SgrA* can only be a black hole. SgrA* is the closest object to the actual Galactic Center. From the parameters of the elliptical orbit of S2 around the black hole, the investigators derived an enclosed mass between 2 to 5 million solar masses. This small volume of space, and large mass, completely excludes the possibility of a massive star cluster. Only a black hole fits the data, and at last settles this issue once and for all.

<u>If you stuck your hand into a black hole, what would happen?</u>

You can't just park outside a black hole and do this experiment because there are no stable orbits possible for objects within an 'arms length' of the black hole's event horizon. Your only option to get close enough is to plot a course that lets you fall into the black hole. General relativity says that as you approach the event horizon, nothing unusual is going on at all. All you would be feeling are the tremendous tidal forces of the black hole, and these could be very substantial across a distance equal to your arm's length. In fact for a black hole produced by a star, by the time you are 40 miles from its horizon, the difference in the gravitational pull between your chest and hand is a thousand Earth gravities. Your hand would be torn from your body. But by then that will be the least of your problems. For the gargantuan black holes that lurk in the cores of many galaxies, the tidal forces near their event horizons are so minor that you might not even realize you had reached the event horizon at all, provided of course that the black hole was not surrounded by a lethal accretion disk spewing out x rays and gamma rays. You would gently pass across the horizon with your body intact, but with a very dismal future awaiting you as you continue to fall into the Singularity located a few billion miles away. In either case because of relativity, distant observers would see your hand 'wink out' followed moments later by the rest of you!

Do black holes ever get full?

No, black holes never get full. The size of a black hole is fixed by the amount of mass it contains. For each increment of mass the size of the Sun, its radius grows by 1.7 miles, so if a black hole has a mass of 10 times the Sun, its radius is 17 miles. The more matter that falls into it, the bigger it gets, so in fact it never fills up inside. If a black hole consumes matter at too ferocious a rate, the radiation pressure generated by the infalling matter provides tremendous resistance to the flow of matter. The rate at which matter can fall into a black hole is regulated to what is called the Eddington Accretion Rate. For a solar-mass black hole, this rate equals the mass of our Sun consumed every 100 million years.

What happens to matter that falls into a black hole?

Outside the black hole, it depends on what form the matter takes. If it happens to be in the form of gas that has been orbiting the black hole in a so-called accretion disk, the matter gets heated to very high temperatures as the individual atoms collide with higher and higher speed producing friction and heat. The closer the gas is to the black hole and its Event Horizon, the more of the gravitational energy of the gas gets converted to kinetic energy and heat. Eventually the atoms collide so violently that they get stripped of their electrons and you then have a plasma. All along, the gas emits light at higher and higher energies, first as optical radiation, then ultraviolet, then X-rays and finally, just before it passes across the Event Horizon, gamma rays.

If the matter is inside a star that has been gravitationally captured by the black hole, the orbit of the star may decrease due to the emission of gravitational radiation over the course of billions of years. Eventually, the star will pass so close to the black hole that its fate is decided by the mass of the black hole. If it is a stellar-mass black hole, the tidal gravitational forces of the black hole will deform the star from a spherical ball, into a football-shaped object, and then eventually the difference in the gravitational force between the side nearest the black hole, and the back side of the star, will be so large that the star can no longer hold itself together. It will be gravitationally shredded by the black hole, with the bulk of the star's mass going into an accretion disk around the black hole. If the black hole has a mass of more than a billion times that of the sun, the tidal gravitational forces of the black hole are weak enough that the star may pass across the Event Horizon without being shredded. The star is, essentially, eaten whole and the matter in the star does not produce a dramatic increase in radiation before it enters the black hole.

Once inside a black hole, beyond the Event Horizon, we can only speculate what the fate of captured matter is. General relativity tells us that there are two kinds of black holes; the kind that do not rotate, and the kind that do. Each of these kinds has a different anatomy inside the Event Horizon. For the non-rotating 'Schwarzschild black hole', there is no way for matter to avoid colliding with the Singularity. In terms of the time registered by a clock moving with this matter, it reaches the Singularity within a few micro seconds for a solar-massed black hole, and a few hours for a supermassive black hole. We can't predict what happens at the Singularity because the theory says we reach a condition of infinite gravitational force. For the rotating ' Kerr Black holes', the internal structure is more complex, and for some ingoing trajectories for matter, you could in principle avoid colliding with the Singularity and possibly reemerge from the black hole somewhere else, or at some very different future time thousands or billions of years after you entered.

Where does matter go after it is squeezed through a Singularity?

That's something that everyone would like to know, but for now we don't have a verifiable theory of gravity (called by some people quantum gravity theory) that can take us beyond the Singularity state predicted by general relativity. The Singularity, a state of infinite gravitation, is a wall that we can't penetrate right now because general relativity which is painting our description of it cannot be tested under these conditions to confirm that it is really giving us correct answers. Physicists are convinced that this state is not real, but only a sign that general relativity has broken down. Some physicists including Stephen Hawking believe that matter entering a black hole in our universe, will emerge as matter spewed forth from a so-called white hole in another universe. The mathematics seem to suggest this, but there are many difficulties in interpreting such theories without knowing whether they are accurate representations of our physical world or not. Our ability to experimentally test a theory has proven itself to be our only sure way of separating truth from mathematical fiction and keeping us on the right course in an astonishingly complex universe. So far, testing these new theories seems almost as hard as creating the theories in the first place.

What happens near the Event Horizon of a black hole?

The region of space very close to a black hole can be a very messy environment. Mathematically it is a very simple region dominated by the black holes' outer boundary called the event horizon. Matter is trying to flow into the black hole at the equator, and through friction, is being heated to thousands and perhaps even millions of degrees. Magnetic fields dragged in by the flow get amplified and concentrated. They eventually pop out of the gas disk like solar prominences and flares, releasing bursts of x-ray, and perhaps even gamma ray, energy. Clumps of clouds and asteroids orbit the black hole in seconds and are shredded by gravity, to produce flickering 'Quasi-Periodic' bursts of x-ray light as they slide into the horizon zone and are finally lost from our universe. Does fusion happen just outside a black hole? Not very easily. To produce thermonuclear reactions you need temperatures in excess of about one million degrees to cause protons to collide with deuterium and produce tritium. Deuterium fusion is the lowest energy fusion reaction we know about. To get to one million degrees, protons have to collide with deuterium nuclei at speeds of about 290,000 miles per hour. Near stellar-mass black holes, gravitational tidal forces are substantially higher and it might be possible for some of the gas to reach these kinds of conditions, at least in a limited volume of space near the horizon, perhaps even in 'solar' flares that pop out of the magnetized accreting matter. A signature of this would be a black hole emitting bursts of gamma rays.

If nothing can escape a black hole, why do they still emit X-rays?

It is true that once matter or energy passes within the so-called Event Horizon of a black hole, that it can never turn around and get back out. However, in the real world, a lot can happen to matter as it approaches the Event Horizon. Commonly, matter falls into what is called an accretion disk which orbits the black hole. Material orbits the black hole within this disk, but if it happens to be gas and dust, this matter experiences friction and the disk heats up as some of the orbital energy of the gas is converted into heat. The closer the disk material is to the black hole, the more rapidly it orbits so that the greater is the heating effect. Just before it reaches the Event Horizon, this disk matter can be heated by friction to thousands of degrees which is enough to produce X-rays. Even higher temperatures approaching a million degrees can occur which can produce gamma rays. This disk radiation, being outside the black hole, is what we detect as we look at black holes.

What exists between the Event Horizon and the Singularity of a black hole?

For black holes formed by collapsing stars, the body of the star itself exists in this zone so far as outside observers are concerned. In fact, as seen from a distance, the surface of the star is 'frozen' just outside the event horizon a few million million millionths of a centimeter from the horizon where the relativistic Doppler factor is enormous.

Mathematically, for black holes old enough that the stellar material has collapsed all the way into the singularity, the region between the horizon and the singularity is occupied by a spacetime where the time and space coordinates are reversed from those of the outside world. What this means in terms of what you experience is unknown. Other more complex conditions can occur of the black hole is rotating. In that case the singularity becomes a ring around the center of the black hole. You can pass through the center, but the tidal gravitational field would be lethal in all likelihood. In nearly all cases there would be gravitational radiation rattling about, and this would cause distortions in spacetime that would probably lead to spectacular optical distortions.

Is the edge of a black hole a sharp, smooth boundary?

As seen from the vantage point of an outside observer, the edge is extremely sharp. It is a mathematically perfect, spherical surface where light gets infinitely redshifted.

To an observer falling into the black hole, the boundary may be much more complicated than our 'axi-symmetric' mathematics would suggest. The horizon could be a turbulent surface rippling with gravitational radiation, or it might dissolve into a fuzzy quantum state at even finer scales of scrutiny. Someone falling into the horizon would experience NOTHING PECULIAR, except that once they cross this mathematical surface by even one millimeter, they can never turn back to escape the black hole. The 'Event Horizon' is a most peculiar concept in physics. Still, it is a theoretical idea which needs to be studied with real data before we can feel confident that we understand it. Getting the data, however, would be fatal! So, what's a physicist to do?

What is space ‘doing’ near the Event Horizon?

So far as local physics is concerned, that is what a freely-falling observer it depends on what kind of black hole it is...whether it is rotating or not.

The details are very complicated, but the common feature is that nothing peculiar really happens. You do not feel a 'lurch' or 'space discontinuity' or something. The gravitational tidal force will continue to grow as you near the horizon and pass through it. Your clocks and meter sticks will still look normal if they are not too big compared to the tidal field. Just outside the horizon, photons will be able to go into orbit at a 'last-stable' orbit radius. Don't worry about what the distant observers see. That's completely irrelevant to what you will be experiencing locally.

Another thing you will discover as you approach the event horizon is that you are no longer able to go into a stable orbit. Because of a relativistic phenomenon known as the Lenz-Thirring Effect or 'frame dragging', space-time is no longer a static thing, but becomes very obviously dynamic in character. It's a bizarre process that basically causes space-time to be dragged along with the rotation of the black hole so that you can never go into an orbit that is close to the horizon. All you can do is be 'dragged' across the event horizon. This exotic process has been detected in careful X-ray studies of nearby black holes as we witness the spiraling-in of matter.

What happens to time as you fall into a black hole?

We are guided in our understanding of the interior of black holes by the theory of general relativity developed by Albert Einstein in 1915, and in particular, the mathematics of the complete, relativistic equation for gravity and space-time. This theory describes in considerable mathematical detail, both those regions of space-time that are accessible to humans, and those that are accessible only by individual observers but not distant observers. For black holes, distant observers will see only the outside of the event horizon, while individual observers falling into the black hole will experience quite another 'reality'. General relativity predicts that for distant observers outside the horizon, they will experience the 3 space-like coordinates and 1 time-like coordinate as they always have.

For someone falling into a black hole and crossing the horizon, this crossing is mathematically predicted to involve the transformation of your single time-like coordinate into a space-like coordinate, and your three space-like coordinates into 3 time-like coordinates. Along any of these 3 former space-like coordinates, they now all terminate on the singularity, and your experiencing them as time-like now means that you have no control over your destiny because all choices always terminate on the singularity...at least in the case of a non-rotating black hole. The coordinate which used to measure external time, now has a space-like character which affords you some wiggle room, but dynamically, in terms of these new reversed space and time coordinates, you find that no stable orbits about the singularity are possible no matter what you try to do. Without any stable orbits, and the inexorable free fall into the singularity, relativists often refer to this as the collapse of space-time geometry.

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