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A Bayesian via Laplace Approximation on Log-gamma Model with Censored Data

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Abstract

Log-gamma distribution is the extension of gamma distribution which is more flexible, versatile and provides a great fit to some skewed and censored data. Objective: In this paper we introduce a solution to closed forms of its survival function of the model which shows the suitability and flexibility towards modelling real life data. Methods: alternatively, Bayesian estimation by MCMC simulation using the Random-walk Metropolis algorithm was applied, using AIC and BIC comparison makes it the smallest and great choice for fitting the survival models and simulations by Markov Chain Monte Carlo Methods. Findings/conclusion: It shows that this procedure and methods are better option in modelling Bayesian regression and survival/reliability analysis integrations in applied statistics, which based on the comparison criterion log-gamma model have the least values. However, the results of the censored data have been clarified with the simulation results.

Keywords: bayesian analysis, censored data, Laplace approximation, log-gamma distribution, simulation, survival analysis

1. Introduction

Bayesian method approach is applied to model censored Survival data analysis its increasingly active research in the last few decades in response to a more refined statistical tools to analysed complex data structures and parameters (Lindley & Smith, 1994). This method is applied to the log-gamma model analytically simulates the model parameters which approximates generally by obtaining the posterior summaries of the density parameters using "LaplacesDemon" package in R software.

The shorthand X~log-gamma (a,b) is used to indicate that the random variable. The Log-gamma distribution (Consul & Jain, 1971) is defined in the following way having a probability density function (PDF) given as:

$$f(t,a,b) = e^{(at)} e^{-(et/a)} \left[a^{(b)} \Gamma(b) \right]^{-1} \tag{1}$$

Survival function is given as:

$$S(t,a,b) = 1 - F(t,a,b) \tag{2}$$

Corresponding the reliability function:

$$R(t,a,b) = \Gamma(a,t/b) \tag{3}$$

The hazard function which is the ratio of the PDF and the survival function is written as:

$$h(t,a,b) = \frac{f(t,a,b)}{S(t,a,b)} = \frac{e^{(at)}e^{-(et/a)}*[a^{(b)}\Gamma(b)]}{1-F(t,a,b)} \tag{4}$$

The likelihood of the log-Gamma survival model is given by:

$$Li = \prod_{i=1}^{n} pr(ti,\delta i) = \prod_{i=1}^{n} [f(t,a,b)]^{\delta i}[S(t,a,b)]^{1-\delta i} \tag{5}$$

$$Li = \prod_{i=1}^{n} pr(ti,\delta i) = \prod_{i=1}^{n} [e^{(at)}e^{-(et/a)} * [a^{(b)}\Gamma(b)]]^{\delta i}[1-F(t,a,b)]^{1-\delta i} \tag{6}$$

Where "a" is the shape parameter and "b" is the scale parameter, and they are all positive greater than zero (Consul & Jain, 1971; Bilal, Khan, Hasan & Khan, 2003; Lindley & Smith, 1994). The corresponding likelihood function for right censored data as where δ_i is a censoring indicator variable which takes value 1 if it's observed while censored for and otherwise. To evaluate characteristics of posterior summaries of the model is actually an intricate case (Koul, Susarla &Van, 1981; Lawless, 2003). Our main objective in this work is to show and illustrate the fitting of Log-gamma model in R programming code which will be used to approximate the posterior probabilities with the Laplace method and simulation in R (Consul & Jain, 1971; Lindley & Smith, 1994; Kalbfleisch & Prentice, 2002). Log-Gamma distribution is the extension of Gamma distribution which is more flexible, versatile and provides a great fit to some skewed and censored data. In this paper we introduce a solution to closed forms of its survival function of the model which shows the suitability of its flexibility by Bayesian estimation of the MCMC simulation using the Random-Walk. Metropolis algorithm was applied. However, the results of the censored data have been clarified with the simulation results.

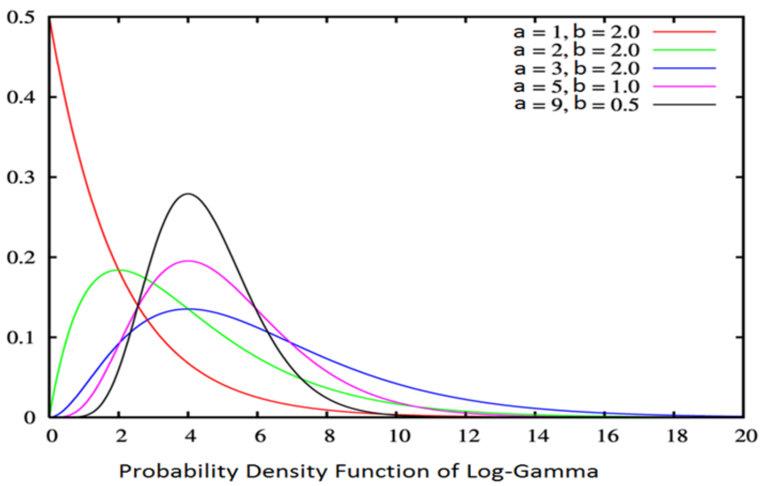

Figure 1.

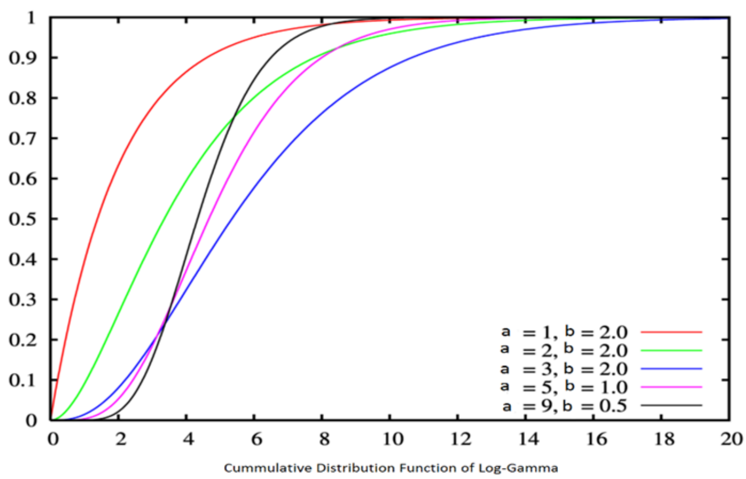

Figure 2.

In this paper, an attempt has been made to the stated objectives below:

(1) To apply a Bayesian analysis model for specifying the prior and likelihood functions.

(2) To developed an R programming code which will be used to approximate the posterior probabilities with the Laplace method and simulation.

(3) To provide a plots and tables for representation of the posterior results.

2. Related Work

According to authors like (Akhtar & Khan, 2014), they proposed an R package for coding and solving high dimensional Bayesian analysis models for modelling real life and censored survival data. They also stated that the Laplace approximation is an attractive numerical approximation and will continue to develop and suggest new methods. In 2003, authors (Khan & Khan, 2013), proposed a procedure and a LaplacesDemon code for solving a Laplace approximation based on Gamma model where the made mention and stated that this procedure generally solves high dimensional attributes and intricate datasets in real life. They also attempted to present a clear true value for the marginal posterior contrary to the asymptotic theory by the frequentist approach. Recently, authors (Bilal, Khan, Hasan & Khan, 2003; Khan & Bhat, 2002; Khan & Khan, 2013), introduces a procedure and present some useful codes where they consider a right censoring in survival regression modelling using Bayesian approach with Laplace method estimating the parameters of the Weibull model and they point out that a survival data are not symmetric in nature but are generally positive skewed following the Weibull properties as very few models violate this wide usage flexible model. Also, authors like (R Development Core Team, 2012; Sheila & Khan, 2013; Schwarz, 1978; Roberts & Rosenthal, 2009; Polson & Scott, 2012; Miller, 1997; Miller, 1976) explain and contribute their quota on how their method estimate some models with censored data in Bayesian survival analysis.

3. Laplace's Approximation

According to (Statistic at, 2013) Bayesian methods depend on non-informative prior models which provides same output with the non-Bayesian procedures. The limit to where a non-informative prior model is validate to be an objective rely on the available in the data provide, as the sample size n increases, for the prior distribution on posterior decreases (Tierney, Kass & Kadana, 1989). The recent most suitable method for evaluating density functions integrals involved in the posterior densities which was computed by Bayes' theorem shown as:

$$\pi(\vartheta \mid y) = P(y \mid \vartheta)\pi(\vartheta)(P(y))^{-1} \ \alpha \ P(y \mid \vartheta)\pi(\vartheta) \quad (7)$$

Where, $P(y) = \int P(y \mid \vartheta)\pi(\vartheta)d\vartheta$ is called the evidence or marginal likelihood. It has been pioneered and examined by "Pierre-Simon, marquis de Laplace in 1806" (Khan & Bhat, 2002) is called the Laplace transformation. (Buckley & James, 1979, Khan & Bhat, 2002) This function is noted by denoting w (θ) be any positive unimodal function with mode. By expanding the w(θ) using a second-order Taylor series about $\hat{\theta}$ will be given as:

$$\log w(\vartheta) \approx \log w(\hat{\vartheta}) - \frac{(\vartheta - \hat{\vartheta})'V(\vartheta - \hat{\vartheta})}{2}$$

$$V_{ij} = -\left[\frac{\delta^2}{\delta_i \vartheta \delta_j \vartheta} \log w(\vartheta)\right]_{\vartheta = \vartheta} \quad (8)$$

By taking the exponent of both sides of equation (9), we obtained:

$$w(\vartheta) \approx w(\hat{\vartheta}) \exp \frac{1}{2}((\vartheta - \hat{\vartheta})'V(\vartheta - \hat{\vartheta}))\, d\vartheta \quad (9)$$

$$\vartheta \mid y \quad N(\vartheta, V^{-1})$$

Though, if w(θ) = P(y|θ)π(θ), then (9) above will be the mode in the obtaining the posterior results by LP method:

$$w(\vartheta) = P(y \mid \vartheta)\pi(\vartheta)$$

Also by integration of (9), it clearly gives us:

$$\int w(\vartheta) \approx \int w(\hat{\vartheta}) \exp \frac{1}{2}((\vartheta - \hat{\vartheta})' V (\vartheta - \hat{\vartheta})) \, d\vartheta = \frac{w(\hat{\vartheta})}{\det(V / 2\pi)} \quad (10)$$

The above equation (10) is a non-negative function is called a **Laplace's method**. In this work, we develop the LaplacesDemon code algorithm using R software by Laplace Approximation function (Akhtar & Khan, 2014; Bernardo, Degroot & Valencia, 1980; Collet, 1994; Kimber, 1990).

3.1 Half-Cauchy Prior Distribution

The half-Cauchy density function with one scale parameter "a" alpha is given by:

$$f(t, a) = \frac{2a}{\pi(t^2 + a^2)}, \quad t, a > 0$$

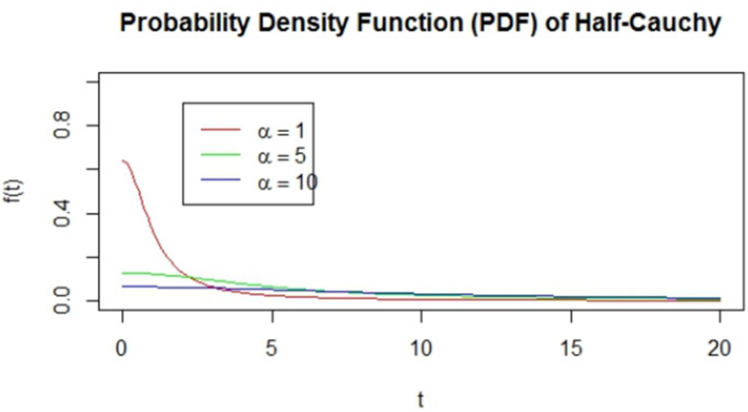

Figure 3.

It is weakly informative prior distribution for a scale parameter. Otherwise, "a" stands for alpha is recommended to be set to be just a little larger than the expected standard deviation, as a weakly informative prior distribution on a standard deviation parameter. Value, The half-Cauchy distribution does not has mean and variance, but its mode is equal to 0 having the" a=25"as a default. (Akhtar, 2014; Bernardo, 1980; Bilal, Khan, Hasan & Khan, 2003) Suggested, the uniform prior distribution, where its compulsory in estimation but half-Cauchy is a better option used as a non-informative prior (Polson & Scott) showing its graph below as follows.

4. Bayesian Analysis: Simulation with Laplace's Demon

Based on some reviews in the area of approximating a Laplace distribution in the literature which has a very effective response for decades and also, in recent years based on Log-gamma estimation of parameters using different approach like Bayes estimate, MLE, Lindley, Newton Raphson's method of optimization etc. (Akhtar, 2014; Bilal, Khan, Hasan & Khan, 2003; Khan & Bhat, 2002; Khan & Khan, 2013). Actually to find the posterior results summaries of such functions with their mean and variances, it is a very intricate case to handle, more especially when more covariate were involved as incorporate variables. In such cases, we use the Bayesian frame-work approach using the Metropolis-Hastings sampling algorithm in MCMC methods to solve and find the posterior result.

As an alternative method to solve intricate integrals using simulation technique by direct method of simulation suggested by (Buckley & James, 1979, Kimber, 1990), in intricate purposes where by MCMC methods is used.

5. Application of Censored Data

The Log-gamma distribution as a parametric family is however used in censored survival modelling, with two parameters, shape and scale parameter. We analyze a data from (R Development Core Team, 2012), known as the leukaemia data having 23 observations with three (3) variables of observation namely: time, status and group (R Development Core Team, 2012).

T ~ Lgamma(a,b), where a,b > 0

The codes are as follows:

```
Failure-time < −C(9, 13, 13, 18, 23, 28, 31, 34, 45, 48, 161, 5, 5, 8, 8, 12, 16, 23, 27, 30, 33, 43, 45)
Censor < −C (1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, Re P (1, 5), 0, Re P (1, 6))
M < −23
K < −1
P < −matrix (1, N-Row = M, N-column = K)
T < −log (failure-time)
Monames < − C(”Lpa”,”Sha pe”).
Parnames < − As. Parm-Names (List (beta = Rep (0, L), logsp = 0)).
My-Dataset < −List (Q = Q, N= N, P = P, Monames =Monames, Parnames = Parnames, Censor =Censor, T=T).
Initial-values < −C (Re P (0, Q), Log (L))
```

6. Model Specification

The Log-gamma model with two parameters alpha and beta is also has almost same properties with the original gamma model stated as its suit the continuous and skewed data having a Weibull model property which is one of its sub-model and also fits a wide range failure-time data quite well. On the other side, it has a very good relationship with its sub-models and also enhances the use of its advantages in-terms of the identically independently distributed (iid) for some exponential variables in inferential statistics (Collet, 1997; Koul, Susarla & Van, 1981).

```
Model <− function (parm, data)
{
”Parameters”
beta<− param [1: dataQ]
shape<- exp (Param[dataQ + 1])
”Log-prior ”
beta-prior < −Sum (dnormv(beta, 0, 1500, Log = True )) shape-prior < −dhalf-Cauchy (Shape, 25, Log = True)
Scalee < −exp(Tcrosss Pr(datax,T(beta)))
”Log-likelihood”
Ll>Sum (dlgamma(data-T, shape, scale = scale, 1, Log = True ))
”Log-Posterior”
LP < −LL+ sum (beta-prior) + shape-prior
Model-Out < −listing (LP = LP, Dev = −2 ∗ LL, Monitor = c(LP, shape),
T-Hat = rlgamma(23, shape, scale = scale, 1), param = param)
Return (Model-out)
}
Init-values < −Giv (Model, My-data, N = 1500)
”Fitting the LP”
M1 <−Lp(Model, Initial-values, My-Data, Iterations = 15000)
Print (M1).
```

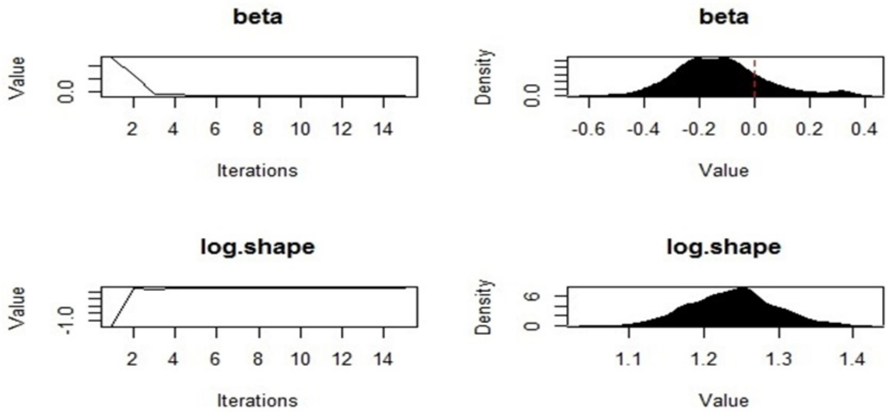

Figure 4.

In Figure 4, they clearly shows the plots of posterior densities of the parameters: beta, log-shape deviance and LP for the Log-gamma model of the survival regression fitting the distribution using Laplace approximation and LaplacesDemon which evidence shows from the plots that Laplace approximation is excellently resemble the model.

Table1. Asymptotic modal estimates for Standard Deviations of the Log-Gamma Parameters.

Parameter	Modal value	Standard Dev.	Lower Bound	Upper Bound
Betas	-0.166	0.144	-0.454	0.122
Log-Shape	1.238	0.054	1.130	1.347

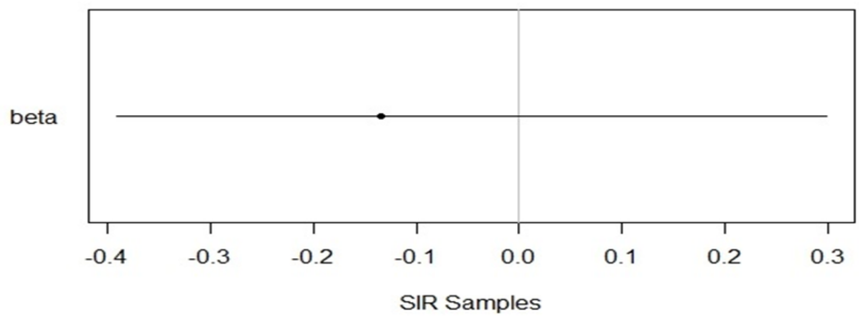

Figure 5.

Figure 5 shows the beta of the shape parameter in the figure for the censoring mechanism.

7. A Simulation Study

We initiate the use Random walk to simulate from the summary and the density of the target population by the proposing the following R codes as follows (Akhtar & Khan, 2014; Bernardo, Degroot & Valencia, 1980; Collet, 1994; Kimber, 1990).

Init-values < −As.Init-values (M1)

”Fitting LaplacesDemon”

M2 < − “LaplacesDemon” (“Model, Mydataa, Init-values”)

7.1 Bayesian Fit for the LaplacesDemon Method

LaplacesDemon package can be used for Bayesian and Non-Bayesian. This function Laplace Approximation maximizes the posterior (Buckley & James, 1979; Khan & Bhat, 2002; Khan & Khan, 2013; Kalbfleisch & Prentice, 2002; Tierney, Kass & Kadane, 1989). LaplacesDemon is an implementation of Markov chain Monte

Carlo tools. To use these functions specifying of a model, is probably the main idea with the prior of model parameters and data or simulation.

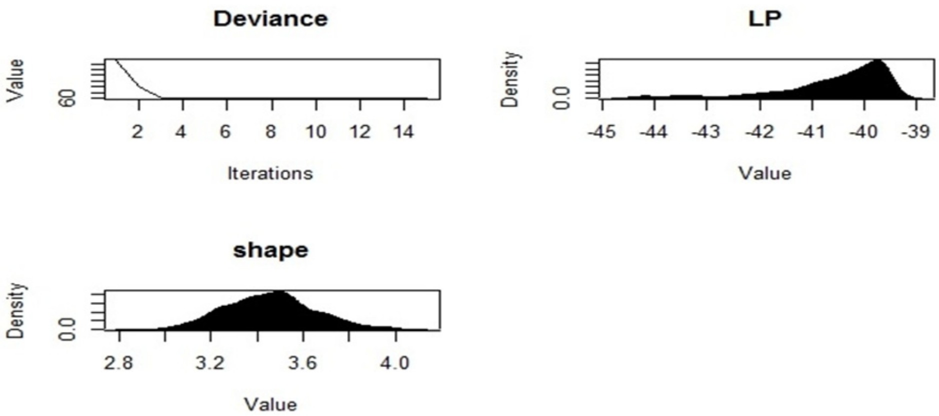

Figure 6.

In Figure 6, based on simulation results, the plots of posterior densities of the parameters: Deviance Shape and LP for the Log-gamma model of the survival regression fitting the distribution using Laplace approximation and Laplace Demon which evidence shows from the plots that Laplace approximation is excellently resemble the model showing the LP relatively goes a bit left-skewed but the shape and deviance were quite great this prove that the mean square error and Bias estimate are relatively small which a very good result.

Talbe 2. Posterior Mean Summaries for the Parameter Estimated By Simulation Using the Sampling Technique

Parameter	Modal value	StandardD	MCSE	ESS	LowerB	Median	UpperB
Betas	-0.113	0.153	0.005	1000	-0.380	-0.129	0.192
Log-Shape	1.238	0.059	0.002	1000	1.113	1.238	1.348
Deviance	62.776	2.128	0.067	1000	60.070	62.070	67.907
L.P	-40.601	1.064	0.034	1000	-43.166	-40.246	-39.525
Shape	3.458	0.203	0.006	1000	3.045	3.449	3.848

Table 2 (above) and Table 3(below) shows the simulation results summaries for the Log-gamma model where the matrices sampling and resampling algorithm LP approximation for effective sample size, and LB, Median, U B are 2.5%, 50%, 97.5% quantiles, respectively.

Table 3. Posterior Mean Summaries for the Parameter Estimated By Simulation Using the Sampling Technique and Stationary Samples

Parameter	Mode	SD	MCSE	ESS	LB	Median	UB
Beta	-0.122	0.150	0.005	882.4950	-0.396	-0.129	0.186
Log-shape	1.236	0.059	0.003	629.579	1.106	1.236	1.348
Deviance	62.714	2.146	0.089	742.106	60.641	62.037	68.706
L.P	-40.571	1.073	0.044	751.234	-43.572	-40.230	-39.534
Shape	3.447	0.202	0.009	630.256	3.024	3.443	3.934

8. Akaike's Information Criterion (AIC) And Bayesian Information Criterion (BIC)

Akaike (1974) suggested and introduced a suitable vast criterion (AIC) with some assumptions attached:

(a) A parametric distribution encompasses a true model.

(b) Its estimate using MLE and other methods, where the least value becomes the best model for selection

(Akaike, 1974), which is given given by:

$$AIC = 2\text{k} - 2\ln(\text{L}) \quad (12)$$

Schwarz (1978) also proposed the BIC criterion following some assumptions that render great impact to statistical methodology as:

(a) It has a constant independent prior vague.

(b) It checks the efficiency and complexity of the parameterized model in terms of intricacy.

(c) BIC [21], has a very close relation to AIC [2], in terms of model selection.

The Bayesian Information Criterion is formally defined as

$$BIC = -2\ln L + \widehat{\widehat{k\ln}}(n) \quad (13)$$

Where,

L= the likelihood function of the estimated model.

x= the observe dataset.

n= the number of samples.

k= the number of free parameters to be estimated.

Talbe 4. Comparison of Parametric sub-models with Log-gamma based on (AIC and BIC)

MODEL	AIC	BIC
Log-Gamma	42.140	41.134
Gamma	42.559	42.167
Weibull	43.246	44.789

The above table 4 shows the result of comparison between the sub models which indicates the Log-gamma model is having the smallest value among them clearly not prove to be the best model but based on the survival data used it makes it superior and better fit.

9. Conclusion

In this research we proposed an Rcode base on simulating and estimating censored survival data and initiate the use of R package Laplace's Demon (Khan & Bhat, 2002) that makes a great impact in Bayesian statistical inference. The log-gamma distribution was used as a Bayesian model to fit the censored data and simulation, where by important techniques were used like: Asymptotic approximation and direct simulation were implemented using the R package LaplacesDemon (Khan & Bhat, 2002). Also, the simulation results shows that the Mean square error of log-gamma model is least compare to other sub-models like (Weibull and gamma models) as well as the AIC and BIC with (42.140 and 41.134) making it the smallest and great choice for fitting the survival models and simulations by Markov Chain Monte Carlo Methods. It shows that this procedure and methods are better option in modelling Bayesian regression and survival/reliability analysis integrations in applied statistics (Lindley & Smith, 1994; Koul, Susarla &Van, 1981).

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