ON SOLVABILITY OF QUASILINEAR ELLIPTIC EQUATIONS IN LARGE DIMENSIONS

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ABSTRACT. We consider some class of quasilinear elliptic equations and show that if the dimension of the domain is large enough then the solutions to these equations exist independently of the right side's increasing rate.

1. Introduction

The theorem we present is obtained by quite standard means and it should not be mentioned here if it were not some strange, from the first look, effect.

As it is well known the problem

$$-\Delta u = |u|^{p-2}u, \quad u\mid_{\partial M}= 0,$$

here M is an m-dimensional star-shaped bounded domain, has nontrivial solutions of $H^1_0(M)$ provided $p < 2m/(m-2)$. And by Pohozaev's identity, if $p > 2m/(m-2)$ then it has not.

These facts convince us that the increasing rate of the equation's right side is critical for existence of a solution.

On the other hand one knows another factor that should impede the existence. This factor is a dimension of the domain M.

We consider some class of quasilinear elliptic equations and show that if the dimension of the domain is large enough then the solutions to these equations exist independently of the right side's increasing rate.

2. Main theorem

Let $M \subset \mathbb{R}^m$ be a bounded domain with smooth boundary $\partial M = \overline{M}\backslash M$.

For $x = (x_1, \dots, x_m)$ introduce the standard Euclidian norm $|x|^2 = \sum_{i=1}^m x_i^2$.

Suppose the domain M to be contained in a ball:

$$M \subseteq B_R(x_0, \mathbb{R}^m) = \{x \in \mathbb{R}^m \mid |x - x_0| < R\}.$$

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Introduce a Banach space

$$C^1_0(\overline{M})=\{v\in C^1(\overline{M})\mid v\mid_{\partial M}=0\}.$$

Assume a function $f:C^1_0(\overline{M})\to L^\infty(M)$ to be continuous.

The main object of our study is the following elliptic problem:

$$-\Delta u=f(u),\quad u\mid_{\partial M}=0. \tag{2.1}$$

Theorem 1. *Suppose there exists a constant λ such that for any*

$$v\in C^1_0(\overline{M}),\quad |v(x)|\le\lambda$$

the inequality

$$|f(v)|\le\frac{2m\lambda}{R^2} \tag{2.2}$$

holds almost everywhere (a.e.) in M.

Then problem (2.1) has a solution

$$u\in\tilde H^{2,r}(M)=H^{1,r}_0(M)\bigcap H^{2,r}(M),\quad r>m.$$

For example one can take $f(u)=(2+\cos(|\nabla u|^2))e^u$.

Let $M_m\subset\mathbb{R}^m$, be bounded domains with smooth boundaries and all these domains are inscribed in corresponding Euclidian balls with the same radius R.

Let a function $g:\mathbb{R}\to\mathbb{R}$ be continuous and define the mapping f as follows: $f(v)=g(v(x))$.

Consider problem (2.1) with given f in the domains M_m. We claim that in such a case problem (2.1) has a solution provided m is sufficiently large. Indeed, take a positive constant λ and observe that the function g is bounded in the closed interval $[-\lambda,\lambda]$, thus inequality (2.2) will certainly be fulfilled, if only the number m is sufficiently large.

To illustrate this effect consider an example:

$$-\Delta u=ce^u,\quad u\mid_{\partial B_1(0,\mathbb{R}^m)}=0, \tag{2.3}$$

constant c is positive.

In one dimensional case equation (2.3) can be integrated explicitly, however the corresponding integrals do not express by the elementary functions. Numerical simulation of these integrals shows that the problem (2.3) has a solution if and only if

$$c\le 0,87845\dots$$

On the other hand, applying Theorem 1 with $|v(x)|\le\lambda$ one has:

$$ce^v\le ce^\lambda\le 2m\lambda. \tag{2.4}$$

If

$$c\le 2e^{-1}m=m\cdot 0,73575\dots$$

then the second inequality of (2.4) has a solution with respect to λ.

So letting for example $c=1$ we see that problem (2.3) has no solutions in one dimensional case, and by Theorem 1 it has a solution for $m\ge 2$. To conclude this, notice that by Proposition 1 (see below), the solution to (2.3) is nonnegative.

3. Proof

Further arguments are quite standard: we use a version of the comparison principle.

Denote by $\Delta^{-1}h$ the solution of the problem

$$\Delta w = h \in H^{s,p}(M), \quad w\mid_{\partial M}= 0, \quad s \geq 0, \quad p > 1.$$

As it is well known the linear mapping $\Delta^{-1} : H^{s,p}(M) \to \tilde{H}^{s+2,p}(M)$ is bounded.

Construct a mapping G as follows:

$$G(v) = -\Delta^{-1} f(v).$$

We look for a fixed point of this mapping.

Previous assumptions imply that $G : C_0^1(\overline{M}) \to \tilde{H}^{2,r}(M)$ is continuous and by virtue of the embeddings:

$$\tilde{H}^{2,r}(M) \sqsubset \tilde{H}^{2-\delta,r}(M) \subset C_0^1(\overline{M}), \quad 0 < \delta < 1, \quad (1-\delta)r > m \tag{3.1}$$

($\sqsubset$ is a completely continuous embedding) the mapping $G : C_0^1(\overline{M}) \to C_0^1(\overline{M})$ is completely continuous.

Consider a function

$$U(x) = \frac{\lambda}{R^2}(R^2 - |x - x_0|^2).$$

This function takes positive values for $x \in B_R(x_0)$, attains its maximum at x_0:

$$\max_{B_R(x_0,\mathbb{R}^m)} U = U(x_0) = \lambda,$$

and satisfy the following Poisson equation:

$$-\Delta U = \frac{2m\lambda}{R^2}. \tag{3.2}$$

Let us recall a version of the maximum principle.

Proposition 1 ([1]). *IF $v \in H^1(M)$ and $\Delta v \geq 0$ then inequality $v(x) \leq 0$ a.e. in ∂M implies that $v(x) \leq 0$ a.e. in M.*

Lemma 1. *The mapping G takes a set*

$$W = \{w \in C_0^1(\overline{M}) \mid |w(x)| \leq \lambda, \quad x \in M\}$$

to itself. Furthermore, the set $G(W)$ is bounded in $\tilde{H}^{2,r}(M)$.

Proof. Since $-\Delta G(w) = f(w)$ by formula (3.2) one has:

$$\Delta(G(w) - U) = -f(w) + \frac{2m\lambda}{R^2} \geq 0$$

a.e. in M. Observing that $(G(w)-U)\mid_{\partial M}=-U\mid_{\partial M}\le 0$ by Proposition 1 we see: $G(w)\le U$ a.e. in M. The same arguments give: $-U\le G(w)$ a.e. in M. Finally a.e. in M we have:

$$|G(w)|\le U\le \max_{B_R(x_0,\mathbb{R}^m)} U=\lambda.$$

By assumption of the Theorem the set $f(W)$ is bounded in $L^\infty(M)$: $|f(W)|\le 2m\lambda/R^2$, consequently the set $\Delta^{-1}f(W)$ is bounded in $\tilde H^{2,r}(M)$. $\square$

Lemma 1 and formula (3.1) imply that the set $G(W)$ is precompact in $C^1_0(\overline{M})$. Observing that W is a convex set, we apply Schauder's fixed point theorem to the mapping $G: W\to W$ and obtain desired fixed point: $u=G(u)\in\tilde H^{2,r}(M)$.

The Theorem is proved.

References

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