An Application of Pontryagin's Maximum Principle in a Linear Quadratic Differential Game

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Abstract

This paper deals with a class of two person zero-sum linear quadratic differential games, where the control functions for both players are subject to integral constraints. The duration of game is fixed. We obtain the necessary conditions of the Maximum Principle and also optimal control by using method of Pontryagin's Maximum Principle. Finally, we discuss an example.

Keywords: Linear quadratic differential games, Optimal control, Pontryagin 's Maximum Principle, Payoff functional

1. Introduction

In this study, a class of two person zero-sum linear quadratic differential games (Basar & Olsder, 1999) is considered. The Pontryagin 's Maximum Principle is a powerful method for the calculation of optimal controls. Because of it has the important advantage not requiring previous evaluation of the payoff functional. Here we describe this method for a non-autonomous linear quadratic differential game and illustrate its use with an example. The present paper is closely related to the following works.

Note a two player linear quadratic differential game was investigated by (Jodar, 1991). In this work an explicit solution of a set of coupled asymmetric Riccati type matrix differential equations was found.

(Amato et al., 2002) considered a class of two player zero-sum linear quadratic differential games. They studied two cases, the finite horizon case and the infinite horizon case. In the finite horizon case, sufficient conditions for the existence of closed loop strategies were obtained based on the existence of the solution of suitable parameterized Riccati equations. Then the infinite horizon case was studied where the closed loop strategies are also required to guarantee asymptotic stability of the whole system.

A problem of linear quadratic optimization was investigated by (Rozonoer, 1999). In this study necessary and sufficient conditions for existence of optimal control for all initial positions were obtained. Using Pontryagin maximum Principle and Bellman's equations, some general hypotheses have been proposed.

(Sussmann and Willems, 1997) gave an historical review of the development of optimal control . They studied the development of the necessary conditions for a minimum, using the Euler-Lagrange equations to the work of Legendre and Weierstrass and, eventually, the maximum principle of optimal control theory.

(Lewis, 2006) studied the linear quadratic optimal control problem by using method of Pontryagin 's Maximum Principlewas in autonomous systems.

(Mou and Yong, 2006) was devoted to a thorough review of general two-person zero-sum linear quadratic games in Hilbert spaces.

In (Wang and Yu, 2010), the Maximum Principle for a new class of non-zero sum stochastic differential games was considered. A necessary and sufficient condition in the form of Maximum Principle for open-loop equilibrium point of the games was established.

The above results can be obtained using two classical approaches: Maximum Principle (Pontryagin et al., 1962) and Dynamic Programming (Bellman, 1957). However our work is base on Maximum Principle of Pontryagin. For a comprehensive review, see (Athans & Falb, 1966).

The paper is organized as follows: in Section 2, we describe the statement of the problem. In Section 3, we give some conditions and Theorem (fixed interval). In Section 4, we derive the main result and one example. Finally the conclusion is given in section 5.

2. Problem formulation

We consider the linear quadratic differential game described by

$$\dot{x}(t) = A(t)x(t) + B(t)(u_1(t) + u_2(t)), \quad x_0 = x(t_0) \tag{1}$$

and by the payoff functional

$$J(x_0, u_1, u_2) = \int_{t_0}^{t_1} L(x, u_1, u_2)dt \tag{2}$$

where

$$L(x, u_1, u_2) = \frac{1}{2}(x(t)^T Q_i(t)x(t) + u_1(t)^T R_{i1}(t)u_1(t) + u_2(t)^T R_{i2}(t)u_2(t)), \tag{3}$$

$x, x_0 \in \mathbb{R}^n$, $u_1, u_2 \in \mathbb{R}^m$, $A(t)$ and $B(t)$ are continuous matrices, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, L is a Lagrangian, $Q_i(t) \in \mathbb{R}^{n\times n}$, $R_{ii}(t) \in \mathbb{R}^{m\times m}$, $i = 1, 2$. Moreover, Q_i is assumed to be symmetric and R_{ii} are symmetric positive definite.

3. Conditions And Scheme Of The Method

In this section we consider a differential game governed by the equation

$$\dot{x}(t) = f(x(t), u_1(t), u_2(t)), \; x \in \mathbb{R}^n, \; u_1 \in \mathbb{R}^m, \; u_2 \in \mathbb{R}^m, \tag{4}$$

where the function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies conditions that ensures existence, uniqueness and extendability of the solution of the initial value problem for (4) with $x(t_0) = x_0$. The corresponding control functions $u_1(t), u_2(t)$ for Pursuer and Evader, respectively, must satisfy the following constraints.

$$\int_{t_0}^{t_1} |u_1(t)|^2 dt \leq \rho^2, \quad \int_{t_0}^{t_1} |u_2(t)|^2 dt \leq \sigma^2 \tag{5}$$

where ρ and σ are positive numbers.

3.1 Boltyanskii Tangent Cone To A Set At A Point

Let S be an arbitrary subset of $\mathbb{R}^n$ and $x_0 \in S$. The vector $v \in \mathbb{R}^n$ is called tangent vector to S at x_0 if and only if there exists a function $\phi : \mathbb{R} \to \mathbb{R}^n$ such that $x_0 + \epsilon v + \phi(\epsilon) \in S$ for all sufficiently small $\epsilon > 0$ and $\frac{\phi(\epsilon)}{\epsilon} \to 0$ as $\epsilon \to 0$. Clearly, if v is a tangent vector to S at x_0 and $k > 0$, then kv is also a tangent vector to S at x_0. We introduce Boltyanskii tangent cone to set S at a point x_0 by $C_S(x_0)$. For simplification, it is denoted by C. The dual cone $C^\perp$ of a cone C in the linear space $\mathbb{R}^n$ is defined as the set

$$C^\perp = \{x^* \in \mathbb{R}^n : \langle x^*, x\rangle \leq 0 \quad \text{for all } x \in \mathbb{R}^n\}.$$

3.2 Transversality Condition

It is an additional necessary condition for optimality for a problem with the terminal constraint. Instead of being of the form $x(t_1) = x_1$ considered form $x(t_1) \in S$, where S is a given subset of $\mathbb{R}^n$.

3.3 The Maximum Principle With Transversality Conditions For Fixed Time Interval Problems

We suppose that the following conditions are valid

C_1. n and m are nonnegative integers.

C_2. t_0 *and* t_1 are real numbers such that $t_0 < t_1$.

C_3. The following continuous functions

$$f : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$$

$$f(t, x, u_1, u_2) = (f_1(t, x, u_1, u_2), ..., f_n(t, x, u_1, u_2)) \in \mathbb{R}^n$$

and

$$L : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

$$(t, x, u_1, u_2) \to L(t, x, u_1, u_2)$$

are given.

C_4. For each $(t, x, u_1, u_2) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ the functions

$$f(t, x, u_1, u_2) = (f_1(t, x, u_1, u_2), ..., f_n(t, x, u_1, u_2)) \in \mathbb{R}^n$$

and

$$x \to L(t, x, u_1, u_2)$$

are continuously differentiable, and their partial derivatives with respect to the x coordinate are continuous functions of (t, x, u_1, u_2).

C_5. x_0 is a given point of $\mathbb{R}^n$ and S is a given subset of $\mathbb{R}^n$.

C_6. The set of all trajectory-controlled triple $\mathcal{TCT}$ define on $[t_0, t_1]$ is the set of all triples $(x(\cdot), u_1(\cdot), u_2(\cdot))$ such that

- $a.$ $u_i : [t_0, t_1] \to \mathbb{R}^m,\ i = 1, 2$, is a measurable bounded function. (see (5))
- $b.$ $x : [t_0, t_1] \to \mathbb{R}^n$, is an absolutely continuous function.
- $c.$ $\dot{x}(t) = f(t, x(t), u_1(t), u_2(t))$ for every $t \in [t_0, t_1]$.

C_7. If L is a Lagrangian, the corresponding payoff functional is

$$J : \mathcal{TCT} \to \mathbb{R}$$

defined by

$$J(x, u_1, u_2) = \int_{t_0}^{t_1} L(x(t), u_1(t), u_2(t))dt.$$

C_8. We assume set $\mathcal{OTCT}$ be set of all optimal control trajectory system (4) such that

- $a.$ $(x^*(\cdot), u_1^*(\cdot), u_2^*(\cdot)) \in \mathcal{OTCT}$
- $b.$ $x^*(t_0) = x_0$ and $x^*(t_1) \in S$
- $c.$ $J(x_0, u_1^*, u_2) \leq J(x_0, u_1^*, u_2^*) \leq J(x_0, u_1, u_2^*)$ for all $(x(\cdot), u_1(\cdot), u_2(\cdot)) \in \mathcal{TCT}$ such that $x(t_0) = x_0$ and $x(t_1) \in S$.

C_9. C is a Boltyanskii tangent cone to S at the point $\hat{x} = x^*(t_1)$.

We use the following Theorem (see Pschenichnii, 1980 and Sussmann, 2006) for proving the main result.

Theorem (fixed interval) *Assume that $m, n, t_0, t_1, f, L, x_0, S$ satisfy the conditions $C_1 - C_5$, $\mathcal{TCT}$ and J are defined by $C_6 - C_7$ and (x^*, u_1^*, u_2^*) holds C_8. $\hat{x}$ is defined by C_9 and C satisfies C_9. Define the Hamiltonian H*

$$H : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

given by

$$H(t, x, u_1, u_2, p) = \langle p, f(t, x, u_1, u_2)\rangle + L(t, x, u_1, u_2).$$

The variable p is the costate and also there exists a pair (λ, λ_0) such that

(i). $[t_0, t_1] \to \lambda(t)$ *is absolutely continuous function;*

(ii). $\lambda_0 \in \{0, 1\}$;

(iii). $(\lambda(t), \lambda_0) \neq (0, 0)$ for every $t \in [t_0, t_1]$ (the nontriviality condition).

(iv). *The adjoint equation holds, i.e.*

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t, x^*, u_1^*, u_2^*, \lambda(t), \lambda_0)$$

for every $t \in [t_0, t_1]$

(v). *The maximum Hamiltonian is the function defined by*

$$H(t, x^*, u_1^*, u_2^*, \lambda(t), \lambda_0) = max\{H(t, x^*, u_1, u_2, \lambda(t), \lambda_0) \mid u_1, u_2 \in \mathbb{R}^m\}$$

for every $t \in [t_0, t_1]$.

(vi). *The transversality condition holds*

$$-\lambda(t_1) \in C^{\perp}.$$

The statement of the problem considered in the Proposition is followed from Theorem (fixed interval) (see Athans & Falb, 1966). The sets S are sub manifolds. Part (vi) of Theorem (fixed interval) give the transversality conditions when the initial and final point are constrained to lie in subsets.

In Proposition, the problem being solved is a very particular case. The initial constraint set is a point, i.e., $S_0 = \{x_0\}$. The final condition, however, is completely unconstrained, i.e., $S_1 = \mathbb{R}^n$. If one applies the transversality condition, this implies that the adjoint vector must vanish at the final time since it must annihilate every vector in $\mathbb{R}^n$.

The main result of this paper is then the following (Athans & Falb, 1966), (Lee & Markus, 1986) and (Pontryagin et al., 1962).

4. Main Result

Proposition. (***The Maximum Principle for A Linear Quadratic Differential Game***) Let (1)-(3) be the linear differential game , $Q_i(t) \in \mathbb{R}^{n\times n}$ and $R_{ii}(t) \in \mathbb{R}^{m\times m}$ with $R_{ii} > 0, i = 1, 2,\ x_0 \in \mathbb{R}^n$, and $t_0, t_1 \in [t_0, t_1]$ satisfy $t_0 < t_1$. If $(\xi_*, \mu_*, \nu_*) \in \mathcal{OTCT}$, then there exists the function $\lambda_i^* : [t_0, t_1] \to \mathbb{R}^n,\ i = 1, 2$, such that ξ_*, μ_* and ν_* satisfy the initial/final value problem

$$\begin{bmatrix} \dot{\xi}_*(t) \\ \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \end{bmatrix} = \begin{bmatrix} A(t) & -S_1(t) & -S_2(t) \\ -Q_1(t) & -A^T(t) & 0 \\ -Q_2(t) & 0 & -A^T(t) \end{bmatrix} \begin{bmatrix} \xi_*(t) \\ \lambda_1^*(t) \\ \lambda_2^*(t) \end{bmatrix}, \qquad \xi_*(t_0) = x_0, \quad \lambda_i^*(t_1) = 0,$$

where $S_i(t) = B(t)R_{ii}^{-1}(t)B^T(t),\ i = 1, 2.$

Proof. By the transversality conditions of the Maximum Principle we have $\lambda_i^*(t_1) = 0,\ i = 1, 2$. By Theorem (fixed interval), the Hamiltonian is

$$\begin{aligned} H(x, u_1, u_2, \lambda) &= \langle \lambda_i(t), A(t)x + B(t)(u_1 + u_2)\rangle + L(x, u_1, u_2) \\ &= \langle \lambda_i(t), A(t)x + B(t)(u_1 + u_2)\rangle + (\frac{1}{2}x^T Q_i(t)x + \frac{1}{2}u_1^T R_{i1}u_1) + \frac{1}{2}u_2^T R_{i2}u_2),\ i = 1, 2 \\ &= \lambda_{ij}(t)(a_{jk}x_k + b_{js}(u_{1l} + u_{2l}) + \frac{1}{2}(x_j q_{ij} x_j + u_{1l} r_{1l} u_l + u_2 r_{2l} u_l), \end{aligned}$$

which is a quadratic function of u_i with a negative-definite second derivative. Thus the unique maximum occurs at the point where the derivative of the Hamiltonian with respect to u_i vanishes. Hamilton's equations are

$$\begin{aligned} \dot{\lambda}_i &= -\frac{\partial H}{\partial x} = -\Sigma(\lambda_{ij}a_{jk} + q_{jk}x_k) \\ \dot{x} &= \frac{\partial H}{\partial \lambda_i} = a_{jk}x_k + b_{js}(u_{1s} + u_{2s}) \\ 0 &= \frac{\partial H}{\partial u_i} = \lambda_{ij}b_{js} + r_{is}u_{is} \end{aligned}$$

where $i = 1, 2,\ \ j = 1, 2, ..., n,\ \ k = 1, 2, ..., n,\ \ s = 1, 2, ..., m$. In matrix form, three equations for optimal control behavior are

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u_i \\ \dot{\lambda}_i &= -Q_i(t)x - A(t)^T \lambda_i(t) \\ 0 &= R_{ii}(t)u_i + B^T(t)\lambda_i(t),\quad i = 1, 2. \end{aligned}$$

These equations are solved under end conditions $x = x_0$ at $t = t_0$ and $\lambda_i(t_1) = \lambda_i^1$ at $t = t_1$. Assuming R_{ii} is regular, i.e., R_{ii}^{-1} exists.

$$u_i = -R_{ii}^{-1}B(t)^T \lambda_i(t),\quad i = 1, 2.$$

By assumption we have $(\xi_*, \mu_*, \nu_*) \in \mathcal{OTCT}$, $\xi_*(t_0) = x_0,\ \lambda_i^*(t_1) = 0,\ \ i = 1, 2.$

That is to say, for almost every $t \in [t_0, t_1]$

$$\mu_{i*}(t) = -R_{ii}^{-1}(t)B(t)^T \lambda_i^*(t),\quad i = 1, 2$$

as may be verified by a direct computation. Since the adjoint equations for the extended systems are

$$\begin{aligned} \dot{\xi}^0(t) &= \frac{1}{2}\xi_*^T(t)Q_i(t)\xi_*(t) + \frac{1}{2}\mu_*^T(t)R_{i1}(t)\mu_*(t) + \frac{1}{2}\nu_*^T(t)R_{i2}(t)\nu_*(t), \\ \dot{\xi}(t) &= A(t)(\xi(t)) + B(t)(\mu(t) + \nu(t)), \\ \dot{\lambda}^0(t) &= 0, \\ \dot{\lambda}(t) &= -Q(t)(\xi(t)) - A^T(t)(\lambda_i(t)). \end{aligned}$$

We substitute the form of the optimal control into the second of these equations to get the differential equations in the statement of the result. It is clear that we must have $\xi_*(t_0) = x_0$. That $\lambda_i^*(t_1) = 0$ follows since the terminal condition is unspecified. □

4.1 Example

We consider a system whose state space is $\mathbb{R}^n$, the describing equations are

$$\dot{x} = \alpha(t)u, \quad x(0) = x_0,$$
$$\dot{y} = \alpha(t)v, \quad y(0) = y_0,$$

where initial position $(x_0, y_0) \in \mathbb{R}^2$, and $\alpha(t)$ is a continuous function. The control function $u(\cdot)$ (respectively, $v(\cdot)$) of the Pursuer (Evader) satisfies the inequalities (5). Terminal condition is that the terminal position x be zero, but we are not imposing any condition on the terminal velocity y. We solve this problem by applying Theorem (fixed interval) and the Proposition. In this case, $(x(\cdot), y(\cdot))$ moves from (x_0, y_0) to some point of the y-axis in minimum time. The dynamical law is denoted by

$$f(x, y, u, v) = (\alpha(t)u, \alpha(t)v)^T$$

and the Lagrangian is identically equal to 1. The terminal constraint is $x(t_1) \in S$, where

$$S = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$

The Hamiltonian H is given by

$$H(x, y, u, v, p_x, p_y) = p_x\alpha(t)u + p_y\alpha(t)v + 1,$$

where p_x and p_y to denote the two components of the variable p. Suppose that $(x^*(\cdot), y^*(\cdot), u^*(\cdot), v^*(\cdot))$ is a solution of our minimum time problem then $(x^*(\cdot), y^*(\cdot), u^*(\cdot), v^*(\cdot)) \in \mathcal{OTCT}$. Hence by Theorem (fixed interval) and the Proposition there exists a pair (λ, λ_0) satisfying all the conditions of the conclusion. Write $\lambda(t) = (\lambda_x(t), \lambda_y(t))$. The adjoint equations implies that

$$\dot{\lambda}_x(t) = 0, \quad \dot{\lambda}_y(t) = 0,$$

for all $t \in [t_0^*, t_1^*]$.

Therefore the function $\lambda_x(t)$ is constant. Then there exist constant a such that $\lambda_x(t) = a$, for all $t \in [t_0^*, t_1^*]$. Then there exists a constant b such that

$$\lambda_y(t) = b, \quad \text{for all } \ t \in [t_0^*, t_1^*]$$

If a and b are both equal to zero, the function λ_x, λ_y would vanish identically and then the Hamiltonian maximization condition would say that the function u is maximized by taking $u = u^*(t)$. By using the conditions of Theorem (fixed interval), we see a and b can not be both vanish. To see this, observe that if $a = b = 0$ then it follows that $\lambda_x = \lambda_y = 0$. But then the nontriviality condition tells us that $\lambda_0 \neq 0$. So the value $H(x^*(\cdot), y^*(\cdot), u^*(\cdot), v^*(\cdot), \lambda_x, \lambda_y, \lambda_0) = 1$. It is clear that the set S itself is a Boltyanskii tangent cone to S at the terminal point $x^*(t_1^*)$. Therefore the tranversality condition says that $\lambda(t_1^*) \in -S^\perp$, i.e.

$$\lambda_y(t_1^*) = 0.$$

So, $\lambda_y(t) = b$ is a linear function which is not identically zero (because a and b do not both vanish) but vanishes at the endpoint t_1^* of the interval $[t_0^*, t_1^*]$. Therefore, $\lambda_y(t)$ never vanishes on (t_0^*, t_1^*). It follows that the optimal control $u^*(\cdot)$ and $v^*(\cdot)$ are either constantly equal to 1 or constantly equal to -1. Thus we have proved that all optimal controls are constant.

5. Conclusion

This paper describes the applications of Pontryagin's Maximum Principle for finding optimal control in a linear quadratic differential games.

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References

Amatoa, F., Mattei, M. & Pironti. A. (2002). Guaranteeing cost strategies for linear quadratic differential games under uncertain dynamics. *Automatica*, 38, 507-515.

Athans, M. & Falb, P. L. (1966). *Optimal control. An introduction to the theory and its applications*. McGraw-Hill Book Co., New York.

Basar, T. & Olsder, G. (1999). *Dynamic noncooperative game theory*. (2nd ed.). Philadelphia: SIAM.

Bellman, R. (1957). *Dynamic programming*. Princeton, NJ: Princeton University Press.

Jodar, L. (1991). Explicit solutions of Riccati equations appearing in differential games. *Appl. Math. Lett,* 4(1), 9-12.

Lee, E. B. & Markus, L. (1986). *Foundations of optimal control theory.* (2nd ed.). Robert E. Krieger Publishing Co. Inc., Melbourne, FL, ISBN 0-89874-807-0.

Lewis, A. D. (2006), The Maximum Principle of Pontryagin in control and in optimal control. Handouts for the course taught at the Universitat Politcnica de Catalunya. Department of Mathematics and Statistics Queen's University.[online] Available: http://www.mast.queensu.ca/ andrew/teaching/courses.shtml (May 5 2006).

Mou, L. & Yong, J. (2006). Two-person zero-sum linear quadratic stochastic differential games by a Hilbert space method. *Journal of industrial and management optimization*, 2(1), 93-115.

Pschenichnii, B. (1980).*Convex ananlysis and extremal problems.* Nauka, Moscow.

Pontryagin, L. S., Boltyansky, V. G., Gamkrelidze, R. V., & Mishenko, E. F. (1962). *The Mathematical theory of optimal processes*. New York. Interscience (translated from Russian 1969).

Rozonoer, L.I. (1999). Linear quadratic optimization and some general hypotheses on optimal control. *Mathematical problems in engineerin*, 5, 275-289.

Sussmann, H. J. (2006). On the validity of the transversality condition for different concepts of tangent cone to a set. *Decision and Control*,45th IEEE Conference on, 241 - 246.

Sussmann, H. J. & Willems, J.C. (1997). 300 years of optimal control: from the brachystochrone to the maximum principle. *Control Systems Magazine, IEEE*, 17(3), 32 - 44.

Wang, G. & Yu, Z. (2010). A Pontryagin's Maximum Principle for non-zero sum differential games of BSDEs with applications. *IEEE Transactions on Automatic Control*, 55 (7), 1742-1747.