Handbook of Differential Equations
3rd edition

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Contents

Preface

Introduction

Introduction to the Electronic Version

How to Use This Book

I.A Definitions and Concepts

1 Definition of Terms . . . 2
2 Alternative Theorems . . . 15
3 Bifurcation Theory . . . 19
4 A Caveat for Partial Differential Equations . . . 27
5 Chaos in Dynamical Systems . . . 29
6 Classification of Partial Differential Equations . . . 36
7 Compatible Systems . . . 43
8 Conservation Laws . . . 47
9 Differential Resultants . . . 50
10 Existence and Uniqueness Theorems . . . 53
11 Fixed Point Existence Theorems . . . 58
12 Hamilton-Jacobi Theory . . . 61
13 Integrability of Systems . . . 65
14 Internet Resources . . . 71
15 Inverse Problems . . . 75
16 Limit Cycles . . . 78
17 Natural Boundary Conditions for a PDE . . . 83
18 Normal Forms: Near-Identity Transformations . . . 86
19 Random Differential Equations . . . 91
20 Self-Adjoint Eigenfunction Problems . . . 95
21 Stability Theorems . . . 101
22 Sturm-Liouville Theory . . . 103
23 Variational Equations . . . 109
24 Well Posed Differential Equations . . . 115
25 Wronskians and Fundamental Solutions . . . 119

26 Zeros of Solutions . . . 123

I.B Transformations
27 Canonical Forms . . . 128
28 Canonical Transformations . . . 132
29 Darboux Transformation . . . 135
30 An Involutory Transformation . . . 139
31 Liouville Transformation - 1 . . . 141
32 Liouville Transformation - 2 . . . 144
33 Reduction of Linear ODEs to a First Order System . . . 146
34 Prufer Transformation . . . 148
35 Modified Prufer Transformation . . . 150
36 Transformations of Second Order Linear ODEs - 1 . . . 152
37 Transformations of Second Order Linear ODEs - 2 . . . 157
38 Transformation of an ODE to an Integral Equation . . . 159
39 Miscellaneous ODE Transformations . . . 162
40 Reduction of PDEs to a First Order System . . . 166
41 Transforming Partial Differential Equations . . . 168
42 Transformations of Partial Differential Equations . . . 173

II Exact Analytical Methods
43 Introduction to Exact Analytical Methods . . . 178
44 Look-Up Technique . . . 179
45 Look-Up ODE Forms . . . 219

II.A Exact Methods for ODEs
46 An Nth Order Equation . . . 224
47 Use of the Adjoint Equation . . . 226
48 Autonomous Equations - Independent Variable Missing . . . 230
49 Bernoulli Equation . . . 235
50 Clairaut's Equation . . . 237
51 Computer-Aided Solution . . . 240
52 Constant Coefficient Linear Equations . . . 247
53 Contact Transformation . . . 249
54 Delay Equations . . . 253
55 Dependent Variable Missing . . . 260
56 Differentiation Method . . . 262
57 Differential Equations with Discontinuities . . . 264
58 Eigenfunction Expansions . . . 268
59 Equidimensional-in-x Equations . . . 275
60 Equidimensional-in-y Equations . . . 278
61 Euler Equations . . . 281
62 Exact First Order Equations . . . 284
63 Exact Second Order Equations . . . 287
64 Exact Nth Order Equations . . . 290
65 Factoring Equations . . . 292

66 Factoring Operators 294
67 Factorization Method 300
68 Fokker-Planck Equation 303
69 Fractional Differential Equations 308
70 Free Boundary Problems 311
71 Generating Functions 315
72 Green's Functions 318
73 Homogeneous Equations 327
74 Method of Images 330
75 Integrable Combinations 334
76 Integral Representation: Laplace's Method 336
77 Integral Transforms: Finite Intervals 342
78 Integral Transforms: Infinite Intervals 347
79 Integrating Factors 356
80 Interchanging Dependent and Independent Variables 360
81 Lagrange's Equation 363
82 Lie Groups: ODEs 366
83 Operational Calculus 379
84 Pfaffian Differential Equations 384
85 Reduction of Order 389
86 Riccati Equations 392
87 Matrix Riccati Equations 395
88 Scale Invariant Equations 398
89 Separable Equations 401
90 Series Solution 403
91 Equations Solvable for x 409
92 Equations Solvable for y 411
93 Superposition 413
94 Method of Undetermined Coefficients 415
95 Variation of Parameters 418
96 Vector Ordinary Differential Equations 421

II.B Exact Methods for PDEs

97 Backlund Transformations 428
98 Method of Characteristics 432
99 Characteristic Strip Equations 438
100 Conformal Mappings 441
101 Method of Descent 446
102 Diagonalization of a Linear System of PDEs 449
103 Duhamel's Principle 451
104 Exact Equations 454
105 Hodograph Transformation 456
106 Inverse Scattering 460
107 Jacobi's Method 464
108 Legendre Transformation 467
109 Lie Groups: PDEs 471

110 Poisson Formula 478
111 Riemann's Method 481
112 Separation of Variables 487
113 Separable Equations: Stackel Matrix 494
114 Similarity Methods 497
115 Exact Solutions to the Wave Equation 501
116 Wiener-Hopf Technique 505

III Approximate Analytical Methods

117 Introduction to Approximate Analysis 510
118 Chaplygin's Method 511
119 Collocation 514
120 Dominant Balance 517
121 Equation Splitting 520
122 Floquet Theory 523
123 Graphical Analysis: The Phase Plane 526
124 Graphical Analysis: The Tangent Field 532
125 Harmonic Balance 535
126 Homogenization 538
127 Integral Methods 542
128 Interval Analysis 545
129 Least Squares Method 549
130 Lyapunov Functions 551
131 Equivalent Linearization and Nonlinearization 555
132 Maximum Principles 560
133 McGarvey Iteration Technique 566
134 Moment Equations: Closure 568
135 Moment Equations: Ito Calculus 572
136 Monge's Method 575
137 Newton's Method 578
138 Pade Approximants 582
139 Perturbation Method: Method of Averaging 586
140 Perturbation Method: Boundary Layer Method 590
141 Perturbation Method: Functional Iteration 598
142 Perturbation Method: Multiple Scales 605
143 Perturbation Method: Regular Perturbation 610
144 Perturbation Method: Strained Coordinates 614
145 Picard Iteration 618
146 Reversion Method 621
147 Singular Solutions 623
148 Soliton-Type Solutions 626
149 Stochastic Limit Theorems 629
150 Taylor Series Solutions 632
151 Variational Method: Eigenvalue Approximation 635
152 Variational Method: Rayleigh-Ritz 638
153 WKB Method 642

IV.A Numerical Methods: Concepts

154 Introduction to Numerical Methods 648
155 Definition of Terms for Numerical Methods 651
156 Available Software 654
157 Finite Difference Formulas 661
158 Finite Difference Methodology 670
159 Grid Generation 675
160 Richardson Extrapolation 679
161 Stability: ODE Approximations 683
162 Stability: Courant Criterion 688
163 Stability: Von Neumann Test 692
164 Testing Differential Equation Routines 694

IV.B Numerical Methods for ODEs

165 Analytic Continuation 698
166 Boundary Value Problems: Box Method 701
167 Boundary Value Problems: Shooting Method 706
168 Continuation Method 710
169 Continued Fractions 713
170 Cosine Method 716
171 Differential Algebraic Equations 720
172 Eigenvalue/Eigenfunction Problems 726
173 Euler's Forward Method 730
174 Finite Element Method 734
175 Hybrid Computer Methods 744
176 Invariant Imbedding 747
177 Multigrid Methods 752
178 Parallel Computer Methods 755
179 Predictor-Corrector Methods 759
180 Runge-Kutta Methods 763
181 Stiff Equations 770
182 Integrating Stochastic Equations 775
183 Symplectic Integration 780
184 Use of Wavelets 784
185 Weighted Residual Methods 786

IV.C Numerical Methods for PDEs

186 Boundary Element Method 792
187 Differential Quadrature 796
188 Domain Decomposition 800
189 Elliptic Equations: Finite Differences 805
190 Elliptic Equations: Monte-Carlo Method 810
191 Elliptic Equations: Relaxation 814
192 Hyperbolic Equations: Method of Characteristics 818
193 Hyperbolic Equations: Finite Differences 824
194 Lattice Gas Dynamics 828

195 Method of Lines 831
196 Parabolic Equations: Explicit Method 835
197 Parabolic Equations: Implicit Method 839
198 Parabolic Equations: Monte-Carlo Method 844
199 Pseudospectral Method 851

Mathematical Nomenclature

Errata

Preface

When I was a graduate student in applied mathematics at the California Institute of Technology, we solved many differential equations (both ordinary differential equations and partial differential equations). Given a differential equation to solve, I would think of all the techniques I knew that might solve that equation. Eventually, the number of techniques I knew became so large that I began to forget some. Then, I would have to consult books on differential equations to familiarize myself with a technique that I remembered only vaguely. This was a slow process and often unrewarding; I might spend twenty minutes reading about a technique only to realize that it did not apply to the equation I was trying to solve.

Eventually, I created a list of the different techniques that I knew. Each technique had a brief description of how the method was used and to what types of equations it applied. As I learned more techniques, they were added to the list. This book is a direct result of that list.

At Caltech we were taught the usefulness of approximate analytic solutions and the necessity of being able to solve differential equations numerically when exact or approximate solution techniques could not be found. Hence, approximate analytical solution techniques and numerical solution techniques were also added to the list.

Given a differential equation to analyze, most people spend only a small amount of time using analytical tools and then use a computer to see what the solution "looks like." Because this procedure is so prevalent, this edition includes an expanded section on numerical methods. New sections on sympletic integration (see page 780) and the use of wavelets (see page 784) also have been added.

In writing this book, I have assumed that the reader is familiar with differential equations and their solutions. The object of this book is not to teach novel techniques but to provide a handy reference to many popular techniques. All of the techniques included are elementary in the usual mathematical sense; because this book is designed to be functional it does not include many abstract methods of limited applicability. This handbook has been designed to serve as both a reference book and as a complement to a text on differential equations. Each technique described is accompanied by several references; these allow each topic to be studied in more detail.

It is hoped that this book will be used by students taking courses in differential equations (at either the undergraduate or the graduate level). It will introduce the student to more techniques than they usually see in a differential equations class and will illustrate many different types of techniques. Furthermore, it should act as a concise reference for the techniques that a student has learned. This book should also be useful for the practicing engineer or scientist who solves differential equations on an occasional basis.

A feature of this book is that it has sections dealing with stochastic differential equations and delay differential equations as well as ordinary differential equations and partial differential equations. Stochastic differential equations and delay differential equations are often studied only in advanced texts and courses; yet, the techniques used to analyze these equations are easy to understand and easy to apply.

Had this book been available when I was a graduate student, it would have saved me much time. It has saved me time in solving problems that arose from my own work in industry (the Jet Propulsion Laboratory, Sandia Laboratories, EXXON Research and Engineering, The MITRE Corporation, BBN).

Parts of the text have been utilized in differential equations classes at the Rensselaer Polytechnic Institute. Students' comments have been used to clarify the text. Unfortunately, there may still be some errors in the text; I would greatly appreciate receiving notice of any such errors.

Many people have been kind enough to send in suggestions for additional material to add and corrections of existing material. There are too many to name them individually, but Alain Moussiaux stands out for all of the checking he has performed. Thank you all!

This book is dedicated to my wife, Janet Taylor.

Boston, Mass. 1997 Daniel Zwillinger
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Introduction

This book is a compilation of the most important and widely applicable methods for solving and approximating differential equations. As a reference book, it provides convenient access to these methods and contains examples of their use.

The book is divided into four parts. The first part is a collection of transformations and general ideas about differential equations. This section of the book describes the techniques needed to determine whether a partial differential equation is well posed, what the "natural" boundary conditions are, and many other things. At the beginning of this section is a list of definitions for many of the terms that describe differential equations and their solutions.

The second part of the book is a collection of exact analytical solution techniques for differential equations. The techniques are listed (nearly) alphabetically. First is a collection of techniques for ordinary differential equations, then a collection of techniques for partial differential equations. Those techniques that can be used for both ordinary differential equations and partial differential equations have a star ($*$) next to the method name. For nearly every technique, the following are given:

- the types of equations to which the method is applicable
- the idea behind the method
- the procedure for carrying out the method
- at least one simple example of the method
- any cautions that should be exercised
- notes for more advanced users
- references to the literature for more discussion or more examples

The material for each method has deliberately been kept short to simplify use. Proofs have been intentionally omitted.

It is hoped that, by working through the simple example(s) given, the method will be understood. Enough insight should be gained from working the example(s) to apply the method to other equations. Further references are given for each method so that the principle may be studied in more detail or so more examples may be seen. Note that not all of the references listed at the end of a method may be referred to in the text.

The author has found that computer languages that perform symbolic manipulations (e.g., Macsyma, Maple, and Mathematica) are very useful for performing the calculations necessary to analyze differential equations. Hence, there is a section comparing the capabilities of these languages and, for some exact analytical techniques, examples of their use are given.

Not all differential equations have exact analytical solutions; sometimes an approximate solution will have to do. Other times, an approximate solution may be *more* useful than an exact solution. For instance, an exact solution in terms of a slowly converging infinite series may be laborious to approximate numerically. The same problem may have a simple approximation that indicates some characteristic behavior or allows numerical values to be obtained.

The third part of this book deals with approximate analytical solution techniques. For the methods in this part of the book, the format is similar to that used for the exact solution techniques. We classify a method as an approximate method if it gives some information about the solution but does not give the solution of the original equation(s) at all values of the independent variable(s). The methods in this section describe, for example, how to obtain perturbation expansions for the solutions to a differential equation.

When an exact or an approximate solution technique cannot be found, it may be necessary to find the solution numerically. Other times, a numerical solution may convey more information than an exact or approximate analytical solution. The fourth part of this book is concerned with the most important methods for finding numerical solutions of common types of differential equations. Although there are many techniques available for numerically solving differential equations, this book has only tried to illustrate the main techniques for each class of problem. At the beginning of the fourth section is a brief introduction to the terms used in numerical methods.

When possible, short Fortran or C programs[1] have been given. Once again, those techniques that can be used for both ordinary differential equations and partial differential equations have a star next to the method name.

This book is not designed to be read at one sitting. Rather, it should be consulted as needed. Occasionally we have used "ODE" to stand for "ordinary differential equation" and "PDE" to stand for "partial differential equation."

This book contains many references to other books. Whereas some books cover only one or two topics well, some books cover all their topics well. The following books are recommended as a first source for detailed understanding of the differential equation techniques they cover; each is broad in scope and easy to read.

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Butkov, E. *Mathematical Physics*. Addison–Wesley Publishing Co., Reading, MA, 1968.

[4] Chester, C. R. *Techniques in Partial Differential Equations*. McGraw–Hill Book Company, New York, 1970.

[5] Collatz, L. *The Numerical Treatment of Differential Equations*. Springer–Verlag, New York, 1966.

[1] We make no warranties, express or implied, that these programs are free of error. The author and publisher disclaim all liability for direct or consequential damages resulting from your use of the programs.

[6] Gear, C. W. *Numerical Initial Value Problems in Ordinary Differential Equations*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1971.

[7] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[8] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis*. Interscience Publishers, Inc., New York, 1958.

Introduction to the Electronic Version

This third edition of *Handbook of Differential Equations* is available both in print form and in electronic form. The electronic version can be used with any modern web browser (such as Netscape or Explorer). Some features of the electronic version include

- Quickly finding a specific method for a differential equation

  Navigating through the electronic version is performed via lists of methods for differential equations. Facilities are supplied for creating lists of methods based on *filters*. For example, a list containing all the differential equation methods that have both a program and an example in the text can be created. Or, a list of differential equation methods that contain either a table or a specific word can be created. It is also possible to apply boolean operations to lists to create new lists.
- Interactive programs demonstrating some of the numerical methods

  For some of the numerical methods, an interactive Java program is supplied. This program numerically solves the example problem described in the text. The parameters describing the numerical solution may be varied, and the resulting numerical approximation obtained.
- Live links to the internet

  The third edition of this book has introduced links to relevant web sites on the internet. In the electronic version, these links are active (clicking on one of them will take you to that site). In the print version, the URLs may be found by looking in the index under the entry “URL.”
- Dynamic rendering of mathematics

  All of the mathematics in the print version is available electronically, both through static `gif` files and via dynamic Java rendering.

How to Use This Book

This book has been designed to be easy to use when solving or approximating the solutions to differential equations. This introductory section outlines the procedure for using this book to analyze a given differential equation.

First, determine whether the differential equation has been studied in the literature. A list of many such equations may be found in the "Look-Up" section beginning on page 179. If the equation you wish to analyze is contained on one of the lists in that section, then see the indicated reference. This technique is the single most useful technique in this book.

Alternatively, if the differential equation that you wish to analyze does not appear on those lists or if the references do not yield the information you desire, then the analysis to be performed depends on the type of the differential equation.

Before any other analysis is performed, it must be verified that the equation is well posed. This means that a solution of the differential equation(s) exists, is unique, and depends continuously on the "data." See pages 15, 53, 101, and 115.

Given an Ordinary Differential Equation

- It may be useful to transform the differential equation to a canonical form or to a form that appears in the "Look-Up" section. For some common transformations, see pages 128–162.
- If the equation has a special form, then there may be a specialized solution technique that may work. See the techniques on pages 275, 278, and 398.
- If the equation is a
  - Bernoulli equation, see page 235.
  - Chaplygin equation, see page 511.
  - Clairaut equation, see page 237.
  - Euler equation, see page 281.
  - Lagrange equation, see page 363.
  - Riccati equation, see page 392.
- If the equation does not depend explicitly on the independent variable, see pages 230 and 411.
- If the equation does not depend explicitly on the dependent variable (undifferentiated), see pages 260 and 409.

- If one solution of the equation is known, it may be possible to lower the order of the equation; see page 389.
- If discontinuous terms are present, see page 264.
- The single most powerful technique for solving analytically ordinary differential equations is through the use of Lie groups; see page 366.

Given a Partial Differential Equation

Partial differential equations are treated in a different manner from ordinary differential equations; in particular, the *type* of the equation dictates the solution technique. First, determine the type of the partial differential equation; it may be hyperbolic, elliptic, parabolic, or of mixed type (see page 36).

- It may be useful to transform the differential equation to a canonical form, or to a form that appears in the "Look-Up" Section. For transformations, see pages 146, 166, 168, 173, 456, and 467.
- The simplest technique for working with partial differential equations, which does not always work, is to "freeze" all but one of the independent variables and then analyze the resulting partial differential equation or ordinary differential equation. Then the other variables may be added back in, one at a time.
- If every term is linear in the dependent variable, then separation of variables may work; see page 487.
- If the boundary of the domain must be determined as part of the problem, see the technique on page 311.
- See all of the exact solution techniques, which are on pages 428–508. In addition, many of the techniques that can be used for ordinary differential equations are also applicable to partial differential equations. These techniques are indicated by a star with the method name.
- If the equation is hyperbolic,
  - In principle, the differential equation may be solved using the method of characteristics; see page 432. Often, though, the calculations are impossible to perform analytically.
  - See the section on the exact solution to the wave equation on page 501.
- The single most powerful technique for analytically solving partial differential equations is through the use of Lie groups; see page 471.

Given a System of Differential Equations

- First, verify that the system of equations is consistent; see page 43.
- Note that many of the methods for a single differential equation may be generalized to handle systems.

- By using differential resultants, it may be possible to obtain a single equation; see page 50.
- The following methods are for systems of equations:
  - The method of generating functions; see page 315.
  - The methods for constant coefficient differential equations; see pages 421 and 449.
  - The finding of integrable combinations; see page 334.
- If the system is hyperbolic, then the method of characteristics will work (in principle); see page 432.
- See also the method for Pfaffian equations (see page 384) and the method for matrix Riccati equations (see page 395).

Given a Stochastic Differential Equation

- A general discussion of random differential equations may be found on page 91.
- To determine the transition probability density, see the discussion of the Fokker–Planck equation on page 303.
- To obtain the moments without solving the complete problem, see pages 568 and 572.
- If the noise appearing in the differential equation is not "white noise," the section on stochastic limit theorems might be useful (see page 629).
- To numerically simulate the solutions of a stochastic differential equation, see the technique on page 775.

Given a Delay Equation

See the techniques on page 253.

Looking for an Approximate Solution

- If exact bounds on the solution are desired, see the methods on pages 545, 551, and 560.
- If the solution has singularities that are to be recovered, see page 582.
- If the differential equation(s) can be formulated as a contraction mapping, then approximations may be obtained in a natural way; see page 58.

Looking for a Numerical Solution

- It is extremely important that the differential equation(s) be well posed before a numerical solution is attempted. See the theorem on page 723 for an indication of the problems that can arise.

- The numerical solution technique must be stable if the numerical solution is to approximate the true solution of the differential equation; see pages 683, 688, and 692.
- It is often easiest to use commercial software packages when looking for a numerical solution; see page 654.
- If the problem is "stiff," then a method for dealing with "stiff" problems will probably be required; see page 770.
- If a low-accuracy solution is acceptable, then a Monte-Carlo solution technique may be used; see pages 810 and 844.
- To determine a grid on which to approximate the solution numerically, see page 675.
- To find an approximation scheme that works on a parallel computer, see page 755.

Other Things to Consider

- Does the differential equation undergo bifurcations? See page 19.
- Is the solution bounded? See pages 551 and 560.
- Is the differential equation well posed? See pages 15 and 115.
- Does the equation exhibit symmetries? See pages 366 and 471.
- Is the system chaotic? See page 29.
- Are some terms in the equation discontinuous? See page 264.
- Are there generalized functions in the differential equation? See pages 318 and 330.
- Are fractional derivatives involved? See page 308.
- Does the equation involve a small parameter? See the perturbation methods (on pages 586, 590, 598, 605, 610, and 614) or pages 538, 642.
- Is the general form of the solution known? See page 415.
- Are there multiple time or space scales in the problem? See pages 538 and 605.
- Always check your results!

Methods Not Discussed in This Book

There are a variety of novel methods for differential equations and their solutions not discussed in this book. These include

1. Adomian's decomposition method (see Adomian [1])
2. Entropy methods (see Baker-Jarvis [2])
3. Fuzzy logic (see Leland [5])
4. Infinite systems of differential equations (see Steinberg [6])
5. Monodromy deformation (see Chowdhury and Naskar [3])
6. p-adic differential equations (see Dwork [4])

References

[1] Adomian, G. *Stochastic Systems*. Academic Press, New York, 1983.

[2] Baker-Jarvis, J. Solution to boundary value problems using the method of maximum entropy. *J. Math. and Physics 30*, 2 (February 1989), 302–306.

[3] Chowdhury, A. R., and Naskar, M. Monodromy deformation approach to nonlinear equations — A survey. *Fortschr. Phys. 36*, 12 (1988), 9399–953.

[4] Dwork, B. *Lectures on p-adic Differential Equations*. Springer–Verlag, New York, 1982.

[5] Leland, R. P. Fuzzy differential systems and Malliavin calculus. *Fuzzy Sets and Systems 70* (1995), 59–73.

[6] Steinberg, S. Infinite systems of ordinary differential equations with unbounded coefficients and moment problems. *J. Math. Anal. Appl. 41* (1973), 685–694.

1. Definition of Terms

Adiabatic invariant When the parameters of a physical system vary slowly under the effect of an external perturbation, some quantities are constant to any order of the variable describing the slow rate of change. Such a quantity is called an adiabatic invariant. This does not mean that these quantities are exactly constant but rather that their variation goes to zero faster than any power of the small parameter.

Analytic A function is analytic at a point if the function has a power series expansion valid in some neighborhood of that point.

Asymptotic equivalence Two functions, $f(x)$ and $g(x)$, are said to be *asymptotically equivalent* as $x \to x_0$ if $f(x)/g(x) \sim 1$ as $x \to x_0$, that is: $f(x) = g(x)\,[1 + o(1)]$ as $x \to x_0$. See Erdélyi [4] for details.

Asymptotic expansion Given a function $f(x)$ and an asymptotic series $\{g_k(x)\}$ at x_0, the formal series $\sum_{k=0}^{\infty} a_k g_k(x)$, where the $\{a_k\}$ are given constants, is said to be an *asymptotic expansion* of $f(x)$ if $f(x) - \sum_{k=0}^{n} a_k g_k(x) = o(g_n(x))$ as $x \to x_0$ for every n; this is expressed as $f(x) \sim \sum_{k=0}^{\infty} a_k g_k(x)$. Partial sums of this formal series are called *asymptotic approximations* to $f(x)$. Note that the formal series need not converge. See Erdélyi [4] for details.

Asymptotic series A sequence of functions, $\{g_k(x)\}$, forms an *asymptotic series* at x_0 if $g_{k+1}(x) = o(g_k(x))$ as $x \to x_0$.

Autonomous An ordinary differential equation is autonomous if the independent variable does not appear explicitly in the equation. For example, $y_{xxx} + (y_x)^2 = y$ is autonomous while $y_x = x$ is not (see page 230).

Bifurcation The solution of an equation is said to undergo a bifurcation if, at some critical value of a parameter, the number of solutions to the equation changes. For instance, in a quadratic equation with real coefficients, as the constant term changes the number of real solutions can change from 0 to 2 (see page 19).

Boundary data Given a differential equation, the value of the dependent variable on the boundary may be given in many different ways.

- **Dirichlet boundary conditions** The dependent variable is prescribed on the boundary. This is also called a boundary condition of the first kind.
- **Homogeneous boundary conditions** The dependent variable vanishes on the boundary.
- **Mixed boundary conditions** A linear combination of the dependent variable and its normal derivative is given on the boundary,

or one type of boundary data is given on one part of the boundary while another type of boundary data is given on a different part of the boundary. This is also called a boundary condition of the third kind.

Neumann boundary conditions The normal derivative of the dependent variable is given on the boundary. This is also called a boundary condition of the second kind.

Sometimes the boundary data also include values of the dependent variable at points interior to the boundary.

Boundary layer A boundary layer is a small region, near a boundary, in which a function undergoes a large change (see page 590).

Boundary value problem An ordinary differential equation, where not all of the data are given at one point, is a boundary value problem. For example, the equation $y'' + y = 0$ with the data $y(0) = 1$, $y(1) = 1$ is a boundary value problem.

Characteristics A hyperbolic partial differential equation can be decomposed into ordinary differential equations along curves known as characteristics. These characteristics are themselves determined to be the solutions of ordinary differential equations (see page 432).

Cauchy problem The Cauchy problem is an initial value problem for a partial differential equation. For this type of problem there are initial conditions but no boundary conditions.

Commutator If $L[\cdot]$ and $H[\cdot]$ are two differential operators, then the commutator of $L[\cdot]$ and $H[\cdot]$ is defined to be the differential operator given by $[L, H] := L \circ H - H \circ L = -[H, L]$. For example, the commutator of the operators $L[\cdot] = x\frac{d}{dx}$ and $H[\cdot] = 1 + \frac{d}{dx}$ is

$$[L, H] = \left(x\frac{d}{dx}\right)\left(1 + \frac{d}{dx}\right) - \left(1 + \frac{d}{dx}\right)\left(x\frac{d}{dx}\right) = -\frac{d}{dx}.$$

See Goldstein [6] for details.

Complete A set of functions is said to be complete on an interval if any other function that satisfies appropriate boundedness and smoothness conditions can be expanded as a linear combination of the original functions. Usually the expansion is assumed to converge in the "mean square," or L_2 sense. For example, the functions $\{u_n(x)\} := \{\sin(n\pi x), \cos(n\pi x)\}$ are complete on the interval $[0, 1]$ because any $C^1[0, 1]$ function, $f(x)$, can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(n\pi x) + b_n \sin(n\pi x)\Big)$$

for some set of $\{a_n, b_n\}$. See Courant and Hilbert [3, pages 51–54] for details.

Complete system The system of nonlinear partial differential equations: $\{F_k(x_1, \dots, x_r, y, p_1, \dots, p_r) = 0 \mid k = 1, \dots, s\}$, in one dependent variable, $y(\mathbf{x})$, where $p_i = dy/dx_i$, is called a complete system if each $\{F_j, F_k\}$, for $1 \le j, k \le r$, is a linear combination of the $\{F_k\}$. Here $\{\,,\,\}$ represents the Lagrange bracket. See Iyanaga and Kawada [8, page 1304].

Conservation form A hyperbolic partial differential equation is said to be in conservation form if each term is a derivative with respect to some variable. That is, it is an equation for $u(\mathbf{x}) = u(x_1, x_2, \dots, x_n)$ that has the form $\frac{\partial f_1(u,\mathbf{x})}{\partial x_1} + \cdots + \frac{\partial f_n(u,\mathbf{x})}{\partial x_n} = 0$ (see page 47).

Consistency There are two types of consistency:

- **Genuine consistency** This occurs when the exact solution to an equation can be shown to satisfy some approximations that have been made in order to simplify the equation's analysis.
- **Apparent consistency** This occurs when the approximate solution to an equation can be shown to satisfy some approximations that have been made in order to simplify the equation's analysis.

When simplifying an equation to find an approximate solution, the derived solution must always show apparent consistency. Even then, the approximate solution may not be close to the exact solution, unless there is genuine consistency. See Lin and Segel [9, page 188].

Coupled systems of equations A set of differential equations is said to be coupled if there is more than one dependent variable and each equation involves more than one dependent variable. For example, the system $\{y' + v = 0,\ v' + y = 0\}$ is a coupled system for $\{y(x), v(x)\}$.

Degree The degree of an ordinary differential equation is the greatest number of times the dependent variable appears in any single term. For example, the degree of $y' + (y'')^2 y + 1 = 0$ is 3, whereas the degree of $y'' y' y^2 + x^5 y = 1$ is 4. The degree of $y' = \sin y$ is infinite. If all the terms in a differential equation have the same degree, then the equation is called equidimensional-in-y (see page 278).

Delay equation A delay equation, also called a differential delay equation, is an equation that depends on the "past" as well the "present." For example, $y''(t) = y(t - \tau)$ is a delay equation when $\tau > 0$. See page 253.

Determined A truncated system of differential equations is said to be determined if the inclusion of any higher order terms cannot affect the topological nature of the local behavior about the singularity.

Differential form A first order differential equation is said to be in differential form if it is written $P(x, y)dx + Q(x, y)dy = 0$.

Dirichlet problem The Dirichlet problem is a partial differential equation with Dirichlet data given on the boundaries. That is, the dependent variable is prescribed on the boundary.

Eigenvalues, eigenfunctions Given a linear operator $L[\cdot]$ with boundary conditions $B[\cdot]$, there will sometimes exist nontrivial solutions to the equation $L[y] = \lambda y$ (the solutions may or may not be required to also satisfy $B[y] = 0$). When such a solution exists, the value of λ is called an eigenvalue. Corresponding to the eigenvalue λ there will exist solutions $\{y_\lambda(x)\}$; these are called eigenfunctions. See Stakgold [12, Chapter 7, pages 411–466] for details.

Elliptic operator The differential operator $\displaystyle\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ is an elliptic differential operator if the quadratic form $\mathbf{x}^{\mathrm{T}} A \mathbf{x}$, where $A = (a_{ij})$, is positive definite whenever $\mathbf{x} \neq \mathbf{0}$. If the $\{a_{ij}\}$ are functions of some variable, say t, and the operator is elliptic for all values of t of interest, then the operator is called *uniformly elliptic*. See page 36.

Euler–Lagrange equation If $u = u(x)$ and $J[u] = \int f(u', u, x)\, dx$, then the condition for the vanishing of the variational derivative of J with respect to u, $\frac{\delta J}{\delta u} = 0$ is given by the Euler–Lagrange equation:

$$\left(\frac{\partial}{\partial u} - \frac{d}{dx} \frac{\partial}{\partial u'} \right) f = 0.$$

If $w = w(x)$ and $J = \int g(w'', w', w, x)\, dx$, then the Euler–Lagrange equation is

$$\left(\frac{\partial}{\partial w} - \frac{d}{dx} \frac{\partial}{\partial w'} + \frac{d^2}{dx^2} \frac{\partial}{\partial w''} \right) g = 0.$$

If $v = v(x, y)$ and $J = \iint h(v_x, v_y, v, x, y)\, dx\, dy$, then the Euler–Lagrange equation is

$$\left(\frac{\partial}{\partial v} - \frac{d}{dx} \frac{\partial}{\partial v_x} - \frac{d}{dy} \frac{\partial}{\partial v_y} \right) h = 0.$$

See page 418 for more details.

First integral: ODE When a given differential equation is of order n and, by a process of integration, an equation of order $n - 1$ involving an arbitrary constant is obtained, then this new equation is known as a first integral of the given equation. For example, the equation $y'' + y = 0$ has the equation $(y')^2 + y^2 = C$ as a first integral.

First integral: PDE A function $u(x, y, z)$ is called a first integral of the vector field $\mathbf{V} = (P, Q, R)$ (or of its associated system: $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$) if at every point in the domain $\mathbf{V}$ is orthogonal to $\operatorname{grad} u$, i.e.,

$$\mathbf{V} \cdot \nabla u = P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} + R \frac{\partial u}{\partial z} = 0.$$

Conversely, any solution of this partial differential equation is a first integral of $\mathbf{V}$. Note that if $u(x, y, z)$ is a first integral of $\mathbf{V}$, then so is $f(u)$.

Fréchet derivative, Gâteaux derivative The Gâteaux derivative of the operator $N[\cdot]$, at the "point" $u(\mathbf{x})$, is the linear operator defined by

$$L[z(\mathbf{x})] = \lim_{\epsilon \to 0} \frac{N[u+\epsilon z] - N[u]}{\epsilon}.$$

For example, if $N[u] = u^3 + u'' + (u')^2$, then $L[z] = 3u^2 z + z'' + 2u'z'$. If, in addition,

$$\lim_{||h|| \to 0} \frac{||N[u+h] - N[u] - L[u]h||}{||h||} = 0$$

(as is true in our example), then $L[u]$ is also called the Fréchet derivative of $N[\cdot]$. See Olver [11] for details.

Fuchsian equation A Fuchsian equation is an ordinary differential equation whose only singularities are regular singular points.

Fundamental matrix The vector ordinary differential equation $\mathbf{y}' = A\mathbf{y}$ for $\mathbf{y}(x)$, where A is a matrix, has the fundamental matrix $\Phi(x)$ if Φ satisfies $\Phi' = A\Phi$ and the determinant of Φ is nonvanishing (see page 119).

General solution Given an nth order linear ordinary differential equation, the general solution contains all n linearly independent solutions, with a constant multiplying each one. For example, the differential equation $y'' + y = 1$ has the general solution $y(x) = 1 + A\sin x + B\cos x$, where A and B are arbitrary constants.

Green's function A Green's function is the solution of a linear differential equation, which has a delta function appearing either in the equation or in the boundary conditions (see page 318).

Harmonic function A function $\phi(\mathbf{x})$ is harmonic if it satisfies Laplace's equation: $\nabla^2\phi = 0$.

Hodograph In a partial differential equation, if the independent variables and dependent variables are switched, then the space of independent variables is called the hodograph space (in two dimensions, the hodograph plane) (see page 456).

Homogeneous equation Used in two different senses:

- An equation is said to be homogeneous if all terms depend linearly on the dependent variable or its derivatives. For example, the equation $y_{xx} + xy = 0$ is homogeneous whereas the equation $y_{xx} + xy = 1$ is not.
- A first order ordinary differential equation is said to be homogeneous if the forcing function is a ratio of homogeneous polynomials (see page 327).

Ill posed problems A problem that is not well posed is said to be ill posed. Typical ill posed problems are the Cauchy problem for the Laplace equation, the initial/boundary value problem for the backward heat equation, and the Dirichlet problem for the wave equation (see page 115).

Initial value problem An ordinary differential equation with all of the data given at one point is an initial value problem. For example, the equation $y'' + y = 0$ with the data $y(0) = 1$, $y'(0) = 1$ is an initial value problem.

Involutory transformation An involutory transformation T is one that, when applied twice, does not change the original system; i.e., T^2 is equal to the identity function.

L_2 function A function $f(x)$ is said to belong to L_2 if $\int_0^\infty |f(x)|^2\,dx$ is finite.

Lagrange bracket If $\{F_j\}$ and $\{G_j\}$ are sets of functions of the independent variables $\{u, v, \dots\}$ then the Lagrange bracket of u and v is defined to be

$$\{u, v\} = \sum_j \left(\frac{\partial F_j}{\partial u}\frac{\partial G_j}{\partial v} - \frac{\partial F_j}{\partial v}\frac{\partial G_j}{\partial u} \right) = -\{v, u\}\,.$$

See Goldstein [6] for details.

Lagrangian derivative The Lagrangian derivative (also called the material derivative) is defined by $\frac{DF}{Dt} := \frac{\partial F}{\partial t} + \mathbf{v} \cdot \nabla F$, where $\mathbf{v}$ is a given vector. See Iyanaga and Kawada [8, page 669].

Laplacian The Laplacian is the differential operator usually denoted by ∇^2 (in many books it is represented as Δ). It is defined by $\nabla^2\phi = \operatorname{div}(\operatorname{grad}\phi)$, when ϕ is a scalar. The *vector Laplacian* of a vector is the differential operator denoted by ✡ (in most books it is represented as ∇^2). It is defined by ✡$\mathbf{v} = \operatorname{grad}(\operatorname{div}\mathbf{v}) - \operatorname{curl}\operatorname{curl}\mathbf{v}$, when $\mathbf{v}$ is a vector. See Moon and Spencer [10] for details.

Leibniz's rule Leibniz's rule states that

$$\frac{d}{dt}\left(\int_{f(t)}^{g(t)} h(t,\zeta)\,d\zeta \right) = g'(t)h(t, g(t)) - f'(t)h(t, f(t)) + \int_{f(t)}^{g(t)} \frac{\partial h}{\partial t}(t,\zeta)\,d\zeta.$$

Lie algebra A Lie algebra is a vector space equipped with a Lie bracket (often called a commutator) $[x, y]$ that satisfies three axioms:

- $[x, y]$ is bilinear (i.e., linear in both x and y separately),
- the Lie bracket is anti-commutative (i.e., $[x, y] = -[y, x]$),
- the Jacobi identity, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, holds.

See Olver [11] for details.

Limit cycle A limit cycle is a solution to a differential equation that is a periodic oscillation of finite amplitude (see page 78).

Linear differential equation A differential equation is said to be linear if the dependent variable appears only with an exponent of 0 or 1. For example, the equation $x^3y''' + y' + \cos x = 0$ is a linear equation, whereas the equation $yy' = 1$ is *nonlinear*.

Linearize To linearize a nonlinear differential equation means to approximate the equation by a linear differential equation in some region. For example, in regions where $|y|$ is "small," the nonlinear ordinary differential equation $y'' + \sin y = 0$ could be linearized to $y'' + y = 0$.

Linearizable Partial differential equations that can be solved either by an appropriate inverse scattering scheme or by a transformation to a linear partial differential equation are said to be linearizable.

Lipschitz condition If $f(x, y)$ is a bounded continuous function in a domain D, then $f(x, y)$ is said to satisfy a Lipschitz condition in y in D if

$$|f(x, y_1) - f(x, y_2)| \le K_y |y_1 - y_2|$$

for some finite constant K_y, independent of x, y_1, and y_2 in D. If, for some finite constant K_x, $f(x, y)$ satisfies

$$|f(x_1, y) - f(x_2, y)| \le K_x |x_1 - x_2|$$

independent of x_1, x_2, and y in D, then $f(x, y)$ satisfies a Lipschitz condition in x in D. If both of these conditions are satisfied and $K = \max(K_x, K_y)$, then $f(x, y)$ satisfies a Lipschitz condition in D, with Lipschitz constant K. This also extends to higher dimensions. See Coddington and Levinson [2] for details.

Maximum principle There are many "maximum principles" in the literature. The most common is "a harmonic function attains its absolute maximum on the boundary" (see page 560).

Mean value theorem This is a statement about the solution of Laplace's equation. It states, "If $\nabla^2 u = 0$ (in N dimensions), then $u(\mathbf{z}) = \int_S u\, dS / \int_S dS$ where S is the boundary of a N-dimensional sphere centered at z." For example, in $N = 2$, we have, "In 2 dimensions, the value of a solution to Laplace's equation at a point is the average of the values on any circle about that point." See Iyanaga and Kawada [8, page 624].

Metaparabolic equation A metaparabolic equation has the form $L[u] + M[u_t] = 0$, where $u = u(\mathbf{x}, t)$, $L[\cdot]$ is a linear differential operator in x of degree n, $M[\cdot]$ is a linear differential operator in x of degree m, and $m < n$. If, conversely, $m > n$, then the equation is called *pseudoparabolic*. See Gilbert and Jensen [5] for details.

Natural Hamiltonian A natural Hamiltonian is one having the form $H = T + V$, where $T = \frac{1}{2}\sum_{k=1}^{n} p_k^2$ and V is a function of the position variables only (i.e., $V = V(\mathbf{q}) = V(q_1, \dots, q_n)$).

Near identity transformation A near-identity transformation is a transformation in a differential equation from the old variables $\{a, b, c, \dots\}$ to the new variables $\{\alpha, \beta, \gamma, \dots\}$ via

$$\begin{aligned} a &= \alpha + A(\alpha, \beta, \gamma, \dots), \\ b &= \beta + B(\alpha, \beta, \gamma, \dots), \\ c &= \gamma + C(\alpha, \beta, \gamma, \dots), \\ &\vdots \end{aligned}$$

where $\{A, B, C, \dots\}$ are strictly nonlinear functions (i.e., there are no linear or constant terms). Very frequently $\{A, B, C, \dots\}$ are taken to be homogeneous polynomials (of, say, degree N) in the variables $\alpha, \beta, \gamma, \dots$, with unknown coefficients. For example, in two variables we might take

$$A(\alpha, \beta) = \sum_{j=0}^{n} A_{j,n-j}\alpha^j \beta^{n-j}, \qquad B(\alpha, \beta) = \sum_{j=0}^{n} B_{j,n-j}\alpha^j \beta^{n-j},$$

for some given value of n (see page 86).

Neumann problem The Neumann problem is a partial differential equation with Neumann data given on the boundaries. That is, the normal derivative of the dependent variable is given on the boundary. See Iyanaga and Kawada [8, page 999].

Normal form An ordinary differential equation is said to be in normal form if it can be solved explicitly for the highest derivative; i.e., $y^{(n)} = G(x,\ y,\ y', \dots, y^{(n-1)})$. A system of partial differential equations (with dependent variables $\{u_1, u_2, \dots, u_m\}$ and independent variables $\{x, y_1, y_2, \dots, y_k\}$) is said to be in normal form if it has the form

$$\frac{\partial^r u_j}{\partial x^r} = F_j\left(x, y_1, \dots, y_k, u_1, \dots, u_m, \frac{\partial u_1}{\partial x}, \dots, \frac{\partial^{r-1} u_m}{\partial x^{r-1}}, \dots, \frac{\partial u_1}{\partial y_1}, \dots, \frac{\partial^r u_m}{\partial {y_k}^r}\right),$$

for $j = 1, 2, \dots, m$. See page 86 or Iyanaga and Kawada [8, page 988].

Normal type An evolution equation is of normal type if it can be written in the form $u_t = u_n + h(u, u_1, \dots, u_m)$ where $n > m$ and $u_j = \partial^j u / \partial x^j$.

Nonlinear A differential equation that is not linear in the dependent variable is nonlinear.

Nonoscillatory The real solution $y(x)$ of $y_{xx} + f(x)y = 0$ is said to be nonoscillatory in the wide sense in $(0, \infty)$ if there exists a finite number c such that the solution has no zeros in $[c, \infty]$.

Order of a differential equation The order of a differential equation is the greatest number of derivatives in any term in the differential equation. For example, the partial differential equation $u_{xxxx} = u_{tt} + u^5$ is of fourth order whereas the ordinary differential equation $v_x + x^2 v^3 + v = 3$ is of first order.

Orthogonal Two vectors, $\mathbf{x}$ and $\mathbf{y}$, are said to be orthogonal with respect to the matrix W if $\mathbf{x}^{\mathrm{T}} W \mathbf{y} = 0$ (often, W is taken to be the identity matrix). Two functions, say $f(x)$ and $g(x)$, are said to be orthogonal with respect to a weighting function $w(x)$ if $(f(x), g(x)) := \int f(x) w(x) \bar{g}(x)\, dx = 0$ over some appropriate range of integration. Here, an overbar indicates the complex conjugate.

Oscillatory Consider the equation $y'' + f(x)y = 0$ and the number of zeros it has in the interval $[0, \infty]$. If the number of zeros is infinite, then the equation (and the solutions) are called *oscillatory*.

Padé approximant A Padé approximant is a ratio of polynomials. The polynomials are usually chosen so that the Taylor series of the ratio is a prescribed function. See page 582.

Particular solution Given a linear differential equation, $L[y] = f(\mathbf{x})$, the general solution can be written as $y = y_p + \sum_i C_i y_i$ where y_p, the particular solution, is any solution that satisfies $L[y] = f(\mathbf{x})$. The y_i are homogeneous solutions that satisfy $L[y] = 0$, and the $\{C_i\}$ are arbitrary constants. If $L[\cdot]$ is an nth order differential operator, then there will be n linearly independent homogeneous solutions.

Poisson bracket If f and g are functions of $\{p_j, q_j\}$, then the Poisson bracket of f and g is defined to be

$$[f, g] = \sum_j \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right) = \sum_j \frac{\partial(f, g)}{\partial(q_j, p_j)} = -\,[g, f]\,.$$

The Poisson bracket is invariant under a change of independent variables. See Goldstein [6] or Olver [11] for details.

Quasilinear equation Used in two different senses:

- A partial differential equation is said to be quasilinear if it is linear in the first partial derivatives. That is, it has the form $\sum_{k=1}^{n} A_k(u, \mathbf{x}) \frac{\partial u}{\partial x_k} = B(u, \mathbf{x})$ when the dependent variable is $u(\mathbf{x}) = u(x_1, \ldots, x_n)$ (see page 432).
- A partial differential equation is said to be quasilinear if it has the form $u_t = g(u) u_{x(n)} + f(u, u_x, y_{x(2)}, \ldots, u_{x(n-1)})$ for $n \geq 2$.

Radiation condition The radiation condition states that a wave equation has no waves incoming from an infinite distance, only outgoing waves.

For example, the equation $u_{tt} = \nabla^2 u$ might have the radiation condition $u(x,t) \simeq A_- \exp(ik(t-x))$ as $x \to -\infty$ and $u(x,t) \simeq A_+ \exp(ik(t+x))$ as $x \to +\infty$. This is also called the Sommerfeld radiation condition. See Butkov [1, page 617] for details.

Riemann's $\boldsymbol{P}$ function Riemann's differential equation (see page 186) is the most general second order linear ordinary differential equation with three regular singular points. If these singular points are taken to be a, b, and c and the exponents of the singularities are taken to be α, α'; β, β'; γ, γ' (where $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$), then the solution to Riemann's differential equation may written in the form of Riemann's P function as

$$y(x) = P \begin{bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{bmatrix}.$$

Robbins problem An elliptic partial differential equation with mixed boundary conditions is called a Robbins problem. See Iyanaga and Kawada [8, page 999].

Schwarzian derivative If $y = y(x)$, then the Schwarzian derivative of y with respect to x is defined to be

$$\{y, x\} \equiv \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2.$$

If $y = y(x)$ and $z = z(x)$, then $\{z, x\} = \{z, y\}\left(\frac{dy}{dx}\right)^2 + \{y, x\}$. Therefore, $\{x, y\} = -\left(\frac{dx}{dy}\right)^2 \{y, x\}$. Note also that $\{y, x\}$ is the unique elementary function of the derivatives, which is invariant under homographic transformations of x; that is, $\{y, x\} = \left\{y, \frac{ax+b}{cx+d}\right\}$, where (a, b, c, d) are arbitrary constants with $ad - bc = 1$. See Ince [7, page 394].

Semi-Hamiltonian A diagonal system of equations having the form $A_i(\mathbf{u})\partial_t u_i = B_i(\mathbf{u})\partial_x u_i$ is called semi-Hamiltonian if the coefficients satisfy $B_i \partial_{u_i} A_k = A_i \partial_{u_i} B_k$ for $i \neq k$.

Semilinear equations A partial differential equation is said to be semilinear if it has the form $u_t = u_{x(n)} + f(u, u_x, y_{x(2)}, \dots, u_{x(n-1)})$ for $n \geq 2$.

Shock A shock is a narrow region in which the dependent variable undergoes a large change. Also called a "layer" or a "propagating discontinuity." See page 432.

Singular point Given the homogeneous nth order linear ordinary differential equation

$$y^{(n)} + q_{n-1}(x)y^{(n-1)} + q_{n-2}(x)y^{(n-2)} + \cdots + q_0(x)y = 0,$$

the point x_0 is classified as being an

Ordinary point: if each of the $\{q_i\}$ are analytic at $x = x_0$.
Singular point: if it is not an ordinary point.
Regular singular point: if it is not an ordinary point and $(x - x_0)^i q_i(x)$ is analytic for $i = 0, 1, \ldots, n$.
Irregular singular point: if it is not an ordinary point and not a regular singular point.

The point at infinity is classified by changing variables to $t = x^{-1}$ and then analyzing the point $t = 0$. See page 403.

Singular solution A singular solution is a solution of a differential equation that is not derivable from the general solution by any choice of the arbitrary constants appearing in the general solution. Only nonlinear equations have singular solutions. See page 623.

Stability The solution to a differential equation is said to be stable if small perturbations in the initial conditions, boundary conditions, or coefficients in the equation itself lead to "small" changes in the solution. There are many different types of stability that are useful.

Stable A solution $\mathbf{y}(x)$ of the system $\mathbf{y}' = f(\mathbf{y}, x)$ that is defined for $x > 0$ is said to be stable if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that any solution $\mathbf{w}(x)$ of the system satisfying $|\mathbf{w}(0) - \mathbf{y}(0)| < \delta$ also satisfies $|\mathbf{w}(x) - \mathbf{y}(x)| < \epsilon$.
Asymptotic stability The solution $\mathbf{u}(x)$ is said to be asymptotically stable if, in addition to being stable, $|\mathbf{w}(x) - \mathbf{u}(x)| \to 0$ as $x \to \infty$.
Relative stability The solution $\mathbf{u}(x)$ is said to be relatively stable if $|\mathbf{w}(0) - \mathbf{u}(0)| < \delta$ implies that $|\mathbf{w}(x) - \mathbf{u}(x)| < \epsilon \mathbf{u}(x)$.

See page 101 or Coddington and Levinson [2, Chapter 13] for details.

Stefan problem A Stefan problem is one in which the boundary of the domain must be solved as part of the problem. For instance, when a jet of water leaves an orifice, not only must the fluid mechanics equations be solved in the stream, but the boundary of the stream must also be determined. Stefan problems are also called free boundary problems (see page 311).

Superposition principle If $u(\mathbf{x})$ and $v(\mathbf{x})$ are solutions to a linear differential equation (ordinary or partial), then the superposition principle states that $\alpha u(\mathbf{x}) + \beta v(\mathbf{x})$ is also a solution, where α and β are any constants (see page 413).

Total differential equation A total differential equation is an equation of the form: $\sum_k a_k(\mathbf{x})\, dx_k = 0$. See page 384.

Trivial solution The trivial solution is the identically zero solution.

Turning points Given the equation $y'' + p(x)y = 0$, points at which $p(x) = 0$ are called turning points. The asymptotic behavior of $y(x)$ can change at these points. See page 645 or Wasow [13].

Weak solution A weak solution to a differential equation is a function that satisfies only an integral form of the defining equation. For example, a weak solution of the differential equation $a(x)y'' - b(x) = 0$ only needs to satisfy $\int_S [a(x)y'' - b(x)]\, dx = 0$ where S is some appropriate region. For this example, the weak solution may not be twice differentiable everywhere. See Zauderer [14, pages 288–294] for details.

Well posed problems A problem is said to be well posed if a unique, stable solution that depends continuously on the data exists. See page 115.

Wronskian Given the smooth functions $\{y_1, y_2, \dots, y_n\}$, the Wronskian is the determinant

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If the Wronskian does not vanish in an interval, then the functions are linearly independent (see page 119).

References

[1] Butkov, E. *Mathematical Physics*. Addison–Wesley Publishing Co., Reading, MA, 1968.

[2] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations*. McGraw–Hill Book Company, New York, 1955.

[3] Courant, R., and Hilbert, D. *Methods of Mathematical Physics*. Interscience Publishers, Inc., New York, 1953.

[4] Erd elyi, A. *Asymptotic Expansions*. Dover Publications, Inc., New York, 1956.

[5] Gilbert, R. P., and Jensen, J. A computational approach for constructing singular solutions of one-dimensional pseudoparabolic and metaparabolic equations. *SIAM J. Sci. Stat. Comput. 3*, 1 (March 1982), 111–125.

[6] Goldstein, H. *Classical Mechanics*. Addison–Wesley Publishing Co., Reading, MA, 1950.

[7] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[8] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics*. MIT Press, Cambridge, MA, 1980.

[9] Lin, C. C., and Segel, L. A. *Mathematics Applied to Deterministic Problems in the Natural Sciences*. The MacMillan Company, New York, 1974.

[10] Moon, P., and Spencer, D. E. The meaning of the vector Laplacian. *J. Franklin Institute 256* (1953), 551–558.

[11] Olver, P. J. *Applications of Lie Groups to Differential Equations*. No. 107 in Graduate Texts in Mathematics. Springer–Verlag, New York, 1986.

[12] Stakgold, I. *Green's Functions and Boundary Value Problems*. John Wiley & Sons, New York, 1979.

[13] Wasow, W. *Linear Turning Point Theory*, vol. 54. Springer–Verlag, New York, 1985.

[14] Zauderer, E. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1983.

2. Alternative Theorems

Applicable to

Linear ordinary differential equations.

Idea

It is often possible to determine when a linear ordinary differential equation has a unique solution. Also, when the solution is not unique, it is sometimes possible to describe the degrees of freedom that make it non-unique.

Procedure

Alternative theorems describe, in some way, the type of solutions to expect from linear differential equations. The most common alternative theorems for differential equations were derived by Fredholm.

Suppose we wish to analyze the nth order linear inhomogeneous ordinary differential equation with boundary conditions

$$\begin{aligned} L[u] &= f(x), \\ B_i[u] &= 0, \qquad \text{for } i = 1, 2, \dots, n, \end{aligned} \tag{2.1}$$

for $u(x)$ on the interval $x \in [a, b]$. First, we must analyze the homogeneous equation and the adjoint homogeneous equation. That is, consider the two problems

$$\begin{aligned} L[u] &= 0, \\ B_i[u] &= 0, \qquad \text{for } i = 1, 2, \dots, n, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} L^*[v] &= 0, \\ B_i^*[v] &= 0, \qquad \text{for } i = 1, 2, \dots, n, \end{aligned} \tag{2.3}$$

where $L^*[\cdot]$ is the adjoint of $L[\cdot]$, and the $\{B_i^*[\cdot]\}$ are the adjoint boundary conditions (see page 95). Then Fredholm's alternative theorem states that

1. If the system in (2.2) has only the trivial solution, that is $u(x) \equiv 0$, then
   (a) the system in (2.1) has a unique solution.
   (b) the system in (2.3) has only the trivial solution.
2. Conversely, if the system in (2.2) has k linearly independent solutions, say $\{u_1, u_2, \dots, u_k\}$, then
   (a) the system in (2.3) has k linearly independent solutions, say $\{v_1, v_2, \dots, v_k\}$.

(b) the system in (2.1) has a solution if and only if the forcing function appearing in (2.1), f, is orthogonal to all solutions to the adjoint system. That is $(f, v_i) := \int_a^b f(x)v_i(x)\,dx = 0$ for $i = 1, 2, \ldots, k$.

(c) the solution to (2.1), if 2(b) is satisfied, is given by $u(x) = \bar{u}(x) + \sum_{j=i}^{k} c_j u_j(x)$ for arbitrary constants $\{c_j\}$, where $\bar{u}(x)$ is any solution to (2.1).

Example 1

Given the ordinary differential equation for $u(x)$

$$\begin{aligned} u' + u &= f(x), \\ u(0) &= 0, \end{aligned} \tag{2.4}$$

we form the homogeneous system

$$\begin{aligned} u' + u &= 0, \\ u(0) &= 0. \end{aligned} \tag{2.5}$$

Because (2.5) has only the trivial solution, we know that the solution to equation (2.4) is unique. By the method of integrating factors (see page 356), the solution to (2.4) is found to be $u(x) = \int_0^x f(t)e^{t-x}dt$.

Example 2

Given the ordinary differential equation for $u(x)$

$$\begin{aligned} u' + u &= f(x), \\ u(0) - eu(1) &= 0, \end{aligned} \tag{2.6}$$

we form the homogeneous system

$$\begin{aligned} u' + u &= 0, \\ u(0) - eu(1) &= 0. \end{aligned} \tag{2.7}$$

In this case, (2.7) has the single non-trivial solution $u(x) = e^{-x}$. Hence, the solution to (2.6) is *not* unique. To find out what restrictions must be placed on $f(x)$ for (2.6) to have a solution, consider the corresponding adjoint homogeneous equation

$$\begin{aligned} v' - v &= 0, \\ -ev(0) + v(1) &= 0. \end{aligned} \tag{2.8}$$

Since (2.8) has a single non-trivial solution, $v(x) = e^x$, we conclude that (2.6) has a solution if and only if

$$\int_0^1 f(t)e^t\,dt = 0. \tag{2.9}$$

If equation (2.9) is satisfied, then the solution of (2.6) will be given by

$$u(x) = Ce^{-x} + \int_0^x f(t)e^{t-x}dt$$

where C is an arbitrary constant.

Example 3

The solution(s) to $xy'' - (1+x)y' + y = 0$ depends on the boundary conditions as follows:

1. With $y(1) = 1$, $y'(1) = 2$, the solution is $y = 3e^{x-1} - (1+x)$.
2. With $y(0) = 1$, $y'(0) = 2$, there is no solution.
3. With $y(0) = 1$, $y'(0) = 1$, there are infinitely many solutions of the form $y = C(e^x - 1 - x) + 1 + x$.

Notes

1. Epstein [1, pages 83 and 111] discusses the Fredholm theorems in the general setting of a Banach space and a Hilbert space.
2. Interpretation of alternative theorems is usually straightforward when the underlying physics are understood. For example, the system
$$-u'' = f(x), \quad 0 < x < 1 \qquad u'(0) = a_1, \qquad -u'(1) = a_2$$
must satisfy the relation $\int_0^1 f(x)\,dx = a_1 + a_2$. This states that for a rod experiencing one-dimensional heat flow, a steady state is possible only if the heat supplied along the rod is removed at the ends.
3. A generalized Green's function is a Green's function (see page 318) for a differential equation that does not have a unique solution. See Greenberg [2] for more details.
4. The Sturm–Liouville problem for $u(x)$ on the interval $x_1 \le x \le x_2$
$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x) \tag{2.10}$$
$$-p(x_1)u'(x_1) + r_1u(x_1) = 0 \qquad p(x_2)u'(x_2) + r_2u(x_2) = 0$$
can be written as
$$\int_{x_1}^{x_2}\left[p(t)u'^2(t) + q(t)u^2(t)\right]\,dt + r_1u^2(x_1) + r_2u^2(x_2)$$
$$= \int_{x_1}^{x_2} f(t)u(t)\,dt + g_1u(x_1) + g_2u(x_2).$$
Hence, if $p(x)$ is positive, $q(x)$, r_1, and r_2 are non-negative and if $\int_{x_1}^{x_2} f(t)u(t)\,dt + g_1u(x_1) + g_2u(x_2) = 0$, then there is a unique solution to (2.10).
5. See also Haberman [3, pages 307–314] and Stakgold [4, pages 82–90, 207–214, and 319–323].

References

[1] Epstein, B. *Partial Differential Equations: An Introduction*. McGraw–Hill Book Company, New York, 1962.

[2] Greenberg, M. D. *Application of Green's Functions in Science and Engineering*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1971.

[3] Haberman, R. *Elementary Applied Partial Differential Equations*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1968.

[4] Stakgold, I. *Green's Functions and Boundary Value Problems*. John Wiley & Sons, New York, 1979.

3. Bifurcation Theory

Applicable to

Nonlinear differential equations.

Idea

Given a nonlinear differential equation that depends on a set of parameters, the number of distinct solutions may change as the parameters change. Points where the number of solutions change are called *bifurcation points*.

Procedure

Although bifurcations occur in all types of equations, we restrict our discussion to ordinary differential equations. Consider the autonomous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x};\boldsymbol{\alpha}), \tag{3.1}$$

where $\mathbf{x}$ and $\mathbf{f}$ are n-dimensional vectors and $\boldsymbol{\alpha}$ is a set of parameters. Define the Jacobian matrix by

$$J(\mathbf{x};\boldsymbol{\alpha}) := \frac{d\mathbf{f}}{d\mathbf{x}} = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{x};\boldsymbol{\alpha}) \mid i,j = 1,\dots,n\right). \tag{3.2}$$

Note that $J(\mathbf{x};\boldsymbol{\alpha})\mathbf{z}$ is the *Fréchet derivative* of $\mathbf{f}$, at the point $\mathbf{x}$ (see page 6). Using the solution $\mathbf{x}(t,\boldsymbol{\alpha})$ of equation (3.1), the values of $\boldsymbol{\alpha}$ where one or more of the eigenvalues of J are zero are defined to be bifurcation points. At such points, the number of solutions to equation (3.1) may change, and the stability of the solutions might also change.

If any of the eigenvalues have positive real parts, then the corresponding solution is unstable. If we are concerned only with the steady-state solutions of equation (3.1), as is often the case, then the bifurcation points will satisfy the simultaneous equations

$$\mathbf{f}(\mathbf{x};\boldsymbol{\alpha}) = \mathbf{0}, \qquad \text{and} \qquad \det J = 0. \tag{3.3}$$

Define the eigenvalues of the Jacobian matrix defined in equation (3.2) to be $\{\lambda_i \mid i = 1,\dots,n\}$. We now presume that equation (3.1) depends on the single parameter α. Suppose that the change in stability is at the point $\alpha = \widehat{\alpha}$, where the real part of a complex conjugate pair of eigenvalues ($\lambda_1 = \overline{\lambda}_2$) pass through zero:

$$\Re\lambda_1(\widehat{\alpha}) = 0, \qquad \Im\lambda_1(\widehat{\alpha}) > 0, \qquad \Re\lambda_1'(\widehat{\alpha}) \neq 0,$$

and, for all values of α near $\widehat{\alpha}$, $\Re\lambda_i(\alpha) < 0$ for $i = 3,\dots,n$.

Then, under certain smoothness conditions, it can be shown that a small amplitude periodic solution exists for α near $\widehat{\alpha}$. Let ϵ measure the

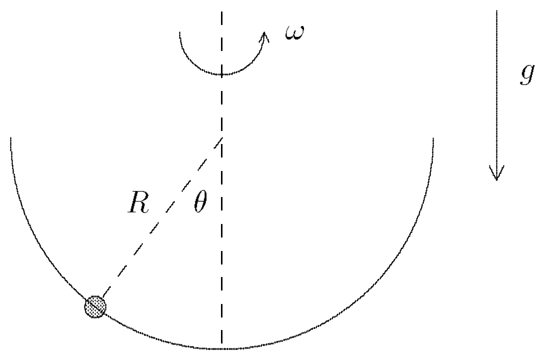

Figure 3.1: A bead on a spinning semi-circular wire.

amplitude of the periodic solution. Then there are functions $\mu(\epsilon)$ and $\tau(\epsilon)$, defined for all sufficiently small, real ϵ, such that $\mu(0) = \tau(0) = 0$ and that the system with $\alpha = \widehat{\alpha} + \mu(\epsilon)$ has a unique small amplitude solution of period $T = 2\pi\left(1 + \tau(\epsilon)\right)/\Im\lambda_1(\widehat{\alpha})$. When expanded, we have $\mu(\epsilon) = \mu_2\epsilon^2 + O(\epsilon^3)$. The sign of μ_2 indicates where the oscillations occur, i.e., for $\alpha < \widehat{\alpha}$ or for $\alpha > \widehat{\alpha}$.

Example 1

The nonlinear ordinary differential equation

$$\frac{du}{dt} = g(u) = u^2 - \lambda_1 u - \lambda_2 \tag{3.4}$$

has steady-state solutions that satisfy $g(u) = u^2 - \lambda_1 u - \lambda_2 = 0$. These steady-state solutions have bifurcation points given by

$$\frac{dg}{du} = 2u - \lambda_1 = 0.$$

Solving these last two equations simultaneously, it can be shown that the bifurcation points of the steady-state solutions are along the curve $4\lambda_2 + \lambda_1^2 = 0$. Further analysis shows that equation (3.4) will have two real steady-state solutions when $4\lambda_2 + \lambda_1^2 > 0$, and it will have no real steady-state solutions when $4\lambda_2 + \lambda_1^2 < 0$.

Example 2

Consider a frictionless bead that is free to slide on a semi-circular hoop of wire of radius R that is spinning at an angular rate ω (see figure 3.1). The equation for $\theta(t)$, the angle of the bead from the vertical, is given by

$$\frac{d^2\theta}{dt^2} + \frac{g\sin\theta}{R}\left(1 - \frac{\omega^2 R}{g}\cos\theta\right) = 0, \tag{3.5}$$

where g is the magnitude of the gravitational force. We define the parameter ν by $\nu = g/\omega^2 R$. We will analyze only the case $\nu \geq 0$.

The three possible steady solutions of equation (3.5) are given by

$$\begin{aligned} &\text{for } \nu \geq 0, \quad \theta(t) = \theta_1 = 0, \\ &\text{for } \nu \leq 1, \quad \theta(t) = \theta_2 = \cos^{-1}\nu, \\ &\text{for } \nu \leq 1, \quad \theta(t) = \theta_3 = -\cos^{-1}\nu. \end{aligned}$$

Therefore, for $\nu > 1$ (which corresponds to slow rotation speeds), the only steady solution is $\theta(t) = \theta_1$. For $\nu \leq 1$, however, there are three possible solutions. The solution $\theta(t) = \theta_1$ will be shown to be unstable for $\nu < 1$.

To determine which solution is stable in a region where there are multiple solutions, a stability analysis must be performed. This is accomplished by assuming that the true solution is slightly perturbed from the given solution, and the rate of change of the perturbation is obtained. If the perturbation grows, then the solution is unstable. Conversely, if the perturbation decays (stays bounded), then the solution is stable (neutrally stable).

First we perform a stability analysis for the solution $\theta(t) = \theta_1$. Define

$$\theta(t) = \theta_1 + \epsilon\phi(t), \tag{3.6}$$

where ϵ is a small number and $\phi(t)$ is an unknown function. Using (3.6) in equation (3.5), and expanding all terms for $\epsilon \ll 1$, results in

$$\frac{d^2\phi}{dt^2} + g\frac{\nu - 1}{\nu}\phi = O(\epsilon). \tag{3.7}$$

The leading order terms in equation (3.7) represent the Fréchet derivative of equation (3.5) at the "point" $\theta(t) = \theta_1$, applied to the function $\phi(t)$. The solution of this differential equation for $\phi(t)$, to leading order in ϵ, is

$$\phi(t) = A\cos\alpha t + B\sin\alpha t, \tag{3.8}$$

where A and B are arbitrary constants and $\alpha = \sqrt{g\left(\frac{\nu-1}{\nu}\right)}$. If $\nu > 1$, then α is real, and the solutions for $\phi(t)$ remain bounded. Conversely, if $\nu < 1$ then α becomes imaginary, and the solution in (3.8) becomes unbounded as t increases. Hence, the solution $\theta(t) = \theta_1$ is unstable for $\nu < 1$.

Now we perform a stability analysis for the solution $\theta(t) = \theta_2$. Writing $\theta(t) = \theta_2 + \epsilon\psi(t)$ and using this form in equation (3.5) leads to the equation for $\psi(t)$:

$$\frac{d^2\psi}{dt^2} + g\frac{1 - \nu^2}{\nu}\psi = O(\epsilon). \tag{3.9}$$

The leading order terms in equation (3.9) represent the Fréchet derivative of equation (3.5) at the "point" $\theta(t) = \theta_2$, applied to the function $\psi(t)$. The

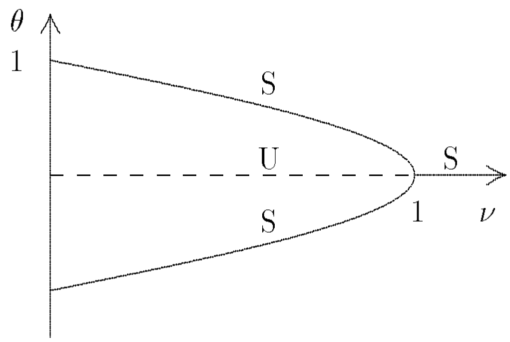

Figure 3.2: Bifurcation diagram for equation 3.6. A branch with the label "S" ("U") is a stable (unstable) branch.

solution of this differential equation for $\psi(t)$ is $\psi(t) = A\cos\beta t + B\sin\beta t$, where A and B are arbitrary constants and $\beta = \sqrt{g\left(\frac{1-\nu^2}{\nu}\right)}$. If $\nu < 1$, then β is real and the solutions for $\psi(t)$ remain bounded. Therefore, the solution $\theta(t) = \theta_2$ is stable for $\nu < 1$. In an exactly analogous manner, $\theta(t) = \theta_3$ is stable for $\nu < 1$.

From what we have found, we can construct the *bifurcation diagram* shown in figure 3.2. In this diagram, the unstable steady solutions are indicated by a dashed line and the letter "U", and the stable steady solutions are indicated by the solid line and the letter "S". In words, this diagram states:

- For no rotation ($\omega = 0$ or $\nu = \infty$), the only solution is $\theta(t) = \theta_1 = 0$.
- As the frequency of rotation increases (and so ν decreases), the solution $\theta(t) = \theta_1$ becomes unstable at the bifurcation point $\nu = 1$.
- For $\nu < 1$, the are two stable solutions, $\theta(t) = \theta_2$ and $\theta(t) = \theta_3$. In this example, there is no way to know in advance which of these two solutions will occur (physically, the bead can slide up either side of the wire).

The formula in (3.3) can be applied to equation (3.5) to determine the location of the bifurcation point without performing all of the above analysis. If we define $x_1 = \theta$ and $x_2 = \frac{d\theta}{dt}$, then equation (3.5) can be written as the system of ordinary differential equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_2 \\ -g\sin x_1\left(1 - \frac{\cos x_1}{\nu}\right) \end{bmatrix},$$

which has the Jacobian matrix

$$J = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ g\cos x_1 + \frac{g}{\nu}\left(\cos^2 x_1 - \sin^2 x_1\right) & 0 \end{bmatrix}.$$

If $\nu > 1$, then no choice of (x_1, x_2) will allow both $\mathbf{f}$ and $\det J$ to be zero simultaneously. For $\nu = 1$, however, $x_1 = x_2 = 0$ make both $\mathbf{f}$ and $\det J$ equal to zero. Hence, a bifurcation occurs at $\nu = 1$.

Example 3

Abelson [1] has developed a computer program in LISP that automatically explores the steady-state orbits of one-parameter families of periodically driven oscillators. The program generates both textual descriptions and schematic diagrams.

For example, consider Duffing's equation in the form $\ddot{x} + 0.1\dot{x} + x^3 = p\cos t$, where the parameter p is in the range $[1, 25]$ and only those solutions with $-5 \leq \dot{x} \leq 5$ and $-10 \leq \ddot{x} \leq 10$ are considered. The program produced the graphical output shown in figure 3.3, along with the following textual description:

> The system was explored for values of p between 1 and 25, and 10 classes of stable periodic orbits were identified.
>
> Class A is already present at the start of the parameter range $p = 1$ with a family of order-1 orbits A_0. Near $p = 2.287$, there is a supercritical-pitchfork bifurcation, and A_0 splits into symmetric families $A_{1,0}$ and $A_{1,1}$, each of order 1. $A_{1,0}$ vanishes at a fold bifurcation near $p = 3.567$. $A_{1,1}$ vanishes similarly.
>
> Class B appears around $p = 3.085$ with a family of order-1 orbits B_0 arising from a fold bifurcation. As the parameter p increases, B_0 undergoes a period doubling cascade, reaching order 2 near $p = 4.876$, and order 4 near $p = 5.441$. Although the cascade was not traced past the order 4 orbit, there is apparently another period-doubling near $p = 5.52$, and a chaotic orbit was observed at $p = 5.688$.
>
> ⋮
>
> Class J appears around $p = 23.96$ as a family of order-5 orbits J_0 arising from a fold bifurcation. J_0 is present at the end of the parameter range at $p = 25$.

This program is capable of recognizing the following types of bifurcations: fold bifurcations, supercritical and subcritical flip bifurcations, supercritical and subcritical Niemark bifurcations, supercritical and subcritical pitchfork bifurcations, and transcritical bifurcations.

Notes

1. There are many different types of bifurcations. See figure 3.4 for diagrams of some of the following bifurcations:

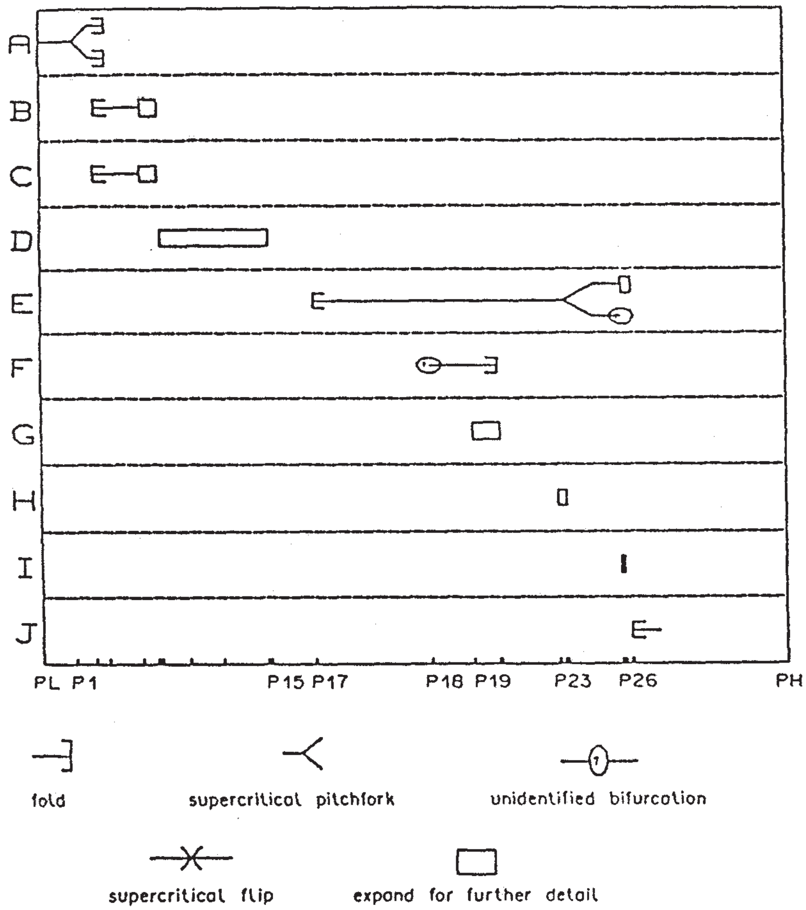

Figure 3.3: Graphical output generated automatically from the Bifurcation Interpreter in Abelson [1]. For Duffing's equation, the evolution of 10 classes of families of periodic orbits and their bifurcations has been traced. The p values along the horizontal axis indicate the parameter value at which the bifurcations occur. (Reprinted with permission from *Comp & Maths. With Appls. 20*, 8, Abelson, H., The bifurcation interpreter: A step towards the automatic analysis of dynamical systems, Copyright 1990, Pergamon Press.)

- *Hopf bifurcation*: a stable steady solution bifurcates into a stable oscillatory solution. That is, there are no stable steady solutions in that particular region of parameter space. This occurs by having some of the eigenvalues of the Jacobian in (3.2) become

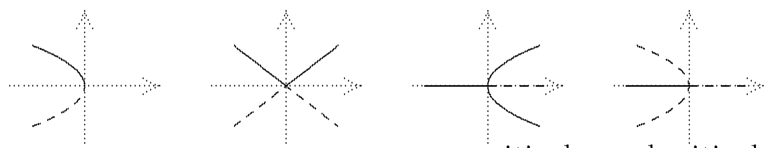

Figure 3.4: Diagrams of some types of bifurcations. Unstable solutions are indicated by dashed lines; stable solutions are indicated by solid lines.

purely imaginary.

- *Fold bifurcation*: on one side of the bifurcation point a stable and an unstable periodic point (of the same order) coexist. On the other side of the bifurcation point, both periodic points have vanished.
- *Flip bifurcation (supercritical)*: a stable periodic point of order n transitions to a stable periodic point of order $2n$ and an unstable periodic point of order n.
- *Flip bifurcation (subcritical)*: an unstable periodic point of order $2n$ and a stable periodic point of order n transition to an unstable periodic point of order n.
- *Niemark bifurcation (supercritical)*: a stable periodic transitions to an unstable periodic point and a stable limit cycle.
- *Niemark bifurcation (subcritical)*: a stable periodic point and unstable limit cycle transition to an unstable periodic point.
- *Pitchfork bifurcation (supercritical)*: a stable periodic point transitions to two stable periodic points and an unstable periodic point, all of the same order.
- *Pitchfork bifurcation (subcritical)*: a stable periodic point and two unstable periodic points transition to an unstable periodic point.
- *Transcritical bifurcation*: a stable periodic point and an unstable periodic point exchange stabilities; on the other side of the bifurcation point, the extrapolated stable point is now unstable, and vice-versa.

2. For a differential equation that is not autonomous, bifurcations can also occur from time-dependent solutions to other time-dependent solutions.
3. For the general finite dimensional mapping, $G(\mathbf{x})$, from R^m to R^n, the Jacobian $J(\mathbf{x}) := \frac{\partial G}{\partial \mathbf{x}}$ need not be square. In this case, the critical points (which include the bifurcation points) are in the set C, with

$$C := \{\mathbf{x} \mid \mathbf{x} \in \mathrm{R}^m,\ \mathrm{rank}\ J(\mathbf{x}) < \min(m, n)\}\,.$$

The *regular points* are $\mathrm{R}^m - C$. The critical values are the values in the set $G(C) := \{\mathbf{y} \mid \mathbf{y} \in \mathrm{R}^n, \mathbf{y} = G(\mathbf{x}) \text{ for some } \mathbf{x} \in C\}$. The regular values are $\mathrm{R}^n - G(C)$.

4. Sacks [8] describes the program POINCARE, which classifies bifurcation points and constructs representative phase diagrams for each type of behavior. The program is available directly from Sacks.
5. Numerical methods for computing bifurcations are described in Guckenheimer *et al.* [3] and Jepson and Spence [6].

References

[1] Abelson, H. The bifurcation interpreter: A step towards the automatic analysis of dynamical systems. *Comp. & Maths. with Appls. 20*, 8 (1990), 13–35.

[2] Guckenheimer, J. Patterns of bifurcations. In *New Approaches to Nonlinear Problems in Dynamics*, P. J. Holmes, Ed. SIAM, Philadelphia, PA, 1980, pp. 71–104.

[3] Guckenheimer, J., Myers, M., and Sturmfels, B. Computing Hopf bifurcations. *SIAM J. Numer. Anal. 34*, 1 (February 1997), 1–21.

[4] Holodniok, M., and Kubcek, M. New algorithms for the evaluation of complex bifurcation points in ordinary differential equations. A comparative numerical study. *Appl. Math. and Comp. 15* (1984), 261–274.

[5] Iooss, G., and Joseph, D. D. *Elementary Stability and Bifurcation Theory.* Springer–Verlag, New York, 1989.

[6] Jepson, A. D., and Spence, A. Numerical methods for bifurcation problems. In *The State of the Art in Numerical Analysis*, A. Iserles and M. J. D. Powell, Eds. Clarendon Press, Oxford, England, 1987, pp. 273–298.

[7] Rand, R. H., and Armbruster, D. *Perturbation Methods, Bifurcation Theory and Computer Algebra.* No. 65 in Applied Mathematical Sciences. Springer–Verlag, New York, 1987.

[8] Sacks, E. Automatic analysis of one-parameter planar ordinary differential equations by intelligent numeric simulation. *Artificial Intelligence 48* (1991), 27–56.

[9] Seydel, R. *From Equilibrium to Chaos: Practical Bifurcation and Stability Analysis.* American Elsevier Publishing Company, New York, 1988.

[10] Wood, E. F., Kempf, J. A., and Mehra, R. K. Bistab: A portable bifurcation and stability analysis package. *Appl. Math. and Comp. 15* (1984), 345–355.

4. A Caveat for Partial Differential Equations

Idea

To solve partial differential equations correctly, a good understanding of the nature of the partial differential equation is required. This requires more than a knowledge of the "physics" of the problem: a thorough understanding of the type of partial differential equation is needed. From Collatz [1, page 260]:

> That an investigation of the situation is absolutely essential is revealed even by quite simple examples; they show that formal calculation applied to partial differential equations can lead to false results very easily and that approximate methods can converge in a disarmingly innocuous manner to values bearing no relation to the correct solution.

Example

Suppose we wish to solve the following wave equation (this example is from Collatz [1])

$$\begin{aligned} u_{xx} &= u_{tt}, \\ u(x,0) &= \cos x, \text{for } |x| < \pi/2, \\ \frac{\partial u(x,0)}{\partial t} &= \cos x, \text{for } |x| < \pi/2, \\ u\left(\pm\frac{\pi}{2}, t\right) &= \sin t, \quad \text{for } t > 0. \end{aligned} \tag{4.1.a-d}$$

We will attempt to solve (4.1) by looking for a series solution of the form

$$u(x,t) = \sum_{n,m=0}^{\infty} a_{mn} x^m t^n. \tag{4.2}$$

Using (4.2) in (4.1.a), we find that

$$a_{m,n+2} = \frac{(m+2)(m+1)}{(n+2)(n+1)} a_{m+2,n}. \tag{4.3}$$

To satisfy (4.1.b), we require $a_{k,1} = 0$. To satisfy (4.1.c), we also require

$$a_{k,0} = \begin{cases} 0, & k = \text{odd}, \\ (-1)^q/(2q)!, & k = \text{even} = 2q. \end{cases} \tag{4.4}$$

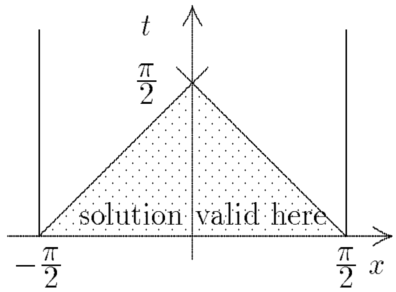

Figure 4.1: Depiction of the characteristics and the range of validity of the solution found for equation 4.1.

Evaluating equation (4.2) at $x = 0$ and using equations (4.3) and (4.4), we find that

$$u(0,t) = \sum_{k=0}^{\infty} a_{0,k} t^k = \sum_{q=0}^{\infty} \frac{(-1)^q}{(2q)!} t^{2q} = \cos t. \tag{4.5}$$

Now the conclusion in equation (4.5) is correct but *only* for $0 \le t \le \pi/2$. This is because the characteristics (see page 432), $t = \pi/2 \pm x$, emanating from the points $(\pi/2, 0)$ and $(-\pi/2, 0)$ do not allow $u(0,t)$ to be determined directly for $t > \pi/2$.

See figure 4.1 for a graphical representation of the characteristics of (4.1) and the region of validity for the solution in (4.5).

References

[1] Collatz, L. *The Numerical Treatment of Differential Equations*. Springer–Verlag, New York, 1966.

[2] Rassias, J. M. *Counter Examples in Differential Equations and Related Topics*. World Scientific, Singapore, 1991.

5. Chaos in Dynamical Systems

Applicable to Nonlinear differential equations.

Yields

Information on whether or not a system is chaotic.

Idea

Chaos is a phenomenon that can appear in solutions to nonlinear differential equations. Chaos is easily defined and can be easily (numerically) found in some equations.

Procedure

For simplicity, we focus on deterministic systems modeled by coupled, autonomous, first order, ordinary differential equations of the form

$$\frac{dx_i}{dt} = g_i(\mathbf{x}; \mathbf{q}) \quad \text{for } i = 1, 2, \ldots, n \tag{5.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the state-space vector and $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ is a set of parameters. This equation determines a set of solutions, each specified by their initial values. We can specify the solution corresponding to the initial condition $\mathbf{p}$ by $\mathbf{x}(t; \mathbf{p})$.

Consider a set of initial conditions contained in a vanishing small volume V. Under the action of equation (5.1), the volume will change as a function of t. Precisely,

$$\frac{dV}{dt} = \underset{V}{\int \cdots \int} \left(\sum_{i=1}^{m} \frac{\partial g_i}{\partial x_i} \right) dx_1 \cdots dx_n.$$

The summation term is the generalized divergence of $\mathbf{g}$ and is called the Lie derivative. Dissipative systems are characterized by contracting volumes; this is equivalent to $dV/dt < 0$. Conservative or Hamiltonian systems, in which equation (5.1) are Hamilton's equations, obey Liouville's theorem: $dV/dt = 0$.

Any trajectory of a dissipative system as $t \to \infty$ will approach a bounded region of phase space called an attractor. An attractor has zero volume in phase space. Attractors include points, limit cycles, and tori. For example, consider an unforced damped pendulum. The attractor for this is a point in phase space, the stable configuration with the pendulum hanging straight down. In this case, starting the pendulum swinging with slightly different initial conditions will lead to close paths in phase space and the same final state.

For nonlinear systems exhibiting chaos, the separation of two nearby trajectories increases exponentially with time. This is referred to as *sensitive dependence on initial conditions*. For dissipative systems, a stretching in one direction has to be accompanied by a more-than-compensating contraction in other directions, so that the volume of an arbitrary droplet of initial conditions will contract with time. The phase-space trajectories for a chaotic system asymptotically approach a *strange attractor*, an attractor with a fractional dimension (i.e., a fractal).

Lyapunov exponents are a measure of the rate of divergence (or convergence) of initially infinitesimally separated trajectories. The ith Lyapunov exponent, λ_i, can be found by considering the evolution of a vanishingly small set of initial conditions that form a hyperellipsoid. We define

$$\lambda_i := \lim_{\substack{t\to\infty \\ \rho_i(0)\to 0}} \left[\frac{1}{t}\left(\frac{\rho_i(t)}{\rho_i(0)}\right)\right] \tag{5.2}$$

where $\rho_i(t)$ is the length of the ith principal axis of the hyperellipsoid at time t, for $i = 1, 2, \ldots, n$. An attractor is chaotic if it has at least one positive Lyapunov exponent.

The Lyapunov exponents can be determined by analyzing the linearized equations corresponding to equation (5.1). For illustrative purposes, we specialize to $n = 3$ for the rest of this section. Consider the two close initial points: $\mathbf{p}_0 = (x_0, y_0, z_0)$ and $\mathbf{p}_1 = \mathbf{p}_0 + \delta\mathbf{x} = (x_0 + \delta x, y_0 + \delta y, z_0 + \delta z)$. We want to find the evolution of the difference $\mathbf{a}(t) := \mathbf{x}(t; \mathbf{p}_1) - \mathbf{x}(t; \mathbf{p}_0)$. Using Taylor series

$$\begin{aligned}
\frac{da_1}{dt} &= \frac{d[x_1(t;\mathbf{p}_1) - x_1(t;\mathbf{p}_0)]}{dt} = \frac{d[g_1(\mathbf{x}(t;\mathbf{p}_0+\delta\mathbf{x})) - g_1(\mathbf{x}(t;\mathbf{p}_0))]}{dt} \\
&\approx \frac{\partial g_1}{\partial x}\delta x + \frac{\partial g_1}{\partial y}\delta y + \frac{\partial g_1}{\partial z}\delta z \\
&= \frac{\partial g_1}{\partial x}a_1 + \frac{\partial g_1}{\partial y}a_2 + \frac{\partial g_1}{\partial z}a_3,
\end{aligned}$$

where the partial derivatives are evaluated at $\mathbf{x}(t; \mathbf{p}_0)$. In general

$$\frac{d\mathbf{a}}{dt} = M(\mathbf{x})\mathbf{a} = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z} \end{bmatrix} \mathbf{a},$$

where M is the Jacobian of the vector $\mathbf{g}$. The Lyapunov exponents are related to the eigenvalues of the matrix M.

In special situations, analytical methods can be used to obtain the Lyapunov spectra, while numerical methods must be used in general. When there is a stationary solution given by $\frac{d\mathbf{x}}{dt} = \mathbf{g}(\mathbf{x}) = \mathbf{0}$, the Jacobian matrix is time independent, and we can analytically obtain the (possibly complex)

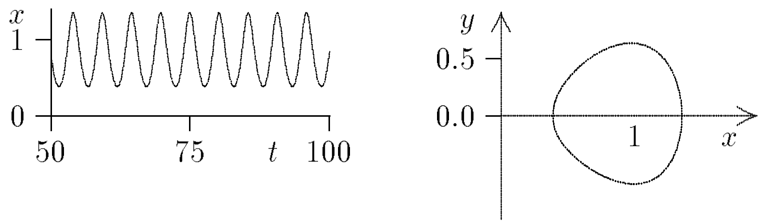

Figure 5.1: Duffing equation with $\Gamma = 0.20$. (Period 1 solution.)

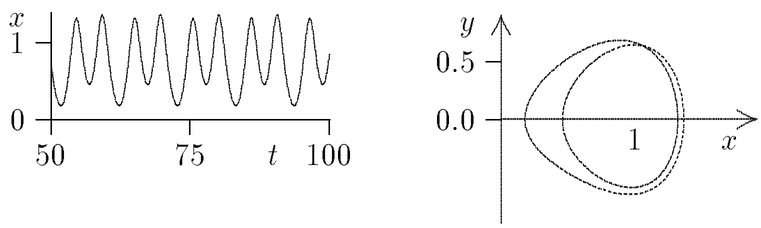

Figure 5.2: Duffing equation with $\Gamma = 0.28$. (Period 2 solution.)

eigenvalues, from which the Lyapunov exponents may be found. In general, there are no stationary solutions and the equations $\frac{d\mathbf{x}}{dt} = \mathbf{g}$ and $\frac{d\mathbf{a}}{dt} = M(\mathbf{x})\mathbf{a}$ must be numerically solved simultaneously. See Wolf *et al.* [12] for a numerical technique for computing Lyapunov exponents.

Example

Consider the Duffing equation: $\ddot{x} + k\dot{x} - x + x^3 = \Gamma\cos\omega t$. This can be converted to an autonomous system as follows:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ -ky + x - x^3 + \Gamma\cos z \\ \omega \end{bmatrix}. \tag{5.3}$$

Figures 5.1–5.3 show the different behavior of this system ($x(t)$ versus t and $x(t)$ versus $y(t)$) when $k = 0.3$, $\omega = 1.2$, and Γ takes on the values 0.20, 0.28 and 0.50. For the numerical simulations shown, the initial conditions used were $\mathbf{x}_0 = (1.3, 0, 0)$, and we began plotting the results when $t = 50$ to remove any initial transients. From deeper analysis, it can be shown that the system has a period 1 (2, 4, 5, 2, 1) solution when $\Gamma = 0.20$ (0.28, 0.29, 0.37, 0.65, 0.73). The solution is chaotic when $\Gamma = 0.50$.

A different set of parameters is shown in figure 5.4. This figure has a plot of the three Lyapunov exponents of equation (5.3) when $\omega = 1.0$, $k = 0.5$, and Γ is varied from 0.2 to 0.9. At low values of Γ, the system is periodic because the largest Lyapunov exponent is zero. The system follows

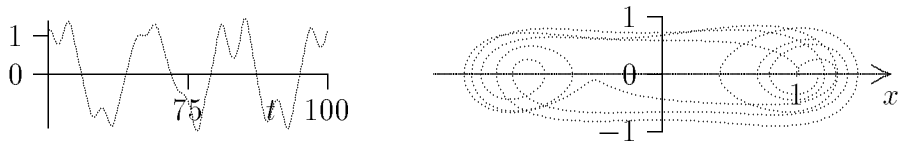

Figure 5.3: Duffing equation with $\Gamma = 0.50$. (Chaotic solution.)

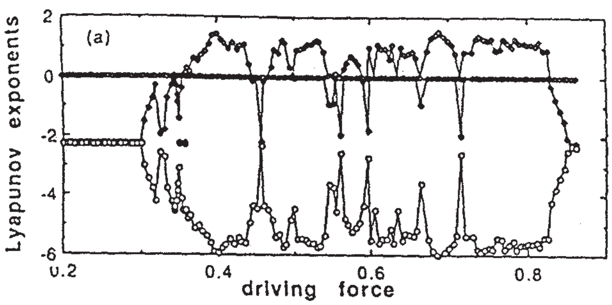

Figure 5.4: The three Lyapunov exponents for Duffing's equation with $\omega = 1.0$ and $k = 0.5$ when Γ is varied from 0.2 to 0.9. (From De Souza-Machado, S., Rollins, R. W., Jacobs, D. T., & Hartman, J. L. Studying chaotic systems using microcomputer simulations and Lyapunov exponents. *Amer. J. Physics 58*, 4, April 1990, 321–329.)

a period doubling route to chaos at $\Gamma \approx 0.36$, when the largest Lyapunov exponent becomes greater than zero. The system remains chaotic until the driving force gets very large ($\Gamma > 0.84$) except for windows of periodicity, which occur throughout the chaotic regime.

Notes

1. There are at least three scenarios in which the regular behavior of a system becomes chaotic. A standard route is via a series of period-doubling bifurcations. Two other routes to chaos that are fairly well understood are via intermittent behavior and through quasiperiodic solutions.
2. Many equations have been shown to be chaotic:

   - Hale and Sternberg [4] have shown that the differential delay equation $\frac{dx(t)}{dt} = ax(t) + b\frac{x(t-\tau)}{1+x^n(t-\tau)}$ is chaotic for certain parameter regimes.

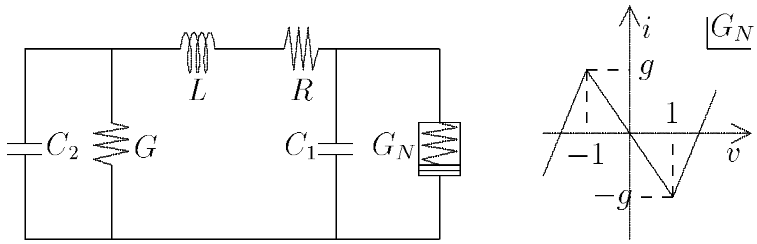

Figure 5.5: The canonical piecewise-linear circuit and the voltage-current characteristic of the nonlinear resistor G_N.

- The equations defining the Lorenz attractor are

$$\begin{aligned}\dot{x} &= 10y - 10x,\\ \dot{y} &= -y - xz + 28x,\\ \dot{z} &= xy - \frac{8}{3}z.\end{aligned} \tag{5.4}$$

- The Rössler equations are

$$\begin{aligned}\dot{x} &= -(y + z),\\ \dot{y} &= x + ay,\\ \dot{z} &= b + xz - cz.\end{aligned} \tag{5.5}$$

  When $a = 0.343$, $b = 1.82$, and $c = 9.75$, this generates the "Rössler funnel." When $a = 0.2$, $b = 0.2$, and $c = 5.7$, this generates "the simple Rössler attractor."

3. For an autonomous electronic circuit to exhibit chaos, it must contain at least three energy storage devices. (Otherwise, the Poincaré–Bendixson theorem states that the limiting set will be a point or a limit cycle, not a strange attractor.) A simple circuit with three energy storage devices that produces chaos is in Matsumoto [7].
   The circuit given in Chua and Lin [2] (see figure 5.5) is almost as simple as that given by Matsumoto and can simulate (by choosing different values for the nonlinear resistor) different chaotic phenomena in a large three-dimensional state space. This circuit contains only six two-terminal elements: Five of them are linear resistors, capacitors, and inductors; and one element (G_N) is a three-segment, piecewise-linear resistor.
4. Different types of dynamical systems can have greater or lesser degrees of randomness. A simple classification of the amount of randomness in dynamical systems is as follows:
   - *Ergodic systems*: this is the "weakest" level of randomness, in which phase averages equal time averages.

- *Mixing systems*: here, no time averaging is required to reach "equilibrium."
- *K-systems*: systems with positive Kolmogorov entropy. This means that a connected neighborhood of trajectories must exhibit a positive average rate of exponential divergence.
- *C-systems*: every trajectory has a positive Lyapunov exponent.
- *Bernoulli systems*: these systems are as random as a fair coin toss.

See Tabor [11] for details.

5. A technical definition of the Lyaponuv exponents is as follows: When $A(t)$ is a bounded coefficient matrix, consider the n-dimensional linear system $\mathbf{y}' = A(t)\mathbf{y}(t)$. Consider n linearly independent solutions of this in the form $\mathbf{y}_i = Y(t)\mathbf{p}_i$, where $Y(t)$ is a fundamental solution matrix with $Y(0)$ orthogonal, and the $\{\mathbf{p}_i\}$ form an orthonormal basis. The characteristic numbers are defined as

$$\lambda_i = \lim_{t\to\infty} \sup \frac{1}{t} \log \left(||Y(t)\mathbf{p}_i|| \right).$$

When the sum of the characteristic numbers is minimized, the orthogonal basis $\{\mathbf{p}_i\}$ is called *normal* and the $\{\lambda_i\}$ are the *Lyapunov exponents*.

6. There are many software packages for numerically computing Lyapunov exponents. See, for example, Parker and Chua [8] and Rollins [9].
7. In this section we have focused on chaos appearing in coupled, first-order, ordinary differential equations. Chaos can also appear in partial differential equations and stochastic equations.
8. The papers by Ablowitz and Herbst [1], Lorenz [6], and and Yamaguti and Ushiki [13] describe and illustrate how numerical discretizations of a differential equation can lead to discrete equations exhibiting chaos.
9. By long-term integration of the equations governing the solar system on speical purpose computers, researchers have found that Pluto's orbit is chaotic, the motion of the Jovian planet subsystem is chaotic, and the motion of comet Halley is chaotic. See, for example, Sussman and Wisdom [10].

References

[1] Ablowitz, M. J., and Herbst, B. M. On homoclinic structure and numerically induced chaos for the nonlinear Schrodinger equation. *SIAM J. Appl. Math. 50*, 2 (April 1990), 339–351.

[2] Chua, L. O., and Lin, G.-N. Canonical realization of Chua's circuit family. *IEEE Trans. Circ. & Syst. 37*, 7 (July 1990), 885–902.

[3] Geist, K., Parlitz, U., and Lauterborn, W. Comparison of different methods for computing Lyapunov exponents. *Progr. Theoret. Phys 83*, 55 (1990), 875–893.

[4] Hale, J. K., and Sternberg, N. Onset of chaos in differential delay equations. *J. Comput. Physics 77* (1988), 221–317.

[5] Hassard, B., Zhang, J., Hastings, S. P., and Troy, W. C. A computer proof that the Lorenz equations have "chaotic" solutions. *Appl. Math. Lett. 7*, 1 (1994), 79–83.

[6] Lorenz, E. N. Computational chaos—A prelude to computational instability. *Physica D 35* (1989), 299–317.

[7] Matsumoto, T. Chaos in electronic circuits. *Proc. IEEE*, 8 (August 1987).

[8] Parker, T. S., and Chua, L. O. Efficient solution of the variational equation for piecewise-linear differential equations. *Circuit Theory and Appl. 14*, 4 (1986), 305–314.

[9] Rollins, R. W. *Chaotic Dynamics Workbench.* Tech. rep., Physics Academic Software, AIP, New York, 1990.

[10] Sussman, G. J., and Wisdom, J. Chaotic evolution of the solar system. *Science 257*, 5066 (1992), 56–62.

[11] Tabor, M. *Chaos and Integrability in Nonlinear Dynamics.* John Wiley & Sons, New York, 1989.

[12] Wolf, A., Swift, J. B., Swinney, H. L., and Vastano, J. A. Determining Lyapunov exponents from a time series. *Physica D 16* (1985), 285–317.

[13] Yamaguti, M., and Ushiki, S. Chaos in numerical analysis of ordinary differential equations. *Physica D 3* (1981), 618–626.

6. Classification of Partial Differential Equations

Applicable to Partial differential equations.

Yields

Knowledge of the type of equation under consideration.

Procedure

Most partial differential equations are of three basic types: elliptic, hyperbolic, and parabolic.

Elliptic equations are often called potential equations. They result from potential problems, where the potential might be temperature, voltage, or a similar quantity. Elliptic equations are also the steady solutions of diffusion equations, and they require boundary values in order to determine the solution.

Hyperbolic equations are sometimes called wave equations, because they often describe the propagation of waves. They require initial conditions (where the waves start from) as well as boundary conditions (to describe how the wave and the boundary interact; for instance, the wave might be scattered or absorbed). These equations can be solved, in principle, by the method of characteristics (see page 432).

Parabolic equations are often called diffusion equations because they describe the diffusion and convection of some substance (such as heat). The dependent variable usually represents the density of the substance. These equations require initial conditions (what the initial concentration of the substance is) as well as boundary conditions (to specify, for instance, whether the substance can cross the boundary or not).

The above classification is most useful for second order partial differential equations. For second order equations, only characteristic curves need to be considered. For equations of higher degree, characteristic surfaces must be considered, see Whitham [8, pages 139–141] or Zauderer [10, pages 78–85 and 91–97] for more details. After two special cases, we specialize the rest of this section to second order partial differential equations.

Special Case 1

The most general second order linear partial differential equation with constant coefficients

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = d,$$

may be placed in the form

$$u_{\xi_1 \xi_1} + \cdots + u_{\xi_n \xi_n} + \lambda u = 0,$$

if the equation is elliptic or may be placed in the form

$$u_{\xi_1\xi_1} - u_{\xi_2\xi_2} - \cdots - u_{\xi_n\xi_n} + \lambda u = 0,$$

if the equation is hyperbolic, for some value of λ. See Garabedian [3, pages 70–76] for details.

Special Case 2

The (real valued) second order partial differential equation in n dimensions

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + f\left(\mathbf{x}, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = 0, \tag{6.1}$$

for $u(\mathbf{x}) = u(x_1, \ldots, x_n)$, where $a_{ij} = a_{ji}$, may be classified at the point $\mathbf{x}_0$ as follows. Let A be the matrix $(a_{ij}(\mathbf{x}_0))$. By means of a linear transformation, the quadratic form $\mathbf{g}^{\mathrm{T}} A \mathbf{g}$ may be reduced to the form

$$\lambda_1 g_1^2 + \lambda_2 g_2^2 + \cdots + \lambda_n g_n^2.$$

The values of $\{\lambda_i\}$, which are the eigenvalues of A, determine the nature of the partial differential equation (6.1). Because A has been assumed to be symmetric, all of the eigenvalues will be real. The classification at the point $\mathbf{x}_0$ is then given by

1. If all of the $\{\lambda_i\}$ are of the same sign, then equation (6.1) is elliptic at $\mathbf{x}_0$.
2. If any of the $\{\lambda_i\}$ are zero, then equation (6.1) is parabolic at $\mathbf{x}_0$.
3. If none of the $\{\lambda_i\}$ are zero and they are not all of the same sign, then equation (6.1) is hyperbolic at $\mathbf{x}_0$.
4. If none of the $\{\lambda_i\}$ are zero and there are at least two that are positive and at least two that are negative, then equation (6.1) is *ultrahyperbolic* at $\mathbf{x}_0$.

If an equation is parabolic along a smooth curve in a domain D, and the equation is hyperbolic on one side of the curve and elliptic on the other side of the curve, then the equation is of *mixed type*. The smooth curve is called the *curve of parabolic degeneracy*.

Special Case 3

We further specialize here and restrict ourselves to second order equations in two independent variables. Consider partial differential equations of second order in two independent variables, of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = \Psi\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, x, y\right), \tag{6.2}$$

where Ψ need not be a linear function.

If $\begin{bmatrix} B^2 - 4AC > 0 \\ B^2 - 4AC = 0 \\ B^2 - 4AC < 0 \end{bmatrix}$ at some point (x, y), then equation (6.2) is $\begin{bmatrix} \text{hyperbolic} \\ \text{parabolic} \\ \text{elliptic} \end{bmatrix}$ at that point. If an equation is of the same type at all points in the domain, then the equation is simply said to be of that type.

Equation 6.2 can be transformed into a canonical form for each of the three types mentioned above. The procedures are as follows.

Hyperbolic Equations

For hyperbolic equations we look for a new set of independent variables $\zeta = \zeta(x, y)$ and $\eta = \eta(x, y)$ for which equation (6.2) may be written in the standard form

$$u_{\zeta\eta} = \phi\left(u, u_\eta, u_\zeta, \eta, \zeta\right). \tag{6.3}$$

Utilizing this change of variables, we can calculate

$$\begin{aligned}
u_x &= u_\eta \eta_x + u_\zeta \zeta_x, \\
u_y &= u_\eta \eta_y + u_\zeta \zeta_y, \\
u_{xx} &= u_{\eta\eta}\eta_x\eta_x + 2u_{\eta\zeta}\eta_x\zeta_x + u_{\zeta\zeta}\zeta_x\zeta_x + u_\eta\eta_{xx} + u_\zeta\zeta_{xx}, \\
u_{xy} &= u_{\eta\eta}\eta_x\eta_y + 2u_{\eta\zeta}(\eta_x\zeta_y + \eta_y\zeta_x) + u_{\zeta\zeta}\zeta_x\zeta_y + u_\eta\eta_{xy} + u_\zeta\zeta_{xy}, \\
u_{yy} &= u_{\eta\eta}\eta_y\eta_y + 2u_{\eta\zeta}\eta_y\zeta_y + u_{\zeta\zeta}\zeta_y\zeta_y + u_\eta\eta_{yy} + u_\zeta\zeta_{yy},
\end{aligned}$$

to find that equation (6.2) transforms into

$$\overline{A}u_{\zeta\zeta} + \overline{B}u_{\zeta\eta} + \overline{C}u_{\eta\eta} = \Phi\left(u, u_\eta, u_\zeta, \eta, \zeta\right), \tag{6.4}$$

where

$$\begin{aligned}
\overline{A} &= A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2, \\
\overline{B} &= A\zeta_x\eta_x + B(\zeta_x\eta_y + \zeta_y\eta_x) + 2C\zeta_y\eta_y, \\
\overline{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2.
\end{aligned}$$

Setting $\overline{A} = \overline{C} = 0$, we can find the following partial differential equations for ζ and η

$$\begin{aligned}
\frac{\zeta_x}{\zeta_y} &= \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \\
\frac{\eta_x}{\eta_y} &= \frac{-B - \sqrt{B^2 - 4AC}}{2A}.
\end{aligned} \tag{6.5.a-b}$$

These equations may be readily solved (in principle) by the method of characteristics. For example, to solve equation (6.5.a) we only need to solve

$$-\frac{dy}{dx} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

for $Q(x, y) = R$, where R is an arbitrary constant. Then ζ will be given by $\zeta = Q(x, y)$.

After ζ and η are determined, then the original equation must be transformed into the new coordinates (see page 168). The resulting equation will then be in standard form.

Note that another standard form for hyperbolic equations (in two independent variables) is obtained from equation (6.3) by the change of variables

$$\alpha = \eta - \zeta, \qquad \beta = \eta + \zeta. \tag{6.6}$$

This results in the equation

$$u_{\alpha\alpha} - u_{\beta\beta} = \phi\left(u, u_\alpha - u_\beta, u_\alpha + u_\beta, \frac{1}{2}(\beta + \alpha), \frac{1}{2}(\beta - \alpha)\right).$$

Example 1

Suppose we have the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0. \tag{6.7}$$

We recognize this equation to be hyperbolic away from the lines $x = 0$ and $y = 0$. To find the new variables ζ and η, we must solve the differential equations in (6.5). For this equation, we have $\{A = y^2,\ B = 0,\ C = -x^2\}$. Therefore (6.5) becomes

$$\frac{\zeta_x}{\zeta_y} = -\frac{x}{y}, \qquad \frac{\eta_x}{\eta_y} = \frac{x}{y},$$

with the solutions $\zeta = y^2 - x^2$, $\eta = y^2 + x^2$. In these new variables, equation (6.7) becomes

$$u_{\zeta\eta} = \frac{\zeta}{2(\zeta^2 - \eta^2)} u_\eta - \frac{\eta}{2(\zeta^2 - \eta^2)} u_\zeta. \tag{6.8}$$

If the change of independent variable in (6.6) is made, then (6.8) becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{2\beta} u_\beta - \frac{1}{2\alpha} u_\alpha.$$

Parabolic Equations

For parabolic equations, we look for a new set of variables $\zeta = \zeta(x, y)$ and $\eta = \eta(x, y)$ in which equation (6.2) can be written in one of the standard forms

$$u_{\zeta\zeta} = \phi(u, u_\eta, u_\zeta, \eta, \zeta), \tag{6.9.a}$$

or

$$u_{\eta\eta} = \phi\left(u, u_\eta, u_\zeta, \eta, \zeta\right). \tag{6.9.b}$$

Utilizing equation (6.4), we see that we need to determine ζ and η in such a way that

$$\overline{B} = 0 = \overline{C}, \qquad \text{corresponding to (6.9.a)}, \tag{6.10.a}$$

or

$$\overline{B} = 0 = \overline{A}, \qquad \text{corresponding to (6.9.b)}, \tag{6.10.b}$$

If $A \neq 0$, then equation (6.10.a) corresponds to the single equation

$$\frac{\zeta_x}{\zeta_y} = -\frac{B}{2A}, \tag{6.11.a}$$

while, if $C \neq 0$, then equation (6.10.b) corresponds to the equation

$$\frac{\zeta_x}{\zeta_y} = -\frac{B}{2C}. \tag{6.11.b}$$

In either case, we have only to solve a single equation to determine ζ. The variable η can then be chosen to be anything linearly independent of ζ. As before, once ζ and η are determined, then the equation needs to be written in terms of these new variables

Example 2

Suppose we have the equation

$$y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} + u_y = 0. \tag{6.12}$$

Since $\{A = y^2, B = -2xy, C = x^2\}$, we find that $B^2 - 4AC = 0$ and so this equation is parabolic. In this case we choose to make $\overline{B} = \overline{C} = 0$. From equation (6.11.a) we must solve $\frac{\zeta_x}{\zeta_y} = \frac{x}{y}$, which has the solution $\zeta = y^2 + x^2$.

We choose $\eta = x$. Using these values of η and ζ, we find that (6.12) becomes

$$u_{\eta\eta} = \frac{2(\zeta + \eta)}{\zeta - \eta^2} u_\zeta + \frac{1}{\zeta - \eta^2} u_\eta.$$

Elliptic Equations

For elliptic equations we look for a new set of variables $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ in which equation (6.2) can be written in the standard form

$$u_{\alpha\alpha} + u_{\beta\beta} = \phi\left(u, u_\alpha, u_\beta, \alpha, \beta\right).$$

The easiest way in which to find α and β is to determine variables $\zeta = \zeta(x, y)$ and $\eta = \eta(x, y)$ that satisfy (6.5) and then form $\alpha = (\eta + \zeta)/2$, $\beta = (\eta - \zeta)/2i$ (where, as usual, $i = \sqrt{-1}$). Note that in this case, the differential equations in (6.5) are complex. However, since ζ and η are conjugate complex functions, the quantities α and β will be real.

Example 3

Suppose we have the equation

$$y^2 u_{xx} + x^2 u_{yy} = 0.$$

We recognize this equation to be elliptic away from the lines $x = 0$ and $y = 0$. To find the new variables ζ and η, we must solve the differential equations in (6.5). For this equation, we have $\{A = y^2,\ B = 0,\ C = x^2\}$. Therefore (6.5) becomes

$$\frac{\zeta_x}{\zeta_y} = -\frac{ix}{y}, \qquad \frac{\eta_x}{\eta_y} = \frac{ix}{y},$$

with the solutions $\zeta = y^2 - ix^2$, $\eta = y^2 + ix^2$. Forming α and β results in

$$\alpha = \frac{\eta + \zeta}{2} = y^2, \qquad \beta = \frac{\eta - \zeta}{2i} = x^2.$$

In these new variables, equation (6.7) becomes

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\alpha} u_\alpha - \frac{1}{2\beta} u_\beta.$$

Notes

1. Equations of mixed type are discussed in Haack and Wendland [4] and Smirnoff [6].
2. Given a partial differential equation in the form of equation (6.1), the characteristic surfaces are defined by the characteristic equation

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}) \left(\frac{\partial u}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} \right) = 0.$$

   The solutions to this equation are the only surfaces across which $u(\mathbf{x})$ may have discontinuities in its second derivatives.
3. The Notes section of the characteristics method (see page 432) describes how to determine when a system of partial differential equations is hyperbolic.
4. See also Farlow [2, pages 174–182 and 331–339], Moon and Spencer [5, pages 137–146], Stakgold [7, pages 467–482], and Young [9, pages 60–70].

References

[1] Bitsadze, A. V. *Equations of the Mixed Type.* The MacMillan Company, New York, 1964.

[2] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers.* John Wiley & Sons, New York, 1982.

[3] Garabedian, P. R. *Partial Differential Equations.* John Wiley & Sons, New York, 1964.

[4] Haack, E. R., and Wendland, W. N. *Lectures on Partial and Pfaffian Differential Equations.* Pergamon Press, New York, 1972. Translated by E. R. Dawson and W. N. Everitt.

[5] Moon, P., and Spencer, D. E. *Partial Differential Equations.* D. C. Heath and Co., Lexington, MA, 1969.

[6] Smirnoff, M. M. *Equations of Mixed Type.* Amer. Math. Soc., Providence, RI, 1978.

[7] Stakgold, I. *Green's Functions and Boundary Value Problems.* John Wiley & Sons, New York, 1979.

[8] Whitham, G. B. *Linear and Nonlinear Waves.* Interscience Publishers, Inc., New York, 1974.

[9] Young, E. C. *Partial Differential Equations.* Allyn and Bacon, Inc., Boston, MA, 1972.

[10] Zauderer, E. *Partial Differential Equations of Applied Mathematics.* John Wiley & Sons, New York, 1983.

7. Compatible Systems

Applicable to

Systems of differential equations.

Yields

Knowledge of whether the equations are consistent.

Procedure 1

The two equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ for $z = z(x, y)$ (where, as usual, $p = z_x$ and $q = z_y$) are said to be *compatible* if every solution of the first equation is also a solution of the second equation, and conversely. These two equations will be compatible if $\{f, g\} = 0$, where

$$\{f, g\} := \frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)},$$

and where $\frac{\partial(u,v)}{\partial(a,b)} = \left|\begin{smallmatrix} u_a & v_a \\ u_b & v_b \end{smallmatrix}\right| = u_a v_b - v_a u_b$ is the usual Jacobian.

Procedure 2

The conditions for consistency of a system of simultaneous partial differential equations of the first order, if the number of equations is an exact multiple of the number of dependent variables involved, is given in Forsyth [3, Part IV, pages 411–419]. To write the consistency conditions, let the unknown dependent variables be $\{z_i \mid i = 1, \ldots, m\}$, let the independent variables be $\{x_j \mid j = 1, \ldots, n\}$, and define $p_{ij} = \partial z_i / \partial x_j$. We presume the system has rm equations (with $r \leq n$) and that these equations can be solved with respect to the p_{ij}. That is

$$p_{ij} = \frac{\partial z_i}{\partial x_j} = f_{ij}\left(\{x_l\}, \{z_k\}, \{p_{\lambda\mu}\}\right),$$

for $i = \langle 1, m\rangle$, $j = \langle 1, n\rangle$, $l = \langle 1, n\rangle$, $\lambda = \langle 1, m\rangle$, $\mu = \langle r+1, n\rangle$. (Here we have introduced the notation $\langle a, b\rangle$ to be the sequence of numbers $a, a+1, a+2, \ldots, b$.) Then, for consistency, the following conditions must be satisfied

$$\begin{aligned} \frac{\partial f_{ij}}{\partial x_a} - \frac{\partial f_{ia}}{\partial x_j} &+ \sum_{\lambda=1}^{m}\left(f_{\lambda a}\frac{\partial f_{ij}}{\partial z_\lambda} - f_{\lambda j}\frac{\partial f_{ia}}{\partial z_\lambda}\right) \\ &+ \sum_{s=1}^{m}\sum_{\mu=r+1}^{n}\left(\frac{\partial f_{ij}}{\partial p_{s\mu}}\frac{\partial f_{sa}}{\partial x_\mu} - \frac{\partial f_{ia}}{\partial p_{s\mu}}\frac{\partial f_{sj}}{\partial x_\mu}\right) \\ &+ \sum_{s=1}^{m}\sum_{\mu=r+1}^{n}\sum_{\lambda=1}^{m}\left[\left(\frac{\partial f_{ij}}{\partial p_{s\mu}}\frac{\partial f_{sa}}{\partial z_\lambda} - \frac{\partial f_{ia}}{\partial p_{s\mu}}\frac{\partial f_{sj}}{\partial z_\lambda}\right)p_{\lambda\mu}\right] = 0, \end{aligned} \tag{7.1}$$

where $i = \langle 1, m\rangle$, $a = \langle j+1, r\rangle$, $j = \langle 1, r-1\rangle$, and

$$\sum_{s=1}^{m} \left(\frac{\partial f_{ij}}{\partial p_{s\mu}} \frac{\partial f_{sa}}{\partial p_{k\tau}} - \frac{\partial f_{ia}}{\partial p_{s\mu}} \frac{\partial f_{sj}}{\partial p_{k\tau}} + \frac{\partial f_{ij}}{\partial p_{s\tau}} \frac{\partial f_{sa}}{\partial p_{k\mu}} - \frac{\partial f_{ia}}{\partial p_{s\tau}} \frac{\partial f_{sj}}{\partial p_{k\mu}} \right) = 0, \tag{7.2}$$

where $i, k = \langle 1, m\rangle$, $a = \langle j+1, r\rangle$, $\mu, \tau = \langle r+1, n\rangle$, $j = \langle 1, r-1\rangle$.

Special Case 1

In the special case of $m = 1$, we have one dependent variable (which we call z) and r equations. Let $p_j = \partial z/\partial x_j = f_j(z, x_1, \ldots, x_n, p_{r+1}, \ldots, p_n)$. In this case, equation (7.2) is automatically satisfied while equation (7.1) becomes

$$\frac{df_j}{dx_a} - \frac{df_a}{dx_j} + \sum_{\mu=r+1}^{n} \left(\frac{\partial f_j}{\partial p_\mu} \frac{df_a}{dx_\mu} - \frac{\partial f_a}{\partial p_\mu} \frac{df_j}{dx_\mu} \right) = 0$$

for $a = \langle 1, j-1\rangle$, $j = \langle 1, r\rangle$, where we have defined $\frac{d}{dx_s} = \frac{\partial}{\partial x_s} + p_s \frac{\partial}{\partial z}$.

Special Case 2

In the special case of $r = n$, the system of mn equations becomes $p_{ij} = f_{ij}(z_1, \ldots, z_m, x_1, \ldots, x_n)$ and the consistency conditions become

$$\frac{\partial f_{ij}}{\partial x_a} - \frac{\partial f_{ia}}{\partial x_j} + \sum_{\lambda=1}^{m} \left(f_{\lambda a} \frac{\partial f_{ij}}{\partial z_\lambda} - f_{\lambda j} \frac{\partial f_{ia}}{\partial z_\lambda} \right) = 0$$

for $i = \langle 1, m\rangle$, $a = \langle 1, j-1\rangle$ and $j = \langle 1, n\rangle$. These are known as Mayer's system of completely integrable equations.

Special Case 3

Consider the special case of $r = 1$, with $\{F_1 = 0, F_2 = 0, \ldots, F_m = 0\}$, where each $F_j = p_j - f_j(z, x_1, \ldots, x_n, p_{r+1}, \ldots, p_n)$ is analytical in each of its arguments. A necessary and sufficient condition for the set of equations to be consistent is that $[F_i, F_j] = 0$, for all combinations of i and j. Here, $[\,,\,]$ represents the usual Poisson bracket.

Example

Suppose we have the two following nonlinear partial differential equations for $z(x, y)$:

$$x z_x = y z_y, \qquad z(x z_x + y z_y) = 2xy. \tag{7.3}$$

From (7.3) we identify

$$f(x, y, z, p, q) = xp - yq, \qquad g(x, y, z, p, q) = z(xp + yq) - 2xy. \tag{7.4}$$

Using (7.4) we can easily calculate

$$\frac{\partial(f, g)}{\partial(x, p)} = 2xy, \qquad \frac{\partial(f, g)}{\partial(z, p)} = -x^2 p,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -2xy, \qquad \frac{\partial(f,g)}{\partial(z,q)} = xyp.$$

Therefore, computing $\{f, g\}$, we find it to be zero. Hence, the two equations in equation (7.3) have identical solution sets.

Because the equations in (7.3) are compatible, we can combine them without changing the solution sets. Solving the equations in (7.3) simultaneously for p and q to obtain $\{z_x = p = y/z,\ z_y = q = x/z\}$. These last two equations can be easily solved we obtain $z^2 = C + 2xy$, where C is an arbitrary constant.

Notes

1. Jacobi's method (see page 464) takes a given partial differential equation and creates a compatible equation and then uses elimination between these two equations.
2. If it is known that a linear homogeneous ordinary differential equation of order n has solutions in common with a linear homogeneous ordinary differential equation of order m (with $m < n$), then it is possible to determine a differential equation of lower degree that has, as its solutions, these common solutions. If the linear homogeneous ordinary differential equations $L_1[u] = 0$ and $L_2[u] = 0$ are defined by

$$\begin{aligned} L_1 &:= p_0 D^n + p_1 D^{n-1} + \cdots + p_{n-1} D + p_n, \\ L_2 &:= q_0 D^m + q_1 D^{m-1} + \cdots + q_{m-1} D + q_m, \end{aligned}$$

where D represents d/dx and each of the functions $\{p_i, q_i\}$ depends on x, define the ordinary differential equation $R_1[u] = 0$ by

$$R_1 := r_0 D^{n-m} + r_1 D^{n-m-1} + \cdots + r_{n-m-1} D + r_{n-m},$$

where the $\{r_i\}$ are defined by

$$\begin{aligned} p_0 &= r_0 q_0, \\ p_1 &= r_1 q_0 + r_0 \left[\binom{n-m}{1} q_0' + q_1 \right], \\ p_2 &= r_2 q_0 + r_1 \left[\binom{n-m-1}{1} q_0' + q_1 \right] \\ &\quad + r_0 \left[\binom{n-m}{2} q_0'' + \binom{n-m}{1} q_1' + q_2 \right], \\ &\ \vdots \\ p_{n-m} &= r_{n-m} q_0 + r_{n-m-1} \left[\binom{1}{1} q_0' + q_1 \right] \\ &\quad + r_{n-m-2} \left[\binom{2}{2} q_0'' + \binom{2}{1} q_1' + \binom{2}{0} q_2 \right] + \ldots, \\ &= r_{n-m} q_0 + r_{n-m-1} \left[q_0' + q_1 \right] + r_{n-m-2} \left[q_0'' + 2q_1' + q_2 \right] + \ldots . \end{aligned}$$

Then the order of the operator $L_3 := L_1 - R_1 L_2$ will be depressed as much as is possible (the order of L_3 will not exceed $m-1$). Note that only a finite number of rational operations and differentiations are required to determine the $\{r_i\}$. From the definition of L_3, we see that all solutions common to both $L_1[u] = 0$ and to $L_2[u] = 0$ will also be solutions to $L_3[u] = 0$. If L_3 is identically zero, then we have found a factorization of L_1 (see page 294). See Ince [4, pages 126–128] or Valiron [6, pages 320–322] for details.

3. Differential resultants can also be used to derive consistency conditions. See Berkovich and Tsirulik [2] for details.
4. Wolf [7] describes an algorithm that determines if an overdetermined system of two equations for one function has any solution. An implementation in FORMAC is mentioned.
5. See also Ames [1, pages 54–65] and Sneddon [5, pages 67–68].

References

[1] Ames, W. F., Ed. *Nonlinear Partial Differential Equations*, vol. 1. Academic Press, New York, 1967.

[2] Berkovich, L. M., and Tsirulik, V. G. Differential resultants and some of their applications. *Differentsial'nye Uravneniya 22*, 5 (May 1986), 750–757.

[3] Forsyth, A. R. *Theory of Differential Equations*. Dover Publications, Inc., New York, 1959.

[4] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[5] Sneddon, I. N. *Elements of Partial Differential Equations*. McGraw–Hill Book Company, New York, 1957.

[6] Valiron, G. *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions*. Math Sci Press, Brookline, MA, 1950.

[7] Wolf, T. An analytic algorithm for decoupling and integrating systems of nonlinear partial differential equations. *J. Comput. Physics 60* (1985), 437–446.

8. Conservation Laws

Applicable to

Partial differential equations.

Yields

Quantities that remain invariant during the evolution of the partial differential equation.

Procedure

Given an evolution equation, which is a partial differential equation of the form

$$u_t = F(u, u_x, u_{xx}, \dots), \tag{8.1}$$

a conservation law is a partial differential equation of the form

$$\frac{\partial}{\partial t} T\Big(u(x,t)\Big) + \frac{\partial}{\partial x} X\Big(u(x,t)\Big) = 0, \tag{8.2}$$

which is satisfied by all solutions of equation (8.1). We define $T(\cdot)$ to be the *conserved density* and $X(\cdot)$ to be the *flux*. An alternative statement of equation (8.2) is that

$$\int T\Big(u(x,t)\Big)\,dx \tag{8.3}$$

is independent of t, for solutions of (8.1) such that the integral converges.

More generally, a partial differential equation of order m in the n independent variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and a single dependent variable u is in *conservation form* if it can be written as

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(\mathbf{x}, u, \partial u, \partial^2 u, \dots, \partial^{m-1} u) = 0. \tag{8.4}$$

Here $\partial^j u$ represents all jth order partial derivatives of u with respect to $\mathbf{x}$.

Example 1

The Korteweg–de Vries equation

$$u_t = u_{xxx} + u u_x \tag{8.5}$$

has an infinite set of conservation laws. The first few, in order of increasing rank, have the conserved densities

$$\begin{aligned} T &= u, \\ T &= u^2, \\ T &= u^3 - 3u_x^2, \\ T &= 5u^4 - 60uu_x^2 - 36u_x u_{xxx}, \\ &\vdots \end{aligned}$$

To demonstrate, for instance, that $T = u^2$ is a conserved density, we compute

$$\frac{\partial T}{\partial t} = \frac{\partial (u^2)}{\partial t} = 2uu_t = 2uu_{xxx} + 2u^2 u_x,$$

where we have used the defining equation in (8.5) to replace the u_t term. Now we must determine a flux X such that equation (8.2) is satisfied. In this case, we find $X = u_x^2 - 2uu_{xx} - \frac{2}{3}u^3$.

Example 2

The Schrödinger equation

$$-\frac{\partial^2 u}{\partial x^2} + V(x)u = i\frac{\partial u}{\partial t}$$

can be expressed in the form of equation (8.2) with

$$T = i\nu(x)u,$$
$$X = \nu(x)\frac{\partial u}{\partial x} - \nu'(x)u,$$

where $\nu(x)$ is defined by $\nu''(x) = V(x)\nu(x)$.

Notes

1. Conservation laws allow estimates of the accuracy of a numerical solution scheme (because the quantity in (8.3) must be invariant in time).
2. Not all partial differential equations have an infinite number of conservation laws; there may be none or a finite number.
3. A conservation law for an evolution equation is called trivial if T is, itself, the x derivative of some expression. If equation (8.1) has an infinite sequence of nontrivial conservation laws, then the equation is *formally integrable*. Infinite sequences of nontrivial conservation laws are given by Cavalcante and Tenenblat [2] for the following equations: Burgers, KdV, mKdV, sine–Gordon, sinh–Gordon.
4. If a given partial differential equation is not written in conservation form, there are a number of ways of attempting to put it in a conserved form. Bluman *er al.* [1] have a short list of techniques.
5. If equation (8.4) is satisfied, then there exists an $(n-1)$-exterior differential form $\mathbf{F}$ such that equation (8.4) can be written $d\mathbf{F} = 0$. This implies that there is an $(n-2)$-form ϕ such that $\mathbf{F} = d\phi$. This, in turn, means that there exists an antisymmetric tensor of rank n, ψ, such that

$$F_i(\mathbf{x}, u, \partial u, \partial^2 u, \ldots, \partial^{m-1} u) = \sum_{i<j\le n} (-1)^j \frac{\partial \psi_{ij}}{\partial x_j} + \sum_{1\le j<i} (-1)^{i-1} \frac{\partial \psi_{ji}}{\partial x_j},$$

for $i = 1, 2, \ldots, n$.

6. A computer program in REDUCE for determining conservation laws is given in Ito and Kako [6]. In Gerdt *et al.* [4] is the description of a computer program in FORMAC that determines conservation laws, determines Lie–Bäcklund symmetries, and also attempts to determine when an evolution equation is formally integrable.
7. Torriani [10] shows how the terms appearing in the expression of the densities and the fluxes for the Korteweg-de Vries equation may be found by combinatorial methods.
8. El-Sherbiny [3] proves that unless a_1/a_2 is a multiple root of order three of the algebraic equation $a_6\lambda^3 - a_5\lambda^2 + a_4\lambda - a_3 = 0$, then the class of nonlinear evolution equations $u_t + u_x + a_1 uu_x + a_2 uu_t + a_3 u_{xxx} + a_4 u_{xxt} + a_5 u_{xtt} + a_6 u_{ttt} = 0$ with the $\{a_i\}$ real numbers has a finite number of conservation laws; otherwise, the class has an infinite number of conservation laws.
9. See also Olver [9, Chapter 4, pages 246–291].

References

[1] Bluman, G. W., Reid, G. J., and Kumei, S. New classes of symmetries for partial differential equations. *J. Math. Physics 29*, 4 (April 1988), 806–811.

[2] Cavalcante, J. A., and Tenenblat, K. Conservation laws for nonlinear evolution equations. *J. Math. Physics 29*, 4 (April 1988), 1044–1049.

[3] El-Sherbiny, H. M. Conservation laws of a class of differential equations II. *Appl. Math. Lett. 6*, 4 (1993), 99–103.

[4] Gerdt, V. P., Shvachka, A. B., and Zharkov, A. Y. Computer algebra applications for classification of integrable non-linear evolution equations. *J. Symbolic Comp. 1* (1985), 101–107.

[5] Ibragimov, N. H. Group theoretical nature of conservation theorems. *Letters Math. Physics 1* (1977), 423–428.

[6] Ito, M., and Kako, F. A REDUCE program for finding conserved densities of partial differential equations with uniform rank. *Comput. Physics Comm. 38* (1985), 415–419.

[7] LeVeque, R. J. *Numerical Methods for Conservation Laws.* Birkhauser, Basel, Switzerland, 1992.

[8] Littinsky, E. Polynomial integrals of evolution equations. *Comm. Math. Physics 121* (1989), 669–682.

[9] Olver, P. J. *Applications of Lie Groups to Differential Equations.* No. 107 in Graduate Texts in Mathematics. Springer–Verlag, New York, 1986.

[10] Torriani, H. H. Conservation laws for the Korteweg-de Vries equation and the theory of partitions. *Physics Letters 113A*, 7 (6 January 1986), 345–348.

[11] Vinokur, M. An analysis of finite-difference and finite-volume formulations of conservations laws. *J. Comput. Physics 81* (1989), 1–52.

9. Differential Resultants

Applicable to Two polynomial ordinary differential equations.

Yields

One ordinary differential equation in one independent variable.

Idea

Given two polynomial equations (in, say, x and y), the classical method of resultants is as follows: The equations can always be written as the system of linear equations $A\mathbf{w} = \mathbf{0}$, where $A = A(y)$ and $\mathbf{w} = \mathbf{w}(x) \neq 0$. Because this system must have $\det A = 0$, a polynomial equation only in y may be determined. The technique for polynomial differential equations is very similar.

Procedure

Resultants have classically been used to eliminate one variable between two polynomial equations. For example, suppose we have the two equations

$$\begin{aligned} x^3 - 3y^2x^2 + x + 5y^2 &= 0, \\ x^3 + 5y^2x^2 - x + 3y^2 &= 0. \end{aligned} \tag{9.1}$$

These equations may be multiplied by powers of x to obtain the system of equations:

$$\begin{array}{ccccccccccccl} x^5 & - & 3y^2x^4 & + & x^3 & + & 5y^2x^2 & & & & & = & 0, \\ & & x^4 & - & 3y^2x^3 & + & x^2 & + & 5y^2x & & & = & 0, \\ & & & & x^3 & - & 3y^2x^2 & + & x & + & 5y^2 & = & 0, \\ & & & & x^3 & + & 5y^2x^2 & - & x & + & 3y^2 & = & 0, \\ & & x^4 & + & 5y^2x^3 & - & x^2 & + & 3y^2x & & & = & 0, \\ x^5 & + & 5y^2x^4 & - & x^3 & + & 3y^2x^2 & & & & & = & 0. \end{array}$$

This system can be written in matrix form as

$$\begin{bmatrix} 1 & -3y^2 & 1 & 5y^2 & 0 & 0 \\ 0 & 1 & -3y^2 & 1 & 5y^2 & 0 \\ 0 & 0 & 1 & -3y^2 & 1 & 5y^2 \\ 0 & 0 & 1 & 5y^2 & -1 & 3y^2 \\ 0 & 1 & 5y^2 & -1 & 3y^2 & 0 \\ 1 & 5y^2 & -1 & 3y^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^5 \\ x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9.2}$$

This last equation is a 6×6 system of the form $A\mathbf{w} = \mathbf{0}$. Because $\mathbf{w} \neq 0$ (because, at least, the last component of $\mathbf{w}$ is non-zero), the determinant of A must vanish. Taking the determinant of the matrix in equation (9.2), we find that y must satisfy the equation

$$32y^2(289y^8 + 16y^4 + 1) = 0. \tag{9.3}$$

All the different values of y, from the solutions of (9.1), must satisfy (9.3).

Differential resultants are the analogue of resultants applied to differential systems. There are two steps analogous to multiplying the original equations by powers of x. They are

- differentiating one of the equations,
- multiplying one of the equations by some term that may involve the independent and/or the dependent variables.

Although there are algorithms published on how to proceed in any given case, as in Mishina and Proskuryakov [3], they are generally written in the language of abstract algebra.

Example

Suppose we have the following two coupled differential equations for $\{y(x), z(x)\}$

$$\begin{aligned} \mathrm{A}:&\quad 3yz + z - y_x = 0, \\ \mathrm{B}:&\quad -z_x + z^2 + y^2 + y = 0. \end{aligned}$$

We seek a single differential equation involving only $z(x)$. Note that we could solve equation (A) for $y(x)$ (by integrating factors) and then substitute this result in equation (B), but this creates an algebraic mess. This, in turn, makes it difficult to obtain a single simple equation for $z(x)$.

If we form the equations $\{A, B, yA, yB, y_xB, \partial_x B, y\partial_x A\}$, then we obtain the system

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 3z & z \\ 1 & 0 & 0 & 0 & 0 & 1 & z^2 - z_x \\ 3z & 0 & 0 & -1 & 0 & z & 0 \\ 1 & 1 & 0 & 0 & 0 & z^2 - z_x & 0 \\ 0 & 0 & z^2 - z_x & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2zz_x - z_{xx} \\ 0 & 0 & 0 & 1 & 2 & 2zz_x - z_{xx} & 0 \end{bmatrix} \begin{bmatrix} y^2 \\ y^3 \\ y_x \\ yy_x \\ y^2y_x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Taking the determinant of the matrix above, we conclude that $z(x)$ is a solution of the single ordinary differential equation

$$\begin{aligned} &z_{xx}^2 + (-16z_x + 12z^2 - 3)zz_{xx} + 64z^2z_x^2 + (23 - 96z^2)z^2z_x \\ &\quad + (36z^4 - 17z^2 + 2)z^2 = 0. \end{aligned}$$

Notes

1. This technique applies directly to systems of partial differential equations and to higher order equations.

2. There are specific technical requirements for when the classical method of resultants (when applied to polynomials) will work. There are similar requirements for when differential resultants will work. See Mishina and Proskuryakov [3] for details.
3. Rubel [5] proves the following theorem, which indicates that elimination is not always possible, at least for algebraic differential equations (ADEs, see page 720):

   > There exists a system of two ADEs, in the two dependent variables u and v which possesses a real-valued $C^{n,m}$ solution $\overline{u}$, $\overline{v}$ on a certain open interval I, but which has no solution u, v on I for which v satisfies an ADE that does not involve u or any derivative of u.

4. By taking equations pairwise a system of, say, 10 equations in 10 different independent variables could, if fortunate, be reduced to a single equation in a single independent variable.
5. The two differential equations considered do not both have to be polynomial for this reduction scheme to work. The two equations have only to be polynomials in one of the dependent variables (the one that will be removed).
6. Any linear second order ordinary differential equation system can be interpreted as the resultant of the elimination of a dependent variable from a pair of conjugate first order Hamilton's equations. See Tolstoy [7] for details.

References

[1] Berkovich, L. M., and Tsirulik, V. G. Differential resultants and some of their applications. *Differentsial'nye Uravneniya 22*, 5 (May 1986), 750–757.

[2] Bocher, M. *Introduction to Higher Algebra.* The MacMillan Company, New York, 1907.

[3] Mishina, A. P., and Proskuryakov, I. V. *Higher Algebra.* Pergamon Press, New York, 1965.

[4] Ritt, J. F. *Differential Algebra.* Dover Publications, Inc., New York, 1966.

[5] Rubel, L. A. An elimination theory for systems of algebraic differential equations. *Houston J. Math. 8*, 2 (1982), 289–295.

[6] Seidenberg, A. An elimination theory for differential algebra. *University of California Press 3*, 2 (1956), 31–66.

[7] Tolstoy, I. Remarks on the linearization of differential equations. *J. Inst. Maths. Applics 20* (1977), 53–60.

[8] Van Der Waerden, B. L. *Modern Algebra.* Frederick Ungar Publishing, New York, 1940.

10. Existence and Uniqueness Theorems

Applicable to Differential equations of all types.

Yields

Knowledge of whether a solution exists and, if so, if the solution is unique.

Idea

There are theorems available for many cases of interest.

Procedure

Corresponding to the difficulty of the subjects involved, there are more theorems applicable to: ordinary differential equations than partial differential equations, linear equations than nonlinear equations, and initial value problems than boundary value problems. In the following we indicate some of the simple theorems that are frequently useful.

The last theorem is applicable to partial differential equations; the rest are applicable to ordinary differential equations. The first and last two theorems are for vector systems; the other theorems are for scalar equations.

Theorem Consider the initial value problem: $d\mathbf{x}/dt = \mathbf{F}(t, \mathbf{x})$ with $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x} = \mathbf{x}(t) = [x_1(t)\ x_2(t)\ \dots\ x_n(t)]^{\mathrm{T}}$. If each of the functions $\{F_i\}$ and $\left\{\frac{\partial F_i}{\partial x_j}\right\}$ are continuous in a region R of $(t, \mathbf{x})$ space containing the point $\mathbf{x}_0$, then there is an interval $|t - t_0| < h$ in which there exists a unique solution to the problem.

Theorem Consider the initial value problem: $y' = f(x, y)$ with $y(x_0) = y_0$. Let the functions f be continuous in some rectangle $a < x < b$, $c < y < d$ containing the point (x_0, y_0). Assume that $f(x, y)$ satisfies a Lipschitz condition in y. Then, in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$, there is a unique solution to the given problem.

Theorem Consider the initial value problem: $y' = f(x, y)$ with $y(x_0) = y_0$. Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $a < x < b$, $c < y < d$ containing the point (x_0, y_0). Then, in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$, there is a unique solution to the given problem.

Theorem Consider the initial value problem: $y'' = f(x, y, y')$ with $y(x_0) = y_0$, $y'(x_0) = y'_0$. Let the functions f, f_y, and $f_{y'}$ be continuous in an open region R of three-dimensional (x, y, y') space. If the point

(x_0, y_0, y'_0) is in R, then there exists some interval about x_0 for which there is a unique solution to the given problem.

Theorem Consider the initial value problem:

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_1(x)y = q(x),$$

with

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad \ldots \quad y^{(n-1)}(x_0) = y_0^{(n-1)}.$$

If the functions $\{p_i(x)\}$ and $q(x)$ are continuous on the open interval $a < x < b$, then there exists a unique solution to the problem.

Theorem Consider the initial value problem:

$$x' = f(x, y, t), \qquad y' = g(x, y, t)$$

with $x(t_0) = x_0$, $y(t_0) = y_0$. If f and g satisfy a Lipschitz condition (with respect to x and y) in the region $\{|t - t_0| \leq A, |x - x_0| \leq B, |y - y_0| \leq C\}$, then the problem has a unique solution in some interval $a < t < b$ about the point t_0.

Theorem Consider the boundary value problem:

$$\begin{aligned} &x'' = f(t, x, x'), \qquad 0 < t < 1, \\ &x(0) = A, \quad x(1) = B. \end{aligned}$$

If f and f_x are continuous and $f_x \geq 0$, then there exists a unique solution.

Theorem Consider the initial value problem

$$\begin{aligned} &y'' + f(x, y, y') = 0, \\ &B_1[y] = y'(a) + Ay(a) - C_1 = 0, \\ &B_2[y] = y'(b) + By(b) - C_2 = 0, \end{aligned} \tag{10.1}$$

where f satisfies a Lipschitz condition, and f_y and $f_{y'}$ are bounded for x in the interval $[a, b]$ and for values of (y, y') of interest. Consider the two comparison equations

$$\begin{aligned} &u_1'' + h_1(x, u_1, u_1') = 0, \quad B_1[u_1] = 0, \qquad B_2[u_1] = 0, \\ &u_2'' + h_2(x, u_2, u_2') = 0, \quad B_1[u_2] = 0, \qquad B_2[u_2] = 0, \end{aligned}$$

with $h_1(x, y, y') \leq f(x, y, y') \leq h_2(x, y, y')$. We assume that the u_1 and u_2 problems have unique solutions. Then there exists at least one solution to (10.1) in the given region, and every solution has the property $u_1(x) \leq y(x) \leq u_2(x)$. (This theorem is one of the major results of the theory of differential inequalities.)

Cauchy–Kowalewski Theorem If the vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}^{\mathrm{T}}$ satisfies

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x, \qquad \mathbf{u}(0,x) = \mathbf{h}(x),$$

where $u_k = u_k(x,t)$, $A(\mathbf{u})$ is an analytic matrix, and $\mathbf{h}(x)$ is an analytic function, then a neighborhood of $t = 0$ can be found in which there is a unique solution $\mathbf{u}$, with each u_k being analytic.

Example 1

The first order initial value problem

$$y' = |y|^{1/3}, \qquad y(x_0) = 0 \tag{10.2}$$

has a right-hand side that is not Lipschitz continuous at $y = 0$. This equation, in fact, has an infinite number of solutions. Let x_1 and x_2 be any two numbers such that $x_1 < x_0 < x_2$. Then the following function

$$f(x) = \begin{cases} -\left(\frac{2}{3}\right)^{3/2}(x_1 - x)^{3/2}, & \text{if } x < x_1, \\ 0, & \text{if } x_1 < x < x_2, \\ \left(\frac{2}{3}\right)^{3/2}(x - x_2)^{3/2}, & \text{if } x_2 < x, \end{cases}$$

is a solution to equation (10.2).

Example 2

The nonlinear second order equation

$$\left(u'^3\right)' + 24(1-u) = 0, \qquad u(0) = 1, \quad u'(0) = 0,$$

has at least three solutions: $u(t) = 1$, $u(t) = 1 - t^2$, and $u(t) = 1 + t^2$.

Notes

1. Differential equations with discontinuities (see page 264) and delay equations (see page 253) do not meet the requirements of the above theorems. They must be investigated separately.
2. It is often possible to determine when a linear ordinary differential equation has a unique solution. When the solution is not unique, it is sometimes possible to describe the degrees of freedom that make it non-unique using alternative theorems (see page 15).
3. Fixed point theorems are a specific method that can be used to prove the existence of a solution (see page 58). The section on well posed differential equations contains some results on existence and uniqueness (see page 115).
4. Bobisud and O'Regan [2] consider existence questions for some second order initial value problems of the form $y'' + F(t,y,y') = 0$, where F is allowed to be suitably singular. For example, $F(t,y,y') = t^{-1/2}y^{-1/2}$ is allowed.

5. The existence of solutions to a differential equation can be critically dependent on the size of the coefficients in the equation. For example, Coddington and Levinson [3] show that the problem

$$\begin{aligned} &\epsilon y'' = -y' - (y')^3, \\ &y(0) = A, \quad y(1) = B \qquad (A \neq B) \end{aligned}$$

does not have a solution for small enough $\epsilon > 0$.

6. The classical problem

$$-\nabla^2 u = u^p \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in R^N, with smooth boundary $\partial\Omega$, has the interesting existence property (see Peletier [8]):

- If $p < \frac{N+2}{N-2}$, then existence of a solution is assured for any domain Ω;
- If $p \geq \frac{N+2}{N-2}$, then there exists no solution in any star-shaped domain.

Similar results are available for the equation $u_t = \nabla^2 u + u^p$; existence of a global positive solution depends on whether p is greater than $1 + 2/N$ (see Fujita [5]).

7. A classic result of Lewy [7] is that the equation

$$-u_x - iu_y + 2(ix - y)u_z = F(x, y, z),$$

where $F(x, y, z)$ is of class C^∞, has no H^1-solution, no matter what open (x, y, z) set is taken as the domain of existence.

8. Waterhouse [11] has the theorem:

Theorem: Consider the homogeneous linear differential equation involving only derivatives of even order and even functions as coefficients, $\left(D^{2n} + a_1 D^{2n-2} + \cdots + a_n\right) f = 0$ with $a_i(x) = a_i(-x)$ and having the symmetric homogeneous boundary conditions $B_1(D)f(s) = \cdots = B_n(D)f(s) = 0 = B_1(D)f(-s) = \cdots = B_n(D)f(-s)$ with $B_i(D) = \sum_j b_{ij} D^j$.

If this boundary value problem has a non-trivial solution, and if each of the vectors $(b_{i0} - b_{i1}, b_{i2} - b_{i3}, \dots)$ is in the span of the vectors $(b_{10}, b_{11}, b_{12}, \dots)$ and $(b_{20}, b_{21}, b_{22}, \dots)$, then this problem has a nontrivial solution that is either even or odd.

9. Agarwal and Sheng [1] provide necessary and sufficient conditions for the existence and uniqueness of solutions of general nth order nonlinear differential equations satisfying Abel–Gontscharoff boundary conditions. These are boundary conditions of the form $y^{(i)}(a_{i+1}) = A_{i+1}$ for $0 \leq i \leq n-1$ where $-\infty < a \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq b < \infty$.

References

[1] Agarwal, R. P., and Sheng, Q. Abel–Gontscharoff boundary value problems. *Math. Comput. Modelling 17*, 7 (1993), 37–55.

[2] Bobisud, L. E., and O'Regan, D. Existence of solutions to some singular initial value problems. *J. Math. Anal. Appl. 133* (1988), 214–230.

[3] Coddington, E. A., and Levinson, N. A boundary value problem for a nonlinear differential equation with a small parameter. *Proc. Amer. Math. Soc. 3* (1952), 73–81.

[4] Feckan, M. A new method for the existence of solution of nonlinear differential equations. *J. Differential Equations 89* (1991), 203–223.

[5] Fujita, H. On the blowing up of solutions of the Cauchy problem for zzzref18refzzz. *J. Fac. Sci. Univ. Tokyo Sect. A. Math 16* (1966), 105–113.

[6] Levine, H. A. The role of critical exponents in blowup theorems. *SIAM Review 32*, 2 (1990), 262–288.

[7] Lewy, H. An example of a smooth linear partial differential equation without solution. *Annals of Math. 66*, 1 (July 1957), 155–158.

[8] Peletier, L. A. Elliptic equations with nearly critical growth. In *Equadiff 1987*, C. M. Dafermos, G. Ladas, and G. Papanicolaou, Eds., no. 118 in Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, New York, 1987, pp. 561–574.

[9] Plum, M. Computer-assisted existence proofs for two-point boundary value problems. *Computing 46* (1991), 19–34.

[10] Redheffer, R. *Differential Equations*. Jones and Bartlett Publishers, Boston, 1991.

[11] Waterhouse, W. C. Some boundary value problems with even or odd solutions. *SIAM Review 38*, 4 (December 1996), 645–646.

11. Fixed Point Existence Theorems

Applicable to Differential equations of all types.

Yields

A statement about the existence of the solution.

Idea

If the statement concerning the existence of a solution to a differential equation can be interpreted as a statement concerning fixed points in a Banach space, then a fixed point theorem might be useful.

Procedure

The Schrauder fixed point theorem states:

> Let X be a non-empty convex set in a Banach space and let Y be a compact subset of X. Suppose $Y = f(X)$ maps X continuously into Y. Then there is a fixed point $x^* = f(x^*)$.

By interpreting a given differential equation as a continuous function in a Banach space, the above theorem indicates the existence of a solution.

Example

Suppose we wish to determine whether a solution exists to the nonlinear boundary value problem

$$\begin{gathered} u'' = -e^{-u(x)}, \\ u(0) = u(1) = 0, \end{gathered} \tag{11.1}$$

on the interval $x \in [0,1]$. We first note that the problem

$$\begin{gathered} v'' = -\phi(x), \\ v(0) = v(1) = 0, \end{gathered}$$

has the solution

$$v(x) = \int_0^1 G(x,z)\phi(z)\,dz,$$

where $G(x,z)$ is the Green's function (see page 321)

$$G(x,z) = \begin{cases} (1-x)z, & \text{for } 0 \le z \le x, \\ (1-z)x, & \text{for } x \le z \le 1. \end{cases}$$

Hence, we can write equation (11.1) in the form of an equivalent integral equation

$$u(x) = f(u(x)) \equiv \int_0^1 G(x,z)e^{-u(z)}\,dz. \tag{11.2}$$

To apply Schrauder's fixed point theorem to equation (11.2), we need to carefully define the Banach space B and the sets X and Y. If we define

$$\begin{aligned} B &= \text{space of continuous functions on } (0,1), \\ X &= \{u(x) \mid 0 \le u(x) \le 1, u(x) \text{ is continuous}\}, \\ Y &= f(X), \end{aligned}$$

then we can apply the theorem. Note that in this example, X is not compact but Y is. Note also that the bounds in X were derived after some analysis of equation (11.1). Finally, then, we conclude that equation (11.1) has a solution.

Notes

1. In the example above we used a fairly standard linearization trick that can be described in more generality. Suppose that an expression $D(f,g)$ (which could involve derivatives of f and/or g) is linear in f. Suppose also that the linear differential equation $D(f,g) = 0$ has a unique solution $f = T[g]$ for each g in some function space. Then to find a solution, in that function space, of the (possibly nonlinear) equation $D(f,f) = 0$ is equivalent to finding a fixed point of the mapping T. Thus a particular nonlinear differential equation can be studied by means of a more general linear differential equation, together with a fixed point problem.
2. Once a differential equation has been formulated as a fixed point statement, numerical methods that search for fixed points in a function space can be used. See, for example, Allgower [1].
3. Interval techniques (see page 545) may also be used to bound the solution of a fixed point statement. See Moore [7, Chapter 15, pages 97–102] for details.
4. A *contraction mapping* is a functional iteration, say $y_{n+1} = N[y_n]$, that converges to the solution of the fixed point equation $y = F[y]$. The Picard iteration (see page 618) is such a mapping.
5. Another fixed point theorem that is of use in differential equations is Krasnoselskii's theorem (see Franklin [3] for details):

   Consider the fixed point equation $\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})$ for $\mathbf{x}$ in a Banach space $\mathcal{B}$. Let X be a non-empty closed convex set in $\mathcal{B}$. Let $\mathbf{f}(\mathbf{x})$ map X continuously into a compact subset $Y \subset X$. Let $\mathbf{g}(\mathbf{x})$ be a contraction mapping on X (note that the range of $\mathbf{g}$ need not be compact). If it is assumed that $\mathbf{y} + \mathbf{g}(\mathbf{x}) \in X$ for $\mathbf{y} \in Y$ and $\mathbf{x} \in X$, then there is a fixed point of $\mathbf{x} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})$.

6. Another fixed point theorem that is of use in differential equations is the Tihonov fixed point theorem (see Iyanaga and Kawada [6, pages 542–543] for details):

Let R be a locally compact topological linear space, A a compact convex subset of R, and T a continuous mapping sending A into itself. Then T has fixed points.

7. Existence theorems for solutions for differential equations may be found on page 53.
8. See also Burton [2, Chapter 3, pages 164–196], Hale [4, Appendix, pages 171–172], Hartman [5, Chapter 12, pages 404–449], Smart [8, Chapter 6, pages 41–52], and Stakgold [9, pages 243–259].

References

[1] Allgower, E. L. Application of a fixed point search algorithm to nonlinear boundary value problems having several solutions. In *Fixed Points: Algorithms and Applications*, S. Karamardian, Ed. Academic Press, New York, 1977.

[2] Burton, T. A. Perturbation and delays in differential equations. *SIAM J. Appl. Math. 29*, 3 (November 1975), 422–438.

[3] Franklin, J. *Methods of Mathematical Economics*. Springer–Verlag, New York, 1980.

[4] Hale, J. K. *Oscillations in Nonlinear Systems*. McGraw–Hill Book Company, New York, 1963.

[5] Hartman, P. *Ordinary Differential Equations*. John Wiley & Sons, New York, 1964.

[6] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics*. MIT Press, Cambridge, MA, 1980.

[7] Moore, R. E. *Interval Analysis*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1966.

[8] Smart, D. R. *Fixed Point Theorems*. Cambridge University Press, New York, 1974.

[9] Stakgold, I. *Green's Functions and Boundary Value Problems*. John Wiley & Sons, New York, 1979.

12. Hamilton–Jacobi Theory

Applicable to Conservative dynamical systems.

Yields

A reformulation of a system of ordinary differential equations.

Idea

A change of variables may lead to more tractable equations.

Procedure

A conservative dynamical system has a Lagrangian L defined by $L = T - V$, where $T(V)$ is the kinetic (potential) energy. If the generalized coordinates in this system are $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, then the equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad \text{for } i = 1, 2, \ldots, n, \tag{12.1}$$

where a dot denotes differentiation with respect to t. The equations in (12.1) are called Lagrange's equations. If we define the generalized momenta by $p_i = \frac{\partial L}{\partial q_i}$ and the Hamiltonian by $H = \mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}} - L$, then Lagrange's equations become

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \\ \frac{\partial L}{\partial t} &= -\frac{\partial H}{\partial t}. \end{aligned} \tag{12.2}$$

These equations are called Hamilton's equations. If we change from the $(H, \mathbf{p}, \mathbf{q})$ variables to the $(J, \mathbf{P}, \mathbf{Q})$ variables via the canonical transformation defined by the generating function $S(\mathbf{P}, \mathbf{q}, t)$ (see page 132), then

$$\begin{aligned} p_i &= \frac{\partial S}{\partial q_i}, \\ Q_i &= \frac{\partial S}{\partial P_i}, \\ J(\mathbf{P}, \mathbf{Q}, t) &= H\Big(\mathbf{p}(\mathbf{P}, \mathbf{Q}, t), \mathbf{q}(\mathbf{P}, \mathbf{Q}, t), t\Big) + \frac{\partial S}{\partial t}. \end{aligned} \tag{12.3}$$

In these new variables, Hamilton's equations may be written

$$\begin{aligned} \dot{Q}_i &= \frac{\partial J}{\partial P_i}, \\ \dot{P}_i &= -\frac{\partial J}{\partial Q_i}. \end{aligned} \tag{12.4}$$

If the canonical transformation is chosen so that $J = 0$, then (12.4) says that $\mathbf{P}$ and $\mathbf{Q}$ are constant. To have J vanish identically, we require (from (12.3))

$$H\left(\frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \dots, \frac{\partial S}{\partial q_n}, q_1, q_2, \dots, q_n, t\right) + \frac{\partial S}{\partial t} = 0.$$

This last equation is known as the Hamilton–Jacobi equation. The procedure is to solve the Hamilton–Jacobi equation for the generating function S, make a canonical change of variables using this generating function, and then solve Hamilton's equation in these new coordinates. This will yield a solution to Lagrange's equations.

Example

Suppose we want to solve the linear constant coefficient ordinary differential equation

$$\ddot{q} + \omega^2 q = 0. \tag{12.5}$$

This differential equation comes from the Hamiltonian $H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$, which, in turn, corresponds to the following Hamilton–Jacobi equation:

$$\frac{1}{2}\left[\left(\frac{\partial S}{\partial q}\right)^2 + \omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0. \tag{12.6}$$

To solve for $S(q,t)$, we use separation of variables (see page 487), and look for a solution in the form $S(q,t) = a(q) + b(t)$, for some unknown functions $a(q)$ and $b(t)$. Using this form for S in equation (12.6) and making the usual argument about which terms must depend upon which variables, we determine that $a(q)$ and $b(t)$ must satisfy

$$\dot{b} = -\alpha, \qquad \left(\frac{da}{dq}\right)^2 + \omega^2 q^2 = 2\alpha,$$

where α is a separation constant. Hence, $S = -\alpha t + \int \sqrt{2\alpha - \omega^2 q^2}\, dq$. If we call $\alpha = P$, then we can compute from equation (12.3)

$$Q = \frac{\partial S}{\partial P} = -t + \int (2P - \omega^2 q^2)^{-1/2}\, dq = -t + \frac{1}{\omega}\sin^{-1}\left(\frac{\omega q}{\sqrt{2P}}\right),$$

which may be inverted to yield $q = \frac{\sqrt{2P}}{\omega}\sin\Big[\omega(t+Q)\Big]$, which is the solution to equation (12.5).

Notes

1. Lagrange's equations can be interpreted as the variational or Euler–Lagrange equations for the functional $J = \int L\, dt$ (see page 418).

2. The functions f and g are said to be *in involution* or to *Poisson commute* if the Poisson bracket $[f, g]$ is identically equal to zero. Liouville's theorem states that a function F is a first integral of a system with Hamiltonian function H if and only if H and F are in involution. See Abraham *et al.* [1, page 471] for details.
3. Poisson's theorem states that the Poisson bracket of two first integrals of a Hamiltonian system is again a first integral. See Goldstein [2, Chapter 9, pages 273–317] for details.
4. Any function $A(\mathbf{p}, \mathbf{q})$ defined along the trajectories of equation (12.2) satisfies
$$\frac{dA}{dt} = [A, H] = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial H}{\partial q_j} \right)$$
where the square brackets denote the Poisson bracket.
5. A general form for a non-conservative system is often taken to be
$$\begin{aligned} \dot{q}_i &= \frac{\partial C}{\partial p_i} + \frac{\partial D}{\partial q_i} \\ \dot{p}_i &= -\frac{\partial C}{\partial q_i} + \frac{\partial D}{\partial p_i} \end{aligned} \tag{12.7}$$
Where $C(\mathbf{p}, \mathbf{q})$ and $D(\mathbf{p}, \mathbf{q})$ are called the conservative and dissipation functions. For $D = 0$, this reduces to equation (12.2). For $C = 0$, this becomes a gradient system. Any function $A(\mathbf{p}, \mathbf{q})$ defined along the trajectories of equation (12.7) satisfies
$$\frac{dA}{dt} = \nabla A \cdot \nabla D + [A, C].$$
Choosing $A = C$ and $A = D$, we obtain the evolution equations for the conservative and dissipative functions
$$\begin{aligned} \frac{dC}{dt} &= \nabla C \cdot \nabla D, \\ \frac{dD}{dt} &= \nabla D \cdot \nabla D + [D, C]. \end{aligned}$$
Note that $\nabla^2 D$ equals the divergence of the vector field of equation (12.7) and that the system is dissipative when $\nabla^2 D < 0$.
6. Given the equations of motion: $\ddot{q}_i = f_i(\mathbf{q}, \dot{\mathbf{q}}, t)$, the inverse problem of classical mechanics is to determine whether these equations are equivalent to the Euler–Lagrange equations based on a Lagrangian L. That is, a matrix $w = w(\mathbf{q}, \dot{\mathbf{q}}, t)$ is desired so that
$$w_{ij}\left(\ddot{q}_j - f_j\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}.$$

The necessary and sufficient conditions for the existence of w and L are called the *Helmholtz conditions*, they are

$$\begin{aligned}
&\frac{\partial w_{ij}}{\partial \dot{x}_k} = \frac{\partial w_{ik}}{\partial \dot{x}_j}, \qquad w_{ij} = w_{ji}, \\
&\mathcal{D} w_{ij} = -\frac{1}{2} w_{ik} \frac{\partial f_k}{\partial \dot{x}_j} - \frac{1}{2} w_{jk} \frac{\partial f_k}{\partial \dot{x}_i} \\
&\frac{1}{2}\mathcal{D}\left(w_{ik} \frac{\partial f_k}{\partial \dot{x}_j} - w_{jk} \frac{\partial f_k}{\partial \dot{x}_i} \right) = w_{ik} \frac{\partial f_k}{\partial x_j} - w_{jk} \frac{\partial f_k}{\partial x_i}
\end{aligned}$$

with $\mathcal{D} = \frac{\partial}{\partial t} + \sum_m \left(\dot{x}_m \frac{\partial}{\partial x_m} + f_m \frac{\partial}{\partial \dot{x}_m} \right)$. See Hojman and Shepley [4].

7. The KdV equation, $u_t = -u_{xxx} + 6uu_x$, can be treated as a Hamiltonian system, $u_t = \{u, \mathcal{H}\}$, with the Hamiltonian and Poisson brackets defined by

$$\mathcal{H} = \frac{1}{2} \int u^2(x)\, dx \qquad \{u(x), u(y)\} = \left[-\partial^3 + 4u\partial + 2u_x \right] \delta(x - y)$$

8. See also Haar [3, Chapter 6, pages 121–145] and Nayfeh [5, pages 179–189].

References

[1] Abraham, R., Marsden, J. E., and Ratiu, T. *Manifolds, Tensor Analysis, and Applications.* Addison–Wesley Publishing Co., Reading, MA, 1983.

[2] Goldstein, H. *Classical Mechanics.* Addison–Wesley Publishing Co., Reading, MA, 1950.

[3] Haar, D. *Elements of Hamiltonian Mechanics.* Pergamon Press, New York, 1971.

[4] Hojman, S. A., and Shepley, L. C. No Lagrangian? No quantization! *J. Math. Physics 32*, 1 (Jan 1991), 142–146.

[5] Nayfeh, A. H. *Perturbation Methods.* John Wiley & Sons, New York, 1973.

[6] Sanz-Serna, J. M. Runge–Kutta schemes for Hamiltonian systems. *BIT 28* (1988), 877–883.

13. Integrability of Systems

Applicable to Systems of differential equations.

Yields

Information about whether a Hamiltonian system is completely integrable.

Idea

The Painlevé test performs a singular point analysis, which gives information about integrability.

Procedure

An autonomous Hamiltonian system is called (Liouville) integrable if there exists another function I such that $[H, I] = 0$. This function must be functionally independent of H, it must exist globally and be single valued, and it must be a complex analytic function of its variables.

If the Hamiltonian system has N degrees of freedom it is called completely integrable if it possesses N independent single valued analytic first integrals $\{I_k\}$ that commute with respect to the Poisson bracket

$$[I_n, I_m] = \sum_{i=1}^{N} \left(\frac{\partial I_n}{\partial q_i} \frac{\partial I_m}{\partial p_i} - \frac{\partial I_n}{\partial p_i} \frac{\partial I_m}{\partial q_i} \right) = 0.$$

One of these first integrals will be the Hamiltonian itself.

Given a Hamiltonian system, there is no known systematic method for determining whether or not that system is integrable. Much recent work has focused on the *Painlevé test*. The test asserts that an equation is integrable if every ordinary differential equation that arises as a similarity reduction of an integrable partial differential equation has the Painlevé property; that is, it has no movable singularities except poles, perhaps after a transformation of variables. For the Painlevé test to be effective, it is necessary to determine the complete symmetry group of the differential equation under consideration. If it passes the test, then it is believed that the original partial differential equation will be solvable by inverse scattering methods (see page 460). The Painlevé test also has applications in determining the stability of systems of ordinary differential equations.

Roughly speaking, a partial differential equation is said to possess the Painlevé property if the only singularities of the general solution on arbitrary non-characteristic surfaces are poles. Singular point analysis is used to determine if differential equations have the Painlevé property. The test consists of substituting

$$u(x) = \sum_{n=0}^{\infty} u_n (x - x_0)^{\alpha + p},$$

for $\alpha < 0$, into the tested equation in the vicinity of a singular point x_0 and investigating whether this expansion is compatible with the equation and contains a sufficient number of undetermined coefficients for the approximation of a general solution.

Example

The motion of the N particle lattice is described by the Hamiltonian

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} e^{q_j - q_{j+1}} \tag{13.1}$$

where $q_{N+1} = q_1$ (which corresponds to cyclic boundary conditions). If $\{a_j, b_j\}$ are defined by

$$a_j := \frac{1}{2}e^{(q_j - q_{j+1})/2}, \qquad b_j := \frac{1}{2}p_j,$$

then the equations of motion are

$$a_j' = a_j(b_j - b_{j+1}), \qquad b_j' = 2(a_{j-1}^2 - a_j^2). \tag{13.2}$$

If the following $N \times N$ matrices are defined:

$$L = \begin{bmatrix} b_1 & a_1 & 0 & \dots & 0 & a_N \\ a_1 & b_2 & a_2 & & 0 & 0 \\ 0 & a_2 & b_3 & & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & & b_{N-1} & a_{N-1} \\ a_N & 0 & 0 & & a_{N-1} & b_N \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -a_1 & 0 & \dots & 0 & a_N \\ a_1 & 0 & -a_2 & & 0 & 0 \\ 0 & a_2 & 0 & & 0 & 0 \\ \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 0 & -a_{N-1} \\ -a_N & 0 & 0 & & a_{N-1} & 0 \end{bmatrix},$$

then equation (13.2) may be written in the form

$$\frac{dL}{dt} = [A, L] = AL - LA.$$

Note we also have $\dfrac{d(L^k)}{dt} = [A, L^k]$ for any positive integer k. From this it follows that the trace of the matrix L^k is constant. Hence, the traces $\{\operatorname{tr}(L),\ \operatorname{tr}(L^2), \dots,\ \operatorname{tr}(L^k), \dots\}$ are first integrals for (13.2). They turn out to be independent and in involution of each other. Hence, the Hamiltonian in equation (13.1) is completely integrable.

Notes

1. Several definitions of "integrability" are in use in the literature. For example, the PDE $N(x,t,u) = 0$ with $u(x,0) = f(x)$ is called completely integrable if there is an integral equation for K of the form
$$K(x,y;t) + F(x,y,;t) + \int_x^\infty K(x,z;t)H(z,y;t)\,dz = 0$$
called the Gelfand–Levitan equation, such that
   - F and H are uniquely determined from $f(x)$
   - the solution of the PDE is given by $u(x,t) = K(x,x;t)$.
2. Completely integrable PDEs are known to possess several remarkable properties including:
   - the existence of soliton solutions (see page 626)
   - the existence of an infinite number of independent conservation laws (see page 47)
   - a Lax representation (see page 460)
   - Bäcklund transformations (see page 428)
3. In general, linear equations only have fixed singularities while nonlinear equations can have both fixed and movable singularities.
   - Consider the linear equation $y'' + p(x)y' + q(x)y = 0$ which has the general solution $y(x) = Ay_1(x) + By_2(x)$ where A and B are arbitrary constants. The location of the singularities of $y(x)$ depend only on $p(x)$ and $q(x)$, not on A or B. The singularities of this equation are *fixed*, since they do not depend upon the constants of integration.
   - Consider the nonlinear equation $y' + y^2 = 0$ which has the general solution $y(x) = (x - x_0)^{-1}$ where x_0 is an arbitrary constant. In this case $y(x)$ has a singularity, a pole, which is *movable* since it depends on the constant of integration x_0.
4. For first order equations of the form $y' = F(y,x)$, where F is rational in y and analytic in x, the only equation which has no movable singularities other than poles is the Riccati equation $y' = p_0(x) + p_1(x)y + p_2(x)y^2$.
For second order equations of the form $y'' = F(y, y', x)$, where F is rational in y and y' and analytic in x, Painlevé *et al.* (see Ince [8]) showed that there are only 50 canonical equations which have no movable singularities except poles. Of these, 44 are integrable in terms of known functions (such as elliptic functions) and the remaining 6 defined new transcendental functions, called the Painlevé transcendents (see page 128).
5. The three-particle Toda lattice has the Hamiltonian $H = \frac{p_1^2+p_2^2+p_3^2}{2} + V$ with the potential energy $V = e^{p_1-p_2} + e^{p_2-p_3} + e^{p_3-p_1}$. The sys-

Parameters	Invariant
$b = 2\sigma$	$(x^2 - 2\sigma z)e^{2\sigma t}$
$b = 0, \sigma = \frac{1}{3}$	$\left(-rx^2 + \frac{1}{3}y^2 + \frac{2}{3}xy + x^2 z - \frac{3}{4}x^4\right) e^{4t/3}$
$b = 1, r = 0$	$\left(y^2 + z^2\right) e^{2t}$
$b = 4, \sigma = 1$	$\left(4(1-r)z + rx^2 + y^2 - 2xy + x^2 z - \frac{1}{4}x^4\right) e^{4t}$
$b = 1, \sigma = 1$	$\left(-rx^2 + y^2 + z^2\right) e^{2t}$
$b = 6\sigma - 2,$ $r = 2\sigma - 1$	$\left(\frac{(2\sigma-1)^2}{\sigma}x^2 + y^2 - (4\sigma - 2)xy + x^2 z - \frac{1}{4\sigma}x^4\right) e^{4\sigma t}$

Table 13.1: First integrals for the Lorenz equations.

tems admits three independent integrals, for instance, the functions

$$\begin{aligned} I_1 &= p_1 + p_2 + p_3 \\ I_2 &= p_1 p_2 + p_2 p_3 + p_3 p_1 - V \\ I_3 &= p_1 p_2 p_3 - p_1 e^{p_2 - p_3} - p_2 e^{p_3 - p_1} - p_3 e^{p_1 - p_2} \end{aligned}$$

These integrals are in involution and they are independent.

6. Consider the Hamiltonian $H = (p_x^2 + p_y^2)/2$ +V:
   - For the Hénon–Heiles potential $V = \frac{\mu}{3}y^3 + x^2 y$, the system is integrable for $\mu = 1$, 6, and 16.
   - For the Holt potential $V = \frac{\mu}{3}y^{4/3} + x^2 y^{-2/3}$, the system is integrable for $\mu = 1$, 6, and 16.
   - For the quartic potential $V = ax^4 + bx^2y^2 + cy^4$, the system is integrable if $a : b : c$ have the ratios a:0:c, 1:2:1, 1:6:1, 1:12:16, 16:12:1, 1:6:8, or 8:6:1.
7. The Lorenz equations (see page 199)

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= -y - xz - rx \\ z' &= xz - bz \end{aligned}$$

have known first integrals for several possible values of the parameters $\{\sigma, r, b\}$. For example, the first integrals in table 13.1 are known.

8. Clarkson *et al.* [4] state that the only third-order semilinear partial differential equations that are linearizable are equivalent to the

following six equations:

$$
\begin{aligned}
u_t &= u_{xxx} + \gamma u_x, \\
u_t &= u_{xxx} + uu_x + \gamma u_x, \\
u_t &= u_{xxx} + u^2 u_x + \gamma u_x, \\
u_t &= u_{xxx} - \frac{1}{8}u_x^3 + \left(\alpha e^u + \beta e^{-u}\right) u_x + \gamma u_x, \\
u_t &= u_{xxx} - \frac{3}{2}u_x u_{xx}^2 \left(1 + u_x^2\right)^{-1} - \frac{3}{2}P(u)(u_x^2 + 1)u_x + \gamma u_x, \\
u_t &= u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 + \alpha u_x^{-1} - \frac{3}{2}P(u)u_x^2 + \gamma u_x,
\end{aligned}
$$

where $P(u)$ is the Weierstrass elliptic function and satisfies

$$\left(\frac{dP}{du}\right)^2 = 4P^3 - \delta P - \epsilon.$$

9. Clarkson *et al.* [4, page 1205] show that the PDE
   - $u_t = u_{xx} + h(u)u_x$, where $h(u)$ is a rational function of u, can pass the Painlevé test only if $h(u)$ is a linear function of u.
   - $u_t = u_{xxx} + (uu_{xx} + u_x^2) + \frac{3}{2}(\alpha - 1)u^2 u_x$, where α is a constant, can pass the Painlevé test only if $\alpha = 0$, $3/2$, or 3.
10. Hereman and Angenent [7] and Rand and Winternitz [13] describe Macsyma programs for determining whether a nonlinear ordinary differential equation has the Painlevé property. (The differential equation must be a polynomial in both the dependent and independent variables and in all derivatives.)
11. The only equations of the form $u_{xt} = f(u)$, where $f(u)$ is a linear combination of exponentials, which pass the Painlevé test are: the sine–Gordan equation $u_{xt} = \sin u$, the Liouville equation $u_{xt} = e^u$, and the Bullough–Dodd equation $u_{xt} = e^u - e^{-2u}$.
12. Polynomial potentials arise in many problems, particularly when truncated Taylor series are used to facilitate analytical study. It is useful to examine the integrability of such potentials. In two dimensions there are only three independent integrable cubic potentials; they are $x^3 + 3xy^2 + \alpha y^3$, $2x^3 + xy^2$, and $16x^3 + 3xy^2$; see Cleary [5].
13. The Mathematica package `DSolveIntegrals` can compute complete integrals of non-linear PDEs. For example, given $yu_y = u + x^2u_x^2$ the integral is determined to be $u = (-a^2 + 4by - 2a\log x - \log^2 x)/4$.
14. The following equations are known to be completely integrable: sine–Gordon equation, Doff–Bullough, Ernst equation, axisymmetric stationary Einstein–Maxwell equation.

References

[1] Albrecht, D. W., Mansfield, E. L., and Milne, A. E. Algorithms for special integrals of ordinary differential equations. *J. Phys. A: Math. Gen. 29* (1996), 973–991.

[2] Buchler, J. R., Ipser, J. R., and Williams, C. A., Eds. *Integrability in Dynamical Systems*. New York Academy of Sciences, New York, 1988.

[3] Chang, Y. F., Tabor, M., and Weiss, J. Analytic structure of the Henon–Heiles Hamiltonian in integrable and nonintegrable regimes. *J. Math. Physics 23*, 4 (April 1982), 531–538.

[4] Clarkson, P. A., Fokas, A. S., and Ablowitz, M. J. Hodograph transformations of linearizable partial differential equations. *SIAM J. Appl. Math. 49*, 4 (August 1989), 1188–1209.

[5] Cleary, P. W. Nonexistence and existence of various order integrals for two- and three-dimensional polynomial potentials. *J. Math. Physics 31*, 6 (June 1990), 1351–1355.

[6] Gerdt, V. P., Zharkov, A. Y., Svinolupov, S. I., and Shabat, A. B. The use of computer algebra to investigate the integrability of non-linear evolution systems. *U.S.S.R. Comput. Maths. Math. Phys. 28*, 6 (1988), 50–57.

[7] Hereman, W., and Angenent, S. The Painleve test for nonlinear ordinary and partial differential equations. *Macsyma Newsletter* (January 1989), 11–18.

[8] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[9] Mel'nikov, V. K. New method for deriving nonlinear integrable systems. *J. Math. Physics 31*, 5 (May 1990), 1106–1113.

[10] Prelle, M. J., and Singer, M. F. Elementary first integrals of differential equations. *Trans. Amer. Math. Soc. 279*, 1 (September 1983), 215–229.

[11] Progrebkov, A. K. On the formulation of the Painleve test as a criterion of complete integrability of partial differential equations. *Inverse Prob. 5* (1989), L7–L10.

[12] Ramani, A., Dorizzi, B., Grammaticos, B., and Bountis, T. Integrability and the Painleve property for low-dimensional systems. *J. Math. Physics 25*, 4 (April 1984), 878–883.

[13] Rand, D. W., and Winternitz, P. ODEPAINLEVE — a Macsyma package for Painleve analysis of ordinary differential equations. *Comput. Physics Comm. 42* (1986), 359–383.

[14] Roekaerts, D., and Schwarz, F. Painleve analysis, Yoshida's theorems and the direct methods in the search for integrable Hamiltonians. *J. Phys. A: Math. Gen. 20* (1987), L127–L133.

14. Internet Resources

Applicable to Many topics related to differential equations.

Procedure

Much information about differential equations is available through the internet. We list next some of these resources:

Symbolic software packages

1. There are a multitude of commercial computer packages available for symbolically solving differential equations (see page 240). These include

   - AXIOM `http://www.nag.co.uk:80`
   - Derive `http://www.derive.com`
   - Macsyma `http://www.macsyma.com`
   - Maple `http://www.maplesoft.com`
   - Mathematica `http://www.wolfram.com`
   - REDUCE `http://www.rrz.uni-koeln.de/REDUCE`

2. The program `CONVODE` will symbolically solve ordinary and partial differential equations across the internet. For example, sending

   ```
   depend y,x;
   CONVODE( {df(y,x,2)+4*y=0}, {y}, {x}, {}, {english});
   ```

   to `convode@riemann.physmath.fundp.ac.be` will have the solution of $y'' + 4y = 0$ returned via email with comments in English (the default is French). See `http://www.physique.fundp.ac.be/physdpt/administration/convode.html`.

3. MathServ provides an interface between the user and Mathematica (a symbolic computational engine). Templates for twelve different types of ODEs are available; the user can specifiy the functions appearing in them. The results are returned directly to your browser. See `http://math.vanderbilt.edu/~pscrooke/detoolkit.html`.

Numerical software packages

There are a multitude of commercial computer packages available for numerically solving differential equations (see page 654). In particular, the Guide to Available Mathematical Software (GAMS) has a taxonomy of software classes, with many representatives of most classes. See `http://gams.nist.gov`. This section lists a few packages that currently may be used freely for non-commercial purposes.

- `Diffpack` is a collection of C++ class libraries aimed at the numerical solution of partial differential equations. The Diffpack home page

```
int main() {
 real r0=.5, r1=1., x0=0., y0=0., theta0=0., theta1=1.; // parameters
 AnnulusMapping annulus(r0,r1,x0,y0,theta0,theta1);     // annulus mapping
 MappedGrid mg(annulus);               // MappedGrid for an annulus
 mg.update();                          // create default variables
 realMappedGridFunction u(mg);         // declare grid function on the grid
 u=1.;                                 // initial condition u=1
 MappedGridOperators op(mg);           // difference operators and BCS
 u.setOperators(op);                   // associate with a grid function
 real t=0, dt=.005, a=1., b=1., nu=.1; // problem parameters
 for( int step=0; step<100; step++ ) { // loop for number of time steps
   u.display("solution");              // print out the solution
   u+=dt*((-a)*u.x()+(-b)*u.y()+nu*(u.xx()+u.yy())); // forward Euler step
   t+=dt;
   u.applyBoundaryCondition(0,BCTypes::dirichlet,BCTypes::allBoundaries,0.);
                                       // apply Boundary condition u=0
   u.finishBoundaryConditions();       // fix up corners, periodic update
 }
 return 0;
}
```

Program 14.1: Overture program for a reaction diffusion problem

is `http://www.oslo.sintef.no/avd/33/3340/diffpack`. The code can be downloaded from `http://www.oslo.sintef.no/diffpack/pub1.4` or from Netlib at `http://www.netlib.org`.

- `DsTool` is *A Dynamical System Toolkit with an Interactive Graphical Interface*. It computes Poincaré sections and bifurcation diagrams and is easily extensible. It was created at Cornell University and runs under X windows. The program and documentation can be obtained via `ftp` from `macomb.cam.cornell.edu` in the `/pub/dstool` directory.

- `KASKADE` is a C++ package that solves elliptic partial differential equations. It is an adaptive multilevel-code for linear scalar elliptic and parabolic problems in 1, 2, and 3 space dimensions. It includes examples for nonlinear methods used in obstacle, porous media, and Stefan problems. It can be obtained via `ftp` from `elib.zib-berlin.de` in the directories `/pub/kaskade/3.x` and `/pub/kaskade/Manuals/3.0`.

- `Overture` is a high level object oriented framework for solving PDEs on structured grids and overlapping grids using finite difference and finite volume methods. Overture is freely available and can be obtained from `http://www.c3.lanl.gov/~henshaw/Overture/Overture.html`. For example, the *entire* program to solve the problem $u_t + au_x + bu_y = \nu(u_{xx} + u_{yy})$ in an annulus A, with $u(t = 0, A) = 1$ and $u\big|_{\partial A} = 0$, using forward Euler's method, is in program 14.1.

Electronic journals

The *Electronic Journal of Differential of Equations* (EJDE) is dedicated to the rapid dissemination of high quality research in mathematics. Publications are available as PostScript, TeX, and DVI files. All topics related to differential equations and their applications are considered for publication. Research articles are refereed under the same standards as those used by the finest-quality printed journals. EJDE may be found at `http://ejde.math.swt.edu`.

Other resources

- `C*ODE*E` is the acronym for the *Consortium of ODE Experiments*. Their goal is to share the rapidly growing wealth of computational instruction techniques with teachers of differential equations. The Consortium publishes a newsletter designed to provide a regular source of ideas, inspiration, and experiments for instructors of ODEs. The newsletter is available on-line and in print format. Their URL is `http://www.math.hmc.edu/codee`.

- `IDEA` is the acronym for *Internet Differential Equations Activities*. This is an interdisciplinary effort to provide students and teachers with computer based activities for differential equations in a wide variety of discplines. This is sponsored by the NSF. It includes a glossary of terms and many other features. Their URL is `http://www.sci.wsu.edu/idea`.

- The American Mathematical Society maintains materials organized by mathematical subject classification at `http://www.ams.org/mathweb/mi-mathbyclass.html`. In this classification, category 34 is "Ordinary differential equations" and category 35 is "Partial differential equations." The AMS Preprint Server for these categories may be found at `http://www.ams.org/preprints/34/msc34-page.html` and `http://www.ams.org/preprints/35/msc35-page.html`.

- Los Alamos maintains a web site on "Exactly Solvable and Integrable Systems", see `http://xxx.lanl.gov/archive/solv-int`.

- The Norwegian University of Science and Technology maintains a "Conservation Laws Preprint Server" at `http://www.math.ntnu.no/conservation`.

- The "Mathematics Archives," see `http://archives.math.utk.edu`, is supported by the NSF, the State of Tennessee, Calvin College, and the University of Tennessee, Knoxville. Their repository of links related to ordinary differential equations and partial differential equations may be found at `http://archives.math.utk.edu/topics/ordinaryDiffEq.html` and `http://archives.math.utk.edu/topics/partialDiffEq.html`.

- The Math/CS Department of Nebraska Wesleyan University has a differential equations resource page documenting course materials (labs and projects) developed as part of an NSF/ILI grant. The URL is `http://brillig.nebrwesleyan.edu/delabs`.
- The Math Department at Oregon State University has developed a web-based study guide for several of its courses. The URL for the ODE home page is `http://iq.orst.edu.mathsg/ode/ode.html`.

Note

1. The URLs in this section are subject to change.

15. Inverse Problems

Applicable to

Inverse problems.

Yields

Information about parameters appearing in a differential equation.

Idea

There are theorems that can be used to determine which inverse problems may be solved.

Procedure

The field of inverse problems is filled with specialized theorems that are useful for specific applications.

Example 1

Consider the eigenvalue problem

$$\begin{aligned} -u'' + q(x)u &= \lambda u, \qquad \text{for } 0 \le x \le 1, \\ u(0)\cos\alpha + u'(0)\sin\alpha &= 0, \\ u(1)\cos\beta + u'(1)\sin\beta &= 0, \end{aligned} \tag{15.1}$$

where λ is a complex parameter, $q(x)$ is a real-valued function that is integrable on the interval $[0, 1]$, and α and β are values in the interval $[0, \pi)$.

One common inverse problem consists of determining the function $q(x)$ from the eigenvalues of equation (15.1). There are many different results in this area. For example

> **Theorem** Suppose that $(\alpha, \beta, q(x))$ give rise to the eigenvalues $\{\lambda_n\}$ and suppose that $(\overline{\alpha}, \overline{\beta}, \overline{q}(x))$ give rise to the eigenvalues $\{\overline{\lambda}_n\}$. If $\lambda_n = \overline{\lambda}_n$ for $n = 0, 1, \dots$; $q(x) = \overline{q}(x)$ for $x \in (0, \frac{1}{2})$; and $\alpha = \overline{\alpha}$, then $q(x) = \overline{q}(x)$ almost everywhere on the interval $(0, 1)$.

Another typical theorem is the following:

> **Theorem** Let $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ be the eigenvalues of the problem $-y'' + q(x)y = \lambda y$ with $y'(0) = y'(\pi) = 0$, where $q(x)$ is a real-valued continuous function. If $\lambda_n = n^2$ for $n = 0, 1, 2, \dots$, then $q(x) = 0$.

Example 2

One common technique to show uniqueness for an inverse problem is to investigate a mapping between the solutions of two equations with different values for the parameter(s) of interest. We have, for example (see Rundell [11]):

Theorem Let $u(x)$ and $v(x)$ satisfy

$$\begin{aligned} u_t &= u_{xx} - a(x)u, & u_x(0,t) &= 0, \\ v_t &= v_{xx} - \overline{a}(x)v, & v_x(0,t) &= 0, \end{aligned}$$

for $0 \le x \le 1$ and $0 \le t < T$. If $u(0,t) = v(0,t)$, then $v(x,t) = u(x,t) + \int_0^x K(x,s)u(s,t)\,ds$, where $K(x,s)$ satisfies the Goursat problem

$$\begin{aligned} K_{ss} - K_{tt} = \left(a(s) - \overline{a}(x)\right) K(x,s), &\qquad \text{for } 0 \le s \le x \le 1, \\ K_s(x,0) = 0 &\qquad \text{for } 0 \le x \le 1, \\ K(x,x) = \frac{1}{2}\int_0^x \left(a(r) - \overline{a}(r)\right)\,dr &\qquad \text{for } 0 \le x \le 1. \end{aligned}$$

In this case it is possible to show that if $\int_0^x K(x,s)f(s)\,ds = 0$ for some positive function $f(x)$, then $a = \overline{a}$.

Notes

1. The numerical methods used to solve inverse problems tend to result in ill-conditioned systems.
2. If the spectra $\{\lambda_i\}$ and $\{\mu_i\}$ are known for the following two problems (with $H \neq \bar{H}$):

$$\begin{aligned} -y'' + q(x)y &= \lambda y, & -y'' + q(x)y &= \mu y, \\ y'(0) - hy(0) &= 0, & y'(0) - hy(0) &= 0, \\ y'(1) - Hy(1) &= 0, & y'(1) - \bar{H}y(1) &= 0, \end{aligned}$$

then $\{q(x), h, H, \bar{H}\}$ are all uniquely determined. See Rundell and Sacks [12].

References

[1] Anger, G. *Inverse Problems in Differential Equations*. Plenum Publishing Corp., New York, 1990.

[2] Barnes, D. C. The inverse eigenvalue problem with finite data. *SIAM J. Math. Anal. 22*, 3 (May 1991), 732–753.

[3] Cannon, J. R., and Lin, Y. An inverse problem of finding a parameter in a semi-linear heat equation. *J. Math. Anal. Appl. 145*, 2 (1990), 470–484.

[4] Castillo, R. D. R. On boundary conditions of an inverse Sturm–Liouville problem. *SIAM J. Appl. Math. 60*, 6 (December 1990), 1745–1751.

[5] Colton, D., Ewing, R., and Rundell, W., Eds. *Inverse Problems in Partial Differential Equations*. SIAM, Philadelphia, PA, 1990.

[6] Eskin, G. Inverse spectral problem for the Schroedinger equation with periodic vector potential. *Comm. Math. Physics 125*, 2 (1989), 263–300.

[7] Hassan, A. A. M., and Abdel-Halim, I. H. Some inverse eigenvalue problems for the Laplacian operator, II. *J. Inst. Math. Comput. Sci. Math. Ser. 2*, 2 (1989), 125–146.

[8] LEVITAN, B. M., AND SARGSJAN, I. S. *Sturm–Liouville and Dirac Operators*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[9] PILANT, M., AND RUNDELL, W. Determining a coefficient in a first-order hyperbolic equation. *SIAM J. Appl. Math. 51*, 2 (April 1991), 494–506.

[10] ROY, D. N. G. *Methods of Inverse Problems in Physics*. CRC, Boca Raton, FL, 1990.

[11] RUNDELL, W. The use of integral operators in undetermined coefficient problems for partial differential equations. *Appl. Analysis 18* (1984), 309–324.

[12] RUNDELL, W., AND SACKS, P. E. Reconstruction techniques for classical inverse Sturm–Liouville problems. *Math. of Comp. 58*, 197 (January 1992), 161–183.

16. Limit Cycles

Applicable to Systems of nonlinear autonomous differential equations.

Yields

Knowledge of whether or not there exist limit cycles.

Idea

Knowing that limit cycles exist for a differential system allows global characterizations of the differential system.

Procedure

A non-constant solution of the system $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$ is called a cycle (or a limit cycle) if there is a positive number T (called the period of the cycle) such that $\mathbf{x}(t+T) = \mathbf{x}(t)$ for all t. It is easy to show that inside of every cycle is at least one critical point (i.e., a point where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, see page 526).

In many systems it is not only true that there are finitely many cycles but also that all solutions tend to one of these cycles. This knowledge permits a concise characterization of the phase plane.

Example 1

The nonlinear autonomous system

$$\begin{aligned}\frac{dx}{dt} &= -y + x(1 - x^2 - y^2),\\ \frac{dy}{dt} &= \quad x + y(1 - x^2 - y^2)\end{aligned}$$

becomes, under the change of variables $\{x = r\cos\theta,\ y = r\sin\theta\}$, the uncoupled system

$$\frac{dr}{dt} = r(1 - r^2), \qquad \frac{d\theta}{dt} = 1.$$

These new equations have the solution

$$r(t) = \frac{1}{\sqrt{1 + Ae^{-2t}}}, \qquad \theta(t) = t + B,$$

where A and B are arbitrary constants. Hence, the solution of the original system is

$$x(t) = \frac{\cos(t+B)}{\sqrt{1 + Be^{-2t}}}, \qquad y(t) = \frac{\sin(t+B)}{\sqrt{1 + Be^{-2t}}}.$$

This states that all solutions tend to the circle $x^2(t) + y^2(t) = 1$ as $t \to \infty$.

Of course, in most circumstances it is not possible to construct explicitly the limit cycle. Generally theorems (such as those below) are used to prove the existence of a limit cycle.

Example 2

The Van der Pol equation

$$\frac{d^2x}{dt^2} - \mu\left(1 - x^2\right)\frac{dx}{dt} + x = 0$$

with $\mu > 0$ has limit cycles. For this equation, there is negative damping for small values of x and positive damping for large values of x. Hence the value of x increases when x is small and it decreases when x is large.

Notes

1. Given a limit cycle Γ and a positive number a, define *the annulus centered on* Γ to be $\{\mathbf{x} \mid \text{distance from } \mathbf{x} \text{ to } \Gamma \text{ is less than } a\}$ where the distance from $\mathbf{x}$ to Γ is defined to be $\min_{\mathbf{u}\in\Gamma}|\mathbf{x} - \mathbf{u}|$.
   A cycle Γ is called *isolated* if there is a positive number a for which the annulus centered on Γ contains no other limit cycles. A cycle is *non-isolated* if every annulus centered of Γ contains at least one other limit cycle. The system

   $$\frac{dx}{dt} = x\sin\left(x^2 + y^2\right) - y, \qquad \frac{dy}{dt} = y\sin\left(x^2 + y^2\right) + x$$

   has infinitely many isolated cycles whereas the system $\{x' = y,\ y' = -x\}$ has infinitely many non-isolated cycles.
2. Part of Hilbert's 16th problem asked for the maximum number of limit cycles of the system $\{x' = A(x, y), y' = B(x, y)\}$ where A and B are polynomials. If A and B are polynomials of degree n, then the maximum number is known as the Hilbert number or the Hilbert function, H_n. It is known that $H_0 = 0$, $H_1 = 0$, $H_2 \geq 4$, $H_3 \geq 8$, $H_n \geq \frac{n-1}{2}$ if n is odd, and $H_n < \infty$.
   The example that demonstrates that $H_2 \geq 4$ (found by Songling [12]) is

   $$\begin{aligned} x' &= ax - y - 10x^2 + (5 + b)xy + y^2, \\ y' &= x + x^2 + (8c - 25 - 9b)xy, \end{aligned}$$

   where $a = -10^{-200}$, $b = -10^{-13}$, and $c = -10^{-52}$. See also James and Lloyd [4].
3. Neto [8] has the two results:

   > **Theorem** The equation $x' = a_2x^2 + a_1x + a_0$, where the $\{a_i\}$ are continuous functions on $[0, 1]$, has at most two closed solutions, if not all solutions in $[0, 1]$ are closed.

   and

   > **Theorem** The equation $x' = a_3x^3 + a_2x^2 + a_1x + a_0$, where the $\{a_i\}$ are continuous functions on $[0, 1]$, has at most three closed solutions.

4. If $f(x)$ and $g(x)$ are continuous, have continuous derivatives, and satisfy the conditions:
   - $xg(x) > 0$ for $x \neq 0$,
   - $f(x)$ is negative in the interval $a < x < b$ (with $a < 0$ and $b > 0$) and positive outside of this interval,
   - $\int_0^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx = \infty$,

   then every nontrivial solution of Liénard's equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \tag{16.1}$$

   is either a limit cycle or a spiral that tends toward a limit cycle as $t \to \infty$. See Birkhoff and Rota [1, pages 135–137] for details.
5. Liénard's theorem states

   If $f(x)$ and $g(x)$ are continuous and satisfy the conditions
   - $F(x) := \int_0^x f(x)\,dx$ is an odd function,
   - $F(x)$ is zero only at $x = 0$, $x = a$, $x = -a$, for some $a > 0$,
   - $F(x) \to \infty$ monotonically for $x > a$,
   - $g(x)$ is an odd function, and $g(x) > 0$ for $x > 0$,

   then equation (16.1) has a unique limit cycle.

   For details, see Jordan and Smith [5]. Note that Van der Pol's equation (see example 2) satisfies Liénard's theorem and, hence, has a unique limit cycle.
6. Bendixson's theorem states (see Simmons [11, pages 338–352])

   If $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$ is continuous and is always positive or always negative in a certain region of the phase plane, then the autonomous system

$$\frac{dx}{dt} = F(x, y), \qquad \frac{dy}{dt} = G(x, y)$$

   has no limit cycles in that region.

   For example, the equation for the Lewis regulator

$$\frac{d^2x}{dt^2} + (1 - |x|)\frac{dx}{dt} + x = 0,$$

   which is equivalent to

$$\frac{dx}{dt} = F(x, y) = y, \qquad \frac{dy}{dt} = G(x, y) = -x - (1 - |x|)y,$$

   has $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = |x| - 1$. Hence, the Lewis regulator has no limit cycles in the strip $-1 < x < 1$.
7. Another statement of Bendixson's theorem, regarding periodic solutions or limit cycles, can be stated as follows:

Consider $\dot{\mathbf{x}} = f(\mathbf{x})$ in a simply connected domain D (in two dimensions). If the gradient of f is not identically zero over any subregion of D and does not change sign in D, then D contains no closed trajectory.

8. The Levinson–Smith theorem states (see Hagedorn [3, page 143])

For the differential equation

$$x'' + f(x, x')x' + g(x) = 0 \tag{16.2}$$

if the following conditions are satisfied:

- $xg(x) > 0$ for all $x > 0$,
- $\int_0^\infty g(x)\,dx = \infty$,
- $f(0, 0) < 0$,
- there exists an $x_0 > 0$ such that $f(x, x') \geq 0$ for $|x| > x_0$, for every x',
- there exists a constant $M > 0$, such that $f(x, x') \geq -M$ for $|x| \leq x_0$,
- there exists an $x_1 > x_0$ such that $\int_{x_0}^{x_1} f(x, v(x))\,dx \geq 10Mx_0$, where $v(x)$ is any arbitrary positive and monotonically decreasing function of x,

then equation (16.2) has at least one limit cycle.

9. Sedaghat [9] shows that *factorable planar systems* (i.e., systems of the form $x' = f(x)h(y)$ and $y' = k(x)g(y)$) do not have limit cycles.

References

[1] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations*. John Wiley & Sons, New York, 1978.

[2] Blows, T. R., and Lloyd, N. G. The number of limit cycles of certain polynomial differential equations. *Proc. Roy. Soc. Edinburgh 98A* (1984), 215–239.

[3] Hagedorn, P. *Non-linear Oscillations*. Clarendon Press, Oxford, England, 1982.

[4] James, E. M., and Lloyd, N. G. A cubic system with eight small-amplitude limit cycles. *IMA J. Appl. Mathematics 47* (1991), 163–171.

[5] Jordan, D. W., and Smith, P. *Nonlinear Ordinary Differential Equations*, second ed. Clarendon Press, Oxford, England, 1987.

[6] Koditschek, D. E., and Narendra, K. S. Limit cycles of planar quadratic differential equations. *J. Differential Equations 54* (1984), 181–195.

[7] Kuang, Y. Finiteness of limit cycles in planar autonomous systems. *Appl. Anal. 32*, 3–4 (1989), 253–264.

[8] Neto, A. L. On the number of solutions of the equation zzzref11refzzz, zzzref12refzzz, for which *x(0)=x(1). Inventiones Mathematicae 59* (1980), 67–76.

[9] Sedaghat, H. Geometric properties of factorable planar systems of differential equations. *SIAM Review 38*, 4 (December 1996), 660–665.

[10] Shahshahani, S. Periodic solutions of polynomial first order differential equations. *Nonlinear Analysis 5*, 2 (1981), 157–165.

[11] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

[12] Songling, S. A concrete example of the existence of four limit cycles for plane quadratic systems. *Sci. Sinica 23* (1980), 153–158.

[13] Yan-Qian, Y. *et al.* *Theory of Limit Cycles*, vol. 66 of *Translations of Mathematical Monographs*. Amer. Math. Soc., Providence, RI, 1986.

17. Natural Boundary Conditions for a PDE

Applicable to Partial differential equations.

Yields

A proper set of boundary conditions.

Idea

Given a partial differential equation it is not always clear what the "correct" boundary conditions are. This is especially true for nonlinear partial differential equations. However, most partial differential equations that arise in mathematical physics have been obtained from a variational principle (see page 418).

If we *start* with the variational principle, then "natural" boundary conditions will be generated while deriving the equation we started with. These boundary condition are, in a sense, the most appropriate boundary conditions for the original equation if there is no physical reason for imposing other conditions.

Procedure

The variational principle that is most often used is $\delta J = 0$, where δ represents a variation and J is a functional given by

$$J[\phi] = \iint_R L(\phi, \phi_t, \phi_{\mathbf{x}})\, dt\, d\mathbf{x}.$$

Here $L(\cdot)$ is a linear or nonlinear functional and $\phi(\mathbf{x}, t)$ is the unknown function to be determined. This variational principle states that the integral $J[\phi]$ should be stationary to small changes in ϕ. If we let $h(\mathbf{x}, t)$ be a continuously differentiable function, that is "small" in magnitude, then we can form

$$J[\phi + h] - J[\phi] = \iint_R \left\{ L_{\phi_t} h_t + L_{\phi_{x_j}} h_{x_j} + L_\phi \right\} dt\, d\mathbf{x} + O(||h||^2),$$

where subscripts on L denote partial derivatives. The variational principle requires that $\delta J := J[\phi + h] - J[\phi] = 0$, or that

$$\iint_R \left\{ L_{\phi_t} h_t + L_{\phi_{x_j}} h_{x_j} + L_\phi \right\} dt\, d\mathbf{x} = 0. \tag{17.1}$$

If R is assumed to be a parallelpiped, then let D_t (D_{x_j}) denote the two parts of the boundary of R on which t (x_j) is constant. By integration by parts, equation (17.1) can be written as

$$\iint\limits_R \left\{ -\frac{\partial}{\partial t} L_{\phi_t} - \frac{\partial}{\partial x_j} L_{\phi_{x_j}} + L_\phi \right\} h \, dt \, d\mathbf{x} = 0, \tag{17.2}$$

where we have assumed that

$$L_{\phi_t}\Big|_{D_t} = 0, \qquad L_{\phi_{x_j}}\Big|_{D_{x_j}} = 0. \tag{17.3}$$

Now $h(\mathbf{x}, t)$ was assumed to be arbitrary, so from equation (17.2) we conclude that

$$\frac{\partial}{\partial t} L_{\phi_t} + \frac{\partial}{\partial x_j} L_{\phi_{x_j}} - L_\phi = 0. \tag{17.4}$$

We conclude that if we can write a given partial differential equation in the form of equation (17.4) for some operator $L(\cdot)$, then equation (17.3) gives the "natural" boundary conditions.

Example

Given the partial differential equation

$$\phi_{tt} - \alpha^2 \nabla^2 \phi + \beta^2 \phi = 0, \tag{17.5}$$

where $\nabla^2 \phi = \sum_{j=1}^N \phi_{x_j x_j}$, we find that

$$L(\phi, \phi_t, \phi_{\mathbf{x}}) = \frac{1}{2}\phi_t^2 - \frac{1}{2}\alpha^2 \sum_{j=1}^N \phi_{x_j}^2 - \frac{1}{2}\beta^2\phi^2 \tag{17.6}$$

makes equations (17.4) and (17.5) identical. Therefore, the "natural" boundary conditions for equation (17.5) are, using equation (17.6) in (17.3),

$$\phi_t\Big|_{D_t} = 0, \qquad \phi_{x_j}\Big|_{D_{x_j}} = 0. \tag{17.7}$$

Equation (17.7) states that the partial differential equation (17.5) requires both initial and boundary conditions. This was to be expected because equation (17.5) is a hyperbolic equation.

For example, if $N = 1$ and R is the region $[0, T] \times [0, \infty)$, then $D_t = \{t = 0\} \cup \{t = T\}$ and $D_{x_1} = \{x_1 = 0\} \cup \{x_1 = \infty\}$. Hence, the natural boundary conditions for equation (17.5) require that $\{\phi_t(0, x_1)$, $\phi_t(T, x_1)$, $\phi_{x_1}(t, 0)$, $\phi_{x_1}(t, \infty)\}$ be specified.

Notes

1. Finding the operator $L(\cdot)$ or, equivalently, finding the variational principle δJ, is a non-trivial task in general. Also, very often one wants a vector variational principle that will encompass, simultaneously, several separate equations.
2. See the section on variational equations (on page 418) for more examples.
3. See also Kantorovich and Krylov [1, Chapter 4, pages 241–357] and Whitham [2].

References

[1] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis*. Interscience Publishers, Inc., New York, 1958.

[2] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

18. Normal Forms: Near-Identity Transformations

Applicable to Systems of ordinary differential equations.

Yields

A reformulation of the differential equations.

Idea

Find a change of variables in the form of an infinite series, so that the original system of differential equations goes into a "normal" (or "simple" or "canonical") form. The normal form is the simplest member of an equivalence class of differential equations, all exhibiting the same qualitative behavior. Normal forms are often useful for stability analyses.

Procedure

Start with the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ such that (without loss of generality) $\mathbf{x} = \mathbf{0}$ is a critical point. Expand this system to obtain

$$\mathbf{x}' = A\mathbf{x} + \mathbf{H}(\mathbf{x}),$$

where $\mathbf{H}(\mathbf{x})$ has *strictly nonlinear* functions (i.e., there are no linear or constant terms).

If $\mathbf{H}(\mathbf{x})$ has nonlinear terms of at least degree n, then make a near-identity transformation using polynomials of degree n with unknown coefficients. By appropriately choosing the unknown coefficients in the near-identity transformation, the original differential equations, when written in the new variables, will have increased the degree of the nonlinear terms by one.

We can summarize the procedure as follows:

- We are given the system of ordinary differential equations $\mathbf{x}' = \mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{H}(\mathbf{x})$, which we wish to analyze near the point $\mathbf{x} = \mathbf{0}$.
- We make the near-identity transformation from $\mathbf{x}$ to $\mathbf{u}$ via $\mathbf{x} = \mathbf{u} + \mathbf{g}(\mathbf{u})$, where $\mathbf{g}(\,)$ is a strictly nonlinear function.
- This change of variables produces the new equation

  $$\mathbf{u}' = [I + J]^{-1}\mathbf{f}(\mathbf{u} + \mathbf{g}(\mathbf{u})) = A\mathbf{u} + \mathbf{K}(\mathbf{u}), \tag{18.1}$$

  where I is the identity matrix and $J = \frac{\partial \mathbf{g}}{\partial \mathbf{u}}$ is the Jacobian of the transformation.
- The function $\mathbf{g}(\,)$ is chosen to eliminate the nonlinear terms in the equation for $\mathbf{u}$ that are of least order.

This procedure can be iterated.

If the critical point is "hyperbolic" (all eigenvalues have non-zero real parts), then the nonlinear terms can always be removed (i.e., one order at a time). Also, the topological nature does not change. See Guckenheimer and Holmes [7, Section 3.3].

Example 1

Suppose we have the system of equations

$$\frac{dx}{dt} = x + y^2,$$
$$\frac{dy}{dt} = y + xy.$$

Defining $\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}$, this system has the form

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} y^2 \\ xy \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \mathbf{H}(\mathbf{x}), \tag{18.2}$$

where $\mathbf{H}(\mathbf{x})$ has quadratic nonlinearities. We now choose to make the near-identity change of variables (of second order)

$$\begin{aligned} x &= u + a_{02}u^2 + a_{11}uv + a_{20}v^2, \\ y &= v + b_{02}u^2 + b_{11}uv + b_{20}v^2, \end{aligned} \tag{18.3}$$

where u and v are functions of t. Combining equation (18.2) and equation (18.3) we find

$$\frac{du}{dt} = u + (1 - a_{02})v^2 - a_{11}uv - a_{20}u^2 + \text{higher order terms},$$
$$\frac{dv}{dt} = v - b_{02}v^2 + (1 - b_{11})uv - b_{20}u^2 + \text{higher order terms}, \tag{18.4}$$

where "higher order terms" means terms that are of order $O(u^3, u^2v, uv^2, v^3)$. To eliminate the second order terms in equation (18.4), we take $\{a_{02} = 1, a_{11} = 0, a_{20} = 0, b_{02} = 0, b_{11} = 1, b_{20} = 0\}$. With these values, the transformation in equation (18.3) becomes

$$x = u + u^2,$$
$$y = v + uv$$

so that the original differential equations in (18.2) becomes

$$\frac{du}{dt} = u + \text{higher order terms},$$
$$\frac{dv}{dt} = v + \text{higher order terms}.$$

This new system now has cubic nonlinearities.

Example 2

The system of ordinary differential equations for $x(t)$ and $y(t)$:

$$\begin{aligned} x' &= y + F(x,y), \\ y' &= G(x,y), \end{aligned} \tag{18.5}$$

where $F(\,)$ and $G(\,)$ are strictly nonlinear, has the normal form

$$\begin{aligned} \theta' &= 1 + D_1 r^2 + D_2 r^4 + D_3 r^6 + \ldots, \\ r' &= \qquad B_1 r^3 + B_2 r^5 + B_3 r^7 + \ldots, \end{aligned}$$

where $u = r\cos\theta$, $v = r\sin\theta$, and $\{u, v\}$ are related, via a near-identity transformation, to $\{x, y\}$. In this example, the linear equations are not sufficient to determine the local behavior. Knowledge of B_1 is needed to determine stability (unless it is zero, in which case B_2 is needed, etc.).

For example, if equation (18.5) has the form

$$\begin{aligned} x' &= y + F_{xx}\frac{x^2}{2} + F_{xy}xy + F_{yy}\frac{y^2}{2} + F_{xxx}\frac{x^3}{6} + F_{xxy}\frac{x^2y}{2} \\ &\quad + F_{xyy}\frac{xy^2}{2} + F_{yyy}\frac{y^3}{6} + \ldots, \\ y' &= G_{xx}\frac{x^2}{2} + G_{xy}xy + F_{yy}\frac{y^2}{2} + G_{xxx}\frac{x^3}{6} + G_{xxy}\frac{x^2y}{2} \\ &\quad + G_{xyy}\frac{xy^2}{2} + G_{yyy}\frac{y^3}{6} + \ldots, \end{aligned}$$

then we find (see Takens [12] for details)

$$\begin{aligned} 16B_1 = G_{yyy} &+ G_{xxy} + F_{xyy} + F_{xxx} + F_{yy}G_{yy} - F_{xx}G_{xx} - G_{xx}G_{xy} \\ &- G_{yy}G_{xy} + F_{xx}F_{xy} + F_{xy}F_{yy}. \end{aligned}$$

Example 3

The system of ordinary differential equations for $x(t)$ and $y(t)$:

$$\begin{aligned} x' &= -y + F(x,y), \\ y' &= x + G(x,y), \end{aligned} \tag{18.6}$$

where $F(\,)$ and $G(\,)$ are strictly nonlinear, has the normal form

$$u' = v + \sum_{n=2}^{\infty} b_n u^n, \qquad v' = \sum_{n=2}^{\infty} a_n u^n, \tag{18.7}$$

where $\{u, v\}$ are related, via a near-identity transformation, to $\{x, y\}$. For example, if equation (18.6) has the form

$$\begin{aligned} x' = -y &+ F_{xx}\frac{x^2}{2} + F_{xy}xy + F_{yy}\frac{y^2}{2} + F_{xxx}\frac{x^3}{6} + F_{xxy}\frac{x^2y}{2} \\ &+ F_{xyy}\frac{xy^2}{2} + F_{yyy}\frac{y^3}{6} + \dots, \\ y' = \quad x &+ G_{xx}\frac{x^2}{2} + G_{xy}xy + F_{yy}\frac{y^2}{2} + G_{xxx}\frac{x^3}{6} + G_{xxy}\frac{x^2y}{2} \\ &+ G_{xyy}\frac{xy^2}{2} + G_{yyy}\frac{y^3}{6} + \dots, \end{aligned}$$

then we find that

$$\begin{aligned} u' &= v + \frac{1}{2}\left(G_{xy} + F_{xx}\right)u^2 + \frac{1}{12}\left(G_{xy}G_{yy} - F_{xx}G_{yy}\right. \\ &\quad \left. + 2F_{xy}G_{xy} + 2G_{xxy} - F_{yy}G_{xx} - G_{xx} + 2F_{xxx}\right)u^3 + \dots, \\ v' &= \frac{1}{2}G_{xx}u^2 + \frac{1}{6}\left(3F_{xy}G_{xx} + G_{xxx} - F_{xx}G_{xy}\right)u^3 + \dots, \end{aligned} \tag{18.8}$$

where C is an arbitrary constant. See Takens [12] for details.

Another normal form for equation (18.6) is given by

$$\begin{aligned} U' &= V, \\ U' &= \sum_{n=2}^{\infty} a_n U^n + \sum_{n=2}^{\infty} n b_n U^{n-1}, \end{aligned}$$

where $\{U, V\}$ are related, via a near-identity transformation, to $\{x, y\}$. See Guckenheimer and Holmes [7] for details.

Notes

1. If $a_2 \neq 0$, then the flow of the system in equation (18.7) is topologically equivalent to the flow of the system $\{u' = v,\ v' = a_2u^2\}$, which can be integrated in terms of elliptic integrals. If $a_2 = 0$, then other conclusions are possible; see Rand and Keith [11] for details.
2. To avoid computing the matrix inverse in equation (18.1), it is sufficient to expand $(I + J)^{-1}$ into $I - J + J^2 - \dots + (-J)^{n-1}$ if only the nonlinear terms of order n are to be removed.
3. The concept of normal forms does not require that the transformations used be near-identity ones, but they are the ones most often used in practice.
4. The computations needed for this technique quickly become unmanageable unless a computer algebra system is used. Macsyma programs for performing the necessary computations are given in Chow *et al.* [3] and in Rand and Keith [10].
5. Abraham and Marsden [1, page 489] have the theorem

Consider the system described by the Lagrangian $L = K - V$ where $K = \frac{1}{2}\sum_{i,j} m_{ij}\dot{q}_i\dot{q}_j$ and $V = \frac{1}{2}\sum_{i,j} c_{ij}q_iq_j$ and the matrices m_{ij} and c_{ij} are symmetric (this is no loss of generality) and m_{ij} is positive definite. Then there is a linear change of coordinates $Q_i = \sum_j a_{ij}q_j$ and $\dot{Q}_i = \sum_j a_{ij}\dot{q}_j$ such that the Lagrangian in the new coordinates is $\bar{L} = \bar{K} - \bar{V}$ where $\bar{K} = \frac{1}{2}\sum_i m_i(\dot{Q}_i)^2$, $\bar{V} = \frac{1}{2}\sum_i c_iQ_iQ_i$, and $m_i > 0$.

The new coordinates $\{Q_1, \dots, Q_n, \dot{Q}_1, \dots, \dot{Q}_n\}$ are called *normal modes* and Lagrange's equations become $\ddot{Q}_i + \lambda_i^2 Q_i = 0$ (for $i = 1, \dots, n$) where $\lambda_i^2 = -c_i/m_i$.

References

[1] Abraham, R., and Marsden, J. E. *Foundations of Mechanics: A Mathematical Exposition of Classical Mechanics.* Addison–Wesley Publishing Co., Reading, MA, 1994.

[2] Ashkenazi, M., and Chow, S.-N. Normal forms near critical points for differential equations and maps. *IEEE Trans. Circ. & Syst. 35*, 7 (July 1988), 850–862.

[3] Chow, S.-N., Byron, B., and Wang, D. Computation of normal forms. *J. Comput. Appl. Math. 29*, 2 (1990), 129–143.

[4] Chua, L. O., and Kokubu, H. Normal forms for nonlinear vector fields—Part I: Theory and algorithm. *IEEE Trans. Circ. & Syst. 35*, 7 (July 1988), 863–880.

[5] Chua, L. O., and Oka, H. Normal forms for constrained nonlinear differential equations—Part I: Theory. *IEEE Trans. Circ. & Syst. 35*, 7 (July 1988), 881–901.

[6] Freire, E., Gamero, E., and Ponce, E. An algorithm for symbolic computation of Hopf bifurcation. In *Computers and Mathematics*, E. Kaltofen and S. M. Watt, Eds. Springer–Verlag, New York, 1990, pp. 109–118.

[7] Guckenheimer, J., and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer–Verlag, New York, 1983.

[8] Nayfeh, A. H. *Method of Normal Forms.* John Wiley & Sons, New York, 1993.

[9] Rand, R. H., and Armbruster, D. *Perturbation Methods, Bifurcation Theory and Computer Algebra.* No. 65 in Applied Mathematical Sciences. Springer–Verlag, New York, 1987.

[10] Rand, R. H., and Keith, W. L. Normal forms and center manifolds calculations on MACSYMA. In *Applications of Computer Algebra*, R. Pavelle, Ed. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1985.

[11] Rand, R. H., and Keith, W. L. Determinacy of degenerate equilibria with linear part *x'=y, y'=0* using MACSYMA. *Appl. Math. and Comp. 21* (1987), 1–19.

[12] Takens, F. Singularities of vector fields. *Publ. Math. Inst. Hautes Etudes Sci. 43* (1974), 47–100.

19. Random Differential Equations

Applicable to Differential equations involving random terms.

Idea

While randomness can appear in differential equations in many ways, most often it appears through "white noise" terms.

Procedure

Suppose that $x(t)$ is a random process that satisfies the stochastic differential equation

$$dx(t) = a[x(t), t]\, dt + b[x(t), t]\, dw(t), \tag{19.1}$$

where $w(t)$ is a standard Wiener process. The Wiener process is a Gaussian random process that has a mean given by its starting point, $\mathrm{E}\,[w(t)] = w_0 = w(t_0)$, a variance of $\mathrm{E}\left[(w(t) - w_0)^2\right] = t - t_0$, and a covariance of $\mathrm{E}\,[w(t)w(s)] = \min(t, s)$. The sample paths of $w(t)$ are continuous but not differentiable. If we define

$$\alpha(x, t) = a(x, t) - \frac{1}{2} b(x, t) \frac{\partial b(x, t)}{\partial x}, \tag{19.2}$$

then the solution to the stochastic differential equation, $x(t)$, can be shown to satisfy (see Gardiner [5])

$$x(t) = x(t_0) + \int_{t_0}^{t} \alpha[x(s), s]\, ds + {}_{\mathcal{S}}\!\int_{t_0}^{t} b[x(s), s]\, dw(s). \tag{19.3}$$

where ${}_{\mathcal{S}}\!\int$ represents the Stratonovich stochastic integral. Hence, an understanding of stochastic integration is required to understand the solutions to stochastic differential equations.

If $w(t)$ is a Wiener process and $G(t, w(t))$ is an arbitrary function, then the stochastic integral $I = \int_{t_0}^{t} G(s, w(s))\, dw(s)$ is defined as a limiting sum. Divide the interval $[t_0, t]$ into n sub-intervals: $t_0 \le t_1 \le \cdots \le t_{n-1} \le t_n = t$, and choose points $\{\tau_i\}$ that lie in each sub-interval: $t_{i-1} \le \tau_i \le t_i$. The stochastic integral I is defined as the limit of partial sums, $I = \lim_{n\to\infty} S_n$, with $S_n = \sum_{i=1}^{n} G(\tau_i, w(\tau_i))[w(t_i) - w(t_{i-1})]$.

Consider, for example, the special case of $G(t) = w(t)$. Then the expectation of S_n is computed as

$$\begin{aligned}\mathrm{E}\left[S_n\right] &= \mathrm{E}\left[\sum_{i=1}^{n} w(\tau_i)[w(t_i) - w(t_{i-1})]\right] \\ &= \sum_{i=1}^{n}[\min(\tau_i, t_i) - \min(\tau_i, t_{i-1})] \\ &= \sum_{i=1}^{n}(\tau_i - t_{i-1}).\end{aligned}$$

If we take $\tau_i = \alpha t_i + (1-\alpha)t_{i-1}$ (where $0 < \alpha < 1$), then $\mathrm{E}\left[S_n\right] = \sum_{i=1}^{n}(t_i - t_{i-1})\alpha = (t - t_0)\alpha$. Hence, the value of S_n depends on α. For consistency, some specific choice must be made for the points $\{\tau_i\}$.

- For the Ito stochastic integral (indicated by ${}_{\mathcal{I}}\int$), we choose $\tau_i = t_{i-1}$ (i.e., $\alpha = 0$ in the above). That is

$$_{\mathcal{I}}\int_{t_0}^{t} G(s, w(s))\, dw(s) = \underset{n\to\infty}{\text{ms-lim}} \left\{\sum_{i=1}^{n} G(t_{i-1}, w(t_{i-1}))[w(t_i) - w(t_{i-1})]\right\}, \tag{19.4}$$

  where ms-lim refers to the mean square limit.

- For the Stratonovich stochastic integral (indicated by ${}_{\mathcal{S}}\int$), we choose $\tau_i = (t_i + t_{i-1})/2$ (i.e., $\alpha = 1/2$ in the above). That is (see Schuss [7])

$$\begin{aligned}&_{\mathcal{S}}\int_{t_0}^{t} G(w(s), x)\, dw(s) \\ &\quad = \underset{n\to\infty}{\text{ms-lim}} \left\{\sum_{i=1}^{n} G\left(t_{i-1}, w\left(\frac{t_i + t_{i-1}}{2}\right)\right)[w(t_i) - w(t_{i-1})]\right\}.\end{aligned} \tag{19.5}$$

The difference in these two integrals can be seen in the evaluation of $\int_{t_0}^{t} w(s)\, dw(s)$. We find that ${}_{\mathcal{I}}\int_{t_0}^{t} w(s)\, dw(s) = \left[w^2(t) - w^2(t_0) - (t - t_0)\right]/2$ while ${}_{\mathcal{S}}\int_{t_0}^{t} w(s)\, dw(s) = \left[w^2(t) - w^2(t_0)\right]/2$.

Notes

1. This book contains several sections for dealing with differential equations containing random terms:
   - To determine the transition probability density, see the discussion of the Fokker–Planck equation on page 303.
   - To obtain the moments without solving the complete problem, see pages 568 and 572.

- If the noise appearing in the differential equation is not "white noise," the section on stochastic limit theorems might be useful (see page 629).
- To numerically simulate the solutions of a stochastic differential equation, see the technique on page 775.

2. It can be shown that the Stratonovich integral has the usual properties of integrals, such as the fundamental theorem of integral calculus:

$$\mathcal{S}\int_{t_0}^{t} f'(w(s))\,dw(s) = f(w(t)) - f(w(t_0)).$$

3. For arbitrary functions G, there is no connection between the Ito and Stratonovich integrals. However, when $x(t)$ satisfies (19.1), then (see Gardiner [5, page 99])

$$\mathcal{S}\int_{t_0}^{t} b[x(s),s]\,dw(s) = \mathcal{I}\int_{t_0}^{t} b[x(s),s]\,dw(s)+\frac{1}{2}\int_{t_0}^{t} b[x(s),s]\frac{\partial b[x(s),s]}{\partial x}\,ds.$$

4. The Black–Scholes PDE for option pricing is obtained using stochastic differential equations (see Black and Scholes [1]). Let S represent the price of a share of stock, and assume S follows a geometric Brownian motion $dS = \mu S\,dt + \sigma S\,d\omega$, where t is time, μ is a constant, and σ is the volatility constant. Let $V(S,t)$ be the price of a derivative security whose payoff is only a function of S and t. Construct a portfolio consisting of V and Δ shares of stock. The value P of this portfolio is $P = V + \Delta S$. The differential of P is given by $dP = dV + \Delta\,dS$. Substituting for dV (using Ito's lemma), and replacing dS by its assumed form results in

$$dP = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \mu\Delta S\right) + \left(\sigma S\frac{\partial V}{\partial S} + \sigma\Delta S\right)\,d\omega.$$

The random component of the portfolio increment can be removed by choosing $\Delta = -\frac{\partial V}{\partial S}$. The concept of arbitrage says that $dP = rP\,dt$, where r is the (constant) risk-free bank interest rate. Combining the above results in the Black–Scholes PDE

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0.$$

References

[1] Black, F., and Scholes, M. The pricing of options and corporate liabilities. *J. Political Economy 81* (1973), 637–659.

[2] Boyce, W. E. Random eigenvalue problems. In *Probabilistic Methods in Applied Mathematics*, A. T. Bharucha-Reid, Ed. Academic Press, New York, 1968, pp. 1–73.

[3] Boyce, W. E. On a conjecture concerning the means of the eigenvalues of random Sturm–Liouville boundary value problems. *Quart. Appl. Math.* (1980), 241–245.

[4] Day, W. B. Asymptotic expansions of eigenvalues and eigenfunctions of random boundary-value problems. *Quart. Appl. Math.* (July 1980), 169.

[5] Gardiner, C. W. *Handbook of Stochastic Methods*. Springer–Verlag, New York, 1985.

[6] Harlow, D. C., and Delph, T. J. The numerical solution of random intial-value problems. *Math. and Computers in Simulation 33* (1991), 243–258.

[7] Schuss, Z. *Theory and Applications of Stochastic Differential Equations*. John Wiley & Sons, New York, 1980.

20. Self-Adjoint Eigenfunction Problems

Applicable to Linear differential operators.

Yields

Information that may be used to show completeness of a set of functions.

Procedure

Many of the differential equations of mathematical physics are related to self-adjoint eigenfunction problems. As a special subcase, Sturm–Liouville equations are often self-adjoint eigenfunction problems. (Sturm–Liouville problems are discussed in more detail on page 103.)

Let $L[\cdot]$ be the nth order linear operator defined by

$$L[y] = p_n(x)\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_0(x)y,$$

where the $\{p_i(x)\}$ are complex valued and analytic and $p_n(x) \neq 0$ on the interval $x \in [a, b]$. Define n boundary conditions by

$$B_j[y] := \sum_{k=1}^{n} \left(M_{jk}\frac{d^{(k-1)}y}{dx^{(k-1)}}(a) + N_{jk}\frac{d^{(k-1)}y}{dx^{(k-1)}}(b) \right) = 0, \qquad j = 1, \ldots, n,$$

where the $\{M_{jk}, N_{jk}\}$ are given complex constants.

The problem we consider is

$$L[y] = \lambda y, \qquad B[y] = 0, \tag{20.1}$$

where $B[y] = 0$ is a shorthand notation for $\{B_j[y] = 0 \mid j = 1, \ldots, n\}$. The system in equation (20.1) will always have the trivial solution, $y(x) = 0$. But, for certain values of λ, called *eigenvalues*, the system in equation (20.1) will have non-trivial solutions. Corresponding to the specific eigenvalue λ_n will be one or more *eigenfunctions*, that is, non-trivial solutions to (20.1) when $\lambda = \lambda_n$.

We represent the complex conjugate of g by $\bar{g}$. Define the *inner product* of $f(x)$ and $g(x)$ by $(f, g) = \int_a^b f(t)\bar{g}(t)\,dt$ and the *norm* of $f(x)$ by $||f|| := \sqrt{(f, f)}$. If $(f, g) = 0$, then f and g are said to be *orthogonal*. If $\{f_1, f_2, \ldots, f_n\}$ are a set of functions with $(f_i, f_j) = 0$ when $i \neq j$, then the $\{f_i(x)\}$ are an *orthogonal family*.

The *adjoint* operator to $L[\cdot]$, called $L^*[\cdot]$, is defined by

$$L^*[y] := (-1)^n \frac{d^{(n)}[\bar{p}_n(x)y]}{dx^{(n)}} + (-1)^{n-1}\frac{d^{(n-1)}[\bar{p}_{n-1}(x)y]}{dx^{(n-1)}} + \cdots + \bar{p}_0 y.$$

Let $u(x)$ be a solution to the system $\{L[u] = 0,\, B[u] = 0\}$, and let $v(x)$ be a solution to the adjoint system $\{L^*[v] = 0,\, B^*[v] = 0\}$, where $\{B^*[y] = 0\}$ is a shorthand notation for $\{B_j^*[y] = 0 \mid j = 1, \ldots, n\}$ and the $B_i^*[\cdot]$ are, for the moment, unspecified. Using the definitions of $u(x)$ and $v(x)$, we can calculate

$$vL[u] - uL^*[v] = \frac{d}{dx} J(u, v), \tag{20.2}$$

where $J(u, v)$ is called the *bilinear concomitant* and is defined by

$$J(u, v) = \sum_{m=1}^{n} \sum_{j+k=m-1} (-1)^k \left(\frac{d^k}{dx^k} (p_m u) \right) \left(\frac{d^j v}{dx^j} \right). \tag{20.3}$$

Integrating equation (20.2) results in

$$\int_a^b (vL[u] - uL^*[v])\, dx = J(u, v) \Big|_a^b = J\Big(u(b), v(b)\Big) - J\Big(u(a), v(a)\Big). \tag{20.4}$$

We now define the $B_i^*[\cdot]$ to be those boundary conditions for which the right-hand side of equation (20.4) vanishes.

If $L = L^*$, then L is said to be *formally self-adjoint*. If $L = L^*$ and $B = B^*$, then L is said to be *self-adjoint*. Note that if $L[\cdot]$ is formally self-adjoint, then $n = 2r$ and $L[\cdot]$ must be of the form

$$L[u] = \frac{d^r}{dx^r} \left(b_r(x) \frac{d^r u}{dx^r} \right) + \cdots + \frac{d}{dx} \left(b_1(x) \frac{du}{dx} \right) + b_0(x) u. \tag{20.5}$$

As we now record, self-adjoint operators have some very useful properties. If $L[\cdot]$ is self-adjoint, then

- The eigenvalues λ_n of equation (20.1) are real.
- The eigenvalues are enumerable (with no cluster point).
- The eigenfunctions $y_n(x)$ corresponding to distinct eigenvalues are orthogonal.
- If $f(x)$ is any analytic function that satisfies the boundary conditions in equation (20.1) (i.e., $B_j[f] = 0$, for $j = 1, \ldots, n$), then, on the interval $[a, b]$, we have the representation $f(x) = \sum_{k=0}^{\infty} \frac{(f, y_k)}{(y_k, y_k)} y_k(x)$.

That is, the $\{y_k(x)\}$ are complete. It is this last statement that is of particular importance in solving differential equations. The method suggested by this statement, the method of eigenfunction expansions, is described on page 268.

Example 1

Suppose we have the linear differential operator

$$L[y] = \frac{d^2}{dx^2}\left(r_2(x)\frac{d^2y}{dx^2}\right) + \frac{d}{dx}\left(r_1(x)\frac{dy}{dx}\right) + r_0(x). \tag{20.6}$$

Because of the form of the operator, we know that $L[\cdot]$ will be formally self-adjoint (see equation (20.5)). For this operator, we can evaluate $J(u, v)$ at the upper and lower limits (from equation (20.3)) to find

$$J(u,v)\Big|_a^b = \Big[v(r_2u'')' - v'r_2u'' + r_2v''u' - u(r_2v'')' + r_1(vu' - uv')\Big]\Big|_a^b. \tag{20.7}$$

To determine whether $L[\cdot]$ is self-adjoint or not, we need to specify $B[y]$. Because equation (20.6) is a fourth order operator, four boundary conditions are required. We will consider three separate cases:

- **Case 1** If $B[y]$ is defined by

$$\begin{aligned} B_1[y] &= y(a), \\ B_2[y] &= y''(a), \\ B_3[y] &= y(b), \\ B_4[y] &= y''(b), \end{aligned} \tag{20.8}$$

  then $J(u, v)$ can be evaluated and equation (20.7) can be simplified to yield

$$r_2v''u' + r_1vu'\Big|_a^b. \tag{20.9}$$

  If we choose $B = B^*$ (i.e., $B_i^*[y] = B_i[y]$), then the quantity in (20.9) is identically zero. Hence, $L[\cdot]$, as defined by equations (20.6) and (20.8) is self-adjoint.

- **Case 2** If $B[y]$ is defined by

$$\begin{aligned} B_1[y] &= y(a), \\ B_2[y] &= y'(a), \\ B_3[y] &= y(b), \\ B_4[y] &= y'(b), \end{aligned} \tag{20.10}$$

  then $J(u, v)$ can be evaluated and equation (20.7) can be simplified to yield

$$v(r_2u'')' - v'r_2u''\Big|_a^b. \tag{20.11}$$

  Once again, if we choose $B = B^*$, then the quantity in (20.11) is identically zero. Hence, $L[\cdot]$, as defined by equations (20.6) and (20.10) is self-adjoint.

- **Case 3** If $B[y]$ is defined by

$$\begin{aligned} B_1[y] &= y(a), \\ B_2[y] &= y'(a), \\ B_3[y] &= y''(a), \\ B_4[y] &= y'''(a), \end{aligned} \tag{20.12}$$

then $J(u, v)$ can be evaluated and equation (20.7) can be simplified to yield

$$v(r_2 u'')' - v' r_2 u'' + r_2 v'' u' - u(r_2 v'')' + r_1(vu' - uv') \Big|_{x=b}. \tag{20.13}$$

If, in this case, we choose $B = B^*$, then the quantity in equation (20.13) does *not* vanish. If $B = B^*$, then no information has been given at the boundary $x = b$, and the quantity in (20.13) is indeterminate. Hence, $L[\cdot]$, as defined by equations (20.6) and (20.12), is not self-adjoint. An initial value problem can never be self-adjoint.

Example 2

The operator

$$L[y] = \frac{d}{dx}\left(a_2(x)\frac{dy}{dx}\right) + a_1(x)\frac{dy}{dx} + a_0(x),$$

with the boundary conditions

$$\begin{aligned} B_1[y] &= y(a), \\ B_2[y] &= y'(b), \end{aligned}$$

is self-adjoint. See the section on Sturm–Liouville theory (page 103).

Example 3

A third order linear ordinary differential equation is formally self-adjoint if it has the form

$$\frac{d^2}{dx^2}\left(P(x)\frac{dy}{dx}\right) + \frac{d}{dx}\left(P(x)\frac{d^2y}{dx^2}\right) + \frac{d}{dx}\Big(Q(x)y\Big) + Q(x)\frac{dy}{dx} = 0. \tag{20.14}$$

The general third order linear ordinary differential equation

$$A(x)\frac{d^3y}{dx^3} + B(x)\frac{d^2y}{dx^2} + C(x)\frac{dy}{dx} + D(x) = 0,$$

will be formally self-adjoint if and only if $B = \frac{3}{2}A'$ and $D = \frac{1}{2}\left(C - \frac{1}{3}B'\right)'$. The self-adjoint third order equation (20.14) has the first integral

$$P\left(2yy'' - (y')^2\right) + P'yy' + Qy^2 = \text{constant}.$$

Example 4

The general fourth order linear ordinary differential equation

$$A(x)y'''' + B(x)y''' + C(x)y'' + D(x)y' + E(x)y = 0,$$

will be formally self-adjoint if and only if $B = 2A'$ and $D = \left(C - \frac{1}{2}B'\right)'$.

Notes

1. Some of the conditions above can be relaxed, and the main results for self-adjoint operators will still be true. See, for instance, Coddington and Levinson [3, Chapter 7].
2. For partial differential equations there are many results analogous to those mentioned above for ordinary differential equations. We enumerate some of them for the Helmholtz equation in two dimensions: For the equation $\nabla^2\phi + \lambda\phi = 0$, in a region R, with the boundary conditions $a\phi + b\nabla\phi \cdot \mathbf{n} = 0$, given on the entire boundary of R (here $\mathbf{n}$ represents the unit normal):
   - All the eigenvalues $\{\lambda_i\}$ are real.
   - There are an infinite number of eigenvalues. There is an eigenvalue of least magnitude but no largest one.
   - The eigenfunctions $\{\phi_i(x,y)\}$ form a complete set: Any analytic function can be represented in the form $f(x,y) = \sum_i a_i\phi_i(x,y)$, for some set of constants $\{a_i\}$.
   - Eigenfunctions belonging to different eigenvalues are orthogonal. That is $\iint_R \phi_i\bar{\phi}_j\,dx\,dy = 0$, if $\lambda_i \neq \lambda_j$.
   - An eigenfunction ϕ is related to it's eigenvalue λ by the Rayleigh quotient
   $$\lambda = \frac{-\oint \phi\nabla\phi\cdot\mathbf{n}\,ds + \iint_R |\nabla\phi|^2\,dx\,dy}{\iint_R \phi^2\,dx\,dy}.$$
3. Many other partial differential equations have very similar properties. See Haberman [5, pages 214–219] for details.
4. Partial differential equations can also be self-adjoint. The elliptic equation $au_{xx}+cu_{yy}+du_x+eu_y+fu = g(x,y)$ is said to be essentially self-adjoint when $N_x = M_y$, where
   $$N := d - \frac{a_x}{a}, \qquad M := e - \frac{c_y}{c}.$$
   In this case, an integrating factor is given by e^ϕ, where $\phi_x = N$, $\phi_y = M$. Multiplying the original equation by this factor puts the equation in self-adjoint form. For example, the equation
   $$u_{xx} + u_{yy} + x^2u_x + y^2u_y + u = 0$$

has $N = x^2$, $M = y^2$, which leads to $\phi = \frac{1}{3}\left(x^3 + y^3\right)$. Multiplying the equation by e^ϕ results in the self-adjoint form of the equation:

$$\left[e^{(x^3+y^3)/3}u_x\right]_x + \left[e^{(x^3+y^3)/3}u_y\right]_y + e^{(x^3+y^3)/3}u = 0.$$

5. See Birkhoff and Rota [1, Chapters 10–11], Butkov [2, Chapter 9, pages 332–404], Dunford and Schwartz [4], Ince [6, Chapters 9–11, pages 204–278], and Stakgold [7, Chapter 3].

References

[1] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations.* John Wiley & Sons, New York, 1978.

[2] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[3] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations.* McGraw–Hill Book Company, New York, 1955.

[4] Dunford, N., and Schwartz, J. *Linear Operators, Part II: Spectral Theory.* John Wiley & Sons, New York, 1958.

[5] Haberman, R. *Elementary Applied Partial Differential Equations.* Prentice–Hall, Inc., Englewood Cliffs, NJ, 1968.

[6] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[7] Stakgold, I. *Green's Functions and Boundary Value Problems.* John Wiley & Sons, New York, 1979.

21. Stability Theorems

Applicable to

Differential equations of all types.

Yields

Knowledge of whether or not there are stable solutions.

Idea

There are theorems available for most cases of interest.

Procedure

There are many theorems that can be used to determine whether the solutions to a differential equation are stable. For example, useful simple theorems include

> **Theorem** Consider the equation $\mathbf{y}' = A\mathbf{y} + \mathbf{f}(t, \mathbf{y})$, where A is a real constant matrix whose eigenvalues all have negative real parts. Let $\mathbf{f}$ be real, continuous for small $|\mathbf{y}|$ and $t \geq 0$, and $\mathbf{f}(t, \mathbf{y}) = o(|\mathbf{y}|)$ as $|\mathbf{y}| \to 0$, uniformly for $t \geq 0$. Then the identically zero solution is asymptotically stable.

> **Theorem** If
>
> 1. Every solution of $\mathbf{y}' = A\mathbf{y}$ approaches zero as $t \to \infty$,
> 2. $||f(\mathbf{z})||/||\mathbf{z}|| \to 0$ as $\mathbf{z} \to 0$,
> 3. $||f(\mathbf{z}_1) - f(\mathbf{z}_2)|| \leq c_1||\mathbf{z}_1 - \mathbf{z}_2||$ for $||\mathbf{z}_1||$ and $||\mathbf{z}_2||$ less than c_2 where $c_1 \to 0$ as $c_2 \to 0$,
>
> then $\mathbf{z} = \mathbf{0}$ is a stable solution of $\mathbf{y}' = A\mathbf{y} + f(\mathbf{y})$.

Example

Consider the equation $y' = -2y + f(t)$. Using the second theorem the solution $y = 0$ is stable for $f(y) = y^n$ when $n > 1$.

Notes

1. Stability is required if a differential equation is to be well posed (see page 115).
2. Floquet theory and Lyapunov functions are two techniques that can determine whether an equation has stable or unstable solutions (see pages 523 and 551).
3. Note that solutions to the equation $\mathbf{y}' = A(t)\mathbf{y}$ can be increasing even if all the eigenvalues of $A(t)$ have negative real parts for any fixed value of t. For example, consider the matrix $A(t) = \begin{bmatrix} -\frac{1}{4(1+t)} & \frac{1}{(1+t)^2} \\ -\frac{1}{4} & -\frac{1}{4(1+t)} \end{bmatrix}$. This matrix has the eigenvalues $\lambda_{1,2} = \dfrac{-1 \pm 2i}{4(1+t)}$,

yet the general solution to $\mathbf{y}' = A(t)\mathbf{y}$ is given by

$$\mathbf{y}(t) = \alpha \begin{bmatrix} (1+t)^{-3/4} \\ -\frac{1}{2}(1+t)^{1/4} \end{bmatrix} + \beta \begin{bmatrix} (1+t)^{-3/4}\log(1+t) \\ (1+t)^{1/4}\left(1 - \frac{1}{2}\log(1+t)\right) \end{bmatrix},$$

where α and β are arbitrary constants.

4. There are many different technical definitions of stability. For the equation

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \tag{21.1}$$

defined when $t \geq t_0$, the solution is said to be

- *Stable* if for each $\epsilon > 0$ there is a corresponding $\delta = \delta(\epsilon) > 0$ such that any solution $\widehat{\mathbf{y}}(t)$ of equation (21.1) that satisfies the inequality $|\widehat{\mathbf{y}}(t_0) - \mathbf{y}(t_0)| < \epsilon$ exists and satisfies the inequality $|\widehat{\mathbf{y}}(t) - \mathbf{y}(t)| < \delta$ for all $t \geq t_0$. A solution that is not stable is said to be *unstable*.
- *Asymptotically stable* if, in addition to the above stability requirements, $|\widehat{\mathbf{y}}(t) - \mathbf{y}(t)| \to 0$ as $t \to \infty$, whenever $|\widehat{\mathbf{y}}(t_0) - \mathbf{y}(t_0)|$ is sufficiently small.
- *Uniformly stable* if for each $\epsilon > 0$ there is a corresponding $\delta = \delta(\epsilon) > 0$ such that any solution $\widehat{\mathbf{y}}(t)$ of equation (21.1) that satisfies the inequality $|\widehat{\mathbf{y}}(t_0) - \mathbf{y}(t_0)| < \delta$ for some $t_1 \geq t_0$ exists and satisfies the inequality $|\widehat{\mathbf{y}}(t) - \mathbf{y}(t)| < \epsilon$ for all $t \geq t_1$.
- *Uniformly asymptotically stable* if, in addition to the requirements for asymptotic stability, there is a $\delta_0 > 0$, and for each $\epsilon > 0$ a corresponding $T = T(\epsilon) > 0$ such that if $|\widehat{\mathbf{y}}(t_1) - \mathbf{y}(t_1)| < \delta_0$ for some $t_1 \geq t_0$, then $|\widehat{\mathbf{y}}(t) - \mathbf{y}(t)| < \epsilon$ for all $t \geq t_1 + T$.
- *Strongly stable* if for each $\epsilon > 0$ there is a corresponding $\delta = \delta(\epsilon) > 0$ such that any solution $\widehat{\mathbf{y}}(t)$ of equation (21.1) that satisfies the inequality $|\widehat{\mathbf{y}}(t_0) - \mathbf{y}(t_0)| < \delta$ for some $t_1 \geq t_0$ exists and satisfies the inequality $|\widehat{\mathbf{y}}(t) - \mathbf{y}(t)| < \epsilon$ for all $t \geq t_0$.

References

[1] Bellman, R. *Stability Theory of Differential Equations.* McGraw–Hill Book Company, New York, 1953.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations.* McGraw–Hill Book Company, New York, 1955.

22. Sturm–Liouville Theory

Applicable to

Second order linear ordinary differential operators.

Yields

Information about whether an operator is self-adjoint.

Procedure

Many of the differential equations of mathematical physics are Sturm–Liouville equations. Sturm–Liouville equations arise naturally, for instance, when separation of variables (see page 487) is applied to the wave equation, the potential equation, or the diffusion equation.

The Sturm–Liouville operator, $\mathcal{L}$, is defined by

$$\mathcal{L} := \frac{1}{s(x)}\left(-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right), \tag{22.1}$$

where p, p', q, and s are real and continuous and $s(x) > 0$ and $p(x) > 0$ on the interval (a, b). The Sturm–Liouville equation is defined by

$$\mathcal{L}[y(x)] = -\lambda y(x), \tag{22.2}$$

or, equivalently,

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda s(x)y = 0, \tag{22.3}$$

for $x \in [a, b]$. The parameter λ is an eigenvalue of the equation. Given a specific set of boundary conditions, there may be specific values of λ for which equation (22.2) has a non-trivial solution. For different types of boundary conditions, different types of behavior are possible.

Many facts are known about Sturm–Liouville systems:

- $\mathcal{L}$, as defined by equation (22.1), is formally self-adjoint (see page 95), with the inner product, $(f, g)_s := \int s(x)f(x)\bar{g}(x)\,dx$.
- $\mathcal{L}$ is self-adjoint (see page 95) when
  - The boundary conditions are *unmixed* (or separated). That is, they are of the form

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) &= 0, \\ \alpha_2 y(b) + \beta_2 y'(b) &= 0. \end{aligned} \tag{22.4}$$

  - The boundary conditions are *periodic*. That is, they are of the form

$$\begin{aligned} y(a) &= y(b), \\ y'(a) &= y'(b). \end{aligned}$$

- When the boundary conditions are given as in equation (22.4), and, in addition, $p(x) > 0$, $q(x) > 0$, $\alpha_1/\beta_1 > 0$, $\alpha_2/\beta_2 > 0$, then
  - $\mathcal{L}$ is a positive definite operator (i.e., $(\mathcal{L}u, u) > 0$, for all $u \neq 0$).
  - The eigenvalues are simple (i.e., each eigenvalue has a single eigenfunction associated with it).
- When the operator $\mathcal{L}$ is not self-adjoint then
  - If λ is a complex eigenvalue of $\mathcal{L}$, then $\bar{\lambda}$ is an eigenvalue of $\mathcal{L}^*$, the adjoint of $\mathcal{L}$.
  - Eigenfunctions of $\mathcal{L}$ are orthogonal to those of $\mathcal{L}^*$.

If the interval $[a, b]$ is finite and $p(x)$ and $s(x)$ are positive at the endpoints, then the problem is said to be *regular*. Otherwise, it is said to be *singular*. For singular Sturm–Liouville problems, problems are subdivided into two cases, the limit-circle case and limit-point case. Consider equation (22.2) when one of the endpoints is regular and the other singular. Define the s-norm of a function $u(x)$ by

$$||u||_s = (u, u)_s = \int_a^b s(x)|u(x)|^2 \, dx.$$

If, for any particular complex number λ, the solution to equation (22.2) satisfies

- $||y||_s < \infty$, then $\mathcal{L}$ is said to be of the *limit-circle type* at infinity. In this case, all solutions of equation (22.2) will satisfy $||y||_s < \infty$, for any value of λ.
- $||y||_s = \infty$, then $\mathcal{L}$ is said to be of the *limit-point type* at infinity.

If both endpoints are singular, we introduce an intermediate point l, $a < l < b$ and then classify $\mathcal{L}$ as being of the limit-point type or the limit-circle type at each endpoint according to the behavior of solutions in $a < x < l$ and in $l < x < b$ (the classification is independent of the choice of l).

For a given real λ, the problem in equation (22.2) is

- *Oscillatory* at $x = a$ if and only if every solution has infinitely many zeros clustering at a.
- *Nonoscillatory* at $x = a$ if and only if no solution has infinitely many zeros clustering at a.

The classification is mutually exclusive for a fixed λ but can vary with λ.

If $\mathcal{L}$ is in the limit-point case at infinity, then there is the following completeness theorem:

Theorem If $g(\lambda) = \int_0^\infty f(x)\Psi(x, \lambda) \, dx$, then $f(x) = \int_{-\infty}^\infty g(\lambda) \, \Psi(x, \lambda) \, d\rho(\lambda)$ for a (computable) density function $\rho(\lambda)$.

A completeness theorem is required for a proof that a separation of variables calculation (see page 487) has been done correctly.

The following theorem and corollaries may help decide the type of the operator $\mathcal{L}$:

Theorem Let M be a positive differentiable function, and let k_1 and k_2 be two positive constants such that for large x,

$$\begin{aligned} &q(x) \geq -k_1 M(x), \\ &\int_x^\infty (p(t)M(t))^{-1/2}\, dt = \infty, \\ &|p^{1/2}(x)M'(x)M^{-3/2}(x)| < k_2, \end{aligned}$$

then $\mathcal{L}$ is in the limit-point case at infinity.

Corollary If $q(x) \geq -k$, where k is a positive constant, and $\int_n^\infty p^{-1/2}(t)\, dt = \infty$ (where n is any finite number), then $\mathcal{L}$ is in the limit-point case at infinity.

Corollary If $p(x) = 1$ for $0 < x < \infty$ and $q(x) \geq -kx^2$ for some positive constant k, then $\mathcal{L}$ is in the limit-point case at infinity.

Example 1

The differential equation and boundary conditions

$$\begin{aligned} -(xy')' &= \lambda xy, \\ u(1) &= 0, \\ u(2) &= 0, \end{aligned}$$

correspond to the Sturm–Liouville operator in equation (22.1) with $p(x) = x$, $q(x) = 0$, and $s(x) = x$. This is a regular Sturm–Liouville problem on the interval $[1, 2]$. The eigenvalues and eigenfunctions are readily computed (see Stakgold [6, page 423]. If we define $\lambda_n = r_n^2$, then the r_n are determined from

$$\frac{J_0(r_n)}{J_0(2r_n)} = \frac{N_0(r_n)}{N_0(2r_n)},$$

and the corresponding eigenfunction is given by

$$y_n(x) = \frac{r_n \pi J_0(2r_n)}{\sqrt{2}\sqrt{J_0(r_n)^2 - J_0(2r_n)^2}} [J_0(r_n)N_0(r_n x) - J_0(r_n x)N_0(r_n)].$$

Example 2

The differential equation with boundary conditions

$$\begin{aligned} -(x^2y')' - \lambda u &= 0, \\ u(1) &= 0, \\ u(e) &= 0, \end{aligned}$$

for $x \in [1, e]$ is a regular Sturm–Liouville problem with unmixed boundary conditions, so the eigenfunctions are complete. In this case we find

$$\lambda_n = n^2\pi^2 + \frac{1}{2}, \qquad y_n = x^{-1/2}\sin(n\pi \log x).$$

22.1 Classification of Sturm–Liouville Problems

Pruess *et al.* [5] have devised a classification scheme and taxonomy for Sturm–Liouville problems on the interval (a, b). They define:

Category 1: Problem (22.2) is nonoscillatory at $x = a$ and $x = b$.

The spectrum is simple, purely discrete, and bounded below.

Category 2: Problem (22.2) is nonoscillatory at one endpoint. At the other endpoint, it is nonoscillatory for $\lambda \in (-\infty, t_0)$ and oscillatory for $\lambda \in (t_0, \infty)$.

The spectrum is simple and bounded below. The point spectrum (if any) is in $(-\infty, t_0)$ *whereas* (t_0, ∞) *is the continuous spectrum.*

Category 3: Problem (22.2) is nonoscillatory at one endpoint. At the other endpoint it is limit-circle and oscillatory.

The spectrum is simple, unbounded both above and below, and purely discrete.

Category 4: Problem (22.2) is nonoscillatory at one endpoint. At the other endpoint, it is limit-point and oscillatory.

The spectrum is simple and purely continuous; the continuous spectrum is the entire real line.

Category 5: Problem (22.2) is limit-circle and oscillatory at $x = a$. It is limit-point and oscillatory at $x = b$.

The spectrum is simple, unbounded both above and below, and purely discrete.

Category 6: Problem (22.2) is limit-point and oscillatory at $x = a$. It is limit-point and oscillatory at $x = b$.

The nature of the spectrum is unknown; a continuous spectrum is likely.

Category 7: Problem (22.2) is limit-point and oscillatory at one endpoint ($x = a$ or $x = b$). At the other endpoint, it is limit-circle and oscillatory.

The spectrum is simple and purely continuous; the continuous spectrum is the entire real line.

Category 8: Problem (22.2) is limit-circle and oscillatory at one endpoint ($x = a$ or $x = b$). At the other endpoint, it is nonoscillatory for $\lambda \in (-\infty, t_0)$ and oscillatory for $\lambda \in (t_0, \infty)$.

The spectrum is simple; the point spectrum (if any) is unbounded below but bounded above by t_0. *The continuous spectrum is in* (t_0, ∞).

Category 9: Problem (22.2) is limit-point and oscillatory at one endpoint ($x = a$ or $x = b$). At the other endpoint, it is nonoscillatory for $\lambda \in (-\infty, t_0)$ and oscillatory for $\lambda \in (t_0, \infty)$.

The spectrum may be nonsimple.

Category 10: At $x = a$ problem (22.2) is nonoscillatory for $\lambda \in (-\infty, t_0)$ and oscillatory for $\lambda \in (t_0, \infty)$. At $x = b$, it is nonoscillatory for $\lambda \in (-\infty, t_1)$ and oscillatory for $\lambda \in (t_1, \infty)$.

The spectrum may be nonsimple. The point spectrum (if any) is in the interval $(-\infty, \min(t_0, t_1))$ *and is bounded below. The continuous spectrum is in* $(\min(t_0, t_1), \infty)$.

Notes

1. For transformations of equation (22.3), see page 157.
2. The regular Sturm–Liouville equation, written in the form

$$\frac{d^2 z}{dt^2} - r(t)z + \lambda z = 0,$$

with the boundary conditions $z(0) = z(L) = 0$, has the asymptotic eigenvalues and eigenfunctions

$$z_n(t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} t\right) + O\left(\frac{1}{n}\right),$$
$$\lambda_n = \frac{n^2 \pi^2}{L^2} + O(1)$$

as $n \to \infty$. (See the Prüfer method on page 150.)

3. For the Sturm–Liouville equation $L[y] = -(py')' + qy - \lambda wy = 0$ on $[a, \infty]$, define θ and ϕ to be solutions satisfying $\{\theta(a) = 0,\, p\theta' \big|_{x=a} = 1\}$ and $\{\phi(a) = -1,\, p\phi' \big|_{x=a} = 0\}$. The Titchmarsh–Weyl function $m(\lambda)$ is defined to be the functions $\{m_\pm\}$, defined on the upper and lower half planes, such that $\int_a^\infty |\theta(x,\lambda) + m_\pm(\lambda)\phi(x,\lambda)|^2 \, dx < \infty$ for all strictly complex values of λ.
4. See also Birkhoff and Rota [1, Chapters 10–11], Coddington and Levinson [2, Chapters 7–12], Levitan and Sargsjan [4, Chapter 6, pages 139–182 and Chapter 12, pages 324–340], Stakgold [6, Chapter 7, pages 411–466], and Zauderer [7, pages 136–159].

References

[1] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations.* John Wiley & Sons, New York, 1978.

[2] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations.* McGraw–Hill Book Company, New York, 1955.

[3] Dunford, N., and Schwartz, J. *Linear Operators, Part II: Spectral Theory.* John Wiley & Sons, New York, 1958.

[4] Levitan, B. M., and Sargsjan, I. S. *Sturm–Liouville and Dirac Operators*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[5] Pruess, S., Fulton, C. T., and Xie, Y. The automatic classification of Sturm–Liouville problems. *Appl. Math. and Comp.*. (submitted for publication).

[6] Stakgold, I. *Green's Functions and Boundary Value Problems*. John Wiley & Sons, New York, 1979.

[7] Zauderer, E. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1983.

23. Variational Equations

Applicable to

Differential equations that arise from variational principles.

Yields

A variational principle.

Procedure

Most differential equations that arise in mathematical physics have been obtained from a variational principle. The variational principle that is most often used is $\delta J = 0$, where δ represent a variation and J is a functional given by

$$J[u] = \iint_R L(\mathbf{x}, \partial_{x_j})u(\mathbf{x})\,d\mathbf{x}. \tag{23.1}$$

Here, $L(\cdot)$ is a linear or nonlinear function of its arguments, and $u(\mathbf{x})$ is the unknown function to be determined. This variational principle states that the integral $J[u]$ should be stationary to small changes in $u(\mathbf{x})$. If we let $h(\mathbf{x})$ be a "small," continuously differentiable function, then we can form

$$J[u+h] - J[u] = \iint_R \left\{ L(\mathbf{x}, \partial_{x_j})(u(\mathbf{x}) + h(\mathbf{x})) - L(\mathbf{x}, \partial_{x_j})u(\mathbf{x}) \right\}\,d\mathbf{x}. \tag{23.2}$$

By integration by parts, equation (23.2) can often be written as

$$J[u+h] - J[u] = \iint_R N(\mathbf{x}, \partial_{x_j})u(\mathbf{x})\,d\mathbf{x} + O(||h||^2),$$

plus some boundary terms (see page 83). The variational principle requires that $\delta J := J[u+h] - J[u]$ vanishes to leading order, or that

$$N(\mathbf{x}, \partial_{x_j})u(\mathbf{x}) = 0. \tag{23.3}$$

Equation (23.3) is called the first variation of equation (23.1) or the *Euler–Lagrange equation* corresponding to equation (23.1). (This is sometimes called the *Euler equation.*) A functional in the form of equation (23.1) determines an Euler–Lagrange equation. Conversely, given an Euler–Lagrange equation, a corresponding functional can sometimes be obtained.

Many approximate and numerical techniques utilize the functional associated with a given system of Euler–Lagrange equations. See, for example, the Rayleigh–Ritz method (page 638) and the finite element method (page 734).

The following collection of examples assumes that the dependent variable in the given differential equation has natural boundary conditions (see page 83). If the dependent variable did not have these specific boundary conditions, then the boundary terms that were discarded in going from equation (23.2) to equation (23.3) would have to be satisfied in addition to the Euler–Lagrange equation.

Example 1

The Euler–Lagrange equation for the functional

$$J[y] = \int_R F\left(x, y, y', \dots, y^{(n)}\right)\, dx, \tag{23.4}$$

where $y = y(x)$ is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0. \tag{23.5}$$

For this equation the natural boundary conditions are given by

$$\begin{aligned} y(x_0) &= y_0, \quad y'(x_0) = y_0', \quad \dots, \quad y^{(n-1)}(x_0) = y_0^{(n-1)}, \\ y(x_1) &= y_1, \quad y'(x_1) = y_1', \quad \dots, \quad y^{(n-1)}(x_1) = y_1^{(n-1)}. \end{aligned}$$

Example 2

The Euler–Lagrange equation for the functional

$$J[u] = \iint_R F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, dx\, dy, \tag{23.6}$$

where $u = u(x, y)$ is

$$\begin{aligned} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) \\ + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial u_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right) = 0. \end{aligned} \tag{23.7}$$

Example 3

The Euler–Lagrange equation for the functional

$$J[u] = \iint_R \left[a\left(\frac{\partial u}{\partial x}\right)^2 + b\left(\frac{\partial u}{\partial y}\right)^2 + cu^2 + 2fu \right] dx\, dy, \tag{23.8}$$

is

$$\frac{\partial}{\partial x}\left(a \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b \frac{\partial u}{\partial y}\right) - cu = f. \tag{23.9}$$

Example 4

For the $2m$th order ordinary differential equation (in formally self-adjoint form)

$$\begin{aligned} &\sum_{k=0}^{m}(-1)^k \frac{d^k}{dx^k}\left(p_k(x)\frac{d^k u}{dx^k}\right) = f(x),\\ &u(a) = u'(a) = \cdots = u^{(m-1)}(a) = 0,\\ &u(b) = u'(b) = \cdots = u^{(m-1)}(b) = 0, \end{aligned} \tag{23.10}$$

a corresponding functional is

$$J[u] = \int_a^b \left(\sum_{k=0}^{m} p_k(x)\left(\frac{d^k u}{dx^k}\right)^2 - 2f(x)u(x)\right) dx. \tag{23.11}$$

Example 5

Consider the system of n second order ordinary differential equations for the unknowns $\{u_k(x) \mid k = 1, \ldots, n\}$

$$\begin{aligned} &-\sum_{k=1}^{n}\left[\frac{d}{dx}\left(p_{jk}(x)\frac{du_k}{dx}\right) + q_{jk}(x)u_k\right] = f_j(x),\\ &u_j(a) = u_j(b) = 0, \end{aligned} \tag{23.12}$$

for $j = 1, 2, \ldots, n$. If $p_{jk} = p_{kj}$, $q_{jk} = q_{kj}$, if the matrix $\{p_{jk}\}$ is bounded and positive definite, and if the matrix $\{q_{jk}\}$ is bounded and non-negative definite, then a functional corresponding to equation (23.12) is

$$J[u] = \int_a^b \left(\sum_{j,k=1}^{n}\left[p_{jk}(x)\frac{du_j}{dx}\frac{du_k}{dx} + q_{jk}(x)u_j u_k\right] - \sum_{j=1}^{n} f_j(x)u_j(x)\right) dx. \tag{23.13}$$

Example 6

If $A_{ij}(x)$ is a symmetric and positive definite matrix, so that the partial differential equation for $u(\mathbf{x}) = u(x_1, \ldots, x_m)$

$$-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) + C(\mathbf{x})u = f(\mathbf{x}), \tag{23.14}$$

is elliptic in Ω, $C(\mathbf{x}) > 0$, and there are Dirichlet boundary conditions

$$u\,\big|_{\partial\Omega} = 0, \tag{23.15}$$

then a corresponding functional is

$$J[u] = \int_\Omega \left(\sum_{i,j=1}^{m} A_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + Cu^2 - 2fu\right) dx, \tag{23.16}$$

where (23.16) is to be minimized over those functions that satisfy equation (23.15).

Example 7

If $A_{ij}(x)$ is a symmetric and positive definite matrix, so that the partial differential equation for $u(\mathbf{x}) = u(x_1, \ldots, x_m)$,

$$-\sum_{i,j=1}^{m} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) + C(\mathbf{x})u = f(\mathbf{x}), \tag{23.17}$$

is elliptic in Ω, $C(\mathbf{x}) > 0$, and there are the boundary conditions

$$\left[\sum_{i,j=1}^{m} A_{ij}\frac{\partial u}{\partial x_j}\cos(\nu, x_i) + \sigma u\right]_{\partial\Omega} = 0, \tag{23.18}$$

where ν is normal to $\partial\Omega$ and σ is a positive function on $\partial\Omega$, then a corresponding functional is

$$J[u] = \int_{\Omega}\left(\sum_{i,j=1}^{m} A_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + Cu^2 - 2fu\right)dx + \int_{\partial\Omega}\sigma u^2\, dS, \tag{23.19}$$

where (23.19) is to be minimized over those functions for which equation (23.18) is satisfied.

Notes

1. Note that two different functionals can yield the same set of Euler–Lagrange equations. For example, $\delta\int J\,dx = \delta\int(J+y+xy')\,dx$. The reason that $\delta\int(y+xy')\,dx = 0$ is because the integrand is an exact differential (i.e., $\int(y+xy')\,dx = \int d(xy)$). Hence, this integral is path independent; its value is determined by the boundary conditions. The Euler–Lagrange equations for the two functionals $\iint u_{xx}u_{yy}\,dx\,dy$ and $\iint (u_{xy})^2\,dx\,dy$ are also the same.
2. If a differential equation can be derived from a variational principle, then admittance of a Lie group is a necessary condition to find conservation laws by Noether's theorem.
3. Even if the boundary conditions given with a differential equation are not natural, a variational principle may sometimes be found. Consider

   $$J[u] = \int_{x_1}^{x_2} F(x,u,u')\,dx - g_1(x,u)\,\Big|_{x=x_1} + g_2(x,u)\,\Big|_{x=x_2},$$

   where $g_1(x,u)$ and $g_2(x,u)$ are unspecified functions. The necessary conditions for u to minimize $J[u]$ are (see Mitchell and Wait [5]).

   $$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} = 0,$$

   $$\left.\frac{\partial F}{\partial u'} + \frac{\partial g_1}{\partial u}\right|_{x=x_1} = 0, \qquad \left.\frac{\partial F}{\partial u'} + \frac{\partial g_2}{\partial u}\right|_{x=x_2} = 0.$$

If g_1 and g_2 are identically zero, then we recover the natural boundary conditions. However, we may choose g_1 and g_2 to suit other boundary conditions. For example, the problem

$$u'' + f(x) = 0,$$

$$u' + \alpha u \,\Big|_{x=x_1} = 0, \qquad u' + \beta u \,\Big|_{x=x_2} = 0$$

corresponds to the functional

$$J[u] = \int_{x_1}^{x_2} \left[\frac{1}{2}\left(u'\right)^2 - f(x)u\right] dx + \frac{\beta u^2}{2}\bigg|_{x=x_2} - \frac{\alpha u^2}{2}\bigg|_{x=x_1}.$$

4. This technique can be used in higher dimensions. For example, consider the functional

$$J[u] = \iint_R F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, dx\, dy + \int_{\partial R} G(x, y, u, u_\sigma, u_{\sigma\sigma}, u_n)\, d\sigma,$$

where $\partial/\partial\sigma$ and $\partial/\partial n$ are partial differential operators in the directions of the tangent and normal to the curve ∂R. Necessary conditions for $J[u]$ to have a minimum are the Euler–Lagrange equations (given in equation (23.7)) together with the boundary conditions:

$$\begin{aligned}
&\left[\frac{\partial F}{\partial u_x} - \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{xx}}\right] y_\sigma - \left[\frac{\partial F}{\partial u_y} - \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{yy}}\right] x_\sigma \\
&\quad - \left[\frac{\partial}{\partial \sigma}\left(\frac{\partial F}{\partial u_{xx}} - \frac{\partial F}{\partial u_{yy}}\right)\right] x_\sigma y_\sigma + \frac{1}{2}\left[\frac{\partial}{\partial \sigma}\frac{\partial F}{\partial u_{xy}}\left(x_\sigma^2 - y_\sigma^2\right)\right] \\
&\quad + \frac{1}{2}\left[\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial u_{xy}}\right) x_\sigma - \left(\frac{\partial}{\partial y}\frac{\partial F}{\partial u_{xy}}\right) y_\sigma\right] \\
&\quad + G_u - \frac{\partial}{\partial \sigma}\frac{\partial G}{\partial u_\sigma} + \frac{\partial^2}{\partial \sigma^2}\frac{\partial G}{\partial u_{\sigma\sigma}} = 0, \\
&\frac{\partial G}{\partial u_n} + \frac{\partial F}{\partial u_{xx}} y_\sigma^2 + \frac{\partial F}{\partial u_{yy}} x_\sigma^2 + \frac{\partial F}{\partial u_{xy}} x_\sigma y_\sigma = 0,
\end{aligned} \tag{23.20}$$

where $x_\sigma = \frac{dx}{d\sigma}$ and $y_\sigma = \frac{dy}{d\sigma}$. See Mitchell and Wait [5] for details.

5. Mathematica has the package `VariationalMethods` which can determine the Euler equations for a general integrand.
6. See also Butkov [1, pages 573–588], Collatz, [2, pages 540–541], Farlow, [3, pages 362–369], and Kantorovich and Krylov [4, Chapter 4, pages 241–357].

References

[1] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[2] Collatz, L. *The Numerical Treatment of Differential Equations.* Springer–Verlag, New York, 1966.

[3] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers.* John Wiley & Sons, New York, 1982.

[4] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis.* Interscience Publishers, Inc., New York, 1958.

[5] Mitchell, A. R., and Wait, R. *The Finite Element Method in Differential Equations.* John Wiley & Sons, New York, 1977.

[6] Yourgrau, W., and Mandelstam, S. *Variational Principles in Dynamics and Quantum Theory.* Dover Publications, Inc., New York, 1979.

24. Well Posed Differential Equations

Applicable to Ordinary and partial differential equations.

Yields

Knowledge of whether the equation is intrinsically well posed.

Idea

Before an attempt is made to determine or approximate the solution of a differential equation, it should be checked to determine if the differential equation problem is intrinsically well posed.

Procedure

A well posed differential equation is one in which

- The solution exists.
- The solution is unique.
- The solution is stable (i.e., the solution depends continuously on the boundary conditions and initial conditions).

If the differential equation is not well posed, it is called an ill posed or improperly posed problem. For such problems, there may not be a solution, there may be more than one solution, or whatever solution is determined (by an approximate scheme) may be unrelated to the actual solution.

For partial differential equations, the third condition (concerning stability) is generally the easiest to check.

Example

Consider the initial value problem for the unknown function $u(x,t)$,

$$\begin{aligned} u_{tt} &= u_{xxxx}, \\ u(x,0) &= g(x). \end{aligned} \tag{24.1}$$

We will show that the solution to this problem is not stable. Suppose that equation (24.1) has a solution, say $u_0(x,t)$. Assume that ϵ is a fixed number, much smaller than one in magnitude, and define $u_1(x,t)$ by

$$u_1(x,t) = u_0(x,t) + \epsilon e^{ikx} e^{\sigma t},$$

where k and σ are also constants. At $t = 0$, $u_1(x,0)$ differs from $g(x)$ by a quantity that has magnitude ϵ, an arbitrarily small amount.

However, using $u_1(x,t)$ in equation (24.1), we determine that $u_1(x,t)$ will satisfy the equation if $\sigma = \pm k^2$. Therefore, at any fixed value of t, say $t = T$, there exists a solution $u_0(x,T)$ and an approximation to the solution $u_1(x,T) = u_0(x,T) + \epsilon e^{ikx} e^{k^2 T}$. The approximation satisfies the same differential equation that the true solution satisfies. But because k is arbitrary, the approximate solution can be arbitrarily larger than the true solution by making k arbitrarily large. Because two different expressions satisfy the same differential equation and initially were arbitrarily close and are arbitrarily different in magnitude at any future time, we conclude that the problem is ill posed.

Note that, with the proper boundary conditions and initial conditions, equation (24.1) would have a unique solution. But the solution would be unstable because the equation is intrinsically ill posed as an initial value problem. Hence, there would be, for instance, no easy way to numerically approximate the solution.

Notes

1. For a discussion of existence and uniqueness theorems, see page 53. For a discussion of stability theorems, see page 101.
2. A standard example of an ill posed problem is Laplace's equation with initial data. For example, the equation $\nabla^2 u = 0$ with the initial data $\frac{\partial u}{\partial y}(x,0) = \frac{1}{n}\sin nx$ has the solution $u(x,y) = \frac{1}{n^2}\sin nx \sinh ny$. As $n \to \infty$, the initial data are becoming arbitrarily small in magnitude whereas the solution (for $y > 0$) is becoming arbitrarily large.
3. Certain classes of equations have been well studied. We can state
   - For Laplace's equation and elliptic equations in general, the Dirichlet problem is well posed. Also, the Neumann problem does not have a unique solution but is otherwise well posed.
   - For the two-dimensional wave equation and hyperbolic equations in general, both are well posed as an initial value problem. Both are, generally, ill posed as boundary value problems.
   - For the heat equation and diffusion equations in general, both are well posed when given Dirichlet data and the time variable is increasing; both are ill posed when the time variable is decreasing. See Beck *et al.* [2] for numerical schemes related to a specific ill posed problem.
4. A backward heat equation (a parabolic equation with decreasing time) is ill posed. It may be made well posed, however, by requiring the solution to satisfy a suitable constraint. Typically, one asks for non-negative solutions or for solutions that satisfy an a priori bound, which is obtained from physical considerations.
5. Payne [9] contains the following non-exhaustive list of methods that have been proposed and used in treating various types of improperly posed Cauchy problems:
   - Function theoretic methods
   - Eigenfunction methods
   - Logarithmic convexity methods

- Weighted energy methods
- Lagrange identity methods
- Quasireversibility methods
- Restriction of data methods
- Numerical and programming methods
- Concavity methods
- Stochastic and probabilistic methods
- Method of generalized inverse in reproducing kernel spaces
- Comparison methods

Payne [9] illustrates several of these methods on a backward heat equation.

6. As Fichera [4] shows, finding the correct boundary conditions for a *degenerate* problem (one in which the type changes) can be difficult in general. Fichera shows, for example, that the first order equation for $u(x,y)$
$$a(x,y)u_x + b(x,y)u_y + cu = f$$
in the rectangle $R = \{-\alpha \leq x \leq \alpha, -\beta \leq y \leq \beta\}$, when a and b satisfy
$$\begin{aligned} a(-\alpha,y) &\geq 0, \qquad a(\alpha,y) \leq 0, \\ b(x,-\beta) &\geq 0, \qquad b(x,\beta) \leq 0, \end{aligned}$$
has *no* boundary conditions! However, the equation,
$$-a(x,y)u_x - b(x,y)u_y + cu = f,$$
in R, with the same conditions on a and b, requires that u be given on the entire boundary of R.
7. See also Garabedian [5, pages 450–457] and Zauderer [10, pages 103–113].

References

[1] Argyros, I. K. On the cardinality of solutions of multilinear differential equations and applications. *Int. J. Math. & Math. Sci. 9*, 4 (1986), 757–766.

[2] Beck, J. V., Blackwell, B., and St. Clair, Jr., C. R. *Inverse Heat Problems*. Wiley, New York, 1985.

[3] Buzbee, B. L., and Carasso, A. On the numerical computation of parabolic problems for preceding times. *Math. of Comp. 27*, 122 (April 1973), 237–266.

[4] Fichera, G. On a unified theory of boundary value problems for elliptic–parabolic equations of second order. In *Boundary Problems in Differential Equations*, R. E. Langer, Ed. University of Wisconsin Press, Madison, Wisconsin, 1960, pp. 97–120.

[5] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[6] Lavrent'ev, M. M., Romanov, V. G., and Shishatskii, S. P. *Ill-Posed Problems of Mathematical Physics and Analysis.* Amer. Math. Soc., Providence, RI, 1986.

[7] Morozov, V. A. *Methods for Solving Incorrectly Posed Problems.* Springer–Verlag, New York, 1984.

[8] Pavlov, N. N. Smoothing of input data in the solution of ill-posed problems. *U.S.S.R. Comput. Maths. Math. Phys. 29*, 5 (1989), 110–114.

[9] Payne, L. E. *Improperly Posed Problems in Partial Differential Equations.* SIAM, Philadelphia, PA, 1975.

[10] Zauderer, E. *Partial Differential Equations of Applied Mathematics.* John Wiley & Sons, New York, 1983.

25. Wronskians and Fundamental Solutions

Applicable to Linear ordinary differential equations.

Yields

A formulation of a linear ordinary differential equation as vector system

Idea

An nth order linear ordinary differential equation can be written as a first order ordinary differential equation for a n element vector.

Procedure

Let $L[\cdot]$ be the linear nth order ordinary differential operator

$$L[y] = \frac{d^n y}{dx^n} + a_1(x)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \cdots + a_n(x)y.$$

The vector equation associated with the linear equation $L[y] = 0$ is given by (see page 146)

$$\mathbf{y}' = A(x)\mathbf{y}, \tag{25.1}$$

where $\mathbf{y} = \begin{bmatrix} y & y' & y'' & \dots & y^{(n-1)} \end{bmatrix}^{\mathrm{T}}$ and A is the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_1 \end{bmatrix}. \tag{25.2}$$

If $\{y_1, y_2, \dots, y_n\}$ is any set of n solutions to the equation $L[y] = 0$, then the matrix

$$\Phi(x) = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}$$

is a *solution matrix* for equation (25.1). It is also called a *fundamental solution*. This matrix satisfies the differential equation $\Phi' = A\Phi$.

The determinant of this matrix, $\det \Phi(x)$, is called the *Wronskian* of $L[y] = 0$ with respect to $\{y_1, y_2, \dots, y_n\}$ and is denoted by $W(y_1, y_2, \dots, y_n)$. Note that the Wronskian is a function of x.

If $\Phi(x)$ satisfies $\Phi' = A\Phi$, then $|\Phi(x)|' = |\Phi| \operatorname{tr} A(t)$, where $\operatorname{tr} A$ denotes the trace of the matrix A. Hence,

$$\det \Phi(x) = \det \Phi(x_0) \exp\left(\int_{x_0}^{x} \operatorname{tr} A(s)\, ds \right).$$

For the matrix in equation (25.2), we have $\operatorname{tr} A = -a_1$ so that

$$W(y_1, \dots, y_n)(x) = \exp\left(- \int_{x_0}^{x} a_1(s)\, ds \right) W(y_1, \dots, y_n)(x_0). \tag{25.3}$$

This is sometimes called *Liouville's formula*.

From equation (25.3), we conclude that either $W(x)$ vanishes for all values for x, or it is never equal to zero. If the Wronskian never vanishes, then the set $\{y_1, y_2, \dots, y_n\}$ is said to be *linearly independent*. A set of n linearly independent solutions to $L[y] = 0$ is called a *basis* or a *fundamental set*.

Alternately, given a set of n linearly independent continuous functions, $\{y_1, y_2, \dots, y_n\}$, it is possible to find a unique homogeneous differential equation of order n (with the coefficient of $y^{(n)}$ being one) for which the set forms a fundamental set. This differential equation is given by

$$(-1)^n \frac{W(y, y_1, y_2, \dots, y_n)}{W(y_1, y_2, \dots, y_n)} = 0. \tag{25.4}$$

Example 1

Given the second order linear ordinary differential equation

$$y'' + y = 0, \tag{25.5}$$

the set $\{\sin x, \cos x\}$ forms a fundamental set because each element in this set satisfies equation (25.5) and also the Wronskian is given by

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1,$$

which does not vanish. Because the Wronskian is constant, we have verified that $a_1(x) = 0$ in equation (25.5) (the $a_1(x)$ term in this equation corresponds to the first derivative term).

Example 2

If we choose the two functions $y_1 = \sin x$ and $y_2 = x$, we can determine the linear second order equation that has these solutions as its fundamental set by constructing equation (25.4). Here, $n = 2$ so we find

$$(-1)^2 \frac{W(y, x, \sin x)}{W(x, \sin x)} = \frac{\begin{vmatrix} y & x & \sin x \\ y' & 1 & \cos x \\ y'' & 0 & -\sin x \end{vmatrix}}{\begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix}},$$

$$= \frac{(x\cos x - \sin x)y'' + (x \sin x)y' - (\sin x)y}{(x\cos x - \sin x)},$$

$$= y'' + \frac{x\sin x}{(x\cos x - \sin x)}y' - \frac{\sin x}{(x \cos x - \sin x)}y.$$

Notes

1. Given the linear partial differential equation

$$L[u] = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu$$

for $u(\mathbf{x})$, let $\Gamma = \Gamma(\mathbf{x}, \boldsymbol{\xi}) = \Gamma(\boldsymbol{\xi}, \mathbf{x})$ be the geodesic distance between the points $\mathbf{x}$ and $\boldsymbol{\xi}$. (For a rectangular coordinate system, $\Gamma(\mathbf{x}, \boldsymbol{\xi}) = ||\mathbf{x} - \boldsymbol{\xi}|| = \sqrt{(x_1 - \xi_1)^2 + \cdots + (x_n - \xi_n)^2}$.) A fundamental solution, $S(\mathbf{x}, \boldsymbol{\xi})$, satisfies $L[S] = 0$ and, near $\mathbf{x} = \boldsymbol{\xi}$, has the form $S = \frac{U}{\Gamma^m} + V \log \Gamma + W$, where U, V, and W are analytic functions and $m = (n-2)/2$. For example, for Laplace's equation in n dimensions with $n > 2$, $\nabla^2 u = 0$, a fundamental solution is given by

$$S = \frac{1}{r^{n-2}}, \quad \text{with} \quad r = \sqrt{(x_1 - \xi_1)^2 + \cdots + (x_n - \xi_n)^2}.$$

See Garabedian [3, pages 152–153] for details.

2. The canonical form of a self-adjoint third order linear homogeneous differential equation is $y''' + 2Ay' + A'y = 0$ (see pages 98 and 163). A fundamental set of solutions for this equation is $\{u^2, uv, v^2\}$, where $u(x)$ and $v(x)$ are any two linearly independent solutions of the second order differential equation $u'' + \frac{1}{2}Au = 0$.
3. Similar to the second example, it is possible to find a single differential equation whose solutions include the products of the solutions of two given linear homogeneous differential equations; see Spigler [6].
4. See also Boyce and DiPrima [1, pages 113–126], Coddington and Levinson [2, pages 67–84], Ince [4, pages 116–121], and Simmons [5, pages 76–80].

References

[1] BOYCE, W. E., AND DIPRIMA, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] CODDINGTON, E. A., AND LEVINSON, N. *Theory of Ordinary Differential Equations*. McGraw–Hill Book Company, New York, 1955.

[3] GARABEDIAN, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[4] INCE, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[5] SIMMONS, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

[6] SPIGLER, R. The linear differential equation whose solutions are the products of solutions of two given differential equations. *J. Math. Anal. Appl. 98* (1984), 130–147.

26. Zeros of Solutions

Applicable to
Linear ordinary differential equations.

Yields
Statements about the zeros of the solutions.

Idea
There are several standard theorems about the zeros of solutions of differential equations.

Procedure
Consider the following equations:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0 \tag{26.1}$$

and

$$\begin{aligned} \frac{d^2y}{dx^2} + p(x)y &= 0 \\ \frac{d^2y}{dx^2} + q(x)y &= 0 \end{aligned} \qquad \text{(26.2.a-b)}$$

and

$$\begin{aligned} \frac{d}{dx}\left(p_1(x)\frac{dy}{dx}\right) + q_1(x)y &= 0 \\ \frac{d}{dx}\left(p_2(x)\frac{dy}{dx}\right) + q_2(x)y &= 0. \end{aligned} \qquad \text{(26.3.a-b)}$$

1. Consider the self-adjoint equation (26.1) in which $p(x) > 0$ and $p(x)$ and $q(x)$ are continuous. Sturm's separation theorem states

   Theorem Let u and v be linearly independent solutions of (26.1). If α and β are successive zeros of u, then v has one and only one zero in the interval (α, β).

   This has been extended by Makay [3] to be

   Theorem Consider the second order equation

   $$F(y'', y', y, x) = 0, \tag{26.4}$$

   where F is continuous. If the two conditions are satisfied

   - If y is a solution of (26.4), then so is cy, for all real c.

- The solution of (26.4) as an initial value problems is unique.

then the results of Sturm's theorem apply to equation (26.4).

2. We have the following result about the interlacing of zeros:

Theorem Let $u(x)$ and $v(x)$ be linearly independent solutions of equation (26.2.a) and assume $u(x)$ has at least two zeros in the interval (a, b). Then, if x_1 and x_2 are two consecutive zeros of $u(x)$, the function $v(x)$ has one, and only one, zero in the interval (x_1, x_2).

Theorem Let $p(x)$ in equation (26.2.a) be continuous in (a, b) with $0 < m \leq p(x) \leq M$. If the solution $u(x)$ of (26.2.a) has two successive zeros x_1 and x_2, then $\frac{\pi}{\sqrt{M}} \leq x_2 - x_1 \leq \frac{\pi}{\sqrt{m}}$.

3. We have the following results about oscillatory solutions:

Theorem Consider the self-adjoint equations in (26.3.a-b). If

- All the solutions of (26.3.a) are oscillatory as $x \to \infty$.
- $q_2(x) \geq q_1(x)$ are continuous functions,
- $p_2(x) \geq p_1(x) > 0$ are continuous functions,

then all solutions of equation (26.3.b) are oscillatory.

Theorem If $p(x) \geq (1 + \epsilon)/4t^2$ and $\epsilon > 0$, then all solutions to equation (26.2.a) are oscillatory.

Theorem If all the solutions to equation (26.2.a) are oscillatory, and if $q(x) \geq p(x)$, then all solutions of equation (26.2.b) are oscillatory.

And we have the converse:

Theorem If $q(x) \geq p(x)$ and some solutions to equation (26.2.b) are nonoscillatory, then some solutions of equation (26.2.a) must be nonoscillatory.

4. The Sturm comparison theorem is

Theorem Consider the self-adjoint equations (26.3.a-b). Let $p_1(x) \geq p_2(x) > 0$ and $q_1(x) \geq q_2(x)$ be continuous functions. Then between any two zeros of a nontrivial solution of equation (26.3.a), there will be at least one zero of every nontrivial solution of (26.3.b).

5. Considering equation (26.1), let $p(x) > 0$, and let p and q be continuous on $[0, \infty]$. Then

Theorem If $\int_1^\infty \frac{dx}{p(x)}$ and $\int_1^\infty q(x)\,dx$ both diverge, then every solution to equation (26.1) has infinitely many zeros on the interval $[1, \infty]$. If, in addition, $\int_0^1 \frac{dx}{p(x)}$ and $\int_0^1 q(x)\,dx$ both diverge to $+\infty$, then every solution to equation (26.1) has infinitely many zeros on the interval $[0, 1]$.

Theorem If $\int_a^\infty \frac{dx}{p(x)}$ converges and if $\left|\int_a^x q(s)\,ds\right|$ is bounded by a constant for $a \leq x \leq \infty$, then every non-trivial solution to equation (26.1) has at most a finite number of zeros on the interval $[a, \infty]$.

6. We also have the following nonoscillation results:

Theorem If lim sup $x^2 p(x) = \gamma^*$ and lim inf $x^2 p(x) = \gamma_*$ then the solution of equation (26.2.a) is

- Nonoscillatory if $\gamma^* < 1/4$
- Oscillatory if $1/4 < \gamma_*$

Theorem For the equations in (26.2): If $P(x) = x \int_x^\infty p(t)\,dt$, $Q(x) = x \int_x^\infty q(t)\,dt$, $0 < Q(x) < P(x)$, and equation (26.2.a) is nonoscillatory in the wide sense, then equation (26.2.b) is nonoscillatory in the wide sense.

Theorem Consider (26.2.a) and define $\lim_{x\to\infty} \sup \left(x \int_x^\infty p(s)\,ds\right) = P^*$ and $\lim_{x\to\infty} \inf \left(x \int_x^\infty p(s)\,ds\right) = P_*$ then

- A necessary condition that the solution to equation (26.2.a) be nonoscillatory is that $P_* \leq 1/4$ and $P^* \leq 1$.
- A sufficient condition that the solution to equation (26.2.a) be nonoscillatory is that $P^* \leq 1/4$.

Notes

1. Makay's [3] theorem applies to equations such as $y''(y')^2 + y^3 = 0$.
2. For the eigenvalue problem $L[u] = \lambda_n u$, let $N(\lambda)$ count the number of eigenvalues less than λ. In one dimension the asymptotics of $N(\lambda)$ can be easily determined because the nth eigenfunction has n zeros. For example, for the Schröedinger equation $-\nabla^2 \psi_n + q(x)\psi_n = \lambda_n \psi_n$

$$\frac{N(\lambda_{n+})}{2} + \frac{N(\lambda_{n-})}{2} = \frac{1}{\pi} \int [\lambda_n - q(x)]_+^{1/2}\,dx + O\left(\frac{1}{n}\right)$$

where $[y]_+ \equiv \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$. The generalization of this formula to k dimensions is (see Newell [4])

$$N(\lambda) = \frac{[1 + o(1)]}{2^k \pi^{k/2} \Gamma(k/2 + 1)} \int [\lambda_n - q(x)]_+^{k/2}\,dx + O\left(\frac{1}{n}\right).$$

References

[1] BANKS, S. B. A note on the location of complex zeros of solutions of linear differential equations. *Bull. Amer. Math. Soc. 18*, 1 (January 1988), 35–38.

[2] ERBE, L. H., KONG, Q., AND ZHANG, B. G. *Oscillation Theory for Functional Differential Equations.* Marcel Dekker, New York, 1994.

[3] MAKAY, G. A simple proof for Sturm's separation theorem. *Amer. Math. Monthly* (March 1992), 218–219.

[4] NEWELL, G. F. Asymptotic distribution of eigenvalues for the multidimensional Schr odinger equation. *J. Math. Physics 21*, 8 (August 1980), 2193–2201.

27. Canonical Forms

Applicable to The ordinary differential equations:

$$\frac{d^2y}{dx^2} + 2\left(\frac{e}{x} + f\right)\frac{dy}{dx} + \left(\frac{p}{x^2} + \frac{2q}{x} + r\right)y = 0, \tag{27.1}$$

$$\frac{d^2y}{dx^2} + 2(e + fx)\frac{dy}{dx} + (px^2 + 2qx + r)y = 0, \tag{27.2}$$

$$(\alpha + \beta x)\frac{d^2y}{dx^2} + (b + mx)\frac{dy}{dx} + (c + nx)y = 0, \tag{27.3}$$

$$\frac{d^2y}{dx^2} = F\left(\frac{dy}{dx}, y, x\right). \tag{27.4}$$

Idea

Each of these equations has certain canonical forms. When approximations and numerical values for these equations are reported in the literature, it is generally for the canonical forms.

Procedure 1

By changing the dependent and independent variables from $y = y(x)$ to $v = v(z)$, via

$$\begin{aligned} y(x) &= \nu z^\lambda e^{\mu z} v(z), \\ x &= \kappa z, \end{aligned}$$

for some choice of the constants $\{\nu, \lambda, \mu, \kappa\}$, equation (27.1) will take the form of one of the following four canonical forms:

$$\begin{aligned} &\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z} + B\right)v = 0, \\ &\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + \frac{2}{z}\right)v = 0, \\ &\frac{d^2v}{dz^2} + \left(\frac{A}{z^2} + 1\right)v = 0, \\ &\frac{d^2v}{dz^2} + \frac{A}{z^2}v = 0, \end{aligned}$$

where A and B are constants.

Procedure 2

By changing the dependent and independent variables from $y = y(x)$ to $v = v(z)$, via

$$\begin{aligned} y(x) &= \nu e^{\mu z+\xi z^2} v(z), \\ x &= \kappa z + \eta, \end{aligned}$$

for some choice of the constants $\{\nu, \mu, \xi, \kappa, \eta\}$, equation (27.2) will take the form of one the following four canonical forms:

$$\begin{aligned} &\frac{d^2v}{dz^2} + \left(z^2 + J\right) v = 0, \\ &\frac{d^2v}{dz^2} - vz = 0, \\ &\frac{d^2v}{dz^2} + v = 0, \\ &\frac{d^2v}{dz^2} = 0, \end{aligned}$$

where J is a constant.

Procedure 3

By changing the dependent and independent variables, equation (27.3) can be reduced to Weiler's canonical form (this is also known as a Kummer equation)

$$z\frac{d^2v}{dz^2} + (b - z)\frac{dv}{dz} - av = 0. \tag{27.5}$$

The transformation used to produce equation (27.5) from equation (27.3) has several different forms depending on the numerical values of the coefficients in equation (27.3), see Bateman [2] for details.

Procedure 4

A critical point is called a moving critical point (or singularity) if its location depends on the initial conditions for the differential equation (and so the location of the critical point is not fixed solely by the coefficients of the differential equation). For example, the nonlinear differential equation $y'' = (y')^2 \frac{2y-1}{y^2+1}$ has the general solution $y(x) = \tan\left[\log(Ax + B)\right]$, where A and B are arbitrary constants. The initial conditions determine A and B and thus determine the location of the singularities of $y(x)$.

Given an ordinary differential equation in the form of equation (27.4), if $F(y', y, x)$ is rational in y', algebraic in y, and analytic in x, and if all of the critical points are fixed, then a change of variables of the form

$$y(x) = \frac{az(x) + b}{cz(x) + d},$$

where a, b, c, d, and w are some functions of x, will transform the equation to one of 50 standard forms. Each of these 50 differential equations is for the unknown function $z(x)$.

Of these standard forms, six have solutions in terms of the Painlevé transcendents and all the others have first integrals that are equations of first order or have elementary integrals. The equations that define the six Painlevé transcendents are

$$\frac{d^2y}{dx^2} = 6y^2 + x,$$

$$\frac{d^2y}{dx^2} = 2y^3 + xy + \alpha,$$

$$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},$$

$$\frac{d^2y}{dx^2} = \frac{1}{2y}\left(\frac{dy}{dx}\right)^2 + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y},$$

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1},$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right).$$

In the above equations, all of the parameters are assumed to be constant.

Notes

1. The first three transformations may be found in Bateman [2, pages 75–79].
2. The transformations for equation (27.4) may be found in Ince [4, Chapter 14, pages 317–355].
3. Even though the Painlevé equations do not have elementary solutions in general, some choices of the parameters will lead to equations solvable in terms of elementary functions. For example, $y = -1/x$ is a solution of the second Painlevé equation when $\alpha = 1$, and $y = -1/x + 3x^2/(x^3 + 4)$ is a solution of the same equation when $\alpha = -2$. See Airault [1] for details.

References

[1] Airault, H. Rational solutions of Painleve equations. *Stud. Appl. Math. 61* (1979), 31–53.

[2] Bateman, H. *Partial Differential Equations of Mathematical Physics*. Dover Publications, Inc., New York, 1944.

[3] Berkovitch, L. M. Canonical forms of ordinary linear differential equations. *Arch. Math. (Brno) 24*, 1 (1988), 25–42.

[4] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[5] Irvine, D., and Savageau, M. A. Efficient solution of nonlinear ordinary differential equations expressed in S-system canonical form. *SIAM J. Numer. Anal. 27*, 3 (1990), 704–735.

[6] Neuman, F. Transformation and canonical forms of functional–differential equations. *Proc. Roy. Soc. Edin. 115A* (1990), 349–357.

28. Canonical Transformations

Applicable to A system of ordinary differential equations that arise from a Hamiltonian.

Yields

A different system of ordinary differential equations that arise from a different Hamiltonian.

Procedure

A Hamiltonian $H(\mathbf{p}, \mathbf{q})$, with $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$, defines the system of ordinary differential equations

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} = -H_{q_i}, \\ \dot{q}_i &= \quad \frac{\partial H}{\partial p_i} = \quad H_{p_i}, \end{aligned}$$

where a dot denotes differentiation with respect to the independent variable t (see page 61). The $\{p_i, q_i\}$ are called the *coordinates* of the Hamiltonian. The transformation to the new system of coordinates $\{P_i, Q_i\}$ via

$$\begin{aligned} p_i &= p_i(\mathbf{P}, \mathbf{Q}), \\ q_i &= q_i(\mathbf{P}, \mathbf{Q}), \end{aligned} \tag{28.1}$$

is (commonly) said to be canonical if Hamilton's equations remain invariant. That is, there exists a new Hamiltonian $K(\mathbf{P}, \mathbf{Q})$ such that the equations

$$\begin{aligned} \dot{P}_i &= -K_{Q_i}, \\ \dot{Q}_i &= \quad K_{P_i}, \end{aligned} \tag{28.2}$$

are valid.

Canonical transformations can be defined implicitly by a *generating function.* For instance, for almost arbitrary $S(\mathbf{p}, \mathbf{Q}, t)$, a canonical transformation is given by

$$\begin{aligned} P_i &= -S_{Q_i}, \\ q_i &= -S_{p_i}, \\ K(\mathbf{P}, \mathbf{Q}) &= H(\mathbf{p}, \mathbf{q}) + S_t, \end{aligned} \tag{28.3.a-c}$$

where equations (28.3.a) and (28.3.b) must be solved to obtain explicit expressions for $\mathbf{q}(\mathbf{P}, \mathbf{Q})$, $\mathbf{p}(\mathbf{P}, \mathbf{Q})$. Note that, for the S_t term, the derivative is taken with respect to the explicit dependence of S on t.

Other functional forms for the generating function are also possible. For example, a function of the form $\overline{S}(\mathbf{q}, \mathbf{P}, t)$ gives rise to the canonical transformation

$$\begin{aligned} p_i &= \overline{S}_{q_i}, \\ Q_i &= \overline{S}_{P_i}, \\ K(\mathbf{P}, \mathbf{Q}) &= H(\mathbf{p}, \mathbf{q}) + \overline{S}_t. \end{aligned} \tag{28.4.a-c}$$

Example

Given the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + a^2(t)q^2\right), \tag{28.5}$$

Hamilton's equations are $\{\dot{p} = -a^2 q,\ \dot{q} = p\}$, which can be combined to yield

$$\ddot{q} + a^2 q = 0. \tag{28.6}$$

Hence, the Hamiltonian in (28.5) defines the second order ordinary differential equation (28.6). Now consider the canonical transformation induced by the generating function $\overline{S}(q, P, t) = q^2 P$. From equation (28.4) we find

$$\begin{aligned} p &= 2qP, \\ Q &= q^2, \\ K(Q, P) &= \frac{1}{2}\left(p^2 + a^2 q^2\right) = \frac{Q}{2}\left(4P^2 + a^2\right). \end{aligned}$$

The equations corresponding to the new Hamiltonian are

$$\begin{aligned} \dot{P} &= -\frac{1}{2}\left(4P^2 + a^2\right), \\ \dot{Q} &= 4PQ. \end{aligned} \tag{28.7.a-b}$$

Equation (28.7.a) is a nonlinear *first* order ordinary differential equation for $P(t)$. After $P(t)$ is determined, equation (28.7.b) can be used to determine $Q(t)$ by quadrature. Hence, this change of variable has changed a second order linear ordinary differential equation into two successive first order ordinary differential equations.

Notes

1. Canonical transformations are sometimes called contact transformations. See page 249 for the correct definition of a contact transformation.
2. Technically, and in more generality, a transformation of the $2n$ variables $\{x_j, p_j \mid j = 1, \ldots, n\}$ to the $2n$ variables $\{X_j, P_j \mid j = 1, \ldots, n\}$

is a canonical transformation if the differential form $\sum_{j=1}^{n}(P_j dX_j - p_j dx_j)$ is exact, i.e., there exists a function $U = U(\mathbf{x}, \mathbf{p})$ such that

$$\sum_{j=1}^{n}(P_j dX_j - p_j dx_j) = dU. \tag{28.8}$$

3. The section on Hamilton–Jacobi theory (see page 61) utilizes canonical transformations to derive the Hamilton–Jacobi equation.
4. Tolstoy [7] shows that any nonlinear ordinary differential equation may be transformed, in principle, by a variable transformation into a linear differential equation or a system of such equations. This is the reverse of the process that was seen in the example.
5. The set of all canonical transformations forms a group.
6. Fouling transformations are canonical transformations in which the $\mathbf{p}$ coordinates in configuration space are preserved (i.e., $\mathbf{P} = \mathbf{p}$, $\mathbf{Q} = \mathbf{Q}(\mathbf{p}, \mathbf{q})$). See Gelman and Saletan [4] for details.
7. A transformation, given by equation (28.1), which allows equation (28.2) to be written, and may or may not satisfy (28.8) is technically called a *canonoid transformation*. The lack of distinction between canonical and canonoid has occasionally led to ambiguity in the literature. See Currie and Saletan [3] or Negri *et al.* [6] for details.
8. See also Caratheodory [1, Chapter 6, pages 79–101], Chester [2, pages 197–206], and Goldstein [5, Chapter 8, pages 237–272].

References

[1] Caratheodory, C. *Calculus of Variations and Partial Differential Equations of the First Order*. Holden–Day, Inc., San Francisco, CA, 1965.

[2] Chester, C. R. *Techniques in Partial Differential Equations*. McGraw–Hill Book Company, New York, 1970.

[3] Currie, D. G., and Saletan, E. J. Canonical transformations and quadratic Hamiltonians. *Nuovo Cimento B 9*, 1 (1972), 143–153.

[4] Gelman, Y., and Saletan, E. J. q-equivalent particle Hamiltonians. II: The two-dimensional classical oscillator. *Nuovo Cimento B 18*, 1 (1973), 53–71.

[5] Goldstein, H. *Classical Mechanics*. Addison–Wesley Publishing Co., Reading, MA, 1950.

[6] Negri, L. J., Oliveira, L. C., and Teixeira, J. M. Canonoid transformations and constants of the motion. *J. Math. Physics 28*, 10 (Oct 1987), 2369–2372.

[7] Tolstoy, I. Remarks on the linearization of differential equations. *J. Inst. Maths. Applics 20* (1977), 53–60.

29. Darboux Transformation

Applicable to Linear second order ordinary differential equations, a single equation or a system.

Yields

A reformulation of the problem.

Procedure

Given the equation

$$y'' = (f(x) + \kappa)y \tag{29.1}$$

for $y(x)$, we say that the transformation

$$z(x) = A(x, \lambda)y + B(x, \lambda)y'$$

is a Darboux transformation if $z(x)$ satisfies a differential equation of the form

$$z'' = (g(x) + \lambda)z. \tag{29.2}$$

For example, if $w(x)$ is a known solution of equation (29.1), then a Darboux transformation is given by

$$z = y' - y\frac{w'}{w}. \tag{29.3}$$

In this case, if y satisfies equation (29.1), then $z(x)$ satisfies equation (29.2) with

$$f(x) = g(x) - 2[\log w(x)]''.$$

That is to say, this transformation changes the potential function appearing in equation (29.1) from $f(x)$ by $\delta f = -2[\log w(x)]''$, where $w(x)$ is an arbitrary solution of equation (29.1). The usefulness of this technique is that equation (29.2) might be easy to solve for $z(x)$; then $y(x)$ may be found from equation (29.3) by a single integration.

For the system of second order ordinary differential equations

$$\mathbf{y}'' = D(x)\mathbf{y}, \tag{29.4}$$

where $D(x)$ is the matrix

$$D(x) = \begin{bmatrix} d_{11}(x) & d_{12}(x) & \dots & d_{1n}(x) \\ d_{21}(x) & d_{22}(x) & \dots & d_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1}(x) & d_{n2}(x) & \dots & d_{nn}(x) \end{bmatrix},$$

we say that

$$\mathbf{z}(x) = A(x)\mathbf{y} + B(x)\mathbf{y}', \tag{29.5}$$

where A and B are matrices, is a Darboux transformation if $\mathbf{z}$ satisfies an equation of the form

$$\mathbf{z}'' = F(x)\mathbf{z}, \tag{29.6}$$

where $F(x)$ is a new matrix. Sometimes Darboux transformations of this type can be used to decouple systems of differential equations. See Humi [2] for details.

Example 1

If the solution of the differential equation

$$y'' = (f(x) + \lambda)y \tag{29.7}$$

is known for each value of λ (call the solutin y_λ), and $w(x) = y_\mu(x)$ is the solution when $\lambda = \mu$, then the general solution of the differential equation

$$z'' = \left(w(x)\frac{d^2}{dx^2}\left(\frac{1}{w(x)}\right) + \lambda - \mu\right) z \tag{29.8}$$

for $z(x)$ is given by (see equation (29.3))

$$z = y_\lambda' - y_\lambda \frac{w'(x)}{w(x)}, \tag{29.9}$$

for $\lambda \neq \mu$. In particular, if we take $f(x) = 0$ in equation (29.7), then $y_0(x) = Ax + B$ when $\lambda = 0$ and $y_\lambda(x) = e^{\pm\sqrt{\lambda}x}$ for $\lambda \neq 0$. If we take $\mu = 0$ and $w(x) = x$, then equation (29.8) becomes

$$z'' = \left(\frac{2}{x^2} + \lambda\right) z,$$

with the solution given by equation (29.9); that is,

$$z(x) = e^{\pm\sqrt{\lambda}x}\left(\pm\sqrt{\lambda} - \frac{1}{x}\right).$$

Example 2

This example is from Humi [2]. Suppose we wish to decouple a system of symmetric equations in the form of equation (29.4) with

$$D(x) = \begin{bmatrix} u_1(x) + \lambda & d(x) \\ d(x) & u_2(x) + \lambda \end{bmatrix}.$$

If we apply a Darboux transformation, we can hope to obtain the form of equation (29.6) with $F(x)$ given by

$$F(x) = \begin{bmatrix} v_1(x) + \lambda & 0 \\ 0 & v_2(x) + \lambda \end{bmatrix}. \tag{29.10}$$

If we choose $B = I$ in equation (29.5), then to obtain equation (29.6), we require

$$\begin{aligned} &A'' + D' + AD = FA, \\ &2A' + D = F. \end{aligned}$$

In our case, with $D(x)$ given by equation (29.7) and $F(x)$ given by equation (29.10) we require that the elements of the matrix $A(x)$ satisfy

$$\begin{aligned} &2a'_{12} = 2a'_{21} = -d, \\ &2a'_{11} + u_1(x) = v_1(x), \\ &2a'_{22} + u_2(x) = v_2(x). \end{aligned} \tag{29.11}$$

It is a simple matter to integrate these equations to obtain

$$\begin{aligned} a_{12}(x) =& a_{21}(x) = c(x), \\ a_{11}(x) =& \frac{1}{2c}\left(\frac{1}{2}d(x) + \alpha + I\right), \\ a_{22}(x) =& \frac{1}{2c}\left(\frac{1}{2}d(x) - \alpha + I\right), \end{aligned}$$

where α is an arbitrary constant and

$$\begin{aligned} c(x) &= -\frac{1}{2}\int^x d(t)\,dt, \\ I(x) &= \int^x c(t)[u_2(t) - u_1(t)]\,dt. \end{aligned}$$

This solution is valid if the consistency constraint

$$u_1 + u_2 = 2c^2 - \left(\frac{d}{2c}\right)' + \frac{1}{2}\left(\frac{d}{2c}\right)^2 + \frac{1}{2c^2}(\alpha + I)^2 \tag{29.12}$$

is satisfied. This constraint was derived in the solution of equation (29.11).

Stated another way, we can choose d and $u_1 - u_2$ as arbitrary functions and then use equation (29.12) to compute the corresponding $u_1 + u_2$ for which the resulting system of equations can be decoupled by the use of a Darboux transformation.

Note

1. See Ince [3, page 182] and Lamb [5, pages 38–41].

References

[1] Darboux, G. *C. R. Acad. Sci Paris* (1882), 1456.

[2] Humi, M. Factorization of systems of differential equations. *J. Math. Physics 27* (Jan 1986), 76–81.

[3] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[4] Konopelchenko, B. G. On exact solutions of nonlinear integrable equations via integral linearising transforms and generalised Backlund–Darboux transformations. *J. Phys. A: Math. Gen. 23* (1990), 3761–3768.

[5] Lamb, G. L. *Elements of Soliton Theory.* John Wiley & Sons, New York, 1980.

[6] Levi, D. Toward a unification of the various techniques used to integrate nonlinear partial differential equations: Backlund and Darboux transformations vs. dressing method. *Rep. Math. Phys. 23*, 1 (1986), 41–56.

[7] Poplavskii, I. V. Generalized Darboux–Crum–Krein transformations. *Theo. Math. Physics 69*, 3 (1986), 1278–1282.

[8] Sall, M. A. Darboux transformations for non-Abelian and nonlocal equations of the Toda chain type. *Theo. Math. Physics 53* (1982), 1092–1099.

[9] Stanek, S., and Vosmansky, J. Transformations of linear second order ordinary differential equations. *Archivum Mathematicum (BRNO) 22*, 1 (1986), 55–60.

[10] Zheng, W. M. The Darboux transformation and solvable double-well potential models for Schrodinger equations. *J. Math. Physics 25*, 1 (Jan 1984), 88–90.

30. An Involutory Transformation

Applicable to Nonlinear partial differential equations of a certain form.

Yields

A reformation of the partial differential equation.

Idea

Inverting the dependent and independent variables might lead to a more tractable equation.

Procedure

Suppose we have a partial differential equation of the form

$$\Phi\left(u; \frac{\partial}{\partial x}; \frac{\partial}{\partial t}\right) := \Phi(u, u_x, u_{xx}, \dots; u_t, u_{tt}, \dots) = 0, \tag{30.1}$$

for $u = u(x,t)$. We introduce the *inverse transformation*

$$T = \begin{cases} u' = x, \\ x' = u, \\ t' = t. \end{cases}$$

Because applying T twice is equivalent to not applying T, the transformation is involutory (i.e., $T^2 = I =$ the identity). Noting that

$$\begin{aligned} D' &:= \frac{\partial}{\partial x} = \frac{1}{\partial u'/\partial x'} \frac{\partial}{\partial x'} \\ \partial' &:= \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \frac{\partial u'/\partial t'}{\partial u'/\partial x'} \frac{\partial}{\partial x'}, \end{aligned}$$

then, under T, equation (30.1) becomes

$$\Phi\left(x; D'; \partial'\right) = 0. \tag{30.2}$$

This transformation may be used to change classes of nonlinear equations with Dirichlet boundary conditions to linear form. For example, the class

$$\begin{aligned} &\frac{\partial u'}{\partial t'} - \gamma(u') \frac{\partial}{\partial x'} \left(\sum_{i=1}^{N} \alpha_i(u', t') {D'}^i x' \right) = 0, \\ &u' = \Psi_1(t') \qquad \text{on } x' = \Phi_1(t'), \\ &u' = \Psi_2(t') \qquad \text{on } x' = \Phi_2(t'), \\ &u' = \Theta(x') \qquad \text{at } t' = 0, \end{aligned} \tag{30.3}$$

transforms, under T, to

$$\begin{aligned} &\frac{\partial u}{\partial t} + \gamma(u)\frac{\partial}{\partial x}\left(\sum_{i=1}^{N} \alpha_i(x,t)\frac{\partial^i u}{\partial x^i}\right) = 0, \\ &u = \Phi_1(t) \qquad \text{on } x = \Psi_1(t), \\ &u = \Phi_2(t) \qquad \text{on } x = \Psi_2(t), \\ &u = \Theta^{-1}(x) \qquad \text{at } t = 0. \end{aligned} \tag{30.4}$$

Example

Given the equation and initial/boundary conditions

$$\begin{aligned} &\frac{\partial u'}{\partial t'} = \frac{\kappa}{\left(\frac{\partial u'}{\partial x'}\right)^2}\frac{\partial^2 u'}{\partial x'^2}, \\ &u' = 0 \qquad \text{on } x' = \Phi_1(t'), \\ &u' = L \qquad \text{on } x' = \Phi_2(t'), \\ &u' = \Theta(x') \qquad \text{at } t' = 0, \end{aligned} \tag{30.5}$$

the transformed equation and initial/boundary conditions become

$$\begin{aligned} &\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \\ &u = \Phi_1(t) \qquad \text{on } x = 0, \\ &u = \Phi_2(t) \qquad \text{on } x = L, \\ &u = \Theta^{-1}(x) \qquad \text{at } t = 0. \end{aligned} \tag{30.6}$$

Then equation (30.6) can be easily solved (by use of, say, Fourier transforms) to yield

$$\begin{aligned} u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \exp\left(-\frac{\kappa n^2\pi^2 t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg[\int_0^L \Theta^{-1}(\sigma)\sin\left(\frac{n\pi\sigma}{L}\right)\,d\sigma \\ + \frac{n\kappa\pi}{L}\int_0^t \exp\left(\frac{\kappa n^2\pi^2\tau}{L^2}\right)\left[\Phi_1(\tau) - (-1)^n\Phi_2(\tau)\right]\,d\tau\Bigg]. \end{aligned}$$

This last relation, can be implicitly solved for $x = x(u,t)$; which (under T) is the solution to equation (30.5) (i.e., $u' = u'(x',t')$).

Note

1. The Hodograph transformation is a different way in which the dependent and independent variables are interchanged (see page 456).

Reference

[1] Rogers, C. Inverse transformations and the reduction of nonlinear Dirichlet problems. *J. Phys. A: Math. Gen. 17* (1984), L681–L685.

31. Liouville Transformation – 1

Applicable to The general Sturm–Liouville problem

$$\begin{aligned} -[p(x)y']' + r(x)y &= \lambda\rho(x)y, \qquad \text{for } a \le x \le b, \\ y'(a) + \alpha y(a) &= 0, \\ y'(b) + \beta y(b) &= 0. \end{aligned} \tag{31.1}$$

Procedure

The Liouville transformation (version 1) is to change the independent variable from $x \in [a, b]$ to $t \in [0, \pi]$ by

$$t = \frac{1}{J}\int_a^x \left(\frac{\rho(x)}{p(x)}\right)^{1/2} dx, \tag{31.2}$$

where J is defined by

$$J = \frac{1}{\pi}\int_a^b \left(\frac{\rho(x)}{p(x)}\right)^{1/2} dx, \tag{31.3}$$

and to change the dependent variable from $y(x)$ to $u(t)$ by

$$u(t) = f(x)y(x) = [\rho(x)p(x)]^{1/4} y(x), \tag{31.4}$$

where we have defined $f(x) = [\rho(x)p(x)]^{1/4}$. With this change of variable, equation (31.1) becomes

$$\begin{aligned} \frac{d^2u}{dt^2} + [k^2 - q(t)]u &= 0, \qquad \text{for } 0 \le t \le \pi, \\ u'(0) + hu(0) &= 0, \\ u'(\pi) + Hu(\pi) &= 0, \end{aligned} \tag{31.5}$$

which is in *Liouville normal form.* The definitions of $\{k, q(t), h, H\}$ are as follows

$$\begin{aligned} k^2 &= J^2\lambda, \\ m(t) &= \frac{r(x)}{\rho(x)}, \\ q(t) &= \frac{f_{tt}}{f} + J^2 m(t), \\ h &= \frac{1}{f^2(a)}[\alpha J p(a) - f(a)f_t(a)], \\ H &= \frac{1}{f^2(b)}[\beta J p(b) - f(b)f_t(b)]. \end{aligned}$$

Note that $q(t)$ may also be written as

$$\begin{aligned} q(t) &= \frac{r}{p} + (p\rho)^{-1/4}\frac{d^2}{dt^2}[(p\rho)^{1/4}], \\ &= \frac{r}{p} + \frac{p}{4\rho}\left[\left(\frac{p'}{p}\right)' + \left(\frac{\rho'}{\rho}\right)' + \frac{3}{4}\left(\frac{p'}{p}\right)^2 + \frac{1}{2}\left(\frac{p'}{p}\right)\left(\frac{\rho'}{\rho}\right) - \frac{1}{4}\left(\frac{\rho'}{\rho}\right)^2\right] \end{aligned}$$

Example

If we have the equation and boundary conditions

$$\begin{aligned} -(xy')' + \frac{1}{x}y &= \lambda xy, \qquad \text{for } \pi \le x \le 2\pi, \\ y'(\pi) &= 0, \\ y'(2\pi) &= 0, \end{aligned}$$

then we identify

$$\begin{aligned} &p(x) = x, \quad r(x) = \frac{1}{x}, \quad \rho(x) = x, \\ &a = \pi, \qquad b = 2\pi, \qquad \alpha = 0, \quad \beta = 0. \end{aligned}$$

A simple calculation results in $J = 1$, $t = x - \pi$, $f(x) = \sqrt{x} = \sqrt{t+1}$, $m(t) = \frac{1}{x^2} = \frac{1}{(t+1)^2}$, $q(t) = \frac{3}{4(t+1)^2}$, $k^2 = \lambda$, $h = -\frac{1}{2}$, and $H = -\frac{1}{2(\pi+1)}$. Hence, we obtain

$$\begin{aligned} u'' + \left(\lambda - \frac{3}{4(t+1)^2}\right)u &= 0, \qquad \text{for } 0 \le t \le \pi, \\ u'(0) - \frac{1}{2}u(0) &= 0, \\ u'(\pi) - \frac{1}{2(\pi+1)}u(\pi) &= 0. \end{aligned} \tag{31.6}$$

Equation (31.6) is in Liouville normal form.

Notes

1. The standard assumptions used with equation (31.1) are that on the interval $[a, b]$: p and q are real-valued, $p > 0$, q does not vanish, and p and q have continuous second derivatives. Boundedness conditions are also required for the new functions.
2. When $\rho = 0$, the transformation

$$\begin{aligned} t &= \int_{x_0}^{x} \sqrt{\frac{|q(z)|}{p(z)}}\, dz, \\ u(t) &= [p(x)|q(x)|]^{1/4}\, y(x), \end{aligned} \tag{31.7}$$

when applied to equation (31.1), results in

$$\frac{d^2u}{dt^2} + [\pm 1 + R(t)]\, u(t) = 0, \tag{31.8}$$

where

$$R(t) = p^{1/4} |q|^{-3/4} \frac{dp(x)}{dx} \frac{d}{dx} \left[p(x)|q(x)|\right]^{-1/4} \Big|_{x=x(t)},$$

and the plus (minus) sign is taken in equation (31.8) if $q(x) > 0$ ($q(x) < 0$). This is also called the Liouville transformation (see Eastham [4]).

3. The two different transformations, the one in equations (31.2) and (31.4), and the one in equation (31.7), are each sometimes called the Liouville–Green transformation.
4. See also Birkhoff and Rota [1, pages 265–267], Boyce [2, pages 20–21], Hille [5, page 340], Lakin and Sanchez [7, pages 36–41], and Valiron [8, page 511].

References

[1] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations.* John Wiley & Sons, New York, 1978.

[2] Boyce, W. E. Random eigenvalue problems. In *Probabilistic Methods in Applied Mathematics*, A. T. Bharucha-Reid, Ed. Academic Press, New York, 1968, pp. 1–73.

[3] Cassell, J. S. An extension of the Liouville–Green asymptotic formula for oscillatory second-order differential equations. *Proc. Roy. Soc .Edin. 100A* (1985), 181–190.

[4] Eastham, M. S. P. Asymptotic formulae of Liouville–Green type for higher-order differential equations. *J. London Math. Soc. 2*, 28 (1983), 507–518.

[5] Hille, E. *Lectures on Ordinary Differential Equations.* Addison–Wesley Publishing Co., Reading, MA, 1969.

[6] Howard, H. C., and Maric, V. An extension of the Liouville–Green approximation. *J. Math. Anal. Appl. 143*, 2 (1989), 548–559.

[7] Lakin, W. D., and Sanchez, D. A. *Topics in Ordinary Differential Equations.* Dover Publications, Inc., New York, 1970.

[8] Valiron, G. *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions.* Math Sci Press, Brookline, MA, 1950.

32. Liouville Transformation − 2

Applicable to The second order linear ordinary differential equation

$$\frac{d^2y}{dt^2} + \lambda m^4(t)y = 0 \tag{32.1}$$

on the finite interval $0 \leq t \leq T$, where λ is a constant and $m(t) > 0$.

Procedure

The Liouville transformation (version 2) is to change the dependent and independent variables in equation (32.1) by

$$\begin{aligned} x &= \frac{1}{J}\int_0^t m^2(z)\,dz, \\ J &= \frac{1}{\pi}\int_0^T m^2(z)\,dz, \\ w(x) &= m(t)y(t). \end{aligned}$$

This transformation changes equation (32.1) into

$$\frac{d^2w}{dx^2} + \left[\lambda J^2 + Q(x)\right] w = 0, \tag{32.2}$$

for $0 \leq x \leq \pi$, where $Q(x)$ is defined by

$$Q(x) = \frac{1}{m(t)}\frac{d^2m(t)}{dx^2} = -\frac{J^2}{m(t)^3}\frac{d^2}{dt^2}\left(\frac{1}{m(t)}\right). \tag{32.3}$$

The inverse transformation, which takes equation (32.2) into equation (32.1), is given by

$$\begin{aligned} t &= J\int_0^x \frac{d\zeta}{[m^*(\zeta)]^2}, \\ J &= T\left(\int_0^\pi \frac{d\zeta}{[m^*(\zeta)]^2}\right)^{-1}, \end{aligned}$$

where $m^*(x) = m(t)$ is any positive solution of the differential equation

$$\frac{d^2m^*}{dx^2} = Q(x)m^*(x).$$

Example

Suppose we have (essentially) Airy's equation

$$\frac{d^2y}{dt^2} + \lambda t y = 0. \tag{32.4}$$

Comparing equation (32.4) to equation (32.1) shows that $m(t) = t^{1/4}$. Using this value for $m(t)$ produces

$$J = \frac{2}{3\pi} T^{3/2},$$
$$x = \pi \left(\frac{t}{T}\right)^{3/2},$$
$$w(x) = t^{1/4} y(t).$$

Under this change of variables, equation (32.4) becomes

$$\frac{d^2w}{dx^2} + \left(\frac{4\lambda}{9\pi^2} T^3 - \frac{5}{36}\frac{1}{x^2}\right) w = 0. \tag{32.5}$$

For large values of x, an approximation to equation (32.5) might be obtained by discarding the second term in the parentheses.

Notes

1. The function $Q(x)$ defined in equation (32.3) will be a constant if and only if $m(t) = (\alpha t^2 + \beta t + \delta)^{-1/2}$. In this case, $Q(x) = -J^2(\alpha\delta - 4\beta^2)$.
2. This transformation is useful when followed by some sort of asymptotic analysis. When the magnitude of λ is large compared to $Q(x)$, then the first order approximation to equation (32.2) will be to discard the $Q(x)$ term.
3. See Magnus and Winkler [1, page 51].

Reference

[1] Magnus, W., and Winkler, S. *Hill's Equation.* Dover Publications, Inc., New York, 1966.

33. Reduction of Linear ODEs to a First Order System

Applicable to Linear ordinary differential equations.

Yields

A first order vector system.

Idea

By introducing variables to represent the derivatives in an nth order linear ordinary differential equation, a first order system of differential equations may be obtained.

Procedure

Given the linear ordinary differential equation

$$\frac{d^n y}{dx^n} = a_{n-1}(x)\frac{d^{(n-1)} y}{dx^{(n-1)}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y + b(x) \tag{33.1}$$

for $y(x)$, introduce the variables $\{z_1, z_2, \ldots, z_n\}$ defined by

$$z_1 = \frac{dy}{dx}, \quad z_2 = \frac{d^2 y}{dx^2}, \quad \ldots, \quad z_n = \frac{d^n y}{dx^n}.$$

Using these new variables, equation (33.1) may be written as

$$\frac{d}{dx}\mathbf{y} = A(x)\mathbf{y} + \mathbf{b}(x), \tag{33.2}$$

where

$$\begin{aligned}
\mathbf{y} &= \begin{bmatrix} y & y^{(1)} & \ldots & y^{(n-1)} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} y & z_1 & z_2 & \ldots & z_{n-1} \end{bmatrix}^{\mathrm{T}}, \\
\mathbf{b} &= \begin{bmatrix} 0 & 0 & \ldots & 0 & b(x) \end{bmatrix}^{\mathrm{T}},
\end{aligned}$$

and A is the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \\ a_0(x) & a_1(x) & a_2(x) & a_3(x) & \ldots & a_{n-1}(x) \end{bmatrix}.$$

If the initial conditions for equation (33.1) were in the form

$$y(x_0) = c_0,\; y'(x_0) = c_1,\; y''(x_0) = c_2,\; \dots,\; y^{(n-1)}(x_0) = c_{n-1},$$

then the initial condition for equation (33.2) is $\mathbf{y}(x_0) = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \end{bmatrix}^{\mathrm{T}}$. To solve an equation in the form of equation (33.2), see the section on vector ordinary differential equations (page 421).

Example

Given the linear ordinary differential equation with initial conditions

$$\begin{aligned} &\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + (\log x)y = \sin x, \\ &y(0) = 3, \qquad y'(0) = 4, \end{aligned}$$

it may easily be changed into the equivalent first order system

$$\frac{d}{dx}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\log x & -x^2 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ \sin x \end{bmatrix},$$

or, equivalently,

$$\frac{d\mathbf{y}}{dt} = A(x)\mathbf{y} + \mathbf{b},$$

where $\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\log x & -x^2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 0 \\ \sin x \end{bmatrix}$.

Notes

1. Many packaged computer programs require the input to be in the form of a first order vector system.
2. The *method of elimination* is the opposite of the method presented here. In the method of elimination, a system of simultaneous equations is converted into a single equation of higher order. See Finizio and Ladas [2, pages 162–170] for details.
3. See also Bronson [1, pages 185–192].

References

[1] Bronson, R. *Modern Introductory Differential Equations*. Schaum's Outline Series. McGraw–Hill Book Company, New York, 1973.

[2] Finizio, N., and Ladas, G. *Ordinary Differential Equations with Modern Applications*. Wadsworth Publishing Company, Belmont, CA, 1982.

34. Prüfer Transformation

Applicable to Linear, homogeneous, second order differential equations.

Yields

An equivalent system of two first order differential equations.

Idea

This transformation changes an equation from Liouville normal form to two successive ordinary differential equations.

Procedure

Suppose we have the Sturm–Liouville equation

$$\frac{d}{dx}\left(P(x)\frac{du}{dx}\right) + Q(x)u = 0, \tag{34.1}$$

defined on $a < x < b$, with $P > 0, P \in C^1$, and Q continuous. If we think of this single second order equation as two first order equations for the unknowns $\{u, u'\}$, then we can change the dependent variables from $\{u, u'\}$ to $R(x)$ and $\theta(x)$ by

$$\begin{aligned} P(x)u'(x) &= R(x)\cos\theta(x), \\ u(x) &= R(x)\sin\theta(x). \end{aligned} \tag{34.2}$$

Using (34.2) in equation (34.1), we obtain two sequential first order ordinary differential equations for the unknowns $R(x)$ and $\theta(x)$

$$\begin{aligned} \frac{d\theta}{dx} &= Q(x)\sin^2\theta + \frac{1}{P(x)}\cos^2\theta, \\ \frac{dR}{dx} &= \left[\frac{1}{P(x)} - Q(x)\right] R(x)\sin 2\theta. \end{aligned} \tag{34.3.a-b}$$

If equation (34.3.a) can be integrated, then equation (34.3.b) can be solved for

$$R(x) = R(a)\exp\left(\int_a^x \left[\frac{1}{P(t)} - Q(t)\right]\sin 2\theta(t)\ dt\right). \tag{34.4}$$

Example

If we have the linear second order homogeneous ordinary differential equation

$$xu'' - u' + x^3u = 0, \tag{34.5}$$

then we can write equation (34.5) in Liouville normal form as

$$\frac{d}{dx}\left(\frac{1}{x}u'\right) + xu = 0,$$

from which we can identify $P(x) = 1/x$, $Q(x) = x$. Therefore, from equation (34.3.a), we have

$$\begin{aligned}\frac{d\theta}{dx} &= x\sin^2\theta + \frac{1}{1/x}\cos^2\theta \\ &= x.\end{aligned}$$

This equation can be solved to yield $\theta(x) = \frac{x^2}{2} + C$, where C is an arbitrary constant. From equation (34.4), we then find $R(x) = R(a)$. Therefore, we conclude that

$$\begin{aligned}u(x) &= R(a)\sin\left(\frac{x^2}{2} + C\right) \\ &= u(a)\frac{\sin(x^2/2 + C)}{\sin(a^2/2 + C)}\end{aligned}$$

is the solution to equation (34.5).

Notes

1. The Prüfer transformation is often used to obtain information about the zeros of $u(x)$.
2. See also Birkhoff and Rota [5, pages 257–266].

References

[1] Adamov a, D., Holrejs i, J., and Ulehla, I. The Atkinson–Prufer transformation and the eigenvalue problem for coupled systems of Schr odinger equations. *J. Phys. A: Math. Gen. 17* (1984), 2621–2631.

[2] Bailey, P. B. Sturm–Liouville eigenvalues via a phase function. *SIAM J. Appl. Math. 14* (1966), 242–249.

[3] Bellman, R., and Wing, G. M. *An Introduction to Invariant Imbedding.* SIAM, Philadelphia, PA, 1992.

[4] Benson, D. C. A Prufer transformation for Lienard's equation. *SIAM J. Numer. Anal. 15*, 4 (July 1984), 656–669.

[5] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations.* John Wiley & Sons, New York, 1978.

[6] Ulehla, I., and Horejsi, J. Generalized Prufer transformation and the eigenvalue problem for radial Dirac equations. *Phys. Lett. A 113*, 7 (1986), 355–358.

35. Modified Prüfer Transformation

Applicable to Linear, homogeneous, second order ordinary differential equations.

Yields

An equivalent system of two first order ordinary differential equations.

Idea

This transformation changes an equation from Liouville normal form to two successive ordinary differential equations.

Procedure

Suppose we have an ordinary differential equation in Liouville normal form

$$u'' + Q(x)u = 0, \tag{35.1}$$

defined on $a < x < b$, with $Q > 0$. We define the *modified amplitude* $R(x)$ and the *modified phase* $\phi(x)$ by

$$\begin{aligned} u(x) &= \frac{R(x)}{Q^{1/4}} \sin \phi(x), \\ u'(x) &= R(x) Q^{1/4} \cos \phi(x). \end{aligned} \tag{35.2.a-b}$$

Using equation (35.2) in equation (35.1), we determine the modified Prüfer system corresponding to equation (35.1) to be

$$\begin{aligned} \frac{d\phi}{dx} &= -Q^{1/2} - \frac{1}{4}\frac{Q'}{Q} \sin 2\phi, \\ \frac{1}{R}\frac{dR}{dx} &= \frac{1}{4}\frac{Q'}{Q} \cos 2\phi. \end{aligned} \tag{35.3}$$

The modified Prüfer transformation is usually used to obtain asymptotic information about the solution to equation (35.1).

Example

If $u(x)$ satisfies

$$u'' + \left(1 - \frac{M}{x^2}\right) u = 0, \tag{35.4}$$

for $0 < x < \infty$, then the exact solution is $u(x) = \sqrt{x} Z_n(x)$, where $Z_n(x)$ is a Bessel function and $n = \pm\sqrt{M + \frac{1}{4}}$. Comparing equation (35.4) to equation (35.1), we identify $Q(x) = 1 - \frac{M}{x^2}$, so that equation (35.3) becomes

$$\begin{aligned} \frac{d\phi}{dx} &= -\sqrt{1 - \frac{M}{x^2}} + \frac{M \sin 2\phi}{2(x^3 - Mx)}, \\ \frac{1}{R}\frac{dR}{dx} &= -\frac{M \cos 2\phi}{2(x^3 - Mx)}. \end{aligned}$$

For $M = O(1)$ and $x \gg 1$, the above expressions can be expanded to yield

$$\begin{aligned} \frac{d\phi}{dx} &\simeq -1 - \frac{1}{2}\frac{M}{x^2} + O\left(\frac{1}{x^3}\right), \\ \frac{1}{R}\frac{dR}{dx} &\simeq O\left(\frac{1}{x^3}\right), \end{aligned}$$

which can be integrated (and then simplified) to yield

$$\begin{aligned} \phi(x) &\simeq \phi_\infty - x - \frac{M}{2x} + O\left(\frac{1}{x^2}\right), \\ R(x) &\simeq R_\infty + O\left(\frac{1}{x^2}\right). \end{aligned} \tag{35.5}$$

Using equation (35.5) and $Q(x)$ in equation (35.2.a) provides an approximation to $u(x)$ for large values of x. This, in turn, provides an approximation to the nth Bessel function.

Notes

1. The modified Prüfer transformation is often used with $Q(x) = \lambda - q(x)$ when λ is large in magnitude compared to $q(x)$.
2. See also Birkhoff and Rota [1, pages 267–277].

References

[1] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations.* John Wiley & Sons, New York, 1978.

[2] Hargrave, B. A. Numerical approximation of eigenvalues of Sturm–Liouville systems. *J. Comput. Physics 20* (1976), 381–396.

36. Transformations of Second Order Linear ODEs – 1

Applicable to The second order linear ordinary differential equation

$$y'' + a(x)y' + b(x)y = 0. \tag{36.1}$$

Transformation 1

If the dependent and independent variables in equation (36.1) are changed by

$$t = \int_{x_0}^{x} \exp\left(-\int_{x_0}^{r} a(z)\,dz\right)\,dr,$$
$$w(t) = y(x),$$

then equation (36.1) becomes

$$\frac{d^2w}{dt^2} + b(x(t))\exp\left(-2\int_{x_0}^{x} a(z)\,dz\right)w = 0. \tag{36.2}$$

Example

For the ordinary differential equation

$$y'' - \frac{3x}{1-x^2}y' + \frac{7}{1-x^2}y = 0,$$

the change of variables becomes $t = x/\sqrt{1-x^2}$ and the equation corresponding to equation (36.2) is $\frac{d^2w}{dt^2} + \frac{7}{(1+t^2)^2}w = 0$.

Transformation 2

If in equation (36.1) the expression

$$\frac{b' + 2ab}{b^{3/2}} \tag{36.3}$$

is found to be a constant, then the change of independent variable given by

$$z = C\int \sqrt{b(x)}\,dx, \tag{36.4}$$

where C is an arbitrary constant, will reduce equation (36.1) to an equation with constant coefficients. Moreover, if the expression in equation (36.3) is not constant, then no change of independent variable alone will reduce equation (36.1) to an equation with constant coefficients.

Example

Given the equation

$$xy'' + (8x^2 - 1)y' + 20x^3 y = 0, \tag{36.5}$$

we note that $a(x) = 8x - 1/x$ and $b(x) = 20x^2$. Hence, the expression in equation (36.3) becomes

$$\frac{b' + 2ab}{b^{3/2}} = \frac{40x + 40x^2(8x - x^{-1})}{20^{3/2}x^3} = \frac{320x^3}{20^{3/2}x^3} = \text{constant.}$$

Therefore, if the independent variable is changed by $z = C\int \sqrt{20}x\,dx$, then equation (36.5), written in terms of z, will be a constant coefficient differential equation. A natural choice for C is $C = 2/\sqrt{20}$ so that the transformation becomes $z = x^2$. Using this new variable in equation (36.5) results in the equation

$$\frac{d^2y}{dz^2} + 4\frac{dy}{dz} + 5y = 0,$$

which has the solution $y = e^{-2z}\left(A\cos z + B\sin z\right)$, where A and B are arbitrary constants. Hence, the general solution to equation (36.5) is

$$y = \left(A\cos x^2 + B\sin x^2\right)\exp\left(-2x^2\right).$$

Transformation 3

If the dependent variable is changed by

$$y(x) = u(x)\exp\left(-\frac{1}{2}\int^x a(z)\,dz\right),$$

then equation (36.1) becomes

$$u'' + I(x)u = 0, \tag{36.6}$$

where

$$I(x) = \left(b - \frac{1}{4}a^2 - \frac{1}{2}\frac{da}{dx}\right). \tag{36.7}$$

Equation (36.6) is said to be the *normal form* for equation (36.1). The quantity $I(x)$ is the *invariant* of equation (36.1).

Two ordinary differential equations that have the same normal form (i.e., $I(x)$ is the same) are said to be *equivalent*. This is because if $y_1(x)$ and $y_2(x)$ satisfy

$$\begin{aligned} y_1'' + p_1 y_1' + q_1 y_1 &= 0, \\ y_2'' + p_2 y_2' + q_2 y_2 &= 0, \end{aligned} \tag{36.8}$$

and if both equations have the same invariant, then

$$y_1(x) = y_2(x) \exp\left(-\frac{1}{2}\int^x \Big(p_1(z) - p_2(z)\Big)\,dz\right). \tag{36.9}$$

Conversely, if y_1 and y_2 are solutions to equation (36.8), and if $y_1(x) = f(x)y_2(x)$ for some $f(x)$, then the invariants of the two equations in equation (36.8) are the same.

Example

Suppose we wish to solve the equation

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \left(a^2 + \frac{2}{x^2}\right) y = 0, \tag{36.10}$$

in which a is a constant. We find that (comparing equation (36.10) with equation (36.1), and using equation (36.7))

$$I(x) = \left(a^2 + \frac{2}{x^2}\right) - \frac{1}{4}\frac{4}{x^2} - \frac{1}{2}\frac{2}{x^2} = a^2.$$

Now, we know the solution of

$$\frac{d^2v}{dx^2} + a^2 v = 0 \tag{36.11}$$

to be $v(x) = A\cos ax + B\sin ax$, where A and B are arbitrary constants. Because equations (36.10) and (36.11) have the same invariant, one can be transformed into the other. Using equation (36.9), we find

$$y(x) = v(x)\exp\left(\int \frac{dx}{x}\right) = xv,$$

and, hence, the solution of equation (36.10) is $y(x) = Ax\cos ax + Bx\sin ax$.

Transformation 4

If, instead of equation (36.1), both sides of

$$y'' + a(x)y' + b(x)y = c(x) \tag{36.12}$$

are multiplied by

$$p(x) = \exp\left(\int_{x_0}^x a(z)\,dz\right),$$

then equation (36.12) is put in the *formally self-adjoint* form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = r(x), \tag{36.13}$$

where

$$q(x) = p(x)b(x),$$
$$r(x) = p(x)c(x).$$

See the method on page 157 for transformations of an equation in the form of equation (36.13).

Transformation 5

Fernández *et al.* [3] suggest transformatng equation (36.1) via $y(x) = \sqrt{x_z}\exp\left(-\int Q(x)\,dx\right) Y(z)$ with $x = x(z)$. This results in the equation $Y_{zz} + R(z)Y = 0$, where

$$R(z) = (x_z)^2 \left\{ b + \frac{x_{zzz}}{2(x_z)^3} - \frac{3(x_{zz})^2}{4x_z} - \frac{a'}{2} - \frac{a^2}{4} \right\}.$$

Example

Suppose we wish to solve the equation

$$(1-x^2)y'' - \gamma x y' + \lambda y = 0.$$

Using $a = -\gamma x/(1-x^2)$, $b = \lambda/(1-x^2)$, and $x(z) = -\cos z$ results in $Y_{zz} + R(z)Y = 0$ with $R(z) = \lambda + \frac{(\gamma-1)^2}{4} - \frac{(\gamma-1)(\gamma-3)}{4\sin^2 z}$.

Notes

1. Note that the invariant of the adjoint of equation (36.1) is equal to the invariant of equation (36.1). That is to say, invariants are preserved under the operation of taking the adjoint.
2. If equation (36.6) has the two linearly independent solutions $u(x)$ and $v(x)$ and if we define $s(x) := u(x)/v(x)$, then $\{s, x\} = 2I(x)$, where $\{\,,\,\}$ denotes the Schwarzian derivative.
3. Kamran and Olver [6] completely solve the equivalence problem, that is, determining when two second order linear differential operators are the same under a change of variable.
4. See also Boyce and DiPrima [2, pages 141–143], Hill [4, pages 42–43], Ince [5, page 394], Murphy [7, pages 88–89], Piaggio [8, pages 91–92], and Rainville [9, pages 7–10 and 15–23].

References

[1] Berkovich, L. M. Canonical forms of ordinary linear differential equations. *Arch. Math. (BRNO) 24*, 1 (1988), 25–42.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Fernandez, F. M., Piñeiro, A. L., and Moreno, B. Alternative factorization of eigenvalue problems in one dimension. *J. Phys. A: Math. Gen. 27* (1994), 5013–5028.

[4] Hill, J. M. *Solution of Differential Equations by Means of One-Parameter Groups*. Pitman Publishing Co., Marshfield, MA, 1982.

[5] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[6] Kamran, N., and Olver, P. J. Equivalence of differential operators. *SIAM J. Math. Anal. 20*, 5 (September 1989), 1172–1185.

[7] Murphy, G. M. *Ordinary Differential Equations and Their Solution*. D. Van Nostrand Company, Inc., New York, 1960.

[8] Piaggio, H. T. H. *An Elementary Treatise on Differential Equations and Their Applications*. G. Bell & Sons, Ltd, London, England, 1926.

[9] Rainville, E. D. *Intermediate Differential Equations*. The MacMillan Company, New York, 1964.

37. Transformations of Second Order Linear ODEs – 2

Applicable to The second order linear ordinary differential equation in formally self-adjoint form

$$L[y] := \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0. \tag{37.1}$$

Transformation 1

If the independent variable in equation (37.1) is changed from x to s by $s = \int \frac{dx}{p(x)}$, and if $p(x) > 0$ for $x > x_0$, and $\int_{x_0}^{\infty} \frac{dx}{p(x)} = \infty$, then equation (37.1) becomes

$$\frac{d^2y}{ds^2} + p(x)q(x)y = 0.$$

Note that, as $x \to \infty$, we have $s \to \infty$. See Courant and Hilbert [1, page 292].

Example

For the ordinary differential equation $(xy')' + y = 0$, we identify $p(x) = x$, $q(x) = 1$, and $x_0 = 0$. Hence, the change of variable $s = \log x$ results in $y_{ss} + e^s y = 0$.

Transformation 2

If the dependent variable in equation (37.1) is changed from $y(x)$ to $w(x)$ by

$$w(x) = \sqrt{p(x)}y(x),$$

then equation (37.1) becomes

$$\frac{d^2w}{dx^2} + \left[\frac{q}{p} - \frac{1}{2}\frac{d}{dx}\left(\frac{p'}{p}\right) - \frac{1}{4}\left(\frac{p'}{p}\right)^2\right] w = 0.$$

Transformation 3

If the range of interest for equation (37.1) is $x_0 < x < \infty$ and if the independent and dependent variables are changed by

$$t = \int_{x_0}^{x} \sqrt{\frac{|q(z)|}{p(z)}}\, dz,$$
$$u(t) = [p(x)|q(x)|]^{1/4}\, y(x),$$

then equation (37.1) becomes

$$\frac{d^2u}{dt^2} + [\pm 1 + R(t)]\, u(t) = 0, \tag{37.2}$$

where

$$R(t) = p^{1/4}|q|^{-3/4} \frac{d}{dx} \left[p(x)|q(x)|\right]^{-1/4} \Big|_{x=x(t)},$$

and the plus (minus) sign is taken in equation (37.2) if $q(x) > 0$ ($q(x) < 0$).

This transformation is sometimes called the Liouville–Green transformation. This transformation is virtually identical to the Liouville transformation (see page 141). See Courant and Hilbert [1, page 292], Eastham [2], and Lakin and Sanchez [3, pages 36–41].

Transformation 4

If the independent and dependent variables are changed in equation (37.1) by

$$\begin{aligned} y(x) = &\mu(x)w(t), \\ t = &\int^x \eta(z)\, dz, \end{aligned}$$

then equation (37.1) becomes

$$\frac{\eta}{\mu}\frac{d}{dt}\left(p\mu^2\eta\frac{dw}{dt}\right) + L[\mu]w = 0. \tag{37.3}$$

Note that the operator $L[\cdot]$ is defined by equation (37.1). If $\eta(z)$ is chosen to be

$$\eta(z) = \frac{1}{p(z)\mu^2(z)},$$

then equation (37.3) simplifies to $\frac{1}{p\mu^3}\frac{d^2w}{dt^2} + L[\mu]w = 0$. See Courant and Hilbert [1, page 292].

References

[1] Courant, R., and Hilbert, D. *Methods of Mathematical Physics.* Interscience Publishers, Inc., New York, 1953.

[2] Eastham, M. S. P. The Liouville–Green asymptotic theory for second-order differential equations: A new approach and extensions. In *Ordinary Differential Equations and Operators*, W. N. Everitt and R. T. Lewis, Eds., no. 1032 in Lecture Notes in Mathematics. Springer–Verlag, New York, 1983.

[3] Lakin, W. D., and Sanchez, D. A. *Topics in Ordinary Differential Equations.* Dover Publications, Inc., New York, 1970.

38. Transformation of an ODE to an Integral Equation

Applicable to Second order linear ordinary differential equations.

Yields

An equivalent integral equation.

Idea

An ordinary differential equation may sometimes be formulated as an integral equation.

Procedure

There is a standard transformation that will allow a linear second order initial value ordinary differential equation to be written as a Volterra integral equation. Given the differential equation with initial conditions for $y(x)$,

$$\begin{aligned}&\frac{d^2y}{dx^2}+A(x)\frac{dy}{dx}+B(x)y=g(x),\\&y(a)=\alpha,\quad y'(a)=\beta,\end{aligned}$$

an equivalent Volterra integral equation is

$$y(x)=f(x)+\int_a^x K(x,\zeta)y(\zeta)\,d\zeta,$$

where

$$\begin{aligned}f(x)&=\int_a^x(x-\zeta)g(\zeta)\,d\zeta+(x-a)\Big(A(a)\alpha+\beta\Big)+\alpha,\\K(x,\zeta)&=(\zeta-x)\Big(B(\zeta)-A'(\zeta)\Big)-A(\zeta).\end{aligned}$$

There is also a standard transformation that will allow a linear second order boundary value ordinary differential equation to be written as a Fredholm integral equation. Given the differential equation and boundary conditions for $w(x)$,

$$\begin{aligned}&\frac{d^2w}{dx^2}+C(x)\frac{dw}{dx}+D(x)w=j(x),\\&w(a)=\gamma,\quad w(b)=\delta,\end{aligned}$$

an equivalent Fredholm integral equation is

$$w(x) = h(x) + \int_a^b H(x,\zeta)w(\zeta)\,d\zeta,$$

where

$$h(x) = \gamma + \int_a^x (x-\zeta)j(\zeta)\,d\zeta + \frac{x-a}{b-a}\left[\delta - \gamma - \int_a^b (b-\zeta)j(\zeta)\,d\zeta\right],$$

$$H(x,\zeta) = \begin{cases} \dfrac{x-b}{b-a}\left[C(\zeta) - (a-\zeta)\Big(C'(\zeta) - D(\zeta)\Big)\right], & \text{for } x > \zeta, \\ \dfrac{x-a}{b-a}\left[C(\zeta) - (b-\zeta)\Big(C'(\zeta) - D(\zeta)\Big)\right], & \text{for } x < \zeta. \end{cases}$$

Example

If $y(x)$ satisfies

$$\begin{aligned} &y'' + y = x, \\ &y(0) = 0, \quad y'(0) = 0, \end{aligned} \tag{38.1}$$

then $y(x)$ satisfies the following Volterra integral equation

$$y(x) = \frac{x^3}{6} + \int_0^x (\zeta - x)y(\zeta)\,d\zeta. \tag{38.2}$$

The solution to equation (38.1), $y = x - \sin x$, satisfies equation (38.2).

Notes

1. There are many other ways in which an ordinary differential equation may be transformed into an integral equation. For example, if $y(x)$ satisfies the nth order ordinary differential equation

$$y^{(n)}(x) = f(x) + \sum_{j=1}^{n} C_j(x)y^{(j-1)}(x)$$

and $u(x) := y^{(n)}(x)$, then $u(x)$ satisfies the integral equation

$$u(x) = F(x) + \int_a^x K(x,t)u(t)\,dt,$$

$$K(x,t) = \sum_{j=1}^{n} C_j(x)\frac{(t-x)^{j-1}}{(j-1)!},$$

where $F(x)$ is $f(x)$ plus a polynomial in $(x-a)$ generated by the initial conditions. See Squire [3, pages 223–227] for more details on this technique, as well as two other techniques.

2. Bose [1] shows that every solution of the nth order linear homogeneous differential equation

$$y^{(n)} = a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y$$

satisfies the integral equation

$$y(x) = y(x_0) + \int_{x_0}^{x} h(u)\,du + \int_{x_0}^{x} \left\{ \int_{x_0}^{u} G(u,v)a_0(v)y(v)\,dv \right\} du,$$

where $h(x)$ is the unique solution to

$$h^{(n-1)} = a_{n-1}(x)h^{(n-2)} + \cdots + a_1(x)h, \qquad (38.3)$$
$$h(x_0) = y'(x_0), \quad h'(x_0) = y''(x_0), \cdots, h^{(n-2)}(x_0) = y^{(n-1)}(x_0),$$

and $G(x,u)$ is the Green's function associated with equation (38.3).

3. See also Jerri [2, pages 60–67].

References

[1] Bose, A. K. An integral equation associated with linear homogeneous differential equations. *Int. J. Math. & Math. Sci. 9*, 2 (1986), 405–408.

[2] Jerri, A. J. *Introduction to Integral Equations with Applications*. Marcel Dekker, New York, 1985.

[3] Squire, W. *Integration for Engineers and Scientists*. American Elsevier Publishing Company, New York, 1970.

39. Miscellaneous ODE Transformations

Applicable to Ordinary differential equations.

Procedure

Many transformations have been developed for equations of specific forms.

Transformation 1

If $y(x)$ is defined by the ordinary differential equation

$$\frac{d^2y}{dx^2} = f(x)y, \tag{39.1}$$

and the dependent variable is changed by

$$w(\zeta) = \sqrt{\zeta'(x)}y(x), \tag{39.2}$$

(for arbitrary $\zeta = \zeta(x)$, or $x = x(\zeta)$), then equation (39.1) becomes

$$\begin{aligned} \frac{d^2w}{d\zeta^2} &= \left[\dot{x}^2 f(x) + \sqrt{\dot{x}}\frac{d^2}{d\zeta^2}\left(\dot{x}^{-1/2}\right)\right] w, \\ &= \left[\dot{x}^2 f(x) - \frac{1}{2}\{x,\zeta\}\right] w, \end{aligned} \tag{39.3}$$

where dots denote differentiation with respect to ζ, and $\{x,\zeta\}$ is the Schwarzian derivative of x with respect to ζ. If we choose $\zeta(x)$ by

$$\zeta(x) = \int^x \sqrt{f(z)}\,dz, \tag{39.4}$$

so that $w(\zeta) = y(x)f^{1/4}(x)$, then equation (39.3) becomes

$$\frac{d^2w}{d\zeta^2} = [1 + \phi(\zeta)]w, \tag{39.5}$$

with

$$\phi(\zeta) = \frac{4ff'' - 5(f')^2}{16f^3} = -\frac{1}{f^{3/4}}\frac{d^2}{dx^2}\left(\frac{1}{f^{1/4}}\right).$$

This is called the Liouville transformation by Olver [7, Chapter 6], and the Liouville–Green transformation by Lakin and Sanchez [6, pages 36–41]. By neglecting $\phi(\zeta)$ in equation (39.5) and solving for $w(\zeta)$, we obtain the first term in the WKB approximation (see page 642).

Example

If we apply this transformation to Airy's equation, $y'' = xy$, for $x > 0$, then we find (using $f(x) = x$)

$$\zeta(x) = \int^x \sqrt{z}\, dz = \frac{2}{3} x^{3/2},$$
$$w(\zeta) = \sqrt{\zeta'(x)}\, y(x) = x^{-1/4} y(x).$$

And so equation (39.5) becomes

$$\frac{d^2 w}{d\zeta^2} - \left(1 + \frac{5}{36\zeta^2}\right) w = 0.$$

This leads to the approximation $w'' - w = 0$ when $\zeta \gg 1$ (which corresponds to $x \gg 1$).

Transformation 2

This transformation removes the $(n-1)$th derivative term in an nth order ordinary differential equation. If $y(x)$ satisfies

$$(-1)^n (p y^{(n)})^{(n)} + L[y] = \lambda q y, \tag{39.6}$$

for $0 \leq x \leq 1$, where $L[y]$ is a linear differential operator of degree less than or equal to $2n-2$ and if the dependent and independent variables are changed from $y(x)$ to $w(t)$ by

$$w(t) = (q^{2n-1} p)^{1/4n} y(x),$$
$$t = \frac{1}{K} \int_0^x \left(\frac{q}{p}\right)^{1/2n} dx,$$
$$K = \int_0^1 \left(\frac{q}{p}\right)^{1/2n} dx,$$

then equation (39.6) is transformed into

$$\frac{d^{2n} w}{dt^{2n}} + H[w] = K^{2n} \lambda w,$$

where $H[w]$ is another linear differential operator of degree less than or equal to $2n-2$. See Boyce [1, page 21].

Transformation 3

The general third order linear homogeneous ordinary differential equation

$$y''' + p_1(x) y'' + p_2(x) y' + p_3(x) y = 0,$$

can be changed to the canonical form

$$w''' + 2Aw' + (A' + b) w = 0, \tag{39.7}$$

by the change of variables

$$w(x) = y(x) \exp\left(- \int_{x_0}^{x} p_1(t)\, dt \right).$$

If we write

$$P_2 = p_2 - p_1^2 - p_1',$$
$$P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1'',$$

then $A(x)$ and $b(x)$ may be written as

$$A(x) = \frac{3}{2} P_2,$$
$$b(x) = P_3 - \frac{3}{2} P_2'.$$

See Greguš [3] for details.

Transformation 4

The general fourth order linear homogeneous ordinary differential equation

$$A(x)y'''' + B(x)y''' + C(x)y'' + D(x)y' + E(x)y = 0,$$

for $y(x)$ can be changed to the canonical form

$$w'''' + a(t)w'' + b(t)w' + c(t)w = 0,$$

for $w(t)$, by the transformation

$$w(t) = \alpha(x)y(x), \qquad t = \beta(x),$$

where $\{\alpha(x), \beta(x)\}$ are chosen to satisfy

$$\alpha {\beta'}^3 = \exp\left[-\frac{1}{2} \int_{x_0}^{x} \frac{B(z)}{A(z)}\, dz \right].$$

Notes

1. If the transformation given by equation (39.2) is applied to the equation
$$\frac{d^2y}{dx^2} = [f(x) + g(x)]y,$$
with ζ defined by equation (39.4), then we obtain
$$\frac{d^2w}{d\zeta^2} = \left(1 + \phi + \frac{g}{f} \right) w.$$

2. The differential equation adjoint to equation (39.7) has the form: $z''' + 2Az' + (A' - b)z = 0$. Hence, the equation in equation (39.7) will be self-adjoint if and only if $b(x) = 0$.

3. Olver [7, pages 190–192] proves that any one-dimensional, first order Hamiltonian differential operator can be put into constant coefficient form by a suitable change of variables.
4. See also Hill [5, pages 44–45].

References

[1] Boyce, W. E. Random eigenvalue problems. In *Probabilistic Methods in Applied Mathematics*, A. T. Bharucha-Reid, Ed. Academic Press, New York, 1968, pp. 1–73.

[2] Gonzalez-Lopez, A. On the linearization of second-order ordinary differential equations. *Lett. Math. Phys. 17*, 4 (1989), 341–349.

[3] Greguš, M. *Third Order Linear Differential Equations*. D. Reidel Publishing Co., Boston, MA, 1987.

[4] Grissom, C., Thompson, G., and Wilkens, G. Linearization of second order ordinary differential equations via Cartan's equivalence method. *J. Differential Equations 77*, 1 (1989), 1–15.

[5] Hill, J. M. *Solution of Differential Equations by Means of One-Parameter Groups*. Pitman Publishing Co., Marshfield, MA, 1982.

[6] Lakin, W. D., and Sanchez, D. A. *Topics in Ordinary Differential Equations*. Dover Publications, Inc., New York, 1970.

[7] Olver, P. J. Darboux theorem for Hamiltonian differential operators. *J. Differential Equations 71* (1988), 10–33.

40. Reduction of PDEs to a First Order System

Applicable to Nonlinear partial differential equations.

Yields

A first order system of partial differential equations.

Idea

By introducing variables to represent the derivatives in a partial differential equation, a first order system may be obtained.

Procedure

Sometimes it is advantageous to reduce a partial differential equation of high order for a single unknown function to a system of several first order equations. This might be done, for instance, to utilize a specific numerical package that requires a partial differential equation to be input as a first order system. This can always be done by introducing an appropriate set of derivatives as unknowns.

The general procedure is to introduce new variables as the derivatives of the desired function and then "discover" relations among these functions. The following derivation for second order equations is from Garabedian [1].

Suppose we have the second order partial differential equation, with boundary conditions

$$\begin{aligned} u_{xx} &= G\left(x, y, u, u_x, u_y, u_{xy}, u_{yy}\right), \\ u(0, y) &= f(y), \\ u_x(0, y) &= g(y), \end{aligned} \tag{40.1}$$

for the unknown $u(x, y)$. We introduce new variables, $\{u_1, \ldots, u_8\}$, which are assumed to depend upon the new independent variables ζ and η, by the definitions

$$\begin{aligned} &u_1 = x, \quad && u_4 = u_x, \quad && u_7 = u_{xy}, \\ &u_2 = y, \quad && u_5 = u_y, \quad && u_8 = u_{yy}, \\ &u_3 = u, \quad && u_6 = u_{xx}. \end{aligned}$$

If we then specify the new independent variables by requiring

$$\begin{aligned} \frac{\partial u_1}{\partial \zeta} &= \frac{\partial u_2}{\partial \eta}, \qquad && \frac{\partial u_2}{\partial \zeta} = 0, \\ u_1(0, \eta) &= 0, \qquad && u_2(0, \eta) = \eta, \end{aligned}$$

then $u_1 = x = \zeta$ and $u_2 = y = \eta$. The purpose of introducing these new independent variables is to eliminate explicit dependence on x and y.

With these new variables, equation (40.1) can be written as the system

$$\frac{\partial u_1}{\partial \zeta} = \frac{\partial u_2}{\partial \eta}, \qquad \frac{\partial u_2}{\partial \zeta} = 0, \qquad \frac{\partial u_3}{\partial \zeta} = u_4 \frac{\partial u_2}{\partial \eta}, \qquad \frac{\partial u_4}{\partial \zeta} = u_6 \frac{\partial u_2}{\partial \eta},$$
$$\frac{\partial u_5}{\partial \zeta} = \frac{\partial u_4}{\partial \eta}, \qquad \frac{\partial u_7}{\partial \zeta} = \frac{\partial u_6}{\partial \eta}, \qquad \frac{\partial u_8}{\partial \zeta} = \frac{\partial u_7}{\partial \eta}. \tag{40.2}$$
$$\frac{\partial u_6}{\partial \zeta} = G_x \frac{\partial u_2}{\partial \eta} + u_4 G_u \frac{\partial u_2}{\partial \eta} + u_6 G_{u_x} \frac{\partial u_2}{\partial \eta} + G_{u_y} \frac{\partial u_4}{\partial \eta} + G_{u_{xy}} \frac{\partial u_6}{\partial \eta} + G_{u_{yy}} \frac{\partial u_7}{\partial \eta}.$$

Most of the above equations are consistency requirements; that is, $(u_x)_y = (u_y)_x$ implies that $(u_5)_\zeta = (u_4)_\eta$. The initial conditions for the variables $\{u_1, \ldots, u_8\}$ are given by

$$\begin{aligned}
u_1(0,\eta) &= 0,\\
u_2(0,\eta) &= \eta,\\
u_3(0,\eta) &= f(\eta),\\
u_4(0,\eta) &= g(\eta),\\
u_5(0,\eta) &= f'(\eta),\\
u_6(0,\eta) &= G(0,\eta,f(\eta),g(\eta),f'(\eta),g'(\eta),f''(\eta)),\\
u_7(0,\eta) &= g'(\eta),\\
u_8(0,\eta) &= f''(\eta).
\end{aligned} \tag{40.3}$$

Note that equation (40.2) is in the general form of a linear first order system

$$\frac{\partial u_j}{\partial \zeta} = \sum_{k=1}^{8} a_{jk}(u_1, \ldots, u_8) \frac{\partial u_k}{\partial \eta},$$

for $j = 1, 2, \ldots, 8$.

To convert the system in equation (40.2) back to the system in equation (40.1) may require the use of the boundary conditions in equation (40.3).

Note

1. Systems of high order partial differential equations can also be made into first order systems by the introduction of enough terms. For instance, the system of equations for $u(x, y)$ and $v(x, y)$

$$F_1(x, y, u, u_x, u_y, v, v_x, v_y) = 0$$
$$F_2(x, y, u, u_x, u_y, v, v_x, v_y) = 0$$

can be written as a first order system, but the resulting system has 12 dependent variables. See Garabedian [1, pages 7–11] for details.

Reference

[1] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

41. Transforming Partial Differential Equations

Applicable to Partial differential equations.

Idea

Changing variables in a partial differential equation is a straightforward process.

Procedure 1

The general procedure is simple: Construct a new function, which depends upon new variables, and then differentiate with respect to the old variables to see how the derivatives transform.

Procedure 2

If a differential equation can be written in terms of coordinate-free expressions (e.g., in terms of the gradient operator), then a change of variables can be avoided by simply using the metric of the new coordinate system. This section contains representations of common coordinate-free expressions for an orthogonal coordinate system. Note that Moon and Spencer [4] list the metric coefficients for 43 different orthogonal coordinate systems. (These consist of 11 general systems, 21 cylindrical systems, and 11 rotational systems.)

In an orthogonal coordinate system, let $\{\mathbf{a}_i\}$ denote the unit vectors in each of the three coordinate directions, and let $\{u_i\}$ denote distance along each of these axes. The coordinate system may be designated by the *metric coefficients* $\{g_{11}, g_{22}, g_{33}\}$, defined by

$$g_{ii} = \left(\frac{\partial x_1}{\partial u_i}\right)^2 + \left(\frac{\partial x_2}{\partial u_i}\right)^2 + \left(\frac{\partial x_3}{\partial u_i}\right)^2, \tag{41.1}$$

where $\{x_1, x_2, x_3\}$ represent rectangular coordinates. Using the metric coefficients defined in equation (41.1), we define $g = g_{11}g_{22}g_{33}$.

When ϕ represents a scalar and $\mathbf{E} = E_1\mathbf{a}_1 + E_2\mathbf{a}_2 + E_3\mathbf{a}_3$ represents a vector, we have

$$\operatorname{grad}\phi = \nabla\phi = \frac{\mathbf{a}_1}{\sqrt{g_{11}}}\frac{\partial\phi}{\partial u_1} + \frac{\mathbf{a}_2}{\sqrt{g_{22}}}\frac{\partial\phi}{\partial u_2} + \frac{\mathbf{a}_3}{\sqrt{g_{33}}}\frac{\partial\phi}{\partial u_3}, \tag{41.2}$$

$$\begin{aligned}\operatorname{div}\mathbf{E} &= \nabla\cdot\mathbf{E}\\ &= \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial u_1}\left(\frac{gE_1}{g_{11}}\right) + \frac{\partial}{\partial u_2}\left(\frac{gE_2}{g_{22}}\right) + \frac{\partial}{\partial u_3}\left(\frac{gE_3}{g_{33}}\right)\right\},\end{aligned} \tag{41.3}$$

$$\operatorname{curl}\mathbf{E} = \nabla\times\mathbf{E} = \mathbf{a}_1\frac{\Gamma_1}{\sqrt{g_{11}}} + \mathbf{a}_2\frac{\Gamma_2}{\sqrt{g_{22}}} + \mathbf{a}_3\frac{\Gamma_3}{\sqrt{g_{33}}}, \tag{41.4}$$

$$\begin{aligned}\nabla^2\phi &= \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial u_1}\left[\frac{\sqrt{g}}{g_{11}}\frac{\partial\phi}{\partial u_1}\right] + \frac{\partial}{\partial u_2}\left[\frac{\sqrt{g}}{g_{22}}\frac{\partial\phi}{\partial u_2}\right] + \frac{\partial}{\partial u_3}\left[\frac{\sqrt{g}}{g_{33}}\frac{\partial\phi}{\partial u_3}\right]\right\}, \\ &= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left[\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right] + \frac{\partial}{\partial u_2}\left[\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right] + \frac{\partial}{\partial u_3}\left[\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right]\right\},\end{aligned} \tag{41.5}$$

$$\operatorname{grad}\operatorname{div}\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) = \frac{\mathbf{a}_1}{\sqrt{g_{11}}}\frac{\partial\Upsilon}{\partial x_1} + \frac{\mathbf{a}_2}{\sqrt{g_{22}}}\frac{\partial\Upsilon}{\partial x_2} + \frac{\mathbf{a}_3}{\sqrt{g_{33}}}\frac{\partial\Upsilon}{\partial x_3}, \tag{41.6}$$

$$\begin{aligned}\operatorname{curl}\operatorname{curl}\mathbf{E} &= \nabla\times(\nabla\times\mathbf{E}) \\ &= \mathbf{a}_1\sqrt{\frac{g_{11}}{g}}\left[\frac{\partial\Gamma_3}{\partial x_2} - \frac{\partial\Gamma_2}{\partial x_3}\right] + \mathbf{a}_2\sqrt{\frac{g_{22}}{g}}\left[\frac{\partial\Gamma_1}{\partial x_3} - \frac{\partial\Gamma_3}{\partial x_1}\right] \\ &\quad + \mathbf{a}_3\sqrt{\frac{g_{33}}{g}}\left[\frac{\partial\Gamma_2}{\partial x_1} - \frac{\partial\Gamma_1}{\partial x_2}\right],\end{aligned} \tag{41.7}$$

$$\begin{aligned}✡\mathbf{E} &= \operatorname{grad}\operatorname{div}\mathbf{E} - \operatorname{curl}\operatorname{curl}\mathbf{E} \\ &= \nabla(\nabla\cdot\mathbf{E}) - \nabla\times(\nabla\times\mathbf{E}) \\ &= \mathbf{a}_1\left\{\frac{1}{\sqrt{g_{11}}}\frac{\partial\Upsilon}{\partial x_1} + \sqrt{\frac{g_{11}}{g}}\left[\frac{\partial\Gamma_2}{\partial x_3} - \frac{\partial\Gamma_3}{\partial x_2}\right]\right\} \\ &\quad + \mathbf{a}_2\left\{\frac{1}{\sqrt{g_{22}}}\frac{\partial\Upsilon}{\partial x_2} + \sqrt{\frac{g_{22}}{g}}\left[\frac{\partial\Gamma_3}{\partial x_1} - \frac{\partial\Gamma_1}{\partial x_3}\right]\right\} \\ &\quad + \mathbf{a}_3\left\{\frac{1}{\sqrt{g_{33}}}\frac{\partial\Upsilon}{\partial x_3} + \sqrt{\frac{g_{33}}{g}}\left[\frac{\partial\Gamma_1}{\partial x_2} - \frac{\partial\Gamma_2}{\partial x_1}\right]\right\},\end{aligned} \tag{41.8}$$

where Υ and $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3)$ are defined by

$$\begin{aligned}\Upsilon &= \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial x_1}\left[E_1\sqrt{\frac{g}{g_{11}}}\right] + \frac{\partial}{\partial x_2}\left[E_2\sqrt{\frac{g}{g_{22}}}\right] + \frac{\partial}{\partial x_3}\left[E_3\sqrt{\frac{g}{g_{33}}}\right]\right\}, \\ \Gamma_1 &= \frac{g_{11}}{\sqrt{g}}\left\{\frac{\partial}{\partial x_2}(\sqrt{g_{33}}E_3) - \frac{\partial}{\partial x_3}(\sqrt{g_{22}}E_2)\right\}, \\ \Gamma_2 &= \frac{g_{22}}{\sqrt{g}}\left\{\frac{\partial}{\partial x_3}(\sqrt{g_{11}}E_1) - \frac{\partial}{\partial x_1}(\sqrt{g_{33}}E_3)\right\}, \\ \Gamma_3 &= \frac{g_{22}}{\sqrt{g}}\left\{\frac{\partial}{\partial x_1}(\sqrt{g_{22}}E_2) - \frac{\partial}{\partial x_2}(\sqrt{g_{11}}E_1)\right\}.\end{aligned} \tag{41.9}$$

Operations for orthogonal coordinate systems are sometimes written in terms of $\{h_i\}$ functions, instead of the $\{g_{ii}\}$ terms. Here, $h_i = \sqrt{g_{ii}}$, so that $\sqrt{g} = h_1 h_2 h_3$. For example

- **Cylindrical Polar Coordinates**

$$\begin{gathered} x_1 = r\cos\phi, \qquad x_2 = r\sin\phi, \qquad x_3 = z \\ g_1 = 1, \qquad g_2 = r^2, \qquad g_3 = 1 \end{gathered} \tag{41.10}$$

- **Elliptic Cylinder Coordinates**

$$\begin{gathered} x_1 = u_1 u_2, \qquad x_2 = \sqrt{(u_1^2 - c^2)(1 - u_2^2)}, \qquad x_3 = u_3 \\ g_1 = \frac{u_1^2 - c^2 u_2^2}{u_1^2 - c^2}, \qquad g_2 = \frac{u_1^2 - c^2 u_2^2}{1 - u_2^2}, \qquad g_3 = 1 \end{gathered}$$

Example 1

Suppose we have the equation

$$f_{xx} + f_{yy} + x f_y = 0, \tag{41.11}$$

and we would like to transform the equation from the $\{x, y\}$ variables to the $\{u, v\}$ variables, where

$$u = x, \qquad v = \frac{x}{y}.$$

Note that the inverse transformation is given by $x = u$, $y = u/v$.

We define $g(u, v)$ to be equal to the function $f(x, y)$ when written in the new variables. That is,

$$f(x, y) = g(u, v) = g\left(x, \frac{x}{y}\right). \tag{41.12}$$

Now we create the needed derivative terms, carefully applying the chain rule. For example, by differentiating equation (41.12) with respect to x, we obtain

$$\begin{aligned} f_x(x, y) &= g_u \frac{\partial}{\partial x}(u) + g_v \frac{\partial}{\partial x}(v) \\ &= g_1 \frac{\partial}{\partial x}(x) + g_2 \frac{\partial}{\partial x}\left(\frac{x}{y}\right) \\ &= g_1 + g_2 \frac{1}{y} \\ &= g_1 + \frac{v}{u} g_2, \end{aligned}$$

where we have used a subscript of "1" ("2") to indicate a derivative with respect to the first (second) argument of the function $g(u, v)$ (i.e., $g_1(u, v) = g_u(u, v)$). Use of this "slot notation" tends to minimize errors.

In a like manner, we find

$$\begin{aligned} f_y(x, y) &= g_u \frac{\partial}{\partial y}(u) + g_v \frac{\partial}{\partial y}(v) \\ &= g_1 \frac{\partial}{\partial y}(x) + g_2 \frac{\partial}{\partial y}\left(\frac{x}{y}\right) \\ &= -\frac{x}{y^2} g_2 \\ &= -\frac{v^2}{u} g_2. \end{aligned}$$

The second order derivatives can be calculated similarly:

$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial}{\partial x}\left(f_x(x, y)\right) \\ &= \frac{\partial}{\partial x}\left(g_1 + \frac{1}{y} g_2\right) \\ &= g_{11} + \frac{2v}{u} g_{12} + \frac{v^2}{u^2} g_{22}, \\ f_{xy}(x, y) &= \frac{\partial}{\partial x}\left(-\frac{x}{y^2} g_2\right) \\ &= -\frac{u^2}{v^2} g_2 - \frac{u^3}{v^3} g_{12} - \frac{u^2}{v^2} g_{22}, \\ f_{yy}(x, y) &= \frac{\partial}{\partial y}\left(-\frac{x}{y^2} g_2\right) \\ &= \frac{2v^3}{u^2} g_2 + \frac{v^4}{u^2} g_{22}. \end{aligned}$$

Finally, then, we can determine what equation (41.11) looks like in the new variables:

$$\begin{aligned} 0 &= f_{xx} + f_{yy} + x f_y \\ &= \left(g_{11} + \frac{2v}{u} g_{12} + \frac{v^2}{u^2} g_{22}\right) + \left(\frac{2v^3}{u^2} g_2 + \frac{v^4}{u^2} g_{22}\right) + (u)\left(-\frac{v^2}{u} g_2\right) \\ &= \frac{v^2(2v - u^2)}{u^2} g_v + g_{uu} + \frac{2v}{u} g_{uv} + \frac{v^2(1 + v^2)}{u^2} g_{vv}. \end{aligned}$$

Example 2

As a simple example of using coordinate-free representations, consider the diffusion equation in rectilinear coordinates:

$$u_t = \kappa \left(u_{xx} + u_{yy} + u_{zz}\right). \tag{41.13}$$

We recognize this to be the same as $u_t = \kappa\nabla^2 u$. Hence, using equation (41.10) in equation (41.5) we find

$$u_t = \kappa\nabla^2 u = \kappa\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

in cylindrical polar coordinates.

Notes

1. A Macsyma program that will perform changes of variables in partial differential equations is described in Steinberg [5].
2. Mathematica has the package `VectorAnalysis` which can compute the divergence, curl, gradient, Laplacian, and the biharmonic operator (∇^4) in 14 different coordinate systems.
3. The Laplacian (∇^2) for 22 different coordinate systems is given starting on page 204.
4. See also Butkov [1, pages 34–39] and Moon and Spencer [3, Chapter 3].

References

[1] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[2] Harper, D. Vector 33: A REDUCE program for vector algebra and calculus in orthogonal curvilinear coordinates. *Comput. Physics Comm. 54* (1989), 295–305.

[3] Moon, P., and Spencer, D. E. *Field Theory for Engineers.* D. Van Nostrand Company, Inc., New York, 1961.

[4] Moon, P., and Spencer, D. E. *Field Theory Handbook.* Springer–Verlag, New York, 1961.

[5] Steinberg, S. Change of variables in partial differential equations. Tech. rep., University of New Mexico, Albuquerque, New Mexico, 1983. Department of Mathematics preprint (AD-A214 702).

42. Transformations of Partial Differential Equations

Applicable to Partial differential equations.

Procedure

Many transformations have been developed for equations of specific forms.

Euler Transformation

Given the first order partial differential equation in two independent variables, $F(x, y, z, p, q) = 0$ (with, as usual, $p = z_x$, $q = z_y$) and $z_{xx} \neq 0$ the transformation

$$\left\{\begin{array}{l} x = Z_X \\ y = Y \\ z = XZ_X - Z \\ p = X \\ q = -Z_Y \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} X = z_x \\ Y = y \\ Z = xz_x - z \\ P = x \\ Q = -z_y \end{array}\right\}, \tag{42.1}$$

is known as the Euler transformation. Note that $Z_Y + z_y = 0$. Under this transformation, the original equation transforms into $F(Z_X, Y, XZ_X - Z, X, -Z_Y) = 0$ (see Kamke [6, section 11.15, pages 100–101]).

As an example, the equation $G(xp - z, y, p, q) = 0$ becomes, under the Euler transformation, $G(Z, Y, X, -Z_Y) = 0$. As another example, the Clairaut partial differential equation $F = z - (xz_x + yz_y + f(z_x, z_y)) = 0$ is transformed into $F = Z - YZ_Y + f(X, -Z_Y) = 0$. Note that this latter equation is really an ordinary differential equation for $Z = Z(Y)$ (the variable X acts as a parameter).

Kirchoff Transformation

Given the elliptic partial differential equation

$$\operatorname{div}[K(\psi) \operatorname{grad} \psi] = \nabla \cdot [K(\psi) \nabla \psi] = 0, \tag{42.2}$$

for $\psi = \psi(\mathbf{x})$, the Kirchoff transformation introduces the new dependent variable, $\Phi(\mathbf{x})$, defined by $\Phi = \int_{\psi_0}^{\psi} K(t)\, dt$, where ψ_0 is some arbitrary reference value. This transforms equation (42.2) into Laplace's equation $\nabla^2 \Phi = 0$; see Ames [1, pages 6–7 and 21–23].

Transformations of Parabolic Differential Equations I

The parabolic partial differential equation

$$u_t = \alpha^2 u_{xx} - \delta u_x + \epsilon u,$$

where $\{\alpha, \delta, \epsilon\}$ are constants, may be transformed into the simple diffusion equation $\phi_t = \alpha^2 \phi_{xx}$, by means of the transformation (see Bateman [2, pages 75–79] or Farlow [3, page 58])

$$u(x,t) = \phi(x,t) \exp\left[\frac{\delta}{2\alpha^2} x + \left(\epsilon - \frac{\delta^2}{4\alpha^2}\right) t\right]. \tag{42.3}$$

Transformations of Parabolic Differential Equations II

The nonlinear parabolic partial differential equation

$$c_t = (D(c) c_x)_x$$

may be transformed, via $v(c,t) = D(c)c_x$, into the following equation with a simpler nonlinearity (see Hill [5, page 148]):

$$D(c) v_t = v^2 v_{cc}.$$

Transformation of Elliptic/Hyperbolic Equations

The linear partial differential equation

$$\alpha(x) \frac{\partial^2 u}{\partial x^2} + \beta(x) \frac{\partial u}{\partial x} + \gamma(x) u = a \frac{\partial u}{\partial t} + b \frac{\partial^2 u}{\partial t^2} \tag{42.4}$$

may be transformed into the equation

$$c(X) \frac{\partial}{\partial X} \left(\frac{1}{c(X)} \frac{\partial v}{\partial X} \right) = a \frac{\partial v}{\partial t} + b \frac{\partial^2 v}{\partial t^2}.$$

Through the transformation

$$X = \int^x \frac{d\tau}{\sqrt{|\alpha(\tau)|}} \qquad v(X,t) = \frac{u(x,t)}{u_0(x)},$$

where $u_0(x)$ is any nonzero "equilibrium" solution of (42.4), and $c(X)$ is a function completely determined by $\{\alpha(x), \beta(x), \gamma(x)\}$. See Varley and Seymour [9].

Removing First Derivative Terms

Linear elliptic equations and hyperbolic equations of second order, all of whose coefficients of the derivative terms are constants, can be transformed so that the first derivative terms no longer appear. For example, we presume that $u(\mathbf{x})$ satisfies

$$\sum_{k=1}^{n} \lambda_k \frac{\partial^2 u}{\partial {x_k}^2} + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} + c(\mathbf{x}) u = 0. \tag{42.5}$$

Note that scaling of the $\{x_k\}$ allows equation (42.5) to be written with each $\{\lambda_k\}$ equal to 0, 1, or −1. If we presume that no λ_k is equal to zero, and we define

$$w(\mathbf{x}) = u(\mathbf{x}) \exp\left[\frac{1}{2}\sum_{k=1}^{n}\left(\frac{b_k}{\lambda_k}\right)x_k\right],$$

then $w(\mathbf{x})$ satisfies (see Garabedian [4, pages 74–75])

$$\sum_{k=1}^{n} \lambda_k \frac{\partial^2 w}{\partial {x_k}^2} + \left(c(\mathbf{x}) - \frac{1}{4}\sum_{i=1}^{n}\frac{b_k^2}{\lambda}\right) w = 0.$$

Von Mises Transformation

For fluid flow with constant viscosity, the Navier–Stokes equations (see page 179) sometimes take the form

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{42.6.a-b}$$

These are called the *boundary layer equations*. A standard procedure for analyzing the Navier–Stokes equations (and equations derived from them) is to introduce the stream function Ψ, defined by

$$u = \frac{\partial \Psi}{\partial y}, \qquad v = -\frac{\partial \Psi}{\partial x}.$$

With this definition, equation (42.6.b) is automatically satisfied. In the Von Mises transformation, Ψ and x are treated as the independent variables, instead of y and x. This transforms equation (42.6.a) into

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \Psi}\left(\nu u \frac{\partial u}{\partial \Psi}\right).$$

See Rosenhead [7], Schlichting [8], or von Mises [10].

Notes

1. If the boundary data are of the Neuman type, then the Kirchoff transformation may introduce nonlinearities in the boundary data for the Φ problem.
2. The Kirchoff transformation is frequently useful in free boundary problems (see page 311), where $K(\psi)$ changes value across the (unknown) boundary.

References

[1] Ames, W. F., Ed. *Nonlinear Partial Differential Equations*, vol. 1. Academic Press, New York, 1967.

[2] Bateman, H. *Partial Differential Equations of Mathematical Physics*. Dover Publications, Inc., New York, 1944.

[3] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[4] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[5] Hill, J. M. *Solution of Differential Equations by Means of One-Parameter Groups*. Pitman Publishing Co., Marshfield, MA, 1982.

[6] Kamke, E. *Differentialgleichungen Losungsmethoden und Losungen*, vol. II. Chelsea Publishing Company, New York, 1947.

[7] Rosenhead, L. *Laminar Boundary Layers*. Clarendon Press, Oxford, England, 1963.

[8] Schlichting, H. *Boundary Layer Theory*. McGraw–Hill Book Company, New York, 1955.

[9] Varley, E., and Seymour, B. A method of obtaining exact solution to partial differential equation with variable coefficients. *Stud. Appl. Math. 78* (1988), 183–225.

[10] von Mises, R. Bemerkungen zur hydrodynamik. *Z. Angnew. Math. Mech. 7* (1927), 425–431.

43. Introduction to Exact Analytical Methods

The methods in this section of the book are for the exact solution of differential equations. The methods have been separated into two parts:

- Methods that can be used for ordinary differential equations and, sometimes, partial differential equations: When a method in this part can be used for a partial differential equation, there is a star ($*$) alongside the method name.
- Methods that can be used only for partial differential equations.

Because many of the common methods for partial differential equations are also useful as methods for ordinary differential equations, the first part of this section should not be overlooked when attempting to find the solution of a partial differential equation.

Listed below are, in the author's opinion, those methods that are the most useful when solving ordinary differential equations and partial differential equations. These are the methods that might be tried first.

Most Useful Methods for ODEs

- Look-Up Technique (page 179)
- Look-Up ODE Forms (page 219)
- Computer-Aided Solution (page 240)
- Constant Coefficient Linear Equations (page 247)
- Eigenfunction Expansions* (page 268)
- Green's Functions* (page 318)
- Integral Transforms: Infinite Intervals* (page 347)
- Integrating Factors* (page 356)
- Series Solution* (page 403)
- Method of Undetermined Coefficients* (page 415)

Most Useful Methods for PDEs

- Look-Up Technique (page 179)
- Eigenfunction Expansions* (page 268)
- Green's Functions* (page 318)
- Integral Transforms: Infinite Intervals* (page 347)
- Method of Characteristics (page 432)
- Conformal Mappings (page 441)
- Lie Groups: PDEs (page 471)
- Separation of Variables (page 487)
- Similarity Methods (page 497)

44. Look-Up Technique

Applicable to Equations of certain forms.

Yields

A reference to the literature that may yield an analytical solution, an approximate analytical solution, or a numerical solution.

Idea

Many functions of mathematical physics have been well studied. If a differential equation can be transformed to a known form, then information about the solution may be obtained by looking in the right reference.

Procedure

Compare the differential equation that you are trying to analyze with the lists on the following pages. If the equation you are investigating appears, see the references cited for that equation.

The equations listed in this section include

- Ordinary differential equations (page 180)
  - First order equations
  - Second order equations
  - Higher order equations
- Partial differential equations (page 189)
  - Linear equations
  - Second order nonlinearity
  - Higher order and variable order nonlinearities
- Systems of differential equations (page 199)
  - Systems of ordinary differential equations
  - Systems of partial differential equations
- The Laplacian in different coordinate systems (page 204)
- Parametrized equations at specific values (page 205)

Notes

1. Realize that the same equation may look different when written in different variables. Some scaling of any given equation may be required to make it look like one of the forms listed.
2. Carslaw and Jaeger [36] have a large collection of exact analytical solutions for parabolic partial differential equations.
3. In Kamke ([90] and [91]), Murphy [123], and Polyanin and Zaitsev [130] are long listings of ordinary differential equations and partial differential equations and their exact solutions.
4. The references follow the listings of differential equations (page 209).

5. A complete list of third-order polynomial evolution equations of not normal type with nontrivial Lie–Bäcklund symmetries is in Fujimoto and Watanabe [60].

44.1 Ordinary Differential Equations

44.1.1 First Order Equations

Abel equation of the first kind (see Murphy [123, page 23]):

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Abel equation of the second kind (see Murphy [123, page 25]):

$$[g_0(x) + g_1(x)y]\, y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Bernoulli equation (see page 235):

$$y' = a(x)y^n + b(x)y$$

Binomial equation (see Hille [80, page 675]):

$$(y')^m = f(x, y)$$

Briot and Bouquet's equation (see Ince [85, page 295]):

$$xy' - \lambda y = a_{10}x + a_{20}x^2 + a_{11}yx + a_{02}y^2 + \dots$$

Clairaut's equation (see page 237):

$$f(xy' - y) = g(y')$$

Elliptic functions (see Gradshteyn and Ryzhik [69, page 917]):

$$y' = \sqrt{(1 - y^2)(1 - k^2y^2)}$$

Euler equation (see Valiron [161, page 201]):

$$y' = \pm\sqrt{\tfrac{ay^4+by^3+cy^2+dy+e}{ax^4+bx^3+cx^2+dx+e}}$$

Euler equation (see Valiron [161, page 212]):

$$y' + y^2 = ax^m$$

Heisenberg equation of motion (see Iyanaga and Kawada [87, page 1083]):

$$\tfrac{dA(t)}{dt} = \tfrac{i}{\hbar}\,[H, A(t)]$$

Jacobi equation (see Ince [85, page 22]):

$$(a_1 + b_1x + c_1y)(xy' - y) - (a_2 + b_2x + c_2y)y' + (a_3 + b_3x + c_3y) = 0$$

Lagrange's equation (see page 363):

$$y = xf(y') + g(y')$$

Löwner's equation (see Iyanaga and Kawada [87, page 1345]):

$$y' = -y\frac{1+\kappa(x)y}{1-\kappa(x)y}$$

Riccati equation (see page 392):

$$y' = a(x)y^2 + b(x)y + c(x)$$

Unnamed equation (see Boyd [30]):

$$y' = -pe^{-q/y}$$

Unnamed equation (see Goldstein and Braun [68, page 42]):

$$g(y)y' = f(x) + h(x)\,G\Big(\int f(x)\,dx - \int g(y)\,dy\Big)$$

Weierstrass function (see Rainville [131, page 312]):

$$y' = \sqrt{4y^3 - g_2 y - g_3}$$

44.1.2 Second Order Equations

Airy equation (see Abramowitz and Stegun [3, Section 10.4.1]):

$$y'' = xy$$

Anger functions (see Gradshteyn and Ryzhik [69, page 989]):

$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right)y = \frac{x-\nu}{\pi x^2}\sin\nu\pi$$

Baer equation (see Moon and Spencer [119, page 156]):

$$(x-a_1)(x-a_2)y'' + \tfrac{1}{2}\left[2x - (a_1+a_2)\right]y' - \left[p^2x + q^2\right]y = 0$$

Baer wave equation (see Moon and Spencer [119, page 157]):

$$(x-a_1)(x-a_2)y'' + \tfrac{1}{2}\left[2x - (a_1+a_2)\right]y' - \left[k^2x^2 - p^2x + q^2\right]y = 0$$

Bessel equation (see Abramowitz and Stegun [3, Section 9.1.1]):

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Bessel equation – modified (see Abramowitz and Stegun [3, Section 9.6.1]):

$$x^2y'' + xy' - (x^2 + n^2)y = 0$$

Bessel equation – spherical (see Abramowitz and Stegun [3, Section 10.1.1]):

$$x^2y'' + 2xy' + \left[x^2 - n(n+1)\right]y = 0$$

Bessel equation – modified spherical (see Abramowitz and Stegun [3, Section 10.2.1]):

$$x^2y'' + 2xy' - \left[x^2 + n(n+1)\right]y = 0$$

Bessel equation – wave (see Moon and Spencer [119, page 154]):

$$x^2y'' + xy' + \left[a^2x^4 + b^2x^2 - c^2\right] y = 0$$

Bôcher equation (see Moon and Spencer [119, page 127]):

$$y'' + \frac{1}{2}\left[\frac{m_1}{x-a_1} + \cdots + \frac{m_{n-1}}{x-a_{n-1}}\right] y' + \frac{1}{4}\left[\frac{A_0 + A_1x + \cdots + A_lx^l}{(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_{n-1})^{m_{n-1}}}\right] y = 0$$

Confluent equation – general (see Abramowitz and Stegun [3, Section 13.1.35]):

$$y''+\left[\frac{2a}{x} + 2f' + \frac{bh'}{h} - h' - \frac{h''}{h'}\right] y' + \left[\left(\frac{bh'}{h} - h' - \frac{h''}{h'}\right)\left(\frac{a}{x} + f'\right) + \frac{a(a-1)}{x^2} + \frac{2af'}{x} + f'' + (f')^2 - \frac{a(h')^2}{h}\right] y = 0$$

Coulomb wave functions (see Abramowitz and Stegun [3, Section 14.1.1]):

$$y'' + \left[1 - \tfrac{2\eta}{x} - \tfrac{L(L+1)}{x^2}\right] y = 0$$

Duffing's equation (see Bender and Orszag [20, page 547]):

$$y'' + y + ay^3 = 0$$

Eckart equation (see Barut *et al.* [18]):

$$y'' + \left[\tfrac{\alpha\eta}{1+\eta} + \tfrac{\beta\eta}{(1+\eta)^2} + \gamma\right] y = 0, \qquad \eta = e^{\delta x}$$

Ellipsoidal wave equation (see Arscott [13]):

$$y'' - (a + bk^2 \operatorname{sn}^2 x + qk^4 \operatorname{sn}^4 x)y = 0$$

Complete elliptic integral (see Gradshteyn and Ryzhik [69, page 907]):

$$\tfrac{d}{dx}\left[x(1-x^2)\tfrac{dy}{dx}\right] - xy = 0$$

Complete elliptic integral (see Gradshteyn and Ryzhik [69, page 907]):

$$(1-x^2)\tfrac{d}{dx}\left(x\tfrac{dy}{dx}\right) + xy = 0$$

Emden equation (see Leach [102]):

$$(x^2y')' + x^2y^n = 0$$

Emden equation – modified (see Leach [102]):

$$y'' + a(x)y' + y^n = 0$$

Emden–Fowler equation (see Rosenau [137]):

$$(x^p y')' \pm x^\sigma y^n = 0$$

Generalized Emden–Fowler equation (see Leach *et al.* [104]):

$$y'' + f(x)y^n = 0$$

Integrals of the error function (see Abramowitz and Stegun [3, Section 7.2.2]):

$$y'' + 2xy' - 2ny = 0$$

Gegenbauer functions (see Infeld and Hull [86]):

$$(1-x^2)y'' - (2m+3)xy' + \lambda y = 0$$

Halm's equation (see Hille [80, page 357]):

$$(1+x^2)^2 y'' + \lambda y = 0$$

Heine equation (see Moon and Spencer [119, page 157]):

$$y'' + \frac{1}{2}\left[\frac{1}{x-a_1} + \frac{2}{x-a_2} + \frac{2}{x-a_3}\right] y + \frac{1}{4}\left[\frac{A_0+A_1x+A_2x^2+A_3x^3}{(x-a_1)(x-a_2)^2(x-a_3)^2} y\right] = 0$$

Hermite polynomials (see Abramowitz and Stegun [3, Section 22.6.21]):

$$y'' - xy' + ny = 0$$

Heun's equation (see Ronveaux [134]):

$$y'' + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} - \frac{\epsilon}{x-a}\right] y' + \frac{\alpha\beta x - q}{x(x-1)(x-a)} y = 0$$

Hill's equation (see Ince [85, page 384]):

$$y'' + (a_0 + 2a_1 \cos 2x + 2a_2 \cos 4x + \dots)\, y = 0$$

Hypergeometric equation (see Abramowitz and Stegun [3, Section 15.5.1]):

$$x(1-x)y'' + [c - (a+b+1)x]\, y' - aby = 0$$

Hyperspherical differential equation (see Iyanaga and Kawada [87, page 1185]):

$$(1-x^2)y'' - 2axy' + by = 0$$

Ince equation (see Athorne [14]):

$$y'' + \frac{\alpha + \beta\cos 2t + \gamma \cos 4t}{(1+a\cos 2t)^2} y = 0$$

Jacobi's equation (see Iyanaga and Kawada [87, page 1480]):

$$x(1-x)y'' + [\gamma - (\alpha+1)x]\, y' + n(\alpha+n)y = 0$$

Kelvin functions (see Abramowitz and Stegun [3, Section 9.9.3]):

$$x^2y'' + xy' - (ix^2 + \nu^2)y = 0$$

Kummer's equation (see Abramowitz and Stegun [3, Section 13.1.1]):

$$xy'' + (b - x)y' - ay = 0$$

Lagerstrom equation (see Rosenblat and Shepherd [138]):

$$y'' + \frac{k}{x}y' + \epsilon yy' = 0$$

Laguerre equation (see Iyanaga and Kawada [87, page 1481]):

$$xy'' + (\alpha + 1 - x)y' + \lambda y = 0$$

Lamé equation (see Moon and Spencer [119, page 157]):

$$y'' + \frac{1}{2}\left[\frac{1}{x-a_1} + \frac{1}{x-a_2} + \frac{1}{x-a_3}\right]y' + \frac{1}{4}\left[\frac{A_0 + A_1 x}{(x-a_1)(x-a_2)(x-a_3)}\right]y = 0$$

Lamé equation (see Ward [167]):

$$y'' + (h - n(n+1)k^2 \operatorname{sn}^2 x)y = 0$$

Lamé equation – wave (see Moon and Spencer [119, page 157]):

$$y'' + \frac{1}{2}\left[\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x-b}\right]y' + \frac{1}{4}\left[\frac{(a^2+b^2)q - p(p+1)x + \kappa x^2}{x(x-a)(x-b)}\right]y = 0$$

Lane–Emden equation (see Seshadri and Na [147, page 193]):

$$y'' + \frac{2}{x}y' + y^k = 0$$

Legendre equation (see Abramowitz and Stegun [3, Section 8.1.1]):

$$(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$$

Legendre equation – wave (see Moon and Spencer [119, page 155]):

$$(1 - x^2)y'' - 2xy' - \left[k^2a^2(x^2 - 1) - p(p+1) - \frac{q^2}{x^2-1}\right]y = 0$$

Lewis regulator (see Hagedorn [71, page 152]):

$$y'' + (1 - |y|)y' + y = 0$$

Liénard's equation (see Villari [163]):

$$y'' + f(y)y' + y = 0$$

Liouville's equation (see Goldstein and Braun [68, page 98]):

$$y'' + g(y)(y')^2 + f(x)y' = 0$$

Lommel functions (see Gradshteyn and Ryzhik [69, page 986]):

$$x^2y'' + xy' + (x^2 - \nu^2)y = x^{\mu+1}$$

Magnetic pole equation (see Infeld and Hull [86]):

$$y'' - \left[\frac{m(m+1) + \frac{1}{4} - \left(m + \frac{1}{2}\right)\cos x}{\sin^2 x} + \left(\lambda + \frac{1}{2}\right)\right]y = 0$$

Mathieu equation (see Abramowitz and Stegun [3, Section 20.1.1]):

$$y'' + (a - 2q\cos 2x)y = 0$$

Mathieu equation – associated (see Ince [85, page 503]):

$$y'' + [(1-2r)\cot x]\, y' + (a + k^2\cos^2 x)y = 0$$

Mathieu equation – modified (see Abramowitz and Stegun [3, Section 20.1.2]):

$$y'' - (a - 2q\cosh 2x)y = 0$$

Morse–Rosen equation (see Barut *et al.* [18]):

$$y'' + \left[\frac{\alpha}{\cosh^2 ax} + \beta\tanh ax + \gamma\right] y = 0$$

Neumann's polynomials (see Gradshteyn and Ryzhik [69, page 990]):

$$x^2y'' + 3xy' + (x^2 + 1 - n^2)y = x\cos^2\frac{n\pi}{2} + n\sin^2\frac{n\pi}{2}$$

Painlevé transcendent – first (see Ince [85, page 345]):

$$y'' = 6y^2 + x$$

Painlevé transcendent – second (see Ince [85, page 345]):

$$y'' = 2y^3 + xy + a$$

Painlevé transcendent – third (see Ince [85, page 345]):

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{x}y' + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

Painlevé transcendent – fourth (see Ince [85, page 345]):

$$y'' = \frac{1}{2y}(y')^2 + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}$$

Painlevé transcendent – fifth (see Ince [85, page 345]):

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{x}y' + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}$$

Painlevé transcendent – sixth (see Ince [85, page 345]):

$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right]$$

Painlevé–Ince – modified (see Abraham-Shrauner [2]):

$$y'' + \sigma yy' + \beta y^3$$

Parabolic cylinder equation (see Abramowitz and Stegun [3, Section 19.1.1]):

$$y'' + (ax^2 + bx + c)y = 0$$

Pinney equation (see Common *et al.* [50, page 908]):

$$y'' + f(x)y + cy^{-3}$$

Poisson–Boltzmann equation (see Chambré [39]):

$$y'' + \frac{k}{x}y' = -\delta e^y$$

Pöschl–Teller equation – first (see Barut *et al.* [18]):

$$y'' - \left[a^2\left(\frac{\kappa(\kappa-1)}{\sin^2 ax} + \frac{\lambda(\lambda-1)}{\cos^2 ax}\right) - b^2\right] y = 0$$

Pöschl–Teller equation – second (see Barut *et al.* [18]):

$$y'' - \left[a^2\left(\frac{\kappa(\kappa-1)}{\sinh^2 ax} + \frac{\lambda(\lambda-1)}{\cosh^2 ax}\right) - b^2\right] y = 0$$

Polytropic differential equation (see Iyanaga and Kawada [87, page 908]):

$$(x^2y')' = -x^2y^n$$

Rayleigh equation (see Birkhoff and Rota [24, page 134]):

$$y'' - \mu\left[1 - (y')^2\right] y' + y = 0$$

Riccati–Bessel equation (see Abramowitz and Stegun [3, Section 10.3.1]):

$$x^2y'' + \left[x^2 - n(n+1)\right] y = 0$$

Richardson's equation (see Binding and Volkmer [23]):

$$-y'' = (\lambda \operatorname{sgn} x + \mu)y$$

Riemann's differential equation (see Abramowitz and Stegun [3, Section 15.6.1]):

$$\begin{aligned} y'' &+ \left[\frac{1-\alpha-\alpha'}{x-a} + \frac{1-\beta-\beta'}{x-b} + \frac{1-\gamma-\gamma'}{x-c}\right] y' \\ &+ \left[\frac{\alpha\alpha'(a-b)(a-c)}{x-a} + \frac{\beta\beta'(b-c)(b-a)}{x-b} + \frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right] \\ &\qquad \times \frac{y}{(x-a)(x-b)(x-c)} = 0 \end{aligned}$$

Spheroidal wave functions (oblate) (see Abramowitz and Stegun [3, Section 21.6.4]):

$$\left[(1-x^2)y'\right]' + \left(\lambda + c^2x^2 - \frac{m^2}{1-x^2}\right) y = 0$$

Spheroidal wave functions radial (see Abramowitz and Stegun [3, Section 21.6.3]):

$$\left[(1+x^2)y'\right]' - \left(\lambda - c^2x^2 - \frac{m^2}{x^2+1}\right) y = 0$$

Struve functions (see Abramowitz and Stegun [3, Section 12.1.1]):

$$x^2y'' + xy' + (x^2 - \nu^2)y = \frac{4\left(\frac{x}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}$$

Symmetric top equation (see Infeld and Hull [86]):

$$y'' - \left[\frac{M^2 - \frac{1}{4} + K^2 - 2MK\cos x}{\sin^2 x} + \left(\sigma + K^2 + \tfrac{1}{4}\right)\right] y = 0$$

Tchebycheff equation (see Abramowitz and Stegun [3, Section 22.6.9]):

$$(1 - x^2)y'' - xy' + n^2y = 0$$

Thomas–Fermi equation (see Bender and Orszag [20, page 25]):

$$y'' = y^{3/2}x^{-1/2}$$

Titchmarsh's equation (see Hille [80, page 617]):

$$y'' + \left(\lambda - x^{2n}\right) y = 0$$

Ultraspherical equation (see Abramowitz and Stegun [3, Section 22.6.5]):

$$(1 - x^2)y'' - (2a + 1)xy' + n(n + 2a)y = 0$$

Van der Pol equation (see Birkhoff and Rota [24, page 134]):

$$y'' - \mu(1 - y^2)y' + y = 0$$

Wangerin equation (see Moon and Spencer [119, page 157]):

$$y'' + \frac{1}{2}\left[\frac{1}{x-a_1} + \frac{1}{x-a_2} + \frac{2}{x-a_3}\right] y' + \frac{1}{4}\left[\frac{A_0 + A_1x + A_2x^2}{(x-a_1)(x-a_2)(x-a_3)^2}\right] y = 0$$

Weber equation (see Moon and Spencer [119, page 153]):

$$y'' + \left(a^2 - \frac{b^2}{4}x^2\right) y = 0$$

Weber functions (see Gradshteyn and Ryzhik [69, page 989]):

$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = -\frac{1}{\pi x^2}\left[x + \nu + (x - \nu)\cos\nu\pi\right]$$

Whittaker's equation (see Abramowitz and Stegun [3, equation 13.1.31]):

$$y'' + \left(-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right) y = 0$$

Whittaker–Hill equation (see Urwin and Arscott [159]):

$$y'' + (A + B\cos 2x + C\cos 4x)y = 0$$

Unnamed equation (see Chrisholm and Common [45]):

$$y'' + (a_0 + a_1y)y' + b_0 + b_1y + b_2y^2 + b_3y^3 = 0$$

Unnamed equation (see Gilding [65]):

$$y'' = -\lambda y^p$$

Unnamed equation (see Latta [101]):
$(1-x^2)y''-2axy'+(b+cx^2)y=0$

Unnamed equation (see Leach *et al.* [103]):
$y''+yy'+\beta y^3=0$

Unnamed equation (see Rubel [140]):
$xyy''+yy'-x(y')^2=0$

Unnamed equation (see Setoyanagi [148]):
$y''+(ax^p+bx^q)y=0$

Unnamed equation (see Tsukamoto [158]):
$y''+e^{at}y^b=0$

44.1.3 Higher Order Equations

Products of Airy functions (see Abramowitz and Stegun [3, equation 10.4.57]):
$y'''-4xy'-2y=0$

Blasius equation (see Meyer [114, page 127]):
$y'''+yy''=0$

Falkner–Skan equation (see Cebeci and Keller [38]):
$y'''+yy''+\beta\left[1-(y')^2\right]=0$

Generalized hypergeometric equation (see Miller [117, page 271]):
$\left(x\frac{d}{dx}+a_1\right)\cdots\left(x\frac{d}{dx}+a_p\right)-\frac{d}{dx}\left(x\frac{d}{dx}+b_1\right)\cdots\left(x\frac{d}{dx}+b_q\right)y=0$

Laplace equations (see Valiron [161, pages 306–315]):
$(a_0x+b_0)y^{(n)}+(a_1x+b_1)y^{(n-1)}+\cdots+(a_nx+b_n)y=0$

Sixth order Onsager equation (see Viecelli [162]):
$\left(e^x\left(e^xy_x\right)_{xx}\right)_{xxx}=f(x)$

Orr–Sommerfeld equation (see Herron [77]):
$\frac{1}{i\alpha R}\left(\frac{d^2}{dx^2}-\alpha^2\right)^2y-\left[(f(x)-c)\left(\frac{d^2}{dx^2}-\alpha^2\right)-f''(x)\right]y=0$

Unnamed equation (see Benguria and Depassier [21]):
$\lambda y'''+y'=f(y)$

Unnamed equation (see Hershenov [78]):
$y'''+a\ xy'+by=0$

Unnamed equation (see Merkin [113]):
$y''' + yy' + \lambda(y')^2 = 0$

Unnamed equation (see Pfeiffer [129]):
$y''' + q(x)y' + r(x)y = 0$

Unnamed equation (see Walker [164]):
$[(ry'')' - py']' + qy = \sigma y$

Unnamed equation (see Watson [168, page 106]):
$y^{(m)} = axy^{-m/2}$

44.2 Partial Differential Equations

44.2.1 Linear Equations

Biharmonic equation (see Kantorovich and Krylov [92, pages 595–615]):
$\nabla^4 u = 0$

Linear Boussinesq equation (see Whitham [170, page 9]):
$u_{tt} - a^2 u_{xx} = b^2 u_{xxtt}$

Busemann equation (see Chaohao [42]):
$(1 - x^2)u_{xx} - 2xyu_{xy} + (1 - y^2)u_{yy} + 2a(xu_x + yu_y) - a(a+1)u = 0$

Chaplygin's equation (see Landau and Lifshitz [99, page 432]):
$u_{xx} + \frac{y^2}{1-y^2/c^2} u_{yy} + yu_y = 0$

Diffusion equation (see Morse and Feshback [122, page 271]):
$\nabla \cdot (\kappa(\mathbf{x}, t) \nabla u) = u_t$

Euler–Darboux equation (see Miller [116]):
$u_{xy} + \frac{1}{x-y}(au_x - bu_y) = 0$

Euler–Poisson–Darboux equation (see Ames [9, Section 3.3]):
$u_{xy} + \frac{N}{x+y}(u_x + u_y) = 0$

Helmholtz equation (see Morse and Feshback [122, page 271]):
$\nabla^2 u + k^2 u = 0$

Klein–Gordon equation (see Morse and Feshback [122, page 272]):
$\nabla^2 u - \frac{1}{c^2} u_{tt} = \mu^2 u$

Kramers equation (see Duck *et al.* [54]):
$P_t = P_{xx} - uP_x + \frac{\partial}{\partial u}\left[(u - F(x))P\right]$

Lambropoulos's equation (see Wilcox [171]):

$u_{xy} + axu_x + byu_y + cxyu + u_t = 0$

Laplace's equation (see Morse and Feshback [122, page 271]):

$\nabla^2 u = 0$

Lavrent'ev–Bitsadze equation (see Chang [41]):

$u_{xx} + (\operatorname{sgn} y)u_{yy} = f(x, y)$

Onsager equation (see Wood and Morton [172]):

$\left(e^x \left(e^x u_{xx}\right)_{xx}\right)_{xx} + B^2 u_{yy} = F(x, y)$

Poisson equation (see Morse and Feshback [122, page 271]):

$\nabla^2 u = -4\pi\rho(\mathbf{x})$

Schröedinger equation (see Morse and Feshback [122, page 272]):

$-\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{x})u = i\hbar u_t$

Spherical harmonics in three dimensions (see Humi [84]):

$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right] Y_{l,m} = 0$

Spherical harmonics in four dimensions (see Humi [84]):

$u_{xx} + 2(\cot x)u_x + \frac{1}{\sin^2 x}\left(u_{yy} + (\cot y)u_y + \frac{1}{\sin^2 y}u_{zz}\right) + (n^2 - 1)u = 0$

Tricomi equation (see Manwell [110]):

$u_{yy} = yu_{xx}$

Wave equation (see Morse and Feshback [122, page 271]):

$u_{tt} = c^2\nabla^2 u$

Weinstein equation – generalized (see Akin [6]):

$\nabla^2 u + \frac{p}{x_{n-1}}u_{x_{n-1}} + \frac{q}{x_n}u_{x_n} = 0$

44.2.2 Second Order Nonlinearity

Benjamin–Bona–Mahony equation (see Avrin and Goldstein [15]):

$u_t - u_{xxx} + uu_x = 0$

Boussinesq equation (see Calogero and Degasperis [34, page 54]):

$u_{tt} - u_{xx} - u_{xxxx} + 3(u^2)_{xx} = 0$

Burgers equation (see Benton and Platzman [22]):

$u_t + uu_x = \nu u_{xx}$

Burgers equation – non-planar (see Sachdev and Nair [142]):

$u_t + uu_x + \frac{Ju}{2t} = \frac{\delta}{2} u_{xx}$

Burgers equation – generalized (see Oliveri [128]):

$u_t + uu_x - u_{xx} + f(t)u = 0$

Ernst equation (see Calogero and Degasperis [34, page 62]):

$(\Re u)\left(u_{rr} + \frac{u_r}{r} + u_{zz}\right) = u_r^2 + u_z^2$

Fisher's equation (see Kaliappan [89]):

$u_t = Du_{xx} + u - u^2$

Convective Fisher's equation (see Shönborn *et al.* [151]):

$u_t = \frac{1}{2} u_{xx} + u(1-u) - \mu u u_x$

Kadomtsev–Petviashvili equation (see Latham [100]):

$(u_t + u_{xxx} - 6uu_x)_x \pm u_{yy} = 0$

Generalized Kadomtsev–Petviashvili–Burgers equation (see Brugarino [31]):

$\left(u_t + \frac{J}{2t}u + J_1 uu_x + J_2 u_{xx} + J_3 u_{xxx}\right)_x + J_4(t) u_{yy} = 0$

Khokhlov–Zabolotskaya equation (see Chowdhury and Nasker [44]):

$u_{xt} - (uu_x)_x = u_{yy}$

Korteweg–de Vries equation (KdV) (see Lamb [98, Chapter 4]):

$u_t + u_{xxx} - 6uu_x = 0$

KdV equation – cylindrical (see Calogero and Degasperis [34, page 50]):

$u_t + u_{xxx} - 6uu_x + \frac{u}{2t} = 0$

KdV equation – generalized (see Boyd [29]):

$u_t + uu_x - u_{xxxxx} = 0$

KdV equation – spherical (see Calogero and Degasperis [34, page 51]):

$u_t + u_{xxx} - 6uu_x + \frac{u}{t} = 0$

KdV equation – transitional (see Calogero and Degasperis [34, page 50]):

$u_t + u_{xxx} - 6f(t)uu_x = 0$

KdV equation – variable coefficient (see Nimala *et al.* [125]):

$u_t + at^n uu_x + bt^m u_{xxx} = 0$

Korteweg–de Vries–Burgers equation (KdVB) (see Canosa and Gazdag [35]):

$u_t + 2uu_x - \nu u_{xx} + \mu u_{xxx} = 0$

Kuramoto–Sivashinksy equation (see Michelson [115]):

$$u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2}|\nabla^2 u|^2 = 0$$

Lin–Tsien equation (see Ames and Nucci [10]):

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0$$

Regularized long-wave equation (RLW) (see Calogero and Degasperis [34, page 49]):

$$u_t + u_x - 6uu_x - u_{txx} = 0$$

Generalized shallow water wave equation (GSWW) (see Clarkson and Mansfield [48]):

$$u_{xxxt} + au_x u_{xt} + bu_t u_{xx} - u_{xt} - u_{xx} = 0$$

Thomas equation (see Rosales [135]):

$$u_{xy} + au_x + bu_y + cu_x u_y = 0$$

Unnamed equation (see Rosen [136]):

$$u_{tt} + 2uu_t - u_{xx} = 0$$

44.2.3 Higher Order/Variable Order Nonlinearities

Affinsphären equation (see Schief and Rogers [146]):

$$\left(\frac{R_u}{R^2 v^2}\right)_u = \left(\frac{R^2 R_v}{v^2}\right)_v$$

Generalized Benjamin–Bona–Mahony equation (see Goldstein and Wichnoski [67]):

$$u_t - \nabla^2 u_t + \nabla \cdot \boldsymbol{\phi}(u)) = 0$$

Benney equation (see Balmforth *et al.* [16]):

$$u_t + (u^n)_x = -u_{xx} - \mu u_{xxx} - u_{xxxx}$$

Born–Infeld equation (see Whitham [170, page 617]):

$$\left(1 - u_t^2\right) u_{xx} + 2u_x u_t u_{xt} - \left(1 + u_x^2\right) u_{tt} = 0$$

Boussinesq equation – modified (see Clarkson [46]):

$$\tfrac{1}{3} u_{tt} - u_t u_{xx} - \tfrac{3}{2} u_x^2 u_{xx} + u_{xxxx} = 0$$

Boussinesq equation – modified (see Clarkson [47]):

$$u_{tt} - u_t u_{xx} - \tfrac{1}{2} u_x^2 u_{xx} + u_{xxxx} = 0$$

Buckmaster equation (see Hill and Hill [79]):

$$u_t = \left(u^4\right)_{xx} + \left(u^3\right)_x$$

Generalized Burgers equation (see Sachdev *et al.* [141]):

$$u_t + u^n u_x + \left(\tfrac{j}{2t} + \alpha\right) u + \left(\beta + \tfrac{\gamma}{x}\right) u^{n+1} = \tfrac{\delta}{2} u_{xx}$$

Generalized Burgers–Huxley equation (see Wang *et al.* [166]):

$$u_t - \alpha u^\delta u_x - u_{xx} = \beta u \left(1 - u^\delta\right)\left(u^\delta - \gamma\right)$$

Cahn–Hilliard equation (see Novick-Cohen and Segel [126]):

$$u_t = \nabla \cdot \left[M(u) \nabla \left(\frac{\partial f}{\partial u} - K \nabla^2 u\right)\right]$$

Calogero–Degasperis–Fokas equation (see Gerdt *et al.* [63]):

$$u_{xxx} - \tfrac{1}{8} u_x^3 + u_x \left(Ae^u + Be^{-u}\right) = 0$$

Caudrey–Dodd–Gibbon–Sawada–Kotera equation (see Aiyer *et al.* [5]):

$$u_t + u_{xxxxx} + 30 u u_{xxx} + 30 u_x u_{xx} + 180 u^2 u_x = 0$$

Clairaut's equation (see Iyanaga and Kawada [87, page 1446]):

$$u = x u_x + y u_y + f(u_x, u_y)$$

Inhomogenous nonlinear diffusion equation (see Saied and Hussein [143]):

$$x^p u_t = (x^m u^n u_x)_x$$

Nonlinear diffusion equation (see King [96]):

$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right)$

Nonlinear diffusion equation (see King [96]):

$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2/3}\frac{\partial u}{\partial x}\right)$

Eckhaus partial differential equation (see Kundu [97]):

$iu_t + u_{xx} + 2\left(|u|^2\right)_x u + |u|^4 u = 0$

Fisher equation – generalized (see Wang [165]):

$u_t - u_{xx} - \frac{m}{u}u_x^2 = u\,(1 - u^\alpha)$

Fisher equation – generalized (see Kaliappan [89]):

$u_t = u_{xx} + u - u^k$

Fisher equation – generalized (see Herrera *et al.* [76]):

$u_t = u_{xx} + u^p - u^{2p-1}$

Gardner equation (see Tabor [155, page 289]):

$u_t = 6(u + a^2u^2)u_x + u_{xxx}$

Ginzburg–Landau equation (see Katou [94]):

$u_t = (1 + ia)u_{xx} + (1 + ic)u - (1 + id)|u|^2u$

Quintic Ginzburg–Landau equation (see Marcq *et al.* [111]):

$A_t = \epsilon A + \alpha_1 A_{xx} - \alpha_3|A|^2A - \alpha_4|A|^4A$

Hamilton–Jacobi equation (see page 61):

$V_t + H(t, \mathbf{x}, V_{x_1}, \ldots, V_{x_n}) = 0$

Harry Dym equation (see Calogero and Degasperis [34, page 53]):

$u_t = u_{xxx}u^3$

Generalized axially symmetric Helmholtz equation (GASHE) (see Lowndes [109, page 96]):

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\alpha}{y}\frac{\partial u}{\partial y} + k^2u = 0$

Generalized biaxially symmetric Helmholtz equation in $(n+1)$ variables (GASHEN) (see Lowndes [109, page 93]):

$\sum_{i=1}^n \frac{\partial^2 u}{\partial {x_i}^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\alpha}{y}\frac{\partial u}{\partial y} + k^2u = 0$

Generalized biaxially symmetric Helmholtz equation (GBSHE) (see Lowndes [109, page 91]):

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\alpha}{x}\frac{\partial u}{\partial x} + \frac{2\beta}{y}\frac{\partial u}{\partial y} + k^2u = 0$

Hirota equation (see Calogero and Degasperis [34, page 56]):

$$u_t + iau + ib(u_{xx} - 2\eta|u^2|u) + cu_x + d(u_{xxx} - 6\eta|u|^2 u_x) = 0$$

Kadomtsev–Petviashvili equation – modified (see Clarkson [46]):

$$u_{xt} = u_{xxx} + 3u_{yy} - 6u_x^2 u_{xx} - 6u_y u_{xx}$$

KdV equation – deformed (see Dodd and Fordy [53]):

$$u_t + \left(u_{xx} - 2\eta u^3 - \tfrac{3}{2}\tfrac{uu_x^2}{\eta + u^2}\right)_x = 0$$

KdV equation – generalized (see Rammaha [133]):

$$u_t + uu_x + p|u|^{p-1}u_x = 0$$

KdV equation – modified (mKdV) (see Calogero and Degasperis [34, page 51]):

$$u_t + u_{xxx} \pm 6u^2 u_x = 0$$

KdV equation – modified modified (see Dodd and Fordy [53]):

$$u_t + u_{xxx} - \tfrac{1}{8}u_x^3 + u_x\left(Ae^{au} + B + Ce^{-au}\right) = 0$$

KdV equation – Schwarzian (see Weiss [169]):

$$\tfrac{u_t}{u_x} + \{u; x\} = \lambda$$

Klein–Gordon equation – nonlinear (see Matsuno [112]):

$$\nabla^2 u + \lambda u^p = 0$$

Klein–Gordon equation – quasilinear (see Nayfeh [124, page 76]):

$$u_{tt} - a^2 u_{xx} + c^2 u = bu^3$$

Kupershmidt equation (see Fuchssteiner *et al.* [59]):

$$u_t = u_{xxxxx} + \tfrac{5}{2}u_{xxx}u + \tfrac{25}{4}u_{xx}u_x + \tfrac{5}{4}u^2 u_x$$

Liouville equation (see Matsuno [112]):

$$\nabla^2 u + e^{\lambda u} = 0$$

Liouville equation (see Calogero and Degasperis [34, page 60]):

$$u_{xt} = e^{\eta u}$$

Molenbroek's equation (see Cole and Cook [49, page 34]):

$$\nabla^2\phi = M_\infty^2 \left\{ \phi_x^2\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + \phi_y^2\phi_{yy} + \frac{\gamma - 1}{2}\left(\phi_x^2 + \phi_y^2 - 1\right) \times \left(\phi_{xx} + \phi_{yy} + \epsilon\frac{\phi_y}{y}\right)\right\}$$

Monge–Ampère equation (see Moon and Spencer [121, page 171]):

$$(u_{xy})^2 - u_x u_y = f(x, y, u, u_x, u_y)$$

Monge–Ampère equation (see Gilbarg and Trudinger [64]):

$$\begin{vmatrix} u_{x_1x_1} & u_{x_1x_2} & \cdots & u_{x_1x_n} \\ u_{x_2x_1} & u_{x_2x_2} & \cdots & u_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_nx_1} & u_{x_nx_2} & \cdots & u_{x_nx_n} \end{vmatrix} = f(u, \mathbf{x}, \nabla u)$$

Nagumo equation (see Zhi-Xiong and Ben-Yu [174]):

$$u_t = u_{xx} + u(u-a)(1-u)$$

Phi–four equation (see Calogero and Degasperis [34, page 60]):

$$u_{tt} - u_{xx} - u + u^3 = 0$$

Plateau's equation (see Bateman [19, page 501]):

$$(1 + u_x^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{yy} = 0$$

Porous-medium equation (see Elliot, Herrero, King, and Ockendon [55]):

$$u_t = \nabla \cdot (u^m \nabla u)$$

Generalized axially symmetric potential equation (GASPE) (see Lowndes [109, page 95]):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\alpha}{y}\frac{\partial u}{\partial y} = 0$$

Generalized biaxially symmetric potential equation (GBSPE) (see Lowndes [109, page 91]):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\alpha}{x}\frac{\partial u}{\partial x} + \frac{2\beta}{y}\frac{\partial u}{\partial y} = 0$$

Generalized biaxially symmetric potential equation in $(n+1)$ variables (GASPEN) (see Lowndes [109, page 92]):

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial {x_i}^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\alpha}{y}\frac{\partial u}{\partial y} = 0$$

Rayleigh wave equation (see Hall [74]):

$$u_{tt} - u_{xx} = a(u_t - u_t^3)$$

Sawada–Kotera equation (see Matsuno [112, page 7]):

$$u_t + 45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx} = 0$$

Schröedinger equation – logarithmic (see Cazenave [37]):

$$iu_t + \nabla^2 u + u \log |u|^2 = 0$$

Schröedinger equation – derivative nonlinear (see Calogero and Degasperis [34, page 56]):

$$iu_t + u_{xx} \pm i\left(|u|^2 u\right)_x = 0$$

Schröedinger equation – derivative nonlinear (see Hayashi and Ozawa [75]):

$$i\partial_t\psi + \partial_x\psi = i\lambda\partial_x(|\psi|^2\psi) + \lambda_1|\psi|^{p_1-1} + \lambda_2|\psi|^{p_2-1}$$

Schröedinger equation – nonlinear (see Calogero and Degasperis [34, page 56]):

$$iu_t + u_{xx} \pm 2|u|^2 u = 0$$

Sine–Gordon equation (see Calogero and Degasperis [34, page 59]):

$$u_{xx} - u_{yy} \pm \sin u = 0$$

Sine–Gordon equation – damped (see Levi *et al.* [105]):

$$u_{tt} + \sigma u_t - u_{xx} + \sin u = 0$$

Sine–Gordon equation – double (see Calogero and Degasperis [34, page 60]):

$$u_{xt} \pm \left[\sin u + \eta \sin\left(\tfrac{u}{2}\right)\right] = 0$$

Sine–Gordon – multidimensional (see Elzoheiry *et al.* [56]):

$$u_{rr} + \tfrac{m-1}{r}u_r - u_{tt} = \sin u$$

Sinh–Gordon equation (see Grauel [70]):

$$u_{xt} = \sinh u$$

Sinh–Poisson equation (see Ting *et al.* [156]):

$$\nabla^2 u + \lambda^2 \sinh u = 0$$

Strongly damped wave equation (see Ang and Dinh [12]):

$$u_{tt} - \nabla^2 u - \nabla^2 u_t + f(u) = 0$$

Tzitzeica equation (see Schief [145]):

$$u_{xy} = e^u - e^{-2u}$$

Unnamed equation (see Aguirre and Escobedo [4]):

$$u_t - \nabla^2 u = u^p$$

Unnamed equation (see Bluman and Kumei [25]):

$$u_t - \tfrac{\partial}{\partial x}\left[\tfrac{au_x}{(u+b)^2}\right] = 0$$

Unnamed equation (see Calogero [32]):

$$u_{xt} + uu_{xx} + F(u_x) = 0$$

Unnamed equation (see Calogero [33]):

$$u_t = u_{xxx} + 3(u_{xx}u^2 + 3u_x^2 u) + 3u_x u^4$$

Unnamed equation (see Daniel and Sahadevan [51]):

$$u_t = u_{xxx} + u^2 u_{xx} + 3uu_x^2 + \tfrac{1}{3}u^4 u_x$$

Unnamed equation (see Fujita [61]):

$$u_t = \nabla^2 u + e^u$$

Unnamed equation (see Fung and Au [62]):

$$u_t + u_{xxx} - 6u^2 u_x + 6\lambda u_x = 0$$

Unnamed equation (see Lin [107]):

$$\nabla^2 u + Ae^{-u} = 0$$

Unnamed equation (see Lindquist [108]):

$$\nabla \cdot (|\nabla u|^p \nabla u) = f$$

Unnamed equation (see Roy and Chowdhury [139]):

$$-iu_t + u_{xx} + \tfrac{2|u_x|^2 u}{1-uu^*} = 0$$

Unnamed equation (see Shivaji [150]):

$$-\nabla^2 u = \lambda \exp\left(\tfrac{\alpha u}{\alpha + u}\right)$$

Unnamed equation (see Trubek [157]):

$$\nabla^2 u + Ku^\sigma = 0$$

Unnamed equation (see Yanagida [173]):

$$\nabla^2 u + K|x||u|^q u = 0$$

Unnamed equation (see Utepbergenov [160]):

$$z^2 u_{zz} + \nabla^2 u + a(z)u = 0$$

Wadati–Konno–Ichikawa–Schimizu equation (see Calogero and Degasperis [34, page 53]):

$$iu_t + \left[\left(1 + |u|^2\right)^{-1/2} u\right]_{xx} = 0$$

Zoomeron equation (see Calogero and Degasperis [34, page 58]):

$$\left(\tfrac{\partial^2}{\partial t^2} - \tfrac{\partial^2}{\partial x^2}\right)\left(\tfrac{u_{xt}}{u}\right) + 2\left(u^2\right)_{xt} = 0$$

44.3 Systems of Differential Equations

44.3.1 Systems of ODEs

Bonhoeffer-van der Pol (BVP) oscillator (see Rajasekar and Lakshmanan [132]):

$$x' = x - \frac{x^3}{3} - y + I(t)$$
$$y' = c(x + a - by)$$

Brusselator (see Hairer *et al.* [73, page 112]):

$$u' = A + u^2 v - (B+1)u$$
$$v' = Bu - u^2 v$$

Full Brusselator (see Hairer *et al.* [73, page 114]):

$$u' = 1 + u^2 v - (w+1)u$$
$$v' = uw - u^2 v$$
$$w' = -uw + \alpha$$

Hamilton's differential equations (see Iyanaga and Kawada [87, page 1005]):

$$\frac{dx_i}{dt} = \quad H_{p_i}(t, \mathbf{x}, \mathbf{p})$$
$$\frac{dp_i}{dt} = -H_{x_i}(t, \mathbf{x}, \mathbf{p})$$

Jacobi elliptic functions (see Hille [80, page 66]):

$$u' = vw$$
$$v' = -uw$$
$$w' = -k^2 uv$$

Kowalevski's top (see Haine and Horozov [72]):

$$\frac{d\mathbf{m}}{dt} = \lambda \mathbf{m} \times \mathbf{m} + \boldsymbol{\gamma} \times \mathbf{l}$$
$$\frac{d\boldsymbol{\gamma}}{dt} = \lambda \boldsymbol{\gamma} \times \mathbf{m}$$

Lorenz equations (see Sparrow [152]):

$$x' = \sigma(y - x)$$
$$y' = rx - y - xz$$
$$z' = xy - bz$$

Lorenz equations – complex (see Flessas [58]):

$$x' = \sigma(y - x)$$
$$y' = rx - y - xz$$
$$z' = -bz + \tfrac{1}{2}(x^* y + x y^*)$$

Lotka–Volterra equations (see Boyce and DiPrima [28, page 494]):

$$u' = u(a - bv)$$
$$v' = v(-c + du)$$

Nahm's equations (see Steeb and Louw [154]):

$$U_t = [V, W]$$
$$V_t = [W, U]$$
$$W_t = [U, V]$$

Toda lattice equation – relativistic (see Ohta *et al.* [127]):

$$\ddot{x}_n = \left(1 + \frac{\dot{x}_{n-1}}{c}\right)\left(1 + \frac{\dot{x}_n}{c}\right)\frac{\exp(x_{n-1} - x_n)}{1 + (1/c^2)\exp(x_{n-1} - x_n)} - \left(1 + \frac{\dot{x}_n}{c}\right)\left(1 + \frac{\dot{x}_{n+1}}{c}\right)\frac{\exp(x_n - x_{n+1})}{1 + (1/c^2)\exp(x_n - x_{n+1})}$$

Toda molecule equation – cylindrical (see Hirota and Nakamura [83]):

$$\left(\partial_{rr} + r^{-1}\partial_r\right)\log V_n - V_{n+1} + 2V_n - V_{n-1} = 0$$

Unnamed equation (see Steeb [153, page 57]):

$$\mathbf{u}_{tt} + c_1|\mathbf{u}|^n\ (\mathbf{u} \times \mathbf{u}_t) + c_2|\mathbf{u}|^m\mathbf{u} = 0$$

44.3.2 Systems of PDEs

Affine Knizhnik–Zamolodchikov equation (see Cherednik [43]):

$$\frac{\partial\phi(\mathbf{z})}{\partial z_i} = k\sum_j \frac{s_{ij}\phi(\mathbf{z})}{z_i - z_j}$$

Beltrami equation (see Iyanaga and Kawada [87, page 1087]):

$$f_{\overline{z}} = \mu(z)f_z$$

Boomeron equation (see Calogero and Degasperis [34, page 57]):

$$u_t = \mathbf{b}\cdot\mathbf{v}_x$$
$$\mathbf{v}_{xt} = u_{xx}\mathbf{b} + \mathbf{a}\times\mathbf{v}_x - 2\mathbf{v}\times[\mathbf{v}\times\mathbf{b}]$$

Carleman equation (see Kaper and Leaf [93]):

$$u_t + u_x = v^2 - u^2$$
$$v_t - v_x = u^2 - v^2$$

Cauchy–Riemann equations (see Levinson and Redheffer [106]):

$$u_x - v_y = 0$$
$$u_y + v_x = 0$$

Chiral field equation (see Calogero and Degasperis [34, page 61]):

$$(U^*U_x)_t + (U^*U_t)_x = 0$$

Davey–Stewartson equations (see Champagne and Winternitz [40]):

$$iu_t + u_{xx} + au_{yy} + bu|u|^2 - uw = 0$$
$$w_{xx} + cw_{yy} + d\left(|u|^2\right)_{yy} = 0$$

Dirac equation in $1+1$ dimensions (see Alvarez *et al.* [7]):

$$u_t + v_x + imu + 2i\lambda\left(|u|^2 - |v|^2\right)u = 0$$
$$v_t + u_x + imv + 2i\lambda\left(|v|^2 - |u|^2\right)v = 0$$

Dispersive long-wave equation (see Boiti *et al.* [27]):

$$u_t = (u^2 - u_x + 2w)_x$$
$$w_t = (2uw + w_x)_x$$

Drinfel'd–Sokolov–Wilson equation (see Hirota *et al.* [82]):

$$u_t = 3ww_x$$
$$w_t = 2w_{xxx} + 2uw_x + u_xw$$

Euler equations (see Landau and Lifshitz [99, page 3]):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \operatorname{grad})\mathbf{v} = -\frac{1}{\rho}\operatorname{grad} P$$

Fitzhugh–Nagumo equations (see Sherman and Peskin [149]):

$$u_t = u_{xx} + u(u-a)(1-u) + w$$
$$w_t = \epsilon u$$

Gross–Neveu model (see Calogero and Degasperis [34, page 62]):

$$iu_x^{(n)} = v^{(n)}\sum_{m=1}^{N}\left(v^{(m)*}u^{(m)} + u^{(m)*}v^{(m)}\right)$$
$$iv_t^{(n)} = u^{(n)}\sum_{m=1}^{N}\left(v^{(m)*}u^{(m)} + u^{(m)*}v^{(m)}\right)$$

Heisenberg ferromagnet equation (see Calogero and Degasperis [34, page 56]):

$$\mathbf{s}_t = \mathbf{s}\times\mathbf{s}_{xx}$$

Hirota–Satsuma equation (see Weiss [169]):

$$u_t = \tfrac{1}{2}u_{xxx} + 3uu_x - 6ww_x$$
$$w_t = -w_{xxx} - 3uw_x$$

Von Kármán equations (see Ames and Ames [8]):

$$\nabla^4 u = E\left[w_{xy}^2 - w_{xx}w_{yy}\right]$$
$$\nabla^4 w = a + b\left[u_{yy}w_{xx} + u_{xx}w_{yy} - 2u_{xy}w_{xy}\right]$$

Kaup's equation (see Dodd and Fordy [53]):

$$f_x = 2fgc(x-t)$$
$$g_t = 2fgc(x-t)$$

KdV equation – super (see Kersten and Gragert [95]):

$$u_t = 6uu_x - u_{xxx} + 3ww_{xx}$$
$$w_t = 3u_x w + 6uw_x - 4w_{xxx}$$

Klein–Gordon–Maxwell equations (see Deumens [52]):

$$\nabla^2 s - (|\mathbf{a}|^2 + 1)s = 0$$
$$\nabla^2 \mathbf{a} - \nabla(\nabla \cdot \mathbf{a}) - s^2\mathbf{a} = \mathbf{0}$$

Landau–Lifshitz equation (see Barouch *et al.* [17]):

$$U_t = U \cdot U_{xx} + U \cdot JU$$

Matrix Liouville equation (see Andreev [11]):

$$\left(U_x U^{-1}\right)_t = U$$

Maxwell's equations (see Jackson [88, page 177]):

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \quad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}$$
$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

Reduced Maxwell–Bloch equations (see Calogero and Degasperis [34, page 59]):

$$E_t - v = 0, \qquad q_x + Ev = 0$$
$$r_x + \omega v = 0, \qquad v_x - \omega r - Eq = 0$$

Nambu–Jona Lasinio–Vaks–Larkin model (see Calogero and Degasperis [34, page 62]):

$$iu_x^{(n)} = v^{(n)} \sum_{m=1}^{N} v^{(m)*}u^{(m)}$$
$$iv_t^{(n)} = u^{(n)} \sum_{m=1}^{N} u^{(m)*}v^{(m)}$$

Navier's equation (see Eringen and Suhubi [57]):

$$(\lambda + 2\mu)\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Navier–Stokes equations (see Landau and Lifshitz [99, page 49]):

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\tfrac{\nabla P}{\rho} + \nu\nabla^2\mathbf{u}$$

Pohlmeyer–Lund–Regge model (see Calogero and Degasperis [34, page 61]):

$$u_{xx} - u_{yy} \pm \sin u\cos u + \left(\frac{\cos u}{\sin^3 u}\right)\left(v_x^2 - v_y^2\right) = 0$$

$$\left(v_x\cot^2 u\right)_x = \left(v_y\cot^2 u\right)_y$$

Vector Poisson equation (see Moon and Spencer [118]):

$$\text{✡}\mathbf{A} = -\operatorname{curl}\mathbf{E}$$

Prandtl's boundary layer equations (see Iyanaga and Kawada [87, page 672]):

$$u_t + uu_x + vu_y = U_t + UU_x + \frac{\mu}{\rho}u_{yy}$$

$$u_x + v_y = 0$$

Sigma-model (see Calogero and Degasperis [34, page 61]):

$$\mathbf{v}_{xt} + (\mathbf{v}_x\mathbf{v}_t)\,\mathbf{v} = 0$$

Massive Thirring model (see Calogero and Degasperis [34, page 62]):

$$iu_x + v + u|v|^2 = 0$$

$$iv_t + u + v|u|^2 = 0$$

Toda equation – 3 + 1-dimensional (see Hirota [81]):

$$\nabla^2\log V_n - V_{n+1} + 2V_n - V_{n-1} = 0$$

Unnamed equation (see Salingaros [144]):

$$\nabla\times\mathbf{u} = k\mathbf{u}$$

Veselov–Novikov equation (see Bogdanov [26]):

$$\left(\partial_t + \partial_z^3 + \partial_{\overline{z}}^3\right)v + \partial_z(uv) + \partial_{\overline{z}}(vw) = 0$$

$$\partial_{\overline{z}}u = 3\partial_z v$$

$$\partial_z w = 3\partial_{\overline{z}}v$$

Yang–Mills equation (see Calogero and Degasperis [34, page 62]):

$$(U^*U_t)_t - (U^*U_x)_{\overline{x}} = 0$$

Anti-self-dual Yang–Mills equation (see Ablowitz *et al.* [1]):

$$\tfrac{\partial}{\partial\overline{x}_1}\left(\Omega^{-1}\tfrac{\partial\Omega}{\partial x_1}\right) + \tfrac{\partial}{\partial\overline{x}_2}\left(\Omega^{-1}\tfrac{\partial\Omega}{\partial x_2}\right) = 0$$

Zakharov equations (see Glassey [66]):

$$iE_t + E_{xx} = NE$$

$$N_{tt} - N_{xx} = \frac{\partial^2}{\partial x^2}(|E|^2)$$

44.4 The Laplacian in Different Coordinate Systems

For ease of recognizing an unknown Laplacian (i.e., ∇^2) in a differential equation, we have frequently used the indeterminates $\{x, y, z\}$ instead of the more customary notation for a specific coordinate system. For details on any of these coordinate systems, see Moon and Spencer [120].

1. rectangular $$u_{xx} + u_{yy} + u_{zz}$$
2. cylindrical polar $$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$
3. elliptic cylinder $$\frac{1}{\cosh^2 x - \cos^2 y}\left[u_{xx} + u_{yy}\right] + u_{zz}$$
4. parabolic cylinder $$\frac{1}{x^2 + y^2}\left[u_{xx} + u_{yy}\right] + u_{zz}$$
5. spherical $$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_\psi$$
6. prolate spheroidal $$\frac{1}{\sinh^2 x + \sin^2 y}\left[u_{xx} + \coth x u_x + u_{yy} + \cot y u_y\right] + \frac{1}{\sinh^2 x \sin^2 y}u_{zz}$$
7. oblate spheroidal $$\frac{1}{\cosh^2 x - \sin^2 y}\left[u_{xx} + \tanh x u_x + u_{yy} + \cot y u_y\right] + \frac{1}{\cosh^2 x \sin^2 y}u_{zz}$$
8. parabolic $$\frac{1}{x^2 + y^2}\left[u_{xx} + \frac{1}{x} + u_{yy} + \frac{1}{y}u_y\right] + \frac{1}{x^2y^2}u_{zz}$$
9. conical $$u_{zz} + \frac{2}{z}u_z + \frac{1}{z^2(x^2 - y^2)}\Big\{(x^2 - b^2)(c^2 - x^2)u_{xx} - x[2x^2 - (b^2 + c^2)]u_x + (b^2 - y^2)(c^2 - y^2)u_{yy} - y[2y^2 - (b^2 + c^2)]u_y\Big\}$$
10. logarithmic-cylinder $$(x^2 + y^2)\left[u_{xx} + u_{yy}\right] + u_{zz}$$
11. tangent-cylinder $$(x^2 + y^2)^2\left[u_{xx} + u_{yy}\right] + u_{zz}$$
12. cardioid-cylinder $$(x^2 + y^2)^3\left[u_{xx} + u_{yy}\right] + u_{zz}$$
13. hyperbolic-cylinder $$2\sqrt{(x^2 + y^2)}\left[u_{xx} + u_{yy}\right] + u_{zz}$$
14. rose-cylinder $$2(x^2 + y^2)^{3/2}\left[u_{xx} + u_{yy}\right] + u_{zz}$$
15. Cassinian-oval $$e^{-2x}\sqrt{e^{2x} + 2e^x\cos y + 1}\left[u_{xx} + u_{yy}\right] + u_{zz}$$
16. inverse Cassinian-oval $$e^{-2x}\left(e^{2x} + 2e^x\cos y + 1\right)^{3/2}\left[u_{xx} + u_{yy}\right] + u_{zz}$$
17. Maxwell-cylinder $$\left(e^{2x} + 2e^x\cos y + 1\right)^{-1}\left[u_{xx} + u_{yy}\right] + u_{zz}$$
18. bi-cylinder $$(\cosh x - \cos y)^2\left[u_{xx} + u_{yy}\right] + u_{zz}$$
19. inverse elliptic-cylinder $$\frac{(\cosh^2 x - \sin^2 y)^2}{(\cosh^2 x - \cos^2 y)}\left[u_{xx} + u_{yy}\right] + u_{zz}$$

20. log tan-cylinder $$(\sinh^2 2x + \sin^2 2y)\,[u_{xx} + u_{yy}] + u_{zz}$$
21. log cosh-cylinder
$$\frac{(\cosh^2 x - \sin^2 y)^2}{(\cosh^2 x \sinh^2 x + (\sinh x \cosh x + \sin y \cos y)^2)^4}\,[u_{xx} + u_{yy}] + u_{zz}$$
22. ellipsoidal
$$\frac{\sqrt{(x^2-b^2)(x^2-c^2)}}{(x^2-y^2)(x^2-z^2)}\,\partial_x\left[\sqrt{(x^2-b^2)(x^2-c^2)}u_x\right]$$
$$+\frac{\sqrt{(y^2-b^2)(c^2-y^2)}}{(x^2-y^2)(y^2-z^2)}\,\partial_y\left[\sqrt{(y^2-b^2)(c^2-y^2)}u_y\right]$$
$$+\frac{\sqrt{(b^2-z^2)(c^2-z^2)}}{(x^2-z^2)(y^2-z^2)}\,\partial_z\left[\sqrt{(b^2-z^2)(c^2-z^2)}u_z\right]$$
23. paraboloidal
$$\sqrt{\frac{(x-b)(x-c)}{(x-y)(x-z)}}\,\partial_x\left[\sqrt{(x-b)(x-c)}u_x\right]$$
$$+\sqrt{\frac{(y-b)(y-c)}{(x-y)(z-y)}}\,\partial_y\left[\sqrt{(b-y)(c-y)}u_y\right]$$
$$+\sqrt{\frac{(z-b)(z-c)}{(z-x)(z-y)}}\,\partial_z\left[\sqrt{(b-z)(z-c)}u_z\right]$$

44.5 Parametrized Equations at Specific Values

1. Polyanin and Zaitsev [130, page 29] tabulate solvable cases of the Abel equation $yy' - y = sx + Ax^m$:

m	s
arbitrary	$-\frac{2(m+1)}{(m+3)^2}$
-7	$15/4$
-4	6
$-5/2$	12
-2	0
-2	2
$-5/3$	$-3/16$
$-5/3$	$-9/100$
$-5/3$	$63/4$
$-7/5$	$-5/36$

m	s
-1	0
$-1/2$	$-2/9$
$-1/2$	$-4/25$
$-1/2$	0
$-1/2$	20
0	arbitrary
$1/2$	$-12/49$
2	$-6/25$
2	$6/25$

Solutions are also tabulated for the Abel equations
$yy' - y = sx + \sigma A\left(\alpha x^{1/2} + \beta A + \gamma A^2 x^{-1/2}\right)$ and
$yy' - y = sx + \alpha A x^p + \beta A^2 x^q$.

2. Polyanin and Zaitsev [130, pages 251–254] tabulate solvable cases of $y'' = A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2}$:
   - Solvable two parameter families (arbitrary m_1 and m_2) include $\{n_1 = 0, n_2 = 0\}$, $\{n_1 = -m-3, n_2 = -m_2 - 3\}$, and $\{n_1 = -1/2(m_1+3), n_2 = -1/2(m_2+3)\}$.

- Several solutions are tabulated where one or both of the A_i are specified.
- Solutions are also available for

m_1	m_2	n_1	n_2
1	-3	arbitrary ($n_1 \neq -2$)	0
-7	-7	4	3
-5	-5	2	0
-3	-7	0	1
			3
	-4	0	0
			1
-2	-3	-2	0
		1	0
	-2	-1	-2
$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$	$\frac{10}{3}$
		$-\frac{4}{3}$	$-\frac{10}{3}$
			$-\frac{7}{3}$
		$-\frac{2}{3}$	$-\frac{4}{3}$
		0	$-\frac{2}{3}$
		2	0
			1
$-\frac{3}{2}$	-2	$-\frac{3}{2}$	-2
		0	1
$-\frac{7}{5}$	$-\frac{7}{5}$	$-\frac{8}{5}$	$\frac{13}{5}$
$-\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{5}{3}$	$-\frac{7}{3}$
		0	1
$-\frac{3}{5}$	$-\frac{7}{5}$	$-\frac{12}{5}$	$-\frac{13}{5}$
		0	1
$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$	$-\frac{7}{2}$
$-\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{8}{3}$	$-\frac{10}{3}$
		$-\frac{8}{3}$	$-\frac{7}{3}$
		$-\frac{8}{3}$	$-\frac{4}{3}$
		0	0
			1
			2

m_1	m_2	n_1	n_2
0	-2	-3	-2
		0	1
	-1	-3	-2
		0	0
	$-\frac{2}{3}$	-3	$-\frac{7}{3}$
		0	0
	$-\frac{1}{2}$	-4	$-\frac{5}{2}$
		-3	$-\frac{7}{2}$
			$-\frac{5}{2}$
			-2
			$-\frac{1}{2}$
		$-\frac{5}{3}$	$-\frac{7}{6}$
		$-\frac{3}{2}$	$-\frac{5}{2}$
		$-\frac{3}{2}$	-2
			$-\frac{1}{2}$
			0
		$-\frac{4}{3}$	$\frac{4}{3}$
		0	-2
			$-\frac{1}{2}$
			0
$\frac{1}{3}$	$-\frac{5}{3}$	$-\frac{10}{3}$	$-\frac{7}{3}$
		0	1
1	-3	-5	0
		1	0
	0	-5	-3
		1	0
2	0	5	-4
			-3
		$-\frac{20}{7}$	$-\frac{13}{7}$
			$-\frac{12}{7}$
		$-\frac{15}{7}$	$-\frac{9}{7}$
		0	0
			1
3	1	-6	-5
		0	1
	2	$-\frac{18}{5}$	$-\frac{14}{5}$
		$-\frac{12}{5}$	$-\frac{11}{5}$

3. Polyanin and Zaitsev [130, pages 304–306] tabulate solvable cases of the modified Emden–Fowler equation $xy'' - ky' = Ax^{n+1}y^m$:
   - Solvable two parameter families include $\{k = {}^{n}\!/_{2}, m \neq -1, n \neq -2\}$, $\{k = -{}^{n+m+3}\!/_{m+1}, m \neq -1, n \neq -2\}$, and $\{k = -{}^{2n+m+3}\!/_{1-m}, m \neq -1, n \neq -2\}$.
   - Solvable one parameter families (with $n = -2$) include $\{k = -1\}$, $\{m = -2\}$, $\{m = -1, k \neq -1\}$, and $\{m = -{}^{1}\!/_{2}, k \neq -1\}$.
   - Solvable one parameter families (with $n \neq -2$) include

$\{m = -7, k = \frac{1}{3}(n-1)\}$	$\{m = -\frac{7}{5}, k = -\frac{1}{3}(5n+13)\}$
$\{m = -7, k = \frac{1}{5}(n-3)\}$	$\{m = -1, k = n+1\}$
$\{m = -4, k = \frac{1}{2}n\}$	$\{m = -1, k = \frac{1}{2}n\}$
$\{m = -4, k = \frac{1}{3}(n-1)\}$	$\{m = -\frac{1}{2}, k = -2n-5\}$
$\{m = -2, k = \frac{1}{3}(n-1)\}$	$\{m = -\frac{1}{2}, k = \frac{1}{2}n)\}$
$\{m = -\frac{5}{2}, k = \frac{1}{2}n\}$	$\{m = -\frac{1}{2}, k = \frac{1}{2}(3n+4)\}$
$\{m = -\frac{5}{2}, k = \frac{1}{3}(2n+1)\}$	$\{m = -\frac{1}{2}, k = \frac{1}{3}(n-1)\}$
$\{m = -\frac{5}{3}, k = -3n-7\}$	$\{m = -\frac{1}{2}, k = \frac{1}{3}(2n+1)\}$
$\{m = -\frac{5}{3}, k = \frac{1}{2}n\}$	$\{m = -\frac{1}{2}, k = -\frac{1}{3}(2n+7)\}$
$\{m = -\frac{5}{3}, k = \frac{1}{2}(3n+4)\}$	$\{m = -\frac{1}{2}, k = \frac{1}{5}(6n+7)\}$
$\{m = -\frac{5}{3}, k = \frac{1}{3}(n-1)\}$	$\{m = \frac{1}{2}, k = \frac{1}{2}n\}$
$\{m = -\frac{5}{3}, k = \frac{1}{3}(2n+1)\}$	$\{m = \frac{1}{2}, k = -\frac{1}{3}(2n+7)\}$
$\{m = -\frac{5}{3}, k = \frac{1}{4}(n-2)\}$	$\{m = 2, k = -7n-15\}$
$\{m = -\frac{5}{3}, k = -\frac{1}{4}(3n+10)\}$	$\{m = 2, k = \frac{1}{2}n\}$
$\{m = -\frac{5}{3}, k = \frac{1}{7}(6n+5)\}$	$\{m = 2, k = -\frac{1}{3}(n+5)\}$
$\{m = -\frac{7}{5}, k = \frac{1}{3}(n-1)\}$	$\{m = 2, k = -\frac{1}{6}(7n+20)\}$

4. Polyanin and Zaitsev [130, pages 278–281] tabulate solvable cases of the Emden–Fowler equation $y'' = Ax^n y^m (y')^k$:
   - Solvable two parameter families include $n = 0$, $m = 0$, and $\{k = {}^{2n+m+3}\!/_{n+m+2}, m \neq -1, n \neq -1\}$.
   - Solvable one parameter families include
     - $\{k \neq 1, 2, m = -1, n = -1\}$
     - $\{k \neq {}^{3}\!/_{2}, m = -\frac{1}{2}, n = -\frac{1}{2}\}$
     - $\{k = \frac{3m+5}{2m+3}, m \neq -\frac{3}{2}, n = -\frac{1}{2}\}$
     - $\{k = \frac{3m+5}{2m+3}, m \neq -\frac{3}{2}, n = 1\}$
     - $\{k = \frac{3n+4}{2n+3}, m = -\frac{1}{2}, n \neq -\frac{3}{2}\}$
     - $\{k = \frac{3n+4}{2n+3}, m = 1, n \neq -\frac{3}{2}\}$
     - $\{k = \frac{3n+4}{2n+3}, m = -n-3, n \neq -\frac{3}{2}\}$
     - $\{k = 1, m \neq -1, 0, n = -1\}$
     - $\{k = 2, m = -1, n \neq -1, 0\}$
     - $\{k = 2, m \neq -2, 0, n = -1\}$
     - $\{k = 3, m = -n-3, n = -1\}$
   - Isolated points at which the solution is tabulated include

k	m	n
$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$
	1	$-\frac{15}{8}$
		$-\frac{20}{13}$
		$-\frac{5}{4}$
		0
$\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{7}{6}$
$\frac{4}{5}$	$-\frac{5}{2}$	$-\frac{1}{2}$
1	-2	1
	-1	-1
$\frac{8}{7}$	1	$-\frac{3}{4}$
		$-\frac{1}{2}$
$\frac{6}{5}$	$-\frac{1}{2}$	$-\frac{2}{3}$
$\frac{5}{4}$	1	$-\frac{1}{2}$
		0
$\frac{9}{7}$	$-\frac{1}{2}$	1
		1
$\frac{13}{10}$	$-\frac{1}{2}$	$-\frac{5}{2}$
$\frac{27}{20}$	$-\frac{1}{2}$	$-\frac{2}{3}$
$\frac{18}{13}$	$-\frac{1}{2}$	$-\frac{7}{2}$
$\frac{7}{5}$	$-\frac{7}{4}$	1
	$-\frac{10}{7}$	1
	$-\frac{2}{3}$	1
	$-\frac{1}{2}$	1
	1	0
		1
	5	1
$\frac{10}{7}$	$-\frac{1}{2}$	$-\frac{5}{2}$
$\frac{22}{15}$	$-\frac{1}{2}$	$-\frac{2}{3}$

k	m	n
$\frac{3}{2}$	-2	$-\frac{1}{2}$
		1
	$-\frac{1}{2}$	-2
		$-\frac{1}{2}$
		1
	1	2
		$-\frac{1}{2}$
$\frac{23}{15}$	$-\frac{2}{3}$	$-\frac{1}{2}$
$\frac{11}{7}$	$-\frac{5}{2}$	$-\frac{1}{2}$
$\frac{8}{5}$	0	1
	1	$-\frac{7}{4}$
		$-\frac{10}{7}$
		$-\frac{2}{3}$
		$-\frac{1}{2}$
		1
		5
$\frac{21}{13}$	$-\frac{7}{2}$	$-\frac{1}{2}$
$\frac{33}{20}$	$-\frac{2}{3}$	$-\frac{1}{2}$
$\frac{17}{10}$	$-\frac{5}{2}$	$-\frac{1}{2}$
$\frac{12}{7}$	1	$-\frac{13}{8}$
		$-\frac{1}{2}$
$\frac{7}{4}$	$-\frac{1}{2}$	1
	0	1
$\frac{9}{5}$	$-\frac{2}{3}$	$-\frac{1}{2}$
$\frac{13}{7}$	$-\frac{3}{4}$	1
	$-\frac{1}{2}$	1
2	-1	-1
	1	-2
$\frac{11}{5}$	$-\frac{1}{2}$	$-\frac{5}{2}$
$\frac{7}{3}$	$-\frac{7}{6}$	$-\frac{1}{2}$

k	m	n
$\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$
	$-\frac{15}{8}$	1
	$-\frac{20}{13}$	1
	$-\frac{5}{4}$	1
	0	1
3	-5	2
	$-\frac{7}{2}$	$-\frac{1}{2}$
	$-\frac{10}{3}$	$-\frac{5}{3}$
	$-\frac{20}{7}$	2
	$-\frac{5}{2}$	$-\frac{1}{2}$
	$-\frac{13}{5}$	$-\frac{7}{5}$
	$-\frac{7}{3}$	$-\frac{5}{3}$
	$-\frac{15}{7}$	2
	-2	-2
		-1
		$-\frac{1}{2}$
		1
	$-\frac{4}{3}$	$-\frac{1}{2}$
	$-\frac{7}{6}$	$-\frac{1}{2}$
	$-\frac{5}{6}$	$-\frac{5}{3}$
	$-\frac{1}{2}$	$-\frac{5}{2}$
		$-\frac{5}{3}$
	0	-4
		$-\frac{5}{2}$
		$-\frac{1}{2}$
		2
	1	-7
		-4
		-2
		$-\frac{5}{3}$
		$-\frac{7}{5}$
		$-\frac{1}{2}$
		0
	2	$-\frac{5}{3}$
	3	-7

5. Polyanin and Zaitsev [130, page 242] tabulate solvable cases of the Emden–Fowler equation $y'' = Ax^n y^m$:

   - Solvable one parameter families include $n = 0$, $n = -m - 3$, $n = -\frac{1}{2}(m + 3)$, $m = 0$, and $m = 1$.

   - Isolated points at which the solution is tabulated include:

m	n
-7	1
-7	3
$-5/4$	$-1/2$
-2	-2
-2	1
$-5/3$	$-10/3$
$-5/3$	$-7/3$
$-5/3$	$-5/6$
$-5/3$	$-1/2$
$-5/3$	1
$-5/3$	2

m	n
$-7/3$	$-13/5$
$-7/3$	1
$-1/2$	$-7/2$
$-1/2$	$-5/2$
$-1/2$	-2
$-1/2$	$-4/3$
$-1/2$	$-7/6$
$-1/2$	$-1/2$
$-1/2$	1
2	-5
2	$-20/7$
2	$-15/7$

6. Solvable cases of the following equations are also tabulated in Polyanin and Zaitsev [130]:
   $(y')^k = Ay^s + Bx^r$ [130, page 106]
   $(y')^k = Ay^s + Be^x$ [130, page 107]
   $(y')^k = Ae^y + Bx^r$ [130, page 107]
   $(y')^k = Ae^y + Be^x$ [130, page 107]
   $y'' = (A_1 x^{n_1} y^{m_1} + A_2 x^{n_2} y^{m_2})(y')^k$ [130, pages 314–319]
   $\alpha y'' = \sigma x^n y^m (y')^k + x^{n-1} y^{m+1} (y')^{k-1}$ [130, pages 349–352]
   $y'' = A_1 x^{n_1} y^{m_1} (y')^{k_1} + A_2 x^{n_2} y^{m_2} (y')^{k_2}$ with $k_1 \neq k_2$ [130, page 367]
   $y''' = Ax^\alpha y^\beta (y')^\gamma (y'')^\delta$ [130, pages 529–535]
   $y''' = Ae^y (y')^\gamma (y'')^\delta$ [130, page 577]
   $y''' = Ay^\beta e^{(y')^2} (y'')^\delta$ [130, page 577]

References

[1] Ablowitz, M. J., Costa, D. G., and Tenenblat, K. Solutions of multidimensional extensions of the anti-self-dual Yang–Mills equation. *Stud. Appl. Math. 77* (1987), 37–46.

[2] Abraham-Shrauner, B. Hidden symmetries and linearization of the modified Painleve–Ince equation. *J. Math. and Physics 34*, 10 (October 1993), 4809.

[3] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington, D.C., 1964.

[4] Aguirre, J., and Escobedo, M. A Cauchy problem for zzzref2refzzz with *0¡p¡1*. Asymptotic behavior of solutions. *Ann. Fac. Sci. Toulouse Math. 8*, 2 (1986/87), 175–203.

[5] Aiyer, R. N., Fuchssteiner, B., and Oevel, W. Solitons and discrete eigenfunctions of the recursion operator of non-linear evolution equations: I. the Caudrey–Dodd–Gibbon–Sawada–Kotera equation. *J. Phys. A: Math. Gen. 19* (1986), 3755–3770.

[6] Akin, O. The integral representation of the positive solutions of the generalized Weinstein equation on a quarter-space. *SIAM J. Appl. Math. 19*, 6 (November 1988), 1348–1354.

[7] Alvarez, A., Pen-Yu, K., and Vazquez, L. The numerical study of a nonlinear one-dimensional Dirac equation. *Appl. Math. and Comp. 13* (1983), 1–15.

[8] Ames, K. A., and Ames, W. F. On group analysis of the Von Karman equations. *Nonlinear Analysis 6*, 8 (1982), 845–853.

[9] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[10] Ames, W. F., and Nucci, M. C. Analysis of fluid equations by group methods. *Journal of Engineering Mathematics 20* (1985), 181–187.

[11] Andreev, V. A. Matrix Liouville equation. *Theo. and Math. Physics 83*, 1 (April 1990), 366–372.

[12] Ang, D. D., and Dinh, A. P. N. On the strongly damped wave equation: zzzref3refzzz. *SIAM J. Appl. Math. 19*, 6 (November 1988), 1409–1418.

[13] Arscott, F. M. The land beyond Bessel: A survey of higher special functions. In *Ordinary and Partial Differential Equations*, W. N. Everitt and B. D. Sleeman, Eds. Springer–Verlag, New York, 1981, pp. 26–45.

[14] Athorne, C. On a subclass of Ince equations. *J. Phys. A: Math. Gen. 23* (1990), L137–L139.

[15] Avrin, J., and Goldstein, J. A. Global existence for the Benjamin–Bona–Mahony equation in arbitrary dimensions. *Nonlinear Analysis 9*, 8 (1985), 861–865.

[16] Balmforth, N. J., Ierley, G. R., and Spiegel, E. A. Chaotic pulse trains. *SIAM J. Appl. Math. 54*, 5 (October 1994), 1291–1334.

[17] Barouch, E., Fokas, A. S., and Papageorgiou, V. G. The bi-Hamiltonian formulation of the Landau–Lifshitz equation. *J. Math. Physics 29*, 12 (Dec 1988), 2628–2633.

[18] Barut, A. O., Inomata, A., and Wilson, R. A new realization of dynamical groups and factorization method. *J. Phys. A: Math. Gen. 20* (1987), 4075–4083.

[19] Bateman, H. *Partial Differential Equations of Mathematical Physics*. Cambridge University Press, New York, 1959.

[20] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[21] Benguria, R. D., and Depassier, M. C. On the existence of monotonic fronts for the class of physical problems described by the equation zzzref17refzzz. *J. Phys. A: Math. Gen. 27* (1994), 1339–1344.

[22] Benton, E. R., and Platzman, G. W. A table of solutions of the one–dimensional Burgers equation. *Quart. Appl. Math.* (July 1972), 195–212.

[23] Binding, P., and Volkmer, H. Eigencurves for two-parameter Sturm–Liouville equations. *SIAM Review 38*, 1 (March 1996), 27–48.

[24] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations*. John Wiley & Sons, New York, 1978.

[25] Bluman, G., and Kumei, S. On the remarkable nonlinear diffusion equation zzzref6refzzz. *J. Math. Physics 21*, 5 (May 1980), 1019–1023.

[26] Bogdanov, L. V. Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation. *Theo. and Math. Physics 70*, 2 (August 1987), 219–233.

[27] Boiti, M., Leon, J. J.-P., and Pempinelli, F. Integrable two-dimensional generalisation of the Sine– and Sinh–Gordon equations. *Inverse Prob. 3* (1987), 37–49.

[28] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[29] Boyd, J. P. Solitons from sine waves: Analytical and numerical methods for non–integrable solitary and cnoidal waves. *Physica D 21* (1986), 227–246.

[30] Boyd, J. P. An analytical solution for a nonlinear differential equation with logarithmic delay. *Advances in Appl. Math. 9* (1988), 358–363.

[31] Brugarino, T. Similarity solutions of the generalized Kadomtsev–Petviashvili–Burgers equations. *Nuovo Cimento B 92*, 2 (1986), 142–156.

[32] Calogero, F. A solvable nonlinear wave equation. *Stud. Appl. Math. 70*, 3 (June 1984), 189–199.

[33] Calogero, F. The evolution partial differential equation zzzref9refzzz. *J. Math. Physics 28*, 3 (March 1987), 538–555.

[34] Calogero, F., and Degasperis, A. *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations*. North–Holland Publishing Co., New York, 1982.

[35] Canosa, J., and Gazdag, J. The Korteweg–de Vries–Burgers equation. *J. Comput. Physics 23* (1977), 393–403.

[36] Carslaw, H. S., and Jaeger, J. C. *Conduction of Heat in Solids*. Clarendon Press, Oxford, England, 1984.

[37] Cazenave, T. Stable solutions of the logarithmic Schrodinger equation. *Nonlinear Analysis 7*, 10 (1983), 1127–1140.

[38] Cebeci, T., and Keller, H. B. Shooting and parallel shooting methods for solving Falkner–Skan boundary layer equation. *J. Comput. Physics 71* (1971), 289–300.

[39] Chambre, P. L. On the solution of the Poisson–Boltzmann equation with application to the theory of thermal explosions. *J. Chem. Physics 20*, 11 (November 1952), 1795–1797.

[40] Champagne, B., and Winternitz, P. On the infinite-dimensional group of the Davey–Stewartson equations. *J. Math. Physics 29*, 1 (Jan 1988), 1–8.

[41] Chang, C. C. On generalized Lavrent'ev–Bitsadze problems. *Mixed Type Equations Teubner, Leipzig* (1987), 55–63.

[42] Chaohao, G. The mixed PDE for amplifying spiral waves. *Lett. Math. Physics 16* (1988), 69–76.

[43] Cherednik, I. Integration of quantum many-body problems by affine Knizhnik–Zamolodchikov equations. *Advances in Mathematics 106* (1994), 65–95.

[44] Chowdhury, A. R., and Nasker, M. Towards the conservation laws and Lie symmetries for the Khokhlov–Zabolotskaya equation in three dimensions. *J. Phys. A: Math. Gen. 19* (1986), 1775–1781.

[45] Chrisholm, J. S. R., and Common, A. K. A class of second-order differential equations and related first-order systems. *J. Phys. A: Math. Gen. 20* (1987), 5459–5472.

[46] Clarkson, P. A. The Painleve property, a modified Boussinesq equation and a modified Kadomtsev–Petviashvili equation. *Physica D 19* (1986), 447–450.

[47] Clarkson, P. A. New similarity solutions for the modified Boussinesq equation. *J. Phys. A: Math. Gen. 22*, 13 (1989), 2355–2367.

[48] Clarkson, P. A., and Mansfield, E. L. On a shallow water wave equation. *Nonlinearity 7* (1994), 975–1000.

[49] Cole, J. D., and Cook, P. *Transonic Aerodynamics*. North–Holland Publishing Co., New York, 1986.

[50] Common, A. K., Hessameddini, E., and Musette, M. The pinney equation and its discretization. *J. Phys. A: Math. Gen. 29* (1996), 6343–6352.

[51] Daniel, M., and Sahadevan, R. On the weak Painleve property and linearization of the evolution equation zzzref13refzzz. *Phys. Lett. A 130*, 1 (1988), 19–21.

[52] Deumens, E. The Klein–Gordon–Maxwell nonlinear system of equations. *Physica D 18* (1986), 371–373.

[53] Dodd, R., and Fordy, A. The prolongation structures of quasi-polynomial flows. *Proc. R. Soc. A. 385* (1983), 389–429.

[54] Duck, P. W., Marshall, T. W., and Watson, E. J. First-passage times for the Uhlenbeck–Ornstein process. *J. Phys. A: Math. Gen. 19* (1986), 3545–3558.

[55] Elliot, C. M., Herrero, M. A., King, J. R., and Ockendon, J. R. The mesa problem: Diffusion patterns for zzzref14refzzz as zzzref15refzzz. *IMA J. Appl. Mathematics 7*, 2, 147–154.

[56] Elzoheiry, H., Iskandar, L., and h. El-Deen Mohamedsin, M. Iterative implicit schemes for the two- and three-dimensional Sine–Gordan equation. *J. Comput. Appl. Math. 34* (1991), 161–170.

[57] Eringen, A. C., and Suhubi, E. S. *Elastodynamics*, vol. 2. Academic Press, New York, 1975.

[58] Flessas, G. P. New exact solutions of the complex Lorenz equations. *J. Phys. A: Math. Gen. 22* (1989), L137–L141.

[59] Fuchssteiner, B., Oevel, W., and Wiwianka, W. Computer-algebra methods for investigation of hereditary operators of high order soliton equations. *Comput. Physics Comm. 44* (1987), 47–55.

[60] Fujimoto, A., and Watanabe, Y. Polynomial evaluation equations of not normal type admitting no trivial solutions. *Phys. Lett. A 136*, 6 (1989), 294–299.

[61] Fujita, H. On the nonlinear equations zzzref19refzzz and zzzref20refzzz. *Bull. Amer. Math. Soc. 75* (1969), 132–135.

[62] Fung, P. C. W., and Au, C. A series of new analytical solutions to the nonlinear equation zzzref21refzzz. I. *J. Math. Physics 25*, 5 (May 1984), 1370–1371.

[63] Gerdt, V. P., Shvachka, A. B., and Zharkov, A. Y. Computer algebra applications for classification of integrable non-linear evolution equations. *J. Symbolic Comp. 1* (1985), 101–107.

[64] Gilbarg, D., and Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order*. Springer–Verlag, New York, 1983.

[65] Gilding, B. H. The first boundary value problem for zzzref22refzzz. *J. Math. Anal. Appl. 128*, 2 (1987), 419–442.

[66] Glassey, R. T. Approximate solutions to the Zakharov equations via finite differences. *J. Comput. Physics 10* (1992), 377–383.

[67] Goldstein, J. A., and Wichnoski, B. J. On the Benjamin–Bona–Mahony equation in higher dimensions. *Nonlinear Analysis 4*, 4 (1980), 665–675.

[68] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations*. U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[69] Gradshteyn, I. S., and Ryzhik, I. M. *Tables of Integrals, Series, and Products*. Academic Press, New York, 1980.

[70] Grauel, A. Sinh–Gordon equation, Painleve property and Backlund transformation. *Physica A 132* (1985), 557–568.

[71] Hagedorn, P. *Non-linear Oscillations*. Clarendon Press, Oxford, England, 1982.

[72] Haine, L., and Horozov, E. A Lax pair for Kowalevski's top. *Physica D 29* (1987), 173–180.

[73] Hairer, E., Nørsett, S. P., and Wanner, G. *Solving Ordinary Differential Equations I.* Springer–Verlag, New York, 1987.

[74] Hall, W. S. The Rayleigh wave equation — an analysis. *Nonlinear Analysis 2*, 2 (1978), 129–156.

[75] Hayashi, N., and Ozawa, T. Finite energy solution of nonlinear Schroder equations of derivative type. *SIAM J. Math. Anal. 25*, 6 (Nov 1994), 1488–1503.

[76] Herrera, J. J. E., Minzoni, A., and Ondarza, R. *Physica D 57* (1992), 290–320.

[77] Herron, I. H. The Orr–Sommerfeld equations on infinite intervals. *SIAM Review 29*, 4 (1987), 597–620.

[78] Hershenov, J. Solutions of the differential equation zzzref23refzzz. *Stud. Appl. Math. 55* (1976), 301–314.

[79] Hill, J. M., and Hill, D. L. High-order nonlinear evolution equations. *IMA J. Appl. Mathematics 45* (1990), 243–265.

[80] Hille, E. *Lectures on Ordinary Differential Equations*. Addison–Wesley Publishing Co., Reading, MA, 1969.

[81] Hirota, R. Exact solutions of the spherical Toda molecule equation. *J. Phys. Soc. Japan 5*, 1 (1988), 66–70.

[82] Hirota, R., Grammaticos, B., and Ramani, A. Soliton structure of the Drinfel'd–Sokolov–Wilson equation. *J. Math. Physics 27*, 6 (June 1986), 1499–1505.

[83] Hirota, R., and Nakamura, A. Exact solutions of the cylindrical Toda molecule equation. *J. Phys. Soc. Japan 56*, 9 (1987), 3055–3061.

[84] Humi, M. Factorisation of separable partial differential equations. *J. Phys. A: Math. Gen. 20* (1987), 4577–4585.

[85] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[86] Infeld, L., and Hull, T. E. The factorization method. *Rev. Mod. Physics 23*, 1 (Jan 1951), 21–68.

[87] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics.* MIT Press, Cambridge, MA, 1980.

[88] Jackson, J. D. *Classical Electrodynamics.* John Wiley & Sons, New York, 1962.

[89] Kaliappan, P. An exact solution for travelling waves of zzzref24refzzz. *Physica D 11* (1984), 368–374.

[90] Kamke, E. *Differentialgleichungen Losungsmethoden und Losungen*, vol. I. Chelsea Publishing Company, New York, 1947.

[91] Kamke, E. *Differentialgleichungen Losungsmethoden und Losungen*, vol. II. Chelsea Publishing Company, New York, 1947.

[92] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis.* Interscience Publishers, Inc., New York, 1958.

[93] Kaper, H. G., and Leaf, G. K. Initial value problems for the Carleman equation. *Nonlinear Analysis 4*, 2 (1980), 343–362.

[94] Katou, K. Asymptotic spatial patterns on the complex time–dependent Ginzburg–Landau equation. *J. Phys. A: Math. Gen. 19* (1986), L1063–L1066.

[95] Kersten, P. H. M., and Gragert, P. K. H. Symmetries for the super-KdV equation. *J. Phys. A: Math. Gen. 21* (1988), L579–L584.

[96] King, J. R. Exact results for the nonlinear diffusion equations zzzref25refzzz and zzzref26refzzz. *J. Phys. A: Math. Gen. 24* (1991), 5721–5745.

[97] Kundu, A. Comments on: The Eckhaus PDE zzzref27refzzz. *Inverse Problems 4*, 4 (November 1988), 1143–1144.

[98] Lamb, G. L. *Elements of Soliton Theory.* John Wiley & Sons, New York, 1980.

[99] Landau, L. D., and Lifshitz, E. M. *Fluid Mechanics.* Pergamon Press, New York, 1959.

[100] Latham, G. A. Solutions of the KP equation associated to rank-three commuting differential operators over a singular elliptic curve. *Physica D 41* (1990), 55–66.

[101] Latta, G. E. Some differential equations of the Mathieu type and related integral equations. *J. Math. and Physics 42* (1963), 139–146.

[102] Leach, P. G. L. First integrals for the modified Emden equation zzzref28refzzz. *J. Math. Physics 26*, 10 (October 1985), 2510–2514.

[103] Leach, P. G. L., Feis, M. R., and Bouquet, S. Analysis and solution of a nonlinear second-order differential equation through rescaling and through a dynamic point of view. *J. Math. Physics 29*, 12 (December 1988), 2563–2569.

[104] Leach, P. G. L., Maartens, R., and Maharaj, S. D. Self-similar solutions of the generalized Emden–Fowler equation. *Int. J. Non-Linear Mechanics 27*, 4 (1992), 575–582.

[105] Levi, M., Hoppensteadt, F. C., and Miranker, W. L. Dynamics of the Josephson junction. *Quart. Appl. Math.* (July 1978), 167–198.

[106] Levinson, N., and Redheffer, R. M. *Complex Variables*. Holden–Day, Inc., San Francisco, CA, 1979.

[107] Lin, S.-S. Symmetry breaking for zzzref29refzzz on a disk with general boundary conditions. *Proc. Roy. Soc. Edinburgh Sect. A 113*, 1–2 (1989), 105–117.

[108] Lindquist, P. Stability for the solutions of divzzzref30refzzz with varying *p*. *J. Math. Anal. Appl. 127* (1987), 93–102.

[109] Lowndes, J. S. On two new operators of fractional integration. In *Fractional Calculus*, A. C. McBride and G. F. Roach, Eds. Pitman Publishing Co., Marshfield, MA, 1985.

[110] Manwell, A. R. *The Tricomi Equation with Applications to the Theory of Plane Transonic Flow*. Pitman Publishing Co., Marshfield, MA, 1979.

[111] Marcq, P., Chate, H., and Conte, R. Exact solutions of the one dimeionsal quintic complex Ginzburg–Landau equation. *Physica D 73* (1994), 305–317.

[112] Matsuno, Y. *Bilinear Transformation Method*. Academic Press, New York, 1984.

[113] Merkin, J. H. A note on the solution of a differential equation arising in boundary-layer theory. *J. Eng. Math. 18* (1984), 31–36.

[114] Meyer, G. H. *Initial Value Methods for Boundary Value Problems: Theory and Application of Invariant Imbedding*. Academic Press, New York, 1973.

[115] Michelson, D. Steady solutions of the Kuramoto–Sivashinsky equation. *Physica D 19* (1986), 89–111.

[116] Miller, Jr., W. Symmetries of differential equations. The hypergeometric and Euler–Darboux equations. *SIAM J. Math. Anal. 4*, 2 (May 1973), 314–328.

[117] Miller, Jr., W. *Symmetry and Separation of Variables*. Addison–Wesley Publishing Co., Reading, MA, 1977.

[118] Moon, P., and Spencer, D. E. The meaning of the vector Laplacian. *J. Franklin Institute 256* (1953), 551–558.

[119] Moon, P., and Spencer, D. E. *Field Theory for Engineers*. D. Van Nostrand Company, Inc., New York, 1961.

[120] Moon, P., and Spencer, D. E. *Field Theory Handbook*. Springer–Verlag, New York, 1961.

[121] Moon, P., and Spencer, D. E. *Partial Differential Equations*. D. C. Heath and Co., Lexington, MA, 1969.

[122] Morse, P. M., and Feshback, H. *Methods of Theoretical Physics*. McGraw–Hill Book Company, New York, 1953.

[123] Murphy, G. M. *Ordinary Differential Equations and Their Solution*. D. Van Nostrand Company, Inc., New York, 1960.

[124] Nayfeh, A. H. *Perturbation Methods*. John Wiley & Sons, New York, 1973.

[125] Nimala, N., Vedan, M. J., and Baby, B. V. A variable coefficient Korteweg–de Vries equation: Similarity analysis and exact solution. II. *J. Math. Physics 27*, 11 (Nov 1986), 2644–2646.

[126] Novick-Cohen, A., and Segel, L. A. Nonlinear aspects of the Cahn–Hilliard equation. *Physica D 10* (1984), 277–298.

[127] Ohta, Y., Kajiwara, K., Matsukidaira, J., and Satsuma, J. Caorati determinant solution to the relativistic Toda lattice equation. *J. Math. and Physics 34*, 11 (November 1993), 5190.

[128] Oliveri, F. Painleve analysis and similarity solutions of Burgers' equation with variable coefficients. *J. of Eng. Mathematics 25* (1991), 317–327.

[129] Pfeiffer, G. W. Asymptotic solutions of *y"'+qy'+ry=0*. *J. Differential Equations 11* (1972), 145–155.

[130] Polyanin, A. D., and Zaitsev, V. F. *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC, Boca Raton, FL, 1995.

[131] Rainville, E. D. *Special Functions*. Chelsea Publishing Company, New York, 1960.

[132] Rajasekar, S., and Lakshmanan, M. Period-doubling bifurcations, chaos, phase-locking and devil's staircase in a Bonhoeffer–Van Der Pol oscillator. *Physica D 32* (1988), 146–152.

[133] Rammaha, M. A. On the asymptotic behavior of solutions of generalized Korteweg-de Vries equations. *J. Math. Anal. Appl. 140*, 1 (1989), 228–240.

[134] Ronveaux, A. *Heun's Differential Equations*. Oxford University Press, New York, 1995.

[135] Rosales, R. R. Exact solutions of some nonlinear evolution equations. *Stud. Appl. Math. 59* (1978), 117–151.

[136] Rosen, G. Solutions of a certain nonlinear wave equation. *J. Math. and Physics 45* (1966), 235–265.

[137] Rosenau, P. A note on integration of the Emden–Fowler equation. *Int. J. Non-Lin. Mech. 19*, 4 (1984), 303–308.

[138] Rosenblat, S., and Shepherd, J. On the asymptotic solution of the Lagerstrom model equation. *SIAM J. Appl. Math. 29*, 1 (July 1975), 110–120.

[139] Roy, S., and Chowdhury, A. R. Prolongation theory. A new nonlinear Schrodinger equations. *Int. J. Theo. Physics 26*, 7 (1987), 707–714.

[140] Rubel, L. A. A differential equation satisfied by all n-nomials. *Nieuw Arch. Wisk. 6*, 3 (1988), 263–267.

[141] Sachdev, P. L., Joseph, K. T., and Vaganan, B. M. Exact N-wave solutions of generalized Burgers equations. *Stud. Appl. Math. 97* (1996), 349–367.

[142] Sachdev, P. L., and Nair, K. R. C. Generalized Burgers equations and Euler-Painleve transcendents. II. *J. Math. Physics 28*, 5 (May 1987), 997–1004.

[143] Saied, E. A., and Hussein, M. M. New classes of similarity solutions of the inhomogenous nonlinear diffusion equations. *J. Phys. A: Math. Gen. 27* (1994), 4867–4874.

[144] Salingaros, N. A. On solutions of the equation zzzref43refzzz: II. The magnetic force–free model. *J. Phys. A: Math. Gen. 19* (1986), L705–L708.

[145] Schief, W. K. The Tzitzeica equation: A Backlund transformation interpreted as truncated Painleve expansion. *J. Phys. A: Math. Gen. 29* (1996), 5153–5155.

[146] Schief, W. K., and Rogers, C. The affinspharen equation. Moutard and Backlund transformations. *Inverse Problems 10* (1994), 711–731.

[147] Seshadri, R., and Na, T. Y. *Group Invariance in Engineering Boundary Value Problems*. Springer–Verlag, New York, 1985.

[148] Setoyanagi, M. Liouvillian solutions of the differential equation *y"+S(x)y=0* with *S(x)* binomial. *Proc. Am. Math. Soc. 100*, 4 (August 1987), 607–612.

[149] Sherman, A. S., and Peskin, C. S. A Monte Carlo method for scalar reaction diffusion equations. *SIAM J. Sci. Stat. Comput. 7*, 4 (October 1986), 1360–1372.

[150] Shivaji, R. A note on the persistence of an S-shaped bifurcation curve. *Appl. Anal. 24*, 3 (1987), 175–179.

[151] Shonborn, O., Desai, R. C., and Stauffer, D. Nonlinear bias and the convective Fischer equation. *J. Phys. A: Math. Gen. 27* (1994), L251–L255.

[152] Sparrow, C. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. Springer–Verlag, New York, 1982.

[153] Steeb, W.-H. *Invertible Point Transformation and Nonlinear Differential Equations*. World Scientific, Singapore, 1993.

[154] Steeb, W.-H., and Louw, J. A. Nahm's equations, singular point analysis, and integrability. *J. Math. Physics 27*, 10 (Oct 198), 2458–2460.

[155] Tabor, M. *Chaos and Integrability in Nonlinear Dynamics*. John Wiley & Sons, New York, 1989.

[156] Ting, A. C., Cheb, H. H., and Lee, Y. C. Exact solutions of a nonlinear boundary value problem: the vortices of the two-dimensional Sinh–Poisson equation. *Physica D* (1987), 37–66.

[157] Trubek, J. Asymptotic behavior of solutions to zzzref46refzzz on zzzref47refzzz for zzzref48refzzz. *Proc. Amer. Math. Soc. 106*, 4 (1989), 953–959.

[158] Tsukamoto, I. On solutions of zzzref49refzzz. *Tokyo J. Math. 12*, 1 (1989), 181–203.

[159] Urwin, K. M., and Arscott, F. M. Theory of the Whittaker–Hill equation. *Proc. R. Soc. Edin. 69* (1970), 28–44.

[160] Utepbergenov, M. Integral representations of solutions of equations with strong degeneration. *Studies in Multidimensional Elliptic Systems of Partial Differential Equations (Akad. Nauk SSSR Sibirsk Otdel., Inst Mat., Novosibirsk)* (1986).

[161] VALIRON, G. *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions*. Math Sci Press, Brookline, MA, 1950.

[162] VIECELLI, J. A. Exponential difference operator approximation for the sixth order Onsager equation. *J. Comput. Physics 50* (1983), 162–170.

[163] VILLARI, G. Periodic solutions of Lienard's equation. *J. Math. Anal. Appl. 86* (1982), 379–386.

[164] WALKER, P. W. Asymptotica of the solutions to zzzref7refzzz. *J. Differential Equations 9* (1971), 108–132.

[165] WANG, X. Y. Exact and explicit solitary wave solutions for the generalised Fisher equation. *Phys. Lett. A 131*, 4–5 (1988), 277–279.

[166] WANG, X. Y., ZHU, Z. S., AND LU, Y. K. Solitary wave solutions of the generalized Burgers–Huxley equation. *J. Phys. A: Math. Gen. 23* (1990), 271–274.

[167] WARD, R. S. The Nahn equations, finite-gap potentials and Lame functions. *J. Phys. A: Math. Gen. 20* (1987), 2679–2683.

[168] WATSON, G. N. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, New York, 1966.

[169] WEISS, J. Modified equations, rational solutions, and the Painleve property for the Kadomtsev–Petviashvili and Hirota–Satsuma equations. *J. Math. Physics 26*, 9 (Sept 1985), 2174–2180.

[170] WHITHAM, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

[171] WILCOX, R. Closed-form solution of the differential equation zzzref50refzzz zzzref51refzzz by normal-ordering exponential operators. *J. Math. Physics 11*, 4 (April 1970), 1235–1237.

[172] WOOD, H. G., AND MORTON, J. B. Onsager's pancake approximation for the fluid dynamics of a gas centrifuge. *J. Math. Physics 101* (1980), 1–31. Part 1.

[173] YANAGIDA, E. Structure of radial solutions to zzzref4refzzz in zzzref5refzzz. *SIAM J. Math. Anal. 27*, 3 (July 1996), 997–1014.

[174] ZHI-XIONG, C., AND BEN-YU, G. Analytic solutions of the Nagumo equation. *IMA J. Appl. Mathematics 48*, 2 (1992), 107–115.

45. Look-Up ODE Forms

Applicable to Ordinary differential equations.

Yields

An idea of whether or not an ordinary differential equation has a closed-form solution.

Idea

An experienced differential equations practitioner can look at many second order ordinary differential equations and readily guess whether or not there is a closed form solution because there are many familiar forms that often appear.

Procedure

Having a listing of familiar differential equation forms will make it possible to recognize these forms. We have tabulated below many of the familiar forms that appear for second order ordinary differential equations.

In the listings below, ( ) represents a term that contains constants. Such a term may or may not be correlated with other terms of the form ( ). For example, equation 22.6.5 in Abramowitz and Stegun [1] is

$$\left(1-x^2\right)y'' - (2\alpha+1)xy' + n(n+2\alpha)y = 0,$$

where α is a real constant and n is an integer. Isolating the x dependence, we list this equation as

$$\left(1-x^2\right)y'' + (\,)xy' + (\,)y = 0$$

and disregard the fact that the hidden values have constraints on them and, in fact, are related.

45.1 Equations of the Form: $y'' + c(x)y = 0$

$c(x) = (\,)$	[1, 22.6.10]
$c(x) = -x$	[1, 10.4.1]
$c(x) = (\,) - x^2$	[1, 22.6.20]
$c(x) = (\,) + (\,)x + (\,)x^2$	[1, 19.1.1]
$c(x) = (\,)x^{(\,)}$	[1, 9.1.51]
$c(x) = (\,) + \frac{(\,)}{x^2}$	[1, 9.1.49]
$c(x) = \frac{(\,)}{x} + \frac{(\,)}{x^2}$	[1, 9.1.50]
$c(x) = (\,) - \frac{(\,)}{x} - \frac{(\,)}{x^2}$	[1, 14.1.1]
$c(x) = (\,) - x^2 + \frac{(\,)}{x^2}$	[1, 13.1.1 and 22.6.8]
$c(x) = (\,)e^{2x} - (\,)$	[1, 9.1.54]
$c(x) = \frac{(\,)}{1-x^2} + \frac{(\,)+x^2}{4(1-x^2)^2}$	[1, 22.6.7]

$c(x) = \frac{()}{1-x^2} + \frac{1}{(1-x^2)^2}$ [1, 22.6.14]

$c(x) = \frac{()}{(1-x)^2} + \frac{()}{(1+x)^2} + \frac{()}{1-x^2}$ [1, 22.6.3]

$c(x) = \frac{()}{x} + \frac{()}{x^2} + ()$ [1, 22.6.17]

$c(x) = () + \frac{()}{\sin^2 x}$ [1, 22.6.8]

$c(x) = () + \frac{()}{\sin^2 \frac{x}{2}} + \frac{()}{\cos^2 \frac{x}{2}}$ [1, 22.6.4]

45.2 Equations of the Form: $y'' + b(x)y' + c(x)y = 0$

$b(x) = -x, \quad c(x) = ()$ [1, 22.6.21]

$b(x) = -2x, \quad c(x) = ()$ [1, 22.6.19]

$b(x) = 2x, \quad c(x) = -()x$ [1, 7.2.2]

$b(x) = 2x, \quad c(x) = x^2 - ()$ [1, 10.1.1]

$b(x) = 2x, \quad c(x) = () - x^2$ [1, 10.2.1]

$b(x) = () - x, \quad c(x) = ()$ [1, 22.6.15]

$b(x) = ()x, \quad c(x) = () + x^{()}$ [1, 9.1.53]

$b(x) = \frac{()}{x}, \quad c(x) = ()$ [1, 9.1.52]

$b(x) = (), \quad c(x) = () - () \cos x$ [1, 20.1.1]

45.3 Equations of the Form: $xy'' + b(x)y' + c(x)y = 0$

$b(x) = () - x\ , \quad c(x) = ()$ [1, 13.1.1]

$b(x) = () + x\ , \quad c(x) = () + \frac{()}{x}$ [1, 22.6.16]

45.4 Equations of the Form: $(1-x^2)y'' + b(x)y' + c(x)y = 0$

$b(x) = (), \quad c(x) = () - ()x^2$ [1, 20.1.8]

$b(x) = -x, \quad c(x) = ()$ [1, 22.6.9]

$b(x) = -x, \quad c(x) = () - ()x^2$ [1, 20.1.7]

$b(x) = -2x, \quad c(x) = ()$ [1, 22.6.13]

$b(x) = -2x, \quad c(x) = () + \frac{()}{1-x^2}$ [1, 8.1.1]

$b(x) = -3x, \quad c(x) = ()$ [1, 22.6.11 and 22.6.12]

$b(x) = ()x, \quad c(x) = ()$ [1, 22.6.5 and 22.6.6]

$b(x) = () + ()x, \quad c(x) = ()$ [1, 22.6.1 and 22.6.2]

45.5 Equations of the Form: $x^2y'' + b(x)y' + c(x)y = 0$

$b(x) = x, \quad c(x) = x^2 - ()$ [1, 9.1.1]

$b(x) = x, \quad c(x) = () - x^2$ [1, 9.6.1]

$b(x) = 2x, \quad c(x) = () + x^2$ [1, 10.1.1]

$b(x) = 2x, \quad c(x) = () - x^2$ [1, 10.2.1]

45.6 Equation of the Form: $x(1-x)y'' + b(x)y' + c(x)y = 0$

$b(x) = () - ()x, \quad c(x) = ()$ [1, 15.5.1]

Note

1. Realize that the same equation may look different when written in different variables. Some scaling of any given equation may be required to make it look like one of the forms listed.

Reference

[1] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington, D.C., 1964.

46. An Nth Order Equation

Applicable to

The equation $\frac{d^n y}{dx^n} = f(x)$.

Yields

Two exact forms of the solution are available.

Idea

The explicit solution can be written analytically.

Procedure

The general solution of the ordinary differential equation for $y(x)$

$$\frac{d^n y}{dx^n} = f(x)$$

can be found by integrating with respect to x a total of n times. This produces

$$\begin{aligned} y(x) = &\int_{x_0}^{x} dx \int_{x_0}^{x} dx \cdots \int_{x_0}^{x} f(x)\, dx + C_1 \frac{(x-x_0)^{n-1}}{(n-1)!} \\ &+ C_2 \frac{(x-x_0)^{n-2}}{(n-2)!} + \cdots + C_{n-1}(x-x_0) + C_n, \end{aligned} \tag{46.1}$$

for any x_0, where the $\{C_j\}$ represent arbitrary constants. This solution can also be written as

$$\begin{aligned} y(x) = &\frac{1}{(n-1)!} \int_{x_0}^{x} (x-t)^{n-1} f(t)\ dt + C_1 \frac{(x-x_0)^{n-1}}{(n-1)!} \\ &+ C_2 \frac{(x-x_0)^{n-2}}{(n-2)!} + \cdots + C_{n-1}(x-x_0) + C_n \end{aligned} \tag{46.2}$$

in which there are no repeated integrals. Sometimes the form in equation (46.2) is more useful than the form in equation (46.1).

Example

The ordinary differential equation

$$\begin{aligned} y^{(4)} &= \sin x, \\ y(0) = 0, &\quad y'(0) = 0, \\ y''(0) = 0, &\quad y'''(0) = 0 \end{aligned}$$

has the solution

$$y(x) = \int_0^x dx \int_0^x dx \int_0^x dx \int_0^x \sin x\, dx. \tag{46.3}$$

This solution may also be written as

$$y(x) = \frac{1}{6}\int_0^x (x-t)^3 \sin t \, dt. \tag{46.4}$$

Sometimes it is easier to evaluate the expression in equation (46.4) (by expanding out $(x-t)^3$ and integrating the four terms) to determine that

$$y(x) = \sin x - x + \frac{x^3}{6}$$

than it is to evaluate the expression in equation (46.3).

Notes

1. When the answer is to be computed numerically, the solution represented by equation (46.2) is more useful than the form in equation (46.1). It is much easier to numerically approximate a one-dimensional integral than a multi-dimensional integral.
2. See Ince [1, page 42].

Reference

[1] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

47. Use of the Adjoint Equation*

Applicable to Linear differential equations.

Yields

A linear differential equation of lower order.

Idea

For every solution of the adjoint equation we can find, we can reduce the order of the original equation by one.

Procedure

If we have the nth order linear differential operator $L[\cdot]$ (shown operating on the function $u(x)$)

$$L[u(x)] = a_0(x)\frac{d^n u}{dx^n} + a_1(x)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{du}{dx} + a_n(x)u, \tag{47.1}$$

then the adjoint of $L[\cdot]$ is defined to be $L^*[\cdot]$, where $L^*[\cdot]$ is given by (shown operating on the function $w(x)$)

$$\begin{aligned} L^*[w(x)] =& (-1)^n \frac{d^n}{dx^n}[a_0(x)w] + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}[a_1(x)w] + \cdots \\ &+ (-1)^1 \frac{d}{dx}[a_{n-1}(x)w] + (-1)^0[a_n(x)w] \end{aligned}$$

(see page 95 for details). The *bilinear concomitant* of $L[\cdot]$ is defined to be

$$B(u,w) = \sum_{k=0}^{n-1}\sum_{m=k}^{n-1}(-1)^{m-k}u^{(n-m-1)}(a_k w)^{(m-k)} \tag{47.2}$$

and satisfies the equation

$$wL[u] - uL^*[w] = \frac{d}{dx}B(u,w), \tag{47.3}$$

for all $u(x)$ and $w(x)$.

Suppose we wish to solve the equation $L[u] = f(x)$. If we can find a solution to $L^*[w] = 0$ and call it $w^*(x)$, then we have (substituting into equation (47.3))

$$w^* L[u] - uL^*[w^*] = \frac{d}{dx}B(u,w^*),$$

or

$$w^*(x)f(x) = \frac{d}{dx}B(u,w^*),$$

or

$$B(u, w^*) = \int^x w^*(x) f(x)\, dx. \tag{47.4}$$

Therefore, to find $u(x)$, we can solve equation (47.4) instead of $L[u] = f(x)$. In other words, $w^*(x)$ is an integrating factor for the equation $L[u] = f(x)$. The original differential equation, $L[u] = f(x)$, is of degree n whereas equation (47.4) is of degree $n - 1$.

Special Case

For $n = 2$ the adjoint equation is important enough to write separately. If the linear operator $L[\cdot]$ is defined by $L[u(x)] = R(x)u'' + S(x)u' + T(x)u$, then the adjoint is $L^*[w(x)] = Rw'' + (2R' - S)w' + (R'' - S' + T)w$, and the bilinear concomitant is $B(u, w) = uSw + u'Rw - u(Rw)'$.

Example

Suppose we wish to solve the equation $L[u] = 1$, where

$$L[u] = (x^2 - x)u'' + (2x^2 + 4x - 3)u' + 8xu.$$

The adjoint, in this case, is the operator

$$L^*[w] = (x^2 - x)w'' + (-2x^2 + 1)w' + (4x - 2)w,$$

and the bilinear concomitant is given by

$$B(u, w) = u(2x^2 + 2x - 2)w + u'(x^2 - x)w - u(x^2 - x)w'. \tag{47.5}$$

A solution to $L^*[w] = 0$, obtained by the method of undetermined coefficients, is $w^*(x) = x^2$. Using this solution in equation (47.4), we obtain (with $f(x) = 1$)

$$B(u, w^*) = \int^x w^*(x) f(x) dx = \int^x x^2 dx = \frac{x^3}{3} + C,$$

where C is an arbitrary constant. Using $w = w^* = x^2$ in equation (47.5) produces

$$B(u, w^*) = (x^4 - x^3)u' + 2x^4 u.$$

Equating these last two equations yields a first order equation for u:

$$(x^4 - x^3)u' + 2x^4 u = \frac{x^3}{3} + C. \tag{47.6}$$

Note that equation (47.6) is a first order equation (the original differential equation was of second order). Because equation (47.6) is a first order linear equation, it can be solved by the use of integrating factors. Multiplying by $\frac{x-1}{x^3}e^{2x}$ and integrating results in

$$\begin{aligned}(x-1)^2 e^{2x} u(x) &= \int^x \left[\frac{x-1}{3}e^{2x} + Ce^{2x}\frac{x-1}{x^3}\right] dx \\ &= \frac{2x-3}{12}e^{2x} + \frac{C}{2x^2}e^{2x} + D,\end{aligned} \tag{47.7}$$

where D is another arbitrary constant. Hence, the final solution is

$$u(x) = \frac{1}{(x-1)^2}\left[\frac{2x-3}{12} + \frac{C}{2x^2} + De^{-2x}\right]. \tag{47.8}$$

Notes

1. If an operator and its adjoint are identical, then the operator is said to be formally self-adjoint (see page 95). In this case, the adjoint method does not help to find a solution of the original differential equation.
2. Similar results hold for linear partial differential equations. For the partial differential operator

$$L[u] = \sum_{i,j=1}^{n} a_{ij}(\mathbf{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x})\frac{\partial u}{\partial x_i} + c(\mathbf{x})u,$$

the adjoint operator is defined by

$$M[w] = \sum_{i,j=1}^{n} \frac{\partial^2 (a_{ij}w)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial (b_i w)}{\partial x_i} + cw.$$

With this definition of the adjoint, we find

$$\int_D \Big(wL[u] - uM[w]\Big)\, d\mathbf{x} + \int_{\partial D} B[u,w]\, dx_1 \cdots \widehat{dx_i} \cdots dx_n = 0, \tag{47.9}$$

where $B[u,w]$ is defined by

$$B[u,w] = \sum_{i=1}^{n} (-1)^i \left\{ \left[b_i - \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j} \right] uw + \sum_{j=1}^{n} a_{ij} \left[w\frac{\partial u}{\partial x_j} - u\frac{\partial w}{\partial x_j} \right] \right\}.$$

In equation (47.9), $dx_1 \cdots \widehat{dx_i} \cdots dx_n$ indicates the product $dx_1 \cdots dx_n$ with the factor dx_i removed. See Garabedian [1, pages 161–162] or Zauderer [5, pages 483–486] for details.
3. If the elliptic operator $L[\cdot]$ is defined by $L[u] = -\nabla \cdot (p\nabla u) + qu$, then

$$wL[u] - uL[w] = \nabla \cdot (-pw\nabla u + pu\nabla w).$$

If the hyperbolic operator $\widetilde{L}[\cdot]$ is defined by $\widetilde{L}[u] = \rho u_{tt} + L[u]$, then

$$w\widetilde{L}[u] - u\widetilde{L}[w] = \widetilde{\nabla} \cdot [-pw\nabla u + pu\nabla w, \rho w u_t - \rho u w_t],$$

where $\widetilde{\nabla} = [\nabla, \partial/\partial t]$ is the space–time gradient operator. If the parabolic operator $\widehat{L}[\cdot]$ is defined by $\widehat{L}[u] = \rho u_t + L[u]$, then

$$w\widehat{L}[u] - u\widehat{L}^*[w] = \widetilde{\nabla} \cdot [-pw\nabla u + pu\nabla w, \rho uw],$$

where the operator $\widehat{L}^*[\cdot]$ is defined by $\widehat{L}^*[u] = -\rho u_t + L[u]$. Each of the last three equations can be integrated to obtain an expression similar to equation (47.9). See Zauderer [5] for details.

4. See also Ince [2, pages 123–125], Kaplan [3, pages 448–453], and Valiron [4, pages 323–324].

References

[1] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[2] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[3] Kaplan, W. *Operational Methods for Linear Systems*. Addison–Wesley Publishing Co., Reading, MA, 1962.

[4] Valiron, G. *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions*. Math Sci Press, Brookline, MA, 1950.

[5] Zauderer, E. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1983.

48. Autonomous Equations – Independent Variable Missing

Applicable to Ordinary differential equations of the form $F(y^{(n)}, y^{(n-1)}, \ldots, y'', y', y) = 0$.

Yields

An ordinary differential equation of lower order.

Idea

An autonomous equation is one left invariant under the transformation $x \to x + a$. Any ordinary differential equation in which the independent variable does not appear explicitly is an autonomous equation. Because we know something about the solution, we can reduce the order of the differential equation.

Procedure

Given the nth order autonomous equation $F(y^{(n)}, y^{(n-1)}, \ldots, y'', y', y) = 0$, change the dependent variable from $y(x)$ to $u(y) = y'(x)$. The resulting ordinary differential equation for $u(y)$ will be of lower order. To find how the higher order derivatives transform, consult table 48.1. After the ordinary differential equation of lower order has been solved for $u(y)$, $y(x)$ can be determined from integrating $u(y) = y'(x)$; i.e., $\displaystyle\int \frac{dy}{u(y)} = x$.

Example

Suppose we want to solve the nonlinear autonomous equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y\frac{dy}{dx}. \tag{48.1}$$

Because there are no explicit occurrences of x in equation (48.1), we recognize the equation to be autonomous. Therefore, we change variables in equation (48.1) by $u(y) = \frac{dy}{dx}$. Using table 48.1, equation (48.1) transforms into $u\frac{du}{dy} - u = 2yu$ or

$$u\left(\frac{du}{dy} - 1 - 2y\right) = 0. \tag{48.2}$$

From equation (48.2), either $u = 0$ or $\frac{du}{dy} - 1 - 2y = 0$. If $u(y) = 0$, then $\frac{dy}{dx} = 0$ and so one solution to equation (48.1) is

$$y(x) = A, \tag{48.3}$$

$$
\begin{aligned}
y_x &= u, \\
y_{xx} &= uu_y, \\
y_{xxx} &= uu_y^2 + u^2 u_{yy}, \\
y_{xxxx} &= uu_y^3 + 4u_{yy}u_y u^2 + u^3 u_{yyy}, \\
y_{xxxxx} &= uu_y^4 + 7u_{yyy}u_y u^3 + 4u^3 u_{yyy}^2 + 11u^2 u_y^2 u_{yy} + u^4 u_{yyyy}, \\
y_{x(5)} &= uu_y^4 + 7u_{y(3)}u_y u^3 + 4u^3 u_{y(3)}^2 + 11u^2 u_y^2 u_{yy} + u^4 u_{y(4)}, \\
y_{x(6)} &= uu_y^5 + 11u_{y(4)}u_y u^4 + 15u^4 u_{y(3)}^2 u_{yy} + 32u^3 u_y^2 u_{y(3)} \\
&\quad +34u^3 u_y u_{yy}^2 + 26u^2 u_y^3 u_{yy} + u^5 u_{y(5)}, \\
y_{x(7)} &= uu_y^6 + 57u^2 u_y^4 u_{yy} + 122u^3 u_y^3 u_{yyy} + 34u^4 u_{yy}^3 + 180u^3 u_y^2 u_{yy}^2 \\
&\quad +76u^4 u_{yy}^3 + 15u^5 u_{y(3)}2 + 192u^4 u_y u_{yy} u_{y(3)} \\
&\quad +26u^5 u_{yy} u_{y(4)} + 16u^5 u_y u_{y(5)} + u^6 u_{y(6)}
\end{aligned}
$$

Table 48.1: How to transform derivatives under the change of independent variable: $u(y) = y_x(x)$. (To simplify notation, we have defined $y_{x(n)}$ to be the nth derivative of y with respect to x. Similarly for $u_{y(n)}$.)

where A is a constant. Conversely, if $u(y) \neq 0$, then equation (48.2) requires that

$$\frac{du}{dy} - 1 - 2y = 0. \tag{48.4}$$

Equation (48.4) can be integrated to obtain

$$u(y) = y^2 + y + B, \tag{48.5}$$

where B is a constant. Using $u(y) = \frac{dy}{dx}$, equation (48.5) can be written as $\frac{dy}{dx} = y^2 + y + B$, so that $\int \frac{dy}{y^2+y+B} = \int dx$, and therefore $\frac{2}{D}\tan^{-1}\left(\frac{2y+1}{D}\right) = x + C$, where $D^2 = 4B - 1$ and C is an additional constant. Inverting this last equation gives y explicitly as a function of x

$$y(x) = E\tan(Ex + F) - \frac{1}{2}, \tag{48.6}$$

where $E = D/2$ and $F = CE$. Hence, the two solutions to equation (48.1) are given by equations (48.3) and (48.6).

Notes

1. This method is derivable from Lie group methods (see page 366).
2. Schwarz's paper [4] describes a REDUCE program that will automatically determine first integrals for an autonomous system of equations.
3. The easiest way to make the necessary transformation in an autonomous differential equation is by replacing every occurrence of

$\frac{d}{dx}$ with $u\frac{d}{dy}$. For instance, writing equation (48.1) in the form

$$\frac{d}{dx}\left(\frac{d}{dx}(y)\right) - \frac{d}{dx}(y) = 2y\frac{d}{dx}(y)$$

leads immediately to equation (48.2) via

$$u\frac{d}{dy}\left(u\frac{d}{dy}(y)\right) - u\frac{d}{dy}(y) = 2yu\frac{d}{dy}(y).$$

4. Sometimes it is advantageous to write a pair of first order autonomous equations as a single first order equation, by dividing the two equations. For example, the non-linear predator–prey equations

$$\frac{dx}{dt} = ax - bxy, \qquad \frac{dy}{dt} = -cy + dxy \tag{48.7}$$

can be written in the form

$$\frac{dx}{dy} = \frac{ax - bxy}{-cy + dxy}. \tag{48.8}$$

Although equation (48.7) cannot be solved explicitly in finite terms, from equation (48.8) we can show that $F(x, y) := dx + by - c\log x - a\log y$ is a constant on the solution curves $\{x(t), y(t)\}$.

5. It is straightforward to create a Macsyma program that will perform the necessary change of variables. Program 48.1 shows a terminal session in which the input equation

$$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 - 1 + \frac{1}{y^3} = 0$$

is transformed into

$$y^3 - u\frac{du}{dy}y^2 + u^2y - 1 = 0.$$

6. Autonomous systems of ordinary differential equations can have *center manifolds*, which are a classification of the solution surface. As a simple example, consider the system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y}' = B\mathbf{x} + \mathbf{g}(\mathbf{x}, \mathbf{y}), \tag{48.9}$$

where A is a constant matrix all of whose eigenvalues are imaginary, B is a constant matrix all of whose eigenvalues have negative real part, and the functions $\mathbf{f}$ and $\mathbf{g}$ and their first derivatives vanish at the point $(\mathbf{0}, \mathbf{0})$. Then, there is a function $\mathbf{h}$ such that

- $\mathbf{h}$ is an invariant manifold under equation (48.9).
- $\mathbf{h}$ and its first derivatives vanish at $(\mathbf{0}, \mathbf{0})$.

```
DEPENDS(Y,X)$
AUTONOMOUS(EQN,Y,X):= BLOCK([NEW,A,U,MAX_DEGREE,J],
    DEPENDS(U,Y),
    MAX_DEGREE:DERIVDEGREE(EQN,Y,X),
    KILL(A),
    A[0]:Y,
    FOR J:1 THRU MAX_DEGREE DO (
            A[J]:EXPAND( SUBST(U,DIFF(Y,X),DIFF(A[J-1],X)) ) ),
    FOR J:1 THRU MAX_DEGREE DO (
            NEW: SUBST( A[J], DIFF(Y,X,J), NEW ) ),
    FACTOR(NEW) )$
EQN: DIFF( DIFF(Y,X)/Y, X) - 1 + 1/Y**3;
                                          2
                             y      (y )
                              x x     x         1
                             ---- - ----- - 1 + --
                              y       2          3
                                     y          y
AUTONOMOUS(EQN,Y,X);
                             3         2    2
                            y  - u u  y  + u  y - 1
                                    y
                            -----------------------
                                        3
                                       y
```

Program 48.1: Macsyma program to change variables.

```
dy[1]= u[y[x]];
dy[2]= D[u[y[x]],x] /. y'[x]->u[y[x]];
dy[n_]:= D[dy[n-1],x] /. y'[x]->u[y[x]]
dy2[n_]:= dy[n] /. {u[y[x]]->u, u'[y[x]]->u', u''[y[x]]->u'',
                          u'''[y[x]]->u''', u''''[y[x]]->u''''}
Table[ {n,dy2[n]}, {n,1,5}] // ColumnForm
```

Program 48.2: Mathematica program to change variables: $u(y) = y_x(x)$.

- The stability of the solution $(\mathbf{0}, \mathbf{0})$ is the same as that of the smaller system $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$.

7. The results in table 48.1 can be obtained with the Mathematica code in program 48.2. The output of that program is

```
{1, u}
{2, u u'}
        2    2
{3, u u'  + u  u''}
        3      2            3  (3)
{4, u u'  + 4 u  u' u'' + u  u   }
        4       2   2           3    2      3     (3)
{5, u u'  + 11 u  u'  u'' + 4 u  u''  + 7 u  u' u    +

   4  (4)
  u  u   }
```

8. See Bender and Orszag [1, pages 24–25] and Rainville and Bedient [3, pages 268–269].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.

[2] Man, Y. K. First integrals of autonomous systems of differential equations and the Prelle–Singer procedure. *J. Phys. A: Math. Gen. 27* (1994), L329–L332.

[3] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations.* The MacMillan Company, New York, 1964.

[4] Schwarz, F. A REDUCE package for determining Lie symmetries of ordinary and partial differential equations. *Comput. Physics Comm. 27* (1982), 179–186.

49. Bernoulli Equation

Applicable to Ordinary differential equations of the form: $y' + P(x)y = Q(x)y^n$.

Yields

An exact solution of the given equation.

Idea

By a change of dependent variable, a Bernoulli equation (which is a nonlinear equation of the form $y' + P(x)y = Q(x)y^n$, where n is not equal to 1) can be transformed to a first order linear equation. This linear equation can be solved by the use of integrating factors.

Procedure

Suppose we have the equation

$$y' + P(x)y = Q(x)y^n, \tag{49.1}$$

which we recognize to be a Bernoulli equation. To solve, we divide the equation by y^n and change the dependent variable from $y(x)$ to $u(x)$ by

$$u(x) = y(x)^{1-n}.$$

This changes equation (49.1) into the first order linear differential equation

$$\frac{1}{1-n}u' + P(x)u = Q(x). \tag{49.2}$$

An exact solution of equation (49.2) can be found by integrating factors (see page 356). The solution is given by

$$u(x) = \exp\left[(n-1)\int^x P(t)\,dt\right]\left\{\int^x \exp\left[(1-n)\int^s P(t)\,dt\right]Q(s)\,ds\right\}. \tag{49.3}$$

Example

Suppose we have the equation

$$y' + y = y^3 \sin x. \tag{49.4}$$

To solve this equation, divide it by y^3 and then define $u(x) = y(x)^{-2}$ so that equation (49.4) becomes

$$-\frac{1}{2}u' + u = \sin x. \tag{49.5}$$

The solution to equation (49.5) (obtained by the method of integrating factors) is

$$u(x) = Ae^{2x} + \frac{2}{5}(\cos x + 2\sin x),$$

where A is an arbitrary constant. Using $y(x) = u(x)^{-1/2}$, the final solution is found to be

$$y(x) = \left\{ Ae^{2x} + \frac{2}{5}(\cos x + 2\sin x) \right\}^{-1/2}.$$

Notes

1. If $n = 1$, then the original equation is in the form of equation (49.2); and it can be solved directly by the use of integrating factors.
2. See also Boyce and DiPrima [1, page 28], Ince [2, page 22], Rainville and Bedient [3, pages 69–71], and Simmons [4, page 49].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[3] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations.* The MacMillan Company, New York, 1964.

[4] Simmons, G. F. *Differential Equations with Applications and Historical Notes.* McGraw–Hill Book Company, New York, 1972.

50. Clairaut's Equation

Applicable to

Differential equations of the form: $f(xy' - y) = g(y')$.

Yields

An exact implicit solution. Sometimes a singular solution may also be obtained.

Idea

A solution of the differential equation $f(xy' - y) = g(y')$ is known.

Procedure

Given the equation

$$f(xy' - y) = g(y'), \tag{50.1}$$

a general solution (for which $y'' = 0$) is given implicitly by

$$f(xC - y) = g(C), \tag{50.2}$$

where C is an arbitrary constant. Equation (50.1) may also have a singular solution. If it does, it can be obtained by differentiating equation (50.1) with respect to x to obtain

$$y'' \left[f'(xy' - y)x - g'(y') \right] = 0. \tag{50.3}$$

If the first term in equation (50.3) is zero, then equation (50.2) is recovered. If the second term in equation (50.3) is zero, then equations (50.1) and (50.2) can be solved together to eliminate y'. The resulting equation for $y = y(x)$ will have no arbitrary constants and so will be a singular solution.

Example 1

Suppose we have the ordinary differential equation

$$(xy' - y)^2 - (y')^2 - 1 = 0. \tag{50.4}$$

Because equation (50.4) is of the same form as equation (50.1) (with $f(x) = x^2$, $g(x) = x^2 - 1$), a general solution can immediately be written down as $(xC - y)^2 = C^2 + 1$ or

$$y = Cx \pm \sqrt{C^2 - 1}, \tag{50.5}$$

where C is an arbitrary constant.

To find the singular solution, we differentiate equation (50.4) with respect to x to obtain

$$y''[2(xy' - 2)x - 2y'] = 0.$$

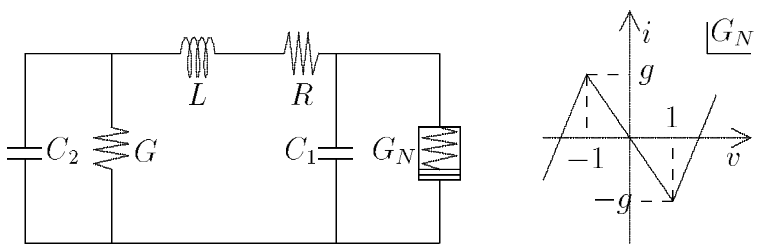

Figure 50.1: Solution curves for the differential equation in Example 2.

If the second term is set equal to zero, then we find

$$y' = \frac{xy}{x^2 - 1}. \tag{50.6}$$

Using equation (50.6) in equation (50.4), we determine the singular solution to be

$$x^2 + y^2 = 1. \tag{50.7}$$

Note that equation (50.7) is not derivable from (50.5) for any choice of C.

Example 2

For the differential equation $xy' - y = g(y')$, with $g(z) = \frac{5}{2}(z^3 - z)$, a set of solution curves is shown in figure 50.1. Because $g(z)$ is a cubic, there are regions where there are three different solutions for a specified x and y. This is clearly shown in the figure.

The singular solution to the above differential equation can be easily shown to be $y = (5 + 2x)^{3/2}/\sqrt{135}$.

Notes

1. The singular solution obtained by this method turns out to be the locus of the solutions in equation (50.2). That is, the envelope of the solutions in equation (50.2), for all possible values of the parameter C, will be the singular solution. See Ford [1, pages 16–18] for details.

2. A generalization of Clairaut's equation is Lagrange's equation (see page 363).
3. Clairaut's partial differential equation $z = \sum_{i=1}^{n} x_i \frac{\partial z}{\partial x_i} + f\left(\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right)$ has the solution $z = \sum_{i=1}^{n} a_i x_i + f(a_1, a_2, \ldots, a_n)$. See Kamke [3, section 13.8, page 123].
4. See also Ince [2, pages 39–40] and Rainville and Bedient [4, pages 263–265].

References

[1] Ford, L. R. *Differential Equations.* McGraw–Hill Book Company, New York, 1955.

[2] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[3] Kamke, E. *Differentialgleichungen Losungsmethoden und Losungen*, vol. II. Chelsea Publishing Company, New York, 1947.

[4] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations.* The MacMillan Company, New York, 1964.

51. Computer-Aided Solution

Applicable to Some classes of ordinary differential equations, most frequently first and second order equations.

Yields

An exact solution.

Idea

Several of the popular computer algebra languages have a symbolic differential equation solver.

Procedure

Find a computer system that runs any of the following commercial computer languages: AXIOM, Derive, FORMAC, Macsyma, Maple, Mathematica, muMath, or REDUCE. Identify the routine that solves differential equations automatically, and use that on your problem. For URLs of these software packages, see page 71.

Nearly all of the symbolic algebra programs have a specialied interface that makes it easy to identify and use the differential equation solver. This interface usually displays the output in a very attractive way; the ascii output shown below is less attractive but represents one output option.

In each of the packages below a different package was asked to solve the simple differential equations $y'' + 4y = 0$ and $y' = xy^2 + y$.

Example 1

The following Macsyma session was run by Jeff Golden. Note that `(c2), (c3), and (c4)` are input lines ("command" lines) and that `(d2), (d3), and (d4)` are output lines ("display" lines). On the first input line, the first equation is defined to be `eqn1`. On the second line, a solution is requested. Note that `%k1` and `%k2` are arbitrary constants in the solution that Macsyma found. The third input line defines the second equation to be `eqn2`, and the fourth line requests the solution (in this case `%c` is the arbitrary constant in the solution).

```
Starting Macsyma math engine with no window system...
This is Macsyma 421.0 for SGI (IRIX) computers.
Copyright (c) 1982 - 1997 Macsyma Inc.  All rights reserved.
Portions copyright (c) 1982 Massachusetts Institute of Technology.
All rights reserved.
Type "DESCRIBE(TRADE_SECRET);" to see important legal notices.
Type "HELP();" for more information.

/usr/macsyma-421/system/init.lsp being loaded.
```

```
(c1) eqn1: 'diff(y,x,2) + 4*y = 0;

                   2
                  d y
(d1)              --- + 4 y = 0
                    2
                  dx

(c2) ode(eqn1, y, x);

/usr/macsyma-421/ode/ode.o being loaded.
/usr/macsyma-421/ode/odeaux.o being loaded.
/usr/macsyma-421/ode/ode2.o being loaded.

(d2)           y = %k1 sin(2 x) + %k2 cos(2 x)

(c3) eqn2: 'diff(y,x) = x*y^2 + y;

                dy      2
(d3)            -- - x y  - y = 0
                dx

(c4) ode(eqn2, y, x);

                               x
                             %e
(d4)               y = ----------------
                                      x
                       %c - (x - 1) %e
```

Example 2

The following MAPLE session was run by the author. Note that input lines begin with a greater than sign. On the first input line, the first equation is defined to be `eqn1`. On the second input line, a solution is requested. Note that `_C1` and `_C2` are arbitrary constants in the solution that MAPLE found. The third input line defines the second equation to be `eqn2`, and the fourth line requests the solution.

```
    |\^/|     Maple V Release 3 (Zwillinger & Associates)
._|\|   |/|_. Copyright (c) 1981-1994 by Waterloo Maple Software and the
 \  MAPLE  /  University of Waterloo. All rights reserved. Maple and Maple V
 <____ ____>  are registered trademarks of Waterloo Maple Software.
      |       Type ? for help.

> eqn1:= diff(y(x),x$2)+4*y(x)=0;

                               /   2      \
                               |  d       |
                       eqn1 := |----- y(x)| + 4 y(x) = 0
                               |   2      |
                               \ dx       /

> dsolve( eqn1, y(x) );
```

```
                    y(x) = _C1 cos(2 x) + _C2 sin(2 x)

> eqn2:= diff(y(x),x)-x*y(x)^2-y(x)=0;

                        /  d      \          2
               eqn2 := |---- y(x)| - x y(x)  - y(x) = 0
                        \ dx      /

> dsolve( eqn2, y(x) );

                       1
                     ---- = - x + 1 + exp(- x) _C1
                     y(x)
```

Example 3

The following Mathematica session was run by Alexei Bocharov. Note that the nth input line is denoted `In[`n`]` and the nth output line is denoted `Out[`n`]`. On the first input line (`In[4]`), the first equation is input and the solution is requested. Note that `C[1]` and `C[2]` are arbitrary constants in the solution that Mathematica found. The next input line defines the second equation and requests the solution.

```
In[4]:= DSolve[y''[x]+4y[x]==0,y[x],x]

Out[4]= {{y[x] -> C[2] Cos[2 x] - C[1] Sin[2 x]}}

In[5]:= DSolve[y'[x]==x*y[x]^2+y[x],y[x],x]

                               1
Out[5]= {{y[x] -> ----------------}}
                                -x
                    1 - x - E   C[1]
```

Example 4

The following MuPAD terminal session was run by Paul Zimmermann. Note that input lines begin with the symbol >>. The first command, `setuserinfo(ode,1)`, tells the system to prints comments. On the second input line, the first equation is input and the solution is requested. Note that `C1`, `C2`, and `C3` are arbitrary constants in the solutions that MuPAD found. The next input line defines the second equation and requests the solution.

```
   *----*    MuPAD 1.4.0 --- Multi Processing Algebra Data Tool
  /|   /|
 *----* |    Copyright (c) 1992-97 by B. Fuchssteiner, Automath
 | *--|-*    University of Paderborn. All rights reserved.
 |/   |/
 *----*      ----------- Developers NSB Version ---------------

>> setuserinfo(ode,1):
>> solve(ode(y'(x)=x*y(x)^2-y(x), y(x)));
```

```
Riccati equation
Riccati method worked

                         {            1        }
                         { 0, ----------------- }
                         {    x + C1 exp(x) + 1 }


>> solve(ode(y''(x)+4*y(x)=0, y(x)));
linear ordinary differential equation of order 2
with constant coefficients

                        {C2 cos(2 x) + C3 sin(2 x)}
```

Example 5

The following Derive terminal session was run by David Stoutemyer. Note that input and output lines begin with an octothorpe (#) and are numbered consecutively. The input was entered in a one-line dialog box that had a Greek toolbar and other capabilities.

```
#2:  DSOLVE2(0, 4, 0, x, y)                          User

#3:  y COS(2 x) + c2 SIN(2 x)                    Simp(#2)

#4:  BERNOULLI_GEN(-1, x, 2, x, y)                   User

      1        x
#5:  --- = c #e  - x + 1                         Simp(#4)
      y
```

Example 5

The following REDUCE terminal session was run by Winfried Neun. Note that all input lines are numbered. The first command tells the system to load the ODE solver. On the second input line the first equation is input and the solution is requested. Note that `arbconstant(1)` –`arbconstant(3)` are arbitrary constants in the solutions that REDUCE found. The next input line defines the second equation and requests the solution.

```
1: load odesolve;

(odesolve)

2: depend y,x;

3:  odesolve(df(y,x,2)+4*y=0,y,x);

{y= - arbconst(2)*sin(2*x) + arbconst(1)*cos(2*x)}

4: odesolve (df(y,x)=x*y^2 +y,y,x);

                      x      x
```

```
  1   arbconst(3) - e *x + e
{---=------------------------}
  y              x
                e
```

Notes

1. A comparative differential equation review of the languages AXIOM, Derive, Macsyma, Maple, Mathematica, MuPad, and REDUCE is maintained by Postel and Zimmermann [13]. Presently, they have 54 equations that they have run though each of the above systems; the input and output files for each are available.
2. Moussiaux [12] has made available the program `CONVODE`, which symbolically solves ordinary and partial differential equations across the internet. For example, sending

   ```
   depend y,x;
   CONVODE( {df(y,x,2)+4*y=0}, {y}, {x}, {}, {english});
   ```

   to `convode@riemann.physmath.fundp.ac.be` will have the solution of $y''+4y=0$ sent to you via email with comments in English (the default is French). See `http://www.physique.fundp.ac.be/physdpt/administration/convode.html`. Note that `CONVODE` is based on REDUCE.
3. REDUCE can be used interactively over the web via the site `http://www.zib-berlin.de/Symbolik/reduce/testreduce.html`.
4. MathServ provides an interface between the user and Mathematica (see `http://math.vanderbilt.edu/~pscrooke/detoolkit.shtml`). Templates for twelve different types of ODEs are available; the user can specify the functions appearing in them.
5. Packages that can handle a wider variety of differential equations are constantly being created. See, for example, Chan [2], Kovacic [9], Schmidt [15], or Watanabe [20]. An example of the use of FORMAC may be found in Hanson *et al.* [5]. Shtokhamer [16] presents a Macsyma program that implements the Prelle–Singer algorithm and gives several examples.
6. All of the programs illustrated above and many others (such as the package by Hubbard and West [7]) can be run on a microcomputer (such as an IBM PC or a Macintosh).
7. Given a homogeneous linear differential equation whose coefficients are in a finite algebraic extension of $\mathbf{Q}[x]$, Singer's [17] paper has a decision procedure to determine a basis for the Liouvillian solutions. Liouvillian functions are essentially those functions that can be built up from rational functions by algebraic operations, taking exponentials and by integration. In detail

   - Let K be a field of functions. The function θ is a *Liouvillian generator* over K if it is:

  - algebraic over K, that is if θ satisfies a polynomial equation with coefficients in K;
  - exponential over K, that is if there is a ζ in K such that $\theta' = \zeta'\theta$, which is an algebraic way of saying that $\theta = \exp\zeta$; or
  - an integral over K, that is if there is a ζ in K such that $\theta' = \zeta$, which is an algebraic way of saying that $\theta = \int \zeta$.
- Let K be a field of functions. An over-field $K(\theta_1, \ldots, \theta_n)$ of K is called a *field of Liouvillian functions over* K if each θ_i is a Liouvillian generator over K. A function is *Liouvillian over* K if it belongs to a Liouvillian field of functions over K.

Then, some of the important theorems in this area are

Theorem There is an algorithm that, given a second order linear differential equation, $y''+ay'+by = 0$ with a and b rational functions of x, either finds two Liouvillian solutions such that every solution is a linear combination with constant coefficients of these two solutions or proves that there is no Liouvillian solution (except zero).

Theorem There is an algorithm that, given a linear differential equation of any order, the coefficients of which are rational or algebraic functions: either finds a Liouvillian solution or proves that there is none.

Theorem Let A be a class of functions containing the coefficients of a linear differential operator L, let g be an element of A, and let us suppose that the equation $L[y] = g$ has an elementary solution over A. Then, either $L[w] = 0$ has an algebraic solution over A, or y belongs to A.

Theorem Let A be a class of functions, that contains the coefficients of a linear differential operator L, let g be an element of A, and let us suppose that the equation $L[y] = g$ has a Liouvillian solution over A. Then either $L[w] = 0$ has a solution $\exp(\int z(x)\,dx)$ with z algebraic over A, or y belongs to A.a

See Davenport *et al.* [4] for details. See also Bronstein [1].

References

[1] Bronstein, M. The transcendental Risch differential equation. *J. Symbolic Comp. 9*, 1 (1990), 49–60.

[2] Chan, W. C. A novel symbolic ordinary differential equation solver. *SIGSAM Bulletin 15*, 3 (August 1981), 9–14.

[3] Char, B. W., Gedded, K. O., Gonnet, G. H., Leong, B. L., Monagan, M. B., and Watt, S. M. *MAPLE V Library Reference Manual.* Springer–Verlag, New York, 1991.

[4] Davenport, J. H., Siret, Y., and Tournier, E. *Systems and Algorithms for Algebraic Computation*. Academic Press, New York, 1988.

[5] Hanson, J. H., Benander, A. C., and Benander, B. A. The computer generated symbolic solution of a system of linear first order differential equations. *Comp. & Maths. with Appls. 19*, 7 (1990), 7–12.

[6] Hearn, A. C. REDUCE 2 user's manual. Tech. Rep. UCP-19, University of Utah, Salt Lake City, 1973. Computational Physics Group Report.

[7] Hubbard, J., and West, B. *MacMath: A Dynamical Systems Software Package*. Springer–Verlag, New York, 1991.

[8] Inference Corp. 3916 S. Sepulveda Blvd., Culver City, CA, 90230.

[9] Kovacic, J. J. An algorithm for solving second order linear homogeneous differential equations. *J. Symbolic Comp. 2* (1986), 3–43.

[10] Macsyma. *Mathematics Reference Manual*. Macsyma, Arlington, MA, 1993.

[11] Meyer, K. R., and Schmidt, D. *Computer Aided Proofs in Analysis*. Springer–Verlag, New York, 1990.

[12] Moussiaux, A. CONVODE: A REDUCE package for solving differential equations. *J. Comput. Appl. Math. 48*, 1–2 (1993), 157–165.

[13] Postel, F., and Zimmermann, P. A review of the ODE solvers of Axiom, Derive, Macsyma, Maple, Mathematica, MuPad, and Reduce. In *Proceedings of the 5th Rhine Workshop On Computer Algebra* (Saint-Louis, France, 1996), ISL. `http://www.loria.fr/~zimmerma/ComputerAlgebra`.

[14] Prelle, M. J., and Singer, M. F. Elementary first integrals of differential equations. *Trans. Amer. Math. Soc. 279*, 1 (September 1983), 215–229.

[15] Schmidt, P. Substitution methods for the automatic symbolic solution of differential equations of first order and first degree. In *EUROSAM 79*, E. Ng, Ed., Lecture Notes in Computer Science 1979. Springer–Verlag, New York, 1979, pp. 164–176.

[16] Shtokhamer, R. Solving first order differential equations using the Prelle–Singer algorithm. Tech. Rep. 88-09, University of Delaware, Newark, DE, 1988.

[17] Singer, M. F. Liouvillian solution of n-th order homogeneous linear differential equations. *Am. J. Math. 103*, 4 (1981), 661–682.

[18] Tournier, E. *Computer Algebra and Differential Equations*. Academic Press, New York, 1990.

[19] Watanabe, S. A technique for solving ordinary differential equations using Riemann's P–functions. In *SYMSAC 81: Proceedings of the 1981 ACM Symposium on Symbolic and Algebraic Computation*, P. S. Wang, Ed. ACM, New York, 1981, pp. 36–43.

[20] Watanabe, S. An experiment towards a general quadrature for second order linear ordinary differential equations by symbolic computation. In *EUROSAM 1984*, J. Fitch, Ed., Lecture Notes in Computer Science. Springer–Verlag, New York, 1984, pp. 13–22.

[21] Wolfram, S. *The Mathematica Book*, third ed. Cambridge University Press, New York, 1996.

[22] Wooff, C., and Hodgkinson, D. *muMATH: A Microcomputer Algebra System*. Academic Press, New York, 1987.

52. Constant Coefficient Linear Equations

Applicable to Homogeneous linear ordinary differential equations with constant coefficients.

Yields

An exact solution.

Idea

Linear constant coefficient ordinary differential equations have exponential solutions. The method of undetermined coefficients can be used to solve this type of equation after a polynomial has been factored.

Procedure

Given the nth order linear equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{52.1}$$

where the $\{a_i\}$ are constants, look for a solution of the form

$$y(x) = Ce^{\lambda x}, \tag{52.2}$$

where C is an arbitrary constant. Substituting equation (52.2) into equation (52.1) yields

$$e^{\lambda x}\left[\lambda^n + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0\right] = 0. \tag{52.3}$$

Hence, equation (52.2) is a solution of equation (52.1) if λ is a root of the *characteristic equation*, defined by

$$\lambda^n + a_{n-1}\lambda^{(n-1)} + \cdots + a_1\lambda + a_0 = 0. \tag{52.4}$$

If equation (52.4) has n different roots $\{\lambda_i\}$, then the general solution to (52.1) is, by use of superposition,

$$y(x) = C_n e^{\lambda_n x} + C_{n-1}e^{\lambda_{n-1}x} + \cdots + C_1 e^{\lambda_1 x},$$

where the $\{C_i\}$ are arbitrary constants. If some of the roots of equation (52.4) are repeated (say $\lambda_1 = \lambda_2 = \cdots = \lambda_m$), then the solution corresponding to these $\{\lambda_i\}$ is

$$y(x) = (C_m x^{m-1} + C_{m-1}x^{m-2} + \cdots + C_2 x + C_1)e^{\lambda_1 x}.$$

Example

Given the linear differential equation

$$y^{(7)} - 14y^{(6)} + 80y^{(5)} - 242y^{(4)} + 419y^{(3)} - 416y'' + 220y' - 48y = 0, \tag{52.5}$$

we substitute $y(x) = e^{\lambda x}$ to find the characteristic equation

$$\lambda^7 - 14\lambda^6 + 80\lambda^5 - 242\lambda^4 + 419\lambda^3 - 416\lambda^2 + 220\lambda - 48 = 0,$$

which factors as

$$(\lambda - 1)^3(\lambda - 2)^2(\lambda - 3)(\lambda - 4) = 0. \tag{52.6}$$

The roots of equation (52.6) are $\{1, 1, 1, 2, 2, 3, 4\}$. The general solution to equation (52.5) is therefore

$$y(x) = \{C_0 + C_1 x + C_2 x^2\}e^x + \{C_3 + C_4 x\}e^{2x} + C_5 e^{3x} + C_6 e^{4x},$$

where $\{C_0, \ldots, C_6\}$ are arbitrary constants.

Notes

1. Using the transformation described on page 146, the system in equation (52.1) can be written in the form $\mathbf{y}' = A\mathbf{y}$, where A is an $n \times n$ constant matrix. Then the techniques for vectors ODEs (see page 421) may be used.
2. See Boyce and DiPrima [1, section 5.3, pages 263–268] and Simmons [2, pages 83–86].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

53. Contact Transformation

Applicable to First order and (occasionally) second order ordinary differential equations.

Yields

A reformulation, which may lead to an exact solution (sometimes in parametric form).

Idea

By changing variables, a different and sometimes easier differential equation may be found.

Procedure

Given a relation between three variables

$$\phi(x, y, p) = 0, \tag{53.1}$$

it will be a first order ordinary differential equation if $dy - p\,dx = 0$. If the variables in equation (53.1) are changed by

$$\begin{aligned} x &= x(X, Y, P), \\ y &= y(X, Y, P), \\ p &= p(X, Y, P), \end{aligned} \tag{53.2}$$

then the transformed equation $\Phi(X, Y, P) = 0$ will also be an ordinary differential equation if $dY - P\,dX = 0$. If this is true, then equation (53.2) is a *contact transformation*. For example, the change of variables

$$\left\{ \begin{aligned} x &= P \\ y &= PX - Y \\ p &= X \end{aligned} \right\} \Longleftrightarrow \left\{ \begin{aligned} X &= p \\ Y &= px - y \\ P &= x \end{aligned} \right\} \tag{53.3}$$

is a contact transformation. It is easy to show this:

$$\begin{aligned} 0 &= dy - p\,dx \\ &= d(PX - Y) - X\,dP \\ &= P\,dX - dY. \end{aligned}$$

If the new differential equation, $\Phi(X, Y, P) = 0$, can be solved, then the solution to $\phi(x, y, p) = 0$ may be determined by eliminating X, Y, and P from the original equation, using the solution found and the transformation rules.

Example

Suppose we have the nonlinear first order ordinary differential equation

$$2y\left(\frac{dy}{dx}\right)^2 - 2x\,\frac{dy}{dx} - y = 0, \tag{53.4}$$

which we may write as

$$2yp^2 - 2xp - y = 0.$$

We utilize the contact transformation in equation (53.3) to obtain, after some algebra, the new first order ordinary differential equation

$$P + Y\left(\frac{1-2X^2}{2X^3-3X}\right) = 0 \qquad \text{or} \qquad \frac{dY}{dX} + Y\left(\frac{1-2X^2}{2X^3-3X}\right) = 0. \tag{53.5}$$

This differential equation can be solved by integrating factors to obtain

$$Y = C\left(2X^3 - 3X\right)^{1/3}, \tag{53.6}$$

where C is an arbitrary constant. Now that we have the solution of the transformed equation, we can find the solution of the original differential equation.

Utilizing $Y = xX - y$ and $P = x$ from equation (53.3), equations (53.5) and (53.6) can be written as

$$\begin{aligned} x + (xX - y)\left(\frac{1-2X^2}{2X^3-3X}\right) &= 0, \\ xX - y &= \left(2X^3 - 3X\right)^{1/3}. \end{aligned} \tag{53.7}$$

Now X can be eliminated between these two equations by, say, the method of resultants (see page 50). This produces the solution to equation (53.4) in the form $f(x,y) = 0$ (there are 21 algebraic terms in this representation). Alternately, we can obtain a parametric representation of the solution by solving equation (53.7) for $x = x(X)$ and $y = y(X)$ and then treating X as a parameter.

Notes

1. Composing two contact transformations or taking the inverse of a contact transformation results in another contact transformation. Because the identity transformation is also a contact transformation, the set of all contact transformations forms an infinite dimensional topological group.
2. This method is derivable from the method of Lie groups (see page 366), where it goes by the name of the *extended group of transformations*. See Ince [4, pages 40–42] or Seshadri and Na [6, pages 18–20].

3. The condition $dy - pdx = 0$ states that, if the point (x, y) is on a curve, then p should be its tangent. The change of variables in this method gives a different parameterization of the same curve. In particular, if two curves touch in the old parameterization, then they also touch in the new parameterization; hence the name of the transformation.
4. Some second order ordinary differential equations also may be solved by this method. If $R = \frac{dP}{dX} = \frac{d^2Y}{dX^2}$ and $\frac{1}{R} = \frac{dp}{dx} = \frac{d^2y}{dx^2}$, then we may use the relation $dP - RdX = dx - Rdp$.
5. In more generality, a transformation of the $2n+1$ variables $\{z, x_j, p_j \mid j = 1, \dots, n\}$ to the $2n + 1$ variables $\{Z, X_j, P_j \mid j = 1, \dots, n\}$ is a contact transformation if the total differential equation

$$dz - p_1 dx_1 - p_2 dx_2 - \cdots - p_n dx_n = 0$$

is invariant under the transformation; that is, if the equality

$$\begin{aligned}(dZ - P_1 dX_1 - P_2 dX_2 - \cdots - P_n dX_n)& \\ = \rho\,(dz - p_1 dx_1 - p_2 dx_2 - \cdots - p_n dx_n)&\end{aligned}$$

holds identically for some nonzero function $\rho(\mathbf{x}, \mathbf{p}, z)$. See Iyanaga and Kawada [5, pages 286 and 1448] for details.
6. A contact transformation is also a canonical transformation (see page 132). The generating function of the canonical transformation, Ω, satisfies the three relations: $\Omega(x, z, X, Z) = 0$, $\frac{\partial\Omega}{\partial X_j} + P_j \frac{\partial\Omega}{\partial Z} = 0$, and $\frac{\partial\Omega}{\partial x_j} + p_j \frac{\partial\Omega}{\partial z} = 0$.
7. Named contact transformations include

   (a) The Legendre transformation (see page 467) is given by $\Omega = Z + z + \sum x_j X_j$, $Z = \sum_j p_j x_j - z$, $X_j = -p_j$, $P_j = -x_j$, and $\rho = -1$.
   (b) The Pedal transformation is given by $\Omega = Z^2 - zZ - \sum x_j X_j + \sum X_j^2$, $X_j = -p_j Z$, $p_j = -\frac{2X_j - x_j}{2Z - z}$, and $\rho = \frac{Z}{2Z-z}$.
   (c) The similarity transformation is given by $\Omega = (Z - z)^2 - a^2 + \sum (X_j - x)j)^2$, $X_j = x_j - ap_j \left(1 + \sum p_j^2\right)^{-1/2}$, $P_j = p_j$, $Z = x_j + a\left(1 + \sum p_j^2\right)^{-1/2}$, and $\rho = 1$.

8. Some other contact transformations are

$$\left\{\begin{aligned} x &= X - YP \\ y &= -Y\sqrt{P^2 - 1} \\ p &= \frac{P}{\sqrt{P^2 - 1}} \end{aligned}\right\} \iff \left\{\begin{aligned} X &= x - yp \\ Y &= y\sqrt{p^2 - 1} \\ P &= -\frac{p}{\sqrt{p^2 - 1}} \end{aligned}\right\} \qquad (53.8)$$

$$\left\{\begin{array}{l} x = X - \dfrac{aP}{\sqrt{1+P^2}} \\ y = Y + \dfrac{a}{\sqrt{1+P^2}} \\ p = P \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} X = x + \dfrac{ap}{\sqrt{1+p^2}} \\ Y = y - \dfrac{a}{\sqrt{1+p^2}} \\ P = p \end{array}\right\}. \tag{53.9}$$

9. See also Bateman [1, pages 81–83], Carathéodory [2, Chapter 7, pages 102–120], and Chester [3, pages 206–207].

References

[1] Bateman, H. *Partial Differential Equations of Mathematical Physics.* Dover Publications, Inc., New York, 1944.

[2] Caratheodory, C. *Calculus of Variations and Partial Differential Equations of the First Order.* Holden–Day, Inc., San Francisco, CA, 1965.

[3] Chester, C. R. *Techniques in Partial Differential Equations.* McGraw–Hill Book Company, New York, 1970.

[4] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[5] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics.* MIT Press, Cambridge, MA, 1980.

[6] Seshadri, R., and Na, T. Y. *Group Invariance in Engineering Boundary Value Problems.* Springer–Verlag, New York, 1985.

54. Delay Equations

Applicable to Ordinary differential delay equations.

Yields

In many cases, an exact analytical solution.

Idea

There are several standard techniques for delay equations.

Procedure

The standard methods for solving delay equations are by the use of

- Laplace transforms
- Fourier transforms
- Generating functions
- General expansion theorems
- The method of steps

For the first two methods, the technique is the same as it is for ordinary differential equations (see page 347). That is, the transform is taken of the delay equation; by algebraic manipulations the transform is explicitly determined; and then an inverse transformation is taken. See Example 1.

For a delay equation with a single delay, the *method of steps* consists of solving the delay equation in successive intervals, whose length is the time delay. In each interval, only an ordinary differential equation needs to be solved. See Example 2.

The method of generating functions is frequently used when only integral values of the variables are of interest. The technique is similar to the technique for integral transforms described above. For generating functions, the integration is replaced by a summation, and the "inverse transformation" is generally a differentiation (see page 315 for more details). See Example 3.

The general expansion theorems are all of the same form; given a delay equation, the solution can be expressed as a sum over the roots of a transcendental equation called the *characteristic equation.*

Example 1

Suppose we have the delay equation

$$y'(t) + ay(t-1) = 0, \tag{54.1}$$

with the boundary conditions

$$y(t) = y_0 \qquad \text{when } -1 \leq t \leq 0, \tag{54.2}$$

where a is a constant. We define the Laplace transform of $y(t)$ to be $Y(s)$ by $Y(s) = \int_0^\infty e^{-st} y(t)\,dt$. Multiplying equation (54.1) by e^{-st} and integrating with respect to t yields

$$\int_0^\infty e^{-st} y'(t)\,dt \,+\, a \int_0^\infty e^{-st} y(t-1)\,dt = 0. \tag{54.3}$$

The first integral in equation (54.3) can be integrated by parts to yield

$$\int_0^\infty e^{-st} y'(t)\,dt = sY(s) - y_0. \tag{54.4}$$

The second integral in equation (54.3) can be evaluated by changing the variable of integration from t to $u = t - 1$:

$$\begin{aligned} a\int_0^\infty e^{-st} y(t-1)\,dt &= a\int_{-1}^\infty e^{-s(u+1)} y(u)\,du \\ &= a\int_0^\infty e^{-s(u+1)} y(u)\,du \,+\, a\int_{-1}^0 e^{-s(u+1)} y(u)\,du \\ &= ae^{-s}Y(s) + ay_0 \frac{1-e^{-s}}{s}. \end{aligned} \tag{54.5}$$

Utilizing equations (54.4) and (54.5) in equation (54.3) results in the algebraic equation

$$sY(s) - y_0 + ae^{-s}Y(s) + ay_0\frac{1-e^{-s}}{s} = 0,$$

which can be solved for $Y(s)$:

$$Y(s) = \frac{y_0}{s} - \frac{ay_0}{s(s+ae^{-s})}. \tag{54.6}$$

If this formula for $Y(s)$ is expanded as

$$Y(s) = \frac{y_0}{s} - y_0 \sum_{n=0}^{\infty} (-1)^n a^{n+1} e^{-ns} s^{-n-2},$$

then an inverse Laplace transform may be taken term by term to conclude that

$$y(t) = y_0 \sum_{n=0}^{\lfloor t \rfloor + 1} (-a)^n \frac{(t-n+1)^n}{n!}, \tag{54.7}$$

where the floor function, $\lfloor t \rfloor$, is the greatest integer less than or equal to t.

Another way of expressing the solution in equation (54.7) is by taking the inverse transform of $Y(s)$, as defined in equation (54.6), directly, and using Cauchy's theorem to evaluate the Bromwich contour integral. This results in

$$y(t) = -ay_0 \sum_r \frac{e^{s_r t}}{s_r(1+s_r)}, \tag{54.8}$$

where the summation is over all roots of the equation

$$s + ae^{-s} = 0. \tag{54.9}$$

All the roots of equation (54.9) will be simple unless $a = e^{-1}$, when there is a double root at $s = -1$. The solution in equation (54.8) can be approximated (for large t) by just using the s_r that has the smallest real part. There exist theorems (see Pinney [15] for instance) that allow the solution of equation (54.1) to be written in the form of equation (54.8) immediately.

Example 2

In the method of steps, only a sequence of ordinary differential equations need to be solved. To illustrate this method, consider equations (54.1) and (54.2). In the interval $0 \le y \le 1$, the solution satisfies

$$\begin{aligned} &y'(t) + ay_0 = 0, \\ &y(0) = y_0. \end{aligned} \tag{54.10}$$

The equation (54.10) has the solution

$$y(t) = y_0(1 - at), \qquad \text{for } 0 \le y \le 1. \tag{54.11}$$

Now we solve for $y(t)$ in the next interval of length one. Using equation (54.11) we find that, in the interval $1 \le y \le 2$, the solution satisfies

$$\begin{aligned} &y'(t) + ay_0[1 - a(t-1)] = 0, \\ &y(0) = y_0(1 - a). \end{aligned} \tag{54.12}$$

The equation (54.12) has the solution

$$y(t) = y_0 \left[1 - at + \frac{1}{2}a^2(t-1)^2\right], \qquad \text{for } 1 \le y \le 2.$$

This process can be repeated indefinitely. The solution obtained is identical to the solution in equation (54.7).

Example 3

This example shows how generating functions may be used to solve delay equations. Consider equations (54.1) and (54.2). Define the generating function associated with $y(t)$, for $0 \le t \le 1$, by

$$Y(t,k) = \sum_{p=0}^{\infty} y(t+p)k^p. \tag{54.13}$$

Once this generating function is known, $y(t)$ may be obtained in either of the two ways

$$\begin{aligned} y(t+p) &= \frac{1}{p!}\left(\frac{\partial^p}{\partial k^p}Y(t,k)\right|_{k=0} \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}} Y(t,k)k^{-p-1}\,dk, \end{aligned}$$

where $\mathcal{C}$ is a closed contour surrounding the origin in the k-plane and lying wholly within the region of analyticity in k of $Y(t,k)$.

By differentiating equation (54.13) with respect to t, multiplying by k, and redefining p, we find that

$$\begin{aligned} Y_t(t,k) &= \sum_{p=0}^{\infty} y'(t+p)k^p, \\ kY(t,k) &= \sum_{p=1}^{\infty} y(t+p+1)k^p. \end{aligned} \tag{54.14}$$

If we now evaluate equation (54.1) when t has the value $t+p$, multiply by k^p, and sum with respect to p from 1 to infinity, we find (using equation (54.14))

$$Y_t(t,k) + a\left(kY(t,k) + y(t-1)\right) = 0$$

or, because $0 \le t \le 1$,

$$Y_t(t,k) + akY(t,k) = -ay_0.$$

This equation is an ordinary differential equation and can be readily solved to yield

$$Y(t,k) = e^{-akt}F(k) - \frac{y_0}{k}, \tag{54.15}$$

where $F(k)$ is some unknown function. We can determine this function by a judicious use of the initial conditions. Evaluating equation (54.13) at $t = 1$, we find

$$\begin{aligned} kY(1,k) &= k\sum_{p=0}^{\infty} y(1+p)k^p \\ &= \sum_{p=0}^{\infty} y(1+p)k^{p+1} \\ &= y(0) + \sum_{p=0}^{\infty} y(p)k^p \\ &= y(0) + Y(0,k). \end{aligned} \tag{54.16}$$

Evaluating equation (54.16) by use of equation (54.15) results in

$$k\left(e^{-ak}F(k) - \frac{y_0}{k}\right) = y_0 + \left(F(k) - \frac{y_0}{k}\right),$$

or

$$F(k) = \frac{y_0}{k\,(1 - e^{-ak})}.$$

This leads to the complete determination of the generating function

$$Y(t,k) = \frac{y_0}{k}\left(\frac{e^{-akt}}{1 - ke^{-ak}} - 1\right).$$

Via some algebraic manipulations, we can obtain

$$Y(t,k) = y_0 \sum_{p=0}^{\infty} k^p \sum_{q=0}^{p+1} \frac{(-a(p+t-q+1))^q}{q!}, \tag{54.17}$$

so that the solution can be read off (compare equation (54.17) with equation (54.13)):

$$y(t) = y_0 \sum_{q=0}^{\lfloor t \rfloor + 1} (-a)^q \frac{(t-q+1)^q}{q!},$$

where the floor function indicates the least integer.

Notes

1. In the literature, equations of the form $y_h'(t) = y_{h-1}(t)$ are often called *differential–difference equations*, whereas equations of the form $y'(t) = y(t-1)$ are called *mixed differential–difference equations*. Delay equations are also known as *functional equations*, *differential–delay equations*, *differential equations with deviating argument*, and *equations with retarded arguments*. *Neutral differential equations* are differential equations in which the highest order derivative of the unknown function is evaluated both at the present state t and at one of more past or future states.
2. The pantograph equation (see Buhmann and Iserles [4]) is $\dot{x}(t) = ax(t) + bx(\theta(t)) + c\dot{x}(\phi(t))$.
3. The Cherwell–Wright differential equation (see Iyanaga and Kawada [12, page 287]) is $\dot{x}(t) = (a - x(t-1))x(t)$.
4. Marsaglia *et al.* [13] numerically evaluate the following functions:
   - Renyi's function: $[(x-1)y(x)]' = 2y(x-1)$
   - Dickman's function: $xy'(x) = -y(x-1)$
   - Buchstab's function: $[xy(x)]' = y(x-1)$
5. Several authors have tried to analyze delay equations by replacing $y(t-r)$ with the first few terms of a Taylor series, say

$$y(t-r) \simeq y(t) - ry'(t) + \frac{1}{2}r^2y''(t) - \cdots + (-1)^m\frac{1}{m!}r^m y^{(m)}(t).$$

This is, in general, a bad idea as the approximations that are obtained are often unrelated to the original equation. See Driver [8, page 235] for more details.

6. The paper by Driver and Driver [7] gives explicit error bounds for the solution of $x'(t) = bx(t-1)$ for a range of b values, when using the first terms in an asymptotic expansion. For example, when $x(t) = 1$ for $t < 0$, and $b = 1$, then $x(t) = x_a(t) + g(t)$ with $x_a(t) = 1.13e^{0.567t}$ and $|g(t)| \leq 0.25e^{-1.47t}$.
7. The book by Pinney [15] contains a large compilation of delay equations that have appeared in the literature. References are cited, and the (then) current knowledge of each of the equations is given.
8. The system of linear delay equations

$$\begin{aligned} \mathbf{u}'(t) &= A\mathbf{u}(t) + B\mathbf{u}(t-d), && \text{for } t \geq t_0, \\ \mathbf{u}(t) &= \mathbf{g}(t), && \text{for } -d \leq t \leq t_0, \end{aligned} \tag{54.18}$$

where $d \geq 0$ is the delay and A and B are constant square matrices has a solution of the form $\mathbf{u}(t) = \mathbf{c}e^{st}$ if and only if s is a zero of the transcendental equation: $\det\left(Is - A - Be^{-ds}\right) = 0$.

9. As an example of the general expansion theorems, the equation

$$au'(t) + bu(t) + cu(t-d) = 0,$$

where a, b, c, and d are all constant and d is positive, is satisfied by

$$u(t) = \sum_r p_r(t)e^{ts_r}, \tag{54.19}$$

where $\{s_r\}$ are complex numbers satisfying $as_r + b + ce^{-ds_r} = 0$, and $p_r(t)$ is a polynomial in t of degree less than the multiplicity s_r (see Bellman and Cooke [3, page 55]). The sum in equation (54.19) is either finite or infinite, with suitable conditions to ensure convergence. In actuality, finding all the solutions to equation (54.15) is very difficult. This technique generalizes to higher order ordinary differential equations and partial differential equations, but the work in obtaining a solution becomes prohibitive unless numerical methods are used.

10. Delay equations are usually solved numerically. A survey of numerical techniques for solving delay equations may be found in Cryer [6]. Nieves's paper [14] contains the description of a computer algorithm that numerically approximates the solutions of functional equations with a minimal amount of user input. Virk's paper [18] extends Runge–Kutta methods to delay-differential equations (the method he presents is compromise between computational efficiency and code complexity).
11. See also Saaty [16, Chapter 5, pages 213–261].

References

[1] Al-Butib, A. N. One-step implicit methods for solving delay differential equations. *Int. J. Comp. Math. 16* (1984), 157–168.

[2] Bakke, V. L., and Jackiewicz, Z. Stability analysis of linear multistep methods for delay differential equations. *Int. J. Math. & Math. Sci. 9*, 3 (1986), 447–458.

[3] Bellman, R. E., and Cooke, K. L. *Differential–Difference Equations.* Academic Press, New York, 1963.

[4] Buhmann, M., and Iserles, A. Stability of the discretized pantograph differential equation. *Math. of Comp. 60*, 202 (April 1993), 575–589.

[5] Burton, T. A. *Stability and Periodic Solutions of Ordinary and Functional Differential Equations.* Academic Press, New York, 1985.

[6] Cryer, C. W. Numerical methods for functional differential equations. In *Delay and Functional Differential Equations and Their Applications*, K. Schmitt, Ed. Academic Press, New York, 1972, pp. 17–101.

[7] Driver, B. K., and Driver, R. D. Simplicity of solutions of *x'(t)=bx(t-1)*. *Journal of Mathematical Analysis and Applications 157*, 2 (15 May 1991), 591–608.

[8] Driver, R. D. *Introduction to Ordinary Differential Equations.* Harper & Row Publishers, New York, 1978.

[9] El'sgol'ts, L. E., and Norkin, S. B. *Introduction to the Theory and Application of Differential Equations with Deviating Arguments.* Academic Press, New York, 1973.

[10] Feldstein, A., Iserles, A., and Levin, D. Embedding of delay equations into an infinite-dimensional ODE system. *J. Differential Equations 117* (1995), 127–150.

[11] Herz, A. V. M. Solutions of zzzref8refzzz approach the Kaplan–Yorke orbits for odd sigmoid *g*. *J. Differential Equations 118* (1995), 36–53.

[12] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics.* MIT Press, Cambridge, MA, 1980.

[13] Marsaglia, G., Zaman, A., and Marsaglia, J. C. W. Numerical solution of some classicial differential-difference equations. *Math. of Comp. 53*, 187 (July 1989), 191–201.

[14] Nieves, K. W. Automatic integration of functional differential equations: An approach. *ACM Trans. Math. Software 1*, 4 (Dec 1975), 357–368.

[15] Pinney, E. *Ordinary Difference–Differential Equations.* University of California Press, Berkeley, CA, 1959.

[16] Saaty, T. L. *Modern Nonlinear Equations.* Dover Publications, Inc., New York, 1981.

[17] Torelli, L. Stability of numerical methods for delay differential equations. *J. Comput. Appl. Math. 25* (1989), 15–26.

[18] Virk, G. S. *IEE Proc. D 132* (1985), 119–123.

[19] Weiner, R., and Strehmel, K. A type insensitive code for delay differential equations basing on adaptive and explicit Runge–Kutta interpolation methods. *Computing 40*, 3 (1988), 255–265.

55. Dependent Variable Missing

Applicable to Ordinary differential equations of the form $G(y^{(n)}, y^{(n-1)}, \ldots, y'', y', x) = 0$.

Yields

An ordinary differential equation of lower order.

Idea

If the dependent variable does not appear explicitly in an ordinary differential equation, then the order of the ordinary differential equation can be reduced by 1.

Procedure

Suppose we have the nth order ordinary differential equation

$$G(y^{(n)}, y^{(n-1)}, \ldots, y'', y', x) = 0. \tag{55.1}$$

Notice that the variable $y(x)$ does not appear explicitly in equation (55.1).

If we define $p(x) = y'(x)$, then equation (55.1) becomes

$$G(p^{(n-1)}, p^{(n-2)}, \ldots, p', p, x) = 0, \tag{55.2}$$

which is an ordinary differential equation of order $(n-1)$ for the dependent variable $p(x)$. After solving equation (55.2) for $p(x)$, $y(x)$ can be found by integrating $p(x)$.

Example

Suppose we have the second order equation

$$y'' + y' = x. \tag{55.3}$$

Using $y'(x) = p(x)$, equation (55.3) can be written as

$$p' + p = x. \tag{55.4}$$

Equation (55.4) can be solved by integrating factors (see page 356) to obtain

$$p(x) = Ae^{-x} + x - 1,$$

where A is an arbitrary constant. Then $p(x)$ can be integrated to obtain $y(x)$

$$y(x) = \int^x p(t)\,dt = B - Ae^{-x} + \frac{x^2}{2} - x,$$

where B is another arbitrary constant.

Notes

1. This solution technique can be derived from Lie group methods (see page 366).
2. See also Boyce and DiPrima [1, pages 111–112], Goldstein and Braun [2, pages 74–76], Ince [3, page 43], and Rainville and Bedient [4, pages 266–268].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations*. U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[3] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[4] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations*. The MacMillan Company, New York, 1964.

56. Differentiation Method

Applicable to

Nonlinear ordinary differential equations.

Yields

An explicit solution.

Idea

Sometimes differentiating an ordinary differential equation will result in an ordinary differential equation that is easier to solve.

Procedure

Given an ordinary differential equation, differentiate it with respect to the independent variable. This will yield a new equation that may sometimes factor (see page 292), or simplify in some other way. By considering each term in this new equation to be equal to zero, several possible solutions may be found.

The general solution of each term must then be used in the original equation, possibly to constrain some of the parameters.

Example

Suppose that we have the nonlinear ordinary differential equation

$$2yy'' - (y')^2 = \frac{1}{3}\left(y' - xy''\right)^2. \tag{56.1}$$

If this equation is differentiated with respect to x, the simplified result is

$$y'''\left(x^2y'' - xy' - 3y\right) = 0,$$

from which we recognize that

$$y''' = 0 \qquad \text{or} \qquad x^2y'' - xy' - 3y = 0. \tag{56.2}$$

In the first case, a candidate for the general solution is

$$y(x) = ax^2 + bx + c.$$

Using this form in the original equation, equation (56.1), we find after some simplification that $3ac = b^2$. Using this equation to determine c, a general solution to equation (56.1) is found to be

$$y(x) = ax^2 + bx + \frac{b^2}{3a}. \tag{56.3}$$

Another possibility is that the second expression in equation (56.2) is equal to zero. This second equation is an Euler equation (see page 281), and so the general solution is found to be

$$y(x) = \alpha x^3 + \frac{\beta}{x}.$$

Using this form in the original equation, equation (56.1), we find after some simplification that $\alpha\beta = 0$. Hence, two different solutions to equation (56.1) are given by

$$y(x) = \alpha x^3 \qquad \text{and} \qquad y(x) = \frac{\beta}{x}. \tag{56.4}$$

Equations (56.3) and (56.4) contain three different solutions to equation (56.1).

Notes

1. The above example is from Bateman [1, pages 66–67].
2. This procedure is used to find the singular solutions to Clairaut's equation (see page 237).

Reference

[1] Bateman, H. *Partial Differential Equations of Mathematical Physics*. Dover Publications, Inc., New York, 1944.

57. Differential Equations with Discontinuities*

Applicable to Equations that contain discontinuous functions.

Yields

An exact solution.

Idea

Equations can be solved locally and then patched together at the points of discontinuity.

Procedure

The following discussion is limited to linear ordinary differential equations, but the general techniques apply to linear and nonlinear ordinary differential equations and partial differential equations.

Suppose we have the equation

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x), \tag{57.1}$$

where the $\{a_i(x)\}$ and $b(x)$ may all be discontinuous. For example, $a_1(x)$ may look like

$$a_1(x) = \begin{cases} x & \text{if } 0 < x < 3, \\ \sin x & \text{if } 3 \le x < 8. \end{cases}$$

We presume that the $\{a_i(x)\}$ and $b(x)$ are discontinuous at only a finite number of points, say $\{x_1, x_2, \dots, x_m\}$, and that we wish to find the solution at the point x_f with $x_0 < x_1 < \cdots < x_m < x_f$. Assume further that the initial data $\{y(x_0), y'(x_0), y''(x_0), \dots, y^{(n-1)}(x_0)\}$ are all given.

The general technique is to divide the interval from x_0 to x_f into m intervals and solve equation (57.1) separately on each interval. Because the equation is continuous on these intervals, we can use any technique known to us to find the solution. Define $y_j(x)$ to be the solution in the interval $[x_j, x_{j+1}]$.

To determine $y_j(x)$ completely, we need to specify the value of $\{y_j(x_j), y_j'(x_j), \dots, y_j^{(n-1)}(x_j)\}$. These can be determined from $y_{j-1}(x)$. Because an equation of nth order (which is what equation (57.1) is) must have continuous derivatives of all orders up to $n-1$, we simply match the values of $y_j(x)$ and its derivatives to the values of $y_{j-1}(x)$ and its derivatives, all at the point x_j.

To illustrate this technique on equation (57.1), we would solve

$$a_n(x)y_j^{(n)} + a_{n-1}y_j^{(n-1)} + \cdots + a_1(x)y_j' + a_0(x)y_j = b(x)$$

in the interval $[x_j, x_{j+1}]$, for $j = 0, 1, 2, \ldots, m$. To obtain the initial values for each equation we take

$$\begin{bmatrix} y_0(x_0) \\ y_0'(x_0) \\ \vdots \\ y_0^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix},$$

and then

$$\begin{bmatrix} y_j(x_j) \\ y_j'(x_j) \\ \vdots \\ y_j^{(n-1)}(x_j) \end{bmatrix} = \begin{bmatrix} y_{j-1}(x_j) \\ y_{j-1}'(x_j) \\ \vdots \\ y_{j-1}^{(n-1)}(x_j) \end{bmatrix}, \qquad \text{for } j = 1, 2, \ldots, m.$$

Finally, the solution at $x = x_f$ will be given by $y_m(x_f)$.

Example

Suppose we want to determine the value of $y(t)$ at $t = T$ when

$$y'' + f(t)y = 0,$$

and $f(t)$ is given by

$$f(t) = \begin{cases} -1 & \text{for } 0 \le t < \tau, \\ 1 & \text{for } \tau \le t \le T, \end{cases}$$

given that $y(0) = 1$, $y'(0) = 0$. (Here, τ and T are fixed constants.) To solve this problem, we break the interval from 0 to T into two intervals; interval I will be from 0 to τ while interval II will be from τ to T.

In interval I, $f(t)$ can be replaced by -1, so we solve

$$y_1'' - y_1 = 0, \qquad y_1(0) = 1, \quad y_1'(0) = 0.$$

This equation has the solution $y_1(t) = \cosh t$. In interval II, $f(t)$ can be replaced by 1, so we solve

$$y_2'' + y_2 = 0 \tag{57.2}$$

in the interval from τ to T. For the *initial* values of $y_2(t)$, we use the *final* values of $y_1(t)$, that is,

$$\begin{aligned} y_2(\tau) &= y_1(\tau) = \cosh\tau, \\ y_2'(\tau) &= y_1'(\tau) = \sinh\tau. \end{aligned} \tag{57.3}$$

The solution of equations (57.2) and (57.3) is

$$y_2(t) = (\sin\tau\cosh\tau + \cos\tau\sinh\tau)\sin t + (\cos\tau\cosh\tau - \sin\tau\sinh\tau)\cos t,$$

and hence, the value of $y(t)$ at $t = T$ is given by

$$y_2(T) = (\sin\tau\cosh\tau + \cos\tau\sinh\tau)\sin T + (\cos\tau\cosh\tau - \sin\tau\sinh\tau)\cos T.$$

Notes

1. When the discontinuities involve the *dependent* variable, then the problem is generally a free boundary problem. See Elliot and Ockendon [3] or Fleishman [6] for a discussion.
2. If the discontinuity appearing in a linear differential equation is a single delta function, which appears as a forcing function, then the solution will be a Green's function (see page 318).
3. If the discontinuities include generalized functions (such as a delta function), then the solution may only exist in the weak sense. See Gear and Østerby [7] for details.
4. There exist computer programs for numerically approximating differential equations with discontinuities. See Enright *et al.* [4] or Gear and Østerby [7].
5. Fleishman [6] analyzes the equation $\dot{\mathbf{x}} = A(t)\mathbf{x} + \operatorname{sgn}(\mathbf{x}) + \mathbf{t}(t)$, where "sgn" represents the signum function.
6. Das *et al.* [2] compare eight different approximations to a one-dimensional steady-state boundary value problem for a general symmetric second order ordinary differential equation with discontinuous leading coefficient.
7. See Leveque and Li [9] for methods for elliptic partial differential equations. See also Boyce and DiPrima [1, Section 6.3.1, pages 304–309].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Das, B., Steinberg, S., Zhang, D., and Robey, T. Comparison of numerical solution methods for differential equations with discontinuous coefficients. *Math. and Computers in Simulation 36* (1994), 57–75.

[3] Elliot, C. M., and Ockendon, J. R. *Weak and Variational Methods for Moving Boundary Problems.* Pitman Publishing Co., Marshfield, MA, 1982.

[4] Enright, W. H., Jackson, K. R., NØorsett, S. P., and Thomsen, P. G. Effective solution of discontinuous IVPs using a Runge–Kutta formula pair with interpolants. *Appl. Math. and Comp. 27* (1988), 313–335.

[5] Filippov, A. F. *Differential Equations with Discontinuous Righthand Sides.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

[6] Fleishman, B. A. Convex superposition in piecewise-linear systems. *J. Math. Anal. Appl. 6*, 2 (April 1963), 182–189.

[7] Gear, C. W., and sterby, O. O. Solving ordinary differential equations with discontinuities. *ACM Trans. Math. Software 10*, 1 (March 1984), 23–44.

[8] Hajj, I. N., and Skelboe, S. Steady-state analysis of piecewise-linear dynamic systems. *IEEE Trans. Circ. & Syst. CAS-28*, 3 (March 1981), 234–241.

[9] LEVEQUE, R. J., AND LI, Z. The immersed interface method for elliptic equations with discontinuus coefficients and singular sources. *SIAM J. Numer. Anal. 31*, 4 (August 1994), 1019–1044.

[10] PAN, H. H., AND HOHENSTEIN, R. M. A method of solution of an ordinary differential equation containing symbolic functions. *Quart. Appl. Math.* (April 1981), 131–136.

[11] PARKER, T. S., AND CHUA, L. O. Efficient solution of the variational equation for piecewise-linear differential equations. *Circuit Theory and Appl. 14*, 4 (1986), 305–314.

[12] STEWART, D. A high accuracy method for solving ODEs with discontinuous right-hand side. *Numer. Math. 58* (1990), 299–328.

[13] WESTREICH, D. Numerical solution of the eigenvalue problem for discontinuous linear ordinary differential equations. *J. Inst. Maths. Applics 25* (1980), 147–160.

58. Eigenfunction Expansions*

Applicable to Linear differential equations with linear boundary conditions.

Yields

An exact solution in terms of an infinite series.

Idea

Any "well-behaved" function can be expanded in a complete set of eigenfunctions. In this method, we expand the dependent variable in a differential equation as a sum of the eigenfunctions with unknown coefficients. From the given equation and boundary conditions, equations can then be determined for the unknown coefficients.

Procedure

We will describe the procedure for ordinary differential equations, but the same procedure can be used for partial differential equations (see Example 2). Assume that we want to solve the inhomogeneous linear ordinary differential equation

$$\begin{aligned} L[y] &:= \sum_{r=1}^{n} p_r(x) \frac{d^r y}{dx^r} = h(x), \\ B_i[y] &:= \sum_{r=1}^{n} \left(c_{ir} \frac{d^r y}{dx^r}(a) + d_{ir} \frac{d^r y}{dx^r}(b) \right) = 0, \qquad i = 1, 2, \dots, n, \end{aligned} \tag{58.1.a-b}$$

for $y(x)$, where $x \in [a, b]$ and $\{c_{ir}, d_{ir}, p_r(x), h(x)\}$ are all known.

Let us suppose that we know a complete set of eigenfunctions $\{u_k(x)\}$ that satisfy the boundary conditions in equation (58.1) and are orthogonal with respect to some weighting function $w(x)$. These could be obtained from a table (e.g., see table 77.1), or we might look for a set that is related to the differential equation in (58.1). A common approach is to choose a set of eigenfunctions $\{u_k\}$ that satisfy

$$\begin{aligned} H[u_k] &= \lambda_k u_k, \\ R_i[u_k] &= 0, \qquad i = 1, 2, \dots, n, \end{aligned} \tag{58.2.a-b}$$

where $H[\cdot]$ is a linear operator related to $L[\cdot]$ in some way, the $R_i[\cdot]$ are linear boundary conditions related to $B_i[\cdot]$ in some way, and λ_k is a constant (λ_k is an eigenvalue of the $(H, \{R_i\})$ system). The orthogonality condition requires that

$$(u_k, u_m) := \int_a^b u_k(x) u_m(x) w(x)\, dx = N_k \delta_{km} = \begin{cases} 0 & \text{for } m \neq k, \\ N_k & \text{for } m = k. \end{cases} \tag{58.3}$$

Frequently the operator $H[\cdot]$ is chosen to be the same as the operator $L[\cdot]$, and the $\{R_i\}$ are chosen to be the same as the $\{B_i\}$. This is not required, nor must the degree of the differential equation in (58.2.a) be n (which is the degree of the differential equation in (58.1.a)).

Because the presumed eigenfunctions are complete, we can write any "sufficiently smooth" function as a linear combination of these functions. In particular, we choose to represent $y(x)$ and $h(x)$ as

$$y(x) := \sum_{k=1}^{\infty} y_k u_k(x), \qquad h(x) := \sum_{k=1}^{\infty} h_k u_k(x). \tag{58.4.a-b}$$

Once the $\{y_k\}$ are known, the problem is solved. The $\{h_k\}$ can be determined, given $h(x)$, by multiplying equation (58.4.b) by $w(x)u_m(x)$ and integrating with respect to x from a to b. This calculation can be written as

$$\begin{aligned}(h(x), u_m(x)) &= \left(\sum_{k=1}^{\infty} h_k u_k(x), u_m(x)\right), \\ &= \sum_{k=1}^{\infty} h_k \left(u_k(x), u_m(x)\right), \\ &= \sum_{k=1}^{\infty} h_k \left(N_k \delta_{km}\right), \\ &= N_m h_m,\end{aligned}$$

where we have utilized equation (58.3). If we take the $\{R_i\}$ to be identical to the $\{B_i\}$ then, from equation (58.2.b), the boundary conditions for $y(x)$ (in equation (58.1.b)) are automatically satisfied. Hence, only equation (58.1.a) needs to be satisfied. Using equation (58.4.a) in equation (58.1.a) results in

$$\begin{aligned}L[y] &= L\left[\sum_{k=1}^{\infty} y_k u_k(x)\right] \\ &= \sum_{k=1}^{\infty} y_k L[u_k] \\ &= h(x).\end{aligned} \tag{58.5}$$

The $\{y_k\}$ can now be determined from equation (58.5) by multiplying equation (58.5) by $w(x)u_m(x)$ and integrating with respect to x from a to b. This produces

$$\sum_{k=1}^{\infty} y_k \left(L[u_k], u_m\right) = (h(x), u_m) = N_m h_m, \quad \text{for } m = 1, 2, \ldots, \tag{58.6}$$

which is an infinite system of linear algebraic equations. In principle, all of the $\{y_k\}$ in equation (58.6) are coupled together.

In practice, if a good choice was made for the eigenfunctions, then equation (58.6) will simplify and y_m can be determined directly from equation (58.6). For instance, if $H[\cdot]$ is chosen to be equal to $L[\cdot]$ then $L[u_n] = \lambda_n u_n$ (from equation (58.2)) and equation (58.6) becomes $\sum_{k=1}^{\infty} y_k \lambda_k (u_k, u_m) = N_m h_m$ or, by orthogonality, $y_m = h_m/\lambda_m$.

Example 1

Suppose we have the fourth order differential equation and boundary conditions

$$\begin{aligned} L[y] &:= y'''' + \alpha y'' + \beta y = h(x), \\ y(0) &= 0, \qquad y(1) = 0, \\ y''(0) &= 0, \qquad y''(1) = 0, \end{aligned} \tag{58.7}$$

to solve for $y(x)$ on the interval $x \in [0, 1]$.

For this case we choose to use the eigenfunctions corresponding to the Sturm–Liouville operator (see page 103)

$$\begin{aligned} H[u] &= u'', \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{58.8}$$

For the operator in equation (58.8), it is easy to determine that the eigenfunctions are $u_k(x) = \sin k\pi x$, the eigenvalues are $\lambda_k = k\pi$, and the weighting function is $w(x) = 1$. Because this is a self-adjoint problem (see page 95), we know that these eigenfunctions are complete. Now that we have a set of eigenfunctions, we observe that they satisfy the four boundary conditions given in equation (58.7).

We write $y(x)$ in terms of these eigenfunctions as

$$y(x) = \sum_{k=1}^{\infty} y_k \sin k\pi x. \tag{58.9}$$

Using equation (58.9) in equation (58.7) and then multiplying by $u_m(x)$ and integrating from $x = 0$ to $x = 1$ results in

$$\begin{aligned} \int_0^1 L[y(x)] u_m(x)\, dx &= \int_0^1 L\left[\sum_{k=1}^{\infty} y_k \sin k\pi x\right] u_m(x)\, dx \\ &= \sum_{k=1}^{\infty} y_k \int_0^1 L[\sin(k\pi x)] u_m(x)\, dx \\ &= \int_0^1 h(x) u_m(x)\, dx. \end{aligned}$$

Equating the last two expressions, using $u_m(x) = \sin m\pi x$ and simplifying gives

$$\sum_{k=1}^{\infty} y_k \int_0^1 \left(k^4\pi^4 - \alpha k^2\pi^2 + \beta\right) \sin k\pi x \sin m\pi x \, dx = \int_0^1 h(x) \sin m\pi x \, dx,$$

or (since $\int_0^1 \sin k\pi x \sin m\pi x \, dx = \frac{1}{2}\delta_{km}$)

$$\frac{1}{2} y_k \left(k^4\pi^4 - \alpha k^2\pi^2 + \beta\right) = \int_0^1 h(x) \sin k\pi x \, dx. \tag{58.10}$$

Hence, solving equation (58.10) for y_k and using this value in equation (58.9) results in the explicit solution

$$y(x) = \sum_{k=1}^{\infty} \left(\frac{2\int_0^1 h(x) \sin k\pi x \, dx}{k^4\pi^4 - \alpha k^2\pi^2 + \beta} \right) \sin k\pi x.$$

If α and β are such that $k^4\pi^4 - \alpha k^2\pi^2 + \beta = 0$, for some value of k, then there will be no solution unless $\int_0^1 h(x) \sin k\pi x \, dx = 0$. Even then, the solution will not be unique; this is because the differential equation $L[u] = 0$, with the boundary conditions in equation (58.7), will have the solution $u(x) = C \sin k\pi x$, where C is arbitrary. See the section on alternative theorems (page 15).

Example 2

Suppose we want to solve the partial differential equation

$$\begin{aligned} \phi_t &= \phi_{xx}, \\ \phi(x,0) &= f(x), \\ \phi(0,t) &= 0, \\ \phi(1,t) &= 0, \end{aligned} \tag{58.11.a-d}$$

for $\phi = \phi(x,t)$. We can use the eigenfunctions in equation (58.8) to solve this problem. In this case, we expand $\phi(x,t)$ as

$$\phi(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x. \tag{58.12}$$

By using this representation for $\phi(x,t)$, the boundary conditions in equation (58.11.b) and equation (58.11.c) are automatically satisfied. By multiplying equation (58.12) by $\sin(m\pi x)$ and integrating from $x = 0$ to $x = 1$, we find that

$$a_n(t) = 2\int_0^1 \phi(z,t) \sin n\pi z \, dz. \tag{58.13}$$

Using the boundary condition from (58.11.b) in equation (58.13) produces the initial values for the $\{a_n(t)\}$

$$a_n(0) = 2\int_0^1 \phi(z,0)\sin n\pi z\,dz = 2\int_0^1 f(z)\sin n\pi z\,dz. \tag{58.14}$$

Now, the correct procedure is to multiply the original equation, equation (58.11.a), by one of the eigenfunctions, $\sin m\pi x$, and integrate from $x = 0$ to $x = 1$ to obtain

$$\int_0^1 \phi_t \sin m\pi x\,dx = \int_0^1 \phi_{xx}\sin m\pi x\,dx. \tag{58.15}$$

After utilizing equation (58.12) for ϕ in equation (58.15), the resulting equation should be integrated by parts, using the information in equation (58.13). This results in

$$a_n'(t) = -n^2\pi^2 a_n(t), \tag{58.16}$$

where a prime denotes a derivative with respect to t. The solution of equation (58.16) is

$$\begin{aligned} a_n(t) &= a_n(0)e^{-n^2\pi^2 t},\\ &= \left(2\int_0^1 f(z)\sin n\pi z\,dz\right)e^{-n^2\pi^2 t}, \end{aligned} \tag{58.17}$$

where we have used equation (58.14). Combining equations (58.12) and (58.17), we determine the final solution to equation (58.11) to be

$$\phi(x,t) = \sum_{n=1}^{\infty}\left(2\int_0^1 f(z)\sin n\pi z\,dz\right)e^{-n^2\pi^2 t}\sin n\pi x.$$

Be aware that it would have been *incorrect*, when trying to obtain an ordinary differential equation for $a_n(t)$, to substitute equation (58.12) into equation (58.11.a) and then multiply by one of the eigenfunctions and perform the integration. Although this would have resulted in the same differential equation and boundary conditions for a_n in this example, it might not work in other cases (see the next example). The proper technique is to multiply the original equation by one of the eigenfunctions and then integrate by parts.

Example 3

Consider solving Laplace's equation in two dimensions in the unit square

$$\begin{aligned} &u_{xx} + u_{yy} = 0,\\ &\quad u(x,1) = u(0,y) = u(1,y) = 0,\\ &\quad u(x,0) = f(x). \end{aligned} \tag{58.18.a-c}$$

Since the functions $\{\sin n\pi y\}$ are complete on the interval $[0,1]$, we choose to represent the solution to equation (58.18) in the form

$$u(x,y) = \sum_{n=1}^{\infty} c_n(x) \sin n\pi y, \tag{58.19}$$

from which we can deduce that

$$c_n(x) = 2 \int_0^1 u(x,y) \sin n\pi y \, dy. \tag{58.20}$$

From the boundary conditions on $u(x,y)$ at $x = 0$ and at $x = 1$, we also find that $c_n(0) = c_n(1) = 0$.

We will show that an incorrect answer is obtained if the $\{c_n\}$ are determined in a naive way. If we substituted the assumed form of the solution (e.g., equation (58.19)), into the equation in (58.18.a), then we would find

$$u_{xx} + u_{yy} = \sum_{n=1}^{\infty} \left(c_n'' - n^2\pi^2 c_n \right) \sin n\pi y = 0.$$

Hence, by orthogonality, we would find that $c_n'' - n^2\pi^2 c_n = 0$. Solving this differential equation with the boundary conditions on c_n (e.g. $c_n(0) = c_n(1) = 0$), we would be led to $c_n(x) = 0$ and so $u(x,y) = 0$. This is clearly *wrong*.

If, instead, the equation (58.18.a) is multiplied by $2 \sin n\pi y$ and integrated with respect to y from 0 to 1, then we obtain

$$\begin{aligned}
0 &= \int_0^1 2 \sin n\pi y (u_{xx} + u_{yy}) \, dy \\
&= \frac{d^2}{dx^2} \int_0^1 2u(x,y) \sin n\pi y \, dy + 2u_y(x,y) \sin n\pi y \,\Big|_0^1 \\
&\qquad - 2n\pi u(x,y) \cos n\pi y \,\Big|_0^1 - n^2\pi^2 \int_0^1 2u(x,y) \sin n\pi y \, dy \\
&= c_n'' + 2n\pi f(x) - n^2\pi^2 c_n,
\end{aligned}$$

where we have integrated by parts twice, used equation (58.20) to substitute for the integral, and used the boundary conditions in equation (58.18.b-c). Solving this last equation for $c_n(x)$, we find

$$c_n(x) = 2n\pi \int_0^1 G(x;t) f(t) \, dt,$$

where $G(x;t)$ is the Green's function $G(x;t) = \frac{\sinh n\pi x_< \sinh n\pi(1-x_>)}{n\pi \sinh n\pi}$ and where $x_>$ ($x_<$) indicates the larger (smaller) of x and t.

This second approach gives the correct solution to this problem. The reason that the first approach would not work is that the series chosen to represent the solution does not have uniform convergence.

Notes

1. Note that the solution in Example 2 would have been obtained in exactly the same form if separation of variables had been used (see page 487).
2. If the chosen eigenfunctions do not come from a self-adjoint operator, then it will be necessary to know the eigenfunctions of the adjoint operator. This is because the orthogonality condition will utilize the eigenfunctions of the adjoint operator.
3. Because the eigenfunctions we used in the examples were just sine functions, the expansions obtained here are identical to the results that would have been obtained from a Fourier sine series (see page 344).
4. To determine that a set of functions is complete, it is not necessary that they be derived from a self-adjoint operator. See Minzoni [6] for an example of a set of functions proved complete by using theorems from analysis.
5. See also Birkhoff and Rota [1, Chapter 11], Butkov [2, pages 304–318], and Farlow [4, Lesson 9, pages 64–71].

References

[1] Birkhoff, G., and Rota, G.-C. *Ordinary Differential Equations.* John Wiley & Sons, New York, 1978.

[2] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[3] Divis, Z. A note on the rate of convergence of Sturm–Liouville expansions. *J. Approx. Theory 50* (1987), 200–207.

[4] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers.* John Wiley & Sons, New York, 1982.

[5] Kobayashi, M. Eigenfunction expansion: A discontinuous version. *SIAM J. Appl. Math. 50*, 3 (June 1990), 910–917.

[6] Minzoni, A. A. On the completeness of the functions zzzref32refzzz zzzref33refzzz for zzzref34refzzz and *p(x)* a zzzref35refzzz periodic function. *Stud. Appl. Math. 75* (1986), 265–269.

[7] Stakgold, I. *Green's Functions and Boundary Value Problems.* John Wiley & Sons, New York, 1979.

[8] Titchmarsh, E. C. *Eigenfunction Expansions Associated with Second–Order Differential Equations.* Clarendon Press, Oxford, England, 1946.

59. Equidimensional-in-x Equations

Applicable to Ordinary differential equations of a certain form.

Yields

An autonomous ordinary differential equation of the same order (which can then be reduced to an ordinary differential equation of lower order).

Idea

An equidimensional-in-x equation is one in which the scaling of the x variable does not change the equation. By a change of independent variable, we can change an equation of this type into an autonomous equation.

Procedure

An equidimensional-in-x equation is one that is left invariant under the transformation $x \to ax$, where a is a constant. That is, if the original equation is an equation for $y(x)$ and the x variable is replaced by the variable ax', then the new equation (in terms of y and x') will be identical to the original equation (which is in terms of y and x). An equation of this type can be converted to an autonomous equation of the same order by changing the independent variable from x to t by the transformation $x = e^t$.

Example

Suppose we have the nonlinear second order ordinary differential equation

$$x\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}. \tag{59.1}$$

First, we will show that this equation is equidimensional-in-x. Substituting ax' for x in equation (59.1) produces

$$(ax')\frac{d^2y}{d(ax')^2} = 2y\frac{dy}{d(ax')}, \tag{59.2}$$

or, multiplying equation (59.2) by the constant a

$$x'\frac{d^2y}{d(x')^2} = 2y\frac{dy}{dx'},$$

which is identical to equation (59.1).

Because we now know that equation (59.1) is equidimensional-in-x, we change variables from $y(x)$ to $y(t)$ by $x = e^t$. Using table 59.1, we find that

$$e^t e^{-2t}(y_{tt} - y_t) = 2y(e^{-t}y_t),$$

$$
\begin{aligned}
y_x &= e^{-t}(y_t),\\
y_{xx} &= e^{-2t}(y_{tt} - y_t),\\
y_{xxx} &= e^{-3t}(y_{ttt} - 3y_{tt} + 2y_t),\\
y_{xxxx} &= e^{-4t}(y_{tttt} - 6y_{ttt} + 11y_{tt} - 6y_t),\\
y_{xxxxx} &= e^{-5t}(y_{ttttt} - 10y_{tttt} + 35y_{ttt} - 50y_{tt} + 24y_t),\\
y_{x(5)} &= e^{-5t}(y_{t(5)} - 10y_{t(4)} + 35y_{ttt} - 50y_{tt} + 24y_t),\\
y_{x(6)} &= e^{-6t}(y_{t(6)} - 15y_{t(5)} + 85y_{t(4)} - 225y_{ttt} + 274y_{tt} - 120y_t),\\
y_{x(7)} &= e^{-7t}(y_{t(7)} - 21y_{t(6)} + 175y_{t(5)} - 735y_{t(4)} + 1624y_{ttt} - 1764y_{tt} + 720y_t).
\end{aligned}
$$

Table 59.1: How to transform derivatives under the change of dependent variable: $x = e^t$. (To simplify notation, define $y_{x(n)}$ to be the nth derivative of y with respect to x, and similarly for $y_{t(n)}$.)

or

$$y_{tt} - y_t = 2yy_t. \tag{59.3}$$

The equation in (59.3) is autonomous (there is no explicit t dependence). Hence, it can be reduced to an ordinary differential equation of order one by the transformation $u(y) = y_t(t)$ (see page 230 for more information).

Carrying out the details (equation (59.3) was the example in the section on autonomous equations), it is easy to derive that either $y(t)$ is a constant for all t, or $y(t)$ satisfies

$$y(t) = E\tan(F + Et) - \frac{1}{2},$$

where E and F are arbitrary constants. Changing the independent variable from t to x we have

$$y(x) = E\tan(F + E\log x) - \frac{1}{2}.$$

Notes

1. This method is derivable from Lie group methods (see page 366).
2. It is straightforward to create a Macsyma program that will perform the necessary change of variables. Program 59.1 shows a terminal session in which the input equation

$$\left(\frac{dy}{dx}\right)^2 - y\frac{d^2y}{dx^2} = 0$$

```
(c1) DEPENDS(Y,X)$
(c2) EQUIDIMENSIONAL_IN_X(EQN,Y,X):= BLOCK([NEW,HOLD,J],
         DEPENDS([U],[T]),
         GRADEF(T, X, %E**(-T) ),
         NEW:SUBST( U, Y, EQN ),
         NEW:EV(NEW, DIFF),
         NEW:SUBST( %E**T, X, NEW),
         NEW:FACTOR(NEW),
         NEW)$
(c3) EQN: DIFF(Y,X)**2-Y*DIFF(Y,X,2);
                                       2
(d3)                             (y )  - y y
                                   x        xx
(c4) EQUIDIMENSIONAL_IN_X(EQN,Y,X);
                             - 2 t              2
(d4)                     - %e      (u u    - (u )  - u u )
                                        t t     t        t
```

Program 59.1: Macsyma program to change variables.

is converted into the second order autonomous equation

$$u\frac{d^2u}{dt^2} - \left(\frac{du}{dt}\right)^2 - u\frac{du}{dt} = 0.$$

This autonomous equation could then be reduced to a first order equation (see page 230).

3. See Bender and Orszag [1, page 25].

Reference

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

60. Equidimensional-in-y Equations

Applicable to Ordinary differential equations of a certain form.

Yields

An ordinary differential equation of lower order.

Idea

An equidimensional-in-y equation is one in which the scaling of the y variable does not change the equation. This information can be used to lower the order of the equation by a change of the dependent variable.

Procedure

An equidimensional-in-y equation is one that is left invariant under the transformation $y \to ay$, where a is a constant. That is, if the original equation is an equation for $y(x)$ and the y variable is replaced by the variable ay', then the new equation (in terms of y' and x) will be identical to the original equation (which is in terms of y and x). An equation of this type can be converted to an equation of lower order by changing the dependent variable from $y(x)$ to $e^{u(x)}$.

Example

Suppose we have the equation

$$(1-x)\left[y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2\right] + x^2y^2 = 0 \tag{60.1}$$

to solve. We can tell by inspection that this equation is equidimensional-in-y because all of the y terms in equation (60.1) all appear to the same power. That is, the y terms in equation (60.1) are all quadratic, the terms being of the form $\{y^2, y_x^2, y_{xx}^2, \dots, yy_x, yy_{xx}, y_x y_{xx}, \dots\}$.

To formally show that equation (60.1) is equidimensional-in-y, substitute ay' for y in equation (60.1) to find

$$(1-x)\left[(ay')\frac{d^2(ay')}{dx^2} - \left(\frac{d(ay')}{dx}\right)^2\right] + x^2(ay')^2 = 0.$$

Or, because a is a non-zero constant,

$$(1-x)\left[y'\frac{d^2y'}{dx^2} - \left(\frac{dy'}{dx}\right)^2\right] + x^2{y'}^2 = 0, \tag{60.2}$$

```
dy[0]= Exp[u[x]];
dy[1]= y[x] u'[x];
dy[n_]:= D[dy[n-1],x]/. {y'[x]->y[x] u'[x]}
dy2[n_]:= dy[n] /. {y[x]->y, u'[x]->u', u''[x]->u'',
                    u'''[x]->u''', u''''[x]->u''''}
Table[ {n,ddy2[n]}, {n,1,4}] // ColumnForm
```

Program 60.1: Mathematica program to change variables: $y(x) = e^{u(x)}$.

which has the same form as equation (60.1). Now, substituting $e^{u(x)}$ for $y(x)$ in equation (60.1) produces

$$(1-x)\left[y^2\left(\frac{d^2u}{dx^2}+\left(\frac{du}{dx}\right)^2\right)-\left(y\frac{du}{dx}\right)^2\right]+x^2y^2=0, \tag{60.3}$$

where table 60.1 has been used to determine how the derivatives transform under this change of variable. For $y \neq 0$, equation (60.3) becomes

$$(1-x)\frac{d^2u}{dx^2}+x^2=0. \tag{60.4}$$

Note that equation (60.4) does not have any explicit y dependence. If it did have any such terms, then the original equation could not have been equidimensional-in-y. The solution to equation (60.3) is (see page 224)

$$\begin{aligned} u(x) &= \int^x \left[\int^w \frac{z^2}{z-1}\,dz\right] dw, \\ &= \frac{x^3}{6}+\frac{x^2}{2}+(x-1)\log(x-1)+Ax+B, \end{aligned}$$

where A and B are arbitrary constants. Hence, the solution of the original equation is

$$y(x) = e^{u(x)} = (x-1)^{(x-1)}\exp\left(\frac{x^3}{6}+\frac{x^2}{2}+Ax+B\right).$$

Notes

1. This method is derivable from Lie group methods (see page 366).
2. Equidimensional-in-y equations are also called *equations homogeneous in y*.
3. The results in table 60.1 can be obtained with the Mathematica code in program 60.1. The output of that program is:

```
{1, y u'}
        2
{2, y u'  + y u''}
        3                (3)
{3, y u'  + 3 y u' u'' + y u   }
        4         2              2            (3)
```

$$
\begin{aligned}
y &= e^u,\\
y_x &= y u_x,\\
y_{xx} &= y(u_{xx} + u_x^2],\\
y_{xxx} &= y(u_{xxx} + 3u_x u_{xx} + u_x^3],\\
y_{xxxx} &= y(u_{xxxx} + 4u_x u_{xxx} + 3u_{xx}^2 + 6u_x^2 u_{xx} + u_x^4].\\
y_{x(4)} &= y(u_{x(4)} + 4u_x u_{xxx} + 3u_{xx}^2 + 6u_x^2 u_{xx} + u_x^4),\\
y_{x(5)} &= y(u_{x(5)} + 5u_x u_{x(4)} + 10u_{xx}u_{xxx} + 10u_x^2 u_{xxx} + 15u_x u_{xx}^2 + 10u_x^3 u_{xx} + u_x^5),\\
y_{x(6)} &= y(u_{x(6)} + 6u_x u_{x(5)} + 15u_{xx}u_{x(4)} + 15u_x^2 u_{x(4)} + 10u_{xxx}^2 + 20u_x^3 u_{xxx}\\
&\quad + 15u_{xx}^3 + 60u_x u_{xx} u_{xxx} + 45u_x^2 u_{xx}^2 + 15u_x^4 u_{xx} + u_x^6).
\end{aligned}
$$

Table 60.1: How to transform derivatives under the change of independent variable: $y(x) = e^{u(x)}$. (To simplify notation, define $y_{x(n)}$ to be the nth derivative of y with respect to x. Similarly for $u_{x(n)}$.)

```
{4, y u'  + 6 y u'  u'' + 3 y u''  + 4 y u' u    +

      (4)
  y u    }
```

4. See Bender and Orszag [1, page 27].

Reference

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.

61. Euler Equations

Applicable to Linear ordinary differential equations of the form $a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = 0$.

Yields

An exact solution.

Idea

An equation of the above type can be turned into a linear constant coefficient ordinary differential equation by a change of independent variable. This new equation can be solved exactly.

Procedure

An Euler equation has the form

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = 0. \tag{61.1}$$

If the independent variable is changed from x to t (via the transformation $x = e^t$), then the resulting equation becomes a linear constant coefficient ordinary differential equation. This type of equation can be solved exactly. (Table 61.1 shows how the derivatives of y with respect to x become derivatives of y with respect to t.)

Alternatively, a solution of the form $y = x^k$ can be tried directly in equation (61.1).

Example 1

Given the Euler equation

$$x^2y_{xx} - 2xy_x + 2y = 0,$$

we change variables by $x = e^t$ to obtain

$$y_{tt} - 3y_t + 2y = 0. \tag{61.2}$$

The standard technique for solving a linear constant coefficient ordinary differential equation is to look for exponential solutions (see page 247). Using $y = e^{\lambda t}$ in equation (61.2), we find the characteristic equation to be $\lambda^2 - 3\lambda + 2 = 0$. The roots of this equation are $\lambda = 1$ and $\lambda = 2$. Therefore, the solution to equation (61.2) is

$$y(t) = C_1e^t + C_2e^{2t},$$

where C_1 and C_2 are arbitrary constants. Writing this solution in the original variables, we determine the final solution

$$y(x) = C_1x + C_2x^2.$$

$$\begin{aligned}
y_x &= e^{-t}(y_t),\\
y_{xx} &= e^{-2t}(y_{tt} - y_t),\\
y_{xxx} &= e^{-3t}(y_{ttt} - 3y_{tt} + 2y_t),\\
y_{xxxx} &= e^{-4t}(y_{tttt} - 6y_{ttt} + 11y_{tt} - 6y_t),\\
y_{xxxxx} &= e^{-5t}(y_{ttttt} - 10y_{tttt} + 35y_{ttt} - 50y_{tt} + 24y_t).\\
y_{x(5)} &= e^{-5t}(y_{t(5)} - 10y_{t(4)} + 35y_{ttt} - 50y_{tt} + 24y_t),\\
y_{x(6)} &= e^{-6t}(y_{t(6)} - 15y_{t(5)} + 85y_{t(4)} - 225y_{ttt} + 274y_{tt} - 120y_t),\\
y_{x(7)} &= e^{-7t}(y_{t(7)} - 21y_{t(6)} + 175y_{t(5)} - 735y_{t(4)} + 1624y_{ttt} - 1764y_{tt} + 720y_t),
\end{aligned}$$

Table 61.1: How to transform derivatives under the change of dependent variable: $x = e^t$ (To simplify notation, define $y_{x(n)}$ to be the nth derivative of y with respect to x. Similarly for $y_{t(n)}$.)

Example 2

Given the Euler equation

$$x^3y''' - x^2y'' - 2xy' - 4y = 0, \tag{61.3}$$

we use $y = x^k$ to find the characteristic equation:

$$k(k-1)(k-2)x^k - k(k-1)x^k - 2kx^k - 4x^k = 0$$

or

$$\left(k^2+1\right)(k-4) = 0.$$

This equation has the roots $k = 4$ and $k = \pm i$. Hence, the general solution to equation (61.3) is

$$y = C_1x^4 + C_2\cos(\log x) + C_3\sin(\log x).$$

Notes

1. This method is also applicable to the equation

   $$a_0(Ax+B)^ny^{(n)} + a_1(Ax+B)^{n-1}y^{(n-1)} + \cdots + a_{n-1}(Ax+B)y' + a_ny = 0,$$

   which is only a trivial modification of an Euler equation.
2. Equations of the form $\frac{dx}{\sqrt{P(x)}} = \pm\frac{dy}{\sqrt{P(y)}}$, where $P(x)$ is a polynomial of degree three or four, have also been called Euler equations (see Valiron [5, pages 201–202]).
3. Euler matrix differential equations (in which the $\{a_i\}$ in equation (61.1) are all matrices) are discussed in Jódar [3].
4. See also Boyce and DiPrima [1, Section 4.4], Finizio and Ladas [2, pages 103–105], and Simmons [4, page 86].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Finizio, N., and Ladas, G. *Ordinary Differential Equations with Modern Applications*. Wadsworth Publishing Company, Belmont, CA, 1982.

[3] Jodar, L. Boundary value problems for second order operator differential equations. *Linear Algebra and Its Appls. 91* (1987), 1–12.

[4] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

[5] Valiron, G. *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions*. Math Sci Press, Brookline, MA, 1950.

62. Exact First Order Equations

Applicable to First order ordinary differential equations.

Yields

An exact solution (generally implicit).

Idea

Some first order ordinary differential equations can be integrated directly.

Procedure

If the given ordinary differential equation has the form

$$\frac{dy}{dx} = \frac{N(x,y)}{M(x,y)} \tag{62.1}$$

and $N(x,y)$ and $M(x,y)$ are such that

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0 \tag{62.2}$$

then equation (62.1) is said to be an exact ordinary differential equation. Such an equation can be solved exactly, though the answer may be in terms of an integral. The (implicit) solution will be of the form

$$\phi(x,y) = C, \tag{62.3}$$

where C is an arbitrary constant. Motivating this is straightforward. Differentiating equation (62.3) with respect to x and rearranging terms gives

$$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y}. \tag{62.4}$$

Comparing equation (62.4) to equation (62.1), we have

$$\phi_x = -N, \quad \phi_y = M, \tag{62.5.a-b}$$

and hence equation (62.2) is satisfied (because $\phi_{xy} = \phi_{yx}$). Conversely, if equation (62.2) is satisfied, then there is a ϕ such that equation (62.5) is satisfied. To solve equation (62.5) for ϕ, integrate equation (62.5.a) with respect to x and integrate equation (62.5.b) with respect to y for

$$\begin{aligned} \phi(x,y) &= -\int N(x,y)\,dx + f(y), \\ \phi(x,y) &= \int M(x,y)\,dy + g(x), \end{aligned} \tag{62.6.a-b}$$

where $f(y)$ and $g(x)$ are unknown functions. Comparing equation (62.6.a) to equation (62.6.b) will determine $f(y)$ and $g(x)$. Knowing either of these, the full solution is then given by equation (62.6.a) or equation (62.6.b).

Example

Suppose we have the equation

$$\frac{dy}{dx} = \frac{3x^2 - y^2 - 7}{e^y + 2xy + 1}. \tag{62.7}$$

In equation (62.7) we identify

$$N(x, y) = 3x^2 - y^2 - 7 \qquad \text{and} \qquad M(x, y) = e^y + 2xy + 1.$$

Following our procedure, we find $M_x = -N_y = 2y$ and so we know that we can solve equation (62.7) exactly. Integrating N and M we find

$$\begin{aligned} \phi(x, y) &= -\int N(x, y)\, dx + f(y) = -(x^3 + y^2 x - 7x) + f(y), \\ \phi(x, y) &= \int M(x, y)\, dy + g(x) = (e^y + y^2 x + y) + g(x). \end{aligned} \tag{62.8.a-b}$$

Comparing equations (62.8.a) and (62.8.b), we deduce that

$$x^3 - y^2 x + 7x + f(y) = e^y + y^2 x + y + g(x)$$

or

$$f(y) - (e^y + y) = g(x) - (7x + x^3). \tag{62.9}$$

From equation (62.9) we conclude that

$$f(y) = e^y + y + A, \quad g(x) = 7x - x^3 + A, \tag{62.10.a-b}$$

where A is an arbitrary constant. Using either equation (62.10.a) in (62.8.a) or equation (62.10.b) in (62.8.b), we conclude

$$\phi(x, y) = -x^3 - y^2 + 7x + e^y + y + A. \tag{62.11}$$

The solution is then given by $\phi(x, y) = C$, where C is an arbitrary constant. Therefore,

$$-x^3 - y^2 + 7x + e^y + y = B \tag{62.12}$$

is the final solution, where $B := A - C$ is a final arbitrary constant. Note that the solution in equation (62.12) is implicit.

Note

1. See Boyce and DiPrima [1, pages 79–84], Rainville and Bedient [2, pages 29–33], and Simmons [3, pages 38–41].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations*. The MacMillan Company, New York, 1964.

[3] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

63. Exact Second Order Equations

Applicable to Some nonlinear second order ordinary differential equations of the form $f(x,y,y')y'' + g(x,y,y') = 0$.

Yields

A first integral (which will be a first order ordinary differential equation).

Idea

Some second order ordinary differential equations can be integrated once.

Procedure

The second order differential equation

$$F(x,y,y',y'') = 0 \tag{63.1}$$

is said to be exact if it is the total differential of some function; i.e., $F = d\phi/dx$ where $\phi = \phi(x,y,y')$. If equation (63.1) is exact, then $\phi = C$ is a solution to equation (63.1), with C an arbitrary constant. Differentiating $\phi = C$ with respect to x, we find

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' + \frac{\partial \phi}{\partial y'} y''. \tag{63.2}$$

Comparing equation (63.2) to equation (63.1), we conclude that, for equation (63.1) to be exact, $F(x,y,y',y'')$ must have the form

$$F(x,y,y',y'') = f(x,y,y')y'' + g(x,y,y'), \tag{63.3}$$

for some functions f and g with

$$f(x,y,y') = \frac{\partial \phi}{\partial y'}, \qquad g(x,y,y') = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y'. \tag{63.4.a-b}$$

By differentiating equation (63.4.a–b) with respect to x, y, and p, (using $p := dy/dx$), all dependence on ϕ can be eliminated between the two equations in equation (63.4) to obtain

$$\begin{aligned} f_{xx} + 2pf_{xy} + p^2 f_{yy} &= g_{xp} + pg_{yp} - g_y, \\ f_{xp} + pf_{yp} + 2f_y &= g_{pp}. \end{aligned} \tag{63.5}$$

If the conditions in equation (63.5) hold, then equation (63.3) is exact. If equation (63.3) is exact, then we can integrate equation (63.4.a) (with respect to p) to determine $\phi(x, y, y')$ as

$$\phi = h(x, y) + \int f(x, y, p)\, dp, \tag{63.6}$$

where $h(x, y)$ is, so far, an arbitrary function of integration. This function will be restricted when equation (63.6) is used in equation (63.4.b).

Example

Given the equation

$$xyy'' + x(y')^2 + yy' = 0, \tag{63.7}$$

which has the form of equation (63.3), we identify: $f = xy$, $g = x(y')^2 + yy' = xp^2 + yp$. It is easy to verify that equation (63.5) holds. Hence, equation equation (63.7) is exact. Equation (63.6) now becomes

$$\begin{aligned} \phi &= h(x, y) + \int xy\, dp \\ &= h(x, y) + xyp. \end{aligned} \tag{63.8}$$

Using equation (63.8) in equation (63.4.b) yields

$$\begin{aligned} g = xp^2 + yp &= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' \\ &= (h_x + yp) + (h_y + xp)p. \end{aligned} \tag{63.9}$$

Hence, if h is constant, say $h = D$, then equation (63.9) will be satisfied.

Therefore a first integral of equation (63.7) is given by $\phi = C$, or

$$\begin{aligned} C &= \phi(x, y, p) \\ &= D + xyp \\ &= D + xy\frac{dy}{dx}. \end{aligned} \tag{63.10}$$

In this example, the first integral equation (63.10) can itself be integrated in closed form (this is often true). A solution to equation (63.7), obtained by solving the ordinary differential equation in equation (63.10), is thus given by

$$\frac{y^2}{2} = (C - D)\log x + E,$$

where E is another arbitrary constant.

Notes

1. The most general solution for $h(x,y)$ in equation (63.9) is $h = h(y-x)$. With this form for h, however, the first integral cannot be integrated to yield an explicit solution.
2. Exact second order linear ordinary differential equations have factorable operators (see page 294).
3. Given the differential equation

$$f(x, y, \ldots, y^{(n)}) = 0, \tag{63.11}$$

define $f_i = \frac{\partial f}{\partial y^{(i)}}$. Then equation (63.11) will be exact if

$$f_0 - \frac{df_1}{dx} + \frac{d^2 f_2}{dx^2} - \cdots + (-1)^n \frac{d^n f_n}{dx^n} = 0. \tag{63.12}$$

If the differential equation (63.11) is exact, then a first integral can be found by a repetitive sequence of steps: First, integrate the highest order term in f and call this result F_1. Then, integrate the highest order term in $f\,dx - dF_1$ and call this result F_2. Continue in this manner until $f\,dx - dF_1 - dF_2 - \cdots = 0$. Then, a first integral is given by $F_1 + F_2 + \cdots =$ constant. For example, given the nonlinear third order equation

$$f = yy''' - y'y'' + y^3 y' = 0, \tag{63.13}$$

we identify $f_3 = y$, $f_2 = -y'$, $f_1 = -y'' + y^3$, $f_0 = y''' + 3y^2 y'$ and verify that equation (63.12) is satisfied. We then calculate $F_1 = yy''$, since the highest order term in f is yy'''. Then, $f\,dx - dF_1 = (-2y'y'' + y^3 y')dx$, and so we take $F_2 = -(y')^2$. Then, $f\,dx - dF_1 - dF_2 = y^3 y'\,dx$, and so $F_3 = \frac{1}{4} y^4$. Finally, then, $f\,dx - dF_1 - dF_2 - dF_3 = 0$, so that

$$yy'' - (y')^2 + \frac{1}{4} y^4 = \text{constant}$$

is a first integral of equation (63.13).
4. See also Goldstein and Braun [1, page 93] and Murphy [2, pages 221–222].

References

[1] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations.* U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[2] Murphy, G. M. *Ordinary Differential Equations and Their Solution.* D. Van Nostrand Company, Inc., New York, 1960.

64. Exact Nth Order Equations

Applicable to Linear nth order ordinary differential equations.

Yields

A first integral.

Idea

Some linear differential equations can be integrated exactly without modifying the equation in any way.

Procedure

The linear nth order ordinary differential equation

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y = R(x), \tag{64.1}$$

is said to be *exact* if it can be integrated once to yield

$$Q_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + Q_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + Q_1(x)\frac{dy}{dx} + Q_0(x)y = \int R(x)\,dx. \tag{64.2}$$

If equation (64.1) is exact, then the $\{Q_i(x)\}$ may be found from

$$\begin{aligned} Q_{n-1} &= P_n, \\ Q_{n-2} &= P_{n-1} - P_n', \\ Q_{n-3} &= P_{n-2} - P_{n-1}' + P_n'', \\ &\vdots \\ Q_0 &= P_1 - P_2' + P_3'' - \cdots + (-1)^{n-1}P_n^{(n-1)}. \end{aligned}$$

A necessary and sufficient condition for equation (64.1) to be exact can be found by differentiating equation (64.2) with respect to x and comparing terms with equation (64.1). This condition is

$$\frac{d^n P_n}{dx^n} - \frac{d^{n-1}P_{n-1}}{dx^{n-1}} + \frac{d^{n-2}P_{n-2}}{dx^{n-2}} - \cdots + (-1)^{n-1}\frac{dP_1}{dx} + (-1)^n P_0 = 0. \tag{64.3}$$

Special Case

The second order linear ordinary differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

will be exact if and only if $P''(x) - Q'(x) + R(x) = 0$.

Example

If we have the linear ordinary differential equation of third order

$$(1+x+x^2)\frac{d^3y}{dx^3} + (3+6x)\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 6x, \qquad (64.4)$$

then we have $P_0 = 0$, $P_1 = 6$, $P_2 = 3+6x$, $P_3 = 1+x+x^2$, and $R(x) = 6x$. It is easy to verify that

$$\frac{d^3P_3}{dx^3} - \frac{d^2P_2}{dx^2} + \frac{dP_1}{dx} - P_0 = 0,$$

and so equation (64.4) is exact. Integrating equation (64.4) directly, we obtain

$$(1+x+x^2)\frac{d^2y}{dx^2} + (2+4x)\frac{dy}{dx} + 2y = 3x^2 + A, \qquad (64.5)$$

where A is an arbitrary constant. Now equation (64.5) is again exact, and so it can be integrated again to yield

$$(1+x+x^2)\frac{dy}{dx} + (1+2x)y = x^3 + Ax + B, \qquad (64.6)$$

where B is an arbitrary constant.

Finally, equation (64.6) is once again exact. It can be integrated to yield the general solution of equation (64.4)

$$(1+x+x^2)y = \frac{x^4}{4} + A\frac{x^2}{2} + Bx + C,$$

where C is an arbitrary constant.

Note

1. See Ford [1, pages 77–78] and Murphy [2, pages 221–222].

References

[1] Ford, L. R. *Differential Equations*. McGraw–Hill Book Company, New York, 1955.

[2] Murphy, G. M. *Ordinary Differential Equations and Their Solution*. D. Van Nostrand Company, Inc., New York, 1960.

65. Factoring Equations*

Applicable to

Ordinary differential equations and partial differential equations.

Yields

Equations of lower degree.

Idea

If a differential equation can be factored into simple terms, then the solution to each of the factors is a solution to the original equation.

Procedure

Given a differential equation, attempt to factor it. If this is possible, then solve each factor separately. Each of the solutions of the different factors will be a solution of the original differential equation.

Example

The nonlinear ordinary differential equation

$$y'(y' + y) = x(x + y) \tag{65.1}$$

for $y(x)$ may be factored into

$$(y' + y + x)(y' - x) = 0. \tag{65.2}$$

Solving each of the factors appearing in equation (65.2) separately, the solutions to equation (65.1) are given by

$$y(x) = \begin{cases} Ae^{-x} + 1 - x, \\ \\ B + \frac{x^2}{2}, \end{cases}$$

where A and B are constants.

Notes

1. The complete solution to the original differential equation may switch from one solution branch to another.
2. See Bateman [2, pages 97–98] and Fogiel [3, pages 1222–1229].

References

[1] Argyros, I. K. On the cardinality of solutions of multilinear differential equations and applications. *Int. J. Math. & Math. Sci. 9*, 4 (1986), 757–766.

[2] Bateman, H. *Partial Differential Equations of Mathematical Physics.* Dover Publications, Inc., New York, 1944.

[3] Fogiel, M. *The Differential Equations Problem Solver.* Research and Education Association, New York, 1978.

[4] Klamkin, M. S. On soluble nth order linear differential equations. *J. Math. Anal. Appl. 84* (1981), 6–11.

66. Factoring Operators*

Applicable to

Ordinary and partial differential equations.

Yields

A sequence of lower order equations to solve.

Idea

If the operator representing a differential equation can be "factored" into two or more operators, it may be easier to find a solution.

Procedure

Suppose we wish to solve the differential equation $Q[u] = 0$ for the quantity $u(\mathbf{x})$, where $Q[\cdot]$ is a differential operator. When possible, "factor" the differential equation $Q[u] = 0$ as $L[H[u]] = 0$, where $L[\cdot]$ and $H[\cdot]$ are also differential operators. Then solve the two equations: $L[v] = 0$ for v, and then $H[u] = v$.

Example 1

The fourth order partial differential equation

$$(\nabla^4 - a^2)u = 0, \tag{66.1}$$

where a is a constant and ∇^2 is the usual Laplacian, may be factored as

$$(\nabla^2 - a)(\nabla^2 + a)u = 0.$$

The general solution of equation (66.1), therefore, is given by the solution of the two successive second order differential equations

$$\begin{aligned}(\nabla^2 - a)v &= 0,\\ (\nabla^2 + a)u &= v.\end{aligned} \tag{66.2}$$

Alternatively, equation (66.1) could have factored equation as

$$(\nabla^2 + a)(\nabla^2 - a)u = 0$$

so that the general solution of equation (66.1) can also be written as the solution of

$$\begin{aligned}(\nabla^2 + a)w &= 0,\\ (\nabla^2 - a)u &= w.\end{aligned} \tag{66.3}$$

Solving equation (66.2) or equation (66.3) as a sequence of two second order differential equations may be easier than solving the fourth order equation (66.1) directly.

Example 2

If we want to solve the nonlinear ordinary differential equation $Q[u] = 0$, where

$$\begin{aligned} Q[u] &= u_{xx}^2 - 2u_x u_{xx} + 2uu_x - u^2 = 0 \\ &= (u_{xx} - u_x)^2 - (u_x - u)^2 = 0, \end{aligned} \tag{66.4}$$

then we might factor the operator $Q[\cdot]$ as $Q[u] = L[H[u]]$, where $L[v] = v_x^2 - v^2$, and $H[u] = u_x - u$. Therefore, the equation $Q[u] = 0$ can be solved by solving the sequence of first order differential equations

$$L[v] = 0, \qquad H[u] = v.$$

The solution of $L[v] = 0$ is $v = Ce^{\pm x}$, where C is an arbitrary constant. The general solution of equation (66.4) can then be determined by solving

$$H[u] = u_x - u = v = Ce^{\pm x}. \tag{66.5}$$

Equation (66.5) can be solved by the use of integrating factors (see page 356) to obtain the two possible forms of the solution

$$u = \begin{cases} (A + Cx)e^x, \\ \\ Ce^{-x} + Be^x, \end{cases}$$

where A and B are also arbitrary constants.

Example 3

The relativistic wave equation

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} + \frac{m^2 c^2}{\hbar^2}\psi = 0$$

was factored by Dirac [4, Chapter 11] using hypercomplex algebra. If $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ represent four of the elements in this algebra that obey the relation $\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 2\delta_{\mu\nu}$, then the factored equation is

$$\left(\frac{1}{c}\frac{d}{dt} - \alpha_1 \frac{d}{dx} - \alpha_2 \frac{d}{dy} - \alpha_3 \frac{d}{dz} - \alpha_4 \frac{imc}{\hbar}\right) \left(\frac{1}{c}\frac{d}{dt} + \alpha_1 \frac{d}{dx} + \alpha_2 \frac{d}{dy} + \alpha_3 \frac{d}{dz} + \alpha_4 \frac{imc}{\hbar}\right)\psi = 0.$$

The first factor led to the correct relativistic theory for the electron, while the second factor led to Dirac's prediction of the positron.

Example 4

The formally self-adjoint homogeneous fourth order operator

$$\frac{d^2}{dx^2}\left(P(x)\frac{d^2y}{dx^2}\right)\frac{d}{dx}\left(Q(x)\frac{dy}{dx}\right)+R(x)y$$

may be factored into $L[\nu(x)L[y]]$, where $L[\cdot]$ is the second order operator

$$L[y]=\frac{d}{dx}\left[\lambda(x)\frac{dy}{dx}\right]+\mu(x)y,$$

where $\{\nu(x),\mu(x),\lambda(x)\}$ satisfy

$$\begin{aligned}
\nu(x)&=\frac{\beta'}{\alpha^2},\\
\lambda(x)&=\alpha^2\beta',\\
\mu(x)&=\frac{\alpha}{\beta'}\left(\alpha''+\frac{1}{2}\gamma\alpha\right),
\end{aligned}$$

and $\{\alpha(x),\beta(x),\gamma(x),\delta(x)\}$ are any solution to

$$\begin{aligned}
P(x)&=\alpha^2{\beta'}^3,\\
Q(x)&=\alpha^2\beta'''+2\alpha\alpha'\beta''+\left(4\alpha\alpha''-2{\alpha'}^2+\gamma\alpha^2\right)\beta',\\
R(x)&=\frac{\alpha}{\beta'}\left(\alpha''''+\alpha\gamma''+\alpha'\gamma'+\alpha\delta\right),
\end{aligned}$$

with $4\delta=2\gamma''+\gamma^2$. See Hill [9] for details.

Notes

1. Note that the equation in example 2 can be directly factored as $Q[u]=(u_{xx}-2u_x+u)(u_{xx}-u)$. In this case, the factorization of the equation simplier than the factorization of the operator (see page 292).
2. It is not true that the number of distinct factorizations is limited by the order of the differential equation. For example, the second order ordinary differential equation

$$(x^2-x^3)u''+(2x^2-4x)u'+(6-2x)u=0,$$

has the three distinct factorizations

$$\begin{aligned}
\left(x\frac{d}{dx}-2\right)\left[(x-x^2)\frac{d}{dx}+2x-3\right]u&=0,\\
\left(x\frac{d}{dx}-3\right)\left[(x-x^2)\frac{d}{dx}+x-2\right]u&=0,\\
\left[(x-x^2)\frac{d}{dx}+x-3\right]\left(x\frac{d}{dx}-2\right)u&=0.
\end{aligned}\tag{66.6}$$

3. The Laplacian in two dimensions admits the factorization:
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial \overline{z}}\right), \tag{66.7}$$
where $i = \sqrt{-1}$. Therefore, using $z = x + iy$, Laplace's equation may be written as $\nabla^2 u = \frac{\partial^2 u}{\partial z \partial \overline{z}} = 0$. This shows that the most general solution to Laplace's equation in two dimensions is $u = f(z) + g(\overline{z})$, where $f(z)$ and $g(\overline{z})$ are arbitrary functions. Also, because the biharmonic equation may be written as $\nabla^4 u = 16\frac{\partial^4 u}{\partial^2 z \partial^2 \overline{z}} = 0$, the general solution of the biharmonic equation is seen to be $u = f(z) + g(\overline{z}) + zh(\overline{z}) + \overline{z}j(z)$. The operators $\partial/\partial z$ and $\partial/\partial\overline{z}$ are known as *Wirtinger derivatives*. In two dimensions, solutions of Poisson's equation may sometimes be found by use of Wirtinger derivatives. See Henrici [8, pages 300–302] for details.
4. It is possible to write down an "explicit" factorization of any nth order linear differential equation. To do so, however, requires explicit knowledge of the n linearly independent solutions. For example, if $L[\cdot]$ is the differential operator
$$L[u] = u'' + p(x)u' + q(x)u,$$
and u_1, u_2 are any two linearly independent solutions of $L[u] = 0$, then
$$L[u] = \frac{W(u_1, u_2)}{u_1}\frac{d}{dx}\left[\frac{u_1^2}{W(u_1, u_2)}\frac{d}{dx}\left(\frac{u}{u_1}\right)\right],$$
where $W(u_1, u_2)$ is the Wronskian of $u_1(x)$ and $u_2(x)$. In the nth order case, consider the differential operator
$$H[u] = u^{(n)} + p_1(x)u^{(n-1)} + p_2(x)u^{(n-2)} + \cdots + p_n(x)u.$$
If $\{u_1, u_2, \ldots, u_n\}$ are n linearly independent solutions of $H[u] = 0$, then define W_k (for $k = 1, 2, \ldots, n$) to be the Wronskian of the first k linearly independent solutions; that is, $W_k := W(u_1, u_2, \ldots, u_k)$. Using this definition, we can write $H[u]$ as
$$H[u] = \frac{W_n}{W_{n-1}}\frac{d}{dx}\left(\frac{W_{n-1}^2}{W_{n-1}W_n}\cdots \right.$$
$$\left.\cdots\frac{d}{dx}\left(\frac{W_2^2}{W_1W_3}\left(\frac{d}{dx}\left(\frac{W_1^2}{W_0W_2}\left(\frac{d}{dx}\left(\frac{u}{W_1}\right)\right)\right)\right)\right)\cdots\right).$$
See Rainville [12, pages 292–299] for details.
5. The factorization
$$\left\{\frac{d}{dt} - q(t)\right\}\left\{\frac{d}{dt} + q(t)\right\}w = \frac{d^2w}{dt^2} + w\left\{\frac{dq}{dt} - q^2\right\}$$
leads to the technique for solving Riccati equations (see page 392).

6. Differential resultants can be used to analyze the factoring of operators for linear differential equations. See Berkovich and Tsirulik [1] for details.
7. Two differential operators P and Q are said to be permutable if $P(Q) = Q(P)$. From Ince [10, page 131], we have

   If P and Q are permutable operators of orders m and n respectively, they satisfy identically an algebraic relation of the form $F(P, Q) = 0$ of degree n in P and of degree m in Q.

   For example, the operators

$$P = \frac{d^2}{dx^2} - \frac{2}{x^2},$$
$$Q = \frac{d^3}{dx^3} - \frac{3}{x^2}\frac{d}{dx} + \frac{3}{x^3},$$

   are permutable because $PQ = QP$. We can also find the algebraic relation $P^3 - Q^2 = 0$, observe

$$P(P(P(f))) = f'''''' - \frac{6}{x^2}f'''' + \frac{24}{x^3}f''' - \frac{72}{x^4}f'' + \frac{144}{x^5}f' - \frac{144}{x^6}f = Q(Q(f)).$$

   This example is due to Ince [10, page 131]. See also Grünbaum [7].
8. Landau [11] gives a (surprising) factorization that depends on an arbitrary parameter a:

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \left(\frac{d}{dx} - \frac{1}{x(1+ax)}\right)\left(\frac{d}{dx} - \frac{1+2ax}{x(1+ax)}\right)y.$$

9. Schwarz [14] has developed an algorithm that will factor ordinary differential equations. As an example, his program derives the factorization

$$y'' - \left(\frac{3}{4x^2} + \frac{5}{2x^3} - \frac{1}{4x^4}\right)y = \left(\frac{d}{dx} - \frac{3}{2x} + \frac{1}{2x^2} + \frac{1}{x - \frac{1}{3}}\right)$$
$$\left(\frac{d}{dx} + \frac{3}{2x} - \frac{1}{2x^2} - \frac{1}{x - \frac{1}{3}}\right)y.$$

References

[1] Berkovich, L. M., and Tsirulik, V. G. Differential resultants and some of their applications. *Differentsial'nye Uravneniya 22*, 5 (May 1986), 750–757.

[2] Brownawell, W. D. On the factorization of partial differential equations. *Can. J. Math. 39*, 4 (1987), 825–834.

[3] Chisholm, J. S. R., and Common, A. K. A class of second-order differential equations and related first-order systems. *J. Phys. A: Math. Gen. 20* (1987), 5459–5472.

[4] Dirac, P. A. M. *The Principle of Quantum Mechanics.* Clarendon Press, Oxford, England, 1974.

[5] Etgen, G. J., Jones, G. D., and Taylor, Jr., W. E. On the factorizations of ordinary linear differential operators. *Trans. Amer. Math. Soc. 297*, 2 (1986), 717–728.

[6] Fordy, A. P., and Gibbons, J. Factorization of operators I. Miura transformations. *J. Math. Physics 21*, 10 (Oct 1980), 2508–2510.

[7] Grunbaum, F. A. Commuting pairs of linear ordinary differential operators of orders four and six. *Physica D 31* (1988), 424–433.

[8] Henrici, P. *Applied and Computational Complex Analysis*, vol. 3. John Wiley & Sons, New York, 1986.

[9] Hill, J. M. *Solution of Differential Equations by Means of One-Parameter Groups.* Pitman Publishing Co., Marshfield, MA, 1982.

[10] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[11] Landau, E. *Journal fur die reine und angewandte Mathematik 124* (1902), 115–120.

[12] Rainville, E. D. *Intermediate Differential Equations.* The MacMillan Company, New York, 1964.

[13] Sandell, D. C., and Stein, F. M. Factorization of operators of second order linear homogeneous ordinary differential equations. *Two Year College Mathematics Journal 8* (1977), 132–141.

[14] Schwarz, F. Efficient factorization of linear ODE's. *ACM-SIGSAM Bulletin 28* (1994), 9–17.

[15] Weston, V. H. Factorization of the wave equation in higher dimensions. *J. Math. Physics 28*, 5 (May 1987), 1061–1068.

67. Factorization Method

Applicable to Eigenvalue/eigenfunction problems for homogeneous linear second order ordinary differential equations.

Yields

An equation from which a single eigenfunction can be used to calculate additional eigenfunctions.

Idea

By "factoring" an ordinary differential equation into a certain form, a ladder of eigenfunctions may be formed.

Procedure

Suppose we have the linear second order ordinary differential equation

$$\frac{d^2y}{dx^2} + r(x,m)y + \lambda y = 0, \tag{67.1}$$

where m is an integer for which we would like to determine the eigenfunctions $\{y\}$ corresponding to a single value of the eigenvalue λ. We denote the eigenfunction by $y(\lambda, m)$ and suppress the x dependence. The equation in (67.1) is said to be *factorizable* if it is equivalent to each of

$$\begin{aligned} H_+^{m+1} H_-^{m+1} y(\lambda, m) &= L(\lambda, m+1) y(\lambda, m), \\ H_-^m H_+^m y(\lambda, m) &= L(\lambda, m) y(\lambda, m), \end{aligned} \tag{67.2.a-b}$$

where $L(\lambda, m)$ is a function and the $H_\pm^m$ are differential operators.

$$H_\pm^m = k(x,m) \pm \frac{d}{dx},$$

For a factorizable equation, finding $L(\lambda, m)$ and the $H_\pm^m$ is a difficult task. Also, not all equations in the form of equation (67.1) are factorizable.

If equation (67.1) is factorizable and if $y(\lambda, m)$ is a solution of equation (67.1), then (see notes)

$$\begin{aligned} y(\lambda, m+1) &= H_-^{m+1} y(\lambda, m), \\ y(\lambda, m-1) &= H_+^m y(\lambda, m), \end{aligned} \tag{67.3.a-b}$$

are also solutions corresponding to the same value of λ, but different values of m. Hence, given one solution of equation (67.1) (for a specific value of λ), a *ladder of solutions* belonging to this value of λ may be formed by repeatedly iterating equation (67.3).

Example 1

The equation for the associated spherical harmonics may be put in the form

$$\frac{d^2y}{d\theta^2} - \frac{m^2 - \frac{1}{4}}{\sin^2\theta} + \left(\lambda + \frac{1}{4}\right) y = 0. \tag{67.4}$$

This equation is factorizable, and we find

$$\begin{aligned} H_\pm^m &= \left(m - \frac{1}{2}\right)\cot\theta \pm \frac{d}{dx}, \\ L(\lambda, m) &= \lambda - \left(m - \frac{1}{2}\right)^2, \end{aligned} \tag{67.5}$$

The eigenvalues of equation (67.4) are of the form $\lambda = l(l+1)$ for $l = m, m+1, \dots$. Some of the eigenfunctions of equation (67.4) are of the form

$$y_l^l(\theta) = \left[\frac{1 \cdot 3 \cdot 5 \cdots (2l+1)}{2 \cdot 2 \cdot 4 \cdots (2l)}\right]^{1/2} \sin^{l+1/2}\theta.$$

All of the remaining eigenfunctions may be found from equation (67.3) and equation (67.5) to be given by

$$\begin{aligned} y_l^{m-1}(\theta) &= \frac{1}{\sqrt{(l+m)(l+1-m)}} \left[\left(m - \frac{1}{2}\right)\cot\theta + \frac{d}{d\theta}\right] y_l^m(\theta), \\ y_l^{m+1}(\theta) &= \frac{1}{\sqrt{(l+m+1)(l-m)}} \left[\left(m + \frac{1}{2}\right)\cot\theta - \frac{d}{d\theta}\right] y_l^m(\theta). \end{aligned}$$

Example 2

As another example, Legendre's differential equation

$$(1 - x^2)\left[(1 - x^2)y_m'\right]' + m(m+1)y_m = 0$$

has the factorizations

$$\begin{aligned} H_-^m H_+^m y_m &= -m^2 y_m, \\ H_+^{m+1} H_-^{m+1} y_m &= -(m+1)^2 y_m, \end{aligned}$$

where $H_\pm^m = (1 - x^2)\frac{d}{dx} \pm mx$. This factorization leads to the ladder of solutions: $y_{m+1} = H_-^m y_m$.

Notes

1. The results in equation (67.3) are straightforward to derive. For example, operating on equation (67.2.b) with H_+^m results in

$$H_+^m H_-^m \left\{H_+^m y(\lambda, m)\right\} = L(\lambda, m)\left\{H_+^m y(\lambda, m)\right\}. \tag{67.6}$$

Because this has the same form as equation (67.2.a), which is by hypothesis equivalent to equation (67.1), it must be that $y = H_+^m y(\lambda, m)$ is a solution of equation (67.1). In equation (67.3), we called this $y(\lambda, m-1)$ because, when equation (67.6) is compared to equation (67.2.a), the parameter m is replaced by $m-1$.

2. The factorization method has been generalized to systems of equations in Humi [4].
3. The operators in equation (67.3) are sometimes called raising and lowering operators. This method is sometimes called the *ladder method*.
4. Infeld and Hull [5] have a large list of equations to which this method applies.
5. The paper by Hermann [3] relates the technique in this section to Lie groups. Sattinger and Weaver [8, pages 49–54] also consider the relation to Lie groups.
6. See also Lamb [6, pages 38–41] and Morse and Feshback [7, pages 788–789].

References

[1] Barut, A. O., Inomata, A., and Wilson, R. A new realization of dynamical groups and factorization method. *J. Phys. A: Math. Gen. 20* (1987), 4075–4083.

[2] Bessis, N., and Bessis, G. Algebraic recursive determination of matrix elements from ladder operator considerations. *J. Phys. A: Math. Gen. 20* (1987), 5745–5754.

[3] Hermann, R. Infeld–Hull factorization, Galois–Picard–Vessiot theory for differential operators. *J. Math. Physics 22*, 6 (June 1981), 1163–1167.

[4] Humi, M. Factorization of systems of differential equations. *J. Math. Physics 27* (Jan 1986), 76–81.

[5] Infeld, L., and Hull, T. E. The factorization method. *Rev. Mod. Physics 23*, 1 (Jan 1951), 21–68.

[6] Lamb, G. L. *Elements of Soliton Theory.* John Wiley & Sons, New York, 1980.

[7] Morse, P. M., and Feshback, H. *Methods of Theoretical Physics.* McGraw–Hill Book Company, New York, 1953.

[8] Sattinger, D. H., and Weaver, O. L. *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*. Springer–Verlag, New York, 1986.

[9] Schrodinger, E. A method of determining quantum-mechanical eigenvalues and eigenfunctions. *Proc. Roy. Irish Acad. A46* (1940), 9–16.

68. Fokker–Planck Equation

Applicable to Linear ordinary differential equations with linearly appearing "white Gaussian noise" terms (a single differential equation or a system).

Yields

A Fokker–Planck equation (which is a parabolic partial differential equation) for the probability density of the solution.

Idea

If a differential equation contains random terms, then the solution to the differential equation can only be described statistically. The solution to the Fokker–Planck equation is the probability density of the solution to the original differential equation.

Procedure

Here we present the technique for constructing the Fokker–Planck equation for a linear system of ordinary differential equations depending on several white noise terms. Consider the linear differential system for the m component vector $\mathbf{x}(t)$

$$\begin{aligned} \frac{d}{dt}\mathbf{x}(t) &= \mathbf{b}(t,\mathbf{x}) + \sigma(t,\mathbf{x})\,\mathbf{n}(t), \\ \mathbf{x}(t_0) &= \mathbf{y}, \end{aligned} \tag{68.1.a-b}$$

where $\sigma(t,\mathbf{x})$ is a real $m \times n$ matrix and $\mathbf{n}(t)$ is a vector of n independent white noise terms. That is,

$$\begin{aligned} E[n_i(t)] &= 0, \\ E[n_i(t)n_j(t+\tau)] &= \delta_{ij}\delta(\tau), \end{aligned} \tag{68.2}$$

where $E[\cdot]$ is the expectation operator, δ_{ij} is the Kronecker delta, and $\delta(\tau)$ is the delta function. The Fokker–Planck equation corresponding to equation (68.1.a) is given by

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}(b_i P) + \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}P), \tag{68.3}$$

where $P = P(t,\mathbf{x})$ is a probability density and the matrix $A = (a_{ij})$ is defined by $A(t,\mathbf{x}) = \sigma(t,\mathbf{x})\sigma^{\mathrm{T}}(t,\mathbf{x})$. The initial conditions for equation (68.3) come from equation (68.1.b); they are

$$P(t_0,\mathbf{x}) = \prod_{i=1}^{m} \delta(x_i - y_i). \tag{68.4}$$

The solution of equations (68.3) and (68.4) is the probability density of the solution to equation (68.1). Any statistical information about $\mathbf{x}(t)$ that could be ascertained from equation (68.1) can be derived from $P(t, \mathbf{x})$. For example, the expected value of some function of $\mathbf{x}$ and t, say $h(\mathbf{x}, t)$, at a time t, can be calculated by

$$E[h(\mathbf{x}(t), t)] = \int_{-\infty}^{\infty} h(\mathbf{x}(t), t) P(t, \mathbf{x})\, d\mathbf{x}.$$

Special Case

In the special case of one dimension, the stochastic differential equation

$$\frac{dx}{dt} = f(x) + g(x)n(t), \tag{68.5}$$

with $x(0) = z$, corresponds to the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(f(x)P) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2(x)P),$$

for $P(t, x)$ with $P(0, x) = \delta(x - z)$.

Example

Consider the *Langevin equation*

$$x'' + \beta x' = N(t), \tag{68.6}$$

with the initial conditions

$$x(0) = 0, \qquad x'(0) = u_0, \tag{68.7}$$

where $N(t)$ satisfies

$$\begin{aligned} E[N(t)] &= 0, \\ E[N(t)N(t+\tau)] &= \delta(\tau). \end{aligned} \tag{68.8}$$

From equation (68.8), we recognize that $N(t)$ is a white noise term. Therefore, we can use the Fokker–Planck equation to determine the probability density of $x(t)$. Because equation (68.6) has second derivative terms, we rewrite equation (68.6) and equation (68.7) as the vector system (see page 146)

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ u \end{bmatrix} &= \begin{bmatrix} u \\ -\beta u \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} n_1(t) \\ n_2(t) \end{bmatrix}, \\ \begin{bmatrix} x \\ u \end{bmatrix}_{t=0} &= \begin{bmatrix} 0 \\ u_0 \end{bmatrix}. \end{aligned} \tag{68.9}$$

The Fokker–Planck equation for $P(t,x,u)$, the joint probability density of x and u at time t, is

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}(uP) + \frac{\partial}{\partial u}(\beta u P) + \frac{1}{2}\frac{\partial^2 P}{\partial u^2}, \\ P(0,x,u) &= \delta(x)\delta(u-u_0).\end{aligned} \tag{68.10}$$

In this example, we can solve equation (68.10) exactly by taking a Fourier transform in x (see page 350) and then using the method of characteristics (see page 432). We eventually determine

$$P(t,x,u) = \frac{1}{\det D}\exp\left(-\begin{bmatrix}x-\mu_x\\u-\mu_u\end{bmatrix} D \begin{bmatrix}x-\mu_x\\u-\mu_u\end{bmatrix}^{\mathrm{T}}\right),$$

where $D = \begin{bmatrix}\sigma_{xx} & \sigma_{xu}\\ \sigma_{xu} & \sigma_{uu}\end{bmatrix}$, and the parameters $\{\mu_x, \mu_u, \sigma_{xx}, \sigma_{xu}, \sigma_{uu}\}$ are given by

$$\begin{aligned}\mu_x &= \frac{u_0}{\beta}\left(1-e^{-\beta t}\right), \\ \mu_u &= u_0 e^{-\beta t}, \\ \sigma_{xx}^2 &= \frac{t}{\beta^2} - \frac{2}{\beta^3}\left(1-e^{-\beta t}\right) + \frac{1}{2\beta^3}\left(1-e^{-2\beta t}\right), \\ \sigma_{xu}^2 &= \frac{1}{\beta^2}\left(1-e^{-\beta t}\right) - \frac{1}{2\beta^2}\left(1-e^{-2\beta t}\right), \\ \sigma_{uu}^2 &= \frac{1}{2\beta}\left(1-e^{-2\beta t}\right).\end{aligned}$$

The details of this calculation are presented in Schuss [7].

Notes

1. With a Fourier transform, the method of characteristics can often solve a Fokker–Planck equation in one dimension.
2. Because a Fokker–Planck equation and the equation for a Green's function (see page 318) both have delta function forcing terms, the solution techniques are similar.
3. Not all noise terms are white Gaussian noise (the requirements in equation (68.2) are very stringent). The book by Srinivasan and Vasudevan [8] has descriptions of several approximate techniques for other types of noise.
4. When the coefficient of the noise term (i.e., $g(x)$ in equation (68.5)) is small, then a singular perturbation problem generally results.
5. The solution of equation (68.1) is a Markov process; the density of its probability transition function is given by the solution to the Fokker–Planck equation and its initial conditions.

6. Another name for the Fokker–Planck equation is the forward Kolmogorov equation.
7. The solution of the Fokker–Planck equation in equation (68.3) (and its initial conditions in equation (68.4)) might be better represented by $P(t, \mathbf{x}; t_0, \mathbf{y})$. The function $P(t, \mathbf{x}; t_0, \mathbf{y})$ also satisfies the backward Kolmogorov equation, which is the adjoint of equation (68.3). This equation

$$\begin{aligned} \frac{\partial P}{\partial t_0} &= -\sum_{i=1}^{m} b_i \frac{\partial P}{\partial y_i} - \frac{1}{2} \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 P}{\partial y_i \partial y_j}, \\ P(t_0, \mathbf{x}; t_0, \mathbf{y}) &= \delta(\mathbf{x} - \mathbf{y}), \end{aligned} \tag{68.11}$$

has as its independent variables the "backward variables" $\{t_0, \mathbf{y}\}$.
8. When only moments of the probability density $P(t, \mathbf{x})$ are required, the method of moments (see page 568) may sometimes be used to calculate these moments without having to solve the Fokker–Planck equation.
9. Another equivalent form of equation (68.1.a) that often appears is

$$d\mathbf{x}(t) = \mathbf{b}(t, \mathbf{x})\, dt + \sigma(t, \mathbf{x})\, d\mathbf{w}(t), \tag{68.12}$$

where $\mathbf{w}(t)$ is a vector of independent standard Wiener processes (see page 91).
10. Consider a particle starting at $\mathbf{y}$ and randomly moving in a domain Ω. If the probability density of the location evolves according to

$$\frac{\partial P}{\partial t} = L[P] = -\sum_{i=1}^{m} b_i(\mathbf{y}) \frac{\partial P}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{m} a_{ij}(\mathbf{y}) \frac{\partial^2 P}{\partial y_i \partial y_j}, \tag{68.13}$$

- Then the expectation of the exit time $w(\mathbf{y})$ is the solution of $L[w] = -1$ in Ω, with $w = 0$ on $\partial\Omega$.
- Then the probability $u(\mathbf{y})$ that the exit occurs on the boundary segment Γ is the solution of $L[u] = 0$ in Ω with $u(\mathbf{y}) = \begin{cases} 1 & \text{for } \mathbf{y} \in \Gamma \\ 0 & \text{for } \mathbf{y} \in \Omega/\Gamma \end{cases}.$

References

[1] Chang, J. S., and Cooper, G. A practical difference scheme for Fokker–Planck equations. *J. Comput. Physics 6* (1970), 1–16.

[2] Dita, P. The Fokker–Planck equation with absorbing boundary. *J. Phys. A: Math. Gen. 18* (1985), 2685–2690.

[3] Gardiner, C. W. *Handbook of Stochastic Methods*. Springer–Verlag, New York, 1985.

[4] Garrido, L., and Masoliver, J. On a class of exact solutions to the Fokker–Planck equations. *J. Math. Physics 23*, 6 (June 1982), 1155–1158.

[5] Harrison, G. W. Numerical solution of the Fokker Planck equation using moving finite elements. *Num. Meth. Part. Diff. Eqs. 4* (1988), 219–232.

[6] Risken, H. *The Fokker–Planck Equation.* Springer–Verlag, New York, 1984.

[7] Schuss, Z. *Theory and Applications of Stochastic Differential Equations.* John Wiley & Sons, New York, 1980.

[8] Srinivasan, S. K., and Vasudevan, R. *Introduction to Random Differential Equations and Their Applications.* American Elsevier Publishing Company, New York, 1971.

69. Fractional Differential Equations*

Applicable to Fractional differential equations.

Yields

An exact solution.

Idea

There are two common ways to solve fractional differential equations; using an integral transform or transforming to an ordinary differential equation.

Procedure

There are two main methods for solving fractional differential equations

- Transformation to an ordinary differential equation
- Using the Laplace transform

To transform to an ordinary differential equation, care must be taken because the ordinary chain rule from calculus does not apply to fractional derivatives.

Example 1

This example will convert a fractional differential equation into an ordinary differential equation. Suppose we wish to solve the fractional differential equation

$$\frac{d^{1/2}f}{dx^{1/2}} + f = 0 \tag{69.1}$$

for $f(x)$. To convert this to an ordinary differential equation, we will differentiate with respect to x one-half time. This will produce a new differential equation that involves $\frac{d^{1/2}f}{dx^{1/2}}$. Eliminating this term between the new equation and equation (69.1), we will have determined an ordinary differential equation.

To differentiate equation (69.1) with respect to x one-half time, we have to use the differentiation rule (from Oldham and Spanier [3, page 155])

$$\frac{d^{1-Q}}{dx^{1-Q}}\frac{d^Q}{dx^Q}f = \frac{df}{dx} + C_1x^{Q-2} + C_2x^{Q-3} + \cdots + C_mx^{Q-m-1},$$

where $0 < Q \leq m < Q+1$, m is an integer and the $\{C_i\}$ are arbitrary constants. Hence, differentiating equation (69.1) one-half time results in

$$\frac{df}{dx} - C_1x^{-3/2} + \frac{d^{1/2}f}{dx^{1/2}} = 0. \tag{69.2}$$

Eliminating the $d^{1/2}/dx^{1/2}$ term between equations (69.1) and (69.2) results in

$$\frac{df}{dx} - f = C_1 x^{-3/2}, \tag{69.3}$$

which is an ordinary differential equation for $f(x)$. Equation (69.3) has the solution (obtained by use of integrating factors)

$$f(x) = De^x - 2C_1 \left[\sqrt{\pi} e^x \operatorname{erf}(\sqrt{x}) + \frac{1}{\sqrt{x}}\right], \tag{69.4}$$

where D is another arbitrary constant. If we now utilize equation (69.4) in equation (69.1), it turns out that D and C_1 are related by $D = 2C_1\sqrt{\pi}$. This is because of the identities

$$\frac{d^{1/2}}{dx^{1/2}} e^x \operatorname{erf}(\sqrt{x}) = e^x, \qquad \frac{d^{1/2}}{dx^{1/2}} \frac{1}{\sqrt{x}} = 0,$$
$$\frac{d^{1/2}}{dx^{1/2}} e^x = \frac{1}{\sqrt{\pi x}} + e^x \operatorname{erf}(\sqrt{x}),$$

from Oldham and Spanier [3, pages 119 and 123]. Therefore, the solution of equation (69.1) is

$$f(x) = D\left[e^x \operatorname{erfc}(\sqrt{x}) - \frac{1}{\sqrt{\pi x}}\right].$$

Example 2

This example will solve a fractional differential equation by use of Laplace transforms. Suppose we wish to solve the fractional differential equation

$$\frac{df}{dx} + \frac{d^{1/2} f}{dx^{1/2}} - 2f = 0. \tag{69.5}$$

The Laplace transform of equation (69.5) is

$$sF(s) - f(0) + \sqrt{s}F(s) - \frac{d^{-1/2} f(0)}{dx^{-1/2}} - 2F(s) = 0, \tag{69.6}$$

where $F(s)$ is defined to be the Laplace transform of $f(x)$; that is, $F(s) = \int_0^\infty f(x) e^{-xs}\, ds$. If we define the constant C by $C = f(0) + d^{-1/2} f(0)/dx^{-1/2}$, then the solution to equation (69.6) is given by

$$F(s) = \frac{C}{(\sqrt{s}-1)(\sqrt{s}+2)} = \frac{C}{3(\sqrt{s}-1)} - \frac{C}{3(\sqrt{s}+2)}, \tag{69.7}$$

and so the final solution to equation (69.5) can be obtained by finding the inverse Laplace transform to equation (69.7), which is

$$f(x) = \frac{C}{3}\left[2e^{4x} \operatorname{erfc}(2\sqrt{x}) + e^x \operatorname{erfc}(-\sqrt{x})\right].$$

Notes

1. Fractional differential equations are also called *extraordinary differential equations*.
2. One of many equivalent definitions for fractional derivatives is the following

$$\frac{d^q}{dx^q} f(x) = \frac{d^n}{dx^n} \left[\frac{1}{\Gamma(n-q)} \int_a^x \frac{f(y)}{(x-y)^{q-n+1}}\, dy \right],$$

   for $n > q \geq 0$.
3. Certain diffusion problems can be reduced to the solution of a semi-differential equation (one in which all the derivatives are either to an integer order or a half integer order). See Oldham and Spanier [3, Chapter 11] for details.
4. A third technique for solving fractional differential equations is by the use of power series (see page 403). For fractional differential equations, a series of the form

$$f(x) = x^p \sum_{k=0}^{\infty} a_k x^{k/n}$$

   is used, where $p > -1$, n is an integer, $a_0 \neq 0$, and the $\{a_i\}$ are unknowns.
5. Erdélyi's paper [1] contains several boundary value problems for ordinary differential equations that are solved by using fractional differential techniques.

References

[1] Erdelyi, A. Axially symmetric potentials and fractional integration. *J. Soc. Indust. Appl. Math. 13*, 1 (March 1965), 216–228.

[2] Nishimoto, K. Applications to the solutions of linear second order differential equations of Fuchs type. In *Fractional Calculus*, A. C. McBride and G. F. Roach, Eds. Pitman Publishing Co., Marshfield, MA, 1985, pp. 140–153.

[3] Oldham, K. B., and Spanier, J. *The Fractional Calculus.* Academic Press, New York, 1974.

[4] Ross, B. *Fractional Calculus and Its Applications (Proceedings of the International Conference at the University of New Haven, June 1974).* No. 457 in Lecture Notes in Mathematics. Springer–Verlag, New York, 1975.

[5] Wyss, W. The fractional diffusion equation. *J. Math. Physics 27*, 11 (1986), 2782–2785.

70. Free Boundary Problems*

Applicable to Systems of differential equations in which the location of the boundary of the domain is one of the unknowns to be determined.

Idea

Sometimes a similarity solution may be used to determine the location of the free boundary. In more difficult problems, a numerical technique may be required.

Procedure

In free boundary problems, a differential equation must be solved in a domain whose size can vary. One of the unknowns to be determined is the size of the domain on which the equation is to be satisfied.

Differential equations of this type are most often solved numerically. In rare cases, an analytical solution may be obtained; these solutions are generally found by use of similarity methods (see page 497).

Example

Consider a mass of water in $x \geq 0$ at time $t = 0$. Initially, the water has the constant temperature $T_H > 0$. If a constant temperature $T_C < 0$ is maintained at the surface $x = 0$, then the boundary of freezing, $x = s(t)$, will move into the fluid. The unknowns to solve for in this problem are the temperature of the water $w(x,t)$, the temperature of the ice $u(x,t)$, and the location of the unknown boundary, $x = s(t)$. See figure 70.1.

The equations that describe the unknowns are

$$\begin{aligned} u_t &= u_{xx}, \qquad \text{for} \quad 0 < x < s(t), \quad t \geq 0, \\ w_t &= w_{xx}, \qquad \text{for} \quad s(t) < x < \infty, \quad t \geq 0, \\ u(0,t) &= T_C, \\ w(x,0) &= T_H, \\ u(s(t),t) &= 0, \\ w(s(t),t) &= 0, \\ u_x(s(t),t) - w_x(s(t),t) &= \lambda s'(t). \end{aligned} \tag{70.1.a-g}$$

Here we have defined the freezing boundary to be the curve along which the temperature is zero, and equation (70.1.g) represents the transfer of latent heat necessary to create the ice. The parameter λ is the latent heat of fusion times the density divided by the coefficient of heat conduction.

Now, we propose the similarity solution. Because diffusion equations often have time scaling as the square of a distance, we assume that a

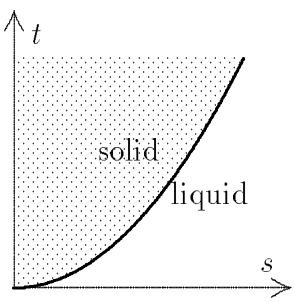

Figure 70.1: This diagram illustrates the location of the freezing boundary for the system given in equation (70.1).

solution to equation (70.1) can be found with

$$u(x,t) = f(\eta) = f\left(\frac{x}{\sqrt{t}}\right), \qquad w(x,t) = g(\eta) = g\left(\frac{x}{\sqrt{t}}\right), \tag{70.2}$$

for some unknown functions $f(\eta)$ and $g(\eta)$. Using these proposed forms in equation (70.1.g) shows that these forms are possible only if the freezing boundary is given by

$$s(t) = \alpha\sqrt{t}, \tag{70.3}$$

for some value of α. Using equations (70.2) and (70.3) in equation (70.1), we find the equivalent system

$$\begin{gathered}
f''(\eta) + \frac{1}{2}\eta f'(\eta) = 0, \qquad \text{for } 0 < \eta < \alpha, \\
g''(\eta) + \frac{1}{2}\eta g'(\eta) = 0, \qquad \text{for } \alpha < \eta < \infty, \\
f(0) = T_C, \quad f(\alpha) = 0, \\
g(\infty) = T_H, \quad g(\alpha) = 0, \\
f'(\alpha) - g'(\alpha) = \frac{\lambda\alpha}{2}.
\end{gathered} \tag{70.4}$$

The ordinary differential equations in equation (70.4) may be solved to determine that

$$\begin{aligned}
f(\eta) &= T_C - T_H \frac{\operatorname{erf}(\eta/2)}{\operatorname{erf}(\alpha/2)}, \\
g(\eta) &= \frac{T_H}{\operatorname{erfc}(\alpha/2)} \left[\operatorname{erf}(\eta/2) - \operatorname{erf}(\alpha/2)\right],
\end{aligned} \tag{70.5}$$

where α satisfies the transcendental equation

$$\frac{T_H}{\operatorname{erf}(\alpha/2)} + \frac{T_C}{\operatorname{erfc}(\alpha/2)} = -\lambda\alpha\frac{\sqrt{\pi}}{2}e^{\alpha^2/4}.$$

Notes

1. In writing equation (70.1.a) and equation (70.1.b), we have assumed that the thermophysical parameters in both the ice and the water are the same (i.e., the Stefan number, which is a ratio of these parameters, is equal to one). In reality, these parameters are different and a constant that cannot be scaled out must be introduced into either equation (70.1.a) or equation (70.1.b).
2. The example illustrated above is described in more detail in Crank [2, Chapter 3].
3. Melting problems for a pure material are also known as *Stefan problems*.
4. Another technique often used in free boundary problems is changing coordinates so that the free boundaries become fixed in the new coordinate space. This is the idea behind the hodograph method (see page 456).
5. Free boundary problems often arise in hydrodynamics, when the flow over an airfoil is being computed. When the flow becomes supersonic, the type of governing equation changes from hyperbolic to elliptic and a different type of numerical scheme is required. Where the equation changes type is not known *a priori*.
6. Some of the popular numerical techniques for solving free boundary problems go by the name of *front tracking methods* or *front fixing methods*. These techniques generally require that the location of the free boundary be approximately known before the computer code is run. A better approach is to use *enthalpy methods*. These methods do not need initial information about the interfaces, and multiple fronts can also occur.
7. The paper by Hill and Dewynne [6] discusses several different approximation techniques applied to a single physical problem involving a free boundary.

References

[1] Charrier, P., and Tessieras, B. On front-tracking methods applied to hyperbolic systems of nonlinear conservation laws. *SIAM J. Numer. Anal. 23*, 3 (June 1986), 461–472.

[2] Crank, J. *Free and Moving Boundary Problems.* Clarendon Press, Oxford, England, 1984.

[3] Duncan, D. B. A simple and effective self-adaptive moving mesh for enthalpy formulations of phase change problems. *IMA J. Num. Analysis 11* (1991), 55–78.

[4] Elliot, C. M., and Ockendon, J. R. *Weak and Variational Methods for Moving Boundary Problems.* Pitman Publishing Co., Marshfield, MA, 1982.

[5] Furzeland, R. M. A comparative study of numerical methods for moving boundary problems. *J. Inst. Maths. Applics 26* (1980), 411–429.

[6] Hill, J. M., and Dewynne, J. N. On the inward solidification of cylinders. *Quart. Appl. Math. 44*, 1 (April 1986), 59–70.

[7] Marshall, G. A front tracking method for one-dimensional moving boundary problems. *SIAM J. Sci. Stat. Comput. 7*, 1 (January 1986), 252–263.

[8] Rubensten, L. I. *The Stefan Problem.* Amer. Math. Soc., Providence, RI, 1971. Translated by A. D. Solomon.

[9] Womble, D. E. A front-tracking method for multiphase free boundary problems. *SIAM J. Numer. Anal. 26*, 2 (April 1989), 380–396.

[10] Wood, A. S. An efficient finite-dimensional scheme for multidimensional stefan problems. *Int. J. Num. Meth. Eng. 23* (1986), 1757–1771.

71. Generating Functions*

Applicable to Systems of differential equations, where each equation has a similar form.

Yields

An exact analytic solution.

Idea

Sometimes a single function can be used to contain the information in several equations.

Procedure

We illustrate the method as it applies to ordinary differential equations. Suppose we have a system of ordinary differential equations for $\{u_k(t)\}$, all of the form

$$\frac{d}{dt}u_N = f(u_{N-m}, \ldots, u_N, \ldots, u_{N+m}, t), \tag{71.1}$$

for $N = 1, 2, \ldots, \infty$ or $N = \pm 1, \pm 2, \ldots, \pm\infty$. We might introduce the ordinary generating function

$$G(s,t) = \sum_k u_k(t)s^k, \tag{71.2}$$

or the exponential generating function

$$H(s,t) = \sum_k u_k(t)\frac{s^k}{k!}. \tag{71.3}$$

Using equation (71.2) (or equations (71.3)) and (71.1), we can sometimes find a partial differential equation for $G(s,t)$ (or $H(s,t)$). After solving the partial differential equation, we can determine the $\{u_k(t)\}$ from either

$$u_k(t) = \frac{1}{k!}\left(\frac{d}{ds}\right)^k G(s,t)\,\Big|_{s=0},$$

or

$$u_k(t) = \left(\frac{d}{ds}\right)^k H(s,t)\,\Big|_{s=0}.$$

After we have solved for the $\{u_k(t)\}$, we must then check that equation (71.2) (or equation (71.3)) converges for the values of t that are of interest.

Example

The classic equations relating to service times are called the *birth and death equations* (see notes). For the special case of "constant death" and "linear birth," these equations have the form

$$\begin{aligned}\frac{d}{dt}P_0(t) &= -\lambda P_0(t) + \mu P_1(t),\\ \frac{d}{dt}P_N(t) &= \lambda P_{N-1}(t) - (\lambda + N\mu)P_N(t) + (N+1)\mu P_{N+1}(t),\end{aligned} \tag{71.4}$$

where μ and λ are constants and $N = 1, 2, \dots, \infty$. The initial conditions for equation (71.4) are

$$P_N(0) = \delta_{Nj}, \tag{71.5}$$

where δ_{Nj} is the Kronecker delta and j is a given positive integer. The ordinary generating function is defined in this case by

$$G(t,s) = \sum_{k=0}^{\infty} P_k(t)s^k. \tag{71.6}$$

Differentiating $G(t,s)$ with respect to t leads to

$$\begin{aligned}\frac{\partial G}{\partial t} &= \sum_{k=0}^{\infty}\left[\frac{d}{dt}P_k(t)\right]s^k\\ &= \left[-\lambda P_0(t) + \mu P_1(t)\right]s^0\\ &\quad + \sum_{k=1}^{\infty}\left(\lambda P_{k-1}(t) - (\lambda + k\mu)P_k(t) + (k+1)\mu P_{k+1}(t)\right)s^k\\ &= \lambda(s-1)\left[P_0 + P_1 s + P_2 s^2 + \cdots\right]\\ &\quad + \mu(1-s)\left[P_1 + 2P_2 s + 3P_3 s^2 + \cdots\right]\\ &= (1-s)\left[-\lambda G + \mu\frac{\partial G}{\partial s}\right].\end{aligned} \tag{71.7}$$

The initial condition for $G(t,s)$, from equations (71.5) and (71.6), becomes

$$G(0,s) = s^j. \tag{71.8}$$

The partial differential equation in (71.7), with the initial condition in (71.8), can be solved by the method of characteristics (see page 432). The solution is

$$G(t,s) = e^{-\lambda(1-s)\left(1-e^{-\mu t}\right)/\mu}\left[1 - (1-s)e^{-\mu t}\right]^j. \tag{71.9}$$

Taking a Taylor series of equation (71.9) with respect to s (see equation (71.6)) allows all of the $\{P_k(t)\}$ to be found. For example

$$\begin{aligned}
P_0(t) &= e^{-\lambda(1-y)/\mu}(1-y)^t, \\
P_1(t) &= e^{-\lambda(1-y)/\mu}\frac{(1-y)^{t-1}}{\mu}\left[\lambda y^2 + (t\mu - 2\lambda)\,y + \lambda\right], \\
P_2(t) &= e^{-\lambda(1-y)/\mu}\frac{(1-y)^{t-2}}{\mu^2}\Big\{\lambda^2 y^4 + (2t\lambda\mu - 4\lambda^2)y^3 \\
&\qquad + [t(t-1)\mu^2 - 2\lambda(2t\mu - 3\lambda)]y^2 + (2t\lambda\mu - 4\lambda^2)y + \lambda^2\Big\},
\end{aligned} \tag{71.10}$$

where $y = e^{-\mu t}$.

Notes

1. In the birth and death equations (see Karlin and Taylor [2, page 135]), $P_k(t)$ is the probability of k unfinished jobs at time t. We also assume: Initially there are M jobs to be finished, the average service time is λ, and the average number of new jobs spawned by an existing job is μ per unit time.
2. For the example given above, Laplace transforms (see page 350) could also have been used to solve equation (71.7) with equation (71.8).
3. Nonlinear systems of differential equations can also be solved by this method. A classic application is to equations describing the aggregation of particles (see Feller [1, Chapter 17, pages 444–482]).
4. See Taylor and Karlin [4, pages 310–316 and 337–338].

References

[1] Feller, W. *An Introduction to Probability Theory and Its Applications.* John Wiley & Sons, New York, 1968.

[2] Karlin, S., and Taylor, H. M. *A First Course in Stochastic Processes.* Academic Press, New York, 1975.

[3] Letessier, J. The numerical resolution of birth and death Kolmogorov equations. *Comp. & Maths. with Appls. 13*, 7 (1987), 595–600.

[4] Taylor, H. M., and Karlin, S. *An Introduction to Stochastic Modeling.* Academic Press, New York, 1984.

72. Green's Functions*

Applicable to Linear differential equations with linear boundary conditions and initial conditions.

Yields

An exact solution, in the form of an integral or an infinite series.

Idea

Initially, the solution of the linear differential equation with a "point source" is determined. Then, using superposition, the "forcing function" (appearing in either the differential equation or the boundary condition) is treated as a collection of point sources.

Procedure

Suppose we have the following linear differential equation for $u(\mathbf{x})$

$$L[u] = f(\mathbf{x}), \tag{72.1}$$

with the linear homogeneous boundary conditions

$$B_i[u] = 0, \tag{72.2}$$

for $i = 1, 2, \ldots, n$. Suppose we can solve for $G(\mathbf{x}; \mathbf{z})$, where $G(\mathbf{x}; \mathbf{z})$ satisfies

$$\begin{aligned} L[G(\mathbf{x}; \mathbf{z})] &= \delta(\mathbf{x} - \mathbf{z}), \\ B_i[G(\mathbf{x}; \mathbf{z})] &= 0 \end{aligned}$$

and $\delta(\mathbf{x})$ is the usual delta function. Then the solution to equations (72.1) and (72.2) can be written as

$$u(\mathbf{x}) = \int G(\mathbf{x}; \mathbf{z}) f(\mathbf{z})\, d\mathbf{z}, \tag{72.3}$$

integrated over some appropriate region.

Conversely, suppose we want to solve the linear homogeneous differential equation

$$\begin{aligned} L[v] &= 0, \\ B[v] &= h(\mathbf{x}). \end{aligned} \tag{72.4}$$

If we can solve

$$\begin{aligned} L[g(\mathbf{x}; \mathbf{z})] &= 0, \\ B[g(\mathbf{x}; \mathbf{z})] &= \delta(\mathbf{x} - \mathbf{z}), \end{aligned}$$

for $g(\mathbf{x};\mathbf{z})$, then the solution to equation (72.4) is given by

$$v(\mathbf{x}) = \int g(\mathbf{x};\mathbf{z}) h(\mathbf{z})\, d\mathbf{z}.$$

Both $G(\mathbf{x};\mathbf{z})$ and $g(\mathbf{x};\mathbf{z})$ are called Green's functions. The functions $f(\mathbf{x})$ and $h(\mathbf{x})$ are often referred to as "forcing functions." If, for example, $f(\mathbf{x}) \equiv 0$, then by equation (72.3) $u(\mathbf{x}) \equiv 0$.

Green's functions can be calculated once, then used repeatedly for different functions $f(\mathbf{x})$ and $h(\mathbf{x})$. Some Green's functions are tabulated in table 72.1. To calculate the Green's function $G(\mathbf{x};\mathbf{z})$, we require:

(a) $L[G(\mathbf{x};\mathbf{z})] = 0,$ except at $\mathbf{x} = \mathbf{z}$.

(b) $B_i[G(\mathbf{x};\mathbf{z})] = 0.$ (72.5)

(c) If $L[\cdot]$ is an nth order ordinary differential equation, then $G(\mathbf{x};\mathbf{z})$ must be continuous (with its derivatives up to order $n-1$) at $\mathbf{x} = \mathbf{z}$.

(d) $\displaystyle\int_{\mathbf{z}^-}^{\mathbf{z}^+} L[G(\mathbf{x};\mathbf{z})]\, d\mathbf{x} = 1.$

The conditions on $g(\mathbf{x};\mathbf{z})$ are very similar:

(a) $L[g(\mathbf{x};\mathbf{z})] = 0.$ (72.6)

(b) $B[g(\mathbf{x};\mathbf{z})] = 0.$ except at $\mathbf{x} = \mathbf{z}$,

(c) If $L[\cdot]$ is an nth order ordinary differential equation, then $g(\mathbf{x};\mathbf{z})$ must be continuous (with its derivatives up to order $n-1$) at $\mathbf{x} = \mathbf{z}$.

(d) $\displaystyle\int_{\mathbf{z}^-}^{\mathbf{z}^+} B[g(\mathbf{x};\mathbf{z})]\, d\mathbf{x} = 1.$

Conditions (72.5.a,d) and (72.6.b,d) follow from the definition of the delta function. Conditions (72.5.c) and (72.6.c) follow from the definition of what a solution to an nth order differential equation means; and conditions (72.5.b) and (72.6.c) follow from the defining equations for $G(\mathbf{x};\mathbf{z})$ and $g(\mathbf{x};\mathbf{z})$.

Many methods can be used to construct a $G(\mathbf{x};\mathbf{z})$ or a $g(\mathbf{x};\mathbf{z})$ that satisfies the above four requirements. We will illustrate two methods for constructing $G(\mathbf{x};\mathbf{z})$ for the special case of a second order linear ordinary differential equation. Then we illustrate the construction process for $g(\mathbf{x};\mathbf{z})$ for a partial differential equation.

In the following, $\mathbf{r} = (x, y, z)$, $\mathbf{r}_0 = (x_0, y_0, z_0)$, $R = |\mathbf{r} - \mathbf{r}_0|$, $P^2 = (x - x_0)^2 + (y - y_0)^2$, and $H(\cdot)$ is the Heaviside function

- For the potential equation $\nabla^2 G + k^2 G = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)$, with the radiation condition (outgoing waves only), the solution is

$$G = \begin{cases} \frac{2\pi i}{k} e^{ik|x-x_0|} & \text{in one dimension,} \\ i\pi H_0^{(1)}(kP) & \text{in two dimensions,} \\ \frac{e^{ikR}}{R} & \text{in three dimensions,} \end{cases}$$

  where $H_0^{(1)}(\cdot)$ is a Hankel function (also called a Bessel function of the third kind).
- For the diffusion equation $\nabla^2 G - a^2 \frac{\partial G}{\partial t} = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)$, with the initial condition $G = 0$ for $t < t_0$, and the boundary condition $G = 0$ at $r = \infty$ in N dimensions, the solution is

$$G = \frac{4\pi}{a^2} \left(\frac{a}{2\sqrt{\pi(t - t_0)}} \right)^N \exp\left(-\frac{a^2 ||\mathbf{r} - \mathbf{r}_0||^2}{4(t - t_0)} \right).$$

- For the wave equation $\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)$, with the initial conditions $G = G_t = 0$ for $t < t_0$, and the boundary condition $G = 0$ at $r = \infty$ the solution is

$$G = \begin{cases} 2c\pi H\left[(t - t_0) - \frac{|x-x_0|}{c}\right] & \text{for one space dimension,} \\ \frac{2c}{\sqrt{c^2(t-t_0)^2 - P^2}} H\left[(t - t_0) - \frac{P}{c}\right] & \text{for two space dimensions,} \\ \frac{1}{R}\delta\left[\frac{R}{c} - (t - t_0)\right] & \text{for three space dimensions.} \end{cases}$$

Table 72.1: Green's functions for common partial differential equations.

Special Case 1

Define the general linear second order ordinary differential equation with linear homogeneous boundary conditions by

$$\begin{aligned} L[u] &:= \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) - s(x)u, \\ B_1[u] &:= \alpha_1 u(a) + \alpha_2 u'(a) = 0, \\ B_2[u] &:= \beta_1 u(a) + \beta_2 u'(b) = 0, \end{aligned}$$

and suppose that we wish to solve $L[u] = f(x)$. If $y_1(x)$ and $y_2(x)$ are non-trivial (i.e., not identically equal to zero) and satisfy

$$\begin{aligned} L[y_1] &= 0, \qquad B_1[y_1] = 0, \\ L[y_2] &= 0, \qquad B_2[y_2] = 0, \end{aligned}$$

then we can write $G(x; z)$ as

$$G(x; z) = \begin{cases} \frac{y_1(x)y_2(z)}{p(z)W(z)} & \text{for } a \le x \le z, \\ \frac{y_2(x)y_1(z)}{p(z)W(z)} & \text{for } z \le x \le b, \end{cases}$$

where $W(z) = \begin{vmatrix} y_1(z) & y_2(z) \\ {y_1}'(z) & {y_2}'(z) \end{vmatrix}$ is the Wronskian of $y_1(x)$ and $y_2(x)$ at the point $x = z$.

Special Case 2

Suppose that $L[\cdot]$ is a self-adjoint operator, so that it has a complete set of orthogonal eigenfunctions (see page 103). Suppose further that we know the eigenvalues $\{\lambda_n\}$ and the eigenfunctions $\{\phi_n\}$ for $\{L, B_1, B_2\}$. That is,

$$\begin{aligned} L[\phi_n] &= \lambda_n \phi_n, \\ B_1[\phi_n] &= 0, \\ B_2[\phi_n] &= 0, \end{aligned}$$

then $G(x; z)$ is found to be

$$G(x; z) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(z)}{\lambda_n \int \phi_n^2(x)\,dx}.$$

Example 1

Suppose we wish to solve

$$\begin{aligned} y'' &= f(x), \\ y(0) &= 0, \quad y(L) = 0. \end{aligned} \tag{72.7}$$

Using the first method, we require the solutions $y_1(x)$ and $y_2(x)$ of

$$\begin{aligned} y_1'' &= 0, \qquad y_1(0) = 0, \\ y_2'' &= 0, \qquad y_2(L) = 0. \end{aligned}$$

The solutions to these equations are

$$y_1(x) = Ax, \qquad y_2(x) = B(x - L),$$

where A and B are arbitrary constants. We compute the Wronskian to be $W(z) = ABL$. Therefore,

$$G(x;z) = \begin{cases} \frac{x(z-L)}{L} & \text{for } 0 \le x \le z, \\ \frac{z(x-L)}{L} & \text{for } z \le x \le L. \end{cases} \tag{72.8}$$

Using the second method, we find the eigenvalues and eigenfunctions to be

$$\lambda_n = \frac{n\pi}{L}, \qquad \phi_n(x) = \sin \lambda_n x = \sin\left(\frac{n\pi x}{L}\right),$$

so that

$$G(x;z) = \frac{2L}{n\pi} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi z}{L}\right). \tag{72.9}$$

Using either of equations (72.8) or (72.9) for $G(x;z)$, the solution to equation (72.7) can be written as

$$y(x) = \int_0^L G(x;z)\; f(z)\, dz. \tag{72.10}$$

For example, using equation (72.8) in equation (72.10), the solution to (72.7) can be written as

$$y(x) = \int_x^L \frac{x(z-L)}{L} f(z)\, dz + \int_0^x \frac{z(x-L)}{L} f(z)\, dz. \tag{72.11}$$

Note the similarity between equation (72.11) and the form of the solution shown in the section on variation of parameters (see page 418).

If, for example, $f(x) = x^3$, then evaluation of equation (72.11) results in

$$y(x) = \frac{x}{20}(x^4 - L^4).$$

The second method yields the same answer. For this example, the second method is equivalent to using finite Fourier series (see page 344).

Example 2

Suppose we are given the parabolic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial u}{\partial t} \tag{72.12}$$

for $u(x,t)$ with the initial and boundary conditions

$$u(x,0) = h(x), \quad u(\pm\infty, t) = 0. \tag{72.13}$$

We choose to write the solution as

$$u(x,t) = \int_{-\infty}^{\infty} g(x,t;z) h(z)\, dz, \tag{72.14}$$

where the Green's function $g(x,t;z)$ satisfies

$$\begin{aligned}&\frac{\partial^2 g}{\partial x^2} = \frac{1}{a^2}\frac{\partial g}{\partial t},\\&g(x,0;z) = \delta(z-x), \quad g(\pm\infty,t;z) = 0.\end{aligned}$$

Taking a Fourier transform (in x) of the equation for $g(x,t;z)$ results in

$$\begin{aligned}&\frac{d\widehat{g}}{dt} = -a^2\omega^2\widehat{g},\\&\widehat{g}(\omega,0;z) = \frac{1}{\sqrt{2\pi}}e^{i\omega z},\end{aligned} \tag{72.15}$$

where $\widehat{g}(\omega,t;z)$ is defined to be the Fourier transform of $g(x,t;z)$; that is,

$$\widehat{g}(\omega,t;z) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x,t;z)e^{i\omega x}\,dx.$$

Solving the ordinary differential equation (72.15) results in

$$\widehat{g}(\omega,t;z) = \frac{1}{\sqrt{2\pi}}e^{i\omega z}e^{-a^2\omega t}.$$

Using the inverse Fourier transform, we then have our solution

$$g(x,t;z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{g}(\omega,t;z)e^{-i\omega x}\,dx.$$

By using the convolution theorem for Fourier transforms, we can determine that

$$g(x,t;z) = \frac{1}{\sqrt{4\pi a^2 t}}e^{-(x-z)^2/4a^2t}.$$

This should be used in equation (72.14) to determine the solution to equations (72.12) and (72.13).

Notes

1. If $\mathbf{z}$ is in a n-dimensional space, then the integrals appearing in equation (72.5.d) and equation (72.6.d) are n single integrals, each one over one of the coordinate axes.
2. Delta functions, in non-rectangular coordinate systems, are easily determined by a change of variables in the defining relation: $\int \delta(\mathbf{z})\,d\mathbf{z} = 1$. In changing variables, the Jacobian of the transformation will then divide the delta function terms. For example
   - In a spherical coordinate system (denoted by the usual coordinates r, θ, and ϕ) the delta function located at the point $\mathbf{x}' = (r',\theta',\phi')$ is given by

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi'),$$

for $r' \neq 0$ and $\theta' \neq 0, \pi$. For a point source at $r = r'$ and $\theta = 0$, the representation $\delta(r - r')\delta(\theta)/2\pi r^2 \sin\theta$ may be used whereas a point source at the origin has the representation $\delta(r)/4\pi r^2$.

- In a cylindrical coordinate system (denoted by the usual coordinates ρ, θ, and z) the delta function located at the point $\mathbf{x}' = (\rho', \theta', z')$ is given by

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{\delta(\rho - \rho')\delta(\theta - \theta')\delta(z - z')}{\rho},$$

for $\rho' > 0$. A point source at the origin has the representation $\frac{\delta(z)\delta(\rho)}{2\pi\rho}$.

3. If $G^*(\mathbf{x}; \mathbf{z})$ satisfies the problem adjoint to $L[\cdot]$ (see page 95), then $G(\mathbf{x}; \mathbf{z}) = G^*(\mathbf{z}; \mathbf{x})$. Therefore, if $L[\cdot]$ and its associated boundary conditions are self-adjoint and $L[G(\mathbf{x}; \mathbf{z})] = \delta(\mathbf{x} - \mathbf{z})$, then $G(\mathbf{x}; \mathbf{z}) = G(\mathbf{z}; \mathbf{x})$. This is called the *reciprocity principle*. It can be observed in our example (see equation (72.9)).
4. When the operator is self-adjoint, the Green's function is sometimes written in terms of the variables $x_<$ and $x_>$ instead of x and z. When this is done, $x_<$ ($x_>$) represents the smaller (larger) of x and z. For example, (72.11) could have been written as $G(x; z) = \frac{x_<(x_> - L)}{L}$.
5. Few analytic solutions of the Helmholtz equation

$$\nabla^2 G + k_0^2 n^2(\mathbf{r}) G = -\delta(\mathbf{r} - \mathbf{r}_0)$$

are known when the *index of refraction*, $n(\mathbf{r})$, is variable. Solutions are known in the following cases:

- (point source) $n = \sqrt{1 + \mathbf{a}^{\mathrm{T}}\mathbf{r} + \mathbf{r}^{\mathrm{T}}B\mathbf{r}}$
- (point source, layered medium) $n = z^{-1}$
- (point source, layered medium) $n = \sqrt{A + Cz + Fz^2}$
- (line source) $n = \sqrt{x}$
- (line source) $n = \sqrt{A + Bx + Cy + Dx^2 + Exy + Fy^2}$

See Li *et al.* [8] for details.

6. As another example, the differential equation with boundary conditions

$$y'' + k^2 y = f(x),$$
$$y(0) = 0, \quad y'(1) = 0$$

has the Green's function $G(x; z) = -\frac{\cos k(1 - x_<) \sin kx_>}{k \cos k}$.

7. Consider the self-adjoint second order operator $L[u] = (p(x)u'(x))' + q(x)u(x)$, and consider the boundary conditions

$$B_1[u] := a_1 u(a) + a_2 u'(a) = 0,$$
$$B_2[u] := b_1 u(b) + b_2 u'(b) = 0. \tag{72.16}$$

Define $\phi(x)$ and $\psi(x)$ to be the solutions to

$$\begin{aligned} L[\phi] &= \lambda r(x)\phi, & B_1[\phi] &= 0, \\ L[\psi] &= \lambda r(x)\psi, & B_2[\psi] &= 0. \end{aligned}$$

Then, the Green's function for the operator $L - \lambda r$, which satisfies the boundary conditions in equation (72.16), is given by $G_\lambda(x; z) = \frac{\phi(x_<)\psi(x_>)}{p(x)W(\phi,\psi)}$, where $W(\phi, \psi)$ represents the Wronskian.

8. There will not exist a Green's function if the solution of the original problem is indeterminate. In this case, a *generalized* Green's function will exist. As an example, consider the system

$$\begin{aligned} y'' &= f(x), \\ y(0) &= y(1), \\ y'(0) &= y'(1). \end{aligned}$$

If $u(x)$ is any solution to the above system, then so is $u(x)+C$ where C is any constant. Because the solution of the original system is indeterminate, an ordinary Green's function cannot be found. See the section on alternative theorems (page 15) or Farlow [5, pages 290–298] for details.

9. Sometimes, in such problems, the specific solution in which the Green's function is symmetric in both $\mathbf{x}$ and $\mathbf{z}$ is chosen. This results in the *modified Green's function.* See Stakgold [10, Chapter 1, pages 215–218] for details.
10. Fokker–Planck equations have delta function initial conditions. The methods used for solving these equations are the same as the methods used for finding Green's functions.
11. Some potential problems can be solved by assuming a continuum of sources. In these cases, the potential outside of the body, which is due to the presence of the body, is represented as the superposition of potentials due to point sources and dipoles lying entirely within the body. See Barshinger [1] for an example.
12. Butkovskiy's book [3] has a comprehensive listing of Green's functions. Any particular Green's function problem is partitioned into one of several separate disjoint groups labeled by a triple of integers: (r, m, n). In this partitioning, r represents the dimension of the spatial domain, m is the order of the highest derivative with respect to t, and n is the order of the highest derivative with respect to the space variables. Over 500 problems are catalogued and solved.
13. See Butkov [2, Chapter 12, pages 503–552] and Zauderer [11, pages 353–449].

References

[1] Barshinger, R. The electrostatic field about a thin oblate dielectric body of revolution. *SIAM J. Appl. Math. 52*, 3 (June 1992), 651–675.

[2] Butkov, E. *Mathematical Physics*. Addison–Wesley Publishing Co., Reading, MA, 1968.

[3] Butkovskiy, A. G. *Green's Functions and Transfer Functions Handbook*. John Wiley & Sons, New York, 1982. Halstead Press.

[4] Courant, R., and Hilbert, D. *Methods of Mathematical Physics*. Interscience Publishers, Inc., New York, 1953.

[5] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[6] Greenberg, M. D. *Application of Green's Functions in Science and Engineering*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1971.

[7] Jordan, K. E., Richter, G. R., and Sheng, P. An efficient numerical evaluation of the Green's function for the Helmholtz operator on periodic structures. *J. Comput. Physics 63*, 1 (1986), 222–235.

[8] Li, Y. L., Liu, C. H., and Franke, S. J. Three-dimensional Green's functions for wave propogation in a linearly inhomogeneous medium—the exact analystical solution. *J. Acoust. Soc. Am. 87*, 6 (June 1990), 2285.

[9] Morse, P. M., and Feshback, H. *Methods of Theoretical Physics*. McGraw–Hill Book Company, New York, 1953.

[10] Stakgold, I. *Green's Functions and Boundary Value Problems*. John Wiley & Sons, New York, 1979.

[11] Zauderer, E. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1983.

73. Homogeneous Equations

Applicable to First order ordinary differential equations of a certain form.

Yields

An exact solution.

Idea

If $P(x, y)$ and $Q(x, y)$ are homogeneous functions of x and y of the same degree, then, by the change of variable $y = vx$, the differential equation $y' = P(x, y)/Q(x, y)$ can be made separable.

Procedure

A function $H(x, y)$ is called homogeneous of degree n if $H(tx, ty) = t^n H(x, y)$. In particular, a polynomial, $P(x, y)$, of two variables is said to be homogeneous of degree n if every term of $P(x, y)$ is of the form $x^j y^{n-j}$ for $j = 0, 1, \dots, n$. A homogeneous function of degree n can be written as $H(x, y) = x^n H(1, y/x)$. Therefore, given an ordinary differential equation of the form

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}, \tag{73.1}$$

where $P(x, y)$ and $Q(x, y)$ are both homogeneous polynomials of degree n, we change variables by $y = vx$ to obtain

$$x\frac{dv}{dx} + v = \frac{P(1, v)}{Q(1, v)}.$$

Because this is a separable equation, it can be integrated to yield (see page 401)

$$\int \frac{dv}{\frac{P(1,v)}{Q(1,v)} - v} = \log x + C,$$

where C is an arbitrary constant.

Example

Suppose we have the ordinary differential equation

$$\frac{dy}{dx} = \frac{2x^3 y - y^4}{x^4 - 2xy^3}. \tag{73.2}$$

Because both the numerator and denominator of the right-hand side of equation (73.2) are homogeneous polynomials of degree four, we set $y = vx$ to obtain

$$x\frac{dv}{dx} + v = \frac{2v - v^4}{1 - 2v^3}$$

or

$$x\frac{dv}{dx} = \frac{v+v^4}{1-2v^3}.$$

This last equation is separable, and the solution is given by

$$\begin{aligned}\int^x \frac{dx}{x} &= \int^v \frac{1-2v^3}{v+v^4}\,dv,\\ \log x &= \int^v \left(\frac{1}{v} - \frac{3v^2}{1+v^3}\right) dv \\ &= \log v - \log(1+v^3) + \log C\end{aligned} \tag{73.3}$$

or $x(1+v^3) = Cv$, where C is an arbitrary constant. Substituting $v = y/x$ in this yields the final solution $x^3 + y^3 = Cxy$.

Notes

1. Equation (73.1) may be made exact (see page 284) by multiplying by the integrating factor $1/(Px - Qy)$.
2. This method is derivable from Lie group methods (see page 366).
3. This method is contained in the method for scale invariant equations (see page 398).
4. Beware that the expression “homogeneous equation” has two entirely different meanings; see the definitions (page 6).
5. It may be simpler to think of homogeneous equations as ordinary differential equations of the form $dy/dx = f\,(y/x)$. This is equivalent to equation (73.1).
6. The equation

$$\frac{dy}{dx} = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \tag{73.4}$$

can always be made homogeneous or separable.

- If $a_1b_2 \neq a_2b_1$, then the change of variables

$$\begin{aligned}x &= X + h,\\ y &= Y + k,\end{aligned}$$

changes equation (73.4) into the homogeneous equation

$$\frac{dY}{dX} = f\left(\frac{a_1X + b_1Y}{a_2X + b_2Y}\right),$$

when h and k satisfy the equations: $\left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} -c_1 \\ -c_2 \end{smallmatrix}\right]$.

- If $a_1b_2 = a_2b_1$, then the change of variables $Y = x + \frac{b_1}{a_1}y = x + \frac{b_2}{a_2}y$ results in the equation

$$\frac{dY}{dx} = 1 + \frac{b_1}{a_1} f\left(\frac{a_1Y + c_1}{a_2Y + c_2}\right).$$

7. See Boyce and DiPrima [1, pages 87–91], Ford [2, pages 40–45], Goldstein and Braun [3, pages 81–84], Ince [4, pages 18–20], and Simmons [5, pages 35–37].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Ford, L. R. *Differential Equations*. McGraw–Hill Book Company, New York, 1955.

[3] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations*. U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[4] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[5] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

74. Method of Images*

Applicable to Differential equations with homogeneous boundary conditions and sources present.

Yields

An exact solution.

Idea

If we know the solution to a free space problem, then we can often use superposition to find a solution in a finite domain with homogeneous boundary conditions.

Procedure

Given a problem with a point source present, solve the free space problem (i.e., disregarding the boundary conditions). By superposition, determine the solution when there are sources at different points of different strengths. Choose the position and strengths of these sources so as to obtain the desired boundary conditions.

The added sources cannot appear in the physical domain of the problem. Symmetry considerations tend to simplify the process of determining where the sources should go.

Example 1

Suppose we wish to find the potential, $\phi(\mathbf{x})$, outside of a grounded sphere of radius R, when there is a point source at position $\mathbf{y}$ (with $||\mathbf{y}|| = \lambda > R$). The equations that represent this problem are

$$\begin{gathered} \nabla^2\phi = \delta(\mathbf{x}-\mathbf{y}), \\ \phi\,\Big|_{||\mathbf{x}||=R} = 0, \qquad \phi\,\Big|_{||\mathbf{x}||=\infty} = 0, \end{gathered} \tag{74.1.a-c}$$

in the region $R < ||\mathbf{x}|| < \infty$ (see figure 74.1). If the boundary condition at $||\mathbf{x}|| = R$ is ignored, then the problem

$$\begin{gathered} \nabla^2\Psi = \delta(\mathbf{x}-\mathbf{y}), \\ \Psi\,\Big|_{||\mathbf{x}||=\infty} = 0, \end{gathered}$$

has the solution (using Green's functions, see table 72.1)

$$\Psi = -\frac{1}{4\pi||\mathbf{x}-\mathbf{y}||}. \tag{74.2}$$

If we place an additional source of strength S at the point $\mathbf{z}$ and solve

$$\begin{gathered} \nabla^2\Phi = \delta(\mathbf{x}-\mathbf{y}) + S\delta(\mathbf{x}-\mathbf{z}), \\ \Phi\,\Big|_{||\mathbf{x}||=\infty} = 0, \end{gathered} \tag{74.3}$$

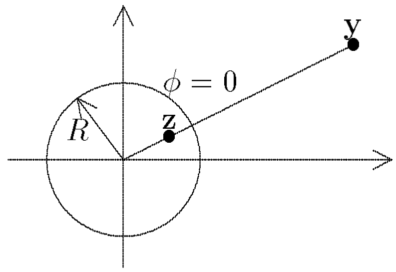

Figure 74.1: Equation (74.1) represents the potential outside of a grounded sphere of radius R, with a source point present.

then we obtain (using equation (74.2) and superposition)

$$\Phi = -\frac{1}{4\pi||\mathbf{x}-\mathbf{y}||} - \frac{S}{4\pi||\mathbf{x}-\mathbf{z}||}. \tag{74.4}$$

Note that the point $\mathbf{z}$ cannot be in the region $R < ||\mathbf{x}|| < \infty$, because then equation (74.3) (whose solution we want to be the solution to equation (74.1)) will not satisfy equation (74.1.a).

To determine the strength and location of the additional source (S and $\mathbf{z}$), we calculate the potential at $\mathbf{x} = \mathbf{p}$, where $||\mathbf{p}|| = R$ (i.e., on the surface of the sphere). We find

$$\Phi\Big|_{\mathbf{x}=\mathbf{p}} = -\frac{1}{4\pi}\left[\frac{1}{||\mathbf{p}-\mathbf{y}||} + \frac{S}{||\mathbf{p}-\mathbf{z}||}\right].$$

For this to be zero (and so $\Phi = \phi$), we require (after some vector algebra)

$$S = -\frac{R^4}{\lambda^4}, \qquad \mathbf{z} = \frac{R^2}{\lambda^2}\mathbf{y}.$$

Hence,

$$\Phi = -\frac{1}{4\pi}\left[\frac{1}{||\mathbf{x}-\mathbf{y}||} - \frac{R^4}{\lambda^4}\frac{1}{||\mathbf{x}-\mathbf{y}R^2/\lambda^2||}\right] \tag{74.5}$$

satisfies equation (74.3) and also equation (74.1.b). Because $||\mathbf{z}|| < R$ (by virtue of $||\mathbf{y}|| = \lambda > R$) the point source, we added is not in the physical domain of the problem. Therefore, the solution to equation (74.1) is given by equation (74.5).

Example 2

Suppose we wish to solve Laplace's equation in the half plane:

$$\begin{aligned} \nabla^2 u &= 0, \qquad \text{for } y > 0, -\infty < x < \infty, \\ u(x,0) &= f(x), \\ u &\to 0, \qquad \text{as } \left|x^2+y^2\right| \to \infty. \end{aligned} \tag{74.6}$$

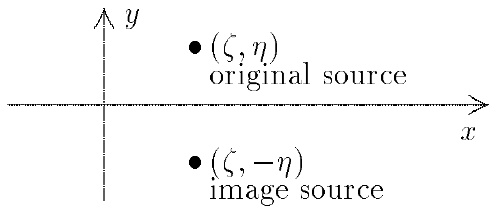

Figure 74.2: The original source and the image source for equation (74.8).

The solution to (74.6) can be obtained by Green's functions (see table 72.1):

$$u(\zeta,\eta) = -\int f(x)\frac{\partial G}{\partial y}(x,0;\zeta,\eta)\,dx, \tag{74.7}$$

where the Green's function $G(x,y;\zeta,\eta)$ satisfies

$$\begin{aligned}\nabla^2 G = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} &= \delta(x-\zeta)\delta(y-\eta),\\ G(x,0;\zeta,\eta) &= 0.\end{aligned} \tag{74.8.a-b}$$

A solution to equation (74.8.a) is given by

$$G(x,y;\zeta,\eta) = \frac{1}{2\pi}\log\sqrt{(x-\zeta)^2+(y-\eta)^2}. \tag{74.9}$$

But this does not satisfy equation (74.8.b). If we place an image source at $(\zeta,-\eta)$, having the opposite sign of the source at (ζ,η) then $G(x,y;\zeta,\eta)$ will vanish along $y=0$ by symmetry. See figure 74.2.

Hence, the solution to (74.8) is

$$G(x,y;\zeta,\eta) = \frac{1}{2\pi}\log\sqrt{(x-\zeta)^2+(y-\eta)^2} - \frac{1}{2\pi}\log\sqrt{(x-\zeta)^2+(y+\eta)^2}.$$

Using this is in equation (74.7), we obtain the solution to equation (74.6):

$$u(\zeta,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{\eta\,dx}{(x-\zeta)^2+\eta^2}.$$

This solution is known as *Poisson's integral*.

Notes

1. The method of images is often used to solve Laplace's equation in hydrodynamics and electrostatics.
2. The method of images can be used for diffusion problems and hyperbolic problems. See, for example, Butkov [1, pages 529–530 and 595–599] or Stakgold [5, pages 72–73 and 491–493].
3. See also Jackson [3, pages 26–29], Kellog [4, pages 228–230], and Zauderer [6, pages 420–432].

References

[1] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[2] Gautesen, A. K. Oblique derivative boundary conditions and the image method for wedges. *SIAM J. Appl. Math. 48*, 6 (December 1988), 1487–1492.

[3] Jackson, J. D. *Classical Electrodynamics.* John Wiley & Sons, New York, 1962.

[4] Kellog, O. D. *Foundations of Potential Theory.* Dover Publications, Inc., New York, 1953.

[5] Stakgold, I. *Green's Functions and Boundary Value Problems.* John Wiley & Sons, New York, 1979.

[6] Zauderer, E. *Partial Differential Equations of Applied Mathematics.* John Wiley & Sons, New York, 1983.

75. Integrable Combinations

Applicable to

Systems of ordinary differential equations.

Yields

One or more ordinary differential equations that can be integrated exactly.

Idea

Sometimes, by combining pieces of a system of differential equations, a combination of the dependent variables can be determined explicitly in terms of the independent variable.

Procedure

Integration of the system of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n), \quad \text{for } i = 1, 2, \ldots, n,$$

is often accomplished by choosing *integrable combinations*. An integrable combination is a differential equation that is derived from a system of differential equations and is readily integrable.

Example 1

Given the two equations

$$\frac{dx}{dt} = y \qquad \text{and} \qquad \frac{dy}{dt} = x, \tag{75.1}$$

an integrable combination can be obtained by adding the two equations to obtain

$$\frac{d(x+y)}{dt} = x + y.$$

This last equation can be integrated (treating $x + y$ as a single variable) to yield

$$x + y = Ae^t, \tag{75.2}$$

where A is an arbitrary constant. For the equations in equation (75.1), another integrable combination may be obtained by subtracting the equations. Integrating this new equation results in

$$x - y = Be^{-t}, \tag{75.3}$$

where B is another arbitrary constant. The explicit solution for $x(t)$ and $y(t)$ may be obtained by combining equations (75.2) and (75.3).

Example 2

Suppose we have the nonlinear system of ordinary differential equations

$$\frac{dx}{dt} = -3yz,$$
$$\frac{dy}{dt} = 3xz,$$
$$\frac{dz}{dt} = -xy.$$

Multiplying the first equation by x, the second by $2y$, and the third by $3z$ and adding, results in

$$x\frac{dx}{dt} + 2y\frac{dy}{dt} + 3z\frac{dz}{dt} = 0.$$

This last equation may be integrated to obtain $x^2+2y^2+3z^2 = C$, where C is an arbitrary constant. For this example, another integrable combination can be found by multiplying the first equation by x, multiplying the second by y, and adding. This new differential equation results in the additional relation $x^2 + y^2 = D$, where D is another arbitrary constant.

Notes

1. Each linearly independent integrable combination yields a first integral of the original system.
2. See El'sgol'ts [1, pages 186–189].

References

[1] El'sgol'ts, L. E. *Differential Equations and the Calculus of Variations.* MIR Publishers, Moscow, USSR, 1970.

76. Integral Representation: Laplace's Method*

Applicable to Linear ordinary differential equations.

Yields

An integral representation of the solution.

Idea

Sometimes the solution of a linear ordinary differential equation can be written as a contour integral. To find such a representation, a lower order differential equation may need to be solved.

Procedure

Let $L_z[\cdot]$ be a linear differential operator with respect to z, and suppose that the ordinary differential equation we wish to solve has the form

$$L_z[u(z)] = 0. \tag{76.1}$$

We look for a solution of equation (76.1) in the form

$$u(z) = \int_{\mathcal{C}} K(z, \xi) v(\xi)\, d\xi, \tag{76.2}$$

for some function $v(\xi)$ and some contour $\mathcal{C}$ in the complex ξ plane. The function $K(z, \xi)$ is called the *kernel*. Some common kernels for Laplace's method are

$$\begin{aligned} \text{Laplace kernel:} \quad & K(z, \xi) = e^{\xi z}. \\ \text{Euler kernel:} \quad & K(z, \xi) = (z - \xi)^N. \end{aligned}$$

We combine equations (76.2) and (76.1) to obtain

$$\int_{\mathcal{C}} L_z[K(z, \xi)] v(\xi)\, d\xi = 0. \tag{76.3}$$

Now we must find a linear differential operator $A_\xi[\cdot]$, operating with respect to ξ, such that $L_z[K(z, \xi)] = A_\xi[K(z, \xi)]$. After $A_\xi[\cdot]$ has been found, then equation (76.3) can be rewritten as

$$\int_{\mathcal{C}} A_\xi[K(z, \xi)] v(\xi)\, d\xi = 0. \tag{76.4}$$

Now we integrate equation (76.4) by parts. The resulting expression will be a differential equation for $v(\xi)$ with some boundary terms. The boundary terms determine the contour $\mathcal{C}$, and the differential equation determines $v(\xi)$. Knowing both $v(\xi)$ and $\mathcal{C}$, the solution to equation (76.1) is given by equation (76.2).

Special Case

For the case where $L_z[\cdot]$ is a linear operator with polynomial coefficients, the solution is easy to find using the Laplace kernel. Let $L_z[\cdot]$ have the form

$$L_z = \sum_{r=0}^{N} \left(\sum_{s=0}^{M} a_{rs} z^s \right) \frac{d^r}{dz^r}, \tag{76.5}$$

where the $\{a_{rs}\}$ are constants. Then define the linear differential operator $M_\xi[\cdot]$ by

$$M_\xi = \sum_{r=0}^{N} \left(\sum_{s=0}^{M} a_{rs} \frac{d^s}{d\xi^s} \right) \xi^r. \tag{76.6}$$

Now define $M_\xi^*[\cdot]$ to be the adjoint of $M_\xi[\cdot]$. Then $L_z[u(z)] = 0$ will have a solution of the form

$$u(z) = \int_{\mathcal{C}} e^{z\xi} v(\xi)\, d\xi,$$

if $v(\xi)$ satisfies

$$M_\xi^*[v(\xi)] = 0, \tag{76.7}$$

and $\mathcal{C}$ is determined by

$$\Big[P\{e^{z\xi}, v(\xi)\} \Big]_{\mathcal{C}} = 0, \tag{76.8}$$

where $P\{e^{z\xi}, v(\xi)\}$ is the bilinear concomitant of $e^{z\xi}$ and $v(\xi)$ (see page 226). Note the order of the original differential operator in equation (76.5) was N while the order of the differential operators in equations (76.6) and (76.7) is M.

Example

Consider Airy's equation

$$u'' - zu = 0. \tag{76.9}$$

We assume that the solution of equation (76.9) has the form

$$u(z) = \int_{\mathcal{C}} e^{z\xi} v(\xi)\, d\xi, \tag{76.10}$$

for some $v(\xi)$ and some contour $\mathcal{C}$. Substituting equation (76.10) into equation (76.9), we find

$$\int_{\mathcal{C}} \xi^2 v(\xi) e^{z\xi}\, d\xi - z \int_{\mathcal{C}} v(\xi) e^{z\xi}\, d\xi = 0. \tag{76.11}$$

The second term in equation (76.11) can be integrated by parts to obtain

$$\int_{\mathcal{C}} \xi^2 v(\xi) e^{z\xi}\, d\xi - \left[v(\xi) e^{z\xi} \right|_{\mathcal{C}} + \int_{\mathcal{C}} v'(\xi) e^{z\xi}\, d\xi = 0$$

or

$$\left[v(\xi) e^{z\xi} \right|_{\mathcal{C}} + \int_{\mathcal{C}} e^{z\xi} \left[\xi^2 v(\xi) + v'(\xi) \right]\, d\xi = 0. \tag{76.12}$$

We choose

$$\xi^2 v(\xi) + v'(\xi) = 0 \tag{76.13}$$

and

$$\left[v(\xi) e^{z\xi} \right|_{\mathcal{C}} = 0. \tag{76.14}$$

With these choices, equation (76.12) is satisfied. From equation (76.13) we can solve for $v(\xi)$

$$v(\xi) = \exp\left(-\frac{\xi^3}{3} \right). \tag{76.15}$$

Using equation (76.15) in equation (76.14), we must choose the contour $\mathcal{C}$ so that

$$\left[v(\xi) e^{z\xi} \right|_{\mathcal{C}} = \left[\exp\left(z\xi - \frac{\xi^3}{3} \right) \right|_{\mathcal{C}} = 0, \tag{76.16}$$

for all real values of z. The only restriction that equation (76.16) places on $\mathcal{C}$ is that the contour start and end in one of the shaded regions in figure 76.1. Finally, the solution to equation (76.9) can now be written

$$u(z) = \int_{\mathcal{C}} e^{(\xi z - \xi^2/3)}\, d\xi. \tag{76.17}$$

Asymptotic methods can be applied to equation (76.17) to determine information about $u(z)$.

For this example, we also could have used the general results in equations (76.6)–(76.8). Identifying equation (76.9) with the operator in equation (76.5), we find

$$L_z = \frac{d^2}{dz^2} - z,$$

so that (from equation (76.6))

$$M_\xi = \xi^2 - \frac{d}{d\xi},$$

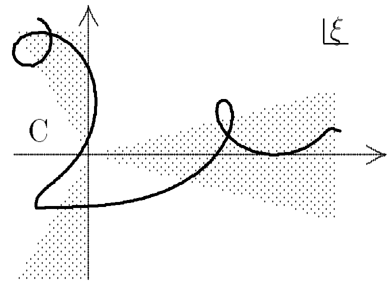

Figure 76.1: A solution to equation (76.9) is determined by any contour $\mathcal{C}$ that starts and ends in the shaded regions. All of the shaded regions extend to infinity. One possible contour is shown.

and also

$$M_\xi^* = \xi^2 + \frac{d}{d\xi}.$$

So, we have to solve (from equation (76.7))

$$M_\xi^*[v(\xi)] = \xi^2 v + v' = 0. \tag{76.18}$$

Because this last equation is identical to equation (76.13), we find the same $v(\xi)$. We compute the bilinear concomitant to be

$$\begin{aligned} P\{e^{z\xi}, v(\xi)\} &= v(\xi)\frac{d}{d\xi}e^{z\xi} - e^{z\xi}\frac{d}{d\xi}v(\xi), \\ &= \left(z+\xi^2\right)\exp\left(-z\xi - \frac{\xi^3}{3}\right), \end{aligned}$$

and we find the same contour $\mathcal{C}$ as before (see (76.16)).

Notes

1. Two linearly independent solutions of Airy's equation are often taken to be

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\,dt, \\ \mathrm{Bi}(x) &= \frac{1}{\pi}\int_0^\infty \left[\exp\left(-\frac{t^3}{3} + xt\right)\cos\left(\frac{t^3}{3} + xt\right)\right]\,dt. \end{aligned}$$

These solutions represent two different choices of the contour in equation (76.17).

2. The Laplace equations

$$(a_0x + b_0)y^{(n)} + (a_1x + b_1)y^{(n-1)} + \cdots + (a_nx + b_n)y = 0$$

have solutions in the form of equation (76.2). Indeed, this was Laplace's original example. See Davies [3, pages 342–367] or Valiron [6, pages 306-319] for details.

3. When the kernel of the transformation is some function of the product $z\xi$, then this method is sometimes called the Mellin transformation. See Ince [4, pages 186–203 and 438–468] for details.
4. Sometimes a double integral is used to find an integral representation. In this case, a solution of the form $u(z) = \iint K(z; s, t) w(s, t)\, ds\, dt$ is proposed. Details may be found in Ince [4, page 197]. As an example, the equation

$$(x^2-1)\frac{d^2y}{dx^2} + (a+b+1)x\frac{dy}{dx} + aby = 0$$

has the two linearly independent solutions

$$y_{\pm}(x) = \int_0^\infty \int_0^\infty \exp\left[\pm xst - \frac{1}{2}(s^2+t^2)\right] s^{a-1} t^{b-1}\, ds\, dt.$$

5. Equations of the form

$$\left[x^n F\left(x\frac{d}{dx}\right) + G\left(x\frac{d}{dx}\right)\right] y = 0,$$

which are sometimes called Pfaffian differential equations, can also be solved by this method. See Bateman [2, Chapter 10, pages 260–264] or Ince [4, page 190] for details.

6. An application of this method to partial differential equations may be found in Bateman [2, pages 268–275].
7. The Mellin–Barnes integral representation for an ordinary differential equation has the form

$$u(z) = \int_{\mathcal{C}} K(z,\xi) z^\xi \left[\frac{\prod_{j=1}^m \Gamma(b_j - \xi) \prod_{j=1}^n \Gamma(1 - a_j + \xi)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \xi) \prod_{j=n+1}^r \Gamma(a_j - \xi)}\right] d\xi.$$

In this representation, only the contour $\mathcal{C}$ and the constants $\{a_i, b_j, m, n, q, r\}$ are to be determined (see Babister [1, pages 24–26] for details).

References

[1] Babister, A. W. *Transcendental Functions Satisfying Nonhomogeneous Linear Differential Equations.* The MacMillan Company, New York, 1967.

[2] Bateman, H. *Differential Equations.* Longmans, Green and Co., 1926.

[3] Davies, B. *Integral Transforms and Their Applications*, second ed. Springer–Verlag, New York, 1985.

[4] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[5] Olver, F. W. J. *Asymptotics and Special Functions*. Academic Press, New York, 1974.

[6] Valiron, G. *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions*. Math Sci Press, Brookline, MA, 1950.

77. Integral Transforms: Finite Intervals*

Applicable to Linear differential equations.

Idea

In order to solve a linear differential equation, it is sometimes easier to transform the equation to some "space," solve the equation in that "space," and then transform the solution back.

Procedure

Given a linear differential equation, multiply the equation by a kernel and integrate over a specified region (see table 77.1 on page 344 for a listing of common kernels and limits of integration). Use integration by parts to obtain an equation for the transform of the dependent variable.

You will have used the "correct" transform (i.e., you have chosen the correct kernel and limits) if the boundary conditions given with the original equation have been utilized. Now solve the equation for the transform of the dependent variable. From this, obtain the solution by multiplying by the inverse kernel and performing another integration. Table 77.1 also lists the inverse kernel.

Example 1

Suppose we have the boundary value problem for $y = y(x)$

$$\begin{aligned} &y_{xx} + y = 1, \\ &y(0) = 0, \quad y(1) = 0. \end{aligned} \tag{77.1.a-c}$$

Because the solution vanishes at both of the endpoints, we suspect that a finite sine transform might be a useful transform to try. Define the finite sine transform of $y(x)$ to be $z(\xi)$, so that

$$z(\xi) := \int_0^1 y(x) \sin \xi x \, dx. \tag{77.2}$$

(See "finite sine transform–2" in table 77.1). Now multiply equation (77.1.a) by $\sin \xi x$ and integrate with respect to x from 0 to 1. This results in

$$\int_0^1 y_{xx}(x) \sin \xi x \, dx + \int_0^1 y(x) \sin \xi x \, dx = \int_0^1 \sin \xi x \, dx. \tag{77.3}$$

If we integrate the first term in equation (77.3) by parts, twice, we obtain

$$\int_0^1 y_{xx}(x) \sin \xi x \, dx = y_x(x) \sin \xi x \Big|_{x=0}^{x=1} - \xi y(x) \cos \xi x \Big|_{x=0}^{x=1} + \xi^2 \int_0^1 y(x) \sin \xi x \, dx. \quad (77.4)$$

Because we are interested only in $\xi = 0, \pi, 2\pi, \ldots$ (see table 77.1), the first term on the right-hand side of equation (77.4) is identically zero. Because of the boundary conditions in equation (77.1.b-c), the second term on the right-hand side of equation (77.4) also vanishes. Because we have used the given boundary conditions to simplify certain terms appearing in the transformed equation, we suspect we have used an appropriate transform. If we had taken a finite cosine transform, instead of the one that we did, the boundary terms from the intergration by parts would not have vanished.

Using equation (77.4), simplified, in equation (77.3) results in

$$\xi^2 \int_0^1 y(x) \sin \xi x \, dx + \int_0^1 y(x) \sin \xi x \, dx = \frac{1 - \cos \xi}{\xi}.$$

Using the definition of $z(\xi)$ (from equation (77.2)), this becomes

$$\xi^2 z(\xi) + z(\xi) = \frac{1 - \cos \xi}{\xi}$$

or

$$z(\xi) = \frac{1 - \cos \xi}{(1 + \xi^2)\xi}.$$

Now that we have found an explicit formula for the transformed function, we can use the summation formula (inverse transform) in table 77.1 to determine that

$$\begin{aligned} y(x) &= \sum_{\xi=0,\pi,2\pi,\ldots} 2 z(\xi) \sin \xi x, \\ &= \sum_{\xi=0,\pi,2\pi,\ldots} 2 \frac{1 - \cos \xi}{(1 + \xi^2)\xi} \sin \xi x, \\ &= \sum_{k=0}^{\infty} 2 \frac{1 - (-1)^k}{(1 + \pi^2 k^2)\pi k} \sin k\pi x, \\ &= \sum_{k=1,3,5,\ldots} \frac{4 \sin k\pi x}{(1 + \pi^2 k^2)\pi k}, \end{aligned} \quad (77.5)$$

where we have defined $k = \xi/\pi$.

The exact solution of equation (77.1) is $y(x) = 1 - \cos x + \frac{\cos 1 - 1}{\sin 1} \sin x$. If this solution is expanded in a finite Fourier series, we obtain the representation in equation (77.5).

Example 2

Suppose we have the following partial differential equation for $\phi(r,t)$ (this corresponds to the temperature of a long circular cylinder whose surface is at a constant temperature)

$$\begin{aligned} &\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=\frac{1}{\kappa}\frac{\partial\phi}{\partial t}, && \text{for } 0\le r<1 \text{ and } t>0,\\ &\phi(1,t)=\phi_0, && \text{for } t>0,\\ &\phi(r,0)=0, && \text{for } 0\le r<1. \end{aligned} \tag{77.6}$$

Multiplying this equation by $rJ_0(pr)$ (where p is positive and satisfies $J_0(p)=0$, see "finite Hankel transform–1" in table 77.1) and integrating with respect to r from 0 to 1, we find

$$p\phi_0 J_0'(p)-p^2\Phi=\frac{1}{\kappa}\frac{d\Phi}{dt}, \tag{77.7}$$

where we have defined $\Phi(p,t)=\int_0^1\phi(r,t)rJ_0(pr)\,dr$. This follows from the relation: $\displaystyle\int_0^1\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\right)rJ_0(pr)\,dr=p\phi_0J_0'(p)-p^2\Phi(p,t)$. The initial condition in equation (77.6) is transformed to $\Phi(p,0)=0$. Using this, we can solve equation (77.7) to find $\Phi(p,t)=\frac{\phi_0}{p}J_0'(p)\left(e^{-\kappa p^2t}-1\right)$. Taking the inverse transform (and noting that $J_0'(p)=-J_1(p)$), we arrive at the final solution to equation (77.6)

$$\phi(r,t)=2\phi_0\sum_p\left(e^{-\kappa p^2t}-1\right)\frac{J_0(pr)}{pJ_1(p)},$$

where the summation is over all positive roots of $J_0(p)=0$.

Table 77.1: Different transform pairs of the form
$$v(\xi_k)=\int_\alpha^\beta K(x,\xi_k)\,u(x)\,dx,\qquad u(x)=\sum_{\xi_k}H(x,\xi_k)\,v(\xi_k).$$

Finite cosine transform − 1, (see Miles [5, page 86]) here l and h are arbitrary, and the $\{\xi_k\}$ satisfy $\xi_k\tan\xi_k l=h$.

$$v(\xi_k)=\int_0^1\cos(x\xi_k)\;u(x)\,dx,\quad u(x)=\sum_{\xi_k}\frac{(2-\delta_{\xi_k0})(\xi_k^2+h^2)\cos(\xi_kx)}{h+l(\xi_k^2+h^2)}\,v(\xi_k).$$

Finite cosine transform − 2, (see Butkov [1, page 161]) this is the last transform with $h=0$, $l=1$, so that $\xi_k=0,\pi,2\pi,\ldots$.

$$v(\xi_k)=\int_0^1\cos(x\xi_k)\;u(x)\,dx,\quad u(x)=\sum_{\xi_k}(2-\delta_{\xi_k0})\cos(\xi_kx)\;v(\xi_k).$$

Finite sine transform − 1, (see Miles [5, page 86]) here l and h are arbitrary, and the $\{\xi_k\}$ satisfy $\xi_k \cot(\xi_k l) = -h$.

$$v(\xi_k) = \int_0^1 \sin(x\xi_k)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} 2 \frac{(\xi_k^2 + h^2)\sin(\xi_k x)}{h + l(\xi_k^2 + h^2)}\, v(\xi_k).$$

Finite sine transform − 2, (see Butkov [1, page 161]) this is the last transform with $h = 0$, $l = 1$, so that $\xi_k = 0, \pi, 2\pi, \ldots$.

$$v(\xi_k) = \int_0^1 \sin(x\xi_k)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} 2\sin(\xi_k x)\, v(\xi_k).$$

Finite Hankel transform − 1, (see Tranter [8, page 88]) here n is arbitrary and the $\{\xi_k\}$ are positive and satisfy $J_n(\xi_k) = 0$.

$$v(\xi_k) = \int_0^1 x J_n(x\xi_k)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} 2 \frac{J_n(x\xi_k)}{J_{m+1}^2(\xi_k)}\, v(\xi_k).$$

Finite Hankel transform − 2, (see Miles [5, page 86]) here n and h are arbitrary and the $\{\xi_k\}$ are positive and satisfy $\xi_k J_n'(a\xi_k) + hJ_n(a\xi_k) = 0$.

$$v(\xi_k) = \int_0^a x J_n(x\xi_k)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} \frac{2\xi_k^2 J_n(x\xi_k)}{\{(h^2 + \xi_k^2)\, a^2 - m^2\}\, J_n^2(a\xi_k)}\, v(\xi_k).$$

Finite Hankel transform − 3, (see Miles [5, page 86]) here $b > a$, the $\{\xi_k\}$ are positive and satisfy $Y_n(a\xi_k)J_n(b\xi_k) = J_n(a\xi_k)Y_n(b\xi_k)$, and $Z_n(x\xi_k) := Y_n(a\xi_k)J_n(x\xi_k) - J_n(a\xi_k)Y_n(x\xi_k)$.

$$v(\xi_k) = \int_a^b x Z_n(x\xi_k)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} \frac{\pi^2}{2} \frac{\xi_k^2 J_n^2(b\xi_k) Z_n(x\xi_k)}{J_n^2(a\xi_k) - J_n^2(b\xi_k)}\, v(\xi_k).$$

Legendre transform, (see Miles [5, page 86]) here $\xi_k = 0, 1, 2, \ldots$.

$$v(\xi_k) = \int_{-1}^1 P_{\xi_k}(x)\, u(x)\, dx, \quad u(x) = \sum_{\xi_k} \frac{2\xi_k + 1}{2} P_{\xi_k}(x)\, v(\xi_k).$$

Note

1. See Butkov [1, Chapter 5 and Section 8.5].

References

[1] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[2] Davies, B. *Integral Transforms and Their Applications*, second ed. Springer–Verlag, New York, 1985.

[3] Erdelyi, A., Ed. *Tables of Integral Transforms.* McGraw–Hill Book Company, New York, 1954. In 3 volumes.

[4] Magnus, W., Oberhettinger, F., and Soni, R. P. *Formulas and Theorems for the Special Functions of Mathematical Physics.* Springer–Verlag, New York, 1966.

[5] Miles, J. W. *Integral Transforms in Applied Mathematics.* Cambridge University Press, New York, 1971.

[6] Sneddon, I. N. *The Use of Integral Transforms.* McGraw–Hill Book Company, New York, 1972.

[7] Stakgold, I. *Green's Functions and Boundary Value Problems.* John Wiley & Sons, New York, 1979.

[8] Tranter, C. J. *Integral Transforms in Mathematical Physics.* Methuen & Co. Ltd., London, England, 1966.

78. Integral Transforms: Infinite Intervals*

Applicable to Linear differential equations.

Idea

In order to solve a linear differential equation, it is sometimes easier to transform the equation to some "space," solve the equation in that "space," and then transform the solution back.

Procedure

Given a linear differential equation, multiply the equation by a kernel and integrate over a specified region (see table 78.1 on page 349 for a listing of common kernels and limits of integration). Use integration by parts to obtain an equation for the transform of the dependent variable.

You will have used the "correct" transform (i.e., you have chosen the correct kernel and limits) if the boundary conditions given with the original equation have been utilized. Now solve the equation for the transform of the dependent variable. From this, obtain the solution by multiplying by the inverse kernel and performing another integration. Table 78.1 also lists the inverse kernel.

Warning

After a solution is obtained by a transform method, it must be checked that the solution satisfies the requirements of the transform. For example, for a function to have a Laplace transform, it must be a L_2 function (i.e., square integrable).

Example 1

Suppose we wish to find the solution to the parabolic partial differential equation

$$u_t = a^2 u_{xx} \tag{78.1}$$

with the initial condition and boundary conditions given by

$$\begin{aligned} u(x,0) &= 0, & & \\ u(0,t) &= u_0, & & \text{for } t > 0, \\ u(\infty,t) &= 0, & & \text{for } t > 0, \end{aligned} \tag{78.2.a-c}$$

where a and u_0 are given constants.

Because this problem is in a semi-infinite domain (i.e., t varies from 0 to ∞), we suspect that a Laplace transform in t may be useful in finding the solution. Let $\mathcal{L}\{\cdot\}$ denote the Laplace transform operator, and define

$$v(x,s) := \mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)\,dt \tag{78.3}$$

to be the Laplace transform of $u(x,t)$. We want to manipulate equation (78.1) into a form such that there are $v(x,s)$ terms present. To obtain this form, multiply equation (78.1) by e^{-st} and integrate with respect to t from 0 to ∞ to obtain

$$\int_0^\infty e^{-st} u_t(x,t)\,dt = a^2 \int_0^\infty e^{-st} u_{xx}(x,t)\,dt. \tag{78.4}$$

The left-hand side of equation (78.4) can be integrated by parts while the x derivatives can be taken out of the integral in the right-hand side to obtain

$$-\left.\frac{u(x,t)e^{-st}}{s}\right|_0^\infty + \int_0^\infty s e^{-st} u(x,t)\,dt = a^2 \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} u(x,t)\,dt.$$

If we assume that $\lim_{t\to\infty} e^{-st}u(x,t) = 0$ and use equation (78.2.a), then we obtain

$$\int_0^\infty s e^{-st} u(x,t)\,dt = a^2 \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} u(x,t)\,dt.$$

Finally, using the definition of $v(x,s)$, from equation (78.3), we obtain

$$s\,v(x,s) = a^2 \frac{\partial^2}{\partial x^2} v(x,s), \tag{78.5}$$

which is essentially an ordinary differential equation in the independent variable x. The boundary conditions for this equation come from taking the Laplace transform of equation (78.2.b–c). We calculate

$$\begin{aligned} v(0,s) &:= \mathcal{L}\{u(0,t)\} = \mathcal{L}\{u_0\} = \int_0^\infty e^{-st} u_0\,dt = \frac{u_0}{s}, \\ v(\infty,s) &:= \mathcal{L}\{u(\infty,t)\} = \mathcal{L}\{0\} = 0. \end{aligned} \tag{78.6}$$

Solving equation (78.5) with the boundary conditions in equation (78.6) results in

$$v(x,s) = \frac{u_0}{s} e^{-x\sqrt{s}/a}. \tag{78.7}$$

A table of inverse Laplace transforms, when applied to equation (78.7), results in

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}\{v(x,s)\} \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} v(x,s)\,ds \\ &= u_0 \left[1 - \operatorname{erf}\left(\frac{x}{2t\sqrt{a}}\right)\right], \end{aligned} \tag{78.8}$$

which is the final solution.

Now that we have the solution, we must either verify that it solves the differential equation and initial condition and boundary conditions that we started with (equation (78.1)), or we must verify that the steps we performed in obtaining the solution are valid. In this case, it means verifying that $\lim_{t\to\infty} e^{-st}u(x,t) = 0$ and that $u(x,t)$ is square integrable. Because each of these are true, the solution found in equation (78.8) is correct.

Example 2

Suppose we have the ordinary differential equation

$$\frac{d^4y}{dx^4} = y + p(x) \tag{78.9}$$

for $y(x)$, for $-\infty < x < \infty$, with the boundary conditions: $y(\pm\infty) = 0$, $y'(\pm\infty) = 0$. Because the equation is on a (doubly) infinite domain, we try to use a Fourier transform in x to find the solution.

Let $\mathcal{F}\{\cdot\}$ denote the Fourier transform operator, and define

$$z(\omega) = \mathcal{F}\{y(x)\} := \int_{-\infty}^{\infty} y(x)e^{i\omega x}\,dx$$

to be the Fourier transform of $y(x)$. If we apply the operator $\mathcal{F}\{\cdot\}$ to equation (78.9) (by multiplying by $e^{i\omega x}$ and integrating with respect to x), we find

$$\int_{-\infty}^{\infty} e^{i\omega x}\frac{d^4y}{dx^4}\,dx = \int_{-\infty}^{\infty} e^{i\omega x}y\,dx + \int_{-\infty}^{\infty} e^{i\omega x}p(x)\,dx.$$

Integrating by parts and using the given boundary conditions, this can be simplified to

$$(i\omega)^4 z(\omega) = z(\omega) + \int_{-\infty}^{\infty} e^{i\omega x}p(x)\,dx.$$

This last expression can be solved to yield

$$z(\omega) = \frac{1}{\omega^4 - 1}\int_{-\infty}^{\infty} e^{i\omega x}p(x)\,dx. \tag{78.10}$$

For any given $p(x)$, the integral in equation (78.10) can be evaluated, and then an inverse Fourier transform can be taken to determine $y(x) = \mathcal{F}^{-1}\{z(\omega)\}$.

Table 78.1: Different integral transform pairs of the form
$v(\xi) = \int_{\alpha}^{\beta} K(x,\xi)u(x)\,dx, \qquad u(x) = \int_{a}^{b} H(x,\xi)v(\xi)\,d\xi.$

Fourier transform, (see Butkov [3, Chapter 7])

$$v(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi}\, u(x)\, dx, \quad u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, v(\xi)\, d\xi.$$

Fourier cosine transform, (see Butkov [3, page 274])

$$v(\xi) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(x\xi)\, u(x)\, dx, \quad u(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos(x\xi)\, v(\xi)\, d\xi.$$

Fourier sine transform, (see Butkov [3, page 274])

$$v(\xi) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(x\xi)\, u(x)\, dx, \quad u(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin(x\xi)\, v(\xi)\, d\xi.$$

Hankel transform, (see Sneddon [22, Chapter 5])

$$v(\xi) = \int_{0}^{\infty} x J_\nu(x\xi)\, u(x)\, dx, \quad u(x) = \int_{0}^{\infty} \xi J_\nu(x\xi)\, v(\xi)\, d\xi.$$

Hilbert transform, (see Sneddon [22, pages 233–238])

$$v(\xi) = \int_{-\infty}^{\infty} \frac{1}{\pi(x-\xi)}\, u(x)\, dx, \quad u(x) = \int_{-\infty}^{\infty} \frac{1}{\pi(\xi-x)}\, v(\xi)\, d\xi.$$

***K*–transform**, (see Erdélyi [7])

$$v(\xi) = \int_{0}^{\infty} K_\nu(x\xi)\sqrt{\xi x}\, u(x)\, dx, \quad u(x) = \frac{1}{\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} I_\nu(x\xi)\sqrt{\xi x}\, v(\xi)\, d\xi.$$

Kontorovich–Lebedev transform, (see Sneddon [22, Chapter 6])

$$v(\xi) = \int_{0}^{\infty} \frac{K_{i\xi}(x)}{x}\, u(x)\, dx, \quad u(x) = \frac{2}{\pi^2} \int_{0}^{\infty} \xi \sinh(\pi\xi) K_{i\xi}(x)\, v(\xi)\, d\xi.$$

Kontorovich–Lebedev transform (alternative form), (see Jones [12])

$$v(\xi) = \int_{0}^{\infty} H_\xi^{(2)}(x)\, u(x)\, dx, \quad u(x) = -\frac{1}{2x} \int_{-i\infty}^{i\infty} \xi J_\xi(x)\, v(\xi)\, d\xi.$$

Laplace transform, (see Sneddon [22, Chapter 3])

$$v(\xi) = \int_{0}^{\infty} e^{-x\xi}\, u(x)\, dx, \quad u(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{x\xi}\, v(\xi)\, d\xi.$$

Mehler–Fock transform of order m, (see Sneddon [22, Chapter 7])

$$v(\xi) = \int_0^\infty \sinh(x) P^m_{i\xi-1/2}(\cosh x)\, u(x)\, dx,$$
$$u(x) = \int_0^\infty \xi \tanh(\pi\xi) P^m_{i\xi-1/2}(\cosh x)\, v(\xi)\, d\xi.$$

Mellin transform, (see Sneddon [22, Chapter 4])

$$v(\xi) = \int_0^\infty x^{\xi-1}\, u(x)\, dx, \quad u(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-\xi}\, v(\xi)\, d\xi.$$

Weber formula, (see Titchmarsh [24, page 75])

$$v(\xi) = \int_a^\infty \sqrt{x}\, [J_\nu(x\xi)Y_\nu(a\xi) - Y_\nu(x\xi)J_\nu(a\xi)]\; u(x)\, dx,$$
$$u(x) = \sqrt{x} \int_0^\infty \frac{J_\nu(x\xi)Y_\nu(a\xi) - Y_\nu(x\xi)J_\nu(a\xi)}{J^2_\nu(a\xi) + Y^2_\nu(a\xi)}\, v(\xi)\, d\xi.$$

Weierstrass transform, (see Hirschman and Widder [10, Chapter 8])

$$v(\xi) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^\infty e^{(\xi-x)^2/4}\, u(x)\, dx, \quad u(x) = \frac{1}{\sqrt{4\pi}} \lim_{T\to\infty} \int_{-T}^T e^{(x-i\xi)^2/4} v(i\xi)\, d\xi.$$

Unnamed transform, (see Naylor [20])

$$v(\xi) = \int_{-\infty}^\infty K_0(|\xi-x|)\, u(x)\, dx, \quad u(x) = -\frac{1}{\pi^2} \left(\frac{d^2}{dx^2} - 1\right) \int_{-\infty}^\infty K_0(|\xi-x|)\, v(\xi)\, d\xi.$$

Unnamed transform, (see Titchmarsh [24, page 83])

$$v(\xi) = \int_{-\infty}^\infty \left[J_{i\sqrt{\xi}}(e^x) + J_{-i\sqrt{\xi}}(e^x)\right]\, u(x)\, dx,$$
$$u(x) = \int_0^\infty \frac{J_{i\sqrt{\xi}}(e^x) + J_{-i\sqrt{\xi}}(e^x)}{4\sinh\left(\pi\sqrt{\xi}\right)}\, v(\xi)\, d\xi.$$

Notes

1. Note that many of the transforms in table 78.1 do not have a standard form. In the Fourier transform, for example, the two $\sqrt{2\pi}$ terms might not be symmetrically placed as we have shown them. Also, a small variation of the K-transform is known as the Meijer transform (see Ditkin and Prudnikov [6, page 75]).
2. There are many tables of transforms available (see Bateman [7] or Magnus *et al.* [14]). It is generally easier to look up a transform than to compute it.

3. Transform techniques may also be used with systems of linear equations.
4. If a function $f(x,y)$ has radial symmetry, then a Fourier transform in both x and y is equivalent to a Hankel transform of $f(r) = f(x,y)$, where $r^2 = x^2 + y^2$. See Sneddon [22, pages 79–83].
5. Integral transforms can be constructed by integrating the Green's function for a Sturm–Liouville eigenvalue problem. This involves explicitly finding an integral representation of the delta function. For example, the relation

$$\delta(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta\nu}\, d\nu \tag{78.11}$$

can be used to derive the Fourier transform. To see this, change η to $x-\xi$ in equation (78.11), multiply by $f(\xi)$ and integrate with respect to ξ to obtain

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\nu}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)e^{i\xi\nu}\, d\xi\right] d\nu$$

For more details, see Davies [5, pages 267–287], or Stakgold [23, Chapter 7, pages 411–466].
6. Many of the transforms in table 78.1 have a *convolution theorem*, which describes how the transform of the product of two functions, is related to the transforms of the individual functions. For example, if $g(t)$ (respectively $h(t)$, $k(t)$) has the Laplace transform $G(s)$ (respectively $H(s)$, $K(s)$), and $G(s) = H(s)K(s)$, then

$$g(t) = \int_0^t h(t-\tau)k(\tau)\, d\tau.$$

This is called a *convolution product* and is often denoted by $g(t) = h(t) * k(t)$. See Miles [16, Table 2.3, page 85].
7. Most of the transforms in table 78.1 have simple formulae relating the transform of the derivative of a function to the transform of the function. For example, if $G(s)$ is the Laplace transform of $g(t)$, then

$$\mathcal{L}\{g^{(n)}(t)\} = s^n G(s) - g^{(n-1)}(0) + sg^{(n-2)}(0) + \cdots + (-1)^n s^{n-1} g(0).$$

8. Two transform pairs that are continuous in one variable and discrete in the other variable, on an infinite interval, are the Hermite transform

$$u(x) = \sum_{n=0}^{\infty} v_n H_n(x) e^{-x^2/2}, \qquad v_n = \frac{1}{(2^n)!\sqrt{\pi}}\int_{-\infty}^{\infty} u(x)H_n(x)e^{-x^2/2}\, dx,$$

where $H_n(x)$ is the nth Hermite polynomialand the Laguerre transform

$$u(x) = \sum_{n=0}^{\infty} v_n L_n^{\alpha}(x)\frac{n!}{\Gamma(n+\alpha+1)}, \qquad v_n = \int_0^{\infty} u(x)L_n^{\alpha}(x)x^{\alpha}e^{-x}\, dx,$$

where $L_n^\alpha(x)$ is the Laguerre polynomial of degree n, and $\alpha \geq 0$. See Haimo [9] for details.

9. Integral transforms are generally created for solving a specific differential equation with a specific class of boundary conditions. For example, the *Mathieu integral transform* (see Inayat-Hussain [11]) has been constructed for the two-dimensional Helmholtz equation in elliptic-cylinder coordinates.
10. The papers by Namias ([17] and [18]) on fractional order Fourier and Hankel transforms contain several examples of how the transforms may be used to solve differential equations.
11. Note that

$$\begin{aligned}\frac{d^r}{dx^r} &= \left[\left(\frac{d}{dx}x\right)\left(\frac{1}{x}\frac{d}{dx}x^2\right)\cdots\left(\frac{1}{x^{r-2}}\frac{d}{dx}x^{r-1}\right)\right]\frac{1}{x^{r-1}}\frac{d}{dx} \\ &= \left[\prod_{i=1}^{r-1}\left(\frac{1}{x^{r-i-1}}\frac{d}{dx}x^{r-i}\right)\right]\frac{d}{x^{r-1}dx}. \qquad (78.12)\end{aligned}$$

Then observe that the $\nu-$transform, defined by

$$\begin{aligned} g(x;\boldsymbol{\nu}) = Z[f(x);\boldsymbol{\nu}] &= \int_0^\infty\cdots\int_0^\infty f\left(x\prod t_i^{1/r}\right)e^{\sum t_i}\prod t_i^{\nu_i}\,dt_i, \\ f(x) &= \frac{1}{(2\pi i)^{r-1}}\int_{-\infty}^{(0+)}\cdots\int_{-\infty}^{(0+)} g\left(x\prod t_i^{-1/r};\boldsymbol{\nu}\right)e^{\sum t_i}\prod t_i^{-\nu_i-1}\,dt_i,\end{aligned}$$

where $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{r-1})$ and i runs from 1 to $r-1$ in each sum and product, can be used with (78.12) to obtain

$$Z\left[\frac{d^r u}{dx^r};\boldsymbol{\nu}_r\right] = \left(\frac{r}{x}\right)^{r-1}\frac{dZ[u;\boldsymbol{\nu}_r]}{dx} - \sum_{i=1}^{r-1}\frac{C_i}{x^{r-i}},$$

where $\boldsymbol{\nu}_r = (-1/r, -2/r, \ldots, -(r-1)/r)$. This transform can be applied, for example, to the equation $y^{(r)} + axy' + by = f(x)$ or to

$$\left(\frac{d^r}{dx^r} + \frac{b_1}{x}\frac{d^{r-1}}{dx^{r-1}} + \cdots + \frac{b_{r-1}}{x^{r-1}}\frac{d}{dx}\right)y + axy' + by = f(x).$$

See Klyuchantsev [13] for details.
12. Classically, the Fourier transform of a function exists only if the function being transformed decays quickly enough at $\pm\infty$. The Fourier transform can be extended, though, to handle generalized functions. For example, the Fourier transform of the nth derivative of the delta function is given by $\mathcal{F}\left(\delta^{(n)}(t)\right) = (i\omega)^n$. Another way to approach the Fourier transform of functions that do not decay quickly enough at either ∞ or $-\infty$ is to use the *one-sided Fourier transforms*. See Chester [4] for details.

13. Many of the transforms listed generalize naturally to n dimensions. For example, in n dimensions we have
    - Fourier transform: $v(\boldsymbol{\xi}) = (2\pi)^{-n/2} \int_{\mathrm{R}^n} e^{i\boldsymbol{\xi}\cdot\mathbf{x}} u(\mathbf{x})\, d\mathbf{x}$, $u(\mathbf{x}) = (2\pi)^{-n/2} \int_{\mathrm{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} v(\boldsymbol{\xi})\, d\boldsymbol{\xi}$.
    - Hilbert transform (see Bitsadze [2]):

$$\frac{\partial f}{\partial x_i} = \frac{\Gamma(n/2)}{\pi^{n/2}} \int_{R^{n-1}} \frac{y_i - x_i}{|\mathbf{y}-\mathbf{x}|^n} \phi(y)\, d\mathbf{y}, \quad i = 1, 2, \ldots, n-1,$$

$$\phi(y) = -\frac{\Gamma(n/2)}{\pi^{n/2}} \int_{R^{n-1}} \frac{(\mathbf{y}-\mathbf{x})\cdot\nabla f}{|\mathbf{y}-\mathbf{x}|^n}\, d\mathbf{y},$$

14. The name *Bessel transform* is given to an integral transform that involves a Bessel function. This class includes Hankel, K, Kontorovich–Lebedev, and many other transforms.
15. Note that, for the Hilbert transform, the integrals in table 78.1 are to be taken in the principal value sense.
16. See also Abramowitz and Stegun [1, pages 1019–1030] and Butkov [3, Chapter 5, pages 179–220 and Section 8.5, pages 299–304].

References

[1] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington, D.C., 1964.

[2] Bitsadze, A. V. The multidimensional Hilbert transform. *Soviet Math. Dokl. 35*, 2 (1987), 390–392.

[3] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[4] Chester, C. R. *Techniques in Partial Differential Equations.* McGraw–Hill Book Company, New York, 1970.

[5] Davies, B. *Integral Transforms and Their Applications*, second ed. Springer–Verlag, New York, 1985.

[6] Ditkin, V. A., and Prudnikov, A. P. *Integral Transforms and Operational Calculus.* Pergamon Press, New York, 1965. Translated by D. E. Brown. English translation edited by I. N. Sneddon.

[7] Erdelyi, A., Ed. *Tables of Integral Transforms.* McGraw–Hill Book Company, New York, 1954. In 3 volumes.

[8] Glaeske, H.-J. Operational properties of a generalized Hermite transformation. *Aequationes Mathematicae 32* (1987), 155–170.

[9] Haimo, D. T. The dual Weierstrass–Laguerre transform. *Trans. AMS 290*, 2 (August 1985), 597–613.

[10] Hirschman, I. I., and Widder, D. V. *The Convolution Transform.* Princeton University Press, Princeton, NJ, 1955.

[11] Inayat-Hussain, A. A. Mathieu integral transforms. *J. Math. Physics 32*, 3 (March 1991), 669–675.

[12] Jones, D. S. The Kontorovich–Lebedev transform. *J. Inst. Maths. Applics 26* (1980), 133–141.

[13] Klyuchantsev, M. I. An integral-transformation method of solving some types of differential equations. *Translated from Differentsial'nye Uravneniya 23*, 10 (October 1987), 1668–1679.

[14] Magnus, W., Oberhettinger, F., and Soni, R. P. *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer–Verlag, New York, 1966.

[15] Marichev, O. I. *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables*. John Wiley & Sons, New York, 1983. translated by L. W. Longdon, Halstead Press.

[16] Miles, J. W. *Integral Transforms in Applied Mathematics*. Cambridge University Press, New York, 1971.

[17] Namias, V. The fractional order Fourier transform and its application to quantum mechanics. *J. Inst. Maths. Applics 25* (1980), 241–265.

[18] Namias, V. Fractionalization of Hankel transforms. *J. Inst. Maths. Applics 26* (1980), 187–197.

[19] Nasim, C. The Mehler–Fock transform of general order and arbitrary index and its inversion. *Int. J. Math.* & *Math. Sci. 7*, 1 (1984), 171–180.

[20] Naylor, D. On an integral transform. *Int. J. Math.* & *Math. Sci. 9*, 2 (1986), 283–292.

[21] Oberhettinger, F., and Higgins, T. P. Tables of Lebedev, Mehler, and generalized Mehler transforms. Tech. rep., Boeing Scientific Research Laboratories, October 1961. Mathematical Note No. 246.

[22] Sneddon, I. N. *The Use of Integral Transforms*. McGraw–Hill Book Company, New York, 1972.

[23] Stakgold, I. *Green's Functions and Boundary Value Problems*. John Wiley & Sons, New York, 1979.

[24] Titchmarsh, E. C. *Eigenfunction Expansions Associated with Second–Order Differential Equations*. Clarendon Press, Oxford, England, 1946.

[25] Tranter, C. J. *Integral Transforms in Mathematical Physics*. Methuen & Co. Ltd., London, England, 1966.

79. Integrating Factors*

Applicable to

Linear first order ordinary differential equations.

Yields

An exact equation that can then be integrated.

Idea

When a given equation is not exact, it may be possible to multiply the equation by a certain term so that it does become exact. The term that is used is called an *integrating factor*.

Procedure

Let us suppose that the nonlinear ordinary differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{79.1}$$

is not exact (see page 284). It may be, however, that if equation (79.1) is multiplied by an integrating factor $u(x,y)$, the resulting equation

$$uM dx + uN dy = 0$$

is exact. For this to be the case, we require $\partial(uM)/\partial y = \partial(uN)/\partial x$, or

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}. \tag{79.2}$$

In general, solving the partial differential equation (79.2) for $u(x,y)$ is more difficult than solving the ordinary differential equation (79.1). But, in certain cases, it may be easier. For example,

1. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, a function of x alone, then $u(x,y) = u(x) = \exp\left(\int^x f(z)\,dz\right)$ is an integrating factor for equation (79.1).
2. If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$, a function of y alone, then $u(x,y) = u(y) = \exp\left(-\int^y g(z)\,dz\right)$ is an integrating factor for equation (79.1).

Example

Suppose we have the general linear first order ordinary differential equation

$$y' + P(x)y = Q(x). \tag{79.3}$$

We recognize that the homogeneous equation corresponding to equation (79.3) is $y' + P(x)y = 0$. Written as $dy + (P(x)y)dx = 0$, we see that the first case applies with $f(x) := P(x)$ (because $M = yP(x)$ and $N = 1$).

Hence we have the integrating factor $u(x) = \exp\big(\int^x P(z)\,dz\big)$, and equation (79.3) can be written as

$$(y' + P(x)y)\exp\left(\int^x P(z)\,dz\right) = Q(x)\exp\left(\int^x P(z)\,dz\right),$$

or

$$\frac{d}{dx}\left[y\exp\left(\int^x P(z)\,dz\right)\right] = Q(x)\exp\left(\int^x P(z)\,dz\right),$$

and therefore (by integrating), we find the solution to be

$$y(x) = \exp\left(-\int^x P(z)\,dz\right)\int^x Q(w)\exp\left(\int^w P(z)\,dz\right)\,dw.$$

Special Case

For a concrete illustration, the equation

$$y' + \frac{1}{x}y = x^2 \tag{79.4}$$

has $\{P(x) = 1/x, Q(x) = x^2\}$, so that

$$\begin{aligned} u(x) &= \exp\left(\int^x \frac{1}{z}\,dz\right) \\ &= \exp(\log x) \\ &= x \end{aligned}$$

is an integrating factor. When equation (79.4) is multiplied by $u(x) = x$, we obtain

$$\begin{aligned} xy' + y &= x^3, \\ \frac{d(xy)}{dx} &= x^3, \\ xy &= \frac{x^4}{4} + C, \end{aligned}$$

or $y = \frac{x^3}{4} + \frac{C}{x}$, where C is an arbitrary constant.

Notes

1. If equation (79.1) admits a one parameter Lie group with generators $\{\xi, \eta\}$ (see page 366), then an integrating factor is given by $u(x,y) = 1/(N\eta - M\xi)$. For example, the differential equation $y(y^2 - x)\,dx + x^2\,dy = 0$ is invariant under the transformation $\{y' = e^{\epsilon/2}y,\ x' = e^{\epsilon}x\}$. Therefore, the infinitesimal operator of the group is described by $\{\eta = \frac{1}{2}y,\ \xi = x\}$. This leads to the integrating factor $u = 2/3xy(x - 2y^2)$, which leads to the solution $y = x/\sqrt{2x + C}$.

2. If $Mx + Ny \neq 0$, and equation (79.1) is homogeneous (see page 327), then an integrating factor is given by $u(x, y) = 1/(Mx + Ny)$. For example, the differential equation $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$ is homogeneous and has the integrating factor $u = 1/xy^2$. This leads to the solution $\frac{x}{y} - \log(x^2y^3) = C$.
3. If $M = M_1(x)y - M_2(x)y^n$ and $N = 1$, then an integrating factor is given by $u(x, y) = y^{-n}\exp\bigl((1-n)/\int M_1\,dx\bigr)$.
4. The differential equation $M_1(x)M_2(y)\,dx + N_1(x)N_2(y)\,dy = 0$ has the integrating factor $u = (M_2N_1)^{-1}$.
5. The differential equation $yf(xy)\,dx + xg(xy)\,dy = 0$, when $f \neq g$, has the integrating factor $u = 1/[xy(f - g)]$. For example, the equation $y(1 - xy)\,dx - x(1 + xy)\,dy = 0$ has $\{f(z) = 1 - z,\ g(z) = -1 - z\}$ so that an integrating factor is given by $u = 1/2xy$. This leads to the implicit solution $ye^{xy} = Cx$.
6. Given equation (79.1), if $z = N - iM$ is an analytic function of x and y (i.e., the Cauchy–Riemann equations $\{N_x = -M_y,\ N_y = M_x\}$ are satisfied), then an integrating factor is given by $1/(N^2 + M^2)$.
   For example, the homogeneous equation
   $$\left(y^2 + 2xy - x^2\right) dy - \left(y^2 - 2xy - x^2\right) dx = 0$$
   has the integrating factor $u = 1/\left[2\left(x^2 + y^2\right)^2\right]$, which leads to the solution $y + x = C(x^2 + y^2)$.
7. Sometimes an integrating factor of the form x^ky^n can be found (for specific values of k and n). This form of the integrating factor will always be adequate for differential equations of the form $x^ay^b(py\,dx + qx\,dy) + x^dy^e(ry\,dx + sx\,dy) = 0$, where $\{a,\ b,\ d,\ e,\ p,\ q,\ r,\ s\}$ are constants.
8. The technique presented here also applies to linear ordinary differential equations of higher order. For example, the second order ordinary differential equation
   $$\sqrt{x}\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 3y = 0$$
   can be made exact (see page 287) by use of the integrating factor $u(x) = \sqrt{x}$. Multiplying equation (8) by $\sqrt{x}$ results in
   $$x\frac{d^2y}{dx^2} + 2x^{3/2}\frac{dy}{dx} + 3y\sqrt{x} = \frac{d}{dx}\left[x\frac{dy}{dx} + (2x^{3/2} - 1)y\right].$$
   Murphy [2, page 165] has a discussion of how to make second order ordinary differential equations exact.
9. When the quasilinear partial differential equation in two independent variables, $M(x, y, u)u_x = N(x, y, u)u_y$, has $M_x = N_y$, then the solution is given implicitly by $\Phi(x, y, u) = 0$, where $M = \Phi_y$ and

$N = \Phi_x$. If, alternately, $M_x \neq N_y$, then it may be possible to find an integrating factor $v(x,y)$ such that $(vM)_x = (vN)_y$. For example, if $(N_y - M_x)/M$ is a function of x alone, then $v(x) = \exp\left(\int \frac{N_y - M_x}{M}\,dx\right)$ will be an integrating factor.

10. For example, the equation $u_x = yu_y$ has the integrating factor $v(x) = e^x$. The solution can then be found to be $u(x,y) = -Cy^3e^{3x}$, where C is an arbitrary constant.
11. See Boyce and DiPrima [1, pages 84–87], Murray [3, pages 22–27], Rainville and Bedient [5, pages 35–37 and 59–66], and Simmons [6, pages 42–46].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Murphy, G. M. *Ordinary Differential Equations and Their Solution*. D. Van Nostrand Company, Inc., New York, 1960.

[3] Murray, J. D. *Asymptotic Analysis*. Springer–Verlag, New York, 1984.

[4] Prelle, M. J., and Singer, M. F. Elementary first integrals of differential equations. *Trans. Amer. Math. Soc. 279*, 1 (September 1983), 215–229.

[5] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations*. The MacMillan Company, New York, 1964.

[6] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

80. Interchanging Dependent and Independent Variables

Applicable to Ordinary differential equations.

Yields

A reformulation of the original equation.

Idea

Sometimes it is easier to solve an ordinary differential equation by interchanging the role of the dependent variable with the role of the independent variable. If this technique works, then the solution is given implicitly by $x = x(y)$ instead of the usual $y = y(x)$.

Procedure

Given the equation

$$\frac{dy}{dx} = f(x, y)$$

to solve, it might be easier to solve the equivalent equation

$$\frac{dx}{dy} = \frac{1}{f(x, y)}.$$

This method can also be used for ordinary differential equations with an order greater than 1. For these cases, table 80.1 can be used to determine how the derivatives $\{y_x, y_{xx}, \dots\}$ transform into the derivatives $\{x_y, x_{yy}, \dots\}$.

Example 1

Suppose the solution is desired to the ordinary differential equation

$$\frac{dy}{dx} = \frac{x}{x^2y^2 + y^5}.$$

Interchanging the dependent and independent variables in this equation produces

$$\frac{dx}{dy} = \frac{x^2y^2 + y^5}{x} = y^2x + \frac{y^5}{x}. \tag{80.1}$$

Equation (80.1) is now a Bernoulli equation with $n = -1$ and can be solved exactly (see page 235). The solution is

$$x(y) = \left(Ae^{2y^3/3} - \frac{y^3}{2} - \frac{3}{4}\right)^{1/2}, \tag{80.2}$$

where A is an arbitrary constant.

$$
\begin{aligned}
y_x &= x_y^{-1}, \\
y_{xx} &= -x_y^{-3} x_{yy}, \\
y_{xxx} &= 3x_y^{-5} x_{yy}^2 - x_y^{-4} x_{yyy}, \\
y_{xxxx} &= -15x_y^{-7} x_{yy}^3 + 10x_y^{-6} x_{yy} x_{yyy} - x_y^{-5} x_{yyyy} \\
y_{xxxxx} &= 105x_y^{-9} x_{yy}^4 - 105x_y^{-8} x_{yy}^2 x_{yyy} + 10x_y^{-7} x_{yyy}^2 \\
&\quad + 15x_y^{-7} x_{yy} x_{yyyy} - x_y^{-6} x_{yyyyy}
\end{aligned}
$$

Table 80.1: How higher order derivatives transform when the dependent and independent variables are switched.

Example 2

The following formidable nonlinear ordinary differential equation

$$y'' + xy(y')^3 = 0 \tag{80.3}$$

becomes, after interchanging the dependent and independent variables, Airy's equation

$$\frac{d^2x}{dy^2} = xy.$$

Hence, the solution to equation (80.3) is given explicitely by

$$x(y) = C_1 \operatorname{Ai}(y) + C_2 \operatorname{Bi}(y),$$

where C_1 and C_2 are arbitrary constants.

Example 3

The nonlinear equation $y'' = (x - y)y'^3$ becomes, after interchanging variables, $x_{yy} = x - y$. This equation has the solution $x = y + Ae^y + Be^{-y}$.

Notes

1. When this method is applied to partial differential equations (and not ordinary differential equations), then the method is called the *hodograph transformation* (see page 456).
2. See Bender and Orszag [1, Section 1.6] and Goldstein and Braun [2, page 107].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.

[2] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations.* U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[3] McAllister, B. L., and Thorne, C. J. *Reverse Differential Equations and Others That Can Be Solved Exactly.* Tech. Rep. 11, University of Utah, Salt Lake City, 1952. Project Number NR-056-239 (Available through NTIS).

81. Lagrange's Equation

Applicable to Equations of the form $y = xF\left(\frac{dy}{dx}\right) + G\left(\frac{dy}{dx}\right)$.

Yields

An exact solution, sometimes given parametrically.

Idea

Equations of this form can be solved by quadratures.

Procedure

Given an equation of the form

$$y = xF\left(\frac{dy}{dx}\right) + G\left(\frac{dy}{dx}\right), \tag{81.1}$$

use p to represent dy/dx so that equation (81.1) can be written as

$$y = xF(p) + G(p). \tag{81.2}$$

Now differentiate equation (81.2) with respect to x to obtain

$$\frac{dy}{dx} \equiv p = F(p) + \frac{dp}{dx}\Big[xF'(p) + G'(p)\Big]. \tag{81.3}$$

Equation (81.3) can be rewritten as

$$\frac{dx}{dp} = x\left(\frac{F'(p)}{p - F(p)}\right) + \left(\frac{G'(p)}{p - F(p)}\right), \tag{81.4}$$

which is now a linear differential equation in x and p. It can be solved by the method of integrating factors (see page 356) to determine

$$x = \phi(p, C), \tag{81.5}$$

where C is an arbitrary constant. Now there are two possibilities:

- Eliminate p between equations (81.2) and (81.5) to obtain the implict solution $\Phi(y, x, C) = 0$.
- Use equation (81.5) in equation (81.2) to obtain the parametric solution

$$x = \phi(P, C),$$
$$y = \phi(P, C)F(P) + G(P),$$

where P is a free parameter.

Example 1

Suppose we have the equation

$$y = 2x\frac{dy}{dx} - a\left(\frac{dy}{dx}\right)^3, \tag{81.6}$$

where a is a constant. Comparing equation (81.6) to equation (81.1), we identify $F(p) = 2p$, $G(p) = ap^3$. Hence, (81.5) becomes

$$\frac{dx}{dp} = -\frac{2x}{p} + 3ap.$$

This last equation has an integrating factor of p^2 and so

$$x = \frac{3a}{4}p^2 + \frac{C}{p^2}, \tag{81.7}$$

where C is an arbitrary constant. Using equation (81.7) in equation (81.6), we can remove the x dependence to obtain

$$y = \frac{a}{2}p^3 + \frac{2C}{p}.$$

Hence, a parametric solution of equation (81.6) is given by

$$\begin{aligned} x &= \frac{3a}{4}P^2 + \frac{C}{P^2}, \\ y &= \frac{a}{2}P^3 + \frac{2C}{P}, \end{aligned} \tag{81.8}$$

where P can have any value. By use of resultants (see page 50), the parameter P can be removed from equation (81.8) to determine the implicit solution

$$(27ay^2 - 16x^3)y^2 + 16a^2x(9ay^2 - 4x^3)C - 128a^3x^2C^2 - 64a^4C^3 = 0.$$

If C is taken to be zero, for example, then the explicit solutions $y = \frac{4}{3\sqrt{3a}}x^{3/2}$ and $y = 0$ are obtained.

Example 2

If we have the equation

$$y = 2x\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2, \tag{81.9}$$

then we make the identification $\{F(p) = 2p, G(p) = -p^2\}$ so that equation (81.4) becomes

$$\frac{dx}{dp} = x\left(-\frac{2}{p}\right) + 2,$$

or (using the integrating factor p^2)

$$x = \frac{2}{3}p^2 + \frac{C}{p^2}, \tag{81.10}$$

where C is an arbitrary constant. Using equation (81.10) in equation (81.9) results in

$$y = \frac{C}{p} - \frac{p^2}{3}.$$

Hence, a parametric solution of equation (81.9) is given by

$$\begin{aligned} x &= \frac{2}{3}P^2 + \frac{C}{P^2}, \\ y &= \frac{C}{P} + \frac{C}{P^2}, \end{aligned} \tag{81.11}$$

where P can have any value. By use of resultants the parameter P can be removed from equation (81.11) to determine the implicit solution

$$y^2(4y - 3x^2) + 6x(2x^2 - 3y)C + 9C^2 = 0.$$

Notes

1. Equation (81.1) is known as d'Alembert's equation and also as an *equation linear in x and y*.
2. If $F \equiv 1$, then equation (81.1) is the same as Clairaut's equation (see page 237).
3. The technique presented in this section is only an application of the more general technique of "solving for y" (see page 411).
4. See Ince [1, pages 38–39], Murphy [2, pages 65–66], and Valiron [3, pages 217–218].

References

[1] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[2] Murphy, G. M. *Ordinary Differential Equations and Their Solution*. D. Van Nostrand Company, Inc., New York, 1960.

[3] Valiron, G. *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions*. Math Sci Press, Brookline, MA, 1950.

82. Lie Groups: ODEs

Applicable to

Linear and nonlinear ordinary differential equations.

Yields

Invariants and symmetries of a differential equation. Often these can be used to solve a differential equation.

Idea

By determining the *transformation group* under which a given differential equation is invariant, we can obtain information about the invariants and symmetries of a differential equation. Sometimes these can be used to solve a given differential equation.

Procedure

A one parameter Lie group of transformations is a family of coordinate transformations of the form

$$\begin{aligned} x_\epsilon &= f(x, y; \epsilon), \\ y_\epsilon &= g(x, y; \epsilon), \end{aligned} \tag{82.1}$$

such that $\epsilon = 0$ gives the identity transformation. It is also required (for the transformations to form a group) that $f(x, y; \epsilon + \delta) = f(x_\epsilon, y_\epsilon; \delta)$, and $f^{-1}(x, y; \epsilon) = f(x, y; -\epsilon)$, with analogous formulae for $g(x, y; \epsilon)$.

Equation (82.1) is called the *global transformation group*. Expanding (82.1) for small values of ϵ yields

$$\begin{aligned} x_\epsilon &= x + \xi(x, y)\epsilon + O(\epsilon^2), \\ y_\epsilon &= y + \eta(x, y)\epsilon + O(\epsilon^2), \end{aligned}$$

where

$$\xi(x, y) = \left(\frac{\partial f}{\partial \epsilon}\right)_{\epsilon=0}, \qquad \eta(x, y) = \left(\frac{\partial g}{\partial \epsilon}\right)_{\epsilon=0}. \tag{82.2}$$

The quantities ξ and η are the *infinitesimal transformations* of the group. Lie's first fundamental theorem states that knowledge of the infinitesimals $\{\xi(x, y), \eta(x, y)\}$ is equivalent to knowing the functions $\{f, g\}$ in (82.1).

An nth order differential equation

$$G\left(x, y, y', \ldots, y^{(n)}\right) = 0 \tag{82.3}$$

is said to be invariant under the group defined by equation (82.1) if the differential equation

$$G\left(x_\epsilon, y_\epsilon, y'_\epsilon, \ldots, y_\epsilon^{(n)}\right) = 0$$

is equivalent to equation (82.3) under the change of variables in (82.1).

The infinitesimal generator (also called the *generator* or *infinitesimal operator*) associated with equation (82.1) is $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$. The *prolongations* of X are defined by

$$X^{(n)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \sum_{l=1}^{n}\xi_l\frac{\partial}{\partial y^{(l)}}, \tag{82.4}$$

where $\xi_0 = \eta$ and $\xi_l = D(\xi_{l-1}) - y^{(l)}D(\xi)$, for $l = 1, 2, \dots, n$, and the *total derivative operator* D is defined by $D := \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \dots$.

The differential equation of nth order in equation (82.3), $G = 0$, will be invariant with respect to the one parameter group defined by (82.1) if

$$X^{(n)}G = 0, \tag{82.5}$$

on the manifold $G = 0$ in the space of the variables $\{x, y, y', \dots, y^{(n)}\}$. Note that equation (82.5) is quasilinear and the method of characteristics may be used to solve it.

If the differential equation $G = 0$ is invariant with respect to the group, then the subsidiary equations of equation (82.5) can be written as (see page 432)

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{d(y')}{\xi_1} = \cdots = \frac{d(y^{(n)})}{\xi_n}.$$

We can sometimes integrate two of these equations to obtain two integrals: $u = u(x, y, y', \dots)$ and $v = v(x, y, y', \dots)$. If the original equation, $G = 0$, is written in terms of these new variables, then the resulting differential equation will be only of order $n - 1$. Hence, we will have reduced the order of the given differential equation.

Special Case

The condition for the equation $F(x, y, y', y'') = 0$ to be invariant under the action of the group defined by equation (82.1) is that $X^{(2)}F\,|_{F=0} = 0$. When $F = y'' - f(x, y, y')$, this determining equation becomes

$$\begin{aligned} \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy}){y'}^2 - {y'}^3\xi_y y \\ + (\eta_y - 2\xi_x - 3y'\xi_y)f - \left[\eta_x + (\eta_y - \xi_x)y' - {y'}^2\right] f_{y'} \\ - \xi f_x - \eta f_y = 0. \end{aligned} \tag{82.6}$$

We emphasize that equation (82.6) is an identity in x, y, and y'. Because η and ξ cannot depend on y', equation (82.6) separates into many simultaneous equations for each type of y' term.

Example 1

Given the class of second order ordinary differential equations

$$G(x, y, y', y'') \equiv xy'' - F\left(\frac{y}{x}, y'\right) = 0, \tag{82.7}$$

we ask if this differential equation is invariant under the magnification group

$$\begin{aligned} x_\epsilon &= xe^\epsilon, \\ y_\epsilon &= ye^\epsilon. \end{aligned} \tag{82.8}$$

If it is, then we should be able to reduce equation (82.7) to a sequence of first order ordinary differential equations. Using (82.8) in the definitions in equations (82.2) and (82.4), we can sequentially calculate

$$\begin{aligned} \xi(x,y) &= x, \qquad \eta(x,y) = y, \\ \xi_0 &= \eta = y, \\ \xi_1 &= D(\xi_0) - y'D(\xi) = D(y) - y'D(x) = 0 \\ \xi_2 &= D(\xi_1) - y''D(\xi) = D(0) - y''D(x) = -y'', \\ X^{(2)} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y''\frac{\partial}{\partial y''}. \end{aligned}$$

Applying $X^{(2)}$ to G, we find

$$\begin{aligned} X^{(2)}G &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y''\frac{\partial}{\partial y''}\right)\left[xy'' - F\left(\frac{y}{x}, y'\right)\right] \\ &= x\left(y'' + \frac{y}{x^2}F_1\right) + y\left(-\frac{1}{x}F_1\right) - y''(x) \\ &= 0, \end{aligned}$$

where F_1 denotes the derivative of F with respect to its first argument. We conclude, then, that $G = 0$ is invariant under the magnification group.

Now we form the subsidiary equations:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy'}{0} = \frac{dy''}{-y''}.$$

From the first equality, $\frac{dx}{x} = \frac{dy}{y}$, we find that y/x is a constant; we write this as $y/x = u$. From the second equality, $\frac{dy}{y} = \frac{dy'}{0}$, we find that y' is a constant; we write this as $y' = v$.

Now we will write the equation $G = 0$ in terms of the "constants" that parameterize the solution space: $\{u, v\}$. To change variables, we will need

$$y'' = \frac{dy'}{dx} = \frac{dv}{dx} = \frac{dv}{du}\frac{du}{dx} = \frac{dv}{du}\left(\frac{y'}{x} - \frac{y}{x^2}\right) = \frac{dv}{du}\frac{v-u}{x}.$$

Hence,

$$G = xy'' - F\left(\frac{y}{x}, y'\right) = (v-u)\frac{dv}{du} - F(u,v) = 0. \tag{82.9}$$

Finally, then, we have transformed the second order differential equation $G = 0$ into a first order differential equation in terms of u and v. After this equation is solved for $v = v(u)$, we then have a first order equation for $y(x)$ (using $u = y/x$ and $v = y'$).

We now illustrate the above result with two special cases:

1. If we choose the special case $F(u, v) = v - u$ (for which equation (82.7) becomes the linear equation $x^2y'' - xy' + y = 0$, with solutions $y = x$ and $y = x \log x$), equation (82.9) becomes $(v-u)\left(\frac{dv}{du} - 1\right) = 0$. The most general solution to this equation is $v = u + C$, where C is an arbitrary constant. Changing to our original variables, this becomes $\frac{dy}{dx} = \frac{y}{x} + C$. This equation has the solution $y = Cx \log x + Dx$, where D is another arbitrary constant.
2. If we choose the special case $F(u, v) = u^2 - v^2$ (for which equation (82.7) becomes the nonlinear equation $x^3y'' + x^2(y')^2 - y^2 = 0$), equation (82.9) becomes $\frac{dv}{du} = -v - u$. This first order equation can be integrated to yield $v = (u^2 - 2u + 2) + Ce^{-u}$, where C is an arbitrary constant. In this case, we cannot integrate again to obtain $y = y(x)$ in closed form.

Example 2

For a given differential equation, the different infinitesimal generators will generate an r-dimensional Lie group (L_r) The following four statements are equivalent (see Ibragimov [10, page 39]):

1. The second order ordinary differential equation

$$y'' = f(x, y, y') \tag{82.10}$$

can be linearized by a change of variables.

2. Equation (82.10) has the form

$$y'' = F_3(x,y){y'}^3 + F_2(x,y){y'}^2 + F_1(x,y)y' + F_0(x,y) = 0$$

with coefficients $\{F_i(x, y)\}$ satisfying the integrability conditions of the following over-determined system:

$$\begin{aligned}
\frac{\partial z}{\partial x} &= z^2 - F_0 w - F_1 z + \frac{\partial F_0}{\partial y} + F_0 F_2, \\
\frac{\partial z}{\partial y} &= -zw + F_0 F_3 - \frac{1}{3}\frac{\partial F_2}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial y}, \\
\frac{\partial w}{\partial x} &= zw - F_0 F_3 - \frac{1}{3}\frac{\partial F_1}{\partial y} + \frac{2}{3}\frac{\partial F_2}{\partial x}, \\
\frac{\partial w}{\partial y} &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3.
\end{aligned} \tag{82.11}$$

3. Equation (82.10) admits the Lie algebra L_8.
4. Equation (82.10) admits the Lie algebra L_2 with a basis $\{X_1, X_2\}$, such that $X_1 \vee X_2 = 0$ (see the notes for the definition of the pseudoscalar product $X_1 \vee X_2 =$).

Examples:

- Consider the equation
$$y'' = f(y'). \tag{82.12}$$
From the above, this will be linearizable if and only if $f(y')$ is a polynomial of the third degree in y'. That is, if equation (82.12) has the form
$$y'' + A_3{y'}^3 + A_2{y'}^2 + A_1y' + A_0 = 0,$$
where the $\{A_i\}$ are constants, then it may be linearized.

- Consider the equation
$$y'' = \frac{f(y')}{x}. \tag{82.13}$$
From the above, this will be linearizable only if $f(y')$ is a polynomial of the third degree in y'. That is, equation (82.13) must have the form
$$y'' + \frac{1}{x}\left(A_3{y'}^3 + A_2{y'}^2 + A_1y' + A_0\right) = 0,$$
where the $\{A_i\}$ are constants. In this case, the integrability conditions in (82.11) become
$$\begin{aligned} A_2(2 - A_1) + 9A_0A_3 &= 0 \\ 3A_3(1 + A_1) - A_2^2 &= 0. \end{aligned} \tag{82.14}$$
If we define $a = -A_3$ and $b = -A_2$, then we can solve equation (82.14) for A_1 and A_2. We conclude: Equation (82.13) may be linearized if and only if it has the form:
$$y'' = \frac{1}{x}\left[a{y'}^3 + b{y'}^2 + \left(1 + \frac{b^2}{3a}\right)y' + \frac{b}{3a} + \frac{b^3}{27a^2}\right].$$

- Consider the equation
$$y'' = F(x, y). \tag{82.15}$$
This matches the above form with $F_1 = F_2 = F_3 = 0$ and $F_0 = F$. In this case, the integrability conditions in equation (82.11) become
$$\begin{aligned} z_x &= z^2 + Fw - F_y, \\ z_y &= -zw, \\ w_x &= zw, \\ w_y &= -w^2. \end{aligned} \tag{82.16}$$

Translation in x $\mathbf{X} = \partial_x$	$x_\epsilon = x + \epsilon$ $y_\epsilon = y$
Translation in y $\mathbf{X} = \partial_y$	$x_\epsilon = x$ $y_\epsilon = y + \epsilon$
Scaling $\mathbf{X} = x\partial_x + y\partial_y$	$x_\epsilon = e^\epsilon x$ $y_\epsilon = e^\epsilon y$
Rotation in the (x, y) plane $\mathbf{X} = -y\partial_x + x\partial_y$	$x_\epsilon = x\cos\epsilon - y\sin\epsilon$ $y_\epsilon = x\sin\epsilon - y\cos\epsilon$

Table 82.1: Some common Lie group generators

Using the first two equations in (82.16) in the identity $z_{xy} = z_{yx}$, we find the compatibility condition $F_{yy} = 0$. This is a necessary condition for the linearizability of equation (82.15).

Notes

1. Lie group analysis is the most useful and general of all the techniques presented in this book. Some common generators are in table 82.1. Many of the other methods presented in this book can be derived from the method of Lie groups. For example

   - Equations with the dependent variable missing (see page 260) are invariant under the translation group $\{x_\epsilon = x, y_\epsilon = y + \epsilon\}$.
   - Equations with the independent variable explicitly missing (see page 230) are invariant under the translation group $\{x_\epsilon = x + \epsilon, y_\epsilon = y\}$.
   - Homogeneous equations (see page 327) are invariant under the affine group $\{x_\epsilon = x, y_\epsilon = ye^\epsilon\}$.
   - Scale invariant equations (see page 398) are invariant under the group $\{x_\epsilon = xe^\epsilon, y_\epsilon = ye^{p\epsilon}\}$.
   - In Kumei and Bluman [13], it is shown that the hodograph transformation (see page 456) and the Legendre transformation (see page 467) are derivable from Lie group methods.
   - Similarity solutions (see page 497) are all derivable from Lie group methods.
   - Contact transformations (see page 249) and the Riccati transformation (see page 392) are also derivable from Lie group methods.

2. Changing variables in an infinitesimal generator is straightforward. Suppose we have the generator $\mathbf{X} = \sum_{i=1}^n b^i \frac{\partial}{\partial x^i}$. To change variables from the $\{x^i\}$ coordinates to the $\{x^{i'}\}$ coordinates (with $x^{i'} = x^{i'}(x^i)$) we find that $\mathbf{X} = \sum_{i=1}^n \left(\mathbf{X}x^i\right) \frac{\partial}{\partial x^{i'}}$. For example, consider the generator for scaling invariance: $\mathbf{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. To change to

the variables $u = y/x$ and $v = xy$, we form

$$\mathbf{X}u = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\frac{y}{x} = 0,$$
$$\mathbf{X}v = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)v = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)xy = 2xy = 2v.$$

Hence, we can write $\mathbf{X}$ in the (u, v) coordinates as $\mathbf{X} = 2v\frac{\partial}{\partial v}$. (Making the further substitution $b = \frac{1}{2}\log v$, we find that $\mathbf{X} = \frac{\partial}{\partial b}$.)

3. In the older literature, transformation groups were found and then classes of equations that were invariant under that group were determined. This was what was done in the first example in this section. For example, it can be shown that the most general second order differential equation invariant under a group of the form

$$x_\epsilon = f(x;\epsilon) = x + \epsilon\xi(x) + O(\epsilon^2),$$
$$y_\epsilon = g(x;\epsilon)y = y + \epsilon\eta(x)y + O(\epsilon^2),$$

has the form

$$y'' + \left(\frac{\xi' - 2\eta}{\xi}\right)y' + \left(\frac{\eta^2 - \xi\eta'}{\xi^2}\right)y = \frac{\Phi(A, B)}{s\xi^2},$$

where Φ is an arbitrary function of its arguments, and $\{A, B, s\}$ are defined by

$$A(x, y) = sy,$$
$$B(x, y) = (\xi x - \eta y)s,$$
$$s(x) = \exp\left(-\int_{x_0}^{x}\frac{\eta(t)}{\xi(t)}\,dt\right).$$

See Hill [9, page 84] for details.

4. Recently, the procedure in the last note has been reversed: Given a differential equation, find a transformation group that leaves the equation invariant. To derive the transformation group, a set of partial differential equations arising from the equation $X^{(n)}G = 0$ must be solved. For example, for the second order ordinary differential equation $\ddot{x} = f(t, x, \dot{x})$ to be invariant under the group

$$x_\epsilon = x + \epsilon\psi(t, x) + O(\epsilon^2),$$
$$t_\epsilon = t + \epsilon\phi(t, x) + O(\epsilon^2),$$

requires that the following equation

$$\begin{aligned}&(2\psi_{xt} - \phi_{tt})\dot{x} + (\psi_{xx} - 2\phi_{xt})\dot{x}^2 - \phi_{xx}\dot{x}^3\\ &+ [(\psi_x - 2\phi_t) - 3\phi_x\dot{x}]\,f(t, x, \dot{x}) - \phi f_t(t, x, \dot{x}) - \psi f_x(t, x, \dot{x})\\ &- \left[\psi_t + (\psi_x - \phi_t)\dot{x} - \phi_x\dot{x}^2\right]f_{\dot{x}}(t, x, \dot{x}) = 0\end{aligned}$$

hold for all $(t, x, \dot{x})$. See Aguirre and Krause [1] for details.

5. The analysis in this section can be obtained from the general results of Lie algebras. For example, if $x(t)$ satisfies the equation $\ddot{x} = f(x, \dot{x})$, where f is in C^∞, and the solution is analytic for all t, then the solution may be obtained from $x_{t+\tau} = e^{t\Omega_\tau} x_\tau$, where

$$\Omega_\tau = v_\tau \left(\frac{\partial}{\partial x_\tau}\right) + f(x_\tau, v_\tau)\left(\frac{\partial}{\partial v_\tau}\right) + \left(\frac{\partial}{\partial \tau}\right),$$

and we have used x_τ to denote $x(\tau)$. For example, for the differential equation $\ddot{x} = 1$, we have $f = 1$ so that $\Omega_\tau = v_\tau \partial_{x_\tau} + \partial_{v_\tau} + \partial_\tau$, and we can calculate

$$\begin{aligned}
\Omega_\tau x_\tau &= v_\tau, \\
\Omega_\tau^2 x_\tau &= \Omega_\tau v_\tau = 1, \\
\Omega_\tau^3 x_\tau &= \Omega_\tau 1 = 0, \\
\Omega_\tau^k x_\tau &= 0, \qquad \text{for } k \geq 3
\end{aligned}$$

Using these calculations, we can then find

$$\begin{aligned}
x_{t+\tau} &= e^{t\Omega_\tau} x_\tau \\
&= \sum_{k=0}^{\infty} \frac{t^k \Omega_\tau^k}{k!} x_\tau \\
&= x_\tau + t v_\tau + \frac{t^2}{2},
\end{aligned}$$

or $x(t+\tau) = x(\tau) + t\dot{x}(\tau) + t^2/2$.
This also generalizes to higher dimensions. For example, the solution of the vector equation $\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}})$ may be written as $\mathbf{x}_{t+\tau} = e^{t\Omega_\tau}\mathbf{x}_\tau$, where

$$\Omega_\tau = \mathbf{v}_\tau \cdot \underset{\mathbf{x}_\tau}{\nabla} + \mathbf{f}(\mathbf{x}_\tau, \mathbf{v}_\tau) \cdot \underset{\mathbf{v}_\tau}{\nabla} + \frac{\partial}{\partial \tau}.$$

6. Note that an arbitrary function of x_ϵ and y_ϵ, $F(x_\epsilon, y_\epsilon)$, can be formally expanded in terms of the generator, x, and y as

$$\begin{aligned}
F(x_\epsilon, y_\epsilon) &= F(x,y) + \epsilon \left(\frac{\partial f}{\partial \epsilon}\frac{\partial}{\partial x} + \frac{\partial g}{\partial \epsilon}\frac{\partial}{\partial y}\right)_{\epsilon=0} F(x,y) + \cdots \\
&= F(x,y) + \epsilon V F(x,y) + \frac{1}{2}\epsilon^2 V^2 F(x,y) + \cdots \\
&= e^{\epsilon V} F(x,y).
\end{aligned}$$

7. If the parameter ϵ appearing in equation (82.1) had been an r-dimensional vector, then there would be r infinitesimal operators $\{X_1, X_2, \ldots, X_r\}$. Lie's second fundamental theorem states that these operators generate an r-dimensional Lie group under commutation $[X_a, X_b] = K_{ab}^c X_c$, where the K's are called *structure constants*

No.	Commutator	Pseudoscalar	Typified by
I	$[X_1, X_2] = 0$	$X_1 \vee X_2 \neq 0$	$\{X_1 = \partial_x, X_2 = \partial_y\}$
II	$[X_1, X_2] = 0$	$X_1 \vee X_2 = 0$	$\{X_1 = \partial_y, X_2 = x\partial_y\}$
III	$[X_1, X_2] = X_1$	$X_1 \vee X_2 \neq 0$	$\{X_1 = \partial_y, X_2 = x\partial_x + y\partial_y\}$
IV	$[X_1, X_2] = X_1$	$X_1 \vee X_2 = 0$	$\{X_1 = \partial_y, X_2 = y\partial_y\}$

Table 82.2: All possible cases for a two-dimensional Lie algebra

and summation occurs over repeated indices. Lie's third fundamental theorem relates the structure constants to one another.

If $r = 1$ in the above, then the order of the original equation can be reduced by 1. If $n \geq 2$ and $r = 2$, then the order of the original equation can be reduced by 2. If $n \geq 3$ and $r \geq 3$, then it does not follow that the order of the original equation can be reduced by more than 2. However, if the r-dimensional Lie algebra has a q-dimensional *solvable subalgebra*, then the order of the original equation can be reduced by q. See Bluman and Kumei [3] for details.

8. Given the two generators $X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}$ and $X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y}$, the pseudoscalar product is $X_1 \vee X_2 = \xi_1\eta_2 - \xi_2\eta_1$ and the commutator is $[X_1, X_2] = X_1X_2 - X_2X_1$. By a suitable choice of basis, any two-dimensional Lie algebra can be reduced to one of four types as shown in table 82.2. Hence, an algorithm for integrating second order ordinary differential equations is given by
   (a) Calculate an admitted Lie algebra L_r.
   (b) Compare r to 2:
      i. If $r < 2$, then the ODE cannot be completely integrated using Lie groups.
      ii. If $r > 2$, then determine a sub-algebra $L_2 \subset L_r$.
   (c) From the commutator and pseudoscalar product change the basis to obtain one of the four cases in table 82.2.
   (d) Integrate the resulting equation.
   (e) Rewrite the solution in the original variables.
9. The generators for some first (second) order ordinary differential equations are in table 82.3 (table 82.4). The Lie groups associated with some second order ordinary differential equations are in table 82.5.
10. The semigroup approach to differential equations starts with the evolution equation $u_t = \mathcal{L}u + \mathcal{N}u$ (where $\mathcal{L}$ and $\mathcal{N}$ are constant coefficient linear and nonlinear operators that do not depend on time) with the initial condition $u(x, t_0) = u_0(x)$ and writes the solution as the nonlinear integral equation

$$u(x,t) = e^{(t-t_0)\mathcal{L}} u_0(x) + \int_{t_0}^{t} e^{(t-t_0)\mathcal{L}} \mathcal{N}(u(x\tau))\, d\tau.$$

Equation	Generator
$y' = F(kx + ly)$	$X = l\partial_x - k\partial_y$
$y' = F\left(\frac{y}{x}\right)$	$X = x\partial_x + y\partial_y$
$y' = \frac{y}{x} + F\left(\frac{y}{x}\right)$	$X = x\partial_y$
$y' = F(x)y$	$X = y\partial_y$

Table 82.3: Generators for some classes of first order ODEs

Equation	Generator
$y'' = F(y, y')$	$X = \partial_x$
$y'' = F(x, y')$	$X = \partial_y$
$y'' = F(x, y - xy')$	$X = x\partial_y$
$y'' = y'^3 F\left(y, \frac{y - xy'}{y'}\right)$	$X = y\partial_y$
$x^3 y'' = F\left(\frac{y}{x}, y - xy'\right)$	$X = x^2\partial_x + xy\partial_y$

Table 82.4: Generators for some classes of second order ODEs

Equation	Lie group L	$\lvert L\rvert$
$y'' = f(y, y')$	$\{\partial_x\}$	1
$y'' = f(y')$	$\{\partial_x, \partial_y\}$	2
$y'' = \frac{f(y')}{x}$	$\{\partial_y, x\partial_x + y\partial_y\}$	2
$y'' = Cy^{-3}$	$\{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y\}$	3
$y'' = Ce^{y'}$	$\{\partial_x, \partial_y, x\partial_x + (x + y)\partial_y\}$	3
$y'' = 0$	$\{\partial_x, \partial_y, x\partial_y, x\partial_x, y\partial_x, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y\}$	8

Table 82.5: Lie groups for some second order ODEs

This representation of the solution is useful for proving existence and uniqueness of solutions and computing estimates of their magnitude, verifying dependence on initial and boundary data, as well as performing asymptotic analysis of the solution (see, e.g., Yosida [22]).

11. Using Lie groups to find symmetries of differential equations can be computationally intensive. Algorithms have been developed for

computerized handling of the calculations, see Azara [2] (for Maple), Bocharov and Bronstein [4], Champagne *et al.* [5] (for Macsyma), Eliseev *et al.* [7] (for REDUCE), or Head [8] (for muMATH).

12. It is also possible to find discrete groups that transform solutions of ordinary differential equations to other solutions, see Zaĭtsev [23]. For example, the generalized Emden–Fowler equation $y'' = Ax^n y^m (y')^l$ is described by the parameters $\mathbf{c} = (n, m, l)$. Under the discrete transformation $\{y = at,\ x = bu\}$, the solution $y = y(x; \mathbf{c})$ is mapped to the solution $u = u(y, \mathbf{c}')$, where $\mathbf{c}' = (n, m, 3 - l)$. Another such discrete transformation is given by $\{y = au^{-1/m},\ x = bt^{1/(n+1)}\}$ for which $\mathbf{c}' = \left(-\frac{n}{n+1}, \frac{1}{1-l}, \frac{2m+1}{m}\right)$. Zaĭtsev [23] illustrates this method by writing the solution of $y'' = x^{-15/8} y\sqrt{y'}$ in terms of the solutions to $u'' = 6u^2$ (which are elliptic functions).
13. Technically, a Lie group is a topological group (i.e., a group that is also a topological space), which is also an analytic manifold on which the group operations are analytic. The tangent space to that manifold is a Lie algebra, which is a linear vector space. See Sattinger and Weaver [16] for an algebraic approach to Lie groups.
14. Easily readable books that explain Lie groups more fully are Bluman and Kumei [3] and Stephani [20]. See also Ince [11, Chapter 4, pages 93–113]. and Olver [14].
15. For the system of second order ordinary differential equations

$$\ddot{y}^a = \omega^a(y^i, \dot{y}^i, t), \qquad a, i = 1, \ldots, N$$

the generalization of equation (82.6) is (using the summation convention, $()_{,t} \equiv \partial()/\partial t$, and $()_{,i} \equiv \partial()/\partial y^i$) (see Stephani [20, page 95]):

$$\begin{aligned} &\xi\omega^a{}_{,t} + \eta^b\omega^a{}_{,b} + \left(\eta^b{}_{,t} + \dot{y}^c\eta^b{}_{,c} - \dot{y}^b\xi_{,t} - \dot{y}^b\dot{y}^c\xi_{,c}\right)\frac{\partial\omega^a}{\partial\dot{y}^b} \\ &\quad + 2\omega^a\left(\xi_{,t} + \dot{y}^b\xi_{,b}\right) + \omega^b\left(\dot{y}^a\xi_b - \eta^a{}_{,b}\right) + \dot{y}^a\dot{y}^b\dot{y}^c\xi_{,bc} \\ &\quad + 2\dot{y}^a\dot{y}^c\xi_{,tc} - \dot{y}^c\dot{y}^b\eta^a{}_{,bc} + \dot{y}^a\xi\ _{,tt} - 2\dot{y}^b\eta^a{}_{,tb} - \eta^a{}_{,tt} = 0. \end{aligned}$$

16. The Blaisus equation $y''' + yy'' = 0$ is invariant under the scaling $y(\eta) = \lambda F(\bar{\eta})$ where $\eta = \bar{\eta}/\lambda$. Hence, if $F(\lambda)$ is a solution, then so is $\lambda F(\lambda\eta)$. Consequently, the solution to the Blaisus equation with the boundary conditions $\{y(0) = y'(0) = 0, y'(\infty) = 2\}$ can be solved by the sequence of two initial value problems

$$\begin{aligned} F''' + FF'' &= 0 \qquad F(0) = F'(0) = 0 \qquad F''(0) = 1 \\ y''' + yy'' &= 0 \qquad y(0) = y'(0) = 0 \qquad y''(0) = [2/F'(\infty)]^{3/2} \end{aligned}$$

This procedure is called *exact shooting*, see Klamkin [12].

As another example, consider the generalized Emden–Fowler equation $N[u] = (t^a u')' + ct^b e^u = 0$ with $u'(0) = 0$ and $u(\infty) = 0$ (for $a + b \neq 2$). If $U(t)$ is a solution of $N[U] = 0$, then so is $u(t) = U\left(te^{\lambda/(b-a+2)}\right) + \lambda$. Hence, the original BVP can be solved by finding U from $\{N[U] = 0, U(0) = U'(0) = 0\}$ and then finding u from $\{N[u] = 0, u(0) = -U(\infty), u'(0) = 0\}$.

References

[1] Aguirre, M., and Krause, J. Infinitesimal symmetry transformations of some one-dimensional linear systems. *J. Math. Physics 25*, 2 (February 1984), 210–218.

[2] Azara, J. L. R. A MAPLE program for the generation of the lie-series solution of systems of non-linear ordinary differential equations. *Comput. Physics Comm. 67* (1992), 537–542.

[3] Bluman, G. W., and Kumei, S. *Symmetries and Differential Equations*. Springer–Verlag, New York, 1989.

[4] Bocharov, A. V., and Bronstein, M. L. Efficiently implementing two methods of the geometrical theory of differential equations: An experience in algorithm and software design. *Acta Appl. Math. 16* (1989), 143–166.

[5] Champagne, B., Hereman, W., and Winternitz, P. The computer calculation of Lie point symmetries of large systems of differential equations. *Comput. Physics Comm. 66* (1991), 319–340.

[6] Dressner, L. *Similarity Solutions of Nonlinear Partial Differential Equations*. Pitman Publishing Co., Marshfield, MA, 1983.

[7] Eliseev, V. P., Fedorova, R. N., and Kornyak, V. V. A REDUCE program for determining point and contact Lie symmetries of differential equations. *Comput. Physics Comm. 36* (1985), 383–389.

[8] Head, A. K. LIE, a PC program for Lie analysis of diffferential equations. *Comput. Physics Comm. 77* (1993), 241–248.

[9] Hill, J. M. *Solution of Differential Equations by Means of One-Parameter Groups*. Pitman Publishing Co., Marshfield, MA, 1982.

[10] Ibragimov, N. H., Ed. *Symmetries, Exact Solution, and Conservation Laws*, vol. 1 of *Lie Group Analysis of Differential Equations*. CRC, Boca Raton, FL, 1994.

[11] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[12] Klamkin, M. S. On the transformation of a class of boundary value problems into initial value problems. *Amer. Math. Monthly 4* (1962), 43–47.

[13] Kumei, S., and Bluman, G. W. When nonlinear differential equations are equivalent to linear differential equations. *SIAM J. Appl. Math. 42*, 5 (October 1982), 1157–1173.

[14] Olver, P. J. *Applications of Lie Groups to Differential Equations*. No. 107 in Graduate Texts in Mathematics. Springer–Verlag, New York, 1986.

[15] Ovsiannikov, L. V. *Group Analysis of Differential Equations*. Academic Press, New York, 1982. Translated by W. F. Ames.

[16] Sattinger, D. H., and Weaver, O. L. *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*. Springer–Verlag, New York, 1986.

[17] Schwarz, F. A REDUCE package for determining Lie symmetries of ordinary and partial differential equations. *Comput. Physics Comm. 27* (1982), 179–186.

[18] Seshadri, R., and Na, T. Y. *Group Invariance in Engineering Boundary Value Problems*. Springer–Verlag, New York, 1985.

[19] Steinberg, S. Lie series and nonlinear differential equations. *J. Math. Anal. Appl. 101* (1984), 39–63.

[20] Stephani, H. *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press, New York, 1989. edited by M. MacCallum.

[21] Winternitz, P. Lie groups and solutions of nonlinear differential equations. In *Nonlinear Phenomena*, K. B. Wolf, Ed., no. 189 in Lecture Notes in Physics. Springer–Verlag, New York, 1983, pp. 263–331.

[22] Yosida, K. *Functional Analysis*. Springer–Verlag, New York, 1980.

[23] Zatsev, V. F. On discrete-group analysis of ordinary differential equations. *Soviet Math. Dokl. 37*, 2 (1988), 403–406.

83. Operational Calculus*

Applicable to Ordinary and partial differential equations.

Yields

A reformulation of the original differential equation.

Idea

It may sometimes be easier to solve a differential equation in a transformed space.

Procedure

Given an ordinary differential equation, transform it to a field of operators, solve the equation in that field, and then transform back. In this field, ordinary functions, generalized functions, and differential operators are all treated as objects in a single algebraic structure.

The operator field that is used has, among other elements, an identity operator ($\mathcal{I}$), a differentiation operator (often denoted by D or s) and an integration operator (often denoted by D^{-1}). The operator D, when applied to the operator corresponding to a function $f(t)$, results in

$$D\{f\} = \{f'\} + \{f(0)\}, . \tag{83.1}$$

The operator D^{-1}, when applied to the operator corresponding to a function $f(t)$ results in

$$D^{-1}\{f\} = \left\{ \int_0^t f(u)\, du \right\}.$$

The braces around the above expressions emphasize that they are operators in the field. In many applications, the operator D is formally treated as being a "large constant."

There are tables of formulae describing how operators interact in their quotient field. For example, because

$$\frac{\mathcal{I}}{D-\alpha} = \left\{ e^{\alpha t} \right\} \tag{83.2}$$

we can calculate

$$\begin{aligned} \frac{\mathcal{I}}{(D-\alpha)^2} &= \frac{\mathcal{I}}{(D-\alpha)} \frac{\mathcal{I}}{(D-\alpha)} \\ &= \left\{ e^{\alpha t} \right\} \left\{ e^{\alpha t} \right\} \\ &= \left\{ \int_0^t e^{\alpha u} e^{\alpha(t-u)}\, du \right\} \\ &= \left\{ t e^{\alpha t} \right\}, \end{aligned}$$

because the "product" of two operators is the operator corresponding to a convolution. The formula in equation (83.2) follows from equation (83.1) when $f(t) = e^{\alpha t}$, because

$$(D - \alpha)\left\{e^{\alpha t}\right\} = \left(\left\{\alpha e^{\alpha t}\right\}\right) + \{1\} - \alpha\left\{e^{\alpha t}\right\} = \mathcal{I}.$$

It is easy to represent generalized functions and non-continuous functions in the field. For example, a square wave of period $2c$ has the operator representation $\frac{\mathcal{I}}{D(\mathcal{I}+e^{-cD})}$.

Example 1

The following ordinary differential equation for $y(t)$

$$y'' + y = 0$$

has the operator representation

$$\left(D^2 + 1\right)\{y\} = 0 \tag{83.3}$$

or $D^2\left(1 + D^{-2}\right)\{y\} = 0$. By applying D^{-2} to the left of the above equation, we obtain

$$\begin{aligned}\left(1 + D^{-2}\right)\{y\} &= D^{-2}\{0\}\\ &= At + B,\end{aligned}$$

where A and B are arbitrary constants. This equation may be formally solved by "dividing" by the operator on the left and expanding terms. We find

$$\begin{aligned}\{y(t)\} &= \left\{\frac{1}{1 + D^{-2}}(At + B)\right\}\\ &= \left\{\left(1 - D^{-2} + D^{-4} - \cdots\right)(At + B)\right\} \qquad (83.4)\\ &= \left\{(At + B) + \left(-\frac{At^3}{6} - \frac{Bt^2}{2}\right) + \left(-\frac{At^5}{120} - \frac{Bt^4}{24}\right) + \cdots\right\}\\ &= \{A\sin t + B\cos t\}.\end{aligned}$$

Hence, $y(t) = A\sin t + B\cos t$.

Really, in this last calculation, there would be many more terms than those illustrated. For instance, when D^{-4} is applied to $(At + B)$, we obtain $\left(-\frac{At^5}{120} - \frac{Bt^4}{24}\right)$ plus some terms of the form $\left(C_1 t^3 + C_2 t^2 + C_3 t + C_4\right)$. When the form of the solution, with all these additional terms, is substituted into the defining equation (83.3), these additional constants turn out to be zero.

Example 2

Consider the constant coefficient linear ordinary differential equation for $z(t)$

$$z'' + 3z' + 2z = f(t),$$
$$z(0) = 1, \quad z'(0) = 0.$$

Because of the formula

$$z^{(n)} = D^n z - \left\{ z^{(n-1)}(0) + Dz^{(n-2)}(0) + \cdots + D^{n-1}z(0) \right\}$$

(which parallels the rule for Laplace transforms), the equation for $z(t)$ has the operator representation

$$\left[D^2 \{z\} - D \right] + 3\left[D\{z\} - \mathcal{I} \right] + 2\{z\} = \{f\}.$$

This operator equation can be manipulated into

$$\begin{aligned} \{z\} &= \frac{D + 3\mathcal{I}}{D^2 + 3D + 2} + \frac{\{f\}}{D^2 + 3D + 2} \\ &= \frac{2\mathcal{I}}{D + 1} - \frac{\mathcal{I}}{D + 2} + \left(\frac{\mathcal{I}}{D + 1} - \frac{\mathcal{I}}{D + 2} \right) \{f\} \\ &= \left\{ 2e^{-t} \right\} - \left\{ e^{-2t} \right\} + \left\{ e^{-t} - e^{-2t} \right\} \{f\}, \end{aligned}$$

and hence,

$$z(t) = 2e^{-t} - e^{-2t} + \int_0^t \left(e^{-u} - e^{-2u} \right) f(u)\, du,$$

which is the same result that would be obtained by use of Laplace transforms.

Notes

1. The operational calculus is also called the Heaviside calculus.
2. The operational calculus, at its simplest level, has a great similarity with Laplace transforms. One school of thought is that any integral transform creates an operational calculus.
3. It is sometimes difficult to justify the formal steps that are employed in using the operation calculus. One solution (see Erdélyi [3]) is to use a more precisely defined operator, such as the primary operator

$$\widehat{D}_\lambda f(t) = f(t) + \lambda \int_0^t e^{\lambda(t-\theta)} f(\theta)\, d\theta,$$

which has the inverse $\widehat{D}_\lambda^{-1} g(t) = g(t) - \lambda \int_0^t g(\theta)\, d\theta$.

4. Infinite order differential equations are often solved by techniques similar to those described above. For example, the ordinary differential equations $\left(\alpha \frac{d}{dx} + 1\right)^{-1} y + (\beta x - a)y = 0$ and $\left[\cosh\left(i \frac{d}{dx}\right) + H(x) - a\right] y = 0$ are infinite order differential equations for $y(x)$ (here $H(x)$ represents the step function). Recent results (as well as the solutions to the two above equations) may be found in Dimitrov [2].
5. The extension of this technique to partial differential equations is straightforward. Using D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial t}$, a partial differential equation can sometimes be written in the form $P(D, D')\{y\} = \{f\}$. The "inversion" process will then proceed in two steps. For example, to obtain a particular solution of $u_{xx} - 6u_{xt} + 9u_{tt} = 12x^2 + 36xt$, a calculation analogous to the one in equation (83.4) might proceed as follows:

$$\begin{aligned}
\{y\} &= \frac{1}{P(D, D')}\{f\} \\
&= \frac{1}{D^2 - 6DD' + 9D'^2}\left(12x^2 + 36xt\right) \\
&= \frac{1}{D^2}\left(1 - \frac{3D'}{D}\right)^{-2}\left(12x^2 + 36xt\right) \\
&= \frac{1}{D^2}\left(1 + 6\frac{D'}{D} + 27\frac{D'^2}{D^2} + \dots\right)\left(12x^2 + 36xt\right) \\
&= \frac{1}{D^2}\left(12x^2 + 36xt\right) + \frac{6}{D^3}\left(36x\right) \\
&= \left(x^4 + 6x^3 t\right) + \left(9x^4\right) = 10x^4 + 6x^3 t.
\end{aligned}$$

6. See Courant and Hilbert [1, Volume 2, pages 507–535] and Kaplan [5, pages 515–538].

References

[1] Courant, R., and Hilbert, D. *Methods of Mathematical Physics*. Interscience Publishers, Inc., New York, 1953.

[2] Dimitrov, H. D. On the solutions of some linear operator non-polynomial differential equations. *J. Phys. A: Math. Gen. 15* (1982), 367–379.

[3] Erdelyi, A. *Operational Calculus and Generalized Functions*. Holt, Rinehart and Winston, New York, 1962.

[4] Glaeske, H.-J. Operational properties of a generalized Hermite transformation. *Aequationes Mathematicae 32* (1987), 155–170.

[5] Kaplan, W. *Operational Methods for Linear Systems*. Addison–Wesley Publishing Co., Reading, MA, 1962.

[6] Mikusinski, J. *Operational Calculus*, fifth ed. Pergamon Press, New York, 1959.

[7] Shtokalo, I. Z. *Operational Calculus*. Pergamon Press, New York, 1976. translated by V. Kumar.

[8] Yosida, K. *Operational Calculus*. Springer–Verlag, New York, 1984.

84. Pfaffian Differential Equations

Applicable to Pfaffian differential equations.

Yields

Knowledge of whether the equation is integrable.

Idea

Pfaffian differential equations are partial differential equations of the form

$$\mathbf{f}(\mathbf{x}) \cdot d\mathbf{x} = \sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n)\, dx_i = 0. \tag{84.1}$$

For equations of this type,

- If $n = 3$, then a necessary and sufficient condition that equation (84.1) be integrable is that

  $$\mathbf{f}(\mathbf{x}) \cdot \operatorname{curl} \mathbf{f}(\mathbf{x}) = 0.$$

- If $n \geq 4$, then a necessary and sufficient condition that equation (84.1) be integrable is that

  $$F_p \left[\frac{\partial F_r}{\partial x_q} - \frac{\partial F_q}{\partial x_r} \right] + F_q \left[\frac{\partial F_p}{\partial x_r} - \frac{\partial F_r}{\partial x_p} \right] + F_r \left[\frac{\partial F_q}{\partial x_p} - \frac{\partial F_p}{\partial x_q} \right] = 0,$$

  where p, q, and r are any three of the integers $1, 2, 3, \ldots, n$.

There exist a number of techniques for integrating Pfaffian equations.

Example

If we have the equation

$$(y^2 + yz)\, dx + (xz + z^2)\, dy + (y^2 - xy)\, dz = 0, \tag{84.2}$$

then we identify $n = 3$ and

$$\mathbf{f}(\mathbf{x}) = (y^2 + yz, xz + z^2, y^2 - xy),$$

so that

$$\operatorname{curl} \mathbf{f}(\mathbf{x}) = \nabla \times \mathbf{f}(\mathbf{x}) = 2(-x + y - z, y, -y).$$

Therefore $\mathbf{f}(\mathbf{x}) \cdot \operatorname{curl} \mathbf{f}(\mathbf{x}) = 0$, and there exists a solution to equation (84.2). The solution is, in fact, given by $y(x + z) = C(y + z)$, where C is any constant.

Procedure 1

If a Pfaffian equation is integrable, then there exists an integrating factor μ such that

$$d\phi = \sum_{i=1}^{n} \mu F_i \, dx_i.$$

By appropriate manipulations of equation (84.1), it may be shown that μ satisfies any of the equations

$$-\frac{d\mu}{\mu} = \sum_{j=1}^{n} \frac{1}{F_i} \left[\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} \right] dx_j, \tag{84.3}$$

for $i = 1, 2, \dots, n$. Any one of these equations may be solved to determine an integrating factor. Alternatively, if two integrating factors can be found, say μ and ν, then a solution to equation (84.1) is given by $\mu/\nu =$ constant.

Example 1

The Pfaffian differential equation

$$y(x^2 - y^2 - yz)\, dx + x(y^2 - x^2 - xz)\, dy + xy(x + y)\, dz = 0 \tag{84.4}$$

can be shown to pass the integrability requirements. Substituting into equation (84.3) results in the three separate equations

$$\begin{aligned} -\frac{d\mu}{\mu} &= \frac{2(x-y)(2x+2y+z)}{y(x^2 - y^2 - yz)}\, dy - \frac{2(x+y)}{x^2 - y^2 - yz}\, dz, \\ &= -\frac{2(x-y)(2x+2y+z)}{x(y^2 - x^2 - xz)}\, dx - \frac{2(x+y)}{y^2 - x^2 - xz}\, dz, \\ &= 2\left(\frac{dx}{x} + \frac{dy}{y} \right), \end{aligned} \tag{84.5}$$

for $j = 1, 2, 3$. The last equation in (84.5) can be integrated to determine $\mu = 1/(xy)^2$. Hence, multiplying equation (84.4) by $1/(xy)^2$ results in

$$d\phi = \left(\frac{x^2 - y^2 - yz}{x^2 y} \right) dx + \left(\frac{y^2 - x^2 - xz}{xy^2} \right) dy + \left(\frac{x+y}{xy} \right) dz,$$

which can be integrated to yield

$$\phi = \frac{x}{y} + \frac{y}{x} + \left(\frac{x+y}{xy} \right) z + C,$$

where C is an arbitrary constant.

Procedure 2

If an integrable Pfaffian differential equation is of the form $Pdx+Qdy+Rdz=0$, where P, Q, and R are homogeneous functions of the same degree, then a solution may be found. First, define $Z = Px + Qy + Rz$. Then, form

$$\frac{Pdx + Qdy + Rdz - dZ}{Z} + \frac{dZ}{Z} = 0 \tag{84.6}$$

and integrate (we have addressed only the case of $Z \neq 0$, although there are special techniques that can be used when $Z = 0$).

Example 2

Given the Pfaffian equation

$$(yz + z^2)\,dx - xz\,dy + xy\,dz = 0,$$

we define $Z = xz(y + z)$. Forming equation (84.6) we obtain

$$\frac{dZ}{Z} - \frac{2(dy + dz)}{y + z} = 0,$$

which can be immediately integrated to yield $Z = C(y + z)^2$ or $xz = C(y + z)$, where C is an arbitrary constant.

Procedure 3

The Pfaffian differential equation $Pdx + Qdy + Rdz = 0$ can sometimes be solved by taking one variable, say z, as a constant. Then, the solution of $Pdx + Qdy = 0$ (because z = constant means that $dz = 0$) will be given by $u(x, y)$ = constant.

We take the "constant" in this last expression to be $f(z)$. Differentiating $u(x, y) = f(z)$ and comparing to the original equation, we may sometimes obtain an ordinary differential equation for $f(z)$.

Example 3

Given the Pfaffian equation

$$2x\,dx + dy + (1 + 2z^2 + 2yz + 2x^2 z)\,dz = 0,$$

we treat z as a constant to obtain $2x\,dx + dy = 0$, which has the solution $x^2 + y$ = constant = $f(z)$. This can be differentiated to obtain

$$2x\,dx + dy + f'(z)\,dz = 0.$$

Comparing this to the original equation, we find that $f(z)$ satisfies the ordinary differential equation: $f' = 1 + 2z^2 + 2zf$. Solving this equation to obtain $f(z) = Ce^{-z^2} - z$, where C is an arbitrary constant, we find the solution to the original equation to be

$$x^2 + y + z = Ce^{-z^2}.$$

Notes

1. Another name for a Pfaffian differential equation is a *total differential equation*.
2. One way to solve Pfaffian differential equations in three dimensions is by the observation: if $\operatorname{curl}\mathbf{f}(\mathbf{x}) = 0$, then $\mathbf{f}(\mathbf{x})$ must be the gradient of a scalar. Hence, the set of partial differential equations

$$f_i(\mathbf{x}) = \frac{\partial v(\mathbf{x})}{\partial x_i}, \quad \text{for } i = 1, \dots, n,$$

may be solvable for $v(\mathbf{x})$. The solution to equation (84.1) would then be given implicitly by $v(\mathbf{x}) = \text{constant}$.
3. If the Pfaffian differential equation is of the form $\sum_{i=1}^{n} f_i(x_i)\, dx_i = 0$, then the *integral surfaces* are defined by $\sum_{i=1}^{n} \int f_i(x_i)\, dx_i = C$, where C is an arbitrary constant.
4. Sometimes a Pfaffian differential equation can be reduced to a system of ordinary differential equations. One such procedure is called Mayer's method. See Carathéodory [1, pages 121–133] for details.
5. Given a system of m Pfaffian differential equations in m dependent variables $\{z_j \mid j = 1, 2, \dots, m\}$ and n independent variables $\{x_k \mid k = 1, 2, \dots, n\}$

$$dz_j = \sum_{k=1}^{n} P_{jk}(\mathbf{x}, \mathbf{z}) dx_k, \qquad j = 1, 2, \dots, m,$$

the condition for complete integrability is given by

$$\frac{\partial P_{jk}}{\partial x_l} + \sum_{i=1}^{m} \frac{\partial P_{jk}}{\partial z_i} P_{il} = \frac{\partial P_{jl}}{\partial x_k} + \sum_{i=1}^{m} \frac{\partial P_{jl}}{\partial z_i} P_{ik},$$

for $j = 1, 2, \dots, m$ and $k, l = 1, 2, \dots, n$. See Iyanaga and Kawada [6] for details on how this system may be solved.
6. Using the notation of exterior calculus, a total differential equation is an equation of the form $\omega = 0$, where ω is a differential 1-form, also called a Pfaffian form, $\sum_{i=1}^{n} a_i(\mathbf{x})\, dx_i$ on a manifold. See Zwillinger [9] for details.
7. See Ford [2, pages 135–141], Ince [5, pages 52–59], Moon and Spencer [7, pages 23–27], and Sneddon [8, pages 18–33].

References

[1] Caratheodory, C. *Calculus of Variations and Partial Differential Equations of the First Order.* Holden–Day, Inc., San Francisco, CA, 1965.

[2] Ford, L. R. *Differential Equations.* McGraw–Hill Book Company, New York, 1955.

[3] Griffiths, P. A., and Jensen, G. R. *Differential Systems and Isometric Imbeddings.* Princeton University Press, Princeton, NJ, 1987.

[4] Haack, E. R., and Wendland, W. N. *Lectures on Partial and Pfaffian Differential Equations.* Pergamon Press, New York, 1972. Translated by E. R. Dawson and W. N. Everitt.

[5] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[6] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics.* MIT Press, Cambridge, MA, 1980.

[7] Moon, P., and Spencer, D. E. *Partial Differential Equations.* D. C. Heath and Co., Lexington, MA, 1969.

[8] Sneddon, I. N. *Elements of Partial Differential Equations.* McGraw–Hill Book Company, New York, 1957.

[9] Zwillinger, D., Ed. *Standard Mathematical Tables and Formulae*, 30 ed. CRC, Boca Raton, FL, 1995.

85. Reduction of Order

Applicable to Linear ordinary differential equations.

Yields

A lower order differential equation, if any non-trivial solution of the homogeneous equation is known.

Idea

For an nth order linear ordinary differential equation, any non-trivial solution of the homogeneous equation can be used to reduce the order of the equation by 1. For the special case of second order linear differential equations, knowing any solution of the homogeneous equation allows the general solution to be found.

Procedure

We choose to illustrate the method for second order equations. If we have the general second order linear ordinary differential equation

$$y'' + p(x)y' + q(x)y = r(x), \tag{85.1}$$

let $z(x)$ be any non-trivial solution to the corresponding homogeneous equation; that is, $z(x)$ satisfies

$$z'' + p(x)z' + q(x)z = 0. \tag{85.2}$$

If we look for a solution of equation (85.1) in the form of $y(x) = z(x)v(x)$, then we can obtain a solvable equation for $v(x)$. Substituting $y(x) = z(x)v(x)$ into equation (85.1) yields

$$zv'' + (2z' + pz)v' + (z'' + pz' + qz)v = r. \tag{85.3}$$

Because $z(x)$ satisfies equation (85.2), equation (85.3) becomes

$$zv'' + (2z' + pz)v' = r. \tag{85.4}$$

If we now let $w(x) = v'(x)$, then equation (85.4) becomes a first order linear ordinary differential equation for $w(x)$. It can be solved by the use of integrating factors (see page 356).

Example

Given the second order linear differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 3, \tag{85.5}$$

we recognize that $z(x) = x$ is a solution of the homogeneous equation. Equation (85.4) becomes

$$x\frac{d^2v}{dx^2} + 2(1 - x^2)\frac{dv}{dx} = 3.$$

This equation may be solved by recognizing that it is a linear first order ordinary differential equation in the unknown dv/dx. Hence, integrating factors can be used to find dv/dx. After dv/dx is determined, it can be integrated directly to yield

$$v(x) = \frac{3}{2x} + A\int^x \frac{e^{t^2}}{t^2}\,dt + B,$$

where A and B are arbitrary constants. Using the relationship $y(x) = z(x)v(x)$, the general solution of equation (85.5) is

$$y(x) = \frac{3}{2} + Ax\int^x \frac{e^{t^2}}{t^2}\,dt + Bx.$$

Notes

1. The general nth order linear ordinary differential equation is treated in Finizio and Ladas [2, pages 108–116] and Rainville and Bedient [3, pages 127–129]. The general result is that

   If $z(x)$ is a solution of the linear homogeneous equation

   $$z^{(n)} + p_1(x)z^{(n-1)} + \cdots + p_n(x)z = 0 \tag{85.6}$$

   and if $y(x) = v(x)z(x)$, then the equation

   $$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = r(x) \tag{85.7}$$

   transforms into

   $$v^{(n)} + q_1(x)v^{(n-1)} + \cdots + q_{n-1}v' = r(x).$$

   This last equation may be reduced in order by defining $w(x) = v'(x)$.

2. More generally, if $\{z_1(x), \ldots, z_p(x)\}$ are linearly independent solutions of equation (85.6), then the substitution

   $$y(x) = \begin{vmatrix} z_1 & \cdots & z_p & v \\ z_1' & \cdots & z_p' & v' \\ \vdots & & \vdots & \vdots \\ z_1^{(p)} & \cdots & z_p^{(p)} & v^{(p)} \end{vmatrix}$$

   reduces equation (85.7) to a linear ordinary differential equation of order $n - p$ for $v(x)$.
3. See also Boyce and DiPrima [1, section 3.4, pages 127–131].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Finizio, N., and Ladas, G. *Ordinary Differential Equations with Modern Applications*. Wadsworth Publishing Company, Belmont, CA, 1982.

[3] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations*. The MacMillan Company, New York, 1964.

86. Riccati Equations

Applicable to Ordinary differential equations of the form $y' = a(x)y^2 + b(x)y + c(x)$.

Yields

A reformulation as a linear second order ordinary differential equation, or a second solution if one solution is already known.

Idea

A change of dependent variable can transform a Riccati equation to a linear second order ordinary differential equation. Also, if one solution to a Riccati equation is known, then the other solution can be written down explicitly.

Procedure 1

Suppose we have the Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x). \tag{86.1}$$

If the dependent variable in equation (86.1) is changed from $y(x)$ to $w(x)$ by

$$y(x) = -\frac{w'(x)}{w(x)}\frac{1}{a(x)}, \tag{86.2}$$

then we obtain the equivalent second order linear ordinary differential equation

$$w'' - \left[\frac{a'(x)}{a(x)} + b(x)\right] w' + a(x)c(x)w = 0. \tag{86.3}$$

It might be easier to solve equation (86.3) than to solve equation (86.1) by other means.

Procedure 2

Suppose we have the Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x), \tag{86.4}$$

and suppose further that one solution to this equation is already known to us, say, $y(x) = z(x)$. If $y(x) = z(x) + u(x)$ is substituted in equation (86.4), then the solvable Bernoulli equation

$$u' = (b + 2az)u + au^2$$

is obtained for $u(x)$. To solve this equation, the new dependent variable $v(x) = 1/u(x)$ should be introduced and then integrating factors should be used (see pages 235 and 356).

Example 1

Suppose we have the Riccati equation

$$y' = e^x y^2 - y + e^{-x} \tag{86.5}$$

to solve. By identifying $a(x) = e^x$, $b(x) = -1$ and $c(x) = e^{-x}$, the change of variables in equation (86.2) becomes

$$y(x) = -\frac{w'(x)}{w(x)} e^{-x}, \tag{86.6}$$

so that equation (86.5) becomes $w'' + w = 0$, which could have been obtained directly from equation (86.3). The solution to this equation is $w(x) = A \sin x + B \cos x$, where A and B are arbitrary constants. Using this solution in equation (86.6) leads to the general solution of equation (86.5)

$$y(x) = -e^{-x} \left(\frac{A \cos x - B \sin x}{A \sin x + B \cos x} \right).$$

There should be only one arbitrary constant in the solution to equation (86.5), because it is a first order ordinary differential equation. In fact, this last equation may be written as

$$y(x) = -e^{-x} \left(\frac{\cos x - C \sin x}{\sin x + C \cos x} \right),$$

where we have defined $C = B/A$ (and assumed $A \neq 0$).

Example 2

Suppose we have the equation

$$y' = y^2 - xy + 1 \tag{86.7}$$

to solve. A solution to equation (86.7), obtained by inspection, is $y(x) = x$. We utilize this solution in forming

$$y(x) = x + u(x), \tag{86.8}$$

and then (using equation (86.8) in equation (86.7)) the equation $u' = u^2 + xu$ is obtained. This Bernoulli equation has the solution $u(x) = \frac{e^{x^2/2}}{A - \int_0^x e^{t^2/2} dt}$, where A is an arbitrary constant. Thus, the second solution to equation (86.7) is

$$y(x) = x + \frac{e^{x^2/2}}{A - \int_0^x e^{t^2/2} dt}.$$

Notes

1. The transformation in equation (86.2) is known as the *Riccati transformation.*
2. The identity

$$\left(\frac{d}{dx} - q(x)\right)\left(\frac{d}{dx} + q(x)\right)u = u'' + \left(q' - q^2\right)u \tag{86.9}$$

   shows that the differential equation $u'' + p(x)u = 0$ can be factored into the form of equation (86.9) if $q' - q^2 = p$, which is a Riccati equation.
3. See Bender and Orszag [1, Section 1.6], Boyce and DiPrima [2, pages 93–94 and 142–143], Goldstein and Braun [3, pages 45–36], Ince [4, pages 23–25 and 295], and Simmons [6, pages 62–63].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations.* U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[4] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[5] Reid, W. T. *Riccati Differential Equations.* Academic Press, New York, 1972.

[6] Simmons, G. F. *Differential Equations with Applications and Historical Notes.* McGraw–Hill Book Company, New York, 1972.

87. Matrix Riccati Equations

Applicable to Systems of quadratic ordinary differential equations.

Yields

An exact solution.

Idea

There is an exact solution available for matrix Riccati differential equations. If a given system of ordinary differential equations can be put in the form of a matrix Riccati equation, then the solution can be found.

Procedure

If $Z(t)$, $A(t)$, and $K(t)$ are all $N \times N$ matrices, then we can use the following theorem:

If $Z(t)$ satisfies the following matrix Riccati differential equation

$$\frac{d}{dt}Z = ZAZ + KZ + ZK^{\mathrm{T}}, \qquad Z(t=0) = Z_0, \tag{87.1}$$

then $Z(t)$ is explicitly given by

$$Z(t) = Q(t)\left[Z_0^{-1} - \int_0^t Q^{\mathrm{T}}(s)A(s)Q(s)\,ds\right]^{-1} Q^{\mathrm{T}}(t), \tag{87.2}$$

where $Q(t)$ is defined to be the solution of

$$\frac{d}{dt}Q(t) = K(t)Q(t), \qquad Q(t=0) = I, \tag{87.3}$$

I is the $N \times N$ identity matrix, and the required matrix inverses are assumed to exist.

If a given system of ordinary differential equations can be placed in the form of equation (87.1), then the solution can be found from equation (87.2).

Example

Suppose we wish to solve the following system of coupled differential equations for $x(t)$ and $y(t)$

$$\begin{aligned} \frac{dx}{dt} &= a(t)(y^2 - x^2) + 2b(t)xy + 2cx, \\ \frac{dy}{dt} &= b(t)(y^2 - x^2) - 2a(t)xy - 2cy, \end{aligned} \tag{87.4}$$

with $x(0) = D$ and $y(0) = E$. If we form the matrices $Z = \left[\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right]$, $K = \left[\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right]$, $Z_0 = \left[\begin{smallmatrix} D & E \\ E & -D \end{smallmatrix}\right]$, and $A = \left[\begin{smallmatrix} -a(t) & b(t) \\ b(t) & a(t) \end{smallmatrix}\right]$, then the equations in (87.4) are the same as those in equation (87.1). The solution for $Q(t)$ from equation (87.3) is $Q(t) = e^{ct} I$. Therefore, the solution for Z is

$$Z(t) = e^{2ct} \left[Z_0^{-1} - \int_0^t e^{2cs} A(s)\, ds \right]^{-1}.$$

If we define

$$\begin{aligned} \alpha(t) &= \int_0^t e^{2cs} a(s)\, ds, \\ \beta(t) &= \int_0^t e^{2cs} b(s)\, ds, \end{aligned}$$

then, by equating the corresponding entries of equation (87.2), we can find $\{x(t), y(t)\}$ in terms of $\{\alpha(t), \beta(t)\}$. We have

$$\begin{aligned} x(t) &= e^{2ct} \left[\alpha(t)(E^2 + D^2) + D \right] / \Delta, \\ y(t) &= e^{2ct} \left[\beta(t)(E^2 + D^2) + D \right] / \Delta, \end{aligned}$$

where $\Delta = \Delta(x)$ is defined by

$$\Delta(x) = \left[\beta^2(t) + \alpha^2(t) \right] \left[E^2 + D^2 \right] - 2\beta(t)E + 2\alpha(t)D + 1.$$

Notes

1. Matrix Riccati equations arise naturally in a number of physical settings. For example, the gains in a Kalman–Bucy filter satisfy a matrix Riccati equation. Also, the deflection of a beam can be described by such equations. They also appear quite often in the context of control theory (see Jodar and Abou-Kandil [3]) and invariant embedding solutions (see page 747).
2. Kerner [7] shows that nonlinear differential systems of arbitrary order

   $$\dot{\zeta}_i = X_i(\zeta_1, \zeta_2, \ldots, \zeta_k, t), \qquad \text{for } i = 1, 2, \ldots, k,$$

   may often be reduced to *Riccati systems*

   $$\begin{aligned} &\dot{x}_i = A_i + B_{i\alpha} x_\alpha + C_{i\alpha\beta} x_\alpha x_\beta, \\ &\text{for } i = 1, 2, \ldots, n, \qquad n \geq k, \qquad \text{and } A, B, C \text{ constant}, \end{aligned}$$

   and then to *elemental Riccati systems*

   $$\dot{z}_i = E_{i\alpha\beta} z_\alpha z_\beta, \qquad \text{for } i = 1, 2, \ldots, p, \qquad p(n) > n,$$

   where each $E_{i\alpha\beta}$ equals 0 or 1. His examples include ordinary differential equation systems that contain exponential functions and elliptic functions.

3. Celletti and Francoise [2] study matrix differential equations of the form $\dot{X} = Y$, $\dot{Y} = -h(X)h'(X)$, where h is a polynomial function.
4. Jodar and Navarro [4] write the solutions of the matrix differential equation $X^{(p)} + A_{p-1}X^{(p-1)} + \cdots + A_0X = 0$ in terms of the matrix algebraic equation $Y^p + A_{p-1}Y^{p-1} + \cdots + A_0 = 0$.

References

[1] Bittanti, S., Laub, A. J., and Willems, J. C., Eds. *The Riccati Equation*. Springer–Verlag, New York, 1991.

[2] Celletti, A., and Francoise, J. P. Matrix-second order differential equations and chaotic Hamiltonian systems. *Z.-Angew.-Math.-Phys. 40* (1989), 925–930.

[3] Jodar, L., and Abou-Kandil, H. A resolution method for Riccati differential systems coupled in their quadratic terms. *SIAM J. Appl. Math. 19*, 6 (November 1988), 1425–1430.

[4] Jodar, L., and Navarro, E. On complete set of solutions for polynomial matrix equations. *Appl. Math. Lett. 3*, 1 (1990), 15–18.

[5] Jones, R. A. Existence theorems for the matrix Riccati equation *W'+WP(t)W+Q(t)=0*. *Int. J. Math.* & *Math. Sci. 1* (1978), 13–19.

[6] Kenney, C. S., and Leipnik, R. B. Numerical integration of the differential matrix Riccati equation. *IEEE Trans. Automat. Control*, 10 (1985), 962–970.

[7] Kerner, E. H. Universal formats for nonlinear ordinary differential equations. *J. Math. Physics 22*, 7 (July 1981), 1366–1371.

[8] Murty, K. N., Prasad, K. R., and Srinivas, M. A. S. Upper and lower bounds for the solution of the general matrix Riccati differential equations. *J. Math. Anal. Appl. 147*, 1 (1990), 12–21.

[9] Rand, D. W., and Winternitz, P. Nonlinear superposition principles: A new numerical method for solving matrix Riccati equations. *Comput. Physics Comm. 33* (1984), 305–328.

[10] Razzaghi, M. A computational solution for the matrix Riccati equation using Laplace transforms. *Int. J. Comp. Math. 11* (1982), 297–304.

[11] Reid, W. T. Solutions of a Riccati matrix differential equation as functions of initial values. *J. Math. Mech. 8* (1959), 221–230.

[12] Reid, W. T. *Riccati Differential Equations*. Academic Press, New York, 1972.

[13] Wilcox, R. M., and Harten, L. P. MACSYMA-generated closed-form solutions to some matrix Riccati equations. *Appl. Math. and Comp. 14* (1984), 149–166.

88. Scale Invariant Equations

Applicable to

Ordinary differential equations of a certain form.

Yields

An equidimensional-in-x ordinary differential equation of the same order (which can then be reduced to an ordinary differential equation of lower order).

Idea

A scale invariant equation is one in which the equation is unchanged when x and y are scaled in a certain way. When an equation is scale invariant, we can convert the equation into an equidimensional-in-x ordinary differential equation of the same order by a change of the dependent variable. This equidimensional-in-x ordinary differential equation can then be changed into an autonomous equation of lower order.

Procedure

A scale invariant equation is one that is left invariant under the transformation $\{x \to ax, y \to a^p y\}$, where a and p are constants. That is, if the original equation is an equation for $y(x)$ and the x variable is replaced by the variable ax' and the y variable is replaced by the variable $a^p y'$, then the new equation (in terms of y' and x') will be identical to the original equation (which is in terms of y and x). The way to determine the value of p is to change variables and then see what value of p leaves the equation unchanged.

A scale invariant equation can be converted to an equidimensional-in-x equation by the substitution for y

$$y(x) = x^p u(x). \tag{88.1}$$

By the techniques on page 275, this equidimensional-in-x equation may then be made autonomous, and then (after another transformation) the order of the equation can be reduced.

Example

Suppose we have the nonlinear second order ordinary differential equation

$$x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} = \frac{1}{y^3 x^4}. \tag{88.2}$$

To determine if this equation is scale invariant, and if so, what the value of p is, we substitute ax' for x and $a^p y'$ for y to obtain

$$(ax')^2 \frac{d^2(a^p y')}{d(ax')^2} + 3(ax') \frac{d(a^p y')}{d(ax')} = \frac{1}{(a^p y')^3 (ax')^4}$$

or

$$a^p x'^2 \frac{d^2 y'}{dx'^2} + 3a^p x' \frac{dy'}{dx'} = a^{(-3p-4)} \frac{1}{y'^3 x'^4}. \tag{88.3}$$

Hence, if we choose p so that $p = -3p - 4$, then the form of equation (88.3) will be the same as the form of equation (88.2). So the equation is scale invariant, with the value $p = -1$. To make this equation equidimensional-in-x, we change variables by equation (88.1): $y(x) = u(x)/x$. Using this change of variables in equation (88.2) produces

$$x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} - u = \frac{1}{u^3}. \tag{88.4}$$

Equation (88.4) is equidimensional-in-x, so we use the substitution $x = e^t$ (see page 275) for

$$\frac{d^2 u}{dt^2} - u = \frac{1}{u^3}. \tag{88.5}$$

Equation (88.5) is autonomous, so we change the independent variable by $v(u) = u'(t)$ (see page 230) for

$$v \frac{dv}{du} - u = \frac{1}{u^3}. \tag{88.6}$$

The solution of equation (88.6) can be found by separating variables (see page 487)

$$v(u) = \pm \sqrt{A - u^2 - \frac{1}{u^2}},$$

where A is an arbitrary constant. To find $u(t)$, we must now solve

$$\frac{du}{dt} = v(u) = \pm \sqrt{A - u^2 - \frac{1}{u^2}}. \tag{88.7}$$

Equation (88.7) is a separable equation whose solution is

$$u(t) = \pm \sqrt{\cosh B + \sinh B \sin(2t + C)},$$

where B and C are arbitrary constants. The last step is to recall that $y(x) = u(x)/x$ and that $x = e^t$. The final solution is therefore

$$y(x) = \pm \frac{1}{x} \sqrt{\cosh B + \sinh B \sin(2 \log x + C)}.$$

Notes

1. This method is derivable from Lie group methods (see page 366). The infinitesimal operator in this case is given by $U = x\frac{\partial}{\partial x} + py\frac{\partial}{\partial y}$.
2. A special case of this method (when $p = 1$) is the method for homogeneous equations (see page 327).
3. Euler equations (see page 281) are scale invariant equations for any value of the parameter p.
4. Scale invariant equations are also called *isobaric equations*.
5. In Rosen's paper [3], a change of variable is proposed, different from the one presented above, that often allows parametric solutions to be obtained.
6. See also Bender and Orszag [1, pages 25–26] and Goldstein and Braun [2, pages 81–84].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations*. U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[3] Rosen, G. Alternative integration procedure for scale-invariant ordinary differential equations. *Int. J. Math. & Math. Sci. 2* (1979), 143–145.

89. Separable Equations

Applicable to

First order ordinary differential equations.

Yields

An exact solution, often implicit.

Idea

First order ordinary differential equations can be solved directly if the forcing term factors into a term involving only the independent variable and a term involving only the dependent variable.

Procedure

Given an equation of the form

$$\frac{dy}{dx} = f(y)g(x), \tag{89.1}$$

both sides can be formally multiplied by $dx/f(y)$ and then integrated to obtain

$$\int \frac{dy}{f(y)} = \int g(x)\,dx. \tag{89.2}$$

The evaluation of equation (89.2) requires only that two integrals be evaluated. An arbitrary constant of integration must be included to obtain the most general solution of equation (89.1).

Example

Suppose we have the equation

$$\frac{dy}{dx} = \frac{9x^8+1}{y^2+1}$$

to solve. Multiplying both sides of equation (89) by $(y^2+1)\,dx$ and then integrating results in

$$\int (y^2+1)\,dy = \int (9x^8+1)\,dx.$$

Evaluating the integrals yields

$$\frac{y^3}{3} + y = x^9 + x + C,$$

where C is an arbitrary constant.

Notes

1. The solution obtained by this method will generally be implicit.
2. The formal procedure of multiplying equation (89.1) by $dx/f(y)$ can be rigorously shown to give the correct answer.
3. See Boyce and DiPrima [1, pages 37–42], Ince [2, pages 17–18], and Simmons [3, pages 35–36].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[3] Simmons, G. F. *Differential Equations with Applications and Historical Notes.* McGraw–Hill Book Company, New York, 1972.

90. Series Solution*

Applicable to Homogeneous linear ordinary differential equations, most frequently second order differential equations.

Yields

An infinite series expansion of the two independent solutions.

Idea

If an infinite series is substituted into a linear equation, the different coefficients may be matched to obtain recurrences for the coefficients of the series. Solving these recurrences results in an explicit solution.

Procedure

Given a homogeneous linear second order ordinary differential equation in the form

$$y'' + P(x)y' + Q(x)y = 0, \tag{90.1}$$

we search for a series solution around the point $x = 0$. There are four different cases to consider.

Clearly, an expansion about any other point, x_0, could be determined by changing the independent variable to $t = x - x_0$ and then analyzing the resulting equation near $t = 0$.

1. If $x = 0$ is an ordinary point of equation (90.1) (the definitions of ordinary points and singular points are given on page 11) then we may assume that $P(x)$ and $Q(x)$ have the known Taylor expansions

$$P(x) = \sum_{n=0}^{\infty} P_n x^n, \qquad Q(x) = \sum_{n=0}^{\infty} Q_n x^n, \tag{90.2}$$

in the region $|x| < \rho$, where ρ represents the minimum of the radii of convergence of the two series in equation (90.2). In this case, equation (90.1) will have two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{90.3}$$

2. Alternately, if $x = 0$ is a regular singular point of equation (90.1) then we may assume that $P(x)$ and $Q(x)$ have the known expansions

$$P(x) = \sum_{n=-1}^{\infty} P_n x^n, \qquad Q(x) = \sum_{n=-2}^{\infty} Q_n x^n, \tag{90.4}$$

in the region $|x| < \rho$. After determining the expansions in equation (90.4), we need to determine the roots to the *indicial equation*

$$\alpha^2 + \alpha(P_{-1} - 1) + Q_{-2} = 0, \tag{90.5}$$

which is obtained by utilizing $y = x^\alpha$ in equation (90.1), along with the expansions in equation (90.4), and then determining the coefficient of the lowest order term. The two roots of this equation are called *the exponents of the singularity.* There are now several cases, depending on the values of the exponents of the singularity:

(a) If $\alpha_1 \neq \alpha_2$ and $\alpha_1 - \alpha_2$ is not equal to an integer, then equation (90.1) will have two linearly independent solutions in the forms

$$\begin{aligned} y_1(x) &= |x|^{\alpha_1}\left(1 + \sum_{n=1}^{\infty} b_n x^n\right), \\ y_2(x) &= |x|^{\alpha_2}\left(1 + \sum_{n=1}^{\infty} c_n x^n\right). \end{aligned} \tag{90.6}$$

(b) If $\alpha_1 = \alpha_2$, then (calling $\alpha = \alpha_1$) equation (90.1) will have two linearly independent solutions in the forms

$$\begin{aligned} y_1(x) &= |x|^{\alpha}\left(1 + \sum_{n=1}^{\infty} d_n x^n\right), \\ y_2(x) &= y_1(x)\log|x| + |x|^{\alpha}\sum_{n=0}^{\infty} e_n x^n. \end{aligned} \tag{90.7}$$

(c) If $\alpha_1 = \alpha_2 + M$, where M is an integer greater than 0, then equation (90.1) will have two linearly independent solutions in the forms

$$\begin{aligned} y_1(x) &= |x|^{\alpha_1}\left(1 + \sum_{n=1}^{\infty} f_n x^n\right), \\ y_2(x) &= h y_1(x)\log|x| + |x|^{\alpha_2}\sum_{n=0}^{\infty} g_n x^n, \end{aligned} \tag{90.8}$$

where the parameter h may be equal to zero.

The procedure in each of the four cases is the same: Substitute the given forms (i.e., the expansions in equation (90.3), (90.6), (90.7), or (90.8)) into the original equation (90.1) and equate the coefficients of the x^j and $x^j \log x$ terms for different values of j. This will yield recurrence relations for the unknown coefficients. Solving these recurrence relations will determine the solution.

In the case of an ordinary point, there will be two unknown coefficients that parameterize the series solutions in equation (90.3). These two coefficients will generate the two linearly independent solutions of equation (90.1).

Example 1

Given the equation

$$y'' + y = 0, \tag{90.9}$$

we easily see that $x = 0$ is an ordinary point. Using equation (90.3) in equation (90.9) we find

$$(2a_2 + a_0) + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + \dots + \left[(n+1)(n+2)a_{n+2} + a_n\right] x^n + \dots = 0.$$

Hence, we must have $a_{n+2} = -\frac{a_n}{(n+1)(n+2)}$. Iterating this relation we find

$$a_{2m} = (-1)^m \frac{1}{(2m)!}, \qquad a_{2m+1} = (-1)^m \frac{1}{(2m+1)!}. \tag{90.10}$$

Hence, using equation (90.10) in equation (90.3),

$$y(x) = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right). \tag{90.11}$$

Of course, the exact solution to equation (90.9) is $y(x) = a_0 \cos x + a_1 \sin x$, which is what equation (90.11) has reproduced.

Example 2

Given the equation

$$y'' + \frac{1+2x}{2x} y' - \frac{1}{2x^2} y = 0, \tag{90.12}$$

we easily see that $x = 0$ is a regular singular point. In this case we have (see equation (90.4)) $P_{-1} = \frac{1}{2}$, $Q_{-2} = -\frac{1}{2}$. Therefore, the indicial equation (from equation (90.5)) becomes

$$\alpha^2 - \frac{1}{2}\alpha - \frac{1}{2} = (\alpha - 1)\left(\alpha - \frac{1}{2}\right) = 0.$$

Because the roots $\alpha_1 = 1$, $\alpha_2 = -\frac{1}{2}$ are unequal and do not differ by an integer, then we have case 2 (a). Using equation (90.6) in equation (90.12), for $\alpha_1 = 1$, and equating powers of x we readily find that

$$\sum_{n\geq 1} (n+1)(n) b_n x^{n-1} + \frac{1+2x}{2x}\left(1 + \sum_{n\geq 1} b_n x^n\right) - \frac{1}{2x^2}\left(x + \sum_{n\geq 1} b_n x^{n+1}\right) = 0.$$

Equating the coefficients for different powers of x, we find that

$$b_1 = -\frac{2}{5}, \qquad b_{j+1} = -\frac{2(j+1)}{2j^2+7j+5} b_j.$$

Hence, one solution of equation (90.12) is of the form

$$y_1(x) = x\left(1 - \frac{2}{5}x + \frac{4}{35}x^2 - \dots\right).$$

The other solution can be obtained by using $\alpha_2 = -\frac{1}{2}$ in equation (90.6) and equation (90.12). For this solution, we find

$$y_2(x) = x^{-1/2}\left(1 - x + \frac{1}{2}x^2 - \dots\right).$$

The general solution of equation (90.12) is a linear combination of $y_1(x)$ and $y_2(x)$.

Notes

1. This method is similar to the method of Taylor series (see page 632) but is different in that
   - It allows for logarithmic terms to be present, as well as fractional powers.
   - The recurrence relations are computed just once.
   - The method applies only to linear ordinary differential equations.
2. The series solution in equations (90.3), (90.6), (90.7) and (90.8) will always converge in the region $|x| < \rho$.
3. The series in equation (90.6) are sometimes called *Frobenius series*. For regular singular points, this method is sometimes called the *method of Frobenius*.
4. When the given linear ordinary differential equation has an irregular singular point, then series solutions are difficult to obtain and they may be slowly convergent. Morse and Feshback [9, pages 667–674] discuss the canonical second order equations that have 1, 2, and 3 regular singular points, 1 regular and 1 irregular singular points, 1 and 2 irregular singular points. See Bender and Orszag [1, Chapter3] or Goldstein and Braun [6, Chapter 9, pages 251–279] for details. Often the WKB method (see page 642) is used to approximate the solution near an irregular singular point.
5. Understanding the nature of the singular points in an ordinary differential equation leads to an understanding of the types of boundary conditions to be expected for that equation. For example, the ordinary differential equation $xy' = 1$ has the solution $y = C + \log x$, where C is an arbitrary constant. Only if $y(x)$ is specified at some point other than $x = 0$ will it be possible to determine the constant C. The point $x = 0$ is a regular singular point of this equation.

6. This method extends easily to the general nth order homogeneous linear ordinary differential equation at a regular singular point x_0. If the differential equation is given by

$$y^{(n)} + \frac{q_{n-1}(x)}{(x-x_0)} y^{(n-1)} + \frac{q_{n-2}(x)}{(x-x_0)^2} y^{(n-2)} + \cdots + \frac{q_0(x)}{(x-x_0)^n} y = 0,$$

where $\{q_0(x), \ldots, q_{n-1}(x)\}$ are analytic at x_0, then the indicial equation for α is given by

$$(\alpha)_n + q_{n-1}(x_0)(\alpha)_{n-1} + q_{n-2}(x_0)(\alpha)_{n-2} + \cdots + q_0(x_0)(\alpha)_0 = 0, \tag{90.13}$$

where $(\alpha)_n := (\alpha)(\alpha-1)\cdots(\alpha-n+1)$ and $(\alpha)_0 := 1$. If the n roots of equation (90.13) do not differ by integers, then there are n linearly independent solutions of the form of equation (90.6). Otherwise, the forms in equation (90.7) and equation (90.8) must be generalized. See Bender and Orszag [1, Chapter 3] for details.
7. Series solutions can also be used to find the solutions of partial differential equations (see Collatz [3, pages 222–226 and 419–422] or Garabedian [5, Chapter 1, pages 1–17]), or to approximate the solution of nonlinear differential equations, see Leavitt [8].
8. Della Dora and Tournier [4] describe a computer package that will symbolically produce the series for singular points.
The computer language Macsyma has the function `SERIES` that will compute the series expansion of a second order ordinary differential equation. Program 90.1 shows a terminal session in which Airy's equation ($y_{xx}+xy=0$) was input and the power series representation of the solution was obtained. Note that the function `fff(n,i)` is defined to be `fff(n,i)`$= (n)_i = n(n-1)\cdots(n-i+1)$ in the Macsyma manual and that `%k1` and `%k1` are arbitrary constants that appear in the general solution.
9. When all of the singular points in an ordinary differential equation are regular, then the equation is said to be of Fuchs's type. A second order Fuchsian equation with 3 regular singular points can be transformed by a linear fractional transformation into the Riemann differential equation:

$$y'' + \left(\frac{A_1}{x} + \frac{A_2}{x-1}\right) + \left(\frac{A_3}{x^2} + \frac{A_4}{(x-1)^2} + \frac{A_5}{x(x-1)}\right) = 0,$$

where the $\{A_i\}$ are constants. This equation can then be changed to a hypergeometric equation by a change of dependent variable.
10. See Boyce and DiPrima [2, Chapter 4, pages 187–256] and Ince [7, Chapter 16, pages 396–437].

```
(c1) DERIVABBREV:TRUE;
(c2) LOAD(SERIES)$
(c3) DEPENDS(Y,X)$
(c4) DIFF(Y,X,2) + X*Y = 0;

(d4)                   y    + x y = 0
                        x x

(c5) NICEINDICES( SERIES(D4,Y,X) );

DIAGNOSIS: ORDINARY POINT

                   inf                          inf
                   ====          i  3 i         ====         i  3 i
                   \        (- 1)  x            \       (- 1)  x
(d5)      y = %k2 x  >    ---------------- + %k1  >   ----------------
                   /          4      i          /         2      i
                   ====  fff (-, i) 9  i!       ====  fff(-, i) 9  i!
                   i = 0      3                 i = 0     3
```

Program 90.1: Macsyma program to produce series solution.

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Collatz, L. *The Numerical Treatment of Differential Equations*. Springer–Verlag, New York, 1966.

[4] Dora, J. D., and Tournier, E. Formal solutions of differential equations in the neighborhood of singular points. In *SYMSAC 81: Proceedings of the 1981 ACM Symposium on Symbolic and Algebraic Computation*, P. S. Wang, Ed. pp. 25–29.

[5] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[6] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations*. U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[7] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[8] Leavitt, J. A. A power series method for solving nonlinear boundary value problems. *Quart. Appl. Math. 27*, 1 (1969), 67–77.

[9] Morse, P. M., and Feshback, H. *Methods of Theoretical Physics*. McGraw–Hill Book Company, New York, 1953.

91. Equations Solvable for x

Applicable to First order ordinary differential equations that are of the first degree in x; that is, equations of the form $x = f(y, y')$.

Yields

An exact solution, sometimes implicit.

Idea

Equations of the form $x = f(y, y')$ can be solved by finding a second equation involving x, y, and y' and then eliminating y' between the two equations.

Procedure

Given an equation of the form

$$x = f\left(y, \frac{dy}{dx}\right), \tag{91.1}$$

define, as usual, $p = \frac{dy}{dx}$, so that equation (91.1) may be written

$$x = f(y, p). \tag{91.2}$$

Now differentiate this with respect to y to obtain

$$\frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

or

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \tag{91.3}$$

for some function ϕ. Now the ordinary differential equation (91.3), for $p = p(y)$, may sometimes be integrated to obtain

$$F(y, p; C) = 0, \tag{91.4}$$

for some function F, where C is an arbitrary constant. By elimination, the p may sometimes be removed from equations (91.2) and (91.4) to determine $y = y(x; C)$. In cases in which it cannot be removed, we obtain a parametric solution.

Example

Suppose we wish to solve the nonlinear ordinary differential equation

$$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 \tag{91.5}$$

for $y(x)$. Solving equation (91.5) for x results in

$$x = -\frac{py}{2} + \frac{y}{2p}, \tag{91.6}$$

where we have used $y' = p$. Differentiating equation (91.6) with respect to y and factoring results in either $p = \pm i$ (leading to the solution $y = \pm ix$) or

$$\left(1 + \frac{1}{p^2}\right)\left(p + y\frac{dp}{dy}\right) = 0.$$

This equation may be integrated to yield

$$py = C. \tag{91.7}$$

Solving equation (91.7) for p and using this in equation (91.5) results in the explicit solution

$$2xC - y^2 + C^2 = 0.$$

Note

1. See Piaggio [1, page 64].

Reference

[1] Piaggio, H. T. H. *An Elementary Treatise on Differential Equations and Their Applications.* G. Bell & Sons, Ltd, London, England, 1926.

92. Equations Solvable for y

Applicable to First order ordinary differential equations that can be explicitly solved for y; i.e., equations of the form $y = f(x, y')$.

Yields

An exact solution, sometimes implicit.

Idea

Equations of the form $y = f(x, y')$ can be solved by finding a second equation involving x, y, and y' and then eliminating the y' term between the two equations.

Procedure

Given an equation of the form

$$y = f\left(x, \frac{dy}{dx}\right), \tag{92.1}$$

define, as usual, $p = \frac{dy}{dx}$, so that equation (92.1) may be written

$$y = f(x, p). \tag{92.2}$$

Now differentiate this with respect to x to obtain

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right), \tag{92.3}$$

for some function ϕ. Now the ordinary differential equation in (92.3), for $p = p(x)$, may sometimes be integrated to obtain

$$F(x, p; C) = 0, \tag{92.4}$$

for some function F, where C is an arbitrary constant. By elimination, the p may sometimes be removed from equations (92.2) and (92.4) to determine $y = y(x; C)$. In cases in which it cannot be removed, we obtain a parametric solution.

Example

Suppose we wish to solve the nonlinear ordinary differential equation

$$x = y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = yp - xp^2 \tag{92.5}$$

for $y(x)$. Differentiating equation (92.5) with respect to x, and using $p = y'$, results in

$$\frac{dp}{dx} = \frac{px}{p^2 - 1}.$$

This last equation may be integrated to determine

$$\frac{1}{2}x^2 = C + \frac{1}{2}p^2 - \log p, \tag{92.6}$$

where C is an arbitrary constant. Together, equations (92.5) and (92.6) constitute a parametric representation of the solution to equation (92.5):

$$x = \sqrt{2C + p^2 - 2\log p}$$
$$y = \frac{x(1+p^2)}{p}.$$

In this representation, p is treated as a running variable.

Notes

1. The technique used for Lagrange's equation is a specialization of the present technique applied to a restricted class of equations (see page 363).
2. See Piaggio [1, page 63].

Reference

[1] Piaggio, H. T. H. *An Elementary Treatise on Differential Equations and Their Applications.* G. Bell & Sons, Ltd, London, England, 1926.

93. Superposition*

Applicable to

Linear differential equations.

Yields

A set of linear differential equations with "easier" initial conditions or boundary conditions. The sum of the solutions to these new equations will produce the solution to the original equation.

Idea

By use of superposition, the solution to an inhomogeneous linear differential equation may be determined in terms of simpler systems.

Procedure

Given a linear differential equation with a forcing term, inhomogeneous initial conditions, or inhomogeneous boundary conditions, construct a set of equations with each equation having more homogeneous parts than the original system. Solve each of these parts separately, and then combine them for the final solution.

Example

Given the linear second order ordinary differential equation

$$L[y] = y'' + a(x)y' + b(x) = f(x), \tag{93.1}$$

we choose $y_1(x)$ and $y_2(x)$ to be any linearly independent solutions of $L[y_i] = 0$. If C_1 and C_2 are any constants, then

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$$

is called the *homogeneous solution* or the *complementary solution* of equation (93.1). We also define $y_p(x)$ to be any solution to $L[y_p] = f(x)$. The function $y_p(x)$ is called a *particular solution.*

Any solution of equation (93.1) (there will be different solutions, depending on what initial conditions or boundary conditions are chosen with equation (93.1)) may be written in the form

$$y(x) = y_c(x) + y_p(x),$$

for some choice of C_1 and C_2.

Notes

1. In fluid dynamics, the influence of an obstacle in a flow can be simulated by a continuous superposition of sources. See, for instance, Homentcovschi [4].
2. There also exist superposition principles for *nonlinear* equations. These are relations that allow new solutions, with arbitrary constants

in them, to be calculated from other solutions. For instance, if y_1, y_2, and y_3 are solutions of the Riccati equation (see page 392), then y will also be solution if it satisfies

$$\frac{y - y_2}{y - y_1} = C \frac{y_3 - y_2}{y_3 - y_1},$$

where C is an arbitrary constant. See Ince [5, pages 23–25] for details.

3. More generally, Lie and Scheffers [7] showed that a necessary and sufficient condition for a system of n first order ordinary differential equations to have a (nonlinear) superposition formula is that the system of equations be of the form $\frac{d\mathbf{y}}{dt} = \sum_{k=1}^{r} f_k(t)\zeta_k(\mathbf{y})$ and that the vector fields $X_k := \sum_{m=1}^{n} \zeta_k^m(\mathbf{y}) \frac{\partial}{\partial y_m}$ generate a finite dimensional Lie algebra. Given a set of vector fields, $Z = \{X_1, \dots, X_r\}$, and a Lie bracket $[\,,\,]$, a Lie algebra is generated by adding to Z all elements of the form $[X_i, X_j]$. This process is repeated with the new, potentially larger, set Z until no new elements enter Z. The resulting Z is closed under the $[\,,\,]$ operation and is a Lie algebra; it may contain a finite or an infinite number of elements.
4. See also Boyce and DiPrima [1, Section 7.4 pages 352–357].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] del Olmo, M. A., Rodriguez, M. A., and Winternitz, P. Superposition formulas for rectangular matrix Riccati equations. *J. Math. Physics 28*, 3 (March 1987), 530–535.

[3] Harnad, J., Winternitz, P., and Anderson, R. L. Superposition principles for matrix Riccati equations. *J. Math. Physics 24*, 5 (May 1983), 1062–1072.

[4] Homentcovschi, D. Uniform asymptotic solutions of the potential field around a thin oblate body of revolution. *SIAM J. Appl. Math. 42*, 1 (February 1982), 44–65.

[5] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[6] Jones, A. S. Quasi-additive solutions of nonlinear differential equations. *J. Austral. Math. Soc. (Series A) 42* (1987), 92–116.

[7] Lie, S., and Scheffers, G. Vorlesungen uber Continuierlichen Gruppen mit geometrischen und anderen Anwendungen, Teubner, Leipzig, 1893.

[8] Shnider, S., and Winternitz, P. Classification of systems of nonlinear ordinary differential equations with superposition principles. *J. Math. Physics 25*, 11 (November 1984), 3155–3165.

94. Method of Undetermined Coefficients*

Applicable to Linear or nonlinear differential equations, a single equation or a system.

Yields

An exact homogeneous solution, an exact particular solution, or both.

Idea

If the general form of the solution of a given differential equation is known (or can be guessed), it can be substituted into the defining equations with unknown coefficients. Then the unknown coefficients can be determined.

Procedure

Very often we can guess the form of a solution to a differential equation. Or, we could just guess blindly. By having several unknown parameters in the assumed form of the solution, the solution should be able to fit the defining equation(s). By forcing the guessed solution to satisfy the equation, we may be able to determine these unknown quantities.

Example 1

Suppose we have the equation

$$y'' - \frac{2}{x^2}y = 7x^4 + 3x^3. \tag{94.1}$$

If we suspect that this equation has a power type solution for $y(x)$, we might search for a solution in the form

$$y(x) = ax^b, \tag{94.2}$$

where a and b are unknowns to be determined. In this example, we presume that a and b are constants (in more complicated problems, the unknowns can be functions to be determined). We try to determine a and b by substituting our guess in the original equation for $y(x)$. Using equation (94.2) in equation (94.1) yields

$$ax^{b-2}(b^2 - b - 2) = 7x^4 + 3x^3. \tag{94.3}$$

This equation must be satisfied for all values of x. There is no single set of values for a and b for which this will be true. However, note the following:

- If $b = 6, a = 1/4$, then the left-hand side of equation (94.3) becomes $7x^4$.

- If $b = 5, a = 1/6$, then the left-hand side of equation (94.3) becomes $3x^3$.
- If $b = -1$, then the left-hand side of equation (94.3) becomes zero.
- If $b = 2$, then the left-hand side of equation (94.3) becomes zero.

The first two facts enable us to write the particular solution of equation (94.1) as

$$y_p(x) = \frac{1}{4}x^6 + \frac{1}{6}x^5.$$

The second two facts tell us that $y(x) = x^2$ and $y(x) = 1/x$ are both solutions to the homogeneous equation

$$y'' - \frac{2}{x^2}y = 0.$$

Therefore, the complete solution to equation (94.1) is

$$y(x) = \frac{1}{4}x^6 + \frac{1}{6}x^5 + Ax^2 + \frac{B}{x},$$

where A and B are arbitrary constants.

Example 2

Suppose we have the partial differential equation

$$\begin{aligned} u_{xx} &= u_t, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= \sin \pi x \end{aligned} \tag{94.4}$$

An appropriate guess for the form of the solution would be

$$u(x,t) = f(t) \sin \pi x,$$

for some unknown function $f(t)$. Using this guess in equation (94.4) results in the system

$$f' + \pi^2 f = 0, \qquad f(0) = 1.$$

Hence, $f(t) = e^{-\pi^2 t}$.

Example 3

A guess for the form of the solution of the nonlinear equation

$$u_t = (uu_x)_x \tag{94.5}$$

might be

$$u(x,t) = f(t) + g(t)x^p \tag{94.6}$$

for some functions $f(t)$ and $g(t)$ and some constant p. Using equation (94.6) in equation (94.5) leads to the choice $p = 2$. With this value, $f(t)$ and $g(t)$ can be determined so that

$$u(x,t) = (C - 6t)^{-1} x^2 + (C - 6t)^{-1/6}.$$

See Ames [1] for more details.

Notes

1. In Table 3.1 of Boyce and DiPrima [2] is a description of general solution forms for a forced linear second order constant coefficient differential equation when the forcing function is a polynomial, a trigonometric function, an exponential function, or a combination of these terms. By utilizing this general form with unknown coefficients, a solution may be obtained.
2. The reason that we suspected equation (94.1) to have a power type solution is that the homogeneous part of equation (94.1) is a Euler equation.
3. See Boyce and DiPrima [2, Section 3.6.1, pages 146–155], Rainville and Bedient [3, pages 115–118], and Simmons [4, pages 87–90].

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations*. The MacMillan Company, New York, 1964.

[4] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

95. Variation of Parameters

Applicable to Forced, linear ordinary differential equations.

Yields

An integral representation of the particular solution.

Idea

If we know the solution to the homogeneous equation, we can write an expression for the particular solution.

Procedure

We illustrate the general technique for the linear ordinary differential equation of second order. Suppose we have the equation

$$y'' + P(x)y' + Q(x)y = R(x), \tag{95.1}$$

and suppose that we know that $\{y_1(x), y_2(x)\}$ are two linearly independent solutions to the homogeneous (unforced) equation

$$y'' + P(x)y' + Q(x)y = 0. \tag{95.2}$$

That is, every solution of equation (95.2) is a linear combination of $y_1(x)$ and $y_2(x)$. We look for the particular solution of equation (95.1) in the form

$$y(x) = v_1(x)y_1(x) + v_2(x)y_2(x), \tag{95.3}$$

where $v_1(x)$ and $v_2(x)$ are to be determined. Differentiating equation (95.3) with respect to x yields

$$y' = (v_1y_1' + v_2y_2') + (v_1'y_1 + v_2'y_2). \tag{95.4}$$

We choose the second term in equation (95.4) to vanish, so that

$$(v_1'y_1 + v_2'y_2) = 0. \tag{95.5}$$

If we now differentiate equation (95.4) with respect to x, and use this expression (with equations (95.3), (95.4) and (95.5)) in equation (95.2) then we obtain

$$v_1'y_1' + v_2'y_2' = R(x). \tag{95.6}$$

Equations (95.5) and (95.6) constitute two algebraic equations for the two unknowns $v_1'(x)$ and $v_2'(x)$. Solving these two algebraic equations yields

$$v_1' = -\frac{y_2(x)R(x)}{W(y_1, y_2)}, \qquad v_2' = \frac{y_1(x)R(x)}{W(y_1, y_2)}, \tag{95.7}$$

where $W(y_1, y_2) := y_1 y_2' - y_1' y_2$ is the usual Wronskian. The equations in (95.7) can be integrated and the results can be used in equation (95.3) for

$$y(x) = -y_1(x) \int \frac{y_2(x)R(x)}{W(y_1, y_2)}\, dx + y_2(x) \int \frac{y_1(x)R(x)}{W(y_1, y_2)}\, dx.$$

Example

Suppose we have the equation

$$y'' + y = \csc x \tag{95.8}$$

to solve. The solutions to the homogeneous equation, $y''+y=0$, are clearly $y_1(x) = \sin x$ and $y_2(x) = \cos x$. Hence, we can compute the Wronskian to be $W(y_1, y_2) = -1$. Using this in equation (95.7) results in

$$\begin{aligned} v_1(x) &= \int \frac{-\cos x \csc x}{-1} dx = \log(\sin x), \\ v_2(x) &= \int \frac{\sin x \csc x}{-1} dx = -x. \end{aligned}$$

Hence, the particular solution to equation (95.8) is $y(x) = \sin x \log(\sin x) - x \cos x$.

Notes

1. In Boyce and DiPrima [1, pages 156–162, 275–277, 391–393] or Finizio and Ladas [3, page 136] may be found the generalization of the analysis presented above for differential equations of higher order. The result is

   If $\{y_1, y_2, \ldots, y_n\}$ form a fundamental system of solutions for the equation

   $$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

   and if the functions $\{u_1, u_2, \ldots, u_n\}$ satisfy the system of equations

   $$\begin{aligned} y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0, \\ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0, \\ y_1'' u_1' + y_2'' u_2' + \cdots + y_n'' u_n' &= 0, \\ &\vdots \\ y_1^{(n-2)} u_1' + y_2^{(n-2)} u_2' + \cdots + y_n^{(n-2)} u_n' &= 0, \\ y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' &= f(x), \end{aligned}$$

   then $y = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$ is a particular solution of

   $$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x).$$

2. This last result could also have been obtained by applying variation of parameters to a system of linear first order ordinary differential equations. Suppose we have the system

$$\begin{aligned} \mathbf{x}' &= P(t)\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{95.9}$$

where $\mathbf{g}(t)$ is a time-dependent vector and $P(t)$ is a time-dependent matrix. Then the solution can be written as

$$\mathbf{x}(t) = \Psi(t)\mathbf{x}_0 + \Psi(t) \int_{t_0}^{t} \Psi^{-1}(s)\mathbf{g}(s)\, ds,$$

where $\Psi(t)$ is a fundamental matrix of the system. This means that $\Psi(t)$ satisfies

$$\Psi' = P(t)\Psi, \quad \Psi(t_0) = I,$$

where I is an identity matrix of appropriate size. See Boyce and DiPrima [1] or Coddington and Levinson [2, pages 87–88] for details.

3. If equation (95.9) is stiff, that is $P(t)$ has eigenvalues with widely separated positive and negative real parts (see page 770), then the fundamental matrix may become numerically singular for $t \gg t_0$. For example, the problem $\mathbf{u}' = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix} \mathbf{u}$ has the fundamental matrix $\begin{bmatrix} \cosh\lambda(t-t_0) & \frac{1}{\lambda}\sinh\lambda(t-t_0) \\ \lambda\sinh\lambda(t-t_0) & \cosh\lambda(t-t_0) \end{bmatrix}$. For $\lambda(t-t_0) \geq 16$, this matrix is numerically singular even in 64-bit arithmetic.

4. See Ince [4, pages 122–123], Rainville and Bedient [5, pages 130–136], and Simmons [6, pages 90–93].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations.* McGraw–Hill Book Company, New York, 1955.

[3] Finizio, N., and Ladas, G. *Ordinary Differential Equations with Modern Applications.* Wadsworth Publishing Company, Belmont, CA, 1982.

[4] Ince, E. L. *Ordinary Differential Equations.* Dover Publications, Inc., New York, 1964.

[5] Rainville, E. D., and Bedient, P. E. *Elementary Differential Equations.* The MacMillan Company, New York, 1964.

[6] Simmons, G. F. *Differential Equations with Applications and Historical Notes.* McGraw–Hill Book Company, New York, 1972.

96. Vector Ordinary Differential Equations

Applicable to A system of constant coefficient linear ordinary differential equations.

Yields

An exact solution is obtained.

Idea

Very often a system of coupled equations with constant coefficients can be transformed to a system of decoupled equations with constant coefficients.

Procedure

Given a system of n ordinary differential equations with constant coefficients, write the system as a vector ordinary differential equation in the following form

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{96.1}$$

where $\mathbf{y}$ is a vector of the unknowns and A is a constant $n \times n$ matrix. Then determine the eigenvectors of A (i.e., those vectors $\mathbf{x}$ that satisfy $A\mathbf{x} = \lambda\mathbf{x}$ for some non-zero value of λ), and construct a diagonalizing matrix S whose columns are the eigenvectors of A. Then change variables by the transformation $\mathbf{y} = S\mathbf{u}$, so that equation (96.1) becomes $(S\mathbf{u})' = A(S\mathbf{u})$, or

$$\mathbf{u}' = S^{-1}AS\mathbf{u}. \tag{96.2}$$

By our choice of S, and assuming that A has n linearly independent eigenvectors, the matrix $S^{-1}AS$ will be diagonal. Hence, the equations in equation (96.2) will decouple and each row of equation (96.2) will be an ordinary differential equation in one dependent variable (u_i). These equations can be solved by the method applicable to linear constant coefficient ordinary differential equations (see page 247). Once $\mathbf{u}$ is known, then $\mathbf{y}$ can be recovered from $\mathbf{y} = S\mathbf{u}$.

Example

Suppose we have the system of equations

$$\begin{aligned}\frac{dy_1}{dt} &= 9y_1 + 2y_2,\\ \frac{dy_2}{dt} &= \ \ y_1 + 8y_2.\end{aligned}$$

This system of equations can be written as a vector ordinary differential equation as follows:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{96.3}$$

or $\mathbf{y}' = A\mathbf{y}$, where $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^{\mathrm{T}}$ and $A = \left[\begin{smallmatrix} 9 & 2 \\ 1 & 8 \end{smallmatrix}\right]$. The eigenvalues of A are $\lambda = 7$ and $\lambda = 10$ with the corresponding eigenvectors $\begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$ and $\begin{bmatrix} 2 & 1 \end{bmatrix}^{\mathrm{T}}$. Therefore, the diagonalizing matrix, S, whose columns are the eigenvectors of A, is $S = \left[\begin{smallmatrix} 1 & 2 \\ -1 & 1 \end{smallmatrix}\right]$. We will also need the inverse of S, which is $S^{-1} = \left[\begin{smallmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{smallmatrix}\right]$. If we change variables by $\mathbf{y} = S\mathbf{u}$, then equation (96.3) attains the form of equation (96.2). Specifically, we find

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ &= \begin{bmatrix} 7 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \end{aligned} \tag{96.4}$$

Equation (96.4) can be expanded as

$$\frac{du_1}{dt} = 7u_1, \qquad \frac{du_1}{dt} = 10u_2.$$

Note that these last equations are decoupled and have constant coefficients. The solutions to these equations are given by

$$u_1 = Be^{7t}, \qquad u_2 = Ce^{10t},$$

where B and C are arbitrary constants. Therefore, using our original transformation, we obtain $\mathbf{y} = S\mathbf{u}$, or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} Be^{7t} \\ Ce^{10t} \end{bmatrix},$$

and therefore

$$\begin{aligned} y_1 &= Be^{7t} + 2Ce^{10t}, \\ y_2 &= -Be^{7t} + Ce^{10t}. \end{aligned} \tag{96.5}$$

The constants B and C may be found by evaluating equation (96.5) at $t = t_0$ and using equation (96.1):

$$\begin{aligned} \mathbf{y}_0 &= B\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{7t_0} + C\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{10t_0}, \\ &= \begin{bmatrix} e^{7t_0} & 2e^{10t_0} \\ -e^{7t_0} & e^{10t_0} \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix}. \end{aligned} \tag{96.6}$$

Notes

1. Of course, some systems of equations that are not of first order can also be reduced to the form of equation (96.1), see page 146.
2. Given the linear matrix differential equation

$$\frac{dR}{dt} = B(t)R, \qquad R(t_0) = I,$$

where R and B are square matrices, note that the determinent of R, $|R|$ satisfies

$$\frac{d|R|}{dt} = \operatorname{trace}(B)\,|R|, \qquad |R|_{t=t_0} = 1.$$

3. For a similar technique applied to partial differential equations, see page 449.
4. Given equation (96.1), a faster technique to find the solution (analogous to the method for constant coefficient linear equations on page 247) is to find the eigenvalues $\{\lambda_i\}$ and eigenvectors $\{\mathbf{x}_i\}$ of A and then write the most general solution in the form

$$\mathbf{y} = \sum_{i=1}^{n} C_i \mathbf{x}_i e^{\lambda_i t}, \tag{96.7}$$

where the $\{C_i\}$ are unknown constants. For the example given, we can directly write the solution as

$$\begin{aligned} \mathbf{x} &= C_1 \mathbf{x}_1 e^{\lambda_1 t} + C_2 \mathbf{x}_2 e^{\lambda_2 t} \\ &= C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{7t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{10t}, \end{aligned}$$

which is identical to equation (96.5).
5. This method is the same as "solving" the system in equation (96.1) by writing $\mathbf{y} = e^{At}\mathbf{y}_0$, where the exponential of a matrix is another matrix. See Coddington and Levinson [4, pages 67–77] or Moler and Van Loan [6] for details.
6. Similar results apply when A is a function of t. The equation $\mathbf{y}' = A(t)\mathbf{y}$, with $\mathbf{y}(t_0) = \mathbf{y}_0$, has the solution $\mathbf{y}(t) = e^{B(t)}\mathbf{y}(t_0)$, where $B(t) := \int_{t_0}^{t} A(t)\,dt$, whenever $BA = AB$.
7. If the matrix A cannot be diagonalized (i.e., if A does not have n linearly independent eigenvectors), then A has *generalized eigenvectors*. If the vector $\mathbf{z}_i^{(m)}$ satisfies $(A - \lambda_i I)^m \mathbf{z}_i^{(m)} = \mathbf{0}$ and $(A - \lambda_i I)^{m-1}\mathbf{z}_i^{(m)} \neq \mathbf{0}$, then $\mathbf{z}_i^{(m)}$ is called a generalized eigenvector of order m. (Note that a generalized eigenvector of order 1 is a usual eigenvector). Given $\mathbf{z}_i^{(m)}$, define $\mathbf{z}_i^{(n-1)} = (A - \lambda_i I)\mathbf{z}_i^{(n)}$ for $n = m, m-1, \dots, 2$, and define

$$\mathbf{y}_{ir} = e^{\lambda_i t}\left(\mathbf{z}_i^{(r)} + t\mathbf{z}_i^{(r-1)} + \frac{t^{r-1}}{(r-1)!}\mathbf{z}_i^{(1)}\right)$$

for $r = 1, 2, \ldots, m$. Then the $\{\mathbf{y}_{ir}\}$ will be a collection of linearly independent vectors and all solutions of equation (96.1) will be of the form $\sum_i \sum_r C_{ir}\mathbf{y}_{ir}$ (as in equation (96.7)). See Campbell [3] for details.

8. An easier method to use when A does not have n linearly independent eigenvectors is by the theorem of Leonard [5]:

   Let A be a constant $n \times n$ matrix with characteristic polynomial $p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. Then $e^{At} = x_1(t)I + x_2(t)A + x_3(t)A^2 + \cdots + x_n(t)A^{n-1}$, where the $x_k(t)$, $1 \le k \le n$, are the solutions to the nth order scalar differential equation

   $$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1 x' + c_0 x = 0$$

   satisfying the following initial conditions:

   $$\left.\begin{array}{r} x_1(0) = 1 \\ x_1'(0) = 0 \\ \vdots \\ x_1^{(n-1)}(0) = 0 \end{array}\right\} \quad \left.\begin{array}{r} x_2(0) = 0 \\ x_2'(0) = 1 \\ \vdots \\ x_2^{(n-1)}(0) = 0 \end{array}\right\} \quad \cdots \quad \left.\begin{array}{r} x_n(0) = 0 \\ x_n'(0) = 0 \\ \vdots \\ x_n^{(n-1)}(0) = 1 \end{array}\right\}.$$

9. Nonhomogeneous systems of linear equations, of the form

   $$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{g}(t),$$

   may also be analyzed (see Boyce and DiPrima [2, Chapter 7, pages 323–395]. The easiest method is a generalization of the method of variation of parameters (see page 418). Alternately, if the nonhomogeneous system is of the form $\mathbf{y}' = A\mathbf{y} + t\mathbf{u}$, where A is a constant matrix and $\mathbf{u}$ is an arbitrary vector, then the system may be rewritten as

   $$\frac{d}{ds}\begin{bmatrix}\mathbf{y} \\ t\end{bmatrix} = \begin{bmatrix}A\mathbf{y} + t\mathbf{u} \\ 1\end{bmatrix} = \begin{bmatrix}A & \mathbf{u} \\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{y} \\ t\end{bmatrix},$$

   which is now in the form of equation (96.1).

10. The solution of

    $$\frac{dX}{dt} = AX + XB, \qquad X(0) = C, \qquad (96.8)$$

    where A, B, C, and X are *all* matrices is $X(t) = e^{At}Ce^{Bt}$. See Bellman [1] for details. When A and B depend on t, we have

    If $U(t)$ is a solution to $U' = A(t)U$ with $U'(0) = I$ and $V(t)$ is a solution to $V' = B^{\mathrm{T}}(t)V$ with $V'(0) = I$, then the solution to (96.8) is given by $X = UCV^{\mathrm{T}}$.

11. For a review of eigenvalues and eigenvectors see Strang [7, Chapter 5, pages 171–230].

References

[1] Bellman, R. *Introduction to Matrix Analysis*. McGraw–Hill Book Company, New York, 1960.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Campbell, S. L. *Singular Systems of Differential Equations*. Pitman Publishing Co., Marshfield, MA, 1980.

[4] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations*. McGraw–Hill Book Company, New York, 1955.

[5] Leonard, I. E. The matrix exponential. *SIAM Review 38*, 3 (September 1996), 507–512.

[6] Moler, C., and Van Loan, C. Nineteen dubious ways to compute the exponential of a matrix. *SIAM Review 20*, 4 (October 1978), 801–836.

[7] Strang, G. *Linear Algebra and Its Applications*. Academic Press, New York, 1976.

97. Bäcklund Transformations

Applicable to Nonlinear partial differential equations.

Yields

If a Bäcklund transformation can be found, then the solution of a nonlinear partial differential equation can be used to obtain either a different solution to the same partial differential equation, or to obtain a solution to a different nonlinear partial differential equation.

Idea

From a solution of a nonlinear partial differential equation, we can sometimes find a relationship that will generate the solution of

- A different partial differential equation (i.e., a Bäcklund transformation)
- The same partial differential equation (i.e., an auto-Bäcklund transformation)

Procedure

The first step (which is extremely difficult) is to determine a Bäcklund transformation between two partial differential equations. There are various methods described in the literature (see the references) that can be utilized for certain classes of equations. This transformation will utilize a solution of one of the partial differential equations to determine a solution to the other partial differential equation.

Example 1

Suppose we wish to determine solutions to the sine–Gordon equation

$$u_{xt} = \sin u. \tag{97.1}$$

An auto-Bäcklund transformation is given by the pair of partial differential equations

$$\begin{aligned} v_x &= u_x + 2\lambda \sin\left(\frac{v+u}{2}\right), \\ v_t &= -u_t + \frac{2}{\lambda} \sin\left(\frac{v-u}{2}\right). \end{aligned} \tag{97.2.a-b}$$

That is, given a solution $u(x,t)$ to equation (97.1), if $v(x,t)$ satisfies equation (97.2), then $v(x,t)$ will also be a solution of equation (97.1). This may be verified by determining v_{xt} both by differentiating equation (97.2.a) with respect to t and by differentiating equation (97.2.b) with respect to x. This results in

$$\begin{aligned} v_{xt} &= \quad u_{xt} + 2\sin\left(\frac{v-u}{2}\right)\cos\left(\frac{v+u}{2}\right), \\ v_{xt} &= -u_{xt} + 2\sin\left(\frac{v+u}{2}\right)\cos\left(\frac{v-u}{2}\right). \end{aligned} \tag{97.3}$$

Equating the two expressions in equation (97.3) results in equation (97.1), while adding them results in

$$v_{xt} = \sin v.$$

Starting with the solution $u(x,t) = 0$ of equation (97.1), we can use the auto-Bäcklund transformation to determine another solution; equation (97.2) becomes

$$v_x = 2\lambda \sin\frac{v}{2}, \qquad v_t = \frac{2}{\lambda}\sin\frac{v}{2}.$$

This system of equations is easily solved to yield a new solution of the sine–Gordon equation

$$\tan\frac{v}{4} = C\exp\left(\lambda t + \frac{x}{\lambda}\right).$$

This solution may be used to determine another solution, and so on.

Example 2

Suppose we wish to find solutions to Burgers's equation

$$u_t + uu_x = \sigma u_{xx}. \tag{97.4}$$

Suppose that a solution of equation (97.4), $w(x,t)$, is already known. If $\phi(x,t)$ is defined to be any solution of the following linear partial differential equation

$$\phi_t + w(x,t)\phi_x = \sigma\phi_{xx}, \tag{97.5}$$

and $v(x,t)$ is defined by

$$v(x,t) = -2\sigma\frac{\phi_x}{\phi} + w, \tag{97.6}$$

then $v(x,t)$ also satisfies Burgers's equation. Hence, one solution of Burgers's equation (i.e., $w(x,t)$) can be used to generate another solution.

For example, a solution to equation (97.4) is clearly $w(x,t) = 0$. Using this in equation (97.5) results in $\phi_t = \sigma\phi_{xx}$. Each solution of this equation results in a new solution of (97.4). For example, one solution is $\phi(x,t) = e^{-x^2/4\sigma t}/\sqrt{4\pi\sigma t}$. Using this in equation (97.6) results in the different solution to Burgers's equation $v(x,t) = x/t$. This solution may be utilized to determine another solution, and the process can be repeated indefinitely.

Notes

1. The transformations in equations (97.5) and (97.6) with
   - w identically equal to zero is the Cole–Hopf transformation (see Whitham [10, pages 97–98])
   - $w \neq 0$ was first found in Fokas [6]
2. The Cole–Hopf transformation may also be written as the set of partial differential equations for the unknown $v(x,t)$
$$v_x = \frac{uv}{2\sigma}, \qquad v_t = \left(2\sigma u_x - u^2\right)\frac{v}{4\sigma}.$$
3. Sometimes a Bäcklund transformation cannot be used to generate an infinite sequence of new solutions; the solutions repeat after some point. See Chan and Zheng [4] for some techniques to find new Bäcklund transformations when this occurs.
4. Sakovich [9] determines all evolution equations (equations of the form $w_t = f(w_x, w_{xx}, \dots, w_{x\dots x})$) and all Klein–Gordon equations (equations of the form $w_{xy} = f(w)$) that admit a Bäcklund autotransformation (i.e., a mapping of the form $\phi = a[w]$, where $a[w]$ includes finite derivatives of w, that maps a solution of an equation to itself). Besides the linear equations, they include only the Liouville equation and the Burgers equation hierarchy.
5. The Miura transformation $u = q_x + q^2$ connects the solution u of the KdV equation $u_t + u_{xxx} + 6uu_x = 0$ and the solution q of the modified KdV equation $q_t + 6q^2 q_x q_{xxx} = 0$.
6. The transformation $\phi = \log(2w_x w_y/w^2)$ connects the solution ϕ of the Liouville equation $\phi_{xy} = e^\phi$ to the solution w of $w_{xy} = 0$.
7. An interesting linearization from Calogero [3] takes the Eckhaus equation, $i\psi_t + \psi_{xx} + \left[|\psi|^4 + 2\left(|\psi|^2\right)_x\right]\psi = 0$, and makes the invertible change of variables
$$\begin{aligned}\phi(x,t) &= \psi(x,t)\exp\left(\int_{-\infty}^{x} |\psi(x',t)|^2\,dx'\right)\\ \psi(x,t) &= \phi(x,t)\left(1 + 2\int_{-\infty}^{x} |\phi(x',t)|^2\,dx'\right)^{-1/2}\end{aligned}$$
   to obtain $i\phi_t + \phi_{xx} = 0$.

References

[1] Anderson, R. L., and Ibragimov, N. H. *Lie–Backlund Transformation in Applications.* SIAM, Philadelphia, PA, 1979.

[2] Bluman, G. W., and Reid, G. J. Sequences of related linear PDEs. *J. Math. Anal. Appl. 144* (1989), 565–585.

[3] Calogero, F. Universal C-integable nonlinear partial differential equation in *N+1* dimensions. *J. Math. Physics 34*, 7 (July 1993), 3197.

[4] Chan, W. L., and Zheng, Y.-K. Backlund transformations for the Caudrey–Dodd–Gibbon–Sawada–Kotera equation and its zzzref10refzzz-modified equation. *J. Math. Physics 30*, 9 (Sep 1989), 2065–2068.

[5] Dodd, R. K., Eilbeck, J. C., and Morris, H. C. *Solitons and Nonlinear Wave Equations*. Academic Press, New York, 1982.

[6] Fokas, A. *Invariants, Lie–Backlund Operators and Bcklund Transformations*. PhD thesis, California Institute of Technology, Pasadena, CA, 1979.

[7] Olver, P. J. *Applications of Lie Groups to Differential Equations*. No. 107 in Graduate Texts in Mathematics. Springer–Verlag, New York, 1986.

[8] Rogers, C., and Shadwick, W. F. *Backlund Transformations and Their Applications*. Academic Press, New York, 1982.

[9] Sakovich, S. Y. On special B acklund autotransformations. *J. Phys. A: Math. Gen. 24* (1991), 401–405.

[10] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

98. Method of Characteristics

Applicable to Systems of quasilinear partial differential equations (i.e., one or more partial differential equations linear in the first derivatives of the dependent variables, with no higher order derivatives present).

Yields

If the initial data are not given along a characteristic, then an exact solution can be obtained (generally implicit).

Idea

A quasilinear partial differential equation of hyperbolic type can be transformed into a set of ordinary differential equations that define the characteristics and a set of ordinary differential equations that describe how the solution changes along any specific characteristic.

Procedure

Suppose we have the quasilinear partial differential equation

$$a_1(\mathbf{x},u)u_{x_1} + a_2(\mathbf{x},u)u_{x_2} + \cdots + a_N(\mathbf{x},u)u_{x_N} = b(\mathbf{x},u) \tag{98.1}$$

for the unknown $u(\mathbf{x}) = u(x_1, x_2, \ldots, x_N)$. If we were to differentiate $u(\mathbf{x})$ with respect to the variable s, then we obtain

$$\frac{du}{ds} = \left(\frac{\partial x_1}{\partial s}\right) u_{x_1} + \left(\frac{\partial x_2}{\partial s}\right) u_{x_2} + \cdots + \left(\frac{\partial x_N}{\partial s}\right) u_{x_N}. \tag{98.2}$$

If we define

$$\frac{\partial x_k}{\partial s} = a_k(\mathbf{x},u), \tag{98.3}$$

for $k = 1, 2, \ldots, N$, then using equation (98.1) in equation (98.2) results in

$$\frac{du}{ds} = b(\mathbf{x},u). \tag{98.4}$$

To determine the solution of the partial differential equation (98.1), we need to integrate the ordinary differential equations given in equation (98.3) and (98.4). (Equation (98.3) may look like a partial differential equation, but it is an ordinary differential equation with respect to s.) To perform this integration, initial conditions are needed in s for the $\{x_k\}$ and for u. Generally, the initial data for equation (98.1) will be given in the form

$$g(\mathbf{x},u) = 0, \tag{98.5}$$

on some manifold in $\mathbf{x}$ space. We identify this surface as corresponding to $s = 0$. If we think of $\mathbf{x}$ and u as depending on the variables $\{s, t_1, t_2, \ldots, t_{N-1}\}$, then the variables $\{t_1, t_2, \ldots, t_{N-1}\}$ can be used to parametrize the initial data in equation (98.5) (the examples will make this clear). That is,

$$\begin{aligned} x_1(s=0) &= h_1(t_1, t_2, \ldots, t_{N-1}), \\ x_2(s=0) &= h_2(t_1, t_2, \ldots, t_{N-1}), \\ &\vdots \\ x_N(s=0) &= h_N(t_1, t_2, \ldots, t_{N-1}), \\ u(s=0) &= v(t_1, t_2, \ldots, t_{N-1}). \end{aligned} \tag{98.6}$$

Hence equation (98.6) supplies the initial conditions for the differential equations in (98.3) and (98.4).

After $\mathbf{x}$ and u are determined from equations (98.3), (98.4), and (98.6), then an implicit solution will have been obtained. If the $\{s, t_1, t_2, \ldots, t_{N-1}\}$ can be analytically eliminated, then an explicit solution will be obtained. It is not always possible to perform this elimination analytically.

The physical picture of the construction of the solution is shown in figure 98.1. The solution u is determined by the ordinary differential equation (98.4) along each characteristic. A characteristic is specified by the $\{t_i\}$ values. The parameter s represents scaled distance along a characteristic. When two characteristics cross, a *shock* is formed.

Note that a shock cannot form if the equation (98.1) is linear; that is, each $\{a_i\}$ is only a function of $\mathbf{x}$ and not of u. At a shock, extra conditions are required. (See Landau and Lifshitz [2, Chapter 9, pages 310–346]) for a discussion of the *Rankine–Hugoniot adiabatic*, which is used in fluid mechanics.)

Example 1

Suppose we want to solve the quasilinear partial differential equation

$$\begin{aligned} u_x + x^2 u_y &= -yu, \\ u &= f(y) \qquad \text{on } x = 0, \end{aligned} \tag{98.7.a-b}$$

where $f(y)$ is a given function. Forming du/ds we have

$$\frac{du}{ds} = \left(\frac{\partial x}{\partial s}\right) u_x + \left(\frac{\partial y}{\partial s}\right) u_y. \tag{98.8}$$

Comparing equation (98.8) to equation (98.7), we take

$$\frac{\partial x}{\partial s} = 1, \qquad \frac{\partial y}{\partial s} = x^2, \qquad \frac{du}{ds} = -yu. \tag{98.9.a-c}$$

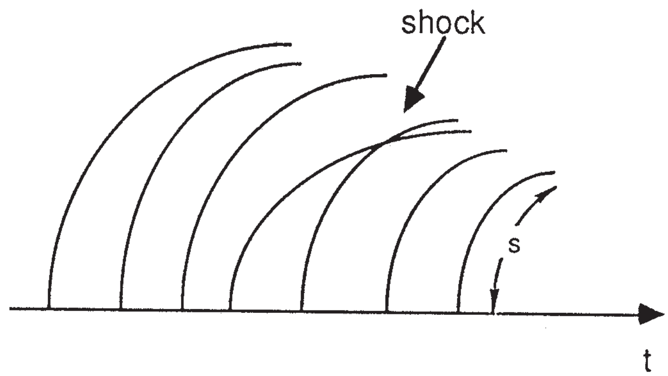

Figure 98.1: Depiction of the characteristics for a quasilinear equation.

The initial data in equation (98.7.b) can be written parametrically as

$$\begin{aligned} x(s=0) &= 0, \\ y(s=0) &= t_1, \\ u(s=0) &= f(t_1). \end{aligned} \tag{98.10.a-c}$$

That is, when $s = 0$, we have $u = f(y)$ and $x = 0$. The solution of (98.9.a) with (98.10.a) is

$$x(s, t_1) = s. \tag{98.11}$$

Therefore, equations (98.9.b) and (98.10.b) can be written as

$$\frac{\partial y}{\partial s} = s^2, \qquad y(s=0) = t_1,$$

with the solution

$$y(s, t_1) = \frac{s^3}{3} + t_1. \tag{98.12}$$

Finally, the equation for u (from equations (98.9.c), (98.10.c), and (98.12)) becomes

$$\frac{du}{ds} = -\left(\frac{s^3}{3} + t_1\right) u, \qquad u(s=0) = f(t_1),$$

with the solution

$$u(s, t_1) = f(t_1) \exp\left(-\frac{s^4}{12} - st_1\right). \tag{98.13}$$

Equations (98.11), (98.12), and (98.13) constitute an implicit solution of equation (98.7).

In this case, it is possible to analytically eliminate the s and t_1 variables to obtain an explicit solution. From equation (98.11) we obtain $s = x$. Using this in equation (98.12) results in $t_1 = y - \frac{x^3}{3}$. Using these two values in equation (98.13) results in the explicit solution

$$u(x, y) = f\left(y - \frac{x^3}{3}\right) \exp\left(\frac{x^4}{4} - xy\right).$$

Example 2

If we have the quasilinear partial differential equation in three dependent variables

$$\begin{aligned} &u_x + u_y + xyu_z = u^2, \\ &u = x^2 \qquad \text{on } y = z, \end{aligned} \tag{98.14}$$

then we can write equations (98.3), (98.4), and (98.6) as

$$\frac{\partial x}{\partial s} = 1, \qquad \frac{\partial y}{\partial s} = 1, \qquad \frac{\partial z}{\partial s} = xy,$$
$$\frac{du}{ds} = u^2,$$
$$x(s=0) = t_1, \quad y(s=0) = t_2, \quad z(s=0) = t_2, \quad u(s=0) = t_1^2.$$

The equations for x and y can be integrated to yield

$$x = s + t_1, \qquad y = s + t_2. \tag{98.15}$$

Using these values for x and y, the equation for z becomes

$$\frac{\partial z}{\partial s} = (s + t_2)(s + t_1),$$

which can be integrated to yield

$$z = \frac{s^3}{3} + \frac{s^2}{2}(t_2 + t_1) + st_2t_1 + t_2. \tag{98.16}$$

The equation for u can also be integrated to obtain

$$u = \frac{t_1^2}{1 - st_1^2}. \tag{98.17}$$

The equations in (98.15) (98.16), and (98.17) constitute an implicit solution to equation (98.14). The variables t_1 and t_2 can be eliminated to yield

$$\begin{aligned} u &= \frac{(x-s)^2}{1 - s(x-s)^2}, \\ z &= -\frac{4s^3}{3} - \frac{s^2}{2}(x + y) + s(xy + 1) + y. \end{aligned} \tag{98.18.a-b}$$

To actually evaluate $u(x, y, z)$ at some given value of x, y, and z requires two steps. First, equation (98.18.b) must be solved for s, and then this value is utilized in equation (98.18.a).

Alternatively, the method of resultants (see page 50) could be used to obtain a single polynomial equation in terms of x, y, z, and u, alone. This results in an equation with many terms; the implicit solution given by equation (98.18) is more useful and more compact.

Notes

1. This technique extends naturally to systems of partial differential equations, with virtually no increase in complexity. This allows a single partial differential equation of higher order (and hyperbolic type) to be analyzed. For example, the wave equation $u_{xx} = u_{tt}$ can be written, in the variables $\{v := u_x,\ w := u_t\}$, as the system of two quasilinear equations $\{v_t = w_x,\ w_t = v_x\}$.
2. The general quasilinear system of N equations for the N unknowns $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in the two independent variables $\{x, t\}$ has the form
$$\sum_{j=1}^{N} A_{ij}(\mathbf{u}, x)\frac{\partial u_j}{\partial t} + \sum_{j=1}^{N} a_{ij}(\mathbf{u}, x)\frac{\partial u_j}{\partial x} + b_i = 0,$$
for $i = 1, 2, \ldots, N$. This equation will be hyperbolic (and hence solvable by the method of characteristics) if there exist N linearly independent real-valued N-dimensional vectors $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(N)}\}$ and N non-zero real-valued two-dimensional vectors $\{\boldsymbol{\alpha}^{(k)}, \boldsymbol{\beta}^{(k)}\}$ such that
$$\sum_{i,j=1}^{N} v_i^{(k)} \left[A_{ij}\boldsymbol{\alpha}^{(k)} - a_{ij}\boldsymbol{\beta}^{(k)}\right] = 0,$$
for $k = 1, \ldots, N$. See Whitham [4, Chapter 5, pages 113–142] for details and several examples using this formalism.
3. Referring to equation (98.1), it turns out that discontinuities in ∇u can propagate along characteristics, but discontinuities in u cannot. In fact, if u satisfies a second order linear hyperbolic partial differential equation in x and y, and if $\{u, u_x, u_y, u_{xx}, u_{xy}\}$ are all continuous across a curve C but u_{yy} suffers a jump upon crossing C, then C is necessarily a characteristic of the partial differential equation.
4. Eliminating the $\{s, \mathbf{t}\}$ variables at the end of the calculation will be possible, in principle, whenever the Jacobian of the transformation does not vanish; that is, $\frac{\partial(u,x_1,x_2,\ldots)}{\partial(s,t_1,t_2,\ldots)} \neq 0$.
5. An equivalent way of writing equation (98.3) is the form
$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \cdots = \frac{dx_N}{a_N},$$
which are called the *subsidiary equations*. When one or more of the a_k are zero, this equation looks peculiar, but it should be interpreted

to be the same as equation (98.3). This form is used in place of equation (98.3) in many older texts. This formulation has been used occasionally in this book.

6. See Farlow [1, Lesson 27, pages 205–212], Moon and Spencer [3, pages 27–29], and Zauderer [5, Chapter 3, pages 78–121].

References

[1] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[2] Landau, L. D., and Lifshitz, E. M. *Fluid Mechanics*. Pergamon Press, New York, 1959.

[3] Moon, P., and Spencer, D. E. *Partial Differential Equations*. D. C. Heath and Co., Lexington, MA, 1969.

[4] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

[5] Zauderer, E. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1983.

99. Characteristic Strip Equations

Applicable to Some partial differential equations in two independent variables.

Yields

When the technique is applicable, an implicit solution.

Idea

This method appears to be a generalization of the method of characteristics, but it can in fact be derived from that method. The formulae presented here are handy to use directly.

Procedure

Given the partial differential equation

$$F(x, y, u, p, q) = 0, \tag{99.1}$$

where $p = u_x$, $q = u_y$, we search for a solution $u = u(x, y)$. The technique is to solve the system of "strip equations" given by

$$\begin{aligned} \frac{\partial x}{\partial s} &= F_p, & \frac{\partial p}{\partial s} &= -F_x - pF_u, \\ \frac{\partial y}{\partial s} &= F_q, & \frac{\partial q}{\partial s} &= -F_y - qF_u, \\ \frac{\partial u}{\partial s} &= pF_p + qF_q, \end{aligned} \tag{99.2}$$

where we now consider $\{x, y, p, q, u\}$ to all be functions of the two variables $\{s, t\}$. The equations in equation (99.2) are also called *Charpit's equations*.

The "initial" values for equation (99.2) (corresponding to $s = 0$) are given in terms of the other independent variable t. It will be possible to give initial values to all of the terms in equation (99.2) because the original equation (99.1) will have data with it that can be parameterized in terms of t.

After we have determined $\{x, y, u\}$ as functions of $\{s, t\}$, we must solve the equations implicitly to obtain the final solution in the form $u = u(x, y)$.

Example

Suppose we have the nonlinear partial differential equation

$$u_x u_y - u = 0, \tag{99.3}$$

with the initial data

$$u = y^2 \qquad \text{on } x = 0. \tag{99.4}$$

By comparing equation (99.3) with equation (99.1), we find that $F = pq - u$. Hence, the equations in equation (99.2) can be written as

$$\begin{aligned} \frac{\partial x}{\partial s} &= q, & \frac{\partial p}{\partial s} &= p, \\ \frac{\partial y}{\partial s} &= p, & \frac{\partial q}{\partial s} &= q, \\ \frac{\partial u}{\partial s} &= 2pq. \end{aligned} \tag{99.5.a-e}$$

The initial conditions for equation (99.5) are given by parameterizing equation (99.4) in terms of the dummy variable t. One such parameterization (there are always infinitely many) is

$$x = 0, \qquad y = t, \qquad u = t^2. \tag{99.6}$$

To determine the initial conditions for p and q, we utilize the chain rule

$$\frac{\partial u}{\partial t} = p\frac{dx}{dt} + q\frac{dy}{dt},$$

which can be evaluated at $s = 0$ (using equation (99.6)) to yield

$$2t = p(0,t) \cdot 0 + q(0,t) \cdot 1$$

or $q(0,t) = 2t$. The original equation, (99.4), can be evaluated at $s = 0$ to determine that $p(0,t) = u(0,t)/q(0,t) = t/2$. Now that we have the initial conditions for all five variables appearing in equation (99.5), we can find the solution.

Equations (99.5.b) and (99.5.d) can be integrated directly to yield

$$p = \frac{1}{2}te^s, \qquad q = 2te^s.$$

Substituting these expressions in equations (99.5.a), (99.5.c), and (99.5.e) and integrating results in

$$\begin{aligned} x &= 2t(e^s - 1), \\ y &= \frac{1}{2}t(e^s + 1), \\ u &= t^2 e^{2s}. \end{aligned} \tag{99.7.a-c}$$

Equations (99.7.a) and (99.7.b) can be inverted to produce s and t as functions of x and y:

$$e^s = \frac{4y + x}{4y - x}, \qquad t = \frac{4y - x}{4}.$$

Using these relations in equation (99.7.c) yields the final answer

$$u(x,t) = \frac{(x+4y)^2}{16}.$$

Notes

1. This method is sometimes called the Lagrange–Charpit method.
2. Frequently, inverting the variables at the end (i.e., finding $s = s(x,y)$ and $t = t(x,y)$) is the only step that cannot be carried out analytically.
3. The variable s really specifies a characteristic, whereas t represents distance along any single characteristic.
4. This technique works, as the example shows, even when the original equation is not quasilinear. That is, the method of characteristics could not have been applied directly to equation (99.3).
5. See also Copson [1, pages 5–9], Garabedian [2, pages 24–31], Sneddon [3, pages 61–66], and Zauderer [4, pages 56–68].

References

[1] Copson, E. T. *Partial Differential Equations.* Cambridge University Press, New York, 1975.

[2] Garabedian, P. R. *Partial Differential Equations.* John Wiley & Sons, New York, 1964.

[3] Sneddon, I. N. *Elements of Partial Differential Equations.* McGraw–Hill Book Company, New York, 1957.

[4] Zauderer, E. *Partial Differential Equations of Applied Mathematics.* John Wiley & Sons, New York, 1983.

100. Conformal Mappings

Applicable to Laplace's equation ($\nabla^2 u = 0$) in two dimensions.

Yields

A reformulation of the original problem.

Idea

Laplace's equation in two dimensions with a given boundary can be transformed to Laplace's equation with a different boundary by a conformal map. The idea is to choose the conformal map in such a way that the new boundary makes the problem easy to solve.

Procedure

Given Laplace's equation in the variables $\{x, y\}$ (i.e., $\nabla^2 u = u_{xx} + u_{yy} = 0$), we define the complex variable $z = x + iy$, where $i = \sqrt{-1}$. All of the boundaries of the original problem can now be described by values of z.

Any analytic transformation between two complex variables, say $\zeta = F(z)$, for which $d\zeta/dz$ is never zero, is said to be *conformal*. It turns out that Laplace's equation is invariant under a conformal map. That is, if $\zeta = \xi + i\eta = F(z)$, $u_{xx} + u_{yy} = 0$, and $F(z)$ is a conformal map, then $u_{\xi\xi} + u_{\eta\eta} = 0$.

In the new variables, $\{\xi, \eta\}$, the boundary might be very simple. If so, then Laplace's equation can be solved in this new domain. Then the solution of Laplace's equation in the original domain can be found by the change of variables induced by the conformal map.

A commonly used conformal map is the *Schwartz–Christoffel transformation*. This maps a closed polygonal figure (with n vertices) into a half plane. The mapping is given by the solution of

$$\frac{dz}{d\zeta} = C(\zeta - \zeta_1)^{\beta_1/\pi - 1}(\zeta - \zeta_2)^{\beta_2/\pi - 1} \cdots (\zeta - \zeta_n)^{\beta_n/\pi - 1} \tag{100.1}$$

for appropriate $\{\beta_1, \beta_2, \ldots, \beta_n\}$ and $\{\zeta_1, \zeta_2, \ldots, \zeta_n\}$. The $\{\beta_i\}$ are the interior angles of the polygon, and the $\{\zeta_i\}$ are the (complex valued) positions of the polygon's vertices.

After the differential equation (100.1) is formulated, it must be solved. The unknown constant C, as well as the arbitrary constant resulting from the integration, will be determined when the $\{\zeta_i\}$ are prescribed. The resulting function $\zeta = F(z)$ is the conformal map that maps the interior of the given polygonal figure into the half plane. See Trefethen [11] for a numerical implementation.

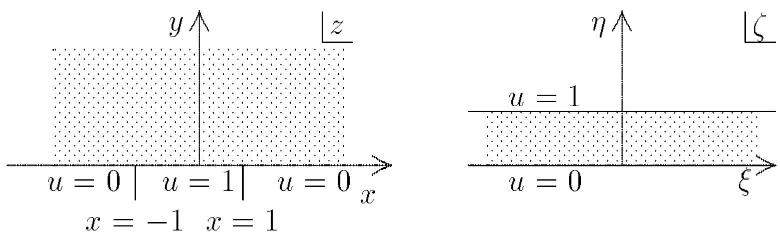

Figure 100.1: The original domain for Laplace's equation and the domain after a conformal mapping has been applied.

Example 1

Suppose we have Laplace's equation ($u_{xx} + u_{yy} = 0$) to solve in the half plane $\mathcal{H} = \{-\infty < x < \infty,\, 0 < y < \infty\}$ with the boundary conditions

$$u(x,0) = \begin{cases} 0 & \text{for } |x| > 1, \\ 1 & \text{for } |x| \le 1. \end{cases}$$

Under the mapping

$$\zeta = \xi + i\eta = F(z) = \log\left(\frac{z-1}{z+1}\right) = \log\left(\frac{x+iy-1}{x+iy+1}\right), \tag{100.2}$$

the half plane $\mathcal{H}$ is mapped into a strip of height π in the $(\xi,\, \eta)$ plane. See figure 100.1 for pictures of the two geometrical regions involved.

In the $(\xi,\, \eta)$ plane the boundary conditions become

$$\begin{aligned} u(\xi, 0) &= 0, \\ u(\xi, \pi) &= 1. \end{aligned}$$

The solution to Laplace's equation in this domain is simply $u(\xi, \eta) = \eta/\pi$. To transform back to $(x,\, y)$ coordinates, the transformation in equation (100.2) must be inverted. After some algebra it can be shown that

$$\eta = \arg\left(\frac{z-1}{z+1}\right) = \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right),$$

so that

$$u(x,y) = \frac{1}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right).$$

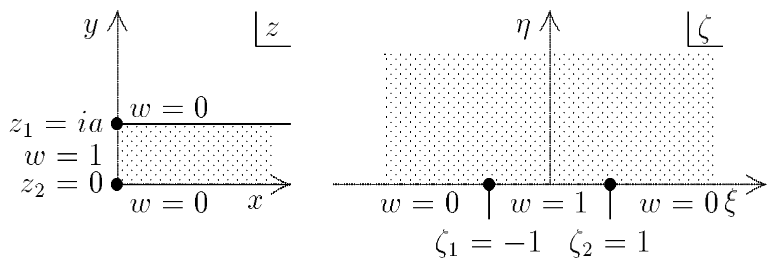

Figure 100.2: The original domain for Laplace's equation and the domain after the Schwartz–Christoffel transformation has been applied.

Example 2

Suppose we have Laplace's equation ($\nabla^2 w = 0$) in the channel open on the right (see figure 100.2), with the boundary conditions

$$\begin{aligned} w(x,0) &= 0 \qquad \text{for } 0 \le x < \infty \\ w(x,a) &= 0 \qquad \text{for } 0 \le x < \infty \\ w(0,y) &= 1 \qquad \text{for } 0 \le y \le 1. \end{aligned}$$

The polygon in which this problem is being solved has vertices at $z_1 = ia$ and $z_2 = 0$, with the corresponding interior angles $\beta_1 = \beta_2 = \pi/2$. Using the Schwartz–Christoffel transformation, we choose the vertices in the z plane to map to the vertices $\zeta_1 = -1$ and $\zeta_2 = 1$ in the ζ plane. The differential equation (100.1) becomes

$$\frac{dz}{d\zeta} = C\,(\zeta+1)^{1/2}\,(\zeta-1)^{1/2}$$

with the solution $z = C\cosh^{-1}\zeta + D$, where D is an arbitrary constant. To determine the constants C and D, we must enforce that the vertices in the z plane mapped to the vertices in the ζ plane. We have the two simultaneous equations:

$$\begin{aligned} z_1 = ia = C\cosh^{-1}(\zeta_1) + D = C\cosh^{-1}(-1) + D = Ci\pi + D, \\ z_2 = 0 = C\cosh^{-1}(\zeta_2) + D = C\cosh^{-1}(1) + D = D, \end{aligned}$$

with the solution $\{D = 0,\ C = a/\pi\}$. Hence, the desired conformal mapping is $\zeta = \cosh\left(\frac{\pi z}{a}\right)$. The problem in the ζ domain is now identical to the problem solved in Example 1.

Notes

1. Conformal mappings are often used in hydrodynamics and electrostatics because, under a conformal mapping, lines of flow and equipotential lines are mapped into lines of flow and equipotential lines.

2. Conformal mappings are often used to obtain an orthogonal coordinate system inside of a two-dimensional body. This may be used, for instance, when a grid is required on which the solution to a partial differential equation will be approximated numerically.
3. The mapping used in this method need not be conformal everywhere; it needs to be conformal only in the domain in which Laplace's equation is being solved. (Very few maps are conformal everywhere.)
4. The *Joukowski transformation*, given by $\zeta = z + a^2/z$, maps an ellipse into a circle or a circle into a strip.
5. Algebraic mappings, given by $\zeta = z^{\beta/\pi}$, with $\beta > 0$, map a corner with angle α to a corner with angle $\alpha\beta/\pi$. For instance, if $\beta = 2\pi$, then a quarter plane ($\alpha = \pi/2$) is mapped to a half plane.
6. Numerical implementation of the Schwartz–Christoffel transformation can fail on some seemingly very simple polygons. Mapping a rectangle with an aspect ration of 20 to 1, or an other region with a similar degree of elongation, onto a half-plane may cause problems because the points in the transformed plane will be very close together. (This is known as the "crowding phenomenon.")
7. The Schwartz–Christoffel transformation can also be used for doubly connected domains, see Iyanaga and Kawada [6, page 1156].
8. Even when an analytic conformal map cannot be found, there are fast numerical techniques for finding an approximate conformal map. Riemann's mapping theorem states that all bounded simply connected plane regions can be conformally mapped onto the unit disk, and all bounded doubly connected plane regions can be conformally mapped onto an annulus. Using Poisson's formula (see page 478) exact solutions can be written down for these two geometries. See Fornberg [5] or Trefethen [12] for details.
9. Kober [8] has a large collection of conformal mappings, with the geometric regions in both the (x, y) and (η, ζ) planes clearly illustrated.
10. Seymour [10] describes a computer package that permits real-time manipulation and display of conformal mappings of one complex plane onto another.
11. If $\nabla^2_{x,y}$ represents the Laplacian in $\{x, y\}$ space, then under the conformal mapping $\zeta = F(z)$ the operator $\nabla^2_{x,y}$ is mapped to the operator $|F'(z)|^2\nabla^2_{\eta,\zeta}$. Hence, the biharmonic equation $\nabla^4 u := \nabla^2_{x,y}\nabla^2_{x,y}u = 0$ becomes $|F'(z)|^2\nabla^2_{\eta,\zeta}\left(|F'(z)|^2\nabla^2_{\eta,\zeta}\right)u = 0$.
12. See also Farlow [3, Lesson 47, pages 379–388], Kantorovich and Krylov [7, Chapters 5 and 6, pages 358–615], and Levinson and Redheffer [9, Chapter 5, pages 259–332].

References

[1] Delillo, T. K. The accuracy of numerical conformal mapping methods: A survey of examples and results. *SIAM J. Numer. Anal. 31*, 3 (June 1994),

788–812.

[2] DeLillo, T. K., and Elcrat, A. R. A comparison of some numerical conformal mapping methods for exterior regions. *SIAM J. Sci. Stat. Comput. 12*, 2 (March 1991), 399–422.

[3] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[4] Floryan, J. M., and Zemach, C. Schwarz–Christoffel mappings: A general approach. *J. Comput. Physics 72* (1987), 347–371.

[5] Fornberg, B. A numerical method for conformal mapping. *SIAM J. Sci. Stat. Comput. 1* (1980), 386–400.

[6] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics*. MIT Press, Cambridge, MA, 1980.

[7] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis*. Interscience Publishers, Inc., New York, 1958.

[8] Kober, H. *Dictionary of Conformal Representations*. Dover Publications, Inc., New York, 1952.

[9] Levinson, N., and Redheffer, R. M. *Complex Variables*. Holden–Day, Inc., San Francisco, CA, 1979.

[10] Seymour, H. R. Conform: A conformal mapping system. In *SYMSAC '86 (Proceedings of the 1986 ACM Symposium on Symbolic and Algebraic Computation)*, B. W. Char, Ed. ACM, New York, 1986, pp. 163–168.

[11] Trefethen, L. N. Numerical computation of the Schwarz–Christoffel transformation. *SIAM J. Sci. Stat. Comput. 1*, 1 (March 1980), 82–102.

[12] Trefethen, L. N. *Numerical Conformal Mapping*. North–Holland Publishing Co., New York, 1986.

[13] Walker, M. *The Schwarz–Christoffel Transformation and Its Applications – A Simple Exposition*. Dover Publications, Inc., New York, 1964.

101. Method of Descent

Applicable to Partial differential equations (most often, wave equations).

Yields

An exact solution.

Idea

For some partial differential equations (in particular, some wave equations) odd dimensional problems are "easier" than even dimensional problems. Hence it is reasonable, when given a $2n$-dimensional problem, to instead solve a $2n+1$-dimensional problem and then "come down one dimension."

Procedure

Given a partial differential equation in n dimensions for the quantity $u(\mathbf{x}) = u(x_1, x_2, \ldots, x_n)$

$$L[u] = 0,$$

it might be easier to solve the $n+1$-dimensional problem

$$L[v] + H[v] = 0,$$

for $v(\mathbf{x}, z) = v(x_1, x_2, \ldots, x_n, z)$, where $H[\cdot]$ is a differential operator with respect to z. Then, when $v(\mathbf{x}, z)$ is known, $u(\mathbf{x})$ can be obtained by either (1) an appropriate integral over z or (2) taking v to be independent of z.

Example

Suppose we are given the two-dimensional wave equation

$$u_{tt} = c^2 \left(u_{xx} + u_{yy} \right), \tag{101.1}$$

with the initial conditions

$$u(0, \mathbf{x}) = f(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = g(\mathbf{x}), \tag{101.2}$$

where $\mathbf{x} = (x, y)$. We might choose to instead solve the three-dimensional wave equation

$$v_{tt} = c^2 \left(v_{xx} + v_{yy} + v_{zz} \right),$$

with the initial conditions

$$v(0, \mathbf{x}, z) = f(\mathbf{x}), \qquad v_t(0, \mathbf{x}, z) = g(\mathbf{x}).$$

The three-dimensional wave equation has the well-known solution (see page 501)

$$v(t, \mathbf{x}, z) = ctM[g] + \frac{\partial}{\partial t}\Big(ctM[f] \Big), \tag{101.3}$$

where $M[\cdot]$ is a functional defined to be the average value of its argument on a circle of radius ct; that is,

$$\begin{aligned} M[h(x,y,z)] &:= \frac{1}{4\pi c^2 t^2} \int_{S(t)} h\, dS \\ &= \frac{1}{4\pi c^2 t^2} \int_0^{\pi} \int_0^{2\pi} h(x + ct\sin\theta\cos\phi, y + ct\sin\theta\sin\phi, z + ct\cos\theta) \\ &\qquad \times \sin\theta\, d\theta\, d\phi, \end{aligned} \tag{101.4}$$

where $S(t)$ is the surface of a sphere with origin at (x, y, z) and radius ct.

To solve the two-dimensional wave equation (101.1), we merely utilize the fact that f and g are independent of the variable z. Performing some algebraic manipulations, equation (101.4) becomes

$$M[h(x,y)] = \frac{1}{2\pi ct} \iint_{\sigma(t)} \frac{h(\zeta,\eta)\, d\zeta\, d\eta}{\sqrt{c^2t^2 - (x-\zeta)^2 - (y-\eta)^2}}, \tag{101.5}$$

where $\sigma(t)$ is the interior of the circle: $(x-\zeta)^2 + (y-\eta)^2 = c^2t^2$. Using equation (101.5) in equation (101.3) results in the solution to equations (101.1) and (101.2).

Notes

1. This method is also called *Hadamard's method of descent*.
2. If the descent step was applied once again, the solution of the one-dimensional wave equation, $w_{tt} = c^2 w_{xx}$, could be obtained from equations (101.3) and (101.5).
3. Note that a line source, in three dimensions, might be viewed as a point source in two dimensions.
4. One reason that odd space dimensional problems are sometimes easier than even dimensional problems is Huygen's principle. Huygen's principle (see Chester [1, pages 154–156] or Garabedian [4, Section 6.3, pages 204–210]) states that the wave equation in an odd number of space dimensions depends only on the initial data (and its derivatives) on the *perimeter* of the domain of dependence. See the section on exact solutions of the wave equation (on page 501).
5. See also Copson [2, pages 95–96], Farlow [3, pages 187–188], Whitham [5, pages 219–235], and Zauderer [6, pages 226–232].

References

[1] Chester, C. R. *Techniques in Partial Differential Equations*. McGraw–Hill Book Company, New York, 1970.

[2] Copson, E. T. *Partial Differential Equations*. Cambridge University Press, New York, 1975.

[3] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[4] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[5] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

[6] Zauderer, E. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1983.

102. Diagonalization of a Linear System of PDEs

Applicable to A linear system of partial differential equations in two independent variables, of the form $\mathbf{u}_t + A\mathbf{u}_x = 0$, where A is a constant matrix.

Yields

A set of uncoupled equations.

Idea

By diagonalizing the coefficient matrix, the equations can be uncoupled and then solved.

Procedure

Given the linear system of differential equations

$$\mathbf{u}_t + A\mathbf{u}_x = 0, \tag{102.1}$$

we change the dependent variables to decouple the system. If the matrix A is $n \times n$ and has the eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ (which we assume to be linearly independent), then we define the matrix S by $S = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n\end{bmatrix}$. Changing variables in equation (102.1) by $\mathbf{u} = S\mathbf{w}$ results in $S\mathbf{w}_t + AS\mathbf{w}_x = 0$, or

$$\mathbf{w}_t + \Lambda\mathbf{w}_x = 0, \tag{102.2}$$

where $\Lambda = S^{-1}AS$ is a diagonal matrix. The equations in (102.2) are now decoupled and can be solved separately for $\{w_1(x,t), w_2(x,t), \ldots, w_n(x,t)\}$. After they have been found, $\mathbf{u}$ may be determined from $\mathbf{u} = S\mathbf{w}$.

Example

Given the system of linear partial differential equations in two independent variables

$$\begin{aligned}\frac{\partial u_1}{\partial t} + 9\frac{\partial u_1}{\partial x} + 2\frac{\partial u_2}{\partial x} &= 0,\\ \frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x} + 8\frac{\partial u_2}{\partial x} &= 0,\end{aligned} \tag{102.3}$$

we define the vector $\mathbf{u} = \left[\begin{smallmatrix}u_1\\ u_2\end{smallmatrix}\right]$ and the matrix $A = \left[\begin{smallmatrix}9 & 2\\ 1 & 8\end{smallmatrix}\right]$ so that equation (102.3) may be written in the form of equation (102.1).

The eigenvalues of A are $\lambda = 7$ and $\lambda = 10$ with the corresponding eigenvectors: $\mathbf{v}_1 = \begin{bmatrix}1 & -1\end{bmatrix}^{\mathrm{T}}$ and $\mathbf{v}_2 = \begin{bmatrix}2 & 1\end{bmatrix}^{\mathrm{T}}$. Hence, the matrix S is given by $S = \left[\begin{smallmatrix}2 & 1\\ 1 & -1\end{smallmatrix}\right]$, which has the inverse $S^{-1} = \begin{bmatrix}1/3 & 1/3\\ 1/3 & -2/3\end{bmatrix}$. Making the change of variables $\mathbf{u} = S\mathbf{w}$ turns equation (102.1) into equation (102.2) with Λ defined by

$$\begin{aligned} \Lambda &= S^{-1}AS, \\ &= \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \\ &= \begin{bmatrix} 10 & 0 \\ 0 & 7 \end{bmatrix}. \end{aligned}$$

The equations in (102.2) can then be separated to obtain

$$\begin{aligned} \frac{\partial w_1}{\partial t} + 10\frac{\partial w_1}{\partial x} &= 0, \\ \frac{\partial w_2}{\partial t} + \ \ 7\frac{\partial w_2}{\partial x} &= 0. \end{aligned}$$

These equations have the solution

$$\begin{aligned} w_1(x,t) &= f(x-10t), \\ w_2(x,t) &= g(x-7t), \end{aligned}$$

where f and g are arbitrary functions of their arguments. Knowing $\mathbf{w}$ we can determine $\mathbf{u} = S\mathbf{w}$ to be

$$\begin{aligned} u_1(x,t) &= 2w_1(x,t) + w_2(x,t) = 2f(x-10t) + g(x-7t), \\ u_2(x,t) &= \ \ w_1(x,t) - w_2(x,t) = \ \ f(x-10t) - g(x-7t). \end{aligned} \tag{102.4}$$

Knowing the general form of the solution, any initial conditions for $u_1(x,t)$ and $u_2(x,t)$ could be utilized. For example, if we had

$$\begin{aligned} u_1(x,0) &= 3\sin 2x, \\ u_2(x,0) &= 0, \end{aligned} \tag{102.5}$$

then utilizing equation (102.4) in equation (102.5) produces

$$\begin{aligned} 2f(x) + g(x) &= 3\sin 2x, \\ f(x) - g(x) &= 0, \end{aligned}$$

and so $f(z) = g(z) = \sin 2z$ and the final solution can be written

$$\begin{aligned} u_1(x,t) &= 2\sin(2x-20t) + \sin(2x-14t), \\ u_2(x,t) &= \ \ \sin(2x-20t) - \sin(2x-14t). \end{aligned}$$

Note

1. See Farlow [1, Lesson 29, pages 223–231]

Reference

[1] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

103. Duhamel's Principle

Applicable to

Linear parabolic and hyperbolic partial differential equations.

Yields

An integral representation in terms of the solution of a more tractable partial differential equation.

Idea

To solve a parabolic partial differential equation with a time-varying source function and time-varying boundary conditions, only a parabolic partial differential equation with a constant source term and constant boundary conditions needs to be solved.

Procedure

Suppose we have the parabolic partial differential equation for $u(\mathbf{x}, t)$

$$\begin{aligned} \frac{\partial}{\partial t} u(\mathbf{x}, t) &= L[u(\mathbf{x}, t)] + F(\mathbf{x}, t), \\ u(\mathbf{y}, t) &= G(\mathbf{y}, t), \qquad \text{for } t > 0, \\ u(\mathbf{x}, 0) &= H(\mathbf{x}), \end{aligned} \tag{103.1}$$

where $L[\cdot]$ is an elliptic operator in $\mathbf{x}$ and $\mathbf{y}$ denotes a point on the boundary. Note that equation (103.1) has a time-dependent source function $F(\mathbf{x}, t)$ and time-dependent surface conditions $G(\mathbf{y}, t)$. Instead of solving equation (103.1) for $u(\mathbf{x}, t)$, we choose to solve the parabolic partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} v(\mathbf{x}, t, \tau) &= L[v(\mathbf{x}, t, \tau)] + F(\mathbf{x}, \tau), \\ v(\mathbf{y}, t, \tau) &= G(\mathbf{y}, \tau), \qquad \text{for } t > 0, \\ v(\mathbf{x}, 0, \tau) &= H(\mathbf{x}), \end{aligned} \tag{103.2}$$

for $v(\mathbf{x}, t, \tau)$. Note that the variable of integration in equation (103.2) is t, while the source term and the surface conditions depend upon the parameter τ. Hence, the equation for $v(\mathbf{x}, t, \tau)$ has (effectively) a *constant* source term and *constant* surface conditions. Thus, it should be easier to determine $v(\mathbf{x}, t, \tau)$ than it was to determine $u(\mathbf{x}, t)$.

Knowing the solution of equation (103.2), the solution to equation (103.1) can be written as

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t} \int_0^t v(\mathbf{x}, t - \tau, \tau)\, d\tau. \tag{103.3}$$

This is easily derived from manipulations of the Laplace transforms of equation (103.1) and equation (103.2). See any of the references for details.

Example

Suppose we want to solve the equations describing the temperature of an initially cool, insulated rod with a temperature $f(t)$ specified at one end

$$\begin{aligned} u_t &= u_{xx}, && \text{for } 0 < x < 1, 0 < t < \infty, \\ u(0,t) &= 0, && \text{for } 0 < t < \infty, \\ u(1,t) &= f(t), && \text{for } 0 < t < \infty, \\ u(x,0) &= 0, && \text{for } 0 \le x \le 1. \end{aligned} \tag{103.4}$$

Instead of solving equation (103.4) for $u(x,t)$ we solve

$$\begin{aligned} v_t &= v_{xx}, && \text{for } 0 < x < 1, 0 < t < \infty, \\ v(0,t,\tau) &= 0, && \text{for } 0 < t < \infty, \\ v(1,t,\tau) &= f(\tau), && \text{for } 0 < t < \infty, \\ v(x,0,\tau) &= 0, && \text{for } 0 \le x \le 1, \end{aligned} \tag{103.5}$$

for $v(x,t,\tau)$. By separation of variables (see page 487), the solution of equation (103.5) is found to be

$$v(x,t,\tau) = f(\tau)\left[x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x\right],$$

which, for notational convenience, we choose to write as $v(x,t,\tau) = f(\tau)g(x,t)$. Using equation (103.3), the solution for $u(x,t)$ can then be written as

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\int_0^t v(x,t-\tau,\tau)\,d\tau \\ &= \frac{\partial}{\partial t}\int_0^t f(\tau)g(x,t-\tau)\,d\tau \\ &= \frac{\partial}{\partial t}\int_0^t f(t-T)g(x,T)\,dT \\ &= f(0)g(x,t) + \int_0^t f'(t-T)g(x,T)\,dT, \end{aligned} \tag{103.6}$$

where we defined $T = t - \tau$ in the above. If, for example, $f(t) = e^{-t}$, then equation (103.6) may be simplified to yield

$$u(x,t) = x - e^{-t} - 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n \sin n\pi x}{n(1-n^2\pi^2)}\left\{n^2\pi^2 e^{-n^2\pi^2 t} - e^{-t}\right\}\right).$$

Notes

1. The procedure for hyperbolic partial differential equations is analogous to the procedure for parabolic partial differential equations. Consider, for example, the hyperbolic equation

$$u_{tt} + L[u] = b(x,t),$$

(where $L[\cdot]$ is uniformly elliptic) with the boundary conditions

$$u(x,0) = u_t(x,0) = 0.$$

If $v(x,t,\tau)$ is defined to be the solution of

$$\begin{aligned} v_{tt} + L[v] &= 0, \qquad \text{for } t > \tau, \\ v(x,\tau,\tau) &= 0, \\ v_t(x,\tau,\tau) &= b(x,\tau), \end{aligned}$$

then we have $u(x,t) = \int_0^t v(x,t,\tau)\,d\tau$. Using this formulation, it can be shown that the solution to $u_{tt} - c^2\nabla^2 u = F(x,y,z,t)$, is given by

$$u(x,y,z,t) = \frac{1}{4\pi c} \iiint\limits_{\xi^2+\eta^2+\zeta^2 \leq c^2t^2} \frac{F(\xi,\eta,\zeta,t-r/c)}{r}\,d\xi\,d\eta\,d\zeta, \tag{103.7}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$. The integrand in equation (103.7) is called the *retarded potential*.

2. See Chester [1, pages 156–158], Courant and Hilbert [2, Volume 2, pages 202–204], Farlow [3, Lesson 14, pages 106–111], Sneddon [4, pages 278–282], and Zauderer [5, pages 159–165].

References

[1] Chester, C. R. *Techniques in Partial Differential Equations*. McGraw–Hill Book Company, New York, 1970.

[2] Courant, R., and Hilbert, D. *Methods of Mathematical Physics*. Interscience Publishers, Inc., New York, 1953.

[3] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[4] Sneddon, I. N. *Elements of Partial Differential Equations*. McGraw–Hill Book Company, New York, 1957.

[5] Zauderer, E. *Partial Differential Equations of Applied Mathematics*. John Wiley & Sons, New York, 1983.

104. Exact Equations

Applicable to

Quasilinear partial differential equations.

Yields

An exact solution.

Idea

Some quasilinear partial differential equations can be integrated directly.

Procedure

Consider the quasilinear partial differential equation

$$M(x, y, u)u_x = N(x, y, u)u_y. \tag{104.1}$$

If this equation satisfies the exactness condition $M_x = N_y$, then an implicit solution to equation (104.1) will be given by $\phi(x, y, u) = 0$, where

$$M = \phi_y, \qquad N = \phi_x. \tag{104.2.a-b}$$

To determine the function ϕ, integrate equation (104.2.a) to obtain

$$\phi = \int M\, dy + g(x, u). \tag{104.3}$$

Then, using equation (104.2.b) we have

$$\int M_x\, dy + g_x(x, u) = N$$

or (solving for g and integrating)

$$g(x, u) = \int \left(N - \int M_x\, dy \right) dx + h(u), \tag{104.4}$$

where $h(u)$ is an arbitrary function. Using equation (104.4) in equation (104.3) results in the final solution.

Example

Consider the equation

$$yu_x = xuu_y,$$

for which $M = y$ and $N = xu$. This equation is exact because $M_x = 0 = N_y$. From equation (104.3), we have

$$\phi = \int M\, dy + g(x, u) = \frac{1}{2}y^2 + g(x, u).$$

From equation (104.2.b), we have $\phi_x = g_x = N = xu$, or $g = \frac{1}{2}x^2u + h(u)$. This leads to the general implicit solution:

$$\phi = \frac{1}{2}\left(y^2 + x^2 u\right) + h(u) = 0.$$

Choosing, for example, $h(u) = \frac{1}{2}(au + b)$ results in the explicit solution

$$u(x, y) = -\frac{b + y^2}{a + x^2}.$$

Note

1. The above example is from Benton [1].

Reference

[1] Benton, Jr., S. H. *The Hamilton–Jacobi Equation.* Academic Press, New York, 1977.

105. Hodograph Transformation

Applicable to Quasilinear partial differential equations, a single equation, or a system of equations.

Yields

A new formulation of the original equations.

Idea

In a partial differential equation, it may be easier to solve the equation with the dependent and independent variables switched.

Procedure

This procedure works on a quasilinear equation or a system of such equations. That is, every term of each equation must have one and only one first derivative term, and there can be no higher order derivative terms in the equations.

Consider the case of two dependent variables (u, v) in two independent variables (x, y). Suppose $L[u, v] = 0$ represents the equation(s) to be solved for $u(x, y)$ and $v(x, y)$. This equation is transformed to the "hodograph" plane by writing $x = x(u, v)$ and $y = y(u, v)$ and transforming $L[u, v] = 0$ into a new equation $H[x, y] = 0$. In this new equation, x and y are treated as the dependent variables.

The solution obtained will, in general, be implicit. After the solution is obtained in the hodograph plane, the transformation must be checked to ensure that it is not singular.

Example 1

Suppose we have a pair of nonlinear equations arising from gas dynamics (from Whitham [9, page 182])

$$\begin{aligned} v_y + uv_x + bvu_x &= 0, \\ u_y + uu_x + \frac{1}{b}vv_x &= 0, \end{aligned} \tag{105.1}$$

where b is a constant. Because the equations in equation (105.1) are quasilinear, the method of characteristics can be used to solve them. However, it is difficult to use that method directly.

The hodograph transformation can be used on equation (105.1) by inverting $u(x, y)$, $v(x, y)$ to find (see, e.g., Kaplan [6, pages 132–135], on how to change variables in this manner)

$$\begin{aligned} x_u &= -v_y/J, \qquad & x_v &= u_y/J, \\ y_u &= v_x/J, \qquad & y_v &= -u_x/J \end{aligned} \tag{105.2}$$

where J is the Jacobian of the transformation, $J = u_y v_x - v_y u_x$. Using equation (105.2) in equation (105.1) results in the equations

$$\begin{aligned} x_u - u y_u + b v y_v &= 0, \\ x_v - u y_v + \frac{1}{b} v y_u &= 0. \end{aligned} \tag{105.3}$$

Because the original equations were linear, the Jacobian factors out of the equations (assuming it never vanishes) and does not appear in (105.3).

The equations in (105.3) are now quasilinear in the dependent variables (x, y). They may easily be solved by the method of characteristics; the details may be found in Whitham [9].

Example 2

An equation that arises in transonic small disturbance theory is

$$\phi_x \phi_{xx} - \phi_{yy} = 0. \tag{105.4}$$

Using $a := \phi_x$ and $b := \phi_y$, equation (105.4) can be written as the system of quasilinear equations:

$$a_y - b_x = 0, \qquad -a a_x + b_y = 0.$$

Using the hodograph transformation, these equations simplify to

$$x_b - y_a = 0, \qquad a y_b - x_a = 0,$$

with $J = x_b y_a - y_b x_a$. Combining these equations results in the familiar Tricomi equation: $a y_{bb} - y_{aa} = 0$.

Notes

1. The hodograph transformation is frequently used in fluid mechanics for problems with unknown boundaries. In many situations, the boundaries become fixed in the hodograph plane.
2. The transformation will be non-singular if the Jacobian of the transformation, J, does not vanish in the region of interest.
3. Ames [1, pages 35–37] shows that the nonlinear equations

   $$u_t - v_x = 0, \qquad v_t - F^2(u) u_x = 0,$$

   become, after applying the hodograph transformation, the linear equations:

   $$x_v - y_u = 0, \qquad x_u - F^2(u) y_v = 0.$$

4. Whitham [9, page 617] shows how the Born–Infeld equation

   $$\left(1 - u_t^2\right) u_{xx} + 2 u_x u_t u_{xt} - \left(1 + u_x^2\right) u_{tt} = 0$$

   may be linearized with the Hodograph transformation.

5. This technique can also be applied to ordinary differential equations; a differential equation for $y(x)$ is inverted to become a differential equation for $x(y)$ (see page 360).
6. Given a PDE for $u(x,t)$ Clarkson *et al.* [3] define a *pure hodograph transformation* to be the change of independent variables $\{\tau = t, \xi = u(x,t)\}$. They define an *extended hodograph transformation* to be the change of independent variables $\{\tau = t, \xi = \int^x \phi(u(z,t))\, dz\}$. Using these definitions they have:

   Theorem: The most general second-order, quasilinear PDE of the form $u_t = g(u)u_{xx} + f(u, u_x)$ with $dg/du \neq 0$, which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form $S_\tau = S_{\xi\xi} + G(S, S_\xi)$ is given by

$$u_t = g(u)u_{xx} + \left(\frac{gg''}{g'} - \frac{g'}{2}\right) u_x^2 + b'(u)u_x$$

   where $' \equiv d/du$, and $g(u)$ and $b(u)$ are arbitrary functions.

   Theorem: The most general third-order, quasilinear PDE of the form $u_t = g(u)u_{xxx} + f(u, u_x, u_{xx})$ with $dg/du \neq 0$, which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form $S_\tau = S_{\xi\xi\xi} + G(S, S_\xi, S_{\xi\xi})$ is given by

$$\begin{aligned} u_t = g(u)u_{xxx} &+ B_u u_x + B_{u_x} u_{xx} \\ &+ \left(\frac{g''}{g'} - \frac{4g'}{3g}\right) B u_x + \left(\frac{gg''}{g'} - \frac{g'}{3}\right) u_x u_{xx} \end{aligned}$$

   where $B_u \equiv \partial B/\partial u$, $B_{u_x} \equiv \partial B/\partial u_x$, $' \equiv d/du$, and $g(u)$ and $B(u, u_x)$ are arbitrary functions.

   Theorem: The most general quasilinear PDE of the form $u_t = g(u)u_{x(n)} + f(u, u_x, \cdots, u_{x(n-1)})$ with $dg/du \neq 0$, which may be transformed via an extended hodograph transformation to a semilinear partial differential equation of the form $S_\tau = S_{\xi(n)} + G(S, S_\xi, \cdots, S_{\xi(n-1)})$ is given by

$$\begin{aligned} u_t &= g(u)u_{x(n)} + \left(\frac{g''}{g'} - \frac{n+1}{n}\frac{g'}{g}\right) B u_x \\ &+ B_u u_x + \sum_{r=2}^{n-1} B_{u_{x(r-1)}} u_{x(r)} + \left(\frac{gg''}{g'} - \frac{g'}{n}\right) u_x u_{x(n-1)} \end{aligned}$$

   where $' \equiv d/du$, and $g(u)$ and $B(u, u_x, \cdots, u_{x(n-2)})$ are arbitrary functions.

7. The Harry Dym equation $u_t = (u^{-1/2})_{xxx}$, written in potential form (i.e., using $u = v_x$) is $v_t = (v_x^{-1/2})_{xx}$. This equation is invariant under a pure hodograph transformation. That is, the transformed equation is $w_\tau = (w_\xi^{-1/2})_{\xi\xi}$.

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Bergman, S. The hodograph method in the theory of compressible fluid. Tech. rep., Brown University, 1942. Supplement to *Fluid Dynamics* by von Mises and Friedrichs.

[3] Clarkson, P. A., Fokas, A. S., and Ablowitz, M. J. Hodograph transformations of linearizable partial differential equations. *SIAM J. Appl. Math. 49*, 4 (August 1989), 1188–1209.

[4] Crank, J. *Free and Moving Boundary Problems*. Clarendon Press, Oxford, England, 1984.

[5] Crank, J., and Ozis, T. Numerical solution of a free boundary problem by interchanging dependent and independent variables. *J. Inst. Maths. Applics 26* (1980), 77–85.

[6] Kaplan, W. *Advanced Calculus*. Addison–Wesley Publishing Co., Reading, MA, 1952.

[7] Manwell, A. R. *The Hodograph Equations: An Introduction to the Mathematical Theory of Plane Transonic Flow*. Hafner, Darien, CT, 1971.

[8] Siddiqui, A. M., Kaloni, P. N., and Chandna, O. P. Hodograph transformation methods in non-Newtonian fluids. *J. Eng. Math. 19* (1985), 203–216.

[9] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

106. Inverse Scattering

Applicable to

Nonlinear evolution equations, a single equation, or a system.

Yields

A reformulation into an inverse problem, which can sometimes result in an exact solution.

Idea

By rewriting the evolution equation, some natural eigenfunction problems emerge.

Procedure

An evolution equation for $u(t, \mathbf{x}) = u(t, x_1, \ldots, x_m)$ may be written in the form

$$u_t = K(u), \tag{106.1}$$

where $K(\cdot)$ denotes a nonlinear differential operator in x. For a system of equations, the u in equation (106.1) represents a vector of unknowns $(u_1, \ldots, u_n)$.

The procedure is to write equation (106.1) in the Lax pair form (this is often the hardest part of the procedure)

$$L_t = i[L, A] = i(LA - AL), \tag{106.2}$$

where L and A are linear differential operators in $\mathbf{x}$, whose coefficients are polynomials in u and its $\mathbf{x}$ derivatives. Here, L_t refers to differentiation of u (and its derivatives) with respect to t in the expression for L. See Example 1 for how equation (106.2) is to be interpreted. Note that, if A were a Hamiltonian, then equation (106.2) would be a Heisenberg equation.

A straightforward calculation now shows that

$$i\phi\lambda_t = (L - \lambda)(A\phi - i\phi_t),$$

for arbitrary $\phi(t, \mathbf{x})$ and λ. If we assume that $\phi(t = 0, \mathbf{x})$ and $\lambda(t)$ are an eigenfunction–eigenvalue pair for L, that is

$$L\phi = \lambda\phi, \tag{106.3}$$

and if the eigenfunctions $\{\phi_j(t, \mathbf{x})\}$ evolve in time as

$$i\phi_t = A\phi, \tag{106.4}$$

then the eigenvalues will be independent of time (i.e., $\lambda_t = 0$).

Hence, the time evolution of the eigenfunctions can be determined from equation (106.4). Using the eigenfunctions $\{\phi_j(t, \mathbf{x})\}$, an inverse problem must be solved; the operator L must be determined from knowledge of its eigenfunctions. Because L depends on u, this might lead to a solution for u. For some problems, the time evolution of the eigenfunctions can be used in the Gelfand–Levitan linear integral equation (see Faddeyev [7]), which may (sometimes) be solved to determine $u(t, \mathbf{x})$.

Given equation (106.1) and the initial conditions $u(t = 0, \mathbf{x})$, the procedure can be summarized as

- Find the Lax pair representation of the evolution equation(s).
- Using $u(t = 0, \mathbf{x})$, evaluate L at $t = 0$ and then determine the eigenvalues $\{\lambda_j\}$ and the initial values of the eigenfunctions $\{\phi_j(0, \mathbf{x})\}$. These are the solutions to equation (106.3).
- Find the time evolution of the eigenfunctions by solving equation (106.4).
- Determine $u(t, \mathbf{x})$ by solving an inverse problem; that is, using $\{\phi_j(t, \mathbf{x})\}$ as the solutions to equation (106.3), determine L for $t > 0$.

Note that the $\{\phi_j(t), \mathbf{x}\}$ are called the *scattering data*. Even if the last step cannot be carried out, useful information may be obtained from the scattering data.

Example 1

For the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0, \tag{106.5}$$

a Lax pair is given by

$$\begin{aligned} L &= \frac{\partial^2}{\partial x^2} - u, \\ A &= -i\left(4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3\frac{\partial u}{\partial x}\right). \end{aligned} \tag{106.6.a-b}$$

This may be verified by calculating, for an arbitrary function $\psi = \psi(x)$,

$$\begin{aligned} L(A(\psi)) &= i(3\psi u_{xxx} + 12\psi_x u_{xx} - 3\psi u u_x + 15\psi_{xx} u_x - 6\psi_x u^2 \\ &\qquad + 10\psi_{xxx} u - 4\psi_{xxxxx}), \\ A(L(\psi)) &= i(4\psi u_{xxx} + 12\psi_x u_{xx} - 9\psi u u_x + 15\psi_{xx} u_x - 6\psi_x u^2 \\ &\qquad + 10\psi_{xxx} u - 4\psi_{xxxxx}), \\ (LA - AL)\psi &= -i\left(u_{xxx} - 6uu_x\right)\psi \end{aligned} \tag{106.7}$$

Using equations (106.6.a) and (106.7), we then determine

$$\begin{aligned} L_t &= -u_t, \\ [L, A] = [LA - AL] &= -i\left(u_{xxx} - 6uu_x\right). \end{aligned} \tag{106.8}$$

When equation (106.8) is used in equation (106.2), the KdV equation (106.5) is the result.

Example 2

For the sine–Gordon equation $u_{xt} = \sin u$, the scattering equations (which determine the initial values of the eigenfunctions) for the vector eigenfunction $\begin{bmatrix}\phi & \psi\end{bmatrix}^{\mathrm{T}}$ may be written as

$$L\begin{bmatrix}\phi \\ \psi\end{bmatrix} = i\begin{bmatrix}\frac{\partial}{\partial x} & \frac{1}{2}\frac{\partial u}{\partial x} \\ \frac{1}{2}\frac{\partial u}{\partial x} & -\frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}\phi \\ \psi\end{bmatrix} = \lambda\begin{bmatrix}\phi \\ \psi\end{bmatrix},$$

whereas the evolution equations for the vector eigenfunction may be written as

$$i\frac{\partial}{\partial t}\begin{bmatrix}\phi \\ \psi\end{bmatrix} = A\begin{bmatrix}\phi \\ \psi\end{bmatrix} = -\frac{1}{4\lambda}\begin{bmatrix}\cos u & \sin u \\ \sin u & -\cos u\end{bmatrix}\begin{bmatrix}\phi \\ \psi\end{bmatrix}.$$

Notes

1. The formulation of inverse scattering presented here is not the only possible formulation. There are other formulations, which may be easier to carry out on specific problems.
2. The paper by Case and Kac [5] discusses a discrete inverse scattering problem; their problem illustrates many of the ideas from scattering theory without all of the mathematical difficulties.
3. The KdV equation is the compatability condition of the linear system

$$\begin{aligned}\left(\partial_x^2 + u - \lambda^2\right)\psi &= 0 \\ \left(\partial_t + 4\partial_x^3 + 6u\partial_x + 3u_x\right)\psi &= 0\end{aligned}$$

   where λ is a spectral parameter.
4. The mKdV equation, $u_t + u_{xxx} - 6u^2u_x = 0$ is the compatability condition of the system

$$\begin{aligned}\left(\partial_x^2 + 2u\partial_x - \lambda^2\right)\psi &= 0 \\ \left(\partial_t + 4\partial_x^3 + 12u\partial_x^2 + 6(u_x + u^2)\partial_x\right)\psi &= 0\end{aligned}$$

5. The Burgers equation, $u_t + u_{xx} + 2uu_x = 0$ is the compatability condition of the system

$$\begin{aligned}\left(\partial_x - u - \lambda\right)\psi &= 0 \\ \left(\partial_t + \partial_x^2 - 2u\partial_x\right)\psi &= 0\end{aligned}$$

References

[1] Ablowitz, M. J., Kaup, D. J., Newell, A. C., and Segur, H. The inverse scattering transform — Fourier analysis for nonlinear problems. *Stud. Appl. Math. 53*, 4 (Dec 1974), 249–315.

[2] Ablowitz, M. J., and Segur, H. *Solitons and the Inverse Scattering Transform.* SIAM, Philadelphia, PA, 1981.

[3] Calogero, F., and Degasperis, A. *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations.* North–Holland Publishing Co., New York, 1982.

[4] Calogero, F., and Nucci, M. C. Lax pairs galore. *J. Math. Physics 32*, 1 (Jan 1991), 72–74.

[5] Case, K. M., and Kac, M. A discrete version of the inverse scattering problem. *J. Math. Physics 14*, 5 (1973), 594–603.

[6] Eckhaus, W., and Harten, A. V. *The Inverse Scattering Transformation and the Theory of Solitons.* North–Holland Publishing Co., New York, 1981.

[7] Faddeyev, L. D. The inverse problem in the quantum theory of scattering. *J. Math. Physics 4* (1963), 72–104.

[8] Ito, M. A REDUCE program for evaluating a Lax pair form. *Comput. Physics Comm. 34* (1985), 325–331.

[9] McLaughlin, J. R. Analytical methods for recovering coefficients in differential equations from spectral data. *SIAM Review 28*, 1 (March 1986), 53–72.

[10] Musetts, M., and Conte, R. Algorithmic method for deriving Lax pairs from the invariant Painleve analysis of nonlinear partial differential equations. *J. Math. Physics 32*, 6 (June 1991), 1450–1457.

[11] Nucci, M. C. Pseudopotentials, Lax equations, and Backlund transformations for non-linear evolution equations. *J. Phys. A: Math. Gen. 21* (1988), 73–79.

[12] Tabor, M. *Chaos and Integrability in Nonlinear Dynamics.* John Wiley & Sons, New York, 1989.

[13] Whitham, G. B. *Linear and Nonlinear Waves.* Interscience Publishers, Inc., New York, 1974.

107. Jacobi's Method

Applicable to First order partial differential equations with three or more dependent variables. In the special case that the dependent variable appears explicitly in the equation, then it also applies to equations with two dependent variables.

Yields

An explicit solution if a certain step can be carried out.

Idea

Given a partial differential equation for $z(\mathbf{x}) = z(x_1, x_2, \ldots, x_n)$, if the set of n first derivatives $\{p_i = \partial z/\partial x_i \mid i = 1, 2, \ldots, n\}$ is explicitly known, then $z(\mathbf{x})$ may be found by integrating the Pfaffian differential equation: $dz = p_1 dx_1 + \cdots + p_n dx_n$. Jacobi's method determines the $\{p_i\}$ from a given partial differential equation.

Procedure

Let us presume that the given partial differential equation for $z = z(\mathbf{x}) = z(x_1, \ldots, x_n)$, with $n = 3$, is of the form

$$F(\mathbf{x}, \mathbf{p}) = 0, \tag{107.1}$$

where $p_i = \partial z/\partial x_i$. If we could find two other equations, that have the same solution as equation (107.1), of the form $\{F_2(\mathbf{x}, \mathbf{p}) = 0,\ F_3(\mathbf{x}, \mathbf{p}) = 0\}$, then we might be able to determine $\{p_1 = p_1(\mathbf{x}), \ldots, p_n = p_n(\mathbf{x})\}$ by combining these three equations. Then we could find $z(\mathbf{x})$ by solving the Pfaffian differential equation (see page 384)

$$dz = p_1\, dx_1 + \cdots + p_n\, dx_n. \tag{107.2}$$

So, we need to determine $\{F_2, F_3\}$ in such a way that their solutions are the same as the solution to equation (107.1). This requirement results in (see the section on compatible systems, page 43)

$$\begin{aligned} [F, F_2] &:= \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial F_2}{\partial x_i} \right) = 0, \\ [F, F_3] &= 0, \\ [F_2, F_3] &= 0, \end{aligned} \tag{107.3}$$

where [,] is the usual Poisson bracket. The characteristic equations for F_2 (or F_3), from equation (107.3), can be written as (see page 432)

$$\frac{dx_1}{-\frac{\partial F}{\partial p_1}} = \frac{dp_1}{\frac{\partial F}{\partial x_1}} = \cdots = \frac{dx_n}{-\frac{\partial F}{\partial p_n}} = \frac{dp_n}{\frac{\partial F}{\partial x_n}}. \tag{107.4}$$

(These are also known as the subsidiary equations.) Hence, the procedure is to solve equation (107.4) for $F_2(\mathbf{x}, \mathbf{p}) = 0$ and $F_3(\mathbf{x}, \mathbf{p}) = 0$. It must be then be verified that $[F_2, F_3] = 0$. Then solving $\{F = 0, F_1 = 0, F_2 = 0\}$ for $p_i = p_i(\mathbf{x})$ and integrating equation (107.2) results in a solution to equation (107.1).

Example

This example is from Piaggio [3, pages 162–170]. Suppose we have the following nonlinear partial differential equation in three independent variables:

$$\begin{aligned} 0 = F(\mathbf{x}, \mathbf{p}) &= 2x_1x_3\frac{\partial z}{\partial x_1} + 3x_3^2\frac{\partial z}{\partial x_2} + \left(\frac{\partial z}{\partial x_2}\right)^2\frac{\partial z}{\partial x_3}, \\ &= 2x_1x_3p_1 + 3x_3^2p_2 + p_2^2p_3. \end{aligned} \tag{107.5}$$

The subsidiary equations in equation (107.4) can be written as

$$\frac{dx_1}{-2x_1x_3} = \frac{dp_1}{2x_3p_1} = \frac{dx_2}{-3x_3^2 - 2p_2p_3} = \frac{dp_2}{0} = \frac{dx_3}{-p_2^2} = \frac{dp_3}{2x_1p_1 + 6x_3p_2}. \tag{107.6}$$

From the first equality in equation (107.6) we have

$$F_2(\mathbf{x}, \mathbf{p}) = p_1x_1 - A_1 = 0, \tag{107.7}$$

where A_1 is an arbitrary constant. From the fourth term in equation (107.6), we have

$$F_3(\mathbf{x}, \mathbf{p}) = p_2 - A_2 = 0, \tag{107.8}$$

where A_2 is another arbitrary constant. Clearly, $[F_2, F_3] = 0$ for our chosen F_2 and F_3. Combining equations (107.7) and (107.8) with the original equation, (107.5), we find that

$$p_3 = -\frac{1}{A_2^2}(2A_1x_3 + 3A_2x_3^2). \tag{107.9}$$

In equations (107.7)–(107.9) we have found expressions for the $\{p_i\}$. Hence,

$$\begin{aligned} dz &= p_1\,dx_1 + p_2\,dx_2 + p_3\,dx_3 \\ &= \frac{A_1}{x_1}\,dx_1 + A_2\,dx_2 - \frac{1}{A_2^2}(2A_1x_3 + 3A_2x_3^2)\,dx_3, \end{aligned}$$

which can be integrated to yield the solution

$$z = A_1\log x_1 + A_2x_2 - \frac{1}{A_2^2}(A_1x_3^2 + A_2x_3^3) + A_3,$$

where A_3 is another arbitrary constant.

Notes

1. If the given partial differential equation has only two independent variables and if the dependent variable z is explicit in the partial differential equation, then we can transform the partial differential equation into the form of equation (107.1). For example, if we have $F(x, y, z, p, q) = 0$ (where, as usual, $p = \partial z/\partial x$, $q = \partial z/\partial y$), suppose that $u(x, y, z) = 0$ is an integral of this equation. If we define $u_1 = \partial u/\partial x$, $u_2 = \partial u/\partial y$, $u_3 = \partial u/\partial z$, then we can write $p = -u_1/u_3, q = -u_2/u_3$. Using these definitions for p and q in the original equation yields an equation of the form $f(x, y, z, u_1, u_2, u_3) = f(\mathbf{x}, \mathbf{p}) = 0$.
2. When $n > 3$, then the only change in the procedure is that we must now determine $\{F_2, F_3, \ldots, F_n\}$ and use these (with F) to solve for the $\{p_i\}$.
3. When this method is specialized to two independent variables, it is often called *Charpit's method*. See Chester [2, page 212, and Chapter 15, pages 315–337] or Piaggio [3] for details.
4. See also Ames [1, pages 54–57] and Sneddon [4, pages 69–73 and 78–80].

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Chester, C. R. *Techniques in Partial Differential Equations*. McGraw–Hill Book Company, New York, 1970.

[3] Piaggio, H. T. H. *An Elementary Treatise on Differential Equations and Their Applications*. G. Bell & Sons, Ltd, London, England, 1926.

[4] Sneddon, I. N. *Elements of Partial Differential Equations*. McGraw–Hill Book Company, New York, 1957.

108. Legendre Transformation

Applicable to Partial differential equations in one dependent variable that are *not* of the form $F(u_{x_1}, u_{x_2}, \ldots, u_{x_n}) = 0$.

Yields

An alternative formulation of the original problem.

Idea

A surface in space may be described by a point or as an envelope of tangent planes. Changing variables from one representation to the other may facilitate finding a solution. After a solution is obtained, it can be transformed back to the original variables.

Procedure

We illustrate the technique for two independent variables; the notes show how the technique may be extended to n independent variables. Given a function $u(x,y)$, we change to the new variables $w(\zeta,\eta)$ by the transformation

$$w(\zeta,\eta) + u(x,y) = x\zeta + y\eta, \tag{108.1}$$

with the following definitions

$$u_x = \zeta, \quad w_\zeta = x, \quad u_y = \eta, \quad w_\eta = y. \tag{108.2}$$

From equation (108.1) and equation (108.2) it is easy to derive that

$$\begin{aligned} u_{xx} &= J w_{\eta\eta}, \\ u_{xy} &= u_{yx} = -J w_{\zeta\eta}, \\ u_{yy} &= J w_{\zeta\zeta}, \end{aligned}$$

where J is the Jacobian of the transformation. The Jacobian may be expressed as

$$J = u_{xx}u_{yy} - (u_{xy})^2 = \frac{1}{w_{\eta\eta}w_{\zeta\zeta} - (w_{\eta\zeta})^2}.$$

To be able to transform from the $\{u,x,y\}$ variables to the $\{w,\zeta,\eta\}$ variables, the Jacobian must not vanish. If $J \neq 0$, then the surface is said to be *developable*. The solutions with $J = 0$ are said to be *non-developable* solutions. The non-developable solutions are not obtainable by the Legendre transformation.

Summary

For the partial differential equation of at most second order in the variables $\{u,x,y\}$,

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{108.3}$$

we make the Legendre transformation to obtain the new equation

$$F\left(w_\zeta, w_\eta, \zeta w_\zeta + \eta w_\eta - w, \zeta, \eta, J w_{\eta\eta}, -J w_{\zeta\eta}, J w_{\zeta\zeta}\right) = 0 \tag{108.4}$$

in the new variables $\{w, \eta, \zeta\}$.

Sometimes equation (108.4) is easier to solve than equation (108.3). After equation (108.4) is solved to determine $w(\eta, \zeta)$, we must change back to the original variables. Changing from the $\{w, \eta, \zeta\}$ variables to the $\{u, x, y\}$ variables can be done (due to the implicit function theorem) but may be difficult.

Example

Consider the nonlinear partial differential equation

$$u_x u_y = x, \tag{108.5}$$

which we want to solve for $u(x, y)$. The Legendre transformation of equation (108.5) is (using the transformations in equations (108.1) and (108.2) or using equation (108.4) directly)

$$w_\zeta = \zeta\eta. \tag{108.6}$$

This has the solution

$$w(\zeta, \eta) = \frac{1}{2}\eta\zeta^2 + f(\eta), \tag{108.7}$$

where $f(\eta)$ is an arbitrary function of η. We have now finished solving the differential equation. Because we have the solution in terms of the new variables, all that remains is to transform to the old variables. This change of variables will utilize the $w(\zeta, \eta)$ that was found.

Using equation (108.6) and $w_\zeta = x$ (from (108.2)), we have

$$x = \eta\zeta. \tag{108.8}$$

Differentiating equation (108.7) with respect to η and using $y = w_\eta$ (from equation (108.2)) yields

$$y = \frac{1}{2}\zeta^2 + f'(\eta). \tag{108.9}$$

Using equations (108.7)–(108.9) in equation (108.1) produces the equation

$$u = x\zeta + y\eta - w(\zeta, \eta) = \eta\zeta^2 + \eta f'(\eta) - f(\eta). \tag{108.10}$$

Solving equation (108.8) for ζ, and then substituting that result in equations (108.9) and (108.10) produces

$$\begin{aligned} y &= \frac{x^2}{2\eta^2} + f'(\eta), \\ u &= \frac{x^2}{\eta} + \eta f'(\eta) - f(\eta). \end{aligned} \tag{108.11.a-b}$$

This is a parametric representation of the solution $u(x, y)$. All of the developable solutions of equation (108.5) are completely characterized by equation (108.11). Given any $f(\eta)$ we can, in principle, find $\eta = \eta(x, y)$ from equation (108.11.a). Using this value for η in equation (108.11.b) then gives u as a function of x and y.

To illustrate this, if we choose

$$f(\eta) = \frac{A}{\eta},$$

where A is an arbitrary constant, then equation (108.11) becomes

$$y = \left(\frac{1}{2}x^2 - A\right)\frac{1}{\eta^2}, \qquad u = \frac{x^2 - 2A}{\eta}. \tag{108.12.a-b}$$

Solving equation (108.12.a) for η and using this expression in equation (108.12.b) produces

$$u(x, y) = \sqrt{2y(x^2 - 2A)}.$$

Now that we have an explicit solution, we must check that the Jacobian does not vanish. In this example, $J \neq 0$.

Notes

1. Observe that

$$u = Dy + \frac{1}{2D}x^2 + C, \tag{108.13}$$

where C and D are constants, is also a solution to equation (108.5), but this solution is not contained in equation (108.11) for any $f(\eta)$. This is because the solution in equation (108.13) is non-developable ($J = 0$).

2. The Legendre transformation may be naturally extended to partial differential equations in n variables. The transformation (from $u(x_1, x_2, \ldots, x_n)$ to $w(\zeta_1, \zeta_1, \ldots, \zeta_n)$) and its inverse is given by

$$\begin{gathered} u(x_1, x_2, \ldots, x_n) = w(\zeta_1, \zeta_1, \ldots, \zeta_n) + x_1\zeta_1 + x_2\zeta_2 + \cdots + x_n\zeta_n, \\ u_{x_1} = \zeta_1, \quad u_{x_2} = \zeta_2, \quad \cdots, \quad u_{x_n} = \zeta_n, \\ w_{\zeta_1} = x_1, \quad w_{\zeta_2} = x_2, \quad \cdots, \quad w_{\zeta_n} = x_n. \end{gathered}$$

See Courant and Hilbert [3, Volume 2, pages 32–39] for more details.

3. Clairaut's equation, $u = xu_x + yu_y + f(u_x, u_y)$, under the Legendre transformation, becomes the simple equation $w = -f(\zeta, \eta)$.

4. The Legendre transformation is an involutory transformation; that is, the Legendre transformation applied twice results in the original equation. The Legendre transformation is also an example of a contact transformation (see page 249).

5. The Legendre transformation is used in mechanics when transforming from the Lagrangian formulation to the Hamiltonian formulation (or vice-versa). See Goldstein [5] for details.
6. The Legendre transformation is used in thermodynamics when transforming the fundamental equation from internal energy (canonical variables are specific volume and specific entropy) to the Gibbs function (canonical variables are pressure and temperature), or to enthalpy (canonical variables are pressure and specific entropy), or to the Helmholtz function (canonical variables are specific volume and temperature). For more details of this application, see Kestin [6].
7. If the Legendre transformation is applied to a partial differential equation of the form $F(u_x, u_y) = 0$, then the algebraic relation $F(\zeta, \eta) = 0$ results. Because $w(\zeta, \eta)$ cannot be determined from this equation, this class of equations cannot be solved by the use of the Legendre transformation.
8. See Ames [1, pages 37–40], Chester [2, pages 209–210], and Epstein [4, pages 65–68].

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Chester, C. R. *Techniques in Partial Differential Equations*. McGraw–Hill Book Company, New York, 1970.

[3] Courant, R., and Hilbert, D. *Methods of Mathematical Physics*. Interscience Publishers, Inc., New York, 1953.

[4] Epstein, B. *Partial Differential Equations: An Introduction*. McGraw–Hill Book Company, New York, 1962.

[5] Goldstein, H. *Classical Mechanics*. Addison–Wesley Publishing Co., Reading, MA, 1950.

[6] Kestin, J. *A Course in Thermodynamics*. Blaisdell Publishing Co., Waltham, MA, 1966.

109. Lie Groups: PDEs

Applicable to Linear and nonlinear partial differential equations.

Yields

Similarity variables that may be used to decrease the number of independent variables in a partial differential equation.

Idea

By determining the transformation group under which a given partial differential equation is invariant, we can obtain information about the invariants and symmetries of that equation. This information, in turn, can be used to determine similarity variables that will reduce the number of independent variables in the system.

Procedure

Some background material about Lie groups may be found in the section "Lie Groups: ODEs" (starting on page 366). We utilize terms that have been defined in that section.

We illustrate the general technique on one partial differential equation in two independent variables. Suppose we would like to solve the partial differential equation

$$N(u, x, y) = 0 \tag{109.1}$$

for $u(x, y)$. We first determine a one parameter Lie group of transformations, under which equation (109.1) is invariant; then we use this group to determine similarity variables. We suppose that the group has the form

$$\begin{aligned} \overline{u} &= u + \epsilon U(u, x, y) + O(\epsilon^2), \\ \overline{x} &= x + \epsilon X(u, x, y) + O(\epsilon^2), \\ \overline{y} &= y + \epsilon Y(u, x, y) + O(\epsilon^2). \end{aligned} \tag{109.2}$$

We want this group to leave equation (109.1) invariant; that is,

$$N(\overline{x}, \overline{y}, \overline{u}) = 0, \tag{109.3}$$

or, equivalently,

$$u(\overline{x}, \overline{y}) = \overline{u}(u, x, y; \epsilon). \tag{109.4}$$

Using the transformations in equation (109.2), the chain rule produces

$$
\begin{aligned}
\frac{\partial x}{\partial \overline{x}} &= 1 - \epsilon\left(X_x + X_u u_x\right) + O\left(\epsilon^2\right),\\
\frac{\partial x}{\partial \overline{y}} &= \quad - \epsilon\left(X_y + X_u u_y\right) + O\left(\epsilon^2\right),\\
\frac{\partial y}{\partial \overline{x}} &= \quad - \epsilon\left(Y_x + Y_u u_x\right) + O\left(\epsilon^2\right),\\
\frac{\partial y}{\partial \overline{y}} &= 1 - \epsilon\left(Y_y + Y_u u_y\right) + O\left(\epsilon^2\right).
\end{aligned}
\tag{109.5}
$$

From equation (109.5), it is conceptually easy (though algebraically intensive) to determine how derivatives in the $\{\overline{u}, \overline{x}, \overline{y}\}$ system transform to derivatives in the $\{u, x, y\}$ system. For instance,

$$
\begin{aligned}
\frac{\partial \overline{u}}{\partial \overline{x}} &= u_x + \epsilon\left(U_x + (U_u - X_x)u_x - Y_x u_y - X_u u_x^2 - Y_u u_x u_y\right) + O\left(\epsilon^2\right),\\
\frac{\partial \overline{u}}{\partial \overline{y}} &= u_y + \epsilon\left(U_y + (U_u - Y_y)u_y - X_y u_x - Y_u u_y^2 - X_u u_y u_x\right) + O\left(\epsilon^2\right),\\
\frac{\partial^2 \overline{u}}{\partial \overline{x}^2} &= u_{xx} + \epsilon\Big(-Y_{uu} u_x^2 u_y - X_{uu} u_x^3 - 2Y_u u_x u_{xy} - (3X_u + 2Y_{yu}) u_x u_y\\
&\qquad - Y_u u_y^2 + (X_{uu} - 2Y_{ux}) U_x^2 - 2Y_y u_{xy} + (U_u - 2X_x Y_{xx}) u_y\\
&\qquad + U_{xx} + (2U_{xu} - Y_{xx}) u_x\Big) + O\left(\epsilon^2\right).
\end{aligned}
\tag{109.6}
$$

The group is then determined (i.e., $\{U, X, Y\}$ are determined) by requiring equation (109.3) to be satisfied.

After the group has been determined, a solution to equation (109.1) may be found from the *invariant surface condition*

$$
U(u, x, y) = X(u, x, y)\frac{\partial u}{\partial x} + Y(u, x, y)\frac{\partial u}{\partial y}, \tag{109.7}
$$

which is just the first order term of equation (109.4) when that equation is expanded for small values of ϵ. The solution of equation (109.7) leads to similarity variables that reduce the number of independent variables in the system. Note that equation (109.7) is quasilinear and that the subsidiary equations may be written as

$$
\frac{du}{U(u, x, y)} = \frac{dx}{X(u, x, y)} = \frac{dy}{Y(u, x, y)}. \tag{109.8}
$$

Example 1

Suppose we wish to analyze the heat equation

$$
u_y = u_{xx}. \tag{109.9}
$$

We take $\overline{u}_{\overline{y}} = \overline{u}_{\overline{x}\overline{x}}$ and substitute for the derivatives from equation (109.6). We also substitute u_y for u_{xx} (from equation (109.9)). This leads to a large expression that must equal zero.

Equating to zero the coefficients of $\{u, u_x, u_y, u_x^2, u_y^2, u_{xy}, u_x u_y, u_x u_{xy}\}$ in this expression leads to eight simultaneous equations involving $\{U, X, Y\}$. The solution to these equations will determine the transformation group. Three of these equations are

$$\begin{aligned} u_x^2 \text{ coefficient:} &\quad Y_u = 0, \\ u_x u_y \text{ coefficient:} &\quad X_u = 0, \\ u_x u_{xy} \text{ coefficient:} &\quad U_{uu} = 0. \end{aligned}$$

These equations produce $X(u, x, y) = X(x, y)$, $Y(u, x, y) = Y(x, y)$ and $U(u, x, y) = f(x, y)u + g(x, y)$, where f and g are functions to be determined. Using this simplification for $\{U, X, Y\}$, the other five equations become

$$\begin{aligned} Y_x &= 0, & f_{xx} - f_y &= 0, \\ 2X_x - Y_y &= 0, & g_{xx} - g_y &= 0, \\ X_y - X_{xx} + 2f_x &= 0. \end{aligned} \tag{109.10}$$

If we take $g = 0$ (just to simplify the algebra), then the equations in equation (109.10) may be solved to determine the transformation group

$$\begin{aligned} X &= 2c_1 y + 4c_2 xy + c_4 + c_5 x, \\ Y &= 4c_2 y^2 + 2c_5 y + c_6, \\ U &= -\left(c_1 x + c_2(x^2 + 2y) + c_3\right) u, \end{aligned} \tag{109.11}$$

where $\{c_1, \ldots, c_6\}$ are arbitrary constants. Now that we have found a transformation group, similarity variables may be found.

Special Case 1

If we take $c_1 = c_2 = c_4 = c_6 = 0$ in equation (109.11), then the subsidiary equations (from equation (109.8)) become

$$\frac{du}{-c_3 u} = \frac{dx}{c_5 x} = \frac{dy}{2c_5 y}.$$

Two solutions to these equations are

$$\text{constant} = \frac{x}{\sqrt{y}}, \qquad \text{constant} = \frac{u}{y^\alpha},$$

where $\alpha = -c_3/2c_5$. From these similarity variables, we propose a solution of the form

$$\eta = \frac{x}{\sqrt{y}}, \qquad h(\eta) = \frac{u}{y^\alpha}. \tag{109.12}$$

That is, $u(x,y) = y^{\alpha} h(x/\sqrt{y})$. Using this form in equation (109.9), we find that $h(\eta)$ satisfies the ordinary differential equation $h'' = \alpha h - \frac{1}{2}\eta h'$. Every solution to this equation will generate a solution to equation (109.9).

Special Case 2

If we take $c_1 = c_2 = c_4 = c_5 = 0$ in equation (109.11), then the subsidiary equations (from equation (109.8)) become

$$\frac{du}{-c_3 u} = \frac{dx}{0} = \frac{dy}{c_6}.$$

Two solutions to these equations are

$$\text{constant} = x, \qquad \text{constant} = \frac{u}{e^{\beta y}},$$

where $\beta = -c_3/c_6$. From these similarity variables we propose a solution of the form

$$\eta = x, \qquad k(\eta) = \frac{u}{e^{\beta y}}.$$

That is, $u(x,y) = e^{\beta y} k(x)$. Using this form in equation (109.9), we find that $k(\eta)$ satisfies the ordinary differential equation $k'' - \beta k = 0$. Every solution to this equation will generate a solution to equation (109.9).

Example 2

Consider similarity solutions of Laplace's equation in two dimensions: $\nabla^2 u = u_{xx} + u_{yy} = 0$. To find the Lie group of transformations that leaves this equation invariant, we consider the group defined in equation (109.2). After extensive algebra we find that, to lowest order, $\{X, Y, U\}$ may be expressed as

$$\begin{aligned} X &= d_1 + d_3 x - d_4 y + d_5(x^2 - y^2) + 2d_6 xy + (\text{cubic terms}), \\ Y &= d_2 + d_3 y + d_4 x + 2d_5 xy + d_6(y^2 - x^2) + (\text{cubic terms}), \\ U &= d_7 u + V(x,y), \end{aligned} \tag{109.13}$$

where $V(x,y)$ is any solution to $\nabla^2 V = 0$ and $\{d_1, d_2, \ldots, d_7\}$ are arbitrary constants. The similarity solutions to $\nabla^2 u = 0$ may now be determined from the subsidiary equations in (109.8). For simplicity, we will take $V = 0$, and investigate two possibilities for the other parameters in equation (109.13).

Special Case 1

If we presume that the only non-zero parameters in equation (109.1) are d_1, d_2, and d_7, then the subsidiary equations become

$$\frac{du}{d_7 u} = \frac{dx}{d_1} = \frac{dy}{d_2}.$$

Using the equation specified by the second equality sign, we determine that $d_2 x - d_1 y$ is constant. Using the equation specified by the first equality sign, we determine that $ue^{-d_7 x/d_1}$ is constant. Hypothezing a solution of the form $u(x,y) = e^{d_7 x/d_1} f(d_2 x - d_1 y)$, and then requiring that $\nabla^2 u = 0$, leads to a constant coefficient ordinary differential equation for f:

$$d_1^2 \left(d_1^2 + d_2^2\right) f'' - 2d_1 d_2 d_7 f' + d_7^2 f = 0.$$

Special Case 2

If we presume that the only non-zero parameters in equation (109.1) are d_3 and d_7, then the subsidiary equations become

$$\frac{dx}{d_3 x} = \frac{dy}{d_3 y} = \frac{du}{d_7 u}.$$

These equations can be solved to determine that $u(x,y) = y^m g(\eta)$, $\eta = y/x$, where $m = d_7/d_3$. By requiring $\nabla^2 u = 0$ to hold, we find the following ordinary differential equation for $g(\eta)$:

$$\left(\eta^2 + \eta^4\right) g'' + 2\eta \left(m + \eta^2\right) g' + m(m-1)g = 0.$$

Notes

1. Lie group analysis is the most useful and general of all the techniques presented in this book.
2. There are other techniques for determining the group under which a given partial differential equation is invariant. A list of techniques is given in Seshadri and Na [12].
3. If $u(x,y)$ is a solution of equation (109.9), then the following transformations also represent solutions:

$$\begin{aligned}
&T_1 : \left\{ \begin{aligned} &x \to x + 2cy \\ &u \to ue^{-c(x+y^2)} \end{aligned} \right\} \\
&T_2 : \left\{ \begin{aligned} &x \to x/(1-4cy) \\ &y \to y/(1-4cy) \\ &u \to u\sqrt{1-4cy}\, \exp\left(-\frac{cx^2}{1-4cy}\right) \end{aligned} \right\} \\
&T_3 : u \to e^c u \\
&T_4 : x \to x + c \\
&T_5 : \left\{ \begin{aligned} &x \to e^c x \\ &y \to e^{2c} y \end{aligned} \right\} \\
&T_6 : y \to y + c.
\end{aligned} \tag{109.14}$$

These transformations were all obtained from the group in equation (109.11). For example, the similarity variable $\eta = x/\sqrt{y}$ in equation (109.12) is equivalent to transformation T_5. If $u = m(x,y)$ is a solution of equation (109.9), then another solution is given by (using all of the transformations listed in (109.14))

$$u = \frac{1}{\sqrt{1+4c_2y}} \exp\left(c_3 - \frac{c_1x + c_2x^2 - c_1^2 y}{1+4c_2y}\right) \times m\left(\frac{e^{-c_5}(x-2c_1y)}{1+4c_2y} - c_4, \frac{e^{-2c_5}y}{1+4c_2y} - c_6\right).$$

See Olver [9, pages 120–123] for details.

4. Using Lie groups to find symmetries of partial differential equations can be computationally intensive. Algorithms have been developed for computerized handling of the calculations. A computer package in FORMAC is described in Fedorova and Kornyak [6], a Macsyma package is in Champagne *et al.* [4], a Maple package is in Mansfield and Clarkson [8], and a REDUCE package is in Schwarz [11].
5. A new technique for finding symmetries of partial differential equations that are neither point symmetries nor Lie–Bäcklund symmetries may be found in Bluman *et al.* [3].
6. The general equation of nonlinear heat conduction takes the form $u_t = (K(u)u_x)_x$. For this equation,
   - If $K(u)$ is constant, then the symmetry group is infinite dimensional.
   - If $K(u) = (au+b)^{-4/3}$, with $a \neq 0$, then there is a five-parameter symmetry group.
   - If $K(u) = (au+b)^m$, for $m \neq -\frac{4}{3}$ and $a \neq 0$, then there is a four-parameter symmetry group.
   - If $K(u) = ce^{au}$, then there is a four-parameter symmetry group.
   - If $K(u)$ does not have one of the forms mentioned above, then there is a three-parameter symmetry group.
7. Olver [9] derives the complete symmetry group for many partial differential equations, including the heat equation, wave equation, Euler equations, and Korteweg-de Vries equation. Ames and Nucci [1] studied the Burgers's equation, Korteweg-de Vries equation (1 and 2 dimensions), Hopf equation, and Lin–Tsien equation.
8. Classical and nonclassical symmetries of the nonlinear heat equation $u_t = u_{xx} + f(u)$ are considered in Clarkson and Mansfield [5].
9. The KdV equation, $u_t = u_{xxx} + 6uu_x$, has the Lie point symmetries $\{\partial_x, \partial_t, -6t\partial_x + \partial_u, x\partial_x + 3t\partial_t - 2u\partial_u\}$.
10. Burgers's equation, $u_t - uu_x - u_{xx} = 0$, has the Lie point symmetries $\{\partial_t, \partial_x, t\partial_x - \partial_u, 2t\partial_t + x\partial_x - u\partial_u, t^2\partial_t + tx\partial_x - (x+tu)\partial_u\}$.
11. The section on similarity methods (beginning on page 497) shows how to find similarity variables of a specific form. The techniques in this section are, of course, much more general and will determine all possible similarity variables.

References

[1] Ames, W. F., and Nucci, M. C. Analysis of fluid equations by group methods. *Journal of Engineering Mathematics 20* (1985), 181–187.

[2] Bluman, G. W., and Kumei, S. *Symmetries and Differential Equations*. Springer–Verlag, New York, 1989.

[3] Bluman, G. W., Reid, G. J., and Kumei, S. New classes of symmetries for partial differential equations. *J. Math. Physics 29*, 4 (April 1988), 806–811.

[4] Champagne, B., Hereman, W., and Winternitz, P. The computer calculation of Lie point symmetries of large systems of differential equations. *Comput. Physics Comm. 66* (1991), 319–340.

[5] Clarkson, P. A., and Mansfield, E. L. Symmetry reductions and exact solution of a class of nonlinear heat equations. *Physica D 70* (1993), 250–288.

[6] Fedorova, R. N., and Kornyak, V. V. Determination of Lie–Backlund symmetries of differential equations using FORMAC. *Comput. Physics Comm. 39* (1986), 93–103.

[7] Hill, J. M. *Solution of Differential Equations by Means of One-Parameter Groups*. Pitman Publishing Co., Marshfield, MA, 1982.

[8] Mansfield, E. L., and Clarkson, P. A. Application of the differential algebra package `diffgrob2` to classical symmetries of differential equations. *J. Symbolic Computation 11* (1994).

[9] Olver, P. J. *Applications of Lie Groups to Differential Equations*. No. 107 in Graduate Texts in Mathematics. Springer–Verlag, New York, 1986.

[10] Reiman, A. Computer-aided closure of the Lie algebra associated with a nonlinear partial differential equation. *Comp. & Maths. with Appls. 7*, 7 (1981), 387–393.

[11] Schwarz, F. Automatically determining symmetries of partial differential equations. *Computing 34* (1985), 91–106.

[12] Seshadri, R., and Na, T. Y. *Group Invariance in Engineering Boundary Value Problems*. Springer–Verlag, New York, 1985.

[13] Steinberg, S. Applications of the Lie algebraic formulas of Baker, Campbell, Hausdorff, and Zassenhaus to the calculation of explicit solutions of partial differential equations. *J. Differential Equations 26* (1977), 404–434.

110. Poisson Formula

Applicable to Laplace's equation ($\nabla^2 u = 0$) in two dimensions with $u(\mathbf{x})$ prescribed on a circle; that is, the Dirichlet problem in a disk.

Yields

An exact solution, given by an integral.

Idea

A simple extension of the Cauchy integral formula (from complex variable theory) allows the solution for Laplace's equation in a circle to be written down analytically.

Procedure

If $u(r, \theta)$ satisfies

$$\begin{aligned} \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0, \qquad \text{for } 0 < r < R, \\ u(R, \theta) &= f(\theta), \qquad \text{for } 0 \leq \theta < 2\pi, \end{aligned} \tag{110.1}$$

then $u(r, \theta)$ for $0 < r < R$ is given by

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(\phi)\, d\phi. \tag{110.2}$$

This is known as the Poisson formula for a circle.

Example

If we have

$$\nabla^2 u = 0, \qquad u(R, \theta) = \sin\theta,$$

then

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \sin\phi \, d\phi \\ &= \frac{r}{R} \sin\theta, \end{aligned}$$

where the integral was carried out by using the method of residues.

Notes

1. By use of conformal mappings (see page 441), Laplace's equation in two dimensions for a non-circular region can often be changed to solving Laplace's equation in a circular region. Poisson's formula can be used for this new problem, and then the mapping can be used to find the solution for the original geometry.

2. The solution to equation (110.1) could also have been obtained by the use of Fourier series (see page 344). Using this technique, the solution to equation (110.1) becomes

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta), \tag{110.3}$$

where $\{a_n, b_n\}$ are defined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta. \tag{110.4}$$

Note that this same solution would have been obtained by utilizing separation of variables. Farlow [2, Lesson 33, pages 262–269] and Young [6, pages 273–285] show that the Poisson formula in equation (110.2) may be derived from the solution in equations (110.3) and (110.4).

3. The Neumann problem for a disk

$$\nabla^2 v = 0, \qquad \frac{\partial v}{\partial n}(R,\theta) = g(\theta), \tag{110.5}$$

may be converted to the Dirichlet problem (equation (110.1)) if we define

$$\begin{aligned} f(\theta) &= \int_0^{\theta} g(\phi)\, d\phi, \\ v(x,y) &= \int^{(x,y)} (u_y\, dx - u_x\, dy); \end{aligned} \tag{110.6}$$

see Young [6, pages 273–285] for details. Note that the periodicity requirement of $f(\theta)$ requires that $g(\theta)$ satisfy $\int_0^{2\pi} g(\phi)\, d\phi = 0$. This must be satisfied if there is to exist any solution to equation (110.5). This requirement is related to the alternative theorems on page 15. (Note that the solution to equation (110.5) is indeterminate with respect to a constant.)

4. The solution to the *exterior* problem

$$\begin{gathered} \nabla^2 w = 0, \\ w(R,\theta) = f(\theta), \quad w \text{ bounded at } r = \infty \end{gathered} \tag{110.7}$$

is given by

$$w(r,\theta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\, f(\phi)\, d\phi, \tag{110.8}$$

which is valid for $r \geq R$. See Kantorovich and Krylov [4, pages 572–575] for details.

5. Other exact solutions to Laplace's equation are also known. For example,
   - If $\nabla^2 u = 0$ in a sphere of radius one and $u(1, \theta, \phi) = f(\theta, \phi)$, then

     $$u(r, \theta, \phi) = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(\Theta, \Phi) \frac{1 - r^2}{(1 - 2r\cos\gamma + r^2)^{3/2}} \sin\Theta \, d\Theta \, d\Phi, \tag{110.9}$$

     where $\cos\gamma := \cos\theta\cos\Theta + \sin\theta\sin\Theta\cos(\phi - \Phi)$.
   - If $\nabla^2 u = 0$ in the half plane, $y \geq 0$, and $u(x, 0) = f(x)$, then

     $$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)y}{(x - t)^2 + y^2} \, dt. \tag{110.10}$$

   - If $\nabla^2 u = 0$ in the half space, $z \geq 0$, and $u(x, y, 0) = f(x, y)$, then

     $$u(x, y, z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(\zeta, \eta)}{\left[(x - \zeta)^2 + (y - \eta)^2 + z^2\right]^{3/2}} d\zeta \, d\eta. \tag{110.11}$$

   - If $\nabla^2 u = 0$ in the annulus, $0 < a \leq r \leq 1$, and $u(1, \theta)$ and $u(a, \theta)$ are given, then an explicit solution is given by Villat's integration formula. See Iyanaga and Kawada [3, page 1450] for details.
6. See also Churchill [1, Chapter 11, pages 242–258] and Levinson and Redheffer [5, page 360].

References

[1] Churchill, R. V. *Complex Variables and Applications.* McGraw–Hill Book Company, New York, 1960.

[2] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers.* John Wiley & Sons, New York, 1982.

[3] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics.* MIT Press, Cambridge, MA, 1980.

[4] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis.* Interscience Publishers, Inc., New York, 1958.

[5] Levinson, N., and Redheffer, R. M. *Complex Variables.* Holden–Day, Inc., San Francisco, CA, 1979.

[6] Young, E. C. *Partial Differential Equations.* Allyn and Bacon, Inc., Boston, MA, 1972.

111. Riemann's Method

Applicable to Linear hyperbolic equations of the second order in two independent variables.

Yields

An exact solution in terms of the solution to the adjoint equation.

Idea

The solution of a non-characteristic initial value problem in two dimensions can be found if the adjoint equation with specified boundary conditions can be solved.

Procedure

Suppose we have the hyperbolic partial differential equation

$$L[u] = u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \tag{111.1}$$

where $u(x,y)$ is specified on the boundary Γ, which is not a characteristic (see figure 111.1). Note that any linear hyperbolic equations of second order in two independent variables can be written in the form of equation (111.1).

We wish to find $u(S) = u(\zeta,\eta)$, where S represents an arbitrary point and is indicated in figure 111.1. If we assume that the initial curve Γ is monotonically decreasing, then we can write the solution as

$$\begin{aligned} u(\zeta,\eta) = \frac{1}{2}R(P;\zeta,\eta)u(P) &+ \frac{1}{2}R(Q;\zeta,\eta)u(Q) \\ &- \int_P^Q B[u(x,y), R(x,y;\zeta,\eta)] \\ &+ \iint\limits_D f(x,y)R(x,y;\zeta,\eta)\,dx\,dy, \end{aligned} \tag{111.2}$$

where

$$B[u,v] = \left(avu + \frac{1}{2}vu_y - \frac{1}{2}v_y u\right)dy + \left(-bvu + \frac{1}{2}vu_x - \frac{1}{2}v_x u\right)dx,$$

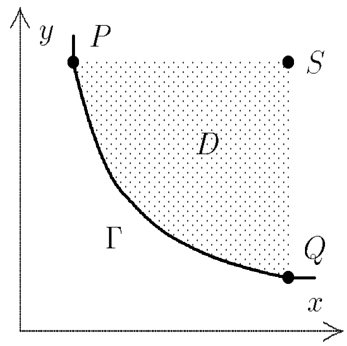

Figure 111.1: Domain in which equation (111.1) is solved.

(note that $B[u,v]$ includes the differential terms dx and dy) and $R(x,y;\zeta,\eta)$ is the Riemann function defined by

$$\begin{aligned} R_{xy} - aR_x - bR_y + (c - a_x - b_y)R = 0,& \\ R(\zeta, y; \zeta, \eta) = \exp\left[\int_\eta^y a(\zeta,\sigma)\,d\sigma\right],& \\ R(x,\eta;\zeta,\eta) = \exp\left[\int_\zeta^x b(\sigma,\eta)\,d\sigma\right],& \\ R(\zeta,\eta;\zeta,\eta) = 1.& \end{aligned} \tag{111.3}$$

In this formulation, PS is a horizontal segment and QS is a vertical segment that contain the domain the dependence D. The derivation of this formula is more detailed than the format of this book allows. See Garabedian [6, pages 127–135] for a full description. A simple motivation for the Riemann function is given in Kreith [9].

Example 1

Suppose we have the partial differential equation

$$\begin{aligned} \alpha^2 w_{\beta\beta} - \beta^2 w_{\alpha\alpha} = 0,& \\ w(\alpha, 1) = f(\alpha),& \\ w_\beta(\alpha, 1) = g(\alpha),& \end{aligned} \tag{111.4.a-c}$$

where $-\infty < \alpha < \infty$, $1 < \beta < \infty$, and $f(\alpha)$ and $g(\alpha)$ are given functions. If we change variables in equation (111.1) from $\{w, \alpha, \beta\}$ to $\{u, x, y\}$ by

$$u(x,y) = w(\alpha,\beta),$$
$$x = \alpha\beta, \qquad y = \frac{\beta}{\alpha},$$

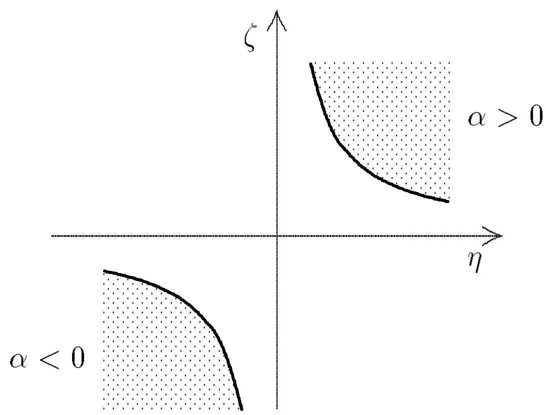

Figure 111.2: Domain in which equation (111.4) is solved.

(see the transformation on page 168), then equation (111.4.a) becomes

$$u_{xy} - \frac{1}{2x} u_y = 0. \tag{111.5}$$

The boundary conditions in equation (111.4) transform to

$$\begin{aligned} u\left(s, \frac{1}{s}\right) &= f(s), \\ s u_x\left(s, \frac{1}{s}\right) + \frac{1}{s} u_y\left(s, \frac{1}{s}\right) &= g(s), \end{aligned} \tag{111.6}$$

where $-\infty < s < \infty$. By manipulations of equation (111.6), we can derive

$$\begin{aligned} u\left(s, \frac{1}{s}\right) &= f(s), \\ u_x\left(s, \frac{1}{s}\right) &= \frac{1}{2}\left[f'(s) + \frac{1}{s} g(s)\right], \\ u_y\left(s, \frac{1}{s}\right) &= \frac{1}{2}\left[s g(s) - s^2 f(s)\right]. \end{aligned} \tag{111.7}$$

The domain in which equations (111.5) and (111.7) are to be solved is shown in figure 111.2.

To solve equations (111.5) and (111.7), we use Riemann's method. Comparing equation (111.5) to equation (111.1) we determine $a = 0$, $b = -1/2x$, $c = 0$, $f = 0$. Hence, the solution (from equation (111.2)) becomes

$$u(\zeta,\eta) = \frac{1}{2}R(P;\zeta,\eta)u(P) + \frac{1}{2}R(Q;\zeta,\eta)u(Q) \qquad (111.8)$$
$$- \int_P^Q \left[\left(\frac{1}{2}Ru_y - \frac{1}{2}R_y u\right)dy - \left(-\frac{1}{2x}Ru + \frac{1}{2}Ru_x - \frac{1}{2}R_x u\right)dx\right].$$

All that remains is to find the Riemann's function. From equation (111.3), $R(x,y;\zeta,\eta)$ satisfies

$$\begin{aligned} R_{xy} + \frac{1}{2x}R_y &= 0, \\ R(\zeta,y;\zeta,\eta) &= 1, \\ R(x,\eta;\zeta,\eta) &= \sqrt{\frac{\zeta}{x}}, \\ R(\zeta,\eta;\zeta,\eta) &= 1. \end{aligned} \qquad (111.9.a\text{-}d)$$

Because equation (111.9.a) can be integrated directly with respect to x and then with respect to y, the general solution to equation (111.9) is easily seen to be of the form

$$R(x,y;\zeta,\eta) = M(x;\zeta,\eta) + \frac{K(y;\zeta,\eta)}{\sqrt{x}}, \qquad (111.10)$$

for some $M(x;\zeta,\eta)$ and some $K(y;\zeta,\eta)$. Using equation (111.10) in the boundary conditions in equation (111.9), the solution is found to be

$$R(x,y;\zeta,\eta) = \sqrt{\frac{\zeta}{x}}. \qquad (111.11)$$

Using equation (111.11) in equation (111.8), we can find $u(\zeta,\eta)$ and hence, $w(\alpha,\beta)$ for any values of α and β.

Example 2

The Riemann's function for the partial differential equation

$$u_{xy} = \frac{1}{4}k^2 u, \qquad (111.12)$$

(when k is a constant) is

$$R(x,y;\zeta,\eta) = I_0\left(k\sqrt{(x-\zeta)(y-\eta)}\right),$$

where I_0 is the usual modified Bessel function of order zero. Hence, the solution to equation (111.12) with the boundary conditions

$$\begin{aligned} u_x &= \psi(x) \qquad \text{when } y = 0, \\ u_y &= \phi(x) \qquad \text{when } x = 0, \end{aligned}$$

is given by

$$u(x,y) = \int_0^y I_0\left(k\sqrt{x(y-\eta)}\right)\phi(\eta)\,d\eta + \int_0^x I_0\left(k\sqrt{y(x-\zeta)}\right)\psi(\zeta)\,d\zeta.$$

Notes

1. Numerical techniques based on this method are called Godunov methods, after Godunov [7]. A comparison of some of these methods can be found in Woodward and Colella [11].
2. Essentially, the Riemann's function is a type of Green's function, the connection is made in Zauderer [12, pages 485–492]. What we have called the Riemann's function is sometimes called a Green's function or a Riemann–Green function.
3. If the operator $L[u]$ in equation (111.1) is self-adjoint, then we have the reciprocity principle: $R(x, y; \zeta, \eta) = R(\zeta, \eta; x, y)$.
4. Numerical methods for solving hyperbolic equations that use the Riemann's function are generally referred to as Godunov-type methods. A comparison of some Godunov-type methods with more classicial methods may be found in Woodward and Colella [11].
5. Copson [3, pages 77–88] suggests that the Riemann's function may often have the form

$$R(x, y; \zeta, \eta) = \sum_{k=0}^{\infty} \frac{G_k \Upsilon^k}{(k!)^2},$$

   where $\Upsilon = (x - \zeta)(y - \eta)$. When this is the case, then only the coefficients $\{G_k\}$ must be found. Copson [3, pages 77–88] gives several examples of this approach.
6. The technique presented here may be extended to higher order equations, for which the Riemann tensor must be determined. See Courant and Hilbert [4, Volume II, pages 450–461].
7. See also Bateman [1, pages 280–285], Chester [2, pages 222–231], Davis [5, pages 75–79], and Sneddon [10, pages 119–122].

References

[1] Bateman, H. *Differential Equations.* Longmans, Green and Co., 1926.

[2] Chester, C. R. *Techniques in Partial Differential Equations.* McGraw–Hill Book Company, New York, 1970.

[3] Copson, E. T. *Partial Differential Equations.* Cambridge University Press, New York, 1975.

[4] Courant, R., and Hilbert, D. *Methods of Mathematical Physics.* Interscience Publishers, Inc., New York, 1953.

[5] Davis, J. L. *Finite Difference Methods in Dynamics of Continuous Media.* The MacMillan Company, New York, 1986.

[6] Garabedian, P. R. *Partial Differential Equations.* John Wiley & Sons, New York, 1964.

[7] Godunov, S. K. Finite difference methods for numerical computations of discontinuous solutions of equations of fluid dynamics. *Mat. Sb.* (1959), 271–295. In Russian.

[8] Iraniparast, N. Green–Riemann functions for a class of hyperbolic focal point problems. *SIAM J. Math. Anal. 20*, 2 (March 1989), 408–414.

[9] Kreith, K. Establishing hyperbolic Green's functions via Leibniz's rule. *SIAM Review 33*, 1 (March 1991), 101–105.

[10] Sneddon, I. N. *Elements of Partial Differential Equations.* McGraw–Hill Book Company, New York, 1957.

[11] Woodward, P., and Colella, P. The numerical simulation of two-dimensional fluid flow with strong shocks. *J. Comput. Physics 54* (1984), 115–173.

[12] Zauderer, E. *Partial Differential Equations of Applied Mathematics.* John Wiley & Sons, New York, 1983.

112. Separation of Variables

Applicable to Most often, linear homogeneous partial differential equations.

Yields

An exact solution, generally in the form of an infinite series.

Idea

We look for a solution to a partial differential equation by separating the solution into pieces, where each piece deals with a single dependent variable.

Procedure

For linear homogeneous partial differential equations, try to represent the solution as a sum of terms in which each term factors into a product of expressions, each expression dealing with a single independent variable. For nonlinear equations, try to represent the solution as a sum of such expressions. In all cases, not only must the equation admit a solution of the proposed form, but the boundary conditions must also have the right form.

In more detail, suppose that $L[u] = 0$ is a linear partial differential equation for $u(\mathbf{x})$ that has the form $L[u] = \sum_i L_i[u]$, where the $L_i[u]$ are differential operators. We look for a solution of this partial differential equation in the form

$$u(\mathbf{x}) = u(x_1, x_2, \cdots, x_n) = X_1(x_1)X_2(x_2)\ldots X_n(x_n),$$

where the functions $\{X_1, X_2, \ldots, X_n\}$ are to be determined. By using the above form in the original equation and reasoning about which terms depend upon which variables, we can often reduce the original partial differential equation into an ordinary differential equation for each of the $\{X_i\}$. In carrying this out, arbitrary constants will be introduced. After the resulting ordinary differential equations are solved, the arbitrary constants can generally be found by physical reasoning.

Because superposition can be used in linear equations, any number of terms (of the form shown above) will also be a solution of the original equation. Also, if each of these terms is multiplied by some constant and then added together, the resulting expression will also be a solution. Hence, the final solution will frequently be a sum or an integral.

This sum will have unknown constants in it due to the constants allowed in the superposition. These constants will be determined from the initial conditions and/or the boundary conditions.

The only time that we can be sure that we have found the most general solution to a given ordinary differential equation by this technique is when there exists a "completeness theorem" for each of the ordinary differential equations that we have found.

Example 1

Suppose we wish to solve the heat equation in a circle

$$\frac{\partial u}{\partial t} = \nabla^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \tag{112.1}$$

for $u(t, r, \theta)$. We try to separate variables in equation (112.1) by proposing a solution of the form

$$u(t, r, \theta) = T(t)R(r)\Theta(\theta). \tag{112.2}$$

Substituting equation (112.2) into equation (112.1) and simplifying yields

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} - \frac{1}{T}\frac{dT}{dt} = 0. \tag{112.3}$$

By the assumption made implicitely in equation (112.2), only the third term in equation (112.3) has any dependence on the variable t. Because the other terms cannot have any t dependence, it must be that the third term also has no t dependence. Therefore, this term must be equal to some (unknown) constant; that is,

$$\frac{1}{T}\frac{dT}{dt} = -\lambda = \text{ some unknown constant.} \tag{112.4}$$

The minus sign in equation (112.4) is taken for convenience later. Using equation (112.4) in equation (112.3) and simplifying, we find

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + r^2\lambda + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0. \tag{112.5}$$

The third term in equation (112.3) is the only one that could depend on θ, but we easily see that it cannot depend on θ because the first two terms in equation (112.5) could not cancel out any θ dependence. Therefore, we must conclude that

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -\rho = \text{ another unknown constant.} \tag{112.6}$$

Using equation (112.6) in equation (112.5), we find

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (-\rho + r^2\lambda)R = 0. \tag{112.7}$$

Note that we have, at this point, found ordinary differential equations that describe each of the terms in the solution proposed in equation (112.2).

But, in doing so, we have introduced two arbitrary constants; λ and ρ. Solving the ordinary differential equations in equations (112.4), (112.6), and (112.7) yields

$$\begin{aligned} T(t) &= Ae^{-\lambda t}, \\ \Theta(\theta) &= B\sin(\sqrt{\rho}\theta) + C\cos(\sqrt{\rho}\theta), \\ R(r) &= DJ_{\sqrt{\rho}}(\sqrt{\lambda}r) + EY_{\sqrt{\rho}}(\sqrt{\lambda}r), \end{aligned} \tag{112.8}$$

where $\{A, B, C, D, E\}$ are arbitrary constants and $\{J_*, Y_*\}$ are Bessel functions. By superposition, the most general solution to equation (112.2) can now be written as

$$\begin{aligned} u(t, r, \theta) = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\rho\ e^{-\lambda t} \Big[B(\lambda, \rho)\sin(\sqrt{\rho}\theta) + C(\lambda, \rho)\cos(\sqrt{\rho}\theta) \Big] \\ \times \Big[D(\lambda, \rho)J_{\sqrt{\rho}}(\sqrt{\lambda}r) + E(\lambda, \rho)Y_{\sqrt{\rho}}(\sqrt{\lambda}r) \Big], \end{aligned} \tag{112.9}$$

where $\{B, C, D, E\}$ may depend on λ and ρ. Now physical reasoning and the initial conditions and boundary conditions must be used to evaluate $\{B, C, D, E\}$.

For example, if the heat equation in (112.1) is being solved in the entire circle, then it must be that the solution is periodic in θ with period 2π. That is, $u(t, r, \theta) = u(t, r, \theta + 2\pi)$. This constraint (which is equivalent to $\Theta(\theta) = \Theta(\theta + 2\pi)$), placed on equation (112.8), restricts $\sqrt{\rho}$ to be an integer. Hence, in this case, the most general solution has the form (using $n^2 = \rho$)

$$\begin{aligned} u(t, r, \theta) = \int_{-\infty}^{\infty} d\lambda \sum_{n=0}^{\infty} e^{-\lambda t} \Big[B(\lambda, n^2)\sin n\theta + C(\lambda, n^2)\cos n\theta \Big] \\ \times \Big[D(\lambda, n^2)J_n(\sqrt{\lambda}r) + E(\lambda, n^2)Y_n(\sqrt{\lambda}r) \Big]. \end{aligned}$$

If the point $r = 0$ was included in the domain of the original problem, then we would require $E(\lambda, n^2) \equiv 0$ because $Y_n(r)$ is unbounded at $r = 0$. Likewise, only those values of $\lambda \geq 0$ will be physically realistic. Hence, in this case, we find

$$u(t, r, \theta) = \int_{0}^{\infty} d\lambda \sum_{n=0}^{\infty} e^{-\lambda t} \Big[B(\lambda, n^2)\sin n\theta + C(\lambda, n^2)\cos n\theta \Big] J_n(\sqrt{\lambda}r). \tag{112.10}$$

More conditions could be placed on the coefficients depending on the exact form of the initial conditions and boundary conditions.

Example 2

Suppose we have the nonlinear equation

$$f(x)u_x^2 + g(y)u_y^2 = a(x) + b(y) \tag{112.11}$$

to solve. We might propose a solution of the form

$$u(x, y) = \phi(x) + \psi(y). \tag{112.12}$$

Using equation (112.12) in equation (112.11) results in the equation

$$f(x)[\phi'(x)]^2 - a(x) = g(y)[\psi'(y)]^2 - b(y). \tag{112.13}$$

The left-hand side of equation (112.13) must be independent of x (because the right-hand side is); hence, we can set

$$f(x)[\phi'(x)]^2 - a(x) = \alpha = \text{ some constant}, \tag{112.14}$$

and then

$$g(y)[\psi'(y)]^2 - b(y) = \alpha. \tag{112.15}$$

Solving equations (112.14) and (112.15), we have determined that a solution to equation (112.11) is given by

$$v(x, y) = \int_{x_0}^{x} \sqrt{\frac{a(\xi) + \alpha}{f(\xi)}}\, d\xi + \int_{y_0}^{y} \sqrt{\frac{b(\eta) - \alpha}{g(\eta)}}\, d\eta + \beta, \tag{112.16}$$

where β is another arbitrary constant. The solution in (112.16) may not be the most general solution to equation (112.11). For nonlinear equations, it is very difficult to determine whether the most general solution has been found.

Notes

1. Note that the solution in equation (112.10) could also have been obtained by use of Fourier series (see page 344). The form of the solution in equation (112.10) (i.e., the $e^{-\lambda t}$ term) suggests that a Laplace transform might also be an appropriate way to analyze equation (112.1).
2. Carslaw and Jaeger [4] have the decompositions (similar to equation (112.9)) for many heat conduction problems.
3. If the equation $L[u] = 0$ can be separated into ordinary differential equations when $u(\mathbf{x}) = \frac{u_1(x_1)u_2(x_2)\cdots u_n(x_n)}{R(\mathbf{x})}$ and $R \neq 1$, then the equation is said to be R separable.
4. Moon and Spencer [11] list 11 common orthogonal coordinate systems in which both Laplace's equation and Helmholtz's equation separate. These coordinate systems are rectangular, circular cylinder, elliptic cylinder, parabolic cylinder, spherical, prolate spheroidal, oblate spheroidal, parabolic, conical, ellipsoidal, and paraboloidal. Also included are the exact decompositions that are obtained (similar to

(112.9)). The above analysis is repeated for 21 different cylindrical coordinate systems that are obtained by translating an orthogonal map in a direction perpendicular to the plane of the map. The above analysis is again carried out for 10 different rotational coordinate systems that are obtained by twirling an orthogonal map in a plane about an axis. In each of these 31 coordinate systems, Laplace's equation or Helmholtz's equation separates (or is R separable).

5. A necessary and sufficient condition for a system with 2 degrees of freedom, with the Hamiltonian $H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y)$, to be separable in elliptic, polar, parabolic, or cartesian coordinates is that the expression

$$\begin{aligned}(V_{yy} - V_{xx})&(-2axy - b'y - bx + d)\\ &+ 2V_{xy}(ay^2 - ax^2 + by - b'x + c - c')\\ &+ V_x(6ay + 3b) + V_y(-6ax - 3b')\end{aligned}$$

vanishes for some constants $(a, b, b', c, c', d) \neq (0, 0, 0, c, c, 0)$. The values of these constants determine in which of the above four coordinate systems the differential equations separate. For 3 degrees of freedom, a similar expression has been devised that determines in which of 11 different coordinate systems the equations separate. For more details, see Marshall and Wojciechowski [9].

6. The equation $(\Box + V(x))u = u_{tt} - u_{xx} + V(x)u = 0$ (where $\Box$ is the D'Alembert operator) can be non-trivially separated if and only if the function $V(x)$ is given (up to an equivalence relation) by one of the following 12 forms (here m, m_1, m_2 are arbitrary real parameters and $m_2 \neq 0$):

(a) $V = (m_1 + m_2 \sin x) \cos^{-2} x$	(g) $V = mx$
(b) $V = (m_1 + m_2 \sinh x) \cosh^{-2} x$	(h) $V = mx^{-2}$
(c) $V = (m_1 + m_2 \cosh x) \sinh^{-2} x$	(i) $V = m \sin^{-2} x$
(d) $V = m_1 e^x + m_2 e^{2x}$	(j) $V = m \sinh^{-2} x$
(e) $V = m_1 + m_2 x^{-2}$	(k) $V = m \cosh^{-2} x$
(f) $V = m$	(l) $V = me^x$

(See Zhdanov *et al.* [13].) Using these forms for $V(x)$, there are 8 inequivalent forms of $(\Box + V)u = 0$ that can be non-trivially separated. These forms and the number of coordinate systems in which they separate are:

(a) 2 systems: $\Box u + mxu = 0$
(b) 9 systems: $\Box u + mx^{-2}u = 0$
(c) 4 systems: $\Box u + (m_1 + m_2 \cos x) \sin^{-2} x\, u = 0$
(d) 4 systems: $\Box u + (m_1 + m_2 \sinh x) \cosh^{-2} x\, u = 0$
(e) 11 systems: $\Box u + (m_1 + m_2 \cosh x) \sinh^{-2} x\, u = 0$

(f) 6 systems: $\Box u + (m_1 + m_2 e^x)e^x\, u = 0$
(g) 6 systems: $\Box u + (m_1 + m_2 x^{-2})u = 0$
(h) 11 systems: $\Box u + mu = 0$

7. The Hartree–Fock approximation is a technique for approximating the eigenfunctions $u(\mathbf{x})$ and eigenvalues λ of the partial differential equation

$$-\nabla^2 u + f(\mathbf{x})u = \lambda u, \tag{112.17}$$

when $f(\mathbf{x})$ is a prescribed function. The technique consists of approximating $f(\mathbf{x})$ by

$$f(\mathbf{x}) \simeq f_1(x_1)f_2(x_2)\cdots f_n(x_n).$$

If $f(\mathbf{x})$ has the form shown above, then equation (112.17) can be solved by separation of variables. The solution will be of the form

$$u(\mathbf{x}) = u_1(x_1)u_2(x_2)\cdots u_n(x_n),$$
$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

In the Hartree–Fock approximation, a variational principle is used to determine what the "best" $\{f_j(x_j)\}$ are. See Fischer [6] for details.

8. Miller [10] contains a group theoretical approach to the method of separation of variables. For many linear differential equations, the separated solutions are easily related to the Lie algebra generated by the equation.
9. See Boyce and DiPrima [3, Chapter 10, pages 513–580].

References

[1] Arscott, F. M., and Darai, A. Curvilinear co-ordinate systems in which the Helmholtz equation separates. *IMA J. Appl. Mathematics 27* (1981), 33–70.

[2] Blum, E. K., and Reid, G. J. On the numerical solution of three-dimensional boundary value problems by separation of variables. *SIAM J. Numer. Anal. 25*, 1 (Februrary 1988), 75–90.

[3] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[4] Carslaw, H. S., and Jaeger, J. C. *Conduction of Heat in Solids*. Clarendon Press, Oxford, England, 1984.

[5] Doyle, P. W. Separation of variables for scalar evolution equations in one space dimension. *J. Phys. A: Math. Gen. 29* (1996), 7581–7595.

[6] Fischer, C. F. Approximate solution of Schr odinger's equation for atoms. In *Numerical Integration of Differential Equations and Large Linear Systems*, J. Hinze, Ed. Springer–Verlag, New York, 1982, pp. 71–81.

[7] Hainzl, J. On a general concept for separation of variables. *SIAM J. Math. Anal. 13*, 2 (March 1982), 208–225.

[8] Kaufman, L., and Warner, D. D. Algorithm 685: A program for solving separable elliptic equations. *ACM Trans. Math. Software 16*, 4 (Dec 1990), 323–351.

[9] Marshall, I., and Wojciechowski, S. When is a Hamiltonian system separable? *J. Math. Physics 29*, 6 (June 1988), 1338–1346.

[10] Miller, Jr., W. *Symmetry and Separation of Variables*. Addison–Wesley Publishing Co., Reading, MA, 1977.

[11] Moon, P., and Spencer, D. E. *Field Theory for Engineers*. D. Van Nostrand Company, Inc., New York, 1961.

[12] Moon, P., and Spencer, D. E. *Field Theory Handbook*. Springer–Verlag, New York, 1961.

[13] Zhdanov, R. Z., Revenko, I. V., and Fushchych, W. I. Orthogonal and non-orthogonal separation of variables in the wave equation zzzref1refzzz. *J. Phys. A: Math. Gen. 26* (1993), 5959–5972.

113. Separable Equations: Stäckel Matrix

Applicable to Helmholtz's or Laplace's equation in some orthogonal coordinate systems.

Yields

An exact solution, generally in the form of an infinite series.

Idea

If certain conditions hold, then it is possible to separate variables in an orthogonal coordinate system for Helmholtz's equation or for Laplace's equation.

Procedure

Suppose we have an orthogonal coordinate system in the variables $\{u^1, u^2, u^3\}$ with the metric $\{g_{ii}\}$. As usual, we define $g = g_{11}g_{22}g_{33}$. Assume that the Stäckel matrix S is defined by

$$S = \begin{bmatrix} \Phi_{11}(u^1) & \Phi_{12}(u^1) & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & \Phi_{22}(u^2) & \Phi_{23}(u^2) \\ \Phi_{31}(u^3) & \Phi_{32}(u^3) & \Phi_{33}(u^3) \end{bmatrix}$$

in which each row only contains functions of one variable. Define the determinant of S to be s

$$s = \begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{vmatrix},$$

and note that the cofactors of the elements in the first column are given by

$$M_{11} = \begin{vmatrix} \Phi_{22} & \Phi_{23} \\ \Phi_{32} & \Phi_{33} \end{vmatrix} \qquad M_{21} = -\begin{vmatrix} \Phi_{12} & \Phi_{13} \\ \Phi_{32} & \Phi_{33} \end{vmatrix} \qquad M_{31} = \begin{vmatrix} \Phi_{12} & \Phi_{13} \\ \Phi_{22} & \Phi_{23} \end{vmatrix},$$

If the following relations hold

$$\begin{aligned} g_{ii} &= \frac{s}{M_{i1}} \\ \frac{\sqrt{g}}{s} &= f_1(u^1)f_2(u^2)f_3(u^3), \end{aligned} \tag{113.1}$$

then the Helmholtz equation $\nabla^2 W + \lambda^2 W = 0$ separates with the solution given by $W = W_1(u^1)W_2(u^2)W_3(u^3)$, where the $\{W_i\}$ are defined by

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i \frac{dW_i}{du^i}\right) + W_i \sum_{j=1}^{3} \alpha_j \Phi_{ij} = 0, \tag{113.2}$$

with $\alpha_1 = \lambda^2$, and α_2 and α_3 arbitrary.

Example

In parabolic coordinates $\{\mu, \nu, \psi\}$, we have the metric coefficients $g_{11} = g_{22} = \mu^2 + \nu^2$ and $g_{33} = \mu^2\nu^2$. Hence, $\sqrt{g} = \mu\nu(\mu^2 + \nu^2)$. The Laplacian in parabolic coordinates is given by

$$\nabla^2\phi = \frac{1}{\mu^2 + \nu^2}\left[\frac{\partial^2\phi}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial\phi}{\partial\mu} + \frac{\partial^2\phi}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial\phi}{\partial\nu}\right] + \frac{1}{\mu^2\nu^2}\frac{\partial^2\phi}{\partial\psi^2}.$$

With this form, it would appear unlikely that the Helmholtz equation $\nabla^2 W + \lambda^2 W = 0$ would separate. But, note that the Stäckel matrix

$$S = \begin{bmatrix} \mu^2 & -1 & -\mu^{-2} \\ \nu^2 & 1 & -\nu^{-2} \\ 0 & 0 & 1 \end{bmatrix},$$

from which we find $s = \mu^2 + \nu^2$, $M_{11} = M_{21} = 1$, and $M_{31} = \mu^{-2} + \nu^{-2}$, satisfies the equations in (113.1) (when we take $f_1 = \mu$, $f_2 = \nu$, $f_3 = 1$). From this we conclude that the Helmholtz equation does separate in parabolic coordinates. The separation equations (corresponding to equation (113.2)) are

$$\begin{aligned}
\frac{1}{\mu}\frac{d}{d\mu}\left(\mu\frac{dW_1}{d\mu}\right) + W_1\left(\alpha_1\mu^2 - \alpha_2 - \frac{\alpha_3}{\mu^2}\right) &= 0 \\
\frac{1}{\nu}\frac{d}{d\nu}\left(\nu\frac{dW_2}{d\nu}\right) + W_2\left(\alpha_1\nu^2 + \alpha_2 - \frac{\alpha_3}{\nu^2}\right) &= 0 \\
\frac{d^2W_3}{d\psi^2} + \alpha_3 W_3 &= 0,
\end{aligned}$$

where $W = W_1(\mu)W_2(\nu)W_3(\psi)$.

Notes

1. The Stäckel matrix is not unique.
2. Not all orthogonal coordinate systems allow separation.
3. All cylindrical coordinate systems in which the Helmoltz equation separates has a Stäckel matrix of the form

$$S = \begin{bmatrix} 0 & \Phi_{12} & \Phi_{13} \\ 0 & \Phi_{22} & \Phi_{23} \\ 1 & 0 & 1 \end{bmatrix}.$$

4. For every rotational coordinate system, the Helmholtz equation separates with a Stäckel matrix of the form

$$S = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Necessary and sufficient conditions for separation of the Laplace equation ($\nabla^2 W = 0$) are

$$\frac{g_{ii}}{g_{jj}} = \frac{M_{j1}}{M_{i1}}$$
$$\frac{\sqrt{g}}{g_{ii}} = f_1(u^1)f_2(u^2)f_3(u^3)M_{i1}.$$

References

[1] Boyer, C. P., Kalnins, E. G., and Miller, Jr, W. Stackel-equivalent integrable Hamiltonian systems. *SIAM J. Math. Anal. 17* (1986), 778–797.

[2] Eisenhart, L. P. Separable systems of Stackel. *Annals of Math. 35* (1934), 284–305.

[3] Kalnins, E. G., Benenti, S., and Miller, Jr, W. Integrability, Stackel spaces, and rational potentials. *J. Math. Physics 38*, 5 (May 1997), 2345–2365.

[4] Kalnins, E. G., and Miller, W. Differential–Stackel matrices. *J. Math. and Physics 26* (1995), 1560–1565.

[5] Kalnins, E. G., and Miller, W. Generalized–Stackel matrices. *J. Math. and Physics 26* (1995), 2168–2173.

[6] Kalnins, E. G., and Miller, Jr, W. The general theory of *R*-separation for Helmholtz equations. *J. Math. Physics 24* (1983), 1047–1053.

[7] Moon, P., and Spencer, D. E. *Field Theory Handbook*. Springer–Verlag, New York, 1961.

[8] Stackel, P. *Uber die integration der Hamilton–Jacobischen differentialgeichung mittels separation der variabeln.* Habilitationschrift, Halle, 1891.

114. Similarity Methods

Applicable to Linear or nonlinear partial differential equations, and also systems of differential equations.

Yields

An equation with one fewer independent variables.

Idea

Sometimes the number of independent variables in a partial differential equation can be reduced by taking algebraic combinations of the independent variables.

Procedure

The idea of this method is to find new independent variables (called *similarity variables*) that are combinations of the old independent variables. The differential equation, when written in the new variables, will not depend on all of the new variables.

One technique for discovering the correct new variables is to choose temporary variables to be a parameter to some (unknown) power times the old variables. After writing the equation in terms of the temporary variables, the powers can be found by requiring homogeneity in the parameter. New variables are then constructed from the old variables in such a way that the parameter does not enter.

Example 1

Suppose the following linear partial differential equation

$$\frac{\partial u}{\partial t} + \frac{u}{2t} = \nu \frac{\partial^2 u}{\partial z^2}, \tag{114.1}$$

for $u(t, z)$ is to be simplified from being a function of the two independent variables $\{t, z\}$ to being a function of only one independent variable. We define the temporary variables u', z', t' and the parameter λ by

$$\begin{aligned} u &= u'\lambda, \\ t &= t'\lambda^m, \\ z &= z'\lambda^n, \end{aligned} \tag{114.2}$$

for some unknown values of n and m. In these temporary variables, equation (114.1) becomes

$$\frac{\partial u'}{\partial t'}\lambda^{1-m} + \frac{u'}{2t'}\lambda^{1-m} = \nu \frac{\partial^2 u'}{\partial (z')^2}\lambda^{1-2n}. \tag{114.3}$$

For the parameter λ to be eliminated from equation (114.3), we require that the exponents of λ in each term of equation (114.3) all be the same.

That is, $1 - m = 1 - 2n$. This equation has the solution $m = 2n$. At this point we know that there are similarity solutions of equation (114.1) but still must determine what they are. Using $m = 2n$ in (114.2), the change of variables becomes

$$\begin{aligned} u &= u'\lambda, \\ t &= t'\lambda^{2n}, \\ z &= z'\lambda^{n}. \end{aligned} \tag{114.4}$$

Combining the original independent variables $\{t, z\}$, we form a new independent variable $\{\eta\}$ whose transformation from the old variables to the temporary variables does not depend on λ:

$$\eta := \frac{z}{\sqrt{t}} = \frac{z'}{\sqrt{t'}}.$$

Now we have to propose the similarity solution. We look for a solution of the form

$$u(t, z) = v\left(\frac{z}{\sqrt{t}}\right) = v(\eta). \tag{114.5}$$

When the form in equation (114.5) is used in equation (114.1), we obtain

$$2\nu \frac{d^2 v}{d\eta^2} + \eta \frac{dv}{d\eta} - v = 0, \tag{114.6}$$

which is now an ordinary differential equation. Every solution of equation (114.6) will generate a solution of equation (114.1).

Example 2

Consider the following nonlinear partial differential equation:

$$\frac{\partial u}{\partial t} + \frac{u}{2t} + \beta u \frac{\partial u}{\partial z} = \nu \frac{\partial^2 u}{\partial z^2} \tag{114.7}$$

for $u(t, z)$. This equation differs from equation (114.1) by the $\beta u u_z$ term. We wish to simplify this equation from being a function of the two independent variables $\{t, z\}$ to being a function of only one independent variable. After we do this, we will find a solution for the $\beta = 0$ case. We define the temporary variables u', z', t', and the parameter λ by equation (114.2). In these temporary variables, equation (114.7) becomes

$$\frac{\partial u'}{\partial t'}\lambda^{1-m} + \frac{u'}{2t'}\lambda^{1-m} + \beta u' \frac{\partial u'}{\partial z'}\lambda^{2-n} = \nu \frac{\partial^2 u'}{\partial (z')^2}\lambda^{1-2n}. \tag{114.8}$$

For the parameter λ to be eliminated from equation (114.8), we require that the exponents of λ in each term of equation (114.8) all be the same. That is,

$$1 - m = 2 - n = 1 - 2n. \tag{114.9}$$

These equations have the unique solution: $n=-1$, $m=-2$. At this point, we know that there is a similarity solution of equation (114.7). Using $n=-1$, $m=-2$ in equation (114.2) changes the variables to $\{u=u'\lambda, t=t'\lambda^{-2}, z=z'\lambda^{-1}\}$. Combining the original independent variables $\{t,z\}$, we form a new independent variable $\{\eta\}$ whose transformation from the old variables to the temporary variables does not depend on λ:

$$\eta := \frac{z}{\sqrt{t}} = \frac{z'}{\sqrt{t'}}.$$

Combining the original dependent variable $\{u\}$ with the original independent variables $\{t,z\}$, we can form a new dependent variable $\{w\}$ whose transformation from the old variables to the temporary variables does not depend on λ:

$$w = \frac{t}{z}u = \frac{t'}{z'}u'. \tag{114.10}$$

Now we have to propose the similarity solution. By solving equation (114.10) for u, we are led to the assumption

$$u(t,z) = \frac{z}{t}w\left(\frac{z}{\sqrt{t}}\right) = \frac{z}{t}w(\eta). \tag{114.11}$$

When the form in equation (114.11) is used in equation (114.7), we obtain

$$2\nu\eta\frac{d^2w}{d\eta^2} + \left(4\nu + \eta^2 - 2\beta\eta^2 w\right)\frac{dw}{d\eta} + (1-2\beta w)\,w = 0. \tag{114.12}$$

If we define $g(\eta)$ by $g(\eta)=\eta w(\eta)$, then equation (114.12) becomes

$$2\nu\frac{d^2g}{d\eta^2} + (\eta - 2\beta g)\frac{dg}{d\eta} = 0. \tag{114.13}$$

Every solution of this ordinary differential equation will lead to similarity solutions of equation (114.7). In the special case of $\beta = 0$ (when equation (114.7) becomes the identical to equation (114.1)), the general solution to equation (114.13) is given by

$$g(\eta) = A + B\,\mathrm{erf}\left(\frac{\eta}{\sqrt{4\nu}}\right),$$

where A and B are arbitrary constants. This results in the solution

$$u(t,z) = \frac{1}{\sqrt{t}}\left[A + B\,\mathrm{erf}\left(\frac{z}{\sqrt{4\nu t}}\right)\right]$$

to equation (114.1). Note that this similarity solution could *not* have been obtained from equation (114.6), because the scalings in equations (114.5) and (114.11) are different.

Notes

1. In general, a partial differential equation may have some similarity solutions and some solutions that are not similarity solutions.
2. This method is sometimes called the *method of one parameter groups*, due to the single parameter λ that was used in equation (114.2). This method is derivable from Lie group methods (see page 471).
3. To solve a differential system (differential equation(s) with boundary condition(s)), the boundary conditions as well as the equation(s) must admit the similarity variable.
4. This method also applies to systems of ordinary differential equations. If $\frac{d\mathbf{u}}{dx} = \mathbf{f}(x, \mathbf{u})$ is a system of first order ordinary differential equations for $\mathbf{u} = (u_1, \dots, u_n)$, and if there exists a one parameter group of symmetries of the system, then there is a change of variables $(y, \mathbf{w}) = \Xi(x, \mathbf{u})$, which takes the system into $\frac{d\mathbf{w}}{dy} = \mathbf{g}(y, w_1, \dots, w_{n-1})$. Hence, the original system reduces to a system of $n-1$ ordinary differential equations for $(w_1, \dots, w_{n-1})$ together with the quadrature $w_n(y) = \int g_n(y, w_1(y), \dots, w_{n-1}(y))\, dy$.
5. For some systems, there are natural similarity variables. For example, in a two-dimensional problem with radial symmetry, the variable r (where $r^2 = x^2 + y^2$) should be a similarity variable if the original equations were written in terms of x and y. Similarly, in a radially symmetric three-dimensional problem, the variable ρ (where $\rho^2 = x^2 + y^2 + z^2$) should be a similarity variable.
6. For diffusion equations, similarity solutions are often of the form $f(x/\sqrt{t})$ or $t^\alpha f(x/\sqrt{t})$.
7. The partial differential equation $F\left(tx, u, \frac{u_t}{x}, \frac{u_x}{t}\right) = 0$ for $u(x,t)$ has the similarity variable $w = tx$. Considering $u = u(w)$, we find the equivalent ordinary differential equation $F(w, u, u_w, u_w) = 0$.
8. See also Ames [1, pages 135–141] and Seshadri and Na [6, pages 39–42].

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Dressner, L. *Similarity Solutions of Nonlinear Partial Differential Equations*. Pitman Publishing Co., Marshfield, MA, 1983.

[3] Kevorkian, J., and Cole, J. D. *Perturbation Methods in Applied Mathematics*. Springer–Verlag, New York, 1981.

[4] King, J. R. Exact similarity solutions to some nonlinear diffusion equations. *J. Phys. A: Math. Gen. 23* (1990), 3681–3697.

[5] Roseneau, P., and Schwarzmeier, J. L. Similarity solutions of systems of partial differential equations. *Comput. Physics Comm. 27* (1982), 179–186.

[6] Seshadri, R., and Na, T. Y. *Group Invariance in Engineering Boundary Value Problems*. Springer–Verlag, New York, 1985.

115. Exact Solutions to the Wave Equation

Applicable to The n-dimensional wave equation.

Yields

An explicit solution in terms of an integral.

Idea

An exact formula is available for the n-dimensional wave equation $u_{tt} = \nabla^2 u$.

Procedure

The n-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u = \frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_n}^2}, \tag{115.1}$$

with the initial data (we use $\mathbf{x} = (x_1, \ldots, x_n)$)

$$u(0, \mathbf{x}) = f(\mathbf{x}), \qquad u_t(0, \mathbf{x}) = g(\mathbf{x}), \tag{115.2}$$

has two different (but similar) forms of the solution, depending on whether n is even or odd. When n is odd the solution is given by

$$\begin{aligned} u(t, \mathbf{x}) = \frac{1}{1 \cdot 3 \cdots (n-2)} \bigg\{ & \frac{\partial}{\partial t} \left(\frac{\partial}{t\,\partial t} \right)^{(n-3)/2} t^{n-2} \omega[f; \mathbf{x}, t] \\ & + \left(\frac{\partial}{t\,\partial t} \right)^{(n-3)/2} t^{n-2} \omega[g; \mathbf{x}, t] \bigg\}, \end{aligned} \tag{115.3}$$

where $\omega[h; \mathbf{x}, t]$ is defined to be the average of the function $h(\mathbf{x})$ over the surface of an n-dimensional sphere of radius t centered at $\mathbf{x}$. That is,

$$\omega[h; \mathbf{x}, t] = \frac{1}{\sigma_n(t)} \int h(0, \boldsymbol{\zeta})\, d\Omega,$$

where $|\boldsymbol{\zeta} - \mathbf{x}|^2 = t^2$, $\sigma_n(t)$ is the surface area of the n-dimensional sphere of radius t, and $d\Omega$ is an element of area. (Note that $\sigma_n(t) = 2\pi^{n/2} t^{n-1} / \Gamma\left(\frac{n}{2}\right)$.)

When n is even the solution to equation (115.1) and equation (115.2) is given by

$$\begin{aligned} u(t, \mathbf{x}) = \frac{1}{2 \cdot 4 \cdots (n-2)} \bigg\{ & \frac{\partial}{\partial t} \left(\frac{\partial}{t\,\partial t} \right)^{(n-2)/2} \int_0^t \omega[f; \mathbf{x}, \rho] \frac{\rho^{n-1}\, d\rho}{\sqrt{t^2 - \rho^2}} \\ & + \left(\frac{\partial}{t\,\partial t} \right)^{(n-2)/2} \int_0^t \omega[g; \mathbf{x}, \rho] \frac{\rho^{n-1}\, d\rho}{\sqrt{t^2 - \rho^2}} \bigg\}, \end{aligned} \tag{115.4}$$

where $\omega[h; \mathbf{x}, t]$ is defined as above. Because the expression in equation (115.4) is integrated over ρ, the values of f and g must be known everywhere in the *interior* of the n-dimensional sphere.

Special Case 1

When $n = 1$, the above formulae produce the D'Alembert solution (see Chester [1, pages 17–23]) of the equation $u_{tt} = c^2 u_{xx}$:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta. \tag{115.5}$$

Special Case 2

When $n = 2$, the above formulae produce the Parseval solution

$$\begin{aligned} u(x,t) = \frac{1}{2\pi}\frac{\partial}{\partial t}\iint\limits_{R(t)} \frac{f(x_1+\zeta_1, x_2+\zeta_2)}{\sqrt{t^2-\zeta_1^2-\zeta_2^2}}\,d\zeta_1\,d\zeta_2 \\ + \frac{1}{2\pi}\iint\limits_{R(t)} \frac{g(x_1+\zeta_1, x_2+\zeta_2)}{\sqrt{t^2-\zeta_1^2-\zeta_2^2}}\,d\zeta_1\,d\zeta_2, \end{aligned}$$

where $R(t)$ is the region $\{(\zeta_1, \zeta_2) \mid \zeta_1^2 + \zeta_2^2 \le t^2\}$.

Special Case 3

When $n = 3$, the above formulae produce the Poisson solution (also known as the Kirchoff solution)

$$u(x,t) = \frac{\partial}{\partial t}\Big(t\omega[f; \mathbf{x}, t]\Big) + t\omega[g; \mathbf{x}, t],$$

where

$$\begin{aligned} \omega[h; \mathbf{x}, t] = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} h(x_1 + t\sin\theta\cos\phi, x_2 + t\sin\theta\sin\phi, x_3 + t\cos\theta) \\ \times \sin\theta\,d\theta\,d\phi. \end{aligned}$$

Example

A string stretched in the shape of a sine wave and then released from rest will have the displacement $u(x,t)$, where

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(x,0) &= \sin x, \\ u_t(x,0) &= 0. \end{aligned}$$

By virtue of equation (115.5), this has the solution $u(x,t) = \frac{1}{2}\Big(\sin(x-t) + \sin(x+t)\Big)$.

Notes

1. The solutions given in equation (115.3) and equation (115.4) may be derived from one another by the method of descent (see page 446).
2. The name "D'Alembert solution" is also applied to the solution of the wave equation in a *semi-infinite* domain

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \\ v(0,t) &= 0, && \text{for } 0 < t < \infty, \\ v(x,0) &= f(x), && \text{for } 0 \le x < \infty, \\ v_t(x,0) &= g(x), && \text{for } 0 \le x < \infty. \end{aligned}$$

   This equation has the solution (see Farlow [2, page 143], page 143)

$$v(x,t) = \begin{cases} \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta, & \text{for } x \ge ct, \\ \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\zeta)\,d\zeta, & \text{for } x < ct. \end{cases}$$

3. Consider the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = F(t,x,y,z),$$

   with the homogeneous initial conditions:

$$u(0,x,y,z) = 0, \qquad u_t(0,x,y,z) = 0.$$

   The solution is given by

$$u(t,x,y,z) = \frac{1}{4\pi} \iiint\limits_{\rho \le t} \frac{F(t-\rho,\zeta,\eta,\xi)}{\rho}\,d\zeta\,d\eta\,d\xi,$$

   with $\rho = \sqrt{(x-\zeta)^2 + (y-\eta)^2 + (z-\xi)^2}$.
4. Another useful formula is for the solution of

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \lambda u, \\ u(0,x,y,z) &= f(x,y,z), \\ u_t(0,x,y,z) &= g(x,y,z), \end{aligned}$$

   where λ is an arbitrary constant. The solution is given by

$$\begin{aligned} u(t,x,y,z) = \frac{\partial}{\partial t}&\left[t\omega[f;\mathbf{x},t] + \lambda \int_0^t \rho^2 \omega[f;\mathbf{x},\rho] I(\lambda t^2 - \lambda\rho^2)\,d\rho\right] \\ &+ t\omega[g;\mathbf{x},t] + \lambda \int_0^t \rho^2 \omega[g;\mathbf{x},\rho] I(\lambda t^2 - \lambda\rho^2)\,d\rho, \end{aligned}$$

   where $I(a) := I_0'(\sqrt{a})/\sqrt{a}$ and I_0 is the usual modified Bessel function.
5. See Farlow [2, Lessons 17 and 18, pages 129–145] and Garabedian [3, pages 191–210].

References

[1] Chester, C. R. *Techniques in Partial Differential Equations.* McGraw–Hill Book Company, New York, 1970.

[2] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers.* John Wiley & Sons, New York, 1982.

[3] Garabedian, P. R. *Partial Differential Equations.* John Wiley & Sons, New York, 1964.

116. Wiener–Hopf Technique

Applicable to Linear partial differential equations on an infinite interval that have different types of boundary data on different parts of the interval.

Yields

An exact solution.

Idea

In some linear partial differential equations, we would like to take a Fourier transform but cannot because the boundary data type changes along the boundary. The Wiener–Hopf technique is to take a Fourier transform anyway and allow part of the data to be "missing." Solving the problem (using Liouville's theorem), we determine the "missing" data and the solution simultaneously.

Procedure

Sometimes a linear partial differential equation has a form amenable to a Fourier transform, but the boundary conditions would seem to preclude it. For example, the reduced wave equation

$$\nabla^2 \phi + k^2 \phi = 0 \tag{116.1}$$

in two dimensions may suggest the use of a Fourier transform in x. But, if the boundary conditions are given by, say,

$$\begin{aligned} \frac{\partial \phi(x,0)}{\partial y} &= 0 \quad \text{for } x \geq 0, \\ \phi(x,0) & \qquad \text{is continuous for } x < 0, \end{aligned} \tag{116.2}$$

then it is not clear how to take such a transform. Generally, we would require $\partial\phi/\partial y$ to be known for all x, before we could take a Fourier transform. The solution technique is to *assume* that $\partial\phi/\partial y$ is known for all x and then take a Fourier transform. The quantity $\partial\phi/\partial y$ for $x < 0$ will be determined when the final solution is determined.

The solution procedure uses Liouville's theorem, one form of which is

> If $E(z)$ is an entire function (i.e., $E(z)$ is analytic in the finite $|z|$ plane) and if $E(z)$ is bounded by a constant as $|z| \to \infty$, then $E(z)$ is identically constant.

(See, e.g., Levinson and Redheffer [4].)

The difficult part of the solution procedure will turn out to be the "factorization" step. That is, given the functions $A(\omega), B(\omega), C(\omega)$ (all analytic in the strip $\alpha < \Im\omega < \beta$), find functions $\Phi_+(\omega)$, $\Psi_-(\omega)$ satisfying

$$A(\omega)\Phi_+(\omega) + B(\omega)\Psi_-(\omega) + C(\omega) = 0, \tag{116.3}$$

where

- Equation (116.3) holds in the strip: $\alpha < \Im\omega < \beta$.
- $\Phi_+(\omega)$ is analytic in the upper-half plane: $\alpha < \Im\omega$.
- $\Psi_-(\omega)$ is analytic in the lower-half plane: $\Im\omega < \beta$.

We will continue to use the following standard notation: a subscript of "+" ("−") indicates a function that is analytic in the upper (lower) half plane $\alpha < \Im\omega$ ($\Im\omega < \beta$).

Example

Suppose we have the linear partial differential equation exterior to the half line $(y = 0, x \geq 0)$

$$\phi_{xx} + \phi_{yy} - \phi_x = 0, \tag{116.4}$$

with the boundary conditions

$$\begin{aligned} \phi \to 0 \quad &\text{as} \quad r = \sqrt{x^2 + y^2} \to \infty, \\ \phi = e^{-x} \quad &\text{on} \quad y = 0,\ x \geq 0. \end{aligned} \tag{116.5.a-b}$$

Define the Fourier transform of $\phi(x, y)$ by $\Phi(\omega, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x, y) e^{i\omega x}\, dx$. If we assume that $\phi_x \to 0$ as $r \to \infty$, then equation (116.4) can be Fourier transformed (by multiplying by $e^{i\omega x}$ and integrating with respect to x) to yield

$$\frac{d^2\Phi}{dy^2} - (\omega^2 - i\omega)\Phi = 0. \tag{116.6}$$

If we extend the definition of $\phi(x, 0)$ in equation (116.5.b) to be

$$\phi(x, 0) = \begin{cases} e^{-x} & \text{for } x \geq 0, \\ u(x) & \text{for } x < 0, \end{cases} \tag{116.7}$$

where $u(x)$ is unknown, then we can transform equation (116.7) to find

$$\Phi(\omega, 0) = U(\omega) + \frac{1}{\sqrt{2\pi}} \frac{1}{1 - i\omega}, \tag{116.8}$$

where $U(\omega)$ is the Fourier transform of $u(x)$ on the semi-infinite interval; that is,

$$U(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} u(x) e^{i\omega x}\, dx.$$

The solution of equation (116.6) (which is an ordinary differential equation in y) using equation (116.8), which vanishes as $|y| \to \infty$, is

$$\Phi(\omega, y) = \left[U(\omega) + \frac{1}{\sqrt{2\pi}} \frac{1}{1 - i\omega} \right] \exp\left(-|y| \sqrt{\omega^2 - i\omega} \right), \tag{116.9}$$

where the square root branch is specified by $\Re\sqrt{\omega^2 - i\omega} \geq 0$.

Once we determine $U(\omega)$, we can (in principle) invert equation (116.9) by taking an inverse Fourier transform. This would yield $\phi(x, y)$. Finding $U(\omega)$ is the hard part of the calculation.

Because the solution of the original problem (and its derivatives) must be continuous across $y = 0$ (for $x < 0$), we define a function $f(x)$ by

$$\begin{aligned} f(x) &:= \phi_y(x, 0^+) - \phi_y(x, 0^-), \\ &= \begin{cases} 0 & \text{for } x < 0, \\ v(x) & \text{for } x > 0, \end{cases} \end{aligned} \tag{116.10.a-b}$$

where 0^+ (0^-) indicates a vanishingly small quantity that is greater (less) than zero and $v(x)$ is an unknown function. Taking the Fourier transform of equation (116.10.b) produces

$$\begin{aligned} F(\omega) :=& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\omega x}\, dx \\ =& \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} v(x) e^{i\omega x}\, dx, \end{aligned} \tag{116.11}$$

whereas the Fourier transform of equation (116.10.a) produces

$$\begin{aligned} F(\omega) &= \Phi_y(\omega, 0^+) - \Phi_y(\omega, 0^-) \\ &= -2 \left[U(\omega) + \frac{1}{\sqrt{2\pi}} \frac{1}{1 - i\omega} \right] \sqrt{\omega^2 - i\omega}, \end{aligned} \tag{116.12}$$

where the solution in equation (116.9) has been used. Using our subscript convention and the definition in equation (116.11), we note that $F(\omega) = F_+(\omega)$, where, for instance, we could take $\alpha = 1/3$.

We now *assume* that $U(\omega) = U_-(\omega)$, for, say, $\beta = 2/3$. This places a constraint on $u(x)$ that has to be verified at the end of the calculation. By algebraic manipulations of equation (116.12), we can obtain (this step should not be trivialized, it is the hardest step in the calculation)

$$-\frac{F_+(\omega)}{2\sqrt{\omega}} - \left[\frac{\sqrt{-i}}{\sqrt{\pi}(1 - i\omega)} \right]_+ = U_-(\omega)\sqrt{\omega - i} + \left[\frac{\sqrt{\omega - i} - \sqrt{-2i}}{\sqrt{2\pi}(1 - i\omega)} \right]_- \tag{116.13}$$

If we define $E(\omega)$ to be the left-hand side of equation (116.13), then $E(\omega)$ is entire. This is because the left-hand side and the right-hand side of equation (116.13) overlap in the strip $\alpha < \Im\omega < \beta$, and these two functions are analytic in their respective half planes. Hence, one side of equation (116.13) supplies the analytic continuation of the other side.

If we now assume that

- $F_+(\omega) \to 0$ as $|\omega| \to \infty$ in $\Im\omega > \beta$,
- $\omega U_-(\omega) \to 0$ as $|\omega| \to \infty$ in $\Im\omega < \alpha$,

then $E(\omega) \to 0$ as $|\omega| \to \infty$. By Liouville's theorem we can conclude that $E(\omega) \equiv 0$ and so from equation (116.13)

$$U(\omega) = U_-(\omega) = -\frac{1}{\sqrt{\omega - i}} \left[\frac{\sqrt{\omega - i} - \sqrt{-2i}}{\sqrt{2\pi}\sqrt{1 - i\omega}} \right].$$

Using this in equation (116.9) and taking an inverse Fourier transform yields $\phi(x, y)$.

Notes

1. The Wiener–Hopf method was originally formulated for the solution of integral equations.
2. The problem in equations (116.1) and (116.2) is analyzed in more detail in Carrier *et al.* [1, pages 376–386]. The same problem, with an incident oblique wave, is solved in Davies [2, pages 288–307].

References

[1] Carrier, G. F., Krook, M., and Pearson, C. E. *Functions of a Complex Variable*. McGraw–Hill Book Company, New York, 1966.

[2] Davies, B. *Integral Transforms and Their Applications*, second ed. Springer–Verlag, New York, 1985.

[3] Heins, A. E. The scope and limitations of the method of Wiener and Hopf. *Communications on Pure and Applied Mathematics 9* (1956), 447–466.

[4] Levinson, N., and Redheffer, R. M. *Complex Variables*. Holden–Day, Inc., San Francisco, CA, 1979.

[5] Noble, B. *Methods Based on the Wiener–Hopf Technique*. Pergamon Press, New York, 1958.

117. Introduction to Approximate Analysis

Sometimes an exact solution cannot be obtained for a differential equation and an approximate solution must be found. Other times, an approximate solution may convey more information than an exact solution.

There are essentially two types of approximations:

- Those that give an approximation over a range of the independent variable
- Those that give an approximation only near a single point

Approximations of the second type are more common.

This section of the book is not broken up into methods for ordinary differential equations and methods for partial differential equations because most of the methods can be used for either type of differential equation.

Listed below are, in the author's opinion, those methods that are the most useful when approximating the solution to ordinary differential equations and partial differential equations. These are the methods that might be tried first.

Most Useful Methods

- Collocation (page 514)
- Dominant Balance (page 517)
- Graphical Analysis: The Phase Plane (page 526)
- Least Squares Method (page 549)
- Lyapunov Functions (page 551)
- Newton's Method (page 578)
- Perturbation Method: Method of Averaging (page 586)
- Perturbation Method: Boundary Layer Method (page 590)
- Perturbation Method: Regular Perturbation (page 610)
- WKB Method (page 642)

118. Chaplygin's Method

Applicable to An initial value problem for a single first order ordinary differential equation.

Yields

Improved upper and lower bounds on the solution.

Idea

Using an upper and lower bound on the solution, a set of tighter bounds can be constructed.

Procedure

For an equation of the form $y' = f(x, y)$, $y(x_0) = y_0$, the method is derived from the following theorem (due to Chaplygin):

Theorem: If the differential inequalities

$$\begin{aligned} u'(x) - f(x, u(x)) &< 0, \\ v'(x) - f(x, v(x)) &> 0, \end{aligned} \tag{118.1}$$

hold for $x > x_0$, with $u(x_0) = y_0$ and $v(x_0) = y_0$, then

$$u(x) < y(x) < v(x) \tag{118.2}$$

holds for all $x > x_0$.

The procedure is to determine (or "guess") a $u(x)$ and a $v(x)$ that satisfy equation (118.1). Then there are two different techniques available for computing $\{u_1(x), v_1(x)\}$, such that

$$u(x) < u_1(x) \leq y(x) \leq v_1(x) < v(x). \tag{118.3}$$

For each of the two techniques, the functions $\{u_1(x), v_1(x)\}$ will be different. The functions obtained, $\{u_1(x), v_1(x)\}$, will also satisfy equation (118.1), and the process may be iterated.

Special Case 1

Let K be the Lipschitz constant of the function $f(x, y)$. Then, if $\{u_1(x), v_1(x)\}$ are defined by

$$\begin{aligned} u_1(x) &= u(x) + \int_{x_0}^{x} e^{-K(x-t)} \left[f(t, u(t)) - u'(t) \right] dt, \\ v_1(x) &= v(x) - \int_{x_0}^{x} e^{-K(x-t)} \left[v'(t) - f(t, v(t)) \right] dt, \end{aligned}$$

then equation (118.3) will be satisfied.

Special Case 2

For this technique, it must be true that $\partial^2 f/\partial y^2$ is of constant sign in the region of interest. Once this has been established, define $\{M(x), N(x), \widehat{M}(x), \widehat{N}(x)\}$ by

$$\begin{aligned} M(x)y + N(x) &= f(x, u(x)) + \frac{f(x, v(x)) - f(x, u(x))}{v(x) - u(x)}(y - u(x)), \\ \widehat{M}(x)y + \widehat{N}(x) &= f(x, u(x)) + f_y(x, u(x))(y - u(x)). \end{aligned} \tag{118.4}$$

(Note that both sides of each equation are linear in the indeterminate y.) Then define $u_1(x)$ to be the solution of

$$y' = M(x)y + N(x), \qquad y(x_0) = y_0. \tag{118.5}$$

and define $v_1(x)$ to be the solution of

$$y' = \widehat{M}(x)y + \widehat{N}(x), \qquad y(x_0) = y_0. \tag{118.6}$$

With these definitions for $u_1(x)$ and $v_1(x)$, equation (118.3) will be satisfied. Note that the equations (118.5) and (118.6) can be solved by the use of integrating factors (see page 356).

Example

Suppose we wish to bound the solution to the equation

$$y' = y^2 + x^2, \qquad y(0) = 0,$$

when x is in the range $[0, 1/\sqrt{2}]$.

First, observe that $u(x) = x^3/3$ and $v(x) = 11x^3/30$ satisfy the conditions of Chaplygin's theorem, so that equation (118.2) holds. Using the first technique, we recognize that $K = \sqrt{2}$ in the region of interest, so that the functions

$$\begin{aligned} u_1(x) &= \frac{x^3}{3} + \frac{1}{9}\int_0^x t^6 e^{-\sqrt{2}(x-t)}\, dt, \\ v_1(x) &= \frac{11}{30}x^3 - \int_0^x \left(\frac{t^2}{10} - \frac{121}{900}t^6\right) e^{-\sqrt{2}(x-t)}\, dt, \end{aligned} \tag{118.7}$$

satisfy the constraint in equation (118.3). Using the second technique, we note that $\partial^2 f/\partial y^2 = 2$ and so we can use the results in equations (118.4), (118.5), and (118.6). It is straightforward to calculate

$$\begin{aligned} M(x) &= \frac{7}{10}x^3, & \widehat{M}(x) &= \frac{2}{3}x^3, \\ N(x) &= x^2 - \frac{11}{90}x^6, & \widehat{N}(x) &= x^2 - \frac{1}{9}x^6. \end{aligned}$$

Solving equations (118.5) and (118.6), we find

$$\begin{aligned} u_1(x) &= e^{x^4/6} \int_0^x \left(z^2 - \frac{1}{9} z^6 \right) e^{-z^4/6} \, dz, \\ v_1(x) &= e^{7x^4/40} \int_0^x \left(z^2 - \frac{11}{90} z^6 \right) e^{-7z^4/40} \, dz. \end{aligned} \tag{118.8}$$

Notes

1. The above example is from Mikhlin and Smolitskiy [4]. The exact solution is given by

$$y(x) = \frac{x \left[Y_{-3/4}(x^2/2) - J_{-3/4}(x^2/2) \right]}{J_{1/4}(x^2/2) - Y_{1/4}(x^2/2)} = \frac{1}{3}x^3 + \frac{1}{63}x^7 + \frac{2}{2079}x^{11} + O\left(x^{15}\right)$$

2. The approximations in equation (118.7) may be expanded about $x = 0$ to obtain

$$u_1(x) = \frac{1}{3}x^3 + \frac{1}{63}x^7 + O\left(x^8\right), \qquad v_1(x) = \frac{1}{3}x^3 + O\left(x^4\right).$$

3. The approximations in equation (118.8) may be expanded about $x = 0$ to obtain

$$\begin{aligned} u_1(x) &= \frac{1}{3}x^3 + \frac{1}{63}x^7 + \frac{2}{2079}x^{11} + O\left(x^{15}\right), \\ v_1(x) &= \frac{1}{3}x^3 + \frac{1}{63}x^7 + O\left(x^{11}\right). \end{aligned}$$

4. Another useful inequality (see McNabb [3]) is the following:

   If $u(t)$, $v(t)$, and $f(t,w)$ satisfy sufficient smoothness conditions on $[a,b]$, if $u(a) < v(a)$, and if $u' - f(t,u) < v' - f(t,v)$ for $a < t \le b$, then $u(t) < v(t)$ on $[a,b]$.

5. This procedure can be implemented numerically.
6. See also Lakshmikantham and Leela [2, pages 64–69] and Mikhlin and Smolitskiy [4, pages 9–12].

References

[1] Fabry, C., and Habets, P. Upper and lower solutions for second-order boundary value problems with nonlinear boundary conditions. *Nonlinear Analysis 10*, 10 (1986), 985–1007.

[2] Lakshmikantham, V., and Leela, S. *Differential and Integral Inequalities.* Academic Press, New York, 1969.

[3] McNabb, A. Comparison theorems for differential equations. *J. Math. Anal. Appl. 119* (1986), 417–428.

[4] Mikhlin, S. G., and Smolitskiy, K. L. *Approximate Methods for Solutions of Differential and Integral Equations.* American Elsevier Publishing Company, New York, 1967.

119. Collocation

Applicable to
Ordinary and partial differential equations.

Yields
An approximation to the solution, valid over an interval.

Idea
An approximation to the solution with some free parameters is proposed. The free parameters are determined by forcing the approximation to exactly satisfy the given equation at some set of points.

Procedure
Suppose we are given the differential equation

$$N[y] = 0, \tag{119.1}$$

for $y(\mathbf{x})$ in some region R, with the boundary conditions

$$B[y] = 0, \tag{119.2}$$

on some portion of the boundary of R. We choose an approximation to $y(\mathbf{x})$ that has several parameters in it, say $y(\mathbf{x}) \simeq w(\mathbf{x}; \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is a vector of parameters. This approximation is chosen in such a way that it satisfies the boundary conditions in equation (119.2). The unknown parameters are determined by requiring the approximation to satisfy equation (119.1) at some collection of points.

Example
Suppose we wish to approximate the solution to the ordinary differential equation

$$\begin{aligned} N[y] &= y'' + y + x = 0, \\ y(0) &= 0, \quad y(1) = 0, \end{aligned} \tag{119.3}$$

by the method of collocation. We choose to approximate the exact solution by

$$y(x) \simeq w(x) = \alpha_1 x(1-x) + \alpha_2 x(1-x^2).$$

Note that $w(x)$ satisfies the boundary conditions for $y(x)$. Using this approximation, we find

$$N[w(x)] = -\alpha_1(2 - x + x^2) - \alpha_2(5x + x^3) + x.$$

Now, we must choose the collocation points. We choose the two points $x = 1/3$ and $x = 2/3$. Requiring $N[w(x)]$ to be zero at these two points results in the simultaneous equations

$$-\frac{48}{27}\alpha_1 - \frac{46}{27}\alpha_2 - \frac{1}{3} = 0,$$
$$-\frac{48}{27}\alpha_1 - \frac{98}{27}\alpha_2 - \frac{2}{3} = 0.$$

The solution to these equations is $\alpha_1 = 9/416$, $\alpha_2 = 9/52$. Hence, our approximation to the solution of equation (119.3) is

$$y(x) \simeq \frac{9}{416}x(1-x) + \frac{9}{52}x(1-x^2). \tag{119.4}$$

Note that the exact solution to equation (119.3) is $y(x) = \frac{\sin x}{\sin 1} - x$. The maximum difference between the approximate solution in equation (119.4) and the exact solution in the range $0 < x < 1$, occurs at $x \simeq 0.7916$ where the error is approximately 0.00081.

Notes

1. This method is an example of a *weighted residual method*.
2. This method is often implemented numerically.
3. There are many choices for the form of the approximation to use. An increasingly popular technique is to use sinc functions; see, for example, Carlson *et al.* [2].
4. Ascher *et al.* [1] contain a review of numerical implementations of the colocation method.

References

[1] Ascher, U., Christiansen, J., and Russell, R. D. Collocation software for boundary–value ODEs. *ACM Trans. Math. Software 7*, 2 (June 1981), 209–222.

[2] Carlson, T. S., Dockery, J., and Lund, J. A sinc-collocation method for initial value problems. *Math. of Comp. 66*, 217 (January 1997), 215–235.

[3] Collatz, L. *The Numerical Treatment of Differential Equations*. Springer–Verlag, New York, 1966.

[4] Hanke, M. On a least-squares collocation method for linear differential-algebraic equations. *Numer. Math. 54* (1988), 79–90.

[5] Houstis, E. N., Christara, C. C., and Rice, J. R. Quadratic-spline collocation methods for two-point boundary value problems. *Internat. J. Numer. Methods Eng. 26*, 4 (1988), 935–952.

[6] Houstis, E. N., Mitchell, W. F., and Rice, J. R. Collocation software for second-order elliptical partial differential equations. *ACM Trans. Math. Software 11*, 4 (Dec 1985), 379–412.

[7] Hussaini, M. Y., Kopriva, D. A., and Patera, A. T. Spectral collocation methods. *Appl. Num. Math. 5* (1989), 177–208.

[8] Lie, I. The stability function for multistep collocation methods. *Numer. Math. 57*, 8 (1990), 779–787.

[9] Madsen, N. K., and Sincovec, R. F. Algorithm 540: PDECOL general collocation software for partial differential equations. *ACM Trans. Math. Software 5*, 3 (Sept 1979), 326–351.

[10] Sakai, M. A collocation method for a singular boundary value problem. *Congr. Numer. 62* (1988), 171–179.

[11] Wright, K., Ahmed, A. H. A., and Seleman, A. H. Mesh selection in collocation for boundary value problems. *IMA J. Num. Analysis 11*, 1 (January 1991), 7–20.

120. Dominant Balance

Applicable to

Linear and nonlinear differential equations.

Yields

An approximation to the solution valid in a region.

Idea

A differential equation with many terms in it might be well determined by only a few of those terms.

Procedure

If there are M terms in a differential equation, try solving the differential equation in a region by only considering 2 (or 3, or 4, ..., or $M-1$) terms to be important in that region. Discard all the other terms and solve this differential equation with fewer terms. After a solution is obtained, check that the discarded terms are actually smaller than the terms that were retained.

Example

Suppose we have the equation

$$y'' - \frac{2}{x^{3/2}} y' = \frac{3}{16x^2}, \tag{120.1}$$

and we would like to find an approximate solution as $x \to 0$. To determine the solution uniquely in this region, we must specify some information about $y(x)$ as $x \to 0$. In this example, we choose the condition: $y \to 0$ as $x \to 0$.

There are three different two-term balances of equation (120.1) that we can take; that is, the first two terms in equation (120.1) can be taken approximately equal, the first and third terms can be taken approximately equal, or the second and third terms can be taken approximately equal. These possibilities yield the following two term balances:

$$y'' - \frac{2}{x^{3/2}} y' \simeq 0, \qquad \text{which requires that} \qquad |y''| \gg \left| \frac{3}{16x^2} \right|, \tag{120.2}$$

or

$$y'' \simeq \frac{3}{16x^2}, \qquad \text{which requires that} \qquad |y''| \gg \left| \frac{2y'}{x^{3/2}} \right|, \tag{120.3}$$

or

$$-\frac{2}{x^{3/2}} y' \simeq \frac{3}{16x^2}, \qquad \text{which requires that} \qquad |y''| \ll \left| \frac{3}{16x^2} \right|. \tag{120.4}$$

We will investigate each of these in turn. The solution to equation (120.2) is

$$y_1(x) = A + B \int \exp\left(-\frac{4}{x^{1/2}}\right) dx,$$

where A and B are arbitrary constants. Note that this solution violates the condition in equation (120.2) because

$$|y_1''| = \frac{2|B|}{x^{3/2}} \exp\left(-\frac{4}{x^{1/2}}\right) \ll \frac{3}{16x^2} \quad \text{as} \quad x \to 0.$$

Therefore equation (120.2) is an *inconsistent balance*.

The solution to equation (120.3) is

$$y_2(x) = -\frac{3}{16} \log x + Cx + D,$$

where C and D are arbitrary constants. But this solution cannot satisfy $y \to 0$ as $x \to 0$, so it must also be discarded.

The solution to equation (120.4) is

$$y_3(x) = -\frac{3}{16}\sqrt{x},$$

where we have already used the fact that $y \to 0$ as $x \to 0$. For this solution, the condition in equation (120.4) is satisfied, because

$$|y''| = \frac{3}{32x^{3/2}} \ll \frac{3}{16x^2} \quad \text{as} \quad x \to 0.$$

Hence, we have found a *consistent balance*. We conclude that

$$y(x) \sim -\frac{3}{16}\sqrt{x} \quad \text{as} \quad x \to 0.$$

Notes

1. Even if a consistent balance has been found, the solution associated with that balance may be unrelated to the true solution of the differential equation(s). This is because a consistent balance has *apparent consistency* but not necessarily *genuine consistency*. Another set of words that express the same ideas are *honest methods* and *dishonest methods*. See Keller [2] or Lin and Segel [4, pages 188–189] for more details.
2. See Bender and Orszag [1, pages 83–88].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.

[2] Keller, J. B. *Wave propagation in Random Media*, vol. 13 of *Proc. Sympos. Appl. Math.* Amer. Math. Soc., Providence, RI, 1960.

[3] Levinson, N. Asymptotic behavior of solutions of non-linear differential equations. *Stud. Appl. Math. 48* (1969), 285–297.

[4] Lin, C. C., and Segel, L. A. *Mathematics Applied to Deterministic Problems in the Natural Sciences.* The MacMillan Company, New York, 1974.

121. Equation Splitting

Applicable to Differential equations.

Yields

An exact solution but usually not the most general form of the solution.

Idea

By equating two parts of a differential equation to a common term, we may be able to find a fairly general solution to the given differential equation.

Procedure

Separate a differential equation into two (or more) terms such that a general solution is available for one of the terms. Use the other term(s) to restrict this general solution.

Example 1

Suppose we have, from fluid dynamics, the stream function form of the boundary layer equations to solve for $\Phi(x, y)$:

$$\Phi_y \Phi_{xy} - \Phi_x \Phi_{yy} = \nu \Phi_{yyy}. \tag{121.1}$$

We split this equation by choosing both the right and the left-hand sides of this equation to be identically equal to zero. That is, we break equation (121.1) into the two simultaneous equations

$$\begin{aligned} \Phi_y \Phi_{xy} - \Phi_x \Phi_{yy} &= 0, \\ \nu \Phi_{yyy} &= 0. \end{aligned} \tag{121.2.a-b}$$

Any solution of equation (121.2) is also a solution of equation (121.1). Note that the converse is *not* true: A solution to equation (121.1) may not satisfy equation (121.2.a) or equation (121.2.b). Hence, the solution that is obtained from equation (121.2) will not be the most general solution.

The general solution to equation (121.2.b) can be easily found because it is essentially an ordinary differential equation in the independent variable y:

$$\Phi(x, y) = a(x)y^2 + b(x)y + c(x), \tag{121.3}$$

for arbitrary coefficient functions $a(x)$, $b(x)$, and $c(x)$. Using equation (121.3) in equation (121.2.a), we conclude that

$$(2ay + b)(2ya' + b') - (a'y^2 + b'y + c')(2a) = 0 \tag{121.4}$$

must hold for all values of x and y. Hence, $a(x)$, $b(x)$, and $c(x)$ can be restricted by equating the coefficients of y^2, y^1, and y^0 in equation (121.4) to zero. This results in

$$\text{coefficient of } y^2: \quad 4aa' - 2aa' = 0, \tag{121.5}$$

$$\text{coefficient of } y^1: \quad (2ab' + ba') - 2ab' = 0, \tag{121.6}$$

$$\text{coefficient of } y^0: \quad bb' - 2ac' = 0. \tag{121.7}$$

Now we solve the equations appearing in equations (121.5), (121.6), and (121.7). Equation (121.5) can be valid only if $a(x)$ is a constant, say A. Then equation (121.6) is valid for any $b(x)$ and equation (121.7) can be rewritten as

$$(b^2)' - 4Ac' = 0. \tag{121.8}$$

Equation (121.8) can be integrated to determine $c(x) = \frac{b^2(x)}{4A} + D$, where D is an arbitrary constant of integration. Now, using what we have found, the solution in equation (121.3) becomes

$$\Phi(x, y) = Ay^2 + b(x)y + \left(\frac{b(x)^2}{4A} + D\right), \tag{121.9}$$

for arbitrary A, D, and $b(x)$.

Example 2

Basarab-Horwath *et. al* [2] present a method, which uses equation splitting, for finding solutions of the d'Alembert equation

$$\Box u \equiv \left(\frac{\partial^2}{\partial {x_0}^2} - \frac{\partial^2}{\partial {x_1}^2} - \cdots - \frac{\partial^2}{\partial {x_n}^2}\right) u = F(u).$$

Choosing $P(w)$ to be an arbitrary polynomial and $\lambda = -1, 0, 1$, they make the change of variable $u = \Phi(w)$, where Φ satisfies the differential equation

$$\lambda \left(\Phi'' + \Phi' \frac{P'}{P}\right) = F(\Phi).$$

Then, using equation splitting, they arrive at the two partial differential equations

$$\begin{aligned} \Box w &= \lambda \frac{P'}{P} \\ \lambda &= \left(\frac{\partial w}{\partial x_0}\right)^2 - \left(\frac{\partial w}{\partial x_2}\right)^2 - \cdots - \left(\frac{\partial w}{\partial x_n}\right)^2. \end{aligned}$$

They demonstrate their method by finding solutions of $\Box u = \sin u$.

Notes

1. Example 1 is from Ames [1, pages 59 and 65–69].
2. Note that for the equations in (121.2) we could have found the general solution of equation (121.2.a) and then used equation (121.2.b) to restrict it. The general solution of equation (121.2.a) is $\Phi(x, y) = F(y + G(x))$, where F and G are arbitrary functions. Using this solution in equation (121.2.b) and determining conditions on F and G results in the solution in equation (121.9).
3. See also Goldstein and Braun [3, page 109] and Whitham [4, page 421].

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Basarab-Horwath, P., Fushchich, W., and Serov, M. A simple method of finding solutions of the nonlinear d'Alembert equation. *J. Phys. A: Math. Gen. 25* (1992), L871–L877.

[3] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations*. U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[4] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

122. Floquet Theory

Applicable to Linear ordinary differential equations with periodic coefficients and periodic boundary conditions.

Yields

Knowledge of whether all solutions are stable.

Idea

If a linear differential equation has periodic coefficients and periodic boundary conditions, then the solutions will generally be a periodic function times an exponentially increasing or an exponentially decreasing function. Floquet theory will determine if the solution is exponentially increasing (and so “unstable”) or exponentially decreasing (and so “stable”).

Procedure

Suppose we have an nth order linear ordinary differential equation whose coefficients are periodic with common period T. The general technique is to write the ordinary differential equation as a first order vector system of dimension n (see page 146), and then solve this vector ordinary differential equation for any set of n linearly independent conditions, for $0 \leq t \leq T$.

This yields a propagator matrix B, such that $\mathbf{y}(t + mT) = B^m \mathbf{y}(t)$, where $m = 1, 2, \dots$. Hence, to determine the stability of the original problem, we need only determine the eigenvalues of B. If any of the eigenvalues are larger than one in magnitude, then the solution is “unstable.”

As an example of the general theory, we consider second order linear ordinary differential equations of the form

$$y'' + q(t)y = 0, \tag{122.1}$$

where $q(t)$ is periodic with period T, i.e., $q(t + T) = q(t)$. We can write equation (122.1) as a vector ordinary differential equation in the form

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}, \qquad \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \mathbf{y},$$

where $\mathbf{y}(0) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$ is known in principle. We now define $u(t)$ and $v(t)$ to be the solutions of

$$\begin{bmatrix} u(t) \\ u'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}, \qquad \begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{122.2}$$

and

$$\begin{bmatrix} v(t) \\ v'(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ v'(t) \end{bmatrix}, \qquad \begin{bmatrix} v(0) \\ v'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{122.3}$$

Then, by superposition, $\mathbf{y}(t) = A(t)\mathbf{y}(0) = \begin{bmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{bmatrix} \mathbf{y}(0)$. Equivalently, $\mathbf{y}(T) = B\mathbf{y}(0)$, where $B = A(T)$. Hence, $\mathbf{y}(2T) = B\mathbf{y}(T) = B^2\mathbf{y}(0)$, $\mathbf{y}(3T) = B^3\mathbf{y}(0)$, etc. The eigenvalues of B are needed to determine stability. By the usual calculation, λ will be an eigenvalue of B if and only if $|B - \lambda I| = 0$. We calculate,

$$\begin{aligned} |B - \lambda I| &= \begin{vmatrix} u(T) - \lambda & v(T) \\ u'(T) & v'(T) - \lambda \end{vmatrix} \\ &= \lambda^2 - \lambda[u(T) + v'(T)] + [u(T)v'(T) - u'(T)v(T)] \\ &= \lambda^2 - \lambda\Delta + 1, \end{aligned} \tag{122.4}$$

where we have defined $\Delta = u(T) + v'(T)$, and we set $u(T)v'(T) - u'(T)v(T)$ equal to one because the Wronskian of equation (122.1) is identically equal to one. Solving equation (122.4) for λ, we determine that $\lambda = \frac{1}{2}\Delta \pm \sqrt{\frac{1}{4}\Delta^2 - 1}$, and so we conclude

- If $|\Delta| < 2$, then, for both values of λ, we have $|\lambda| \leq 1$ and so all of the solutions to equation (122.1) are stable.
- If $|\Delta| > 2$, then there is least one value of λ with $|\lambda| > 1$ and so the solutions to equation (122.1) are unstable.

Example

Suppose we have the equation

$$y'' + f(t)y = 0, \tag{122.5}$$

where $f(t)$ is a square wave function of period T

$$f(t+T) = f(t) = \begin{cases} -1 & \text{for } 0 \leq t < T/2, \\ 1 & \text{for } T/2 \leq t \leq T. \end{cases} \tag{122.6}$$

Note that f(t) is *not* continuous. This does not change any of the analysis. We can solve equation (122.5) and equation (122.6) by using $f(t) = -1$ and solving for $\{u(t), v(t)\}$ in the interval $0 \leq t < T/2$. Then we set $f(t) = 1$ and solve for $\{u(t), v(t)\}$ in the interval $T/2 < t \leq T$, using as initial conditions the values calculated when we took $f(t) = -1$. See the section on solving equations with discontinuities (page 264).

The solutions of equations (122.2) and (122.3) are found to be (for $T/2 < t \leq T$)

$$u(t) = (\sinh\tau \sin\tau + \cosh\tau \cos\tau)\sin t + (\sinh\tau\cos\tau + \cosh\tau\sin\tau)\cos t,$$

and

$$v(t) = (\cosh\tau \sin\tau + \sinh\tau \cos\tau)\sin t + (\cosh\tau\cos\tau - \sinh\tau\sin\tau)\cos t,$$

where $\tau = T/2$. From these equations, we determine Δ to be

$$\Delta = u(T) + v'(T) = 2\cosh\tau\cos\tau. \tag{122.7}$$

The conclusion is that the solutions to equation (122.5) will be stable or unstable depending on whether the magnitude of Δ, as given by equation (122.7), is greater than or smaller than 2. Different values of T will give different conclusions. For example,

- If T=17 or $T = e^2$, then $|\Delta| > 2$ and some unstable solutions to equation (122.5) exist.
- If T=1 or $T = \pi$, then $|\Delta| < 2$ and all to the solutions to equation (122.5) are stable.

Notes

1. Mathematicians call this technique Floquet theory, whereas physicists call it Bloch wave theory. Solid state physicists use this technique to determine band gap energies.
2. Note that the periodicity of $f(t)$ in equation (122.5) does *not*, by itself, insure that $y(t)$ has a periodic solution. If, however, $f(t)$ is periodic and has mean zero, then equation (122.5) will have a periodic solution of the same period.
3. The linear system $\mathbf{y}' = B(t)\mathbf{y}$ is said to be *noncritical* with respect to T if it has no periodic solution of period T except the trivial solution $\mathbf{y} = \mathbf{0}$. Otherwise, the system is said to be critical.
4. See also Coddington and Levinson [1, pages 78–81], Kaplan [3, pages 472–490], Lukes [5, Chapter 8, pages 162–179], and Magnus and Winkler [6, pages 3–10].

References

[1] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations*. McGraw–Hill Book Company, New York, 1955.

[2] Hassan, H. S. Floquet solutions of nonlinear ordinary differential equations. *Proc. Roy. Soc. Edin. Sect. A 106*, 3–4 (1987), 267–275.

[3] Kaplan, W. *Operational Methods for Linear Systems*. Addison–Wesley Publishing Co., Reading, MA, 1962.

[4] Kuchment, P. *Floquet Theory for Partial Differential Equations*, vol. 60 of *Operator Theory Advances and Appliations*. Birkhauser, Basel, Switzerland, 1993.

[5] Lukes, D. L. *Differential Equations: Classical to Controlled*. Academic Press, New York, 1982.

[6] Magnus, W., and Winkler, S. *Hill's Equation*. Dover Publications, Inc., New York, 1966.

[7] Sleeman, B. D., and Smith, P. D. Double periodic Floquet theory for a second order system of ordinary differential equations. *Quart. J. Math. Oxford Ser. 37*, 147 (1986), 347–356.

123. Graphical Analysis: The Phase Plane

Applicable to Two coupled autonomous first order ordinary differential equations or an autonomous second order ordinary differential equation.

Yields

A graphical representation of the solution.

Idea

The qualitative features of the solution of two coupled autonomous first order ordinary differential equations may be ascertained from the phase plane.

Procedure

Suppose we have the set of two coupled autonomous first order ordinary differential equations

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y). \tag{123.1}$$

As t increases, $x(t)$ and $y(t)$ will describe a path in (x,y) space. This will not be the case at those points (x_0, y_0), where

$$f(x_0,y_0) = 0, \quad g(x_0,y_0) = 0.$$

At these points, the value does not change with t: $x(t) = x_0$ and $y(t) = y_0$. These points are called *critical points*. (They are also called *equilibrium points* or *singular points*).

To analyze the motion near a single critical point, we linearize equation (123.1) about that point. By a linear change of variables, we can place the critical point at the origin $(x,y) = (0,0)$. Near a critical point at the origin, equation (123.1) can be written as

$$\begin{aligned} \frac{dx}{dt} &= ax + by + \widehat{f}(x,y), \\ \frac{dy}{dt} &= cx + dy + \widehat{g}(x,y), \end{aligned} \tag{123.2}$$

where $\widehat{f}(x,y) = o(|x| + |y|)$ and $\widehat{g}(x,y) = o(|x| + |y|)$ as $x \to 0$, $y \to 0$. We assume that a, b, c, d are real numbers and they are not all equal to zero. If we discard the $\widehat{f}$ and $\widehat{g}$ terms in equation (123.2) and look for solutions of the form

$$x(t) = Ae^{\lambda t}, \quad y(t) = Be^{\lambda t},$$

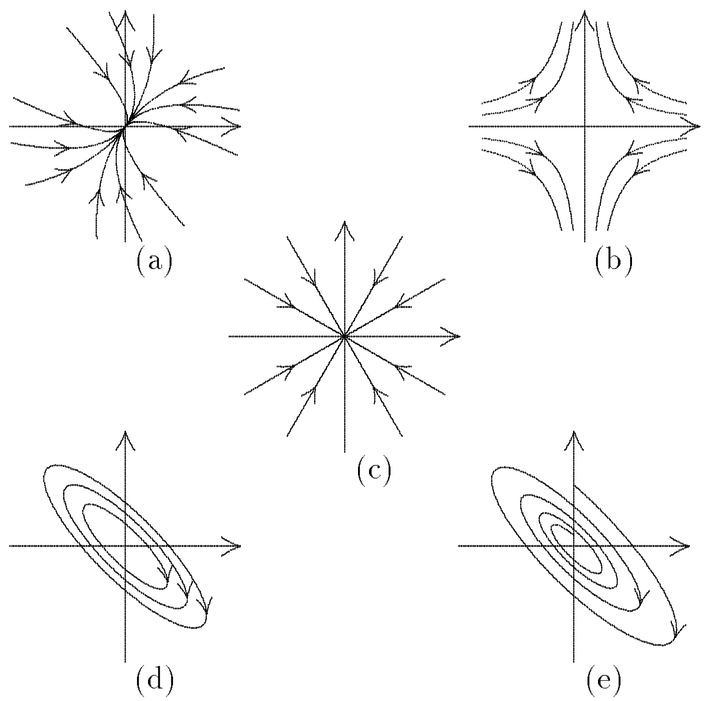

Figure 123.1: The different types of behavior in the phase plane: (a) and (c) are nodes, (b) is a saddle point, (d) is a center, and (e) is a spiral.

then we find that λ must be an eigenvalue of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. That is, λ must satisfy

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0. \tag{123.3}$$

There are five different types of behavior that can be observed near the critical point $(0, 0)$, based on the roots of equation (123.3). If the roots of equation (123.3) are

- Real, distinct, and of the same sign, then the critical point is called a *node*. (See figure 123.1.a for a typical picture.) Note that the symmetry axes are determined by the eigenvectors of the 2×2 matrix shown above.
- Real, distinct, and of opposite signs, then the critical point is called a *saddle point*. (See figure 123.1.b for a typical picture.)
- Real and equal, then the critical point is again a node. (See figure 123.1.c for a typical picture.)
- Pure imaginary, then the critical point is called a *center*. (See figure 123.1.d for a typical picture.)

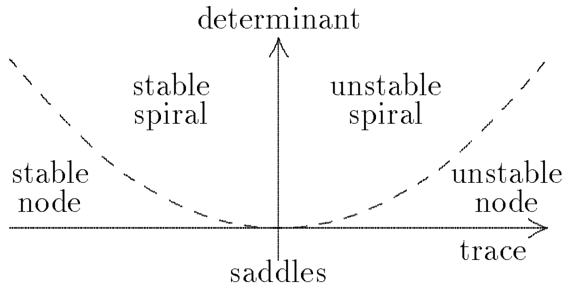

Figure 123.2: The different types of behavior in the phase plane, as a function of the trace and determinant of the 2×2 matrix.

- Conjugate complex numbers but not pure imaginary, then the critical point is called a *spiral* or a *focus*. (See figure 123.1.e for a typical picture.)

In each of the figures, an arrow points in the direction of increasing t. For each case illustrated, there exist systems in which the arrows are pointing in the opposite direction from what we have illustrated. Each solution of equation (123.2) (corresponding to different initial conditions) describes a single trajectory. Every trajectory must

- Go to infinity or
- Approach a limit cycle (see page 78) or
- Tend to a critical point.

If the solution goes to infinity, then the solution is said to be *unstable*, otherwise it is said to be *stable*.

Example 1

Consider the simple linear differential equation system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}.$$

For this equation, the eigenvalues satisfy equation (123.3), which we write in the form $\lambda^2 - T\lambda + \Delta = 0$, where T is the trace of the matrix ($T = a + d$) and Δ is the determinant ($\Delta = ad - bc$). The eigenvalues, and the qualitative picture of the phase plane, can be deduced from T and Δ. Figure 123.2 shows the type of behavior to expect for different values of T and Δ. The curve figure 123.2 is given by determinant= $(\text{trace})^2$; only centers can occur along this curve.

Example 2

Consider the nonlinear autonomous second order ordinary differential equation

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \omega^2 \sin x = 0, \tag{123.4}$$

which can be written as the coupled system

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\beta y - \omega^2 \sin x. \end{aligned} \tag{123.5}$$

For the equations in equation (123.5) there are infinitely many critical points at the locations $\{x = n\pi,\ y = 0 \mid n = 0, 1, 2, \dots\}$. To analyze the behavior near the point $(k\pi, 0)$ the new variables $\widetilde{y} = y$, $\widetilde{x} = x - k\pi$ are introduced. In these new variables, the system in equation (123.5) can be approximated by

$$\begin{aligned} \frac{d\widetilde{x}}{dt} &= \widetilde{y}, \\ \frac{d\widetilde{y}}{dt} &= -\beta\widetilde{y} + (-1)^{k+1}\omega^2\widetilde{x}, \end{aligned} \tag{123.6}$$

when $\widetilde{x}$ and $\widetilde{y}$ are both small. From equation (123.3) the characteristic equation for equation (123.6) becomes

$$\lambda^2 + \beta\lambda + \omega^2(-1)^k = 0,$$

with the roots

$$\lambda_1 = \frac{-\beta + \sqrt{\beta^2 + (-1)^{k+1}4\omega^2}}{2}, \quad \lambda_2 = \frac{-\beta - \sqrt{\beta^2 + (-1)^{k+1}4\omega^2}}{2}.$$

If we now assume that $\beta > 0$ and $\beta^2 > 4\omega^2$, then

- For k even, $\lambda_1 < 0$ and $\lambda_2 < 0$. Hence, the point is a node.
- For k odd, $\lambda_1 > 0$ and $\lambda_2 < 0$. Hence, the point is a saddle point.

With this information, we can draw the phase plane for the system in equation (123.5) (see figure 123.3). Because the system in equation (123.4) is dissipative (i.e., the total "energy" decays), all of the different possible solutions approach one of the nodes in infinite time. The trajectories in the phase plane clearly show this.

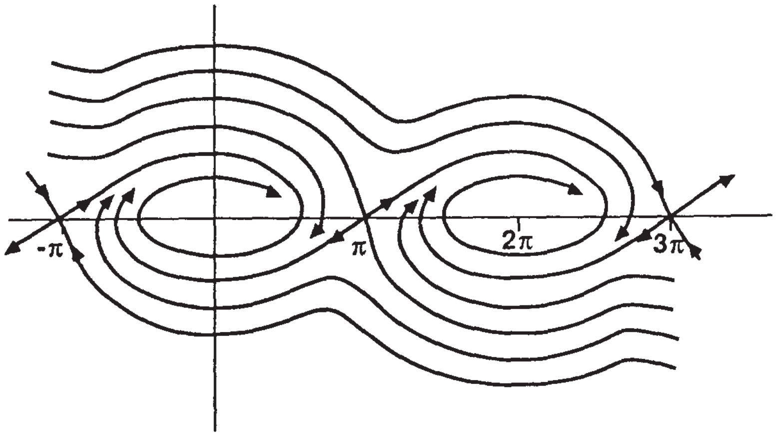

Figure 123.3: Phase plane for equation (123.4).

Notes

1. In the above, we have presumed that the critical points are *isolated*; that is, each critical point has a neighborhood around it in which no other critical points are present.
2. If, in equation (123.2), $ad-bc$ were equal to zero, then second degree (or higher) terms in the Taylor series of f and g would be required to determine the behavior near that critical point. See Boyce and DiPrima [3, pages 456–486] for details.
   If $ad-bc \neq 0$, then the solution curves of the nonlinear system in equation (123.1) will be qualitatively similar to the solution curves of the linear system in equation (123.2), with the single exception that a center for equation (123.2) may be either a center or a spiral for the system in equations (123.1).
3. A second order autonomous ordinary differential equation can always be written as a first order system (see page 146). Also, the general equation of first order $M(x,y)\,dx+N(x,y)\,dy=0$ may be written as a system in the form of equation (123.1); i.e.,

$$\frac{dx}{dt}=N(x,y), \qquad \frac{dy}{dt}=-M(x,y).$$

4. The point at infinity may be analyzed by changing variables by

$$x_1=\frac{x}{x^2+y^2}, \qquad y_1=\frac{-y}{x^2+y^2}$$

and then analyzing the point $(0, 0)$ in the x_1, y_1-plane. This corresponds to the substitution $z_1 = 1/z$, when $z = x + iy$ is treated as a complex variable.

5. Kath [9] describes a method that combines phase plane techniques with matched asymptotic expansions. This method can be used to analyze second order, nonlinear, non-autonomous, singular boundary value problems.
6. Two different graphing programs for showing phase planes on a Macintosh computer are *DEGraph* and *Phase Portraits*. A review of these programs is in Hartz [5]. A program that runs on IBM personal computers (and compatibles) is *Phaser*; see Margolis [10] for a review.
7. A large collect of phase portraits may be found in Borrelli *et al.* [2].
8. See also Bender and Orszag [1, pages 171–197], Coddington and Levinson [4, Chapter 15, pages 371–388], and Huntley and Johnson [7, Chapter 8, pages 114–133].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Borrelli, R. L., Coleman, C. S., and Boyce, W. E. *Differential Equations Laboratory Workbook*. John Wiley & Sons, New York, 1992.

[3] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[4] Coddington, E. A., and Levinson, N. *Theory of Ordinary Differential Equations*. McGraw–Hill Book Company, New York, 1955.

[5] Hartz, D. Degraph and phase portraits. *Notices of the American Mathematical Society 36*, 5 (May/June 1989), 559–561.

[6] Hubbard, J., and West, B. *MacMath: A Dynamical Systems Software Package*. Springer–Verlag, New York, 1991.

[7] Huntley, I., and Johnson, R. M. *Linear and Nonlinear Differential Equations*. Halstead Press, New York, 1983.

[8] Jordan, D. W., and Smith, P. *Nonlinear Ordinary Differential Equations*, second ed. Clarendon Press, Oxford, England, 1987.

[9] Kath, W. L. Slowly varying phase planes and boundary-layer theory. *Stud. Appl. Math. 72* (1985), 2221–239.

[10] Margolis, M. S. Phaser. *Notices of the American Mathematical Society 37*, 4 (April 1990), 430–434.

[11] Wang, D. Computer algebraic methods for investigating plane differential systems of center and focus type. In *Computers and Mathematics*, E. Kaltofen and S. M. Watt, Eds. Springer–Verlag, New York, 1990, pp. 91–998.

124. Graphical Analysis: The Tangent Field

Applicable to First order ordinary differential equations.

Yields

A graphical representation of the solutions corresponding to different initial conditions.

Idea

The qualitative features of the solution of a first order ordinary differential equation may be ascertained from the tangent field.

Procedure

Given a first order ordinary differential equation in the form

$$\frac{dy}{dx} = f(x, y), \tag{124.1}$$

the procedure is to draw small line segments in the (x, y) plane, such that the line segment that goes through the point (x_0, y_0) has the slope $f(x_0, y_0)$. Note that a slope of m corresponds to an angle of $\tan^{-1} m$. After a region of (x, y) space has been covered with these small line segments, it should be apparent how the solution curves of equation (124.1) behave. An approximate solution may then be drawn by "connecting up" the line segments that originate from a given point.

Constructing the tangent field by hand is often facilitated by the *method of isoclines*. In this method, a few curves of the form $f(x, y) = C$, with C being a constant, are constructed. Along each one of these curves, dy/dx is equal to the constant C. Hence, at every point on these curves, the small line segments all have the same slope.

Example 1

Suppose we have the nonlinear ordinary differential equation

$$\frac{dy}{dx} = 1 - xy^2. \tag{124.2}$$

It is straightforward to construct the tangent field, which is shown in figure 124.1.

Every solution of equation (124.2) must be tangent to whatever line segments it passes near. For example, if equation (124.2) had the initial condition $y(0) = 1$, then the solution can be approximately traced by starting at the point $(0, 1)$ and drawing a line that remains tangent to the line segments. For this equation and initial condition, y tends to zero as x tends to infinity. This behavior can be seen in figure 124.1.

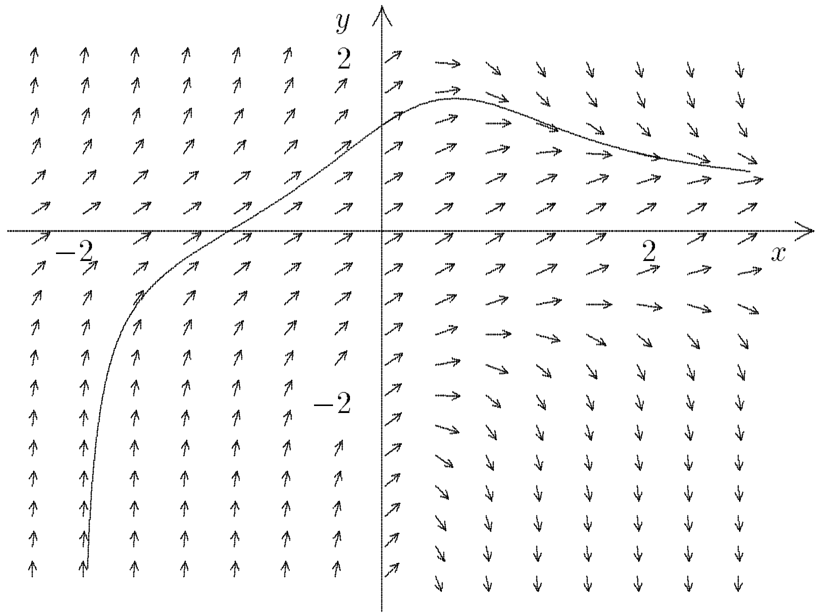

Figure 124.1: Tangent field for equation (124.2).

Example 2

Given the differential equation

$$\frac{dy}{dx} = 2x + y, \tag{124.3}$$

we find that the isoclines are the straight lines $2x + y = C$. Figure 124.2 shows the isoclines, with small line segments superposed, as well as three solutions to equation (124.3).

The exact solution to equation (124.3) is $y = 2(1-x) + Ae^{-x}$, where A is an arbitrary constant. The linear behavior for $x \gg 0$ and the exponential behavior for $x < 0$ can be identified in this figure.

Notes

1. Consider drawing a small circle Γ in the (x, y) plane that surrounds the point (x_0, y_0). Traversing the circle counter-clockwise, the direction field will change. In every case, the change in angle must be a multiple of 2π: $[\text{angle}]_\Gamma = 2\pi I_\Gamma$, where I_Γ is an integer called the *index of the vector field*. Suppose the number of times the slope dy/dx changes from $+\infty$ to $-\infty$ is m and number of times it changes from $-\infty$ to $+\infty$ is n. Then the index is equal to $(m-n)/2$. The index may be positive, negative, or zero. If Γ surrounds no critical points, then the index is zero. If Γ surrounds a saddle point, then the

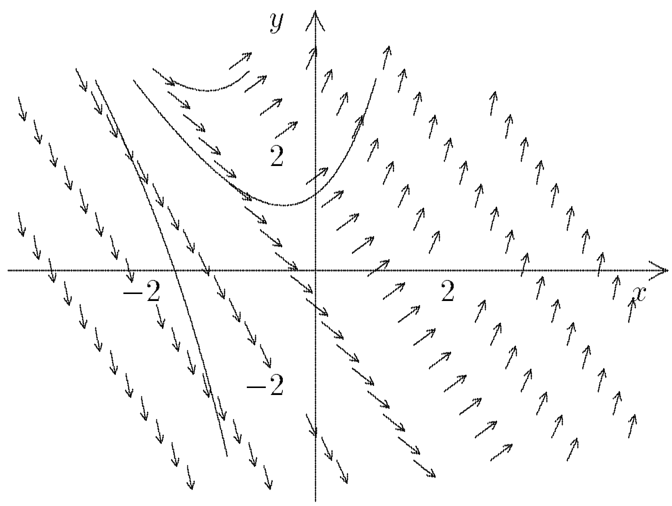

Figure 124.2: Tangent field for equation (124.3).

index is −1. If Γ surrounds a center, spiral, or node, then the index is +1. If Γ surrounds more than one critical point, then the index is the sum of the indices for each critical point.

Equation (124.1) sometimes arises from the autonomous system $\{\dot{x} = F(x,y),\ \dot{y} = G(x,y)\}$, via $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$. In this case, we have $I_\Gamma = \dfrac{1}{2\pi}\oint_\Gamma \dfrac{F\,dG - G\,dF}{F^2+G^2}$. See Jordan and Smith [3] for details.

2. Mathematica has the packages `PlotField` and `PlotField3D` which can plot two- and three-dimensional vector fields. They contain functions for plotting gradient and Hamiltonian vector fields.
3. Even rough hand construction of the tangent field can produce useful qualitative information.
4. See also Bender and Orszag [1, pages 148–149] and Boyce and DiPrima [2, pages 34–35].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Jordan, D. W., and Smith, P. *Nonlinear Ordinary Differential Equations*, second ed. Clarendon Press, Oxford, England, 1987.

125. Harmonic Balance

Applicable to Nonlinear ordinary differential equations with periodic solutions.

Yields

An approximate solution valid over the entire period. There is a specified procedure for increasing the number of terms and, hence, for increasing the accuracy.

Idea

Harmonic balance is a way of looking for periodic solutions in nonlinear systems by trying to fit a truncated Fourier series and choosing the frequency, amplitude, and phases so that any error occurs only in the discarded harmonics.

Procedure

Suppose we have a differential equation of the form

$$f(x, x_t, x_{tt}, t) = 0, \tag{125.1}$$

and we wish to find a periodic solution of period T. We look for an approximation to equation (125.1) in the form of a truncated Fourier series

$$x(t) \simeq y(t) := a_0 + \sum_{j=1}^{N} a_j \cos j\omega t + b_j \sin j\omega t,$$

where $\omega = 2\pi/T$. The unknowns to be determined are $\{a_0, a_j, b_j \mid j = 1, \dots, N\}$ and possibly T.

If T is known, then we require the $2N+1$ unknowns to satisfy the $2N+1$ algebraic equations

$$\begin{aligned} \int_0^T f(y, y_t, y_{tt}, t) \sin k\omega t \, dt &= 0, \\ \int_0^T f(y, y_t, y_{tt}, t) \cos k\omega t \, dt &= 0, \end{aligned} \tag{125.2.a-b}$$

for $k = 0, 1, \dots, N$.

If the period T is unknown, then there are $2N+2$ unknowns to be determined. To find algebraic equations for these unknowns, we require equation (125.2) to hold for $k = 0, 1, \dots, N$ and, say, equation (125.2.a) for $k = N+1$.

Example 1

Given the equation

$$\frac{d^2x}{dt^2} + x + \alpha \left(\frac{dx}{dt}\right)^2 = \sin t, \tag{125.3}$$

where α is a given constant, we search for a 2π periodic solution. If we take $T = 2\pi$ and $N = 2$, then we are assuming that

$$x(t) \simeq y(t) = a_0 + a_1 \cos t + a_2 \cos 2t + +b_1 \sin t + b_2 \sin 2t. \tag{125.4}$$

Using equation (125.3) and equation (125.4) in equation (125.2) produces the set of simultaneous algebraic equations

$$\begin{aligned} \alpha\left(4b_2^2 + b_1^2 + 4a_2^2 + a_1^2\right) + 2a_0 &= 0,\\ \alpha\left(b_1 b_2 + a_1 a_2\right) &= 0,\\ \alpha\left(b_1^2 - a_1^2\right) - 6a_2 &= 0,\\ 2\alpha\left(a_1 b_1 - a_2 b_1\right) - 1 &= 0,\\ 3b_2 + \alpha a_1 b_1 &= 0. \end{aligned}$$

These equations have the unique solution $\{a_0 = -(\alpha^{2/3} + 3^{4/3})/2(9\alpha)^{1/3}$, $a_1 = 0$, $a_2 = 1/2(3\alpha)^{1/3}$, $b_1 = -3^{1/3}/\alpha^{2/3}$, $b_2 = 0\}$. Hence, the approximation (for $N = 2$) becomes

$$x(t) \simeq -\left(\frac{3}{\alpha^2}\right)^{1/3} \sin t + \frac{1}{2(3\alpha)^{1/3}}(\cos 2t - 3) - \frac{(3\alpha)^{1/3}}{6}. \tag{125.5}$$

Note that this approximation indicates the qualitatively correct behavior, at least for small values of α. When α is small, equation (125.3) is a harmonic oscillator being forced near resonance. This would lead to a large magnitude solution, which is what equation (125.5) indicates.

Example 2

Given the equation

$$\frac{d^2x}{dt^2} + x = c(x^2 + \cos t),$$

we choose $N = 1$ and look for solutions of period $T = 2\pi$. Using the approximation

$$x(t) \simeq y(t) = a_0 + a_1 \cos t + b_1 \sin t,$$

we find that $b_1 = 0$, $a_1 = -1/2a_0$ and $a_0 = c^{1/3}z/2$, where z satisfies the cubic equation $c^{4/3}z^4 - 2z^3 + 2 = 0$. Here, the analytical solution for a_0 is available (implicitly) but is not very informative. However, if we assume that $|c| \ll 1$, then it can be shown that $a_0 = \frac{c^{1/3}}{2}\left[1 + \frac{c^{1/3}}{6} + O(c^{8/3})\right]$.

Example 3

The requirements in equation (125.2) are not the only way in which to obtain useful approximations. Consider the Duffing equation, $\ddot{x} + x = \epsilon x^3$, with $\dot{x}(0) = 0$. If we presume that $x = A\cos\omega t$, then

$$x - \epsilon x^3 = A\cos\omega t\left(1 - \frac{3}{4}\epsilon A^2\right) - \frac{1}{4}\epsilon A^3 \cos 3\omega t.$$

If we disregard the last, higher order term, then we may write $x - \epsilon x^3 \approx x\left(1 - \frac{3}{4}\epsilon A^2\right)$. With this approximation, the original equation becomes $\ddot{x} - \left(1 - \frac{3}{4}\epsilon A^2\right) x \approx 0$. Because we have presumed that $x = A\cos\omega t$, we can immediately identify the frequency: $\omega^2 \approx 1 - \frac{3}{4}\epsilon A^2$. Hence, to leading order, our approximate solution becomes $x \approx A\cos\left(1 - \frac{3}{8}\epsilon A^2\right) t$.

Notes

1. This technique is known in the engineering literature as the *describing function method*.
2. Strictly speaking, this method may also be used to obtain approximations to differential equations that do not have periodic solutions.
3. This technique applies, in principle, to equations in which there is no small parameter. However, it may prove that the algebraic equations generated by equation (125.2) are not solvable in closed form unless a perturbation expansion is used (as in Example 2).
4. Mees [7] has a very extensive bibliography, separated into categories (applications, theory, background theory, Hopf bifurcation, and harmonic balance). See also MacDonald [6].
5. When this method is implemented numerically, it is known as the *spectral method* (see page 851 or see Gottlieb and Orszag [2] for details).
6. See also Ferri [1], Groves [3], Huntley and Johnson [4, Chapter 12, pages 166–168] and Kundert *et al.* [5].

References

[1] Ferri, A. A. On the equivalence of the incremental harmonic balance method and the harmonic balance–Newton Raphson method. *J. Appl. Mech. 53* (June 1986), 455–457.

[2] Gottlieb, D., and Orszag, S. A. *Numerical Analysis of Spectral Methods: Theory and Applications.* SIAM, Philadelphia, PA, 1977.

[3] Groves, Jr., F. R. Numerical solution of nonlinear differential equations using computer algebra. *Int. J. Comp. Math. 13* (1983), 301–309.

[4] Huntley, I., and Johnson, R. M. *Linear and Nonlinear Differential Equations.* Halstead Press, New York, 1983.

[5] Kundert, K. S., Sorkin, G. B., and Sangiovanni-Vincentelli, A. Applying harmonic balance to almost-periodic circuits. *IEEE Trans. Microwave Theory and Tech. 36*, 2 (February 1988), 366–378.

[6] MacDonald, N. Choices in the harmonic balance technique. *J. Phys. A: Math. Gen. 26* (1993), 6367–6377.

[7] Mees, A. I. Describing functions: Ten years on. *IMA J. Appl. Mathematics 32* (1984), 221–233.

126. Homogenization

Applicable to

"Microscopic" differential equations.

Yields

"Macroscopic" differential equations.

Idea

By averaging microscopic differential equations, differential equations for macroscopic quantities may be determined.

Procedure

In many fields, the ("microscopic") equations of motion contain more information than is needed by a practitioner who is solving a specific problem. For instance, in a fluid flow problem, it may be that only the mass flow is required, rather than a detailed analysis of the flow field. Consequently, it is of interest to take an "average" of the "microscopic" differential equations to obtain a set of differential equations that describe the "macroscopic" quantities of interest. The average taken could be a time average, a space average, an ensemble average, or an average of some other type.

In the homogenization method, it is usually assumed that there is a fast time (or a short length) scale, on which the "microscopic" differential equations vary. The dependence on this fast scale is usually assumed to be either periodic or random. In mechanics problems, the small length scale is often the length scale of the inclusions or heterogeneities.

Often, a formal procedure for analyzing problems via homogenization is by a multiscaling procedure (see page 605).

Example 1

As an example of the general procedure, consider the elliptic problem

$$-\sum_{i,j} \frac{\partial}{\partial x_i}\left[a_{ij}^{\epsilon}(\mathbf{x})\frac{\partial u}{\partial x_j}\right] = f(\mathbf{x}), \tag{126.1}$$

in some domain Ω. The equation (126.1) probably came from a system of the form

$$\begin{aligned} -\frac{\partial p_i}{\partial x_i} &= f(\mathbf{x}), \\ p_i &= a_{ij}^{\epsilon}(\mathbf{x})\frac{\partial u}{\partial x_j}, \end{aligned} \tag{126.2}$$

via Hamilton's equations. In equations (126.1) and (126.2), it is now assumed that $a_{ij}^{\epsilon}(\mathbf{x})$ is of the form $a_{ij}(\mathbf{x}/\epsilon)$ and that $a_{ij}^{\epsilon}(\mathbf{x})$ is periodic in $\mathbf{x}$, with the period in the x_i variable being L_i. A formal two scale procedure can be defined by (see page 605)

$$\begin{aligned} y_i &= \frac{x_i}{\epsilon}, \\ u(\mathbf{x}) &= u_0(\mathbf{x},\mathbf{y}) + \epsilon u_1(\mathbf{x},\mathbf{y}) + \epsilon^2 u_2(\mathbf{x},\mathbf{y}) + \cdots, \\ \mathbf{p}(\mathbf{x}) &= \mathbf{p}^0(\mathbf{x},\mathbf{y}) + \epsilon \mathbf{p}^1(\mathbf{x},\mathbf{y}) + \epsilon^2 \mathbf{p}^2(\mathbf{x},\mathbf{y}) + \cdots, \end{aligned}$$

where $\mathbf{p} = (p_1, p_2, \dots)$. In this case, we choose to define the average of some arbitrary function of $\mathbf{x}$ and $\mathbf{y}$ to be

$$\bar{A}(\mathbf{x}) := \frac{1}{L_1 L_2 \cdots L_n} \int A(\mathbf{x},\mathbf{y})\, d\mathbf{y}. \tag{126.3}$$

We integrate over $\mathbf{y}$ in equation (126.3) to average over the high frequency component of a function that depends on both $\mathbf{x}$ and $\mathbf{y}$. For our example, it is straightforward to show that

$$-\frac{\partial \bar{p}_i^0}{\partial x_i} = f(\mathbf{x}), \tag{126.4}$$

where $\bar{\mathbf{p}}^0 = (\bar{p}_1^0, \bar{p}_2^0, \dots)$. Now, if an $a_{ij}^h(\mathbf{x})$ can be found such that

$$\bar{p}_i^0 = a_{ij}^h(\mathbf{x}) \frac{\partial u_0}{\partial x_j}, \tag{126.5}$$

then $a_{ij}^h(\mathbf{x})$ is said to be the homogenized coefficient, and equations (126.4) and (126.5) are the homogenized equations.

Example 2

For a more detailed example, consider the equation

$$A_\epsilon u_\epsilon := -\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij}\left(\frac{\mathbf{x}}{\epsilon}\right) \frac{\partial u_\epsilon}{\partial x_j} + a_0\left(\frac{\mathbf{x}}{\epsilon}\right) u_\epsilon \right) = f(\mathbf{x}), \tag{126.6}$$

where $a_{ij}(\mathbf{y})$ and $a_0(\mathbf{y})$, with $\mathbf{y} := \mathbf{x}/\epsilon$, are periodic on the unit cube Y. We assume that the solution can be expanded in the form

$$\begin{aligned} u_\epsilon &= u_0\left(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}\right) + \epsilon u_1\left(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}\right) + \dots \\ &= u_0(\mathbf{x},\mathbf{y}) + \epsilon u_1(\mathbf{x},\mathbf{y}) + \dots. \end{aligned} \tag{126.7}$$

Using the chain rule (i.e., ∂_{x_i} becomes $\partial_{x_i} + \frac{1}{\epsilon}\partial_{y_i}$), inserting equation (126.7) into equation (126.6) and equating powers of ϵ results in

$$\begin{aligned} A_1 u_0 &= 0, \\ A_1 u_1 &= A_2 u_0, \\ A_1 u_2 &= A_2 u_1 + A_3 u_0 + f, \end{aligned} \tag{126.8.a-c}$$

where

$$A_1 = -\sum_{i,j} \frac{\partial}{\partial y_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right),$$

$$A_2 = \sum_{i,j} \frac{\partial}{\partial y_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right),$$

$$A_3 = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial x_j}\right) + a_0(\mathbf{y}).$$

If we define an averaging operator by

$$M[v] = \frac{1}{|Y|}\int_Y v(\mathbf{y})\, d\mathbf{y},$$

then it can be shown that the equation $A_1 v = h$ will have a unique solution only if $M[h] = 0$ (see the section on alternative theorems, page 15). This condition, applied to equation (126.8.b), indicates that $u_0 = u_0(\mathbf{x})$. This fact simplifies equation (126.8.b) to

$$A_1 u_1 = -\sum_{i,j} \frac{\partial}{\partial y_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right) u_1 = \sum_{i,j}\left(\frac{\partial a_{ij}(\mathbf{y})}{\partial y_i}\right)\frac{\partial u_0(\mathbf{x})}{\partial x_j} = A_2 u_0.$$

Using separation of variables on this results in $u_1(\mathbf{x},\mathbf{y}) = \sum_k z_k(\mathbf{y})\frac{\partial u_0(\mathbf{x})}{\partial x_k}$, where $z_k(\mathbf{y})$ is the unique periodic solution of

$$A_1 z_k = -\sum_{i,j} \frac{\partial}{\partial y_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right) z_k = \sum_i \frac{\partial a_{ik}(\mathbf{y})}{\partial y_i}. \tag{126.9}$$

Equation (126.9) is known as the *cell problem.*

To finally obtain a solution, we require from equation (126.8.c) that $M[A_2 u_1 + A_3 u_0 + f] = 0$. This results in

$$-\sum_{i,j} p_{ij}(\mathbf{x})\frac{\partial^2 u_0(\mathbf{x})}{\partial x_i \partial x_j} + M[a_0]\, u_0(x) = f(\mathbf{x}), \tag{126.10}$$

where $p_{ij}(\mathbf{x}) := M[a_{ij}] - M\left[\sum_k a_{ik}\frac{\partial z_j}{\partial y_k}\right]$.

Notes

1. Homogenization techniques are often used in fluid mechanics (two phase flow in particular), electric field theory, and solid mechanics.
2. Homogenization is often the method used in ad hoc "mean field" theories, "effective media" theories, and "averaged equations."

3. Homogenization seems to be related to renormalization group theory. Renormalization group methods study the asymptotic behavior of a system (i.e., the macroscopic behavior) when the scale of observation is much larger than the scale of microscopic description. See Goldenfeld *et al.* [4] or Nunes da Silva [6].
4. In Persson and Wyller [7], it is shown that, for a sample problem, homogenization is equivalent to Whitham's averaged Lagrangian method.
5. Averages, denoted by $\langle \cdot \rangle$, are generally required to satisfy "Reynold's rules"

$$\begin{aligned} \langle f+g\rangle &= \langle f\rangle + \langle g\rangle\,, \\ \langle \langle f\rangle\, g\rangle &= \langle f\rangle\,\langle g\rangle\,, \\ \langle c\rangle &= c, \end{aligned}$$

when f and g are random or periodic functions and c is a constant. It is also often required that

$$\left\langle \frac{\partial f}{\partial t}\right\rangle = \frac{\partial \langle f\rangle}{\partial t}$$

be satisfied for functions f that are "well behaved."

References

[1] Avellaneda, M. Iterated homogenization: Differential effective medium theory and applications. *Comm. Pure Appl. Math 60*, 5 (September 1987), 527–554.

[2] Burgers, J. M. On some problems of homogenization. *Quart. Appl. Math. 35*, 4 (January 1978), 421–434.

[3] Ericksen, J., Kinderlehrer, D., Kohn, R., and Lions, J.-L., Eds. *Homogenization and Effective Moduli of Materials and Media*. Springer–Verlag, New York, 1986.

[4] Goldenfeld, N., Martin, O., and Oono, Y. Asymptotics of partial differential equations and the renormalization group. In *Asymptotics Beyond All Orders*, Segur, H. *et al.*, Ed. Plenum Publishing Corp., New York, 1991, pp. 375–383.

[5] Larsen, E. W. Two types of homogenization. *SIAM J. Appl. Math. 36*, 1 (February 1979), 26–33.

[6] Nunes da Silva, J. M. Renormalized vibrations of a loaded spring. *Am. J. Phys. 62*, 5 (May 1994), 423–426.

[7] Persson, L., and Wyller, J. A note on Whithams method and the homogenization procedure. *Physica Scripta 38* (1988), 774–776.

[8] Sanchez-Palencia, E. Homogenization method for the study of composite media. In *Asymptotic Analysis*, J. D. Murray, Ed. Springer–Verlag, New York, 1984, pp. 192–214.

[9] Weinan, E. Homogenization of linear and nonlinear transport equations. *Comm. Pure Appl. Math 65* (1992), 301–326.

127. Integral Methods

Applicable to

Linear and nonlinear partial differential equations.

Yields

An approximation of the solution.

Idea

A sequence of physical approximations may lead to an approximate solution.

Procedure

There are generally three separate steps in using the common integral approximation techniques:

- A physical boundary (either natural or imposed mathematically) is assumed to be at some finite distance.
- A weak form of the equations is assumed to hold, up to the boundary described above.
- The form of the solution is guessed by the method of undetermined coefficients.

These concepts are made clear in the following example.

Example

Suppose we want to approximate the solution of the linear parabolic partial differential equation

$$\begin{aligned} u_t &= \alpha u_{xx}, \quad \text{for } x > 0, t > 0, \\ u(0,x) &= u_0, \\ \frac{\partial u}{\partial x}(t,0) &= f(t), \end{aligned} \tag{127.1.a-c}$$

where $f(t)$ is some prescribed function. Note that the value of $u(t,x)$ (which physically might represent a temperature) is initially u_0. For the first approximation, we suppose that there is a finite distance $\beta(t)$ that varies with time, beyond which the temperature is still u_0.

This assumption is contrary to fact; we know that the diffusion equation has an infinite propagation speed, and the value of u at *all* points is immediately changed from u_0. But the change from u_0 will be exponentially small at large distances, so we assume it is zero for $x \geq \beta(t)$. This adds the boundary conditions

$$\begin{aligned} u(t,\beta(t)) &= u_0, \\ u_x(t,\beta(t)) &= 0. \end{aligned} \tag{127.2.a-b}$$

Equation (127.2.a) states that the temperature at the boundary $x = \beta(t)$ is always equal to u_0. Equation (127.2.b) states that there is no heat flux across $x = \beta(t)$; if there was such a flux, then the region beyond $x = \beta(t)$ would not maintain the temperature $u = u_0$.

The second approximation is to assume that a weak form of the differential equation will hold. To obtain this weak form, we integrate equation (127.1.a) with respect to x from $x = 0$ to $x = \beta(t)$ to obtain

$$\int_0^{\beta(t)} u_t \, dx = \alpha \int_0^{\beta(t)} u_{xx} \, dx.$$

This expression can be integrated by parts to obtain

$$\begin{aligned} \frac{d}{dt} \int_0^{\beta(t)} u \, dx - u(t, \beta(t)) \frac{d\beta(t)}{dt} &= \alpha \Big[u_x(t, \beta(t)) - u_x(t, 0) \Big] \\ &= -\alpha f(t), \end{aligned} \tag{127.3}$$

where we have used equation (127.2.b) and equation (127.1.c). If we define

$$w(t) = \int_0^{\beta(t)} u \, dx, \tag{127.4}$$

then equation (127.3) can be written as the ordinary differential equation

$$\frac{d}{dt}\Big(w - u_0 \beta \Big) = -\alpha f(t). \tag{127.5}$$

Note that, from equation (127.4), the average value of $u(t, x)$ in the region $0 \le x \le \beta(t)$ is given by $w(t)/\beta(t)$.

Now we must determine $\beta(t)$ from equation (127.5). Before we can solve for $\beta(t)$, however, we need to determine $w(t)$. To determine $w(t)$, we presume some form of the general solution for $u(t, x)$. By the use of undetermined coefficients, we suppose that $u(t, x)$ has the form

$$u(t, x) = \begin{cases} a(t) + b(t)x + c(t)x^2, & \text{for } 0 < x < \beta(t), \\ u_0, & \text{for } x > \beta(t), \end{cases}$$

where $a(t)$, $b(t)$ and $c(t)$ are all unknowns. If this form is to satisfy equation (127.1.c) and equation (127.2), then it must be restricted to be of the form

$$u(t, x) = \begin{cases} u_0 - \frac{f(t)}{2\beta(t)} [\beta(t) - x]^2, & \text{for } 0 \le x < \beta(t), \\ u_0, & \text{for } x \ge \beta(t). \end{cases} \tag{127.6}$$

Using this form in equation (127.4) results in $w(t) = u_0 \beta(t) - \frac{\beta^2(t) f(t)}{6}$. Using this value for $w(t)$ in equation (127.5) results in $\frac{d}{dt}\left[\frac{\beta^2(t) f(t)}{6}\right] = \alpha f(t)$.

The solution of this ordinary differential equation is

$$\beta(t) = \sqrt{\frac{6\alpha}{f(t)} \int_0^t f(s)\, ds}. \tag{127.7}$$

Using this form for $\beta(t)$ in equation (127.6) completes the determination of the approximate solution.

For comparison purposes, if $f(t)$ is the constant F, then the temperature at $x = 0$ is given by (using equations (127.7) and (127.6))

$$u(t, 0) \simeq u_0 - \sqrt{\frac{3}{2}\alpha t}F. \tag{127.8}$$

Conversely, the exact solution of equation (127.1) can be found by the use of Laplace transforms to be $u(t,x) = u_0 - \sqrt{\frac{\alpha}{\pi}} \int_0^t \frac{f(t-\tau)}{\sqrt{\tau}} e^{-x^2/4\alpha\tau}\, d\tau$, and so, when $f(t)$ is the constant F, the exact solution becomes $u(t, 0) = u_0 - \sqrt{\frac{4}{\pi}\alpha t}F$. The difference between this exact solution and the approximation in equation (127.8) is about 9%.

Notes

1. This method, in fluid mechanics, is known as the Kármán–Pohlausen technique. The distance $\beta(t)$ then represents the thickness of a boundary layer.
2. When this technique is used, it is most often with partial differential equations that have only a single space variable.
3. This technique is often used in free boundary problems (see page 311).
4. See Ames [1, pages 271–278].

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Goodman, T. R. Application of integral methods to transient nonlinear heat transfer. In *Advances in Heat Transfer*, T. F. Irvine, Jr. and J. P. Hartnett, Eds. Academic Press, New York, 1964, pp. 51–122.

[3] Riley, D. S., and Duck, P. W. Application of the heat-balance integral method to the freezing of a cuboid. *Int. J. Heat Mass. Transfer 20* (1977), 294–296.

128. Interval Analysis

Applicable to Ordinary and partial differential equations.

Yields

An analytical approximation with an *exact* bound on the error.

Idea

Initially, we bound the solution between an upper and lower bound. Then, iterating a contraction mapping, we generate a sequence of approximations in which the upper bound decreases and the lower bound increases.

Procedure

We use the interval notation $[a, b]$ to indicate some number between the values of a and b. We allow the coefficients of polynomials to be intervals. For example, the interval polynomial

$$Q(x) = 1 + [2, 3]x^2 + [-1, 4]x^3,$$

evaluated at the point $x = y$, means that

$$\min_{\substack{2\le\eta\le3 \\ -1\le\zeta\le4}} \left(1 + \eta y^2 + \zeta y^3\right) \le Q(y) \le \max_{\substack{2\le\eta\le3 \\ -1\le\zeta\le4}} \left(1 + \eta y^2 + \zeta y^3\right).$$

There exists an algebra of interval polynomials. For example

$$\left(x + [2, 3]x^3\right) + \left([1, 2]x + [1, 4]x^3\right) = [2, 3]x + [3, 7]x^3,$$
$$\left([1, 3] + [-1, 2]x\right)^2 = [1, 9] + [-6, 12]x + [0, 4]x^2.$$

If $P(x)$ and $Q(x)$ are interval polynomials, then at any point y we can write $P(y) \in [P_L, P_U]$ and $Q(y) \in [Q_L, Q_U]$. We say that $P(x)$ contains $Q(x)$ on some interval $[c, d]$ if $P_L \le Q_L$ and $Q_U \le P_U$ for all $y \in [c, d]$. This is denoted by $Q(x) \subset P(x)$.

To approximate the solution of an ordinary differential equation, we search for a contraction mapping (see page 58) that has the form $P_{k+1} = F[P_k]$, where $F[\cdot]$ is a functional, $P_{k+1} \subset P_k$, and P_k tend to the solution of the differential equation as $k \to \infty$.

Example

Suppose we want to approximate the solution of

$$y' = y^2, \qquad y(0) = 1, \tag{128.1}$$

for values of x in the interval $[0, 1/4]$. Equation (128.1) can be written as the equivalent integral equation

$$y(x) = 1 + \int_0^x y^2(z)\,dz. \tag{128.2}$$

It is easy to see that the solution of equation (128.2) must lie in the interval $[1, 2]$ when $x \in [0, 1/4]$. This is because y' is always positive, so y cannot be smaller than 1 (which is what $y(0)$ is) and if it is assumed that $y(z_0) = 2$ for some $z_0 \in (0, 1/4)$, then a contradiction can be reached by using equation (128.2). We now define the iteration sequence (the contraction mapping) by

$$P_{k+1}(x) = 1 + \int_0^x P_k^2(z)\, dz,$$

for $k = 0, 1, 2, \dots$, which is just Picard's integral formula (see page 618). We start the sequence off by $P_0(x) = [1, 2]$ and then calculate

$$\begin{aligned}
P_1(x) &= 1 + \int_0^x [1,2]^2\, dz, \\
&= 1 + [1,4]x, \\
P_2(x) &= 1 + \int_0^x (1 + [1,4]z)^2\, dz, \\
&= 1 + \int_0^x (1 + [2,8]z + [1,16]z^2)\, dz, \\
&= 1 + x + [1,4]x^2 + \left[\frac{1}{3}, \frac{16}{3}\right] x^3, \\
P_3(x) &= 1 + x + x^2 + x^3 + [1,2]x^4 + \cdots, \\
P_4(x) &= 1 + x + x^2 + x^3 + x^4 + \left[1, \frac{7}{5}\right] x^5 + \cdots.
\end{aligned}$$

It is easy to show that $P_{k+1}(x) \subset P_k(x)$ and that $\{P_k(x)\}$ converges to the exact solution $y(x)$ of equation (128.1). Note that, from the $P_k(x)$, *exact* estimates of the solution are available. For example, from $P_2(x)$, we find, $1.141 < y(1/8) < 1.198$.

Notes

1. The exact solution to the system in equation (128.1) is $y(x) = 1/(1-x)$, which has the Taylor series: $y(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$.
2. To avoid dealing with polynomials of large degree, as in the example, we could observe that

$$x^n \subset \left[0, \left(\frac{1}{4}\right)^{n-m}\right] x^m,$$

   for x in the interval $[0, \frac{1}{4}]$. This allows us to replace x^n by x^m, with a coarsening of the bounds.
3. The real power of this method is that it can be applied to differential equations whose coefficients are given by intervals. For example, this would be the case in a problem in which a parameter appearing in a differential equation is known only approximately.

4. The paper by Ames and Nicklas [2] describes the solution of elliptic partial differential equations, using interval analysis to solve the finite difference equations produced by a numerical approximation. Schwandt [10] addresses the same issue, but with the use of a vector computer.
5. When solving ordinary differential equations numerically, using interval techniques, the error bounds often exhibit spurious exponential growth due to the differential equation solver used. Numerical methods have been developed that prevent spurious exponential growth of the intervals for linear systems, see Gambill and Skeel [5] for details.
6. The techniques presented in this section can be implemented numerically. Interval arithmetic packages are available in Algol (see Guenther and Marquardt [6]), Fortran (see Yohe [12]), and PASCAL (see Rall [9]).
7. There is an interval computations web page at `http://cs.utep.edu/interval-comp/main.html`. The book by Eijgenraam [4] contains several worked examples. The journal *Interval Computations* is a useful reference.

References

[1] Alefeld, G., and Herzberger, J. *Introduction to Interval Computations.* Academic Press, New York, 1983.

[2] Ames, W. F., and Nicklas, R. C. Accurate elliptic differential equation solver. In *Accurate Scientific Computations*, W. L. Miranker and R. A. Toupin, Eds., no. 235 in Lecture Notes in Computer Science. Springer–Verlag, New York, 1986, pp. 70–85.

[3] Corliss, G. F. Survey of interval algorithms for ordinary differential equations. *Appl. Math. and Comp. 31* (1989), 112–120.

[4] Eijgenraam, P. The solution of initial value problems using interval arithmetic. Tech. rep., Mathematisch Centrum, Amsterdam, The Netherlands, 1981.

[5] Gambill, T. N., and Skeel, R. D. Logarithmic reduction of the wrapping effect with application to ordinary differential equations. *SIAM J. Numer. Anal. 25*, 1 (February 1988), 153–162.

[6] Guenther, G., and Marquardt, G. A programming system for interval arithmetic. In *Interval Mathematics 1980*, K. Nickel, Ed. Academic Press, New York, 1980, pp. 355–366.

[7] Moore, R. E., and Zuhe, S. An interval version of Chebyshev's method for nonlinear operator equations. *Nonlinear Analysis 7*, 1 (1983), 21–34.

[8] Oppenheimer, E. P., and Michel, A. N. Application of interval analysis techniques to linear systems: Part III—Initial value problems. *IEEE Trans. Circuits and Systems 35*, 10 (October 1988), 1243–1256.

[9] Rall, L. B. An introduction to the scientific computing language Pascal–SC. *Comp. & Maths. with Appls. 14*, 1 (1987), 53–69.

[10] Schwandt, H. Newton-like interval methods for large nonlinear systems of equations on vector computers. *Comput. Physics Comm. 37* (1985), 223–232.

[11] Schwandt, H. Interval arithmetic methods for systems of nonlinear equations arising from discretizations of quasilinear elliptic and parabolic partial differential equations. *Applied Numerical Math. 3* (1987), 257–287.

[12] Yohe, J. M. Software for interval arithmetic: A reasonable portable package. *ACM Trans. Math. Software 5*, 1 (March 1979), 50–63.

129. Least Squares Method

Applicable to Ordinary and partial differential equations.

Yields

An approximation to the solution.

Idea

A variational principle is created for a given differential equation, and then an approximation to the solution with some free parameters is proposed. By use of the variational principle, the free parameters are determined.

Procedure

Given the differential equation

$$N[u] = 0, \tag{129.1}$$

for $u(\mathbf{x})$ in some region of space R, with the homogeneous boundary conditions

$$B[u] = 0, \tag{129.2}$$

on some portion of the boundary of R, we define the functional

$$J[v(\mathbf{x})] = \int_R \Big(N[v(\mathbf{x})] \Big)^2 \, d\mathbf{x}. \tag{129.3}$$

Notice that $J[v(\mathbf{x})] \geq 0$, for all functions $v(\mathbf{x})$.

The solution to equations (129.1) and (129.2) clearly satisfies $J[u] = 0$ because the integrand is identically equal to zero in this case. Hence, the solutions to equations (129.1) and (129.2) represents a minimum of the functional $J[\cdot]$.

Now we choose an approximation to $u(\mathbf{x})$ that has several parameters in it, say $u(\mathbf{x}) \simeq w(\mathbf{x}; \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is a vector of parameters. This approximation is chosen in such a way that it satisfies the conditions in equation (129.2). The parameters in $w(\mathbf{x}; \boldsymbol{\alpha})$ are determined by minimizing $J[w(\mathbf{x}, \boldsymbol{\alpha})]$; i.e., by solving the simultaneous system of equations

$$\frac{\partial}{\partial \alpha_k} J[w(\mathbf{x}, \boldsymbol{\alpha})] = 0, \qquad \text{for } k = 1, 2, \ldots. \tag{129.4}$$

Example

Suppose we wish to approximate the solution of the two point boundary value problem

$$\begin{aligned} &u'' + u + x = 0, \\ &u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{129.5}$$

(Note that the exact solution of equation (129.5) is $y(x) = \frac{\sin x}{\sin 1} - x$.) In this case, we may define $J[v(x)]$ to be

$$J[v(x)] = \int_0^1 (v'' + v + x)^2 \, dx.$$

We choose to approximate the solution of equation (129.5) by

$$u(x) \simeq w(x) = \alpha_1(x - x^2) + \alpha_2(x - x^3).$$

This approximation has been chosen in such a way that the boundary conditions for $u(x)$ are satisfied. Using $w(x)$ in the functional results in

$$J[w(x)] = \frac{1}{210}\left[707\alpha_1^2 + 2121\alpha_1\alpha_2 + 2200\alpha_2^2 - 385\alpha_1 - 784\alpha_2 + 70\right].$$

Forming equation (129.4) for $k = 1, 2$, we determine that α_1 and α_2 must satisfy the simultaneous algebraic equations

$$\begin{aligned}\frac{\partial J[w(x)]}{\partial \alpha_1} &= \frac{1}{210}\left[1414\alpha_1 + 2121\alpha_2 - 385\right] = 0,\\ \frac{\partial J[w(x)]}{\partial \alpha_2} &= \frac{1}{210}\left[2121\alpha_2 + 4400\alpha_2 - 784\right] = 0.\end{aligned}$$

These equations have the solution: $\{\alpha_1 = \frac{4448}{246137} \simeq 0.0181, \alpha_2 = \frac{413}{2437} \simeq 0.1694\}$. The function $w(x)$, with these values, becomes our approximation. The greatest difference between the exact solution and the approximate solution, in the range $0 < x < 1$, is at $x \simeq 0.5215$, where the difference is approximately 0.0016.

Notes

1. Note that for the functional in equation (129.3) there may exist, in general, several different functions $\{v_k(\mathbf{x})\}$ that satisfy $J[v_k(\mathbf{x})] = 0$.
2. This method is similar to the Rayleigh–Ritz method (see page 638) in that an approximation is utilized in a variational equation.
3. This method is an example of a *weighted residual method* (see page 786).
4. This technique is often implemented numerically.
5. See Collatz [2, pages 184 and 220–221].

References

[1] Chang, C. L., and Gunzburger, M. D. A subdomain-Galerkin/least squares method for first-order elliptic systems in the plane. *SIAM J. Numer. Anal. 27*, 5 (1990), 1197–1211.

[2] Collatz, L. *The Numerical Treatment of Differential Equations*. Springer–Verlag, New York, 1966.

[3] Hanke, M. On a least-squares collocation method for linear differential-algebraic equations. *Numer. Math. 54* (1988), 79–90.

130. Lyapunov Functions

Applicable to Ordinary and partial differential equations.

Yields

Bounds on the solution in phase space.

Idea

Even without solving a given differential equation, sometimes we can restrict the solution to be in a certain portion of phase space.

Procedure

Given a differential equation, find a non-negative functional of the solution, which has a non-positive derivative. Then the solution of the differential equation will remain in a region described by the functional and the initial conditions. Most often, the functional will involve the dependent variable and some of its derivatives.

Example 1

Suppose we wish to bound the solution of a damped harmonic oscillator

$$\begin{aligned} &x_{tt} + \beta x_t + \omega^2 x = 0, \\ &x(0) = A, \qquad x_t(0) = B, \end{aligned} \tag{130.1}$$

with $\beta > 0$. In this case, we define the Lyapunov functional to be

$$L[x(t), x_t(t), x_{tt}(t), t] = \omega^2 x^2(t) + x_t^2(t).$$

Because $L[\cdot]$ is a sum of squares, it cannot be negative. Differentiating $L[\cdot]$ with respect to t produces

$$\begin{aligned} L_t[x, x_t, x_{tt}, t] &= (\omega^2 x^2 + x_t^2)_t \\ &= 2\omega^2 x x_t + 2 x_t x_{tt} \\ &= -2\beta x_t^2, \end{aligned} \tag{130.2.a-c}$$

where we have used the original differential equation equation (130.1) to replace the x_{tt} term in equation (130.2.b). Because β is positive, L_t is non-positive. Therefore, $L[\cdot]$ is a non-increasing function of t. Hence,

$$\begin{aligned} \omega^2 x^2(t) + x_t^2(t) &= L[x(t), x_t(t), x_{tt}(t), t] \\ &\leq L[x(0), x_t(0), x_{tt}(0), 0] \\ &\leq \omega^2 x^2(0) + x_t^2(0) \\ &\leq \omega^2 A^2 + B^2 \\ &\leq \text{ a prescribed constant.} \end{aligned} \tag{130.3}$$

Therefore, we have found an upper bound for $\omega^2 x^2(t) + x_t^2(t)$ without solving the original equation.

Example 2

Suppose we have the wave equation on a finite domain ($0 \leq x \leq L$)

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \\ &u_x(0,t) = u_x(L,t) = 0, \quad u(x,0) = g(x), \end{aligned} \tag{130.4}$$

where c is a given constant and $g(x)$ is given. In this case, we choose the Lyapunov functional to be

$$V(t) = \frac{1}{2}\int_0^L [u_t^2 + c^2 u_x^2]\,dx. \tag{130.5}$$

Because $V(t)$ is the integral of a non-negative quantity, $V(t)$ is also non-negative. Differentiating $V(t)$ with respect to t produces

$$\begin{aligned} V_t &= \int_0^L [u_t u_{tt} + c^2 u_x u_{xt}]\,dx \\ &= \int_0^L [u_t(c^2 u_{xx}) + c^2 u_x u_{xt}]\,dx \\ &= c^2 \int_0^L [u_t u_{xx} + u_x u_{xt}]\,dx. \end{aligned} \tag{130.6.a-c}$$

Integration of the second term in equation (130.6.c) by parts yields

$$\begin{aligned} V_t &= c^2 \int_0^L [u_t u_{xx} - u_{xx} u_t]\,dx + c^2 u_x u_t \bigg|_{x=0}^{x=L} \\ &= c^2 \left[u_x(L,t)u_t(L,t) - u_x(0,t)u_t(0,t)\right], \end{aligned}$$

or, using the initial conditions in equation (130.4),

$$V_t = 0.$$

We conclude that $V(t) = V(0)$, for all values of t. This statement is essentially an "energy" statement: The energy (described by equation (130.5)) carried by a wave (described by equation (130.4)) remains constant.

Notes

1. Lyapunov functionals are often devised from physical considerations. The Lyapunov functionals in both of these examples represent the "energy" of the system in a mathematical way.
2. Finding Lyapunov functionals is, in general, a difficult task. It is often made easier by considering conservation laws: energy, momentum, etc. The "energy" in example one is *not* held constant because of the

dissipation due to the β term. If $\beta = 0$, then $L_t = 0$ and so equation (130.3) becomes

$$\omega^2 x^2(t) + x_t^2(t) = \omega^2 A^2 + B^2.$$

In this case, the energy is constant.

3. There is a constructive method, due to Zubov [10], for obtaining Lyapunov functionals for systems of ordinary differential equations. The procedure requires the solution of a partial differential equation, which is derived from the given system of ordinary differential equations. See Hahn [3, pages 78–82] or Willems [9, pages 42–43] for details. Hahn [3] gives an example: A Lyapunov function for the system $\{\dot{x} = -x + 2x^2 y,\ \dot{y} = -y\}$ is $L = -1 + \exp\left(-\frac{y^2}{2} - \frac{x^2}{2(1-xy)}\right)$.
4. A different constructive method is described in Oğuztöreli *et al.* [7]. A detailed algorithm is given for systems of ordinary differential equations of the form: $\{\dot{x} = f(t,x,y),\ \dot{y} = g(t,x,y)\}$. The Lyapunov function for a modification of the Mathieu differential equation, $\ddot{x} = (\alpha + 2\beta\cos 2t)x + \epsilon x^2$, is derived for the region $\left(x^2 + y^2 < \rho^2\right)$, where ρ is a sufficiently small number.
5. Consider the nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and the Jacobian matrix $J(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$. If a constant, symmetric, positive definite matrix P can be found such that $PJ(\mathbf{x}) + J^{\mathrm{T}}(\mathbf{x})P$ is negative definite, then $V = \mathbf{x}^{\mathrm{T}} P \mathbf{x}$ is a Lyapunov function (with $V' = \mathbf{x}^{\mathrm{T}} \left\{\int \left[J^{\mathrm{T}}(z\mathbf{x})P + PJ(z\mathbf{x})\right] dz\right\} \mathbf{x}$).If P is chosen to be the identity matrix, then $V = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ will be a Lyapunov function if all of the eigenvalues of the matrix $J(\mathbf{x}) + J^{\mathrm{T}}(\mathbf{x})$ are negative. This is known as Krasovskii's theorem.
6. Burton [2] describes how Lyapunov functions may be constructed for delay differential equations.
7. "Lyapunov" is sometimes written "Liapunov."
8. See Boyce and DiPrima [1, pages 502–512] and Simmons [8, pages 316–322].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Burton, T. A. Perturbation and delays in differential equations. *SIAM J. Appl. Math. 29*, 3 (November 1975), 422–438.

[3] Hahn, W. *Theory and Application of Liapunov's Direct Method*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1963.

[4] Jordan, D. W., and Smith, P. *Nonlinear Ordinary Differential Equations*, second ed. Clarendon Press, Oxford, England, 1987.

[5] Kalman, R. E., and Bertram, J. E. Control system analysis and design via the 'second method' of Lyapunov, I: Continuous-time systems. *J. Basic Engrg. Trans. ASME 82*, 2 (1960), 371–393.

[6] LASALLE, J., AND LEFSCHETZ, S. *Stability by Liapunov's Direct Method with Applications*. Academic Press, New York, 1961.

[7] OGUZTORELI, M. N., LAKSHMIKANTHAM, V., AND LEELA, S. An algorithm for the construction of Liapunov functions. *Nonlinear Analysis 5*, 11 (1981), 1195–1212.

[8] SIMMONS, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

[9] WILLEMS, J. L. *Stability Theory of Dynamical Systems*. John Wiley & Sons, New York, 1970.

[10] ZUBOV, V. I. *Methods of A. M. Lyapunov and their Application*. P. Noordhoff, The Netherlands, 1964.

131. Equivalent Linearization and Nonlinearization

Applicable to Nonlinear ordinary differential equations. This technique is most frequently used for ordinary differential equations with periodic solutions.

Yields

An approximate periodic solution.

Idea

We model the given equation by a linear or nonlinear equation for which the exact solution can be found.

Procedure

Suppose we want to approximate the solution to the nonlinear ordinary differential equation

$$D[x(t), t] = 0, \tag{131.1}$$

where $D[\cdot]$ is a differential operator. We represent the initial conditions and boundary conditions for $x(t)$ as $B[x(t)] = 0$, and assume that $x(t)$ is periodic on some interval, say for t from 0 to T. We do not need to know T *a priori*.

We model equation (131.1) by choosing a $D^*[\cdot]$ that has properties that are "similar" to the properties of $D[\cdot]$. This can be done by any technique. To allow some generality, we assume that $D^*[\cdot]$ depends on a set of parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Now we look for a solution $y(t; \boldsymbol{\alpha})$ of

$$D^*[y(t; \boldsymbol{\alpha}), t; \boldsymbol{\alpha}] = 0, \quad B[y(t; \boldsymbol{\alpha})] = 0 \tag{131.2}$$

that is periodic on the interval $[0, T]$. We will approximate the solution to equation (131.1), $x(t)$, by the solution to equation (131.2), $y(t; \boldsymbol{\alpha})$. For this to be a good approximation, the error made must be small. We define the error made in using $y(t; \boldsymbol{\alpha})$ for $x(t)$ to be

$$\mathcal{E}(t, \boldsymbol{\alpha}) := D[y(t; \boldsymbol{\alpha}), t].$$

The claim is that $x(t) \simeq y(t; \alpha)$ if the "total error" is "small" in some sense. The "total error" could be measured as

$$\frac{1}{T} \int_0^T |\mathcal{E}(t, \boldsymbol{\alpha})|^2 \, dt \qquad \text{"mean square error,"}$$

or

$$\frac{1}{T}\int_0^T |\mathcal{E}(t,\boldsymbol{\alpha})|\,dt \qquad \text{“mean modulus,”}$$

or

$$\max|\mathcal{E}(t,\boldsymbol{\alpha})| \qquad \text{“extremum.”}$$

The “total error” is minimized by choosing the $\boldsymbol{\alpha}$. This is accomplished by differentiating the total error with respect to α_i and setting the resulting expression to zero (for $i = 1, 2, \ldots, n$). Solving these simultaneous algebraic equations yields the desired values of the α_i.

Example 1

Suppose we wish to approximate the periodic solution of the nonlinear ordinary differential equation

$$\begin{gathered} D[x(t),t] = x'' + ax + bx^3 + cx^5 = 0, \\ x(0) = A, \qquad x'(0) = 0. \end{gathered} \tag{131.3}$$

Here $\{a, b, c, A\}$ are all known, fixed constants. We choose to approximate the solution of equation (131.3) by the solution of the linear ordinary differential equation

$$\begin{aligned} &D^*[y(t),t;\omega] = y'' + \omega^2 y = 0, \\ &y(0) = A, \qquad y'(0) = 0, \end{aligned} \tag{131.4}$$

for some (unknown) value of ω. In this example, the vector of unknown parameters $\boldsymbol{\alpha}$ is the single variable ω. The solution to equation (131.4) is

$$y(t) = A\cos\omega t. \tag{131.5}$$

The error in using equation (131.5) for the solution of equation (131.3) is

$$\begin{aligned} \mathcal{E}(t,\omega) &= D[y(t),t], \\ &= y'' + ay + by^3 + cy^5, \\ &= (a-\omega^2)\cos\omega t + b\cos^3\omega t + c\cos^5\omega t. \end{aligned}$$

We choose, in this example, to minimize the mean square error. Hence, we define the total error, $E(\omega)$, by

$$\begin{aligned} E(\omega) &= \frac{1}{T}\int_0^T |\mathcal{E}(t,\omega)|^2\,dt \\ &= \frac{1}{T}\int_0^T [(a-\omega^2)\cos\omega t + b\cos^3\omega t + c\cos^5\omega t]^2\,dt. \end{aligned} \tag{131.6}$$

Now, what is T? For equation (131.3), we do not know the true period of the solution. But, we are using the solution of equation (131.4) to approximate the solution of equation (131.3). And, for equation (131.4), the solution has period $T = 2\pi/\omega$ (see equation (131.5)). Hence, to evaluate equation (131.6), we use $T = 2\pi/\omega$ to obtain

$$E(\omega) = \big[128\omega^4 - (160c + 192b + 256a)\omega^2 + 63c^2 + (140b + 160a)c + 80b^2 + 192ab + 128a^2\big]/256. \tag{131.7}$$

Now, the goal is to minimize the total error. If equation (131.7) is differentiated with respect to ω, and the resulting equation is solved for ω, then

$$\omega^2 = a + \frac{3}{4}bA^2 + \frac{5}{8}cA^4 \qquad \text{or} \qquad \omega = 0. \tag{131.8}$$

Therefore, an approximation to the solution of equation (131.3) is found by using equation (131.8) in equation (131.5):

$$x(t) \simeq A\cos\left[t\sqrt{a + \frac{3}{4}bA^2 + \frac{5}{8}cA^4}\right].$$

Example 2

Suppose we wish to approximate the periodic solution of the undamped Duffing equation

$$D[x(t), t] = x'' + ax + bx^3 = B\cos\omega t, \tag{131.9}$$

where $\{a, b, B, \omega\}$ are all known constants. We choose to model the equation (131.9) by the nonlinear equation

$$D^*[y(t), t] = y'' + ay + by^3 = \gamma\,\mathrm{cn}(\eta t, k), \tag{131.10}$$

where $\mathrm{cn}(\eta t, k)$ is the Jacobian elliptic cosine function with modulus k. The a and b in equation (131.10) are the same as the a and b in equation (131.9). The three remaining parameters in equation (131.10) that are under our control are $\{\gamma, \eta, k\}$. The solution to equation (131.10) is known to be (see the look-up solution technique, on page 179)

$$y(t) = \beta\,\mathrm{cn}(\eta t, k), \tag{131.11}$$

where β, γ, η, and k are related by

$$b\beta^3 + (a - \eta^2)\beta = \gamma, \qquad k^2 = \frac{b\beta^2}{2\eta^2}. \tag{131.12}$$

These equations determine β and k (in principle) in terms of γ and η. For the period of the forcing function in equation (131.10) to match the period of the forcing function in equation (131.9) (which is $2\pi/\omega$), we also require

$$\eta = \frac{2K(k)\omega}{\pi}, \tag{131.13}$$

where $K(k)$ is the complete elliptic integral of the first kind with modulus k. We will use equation (131.13) to determine η. This leaves us with one adjustable parameter, γ, with which to effect the minimization of the total error.

Now we calculate

$$\mathcal{E}(t,\gamma) = D[y(t)] = B\cos\omega t - \gamma\,\mathrm{cn}(\eta t, k). \tag{131.14}$$

If we choose to use the mean square error, with $T = 2\pi/\omega$, we find that the total error is minimized for

$$\gamma = \frac{B\pi K(k)}{2\left[E(k) - {k'}^2 K(k)\right]}\,\mathrm{sech}\left(\frac{\pi K(k')}{2K(k)}\right), \tag{131.15}$$

where $E(k)$ is the complete elliptic integral of the second kind and k', given by ${k'}^2 = 1 - k^2$, is the complementary modulus.

Using equations (131.13), (131.14), and (131.15) in equation (131.11) results in the final approximation to the steady-state periodic solution of equation (131.9).

Notes

1. Note that in example 1 the effective frequency of the approximate solution depends on the initial conditions. This is generally expected in nonlinear problems.
2. For example 2, the approximate solution $y(t)$ correctly tracks the frequency change of the solution when the magnitude of the forcing function is changed. More details on this example may be found in Iwan and Patula [4].
3. This technique also works well for stochastic equations. In this application, the definition of the total error should include expectations taken over all of the random variables. This is sometimes called "statistical linearization." See Beaman [1] for details.
4. This technique extends naturally to systems of equations. In this case, there will be an error associated with each equation $\{E_i(t,\boldsymbol{\alpha})\}$, and we can define the total error by $E(t,\boldsymbol{\alpha}) = \sum_i |E_i(t,\boldsymbol{\alpha})|^2$.
5. This technique can also be used for problems that do not have periodic solutions. The technique often used in this case is to minimize the integral of $|\mathcal{E}|^2$ from 0 to ∞.
6. Differential operators representing differential equations may also be linearized directly, without minimizing some error functional. We have the definition:

   > The operator $A[\cdot]$ is linearizable at u_0 if there exists a bounded linear operator $L[\cdot]$ such that $A[u] - A[u_0] = L[h] + r$, with $\lim\limits_{h\to 0}\frac{||r||}{||h||} = 0$, when $h = u - u_0$.

   See Stakgold [8, pages 578–581] for details.

7. See also Hagedorn [3, pages 14–16] and McLachlan [7, Chapter 6, pages 103–112].

References

[1] Beaman, J. J. Accuracy of statistical linearization. In *New Approaches to Nonlinear Problems in Dynamics*, P. J. Holmes, Ed. SIAM, Philadelphia, PA, 1980, pp. 195–207.

[2] Caughey, T. Equivalent linearization techniques. *J. Acoust. Soc. of America 35*, 11 (November 1963), 1706–1711.

[3] Hagedorn, P. *Non-linear Oscillations.* Clarendon Press, Oxford, England, 1982.

[4] Iwan, W. D., and Patula, E. J. The merit of different error minimization criteria in approximate analysis. *J. Appl. Mech.* (March 1972), 257–262.

[5] Iwan, W. D., and Yang, I.-M. Application of statistical linearization techniques to nonlinear multidegree - of - freedom systems. *J. Appl. Mech.* (June 1972), 545–550.

[6] Kroger, H. Linearization of nonlinear differential equations by means of Cauchy's integral. *J. Math. Physics 26*, 5 (May 1985), 929–940.

[7] McLachlan, N. W. *Ordinary Non-Linear Differential Equations in Engineering and Physcial Sciences.* Oxford University Press, New York, 1950.

[8] Stakgold, I. *Green's Functions and Boundary Value Problems.* John Wiley & Sons, New York, 1979.

132. Maximum Principles

Applicable to Linear ordinary differential equations and linear partial differential equations.

Yields

Upper or lower bounds on the solution.

Idea

By the use of a maximum theorem, we can find bounds on certain types of equations.

Procedure

There are many theorems applicable to specialized equations and boundary conditions, which lead to bounds on the solutions. Maximum principles exist for all types of partial differential equations (hyperbolic, elliptic, and parabolic) as well as for ordinary differential equations. We choose to illustrate two theorems.

Example 1

A theorem from calculus is

> **Theorem** A continuous real-valued function on a bounded closed interval attains its maximum and minimum on the interval.

We will use this theorem to bound the solution to an ordinary differential equation. Consider the equation

$$e^x y'' + x(1-x)y' = (1+x^2)y, \tag{132.1}$$

where $|y(a)| \leq M$ and $|y(b)| \leq M$. We claim that, for all x in the finite interval $[a, b]$, $y(x)$ is bounded in magnitude by M.

Suppose that $y(x)$ exceeded M in some region within the interval $[a, b]$. Then there would be maximum value of y on the interval; say it occurs at the point $x = c$. Because y is a maximum at $x = c$, we require $y'(c) = 0$ and $y''(c) \leq 0$. But this, with equation (132.1), implies that $e^c y''(c) = (1 + c^2)y(c)$. This cannot be correct; the right side is positive, but the left side cannot be. Hence, y does not exceed M in the interval. It can similarly be shown that y cannot be less than $-M$. Hence $|y(x)| \leq M$ for x in the interval.

Example 2

Ames [1, page 181] has the theorem:

Let $u(x)$ be a solution of the ordinary differential equation

$$\begin{aligned} L[u] &= u'' + H(x, u, u') = 0, \qquad \text{for } a < x < b, \\ B_1[u] &= -u'(a)\cos\theta + u(a)\sin\theta = \gamma_1, \\ B_2[u] &= -u'(b)\cos\phi + u(b)\sin\phi = \gamma_2, \end{aligned} \tag{132.2}$$

where $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$, θ and ϕ are not both zero, H, H_u, $H_{u'}$ are all continuous, and $H_u \le 0$. If z_1 and z_2 satisfy

$$\begin{aligned} L[z_1] &\le 0, \qquad \text{for } a < x < b, \\ B_1[z_1] &\ge \gamma_1, \\ B_2[z_1] &\ge \gamma_2, \end{aligned} \tag{132.3}$$

and

$$\begin{aligned} L[z_2] &\ge 0, \qquad \text{for } a < x < b, \\ B_1[z_2] &\le \gamma_1, \\ B_2[z_2] &\le \gamma_2, \end{aligned} \tag{132.4}$$

then we can conclude

$$z_2(x) \le u(x) \le z_1(x), \tag{132.5}$$

for $a < x < b$.

Hence, the solutions to equations (132.3) and (132.4) form bounds on the solution of equation (132.2).

As an illustration of this theorem, suppose we want to approximate the solution of the ordinary differential equation

$$\begin{aligned} u'' - u^3 &= 0, \qquad \text{for } 0 < x < 1, \\ u(0) &= 0, \\ u(1) &= 1. \end{aligned}$$

This is in the form of equation (132.2) with $a = 0$, $b = 1$, $\theta = \phi = \pi/2$, $\gamma_1 = 0$, $\gamma_2 = 1$. We note that $z_1(x) = x$ satisfies equation (132.3) because

$$\begin{aligned} z_1'' - z_1^3 &= -x^3 \le 0, \\ z_1(0) &= 0, \\ z_1(1) &= 1. \end{aligned}$$

We now search for a $z_2(x)$ of the form x^α. Using $z_2(x) = x^\alpha$ in equation (132.4) yields

$$\begin{aligned} \alpha(\alpha-1)x^{\alpha-2} - x^{3\alpha} &\ge 0, \\ B_1[z_2] &= 0 \text{ if } \alpha > 0, \\ B_2[z_2] &= 1. \end{aligned} \tag{132.6.a-c}$$

Because of equation (132.6.b), we restrict our search to $\alpha > 0$. With this assumption, $x^{2(\alpha+1)} \leq 1$ for x between 0 and 1. Hence, equation (132.6.a) will be satisfied if

$$\alpha(\alpha - 1) \geq 1. \tag{132.7}$$

We choose $\alpha = (1+\sqrt{5})/2$, so that equation (132.7) is satisfied. Hence, we can conclude, from equation (132.5)

$$x^{(1+\sqrt{5})/2} \leq u(x) \leq x, \qquad \text{for } 0 < x < 1. \tag{132.8}$$

Notes

1. Any value of α larger than $(1+\sqrt{5})/2$ would also have yielded a bound for $u(x)$ in equation (132.8). The best bound corresponds to the minimal value of α, which was the one used.
2. Some of the "classical" maximum principles are (see Protter and Weinberger [5] or Sperb [7, pages 12–21])
   - If $u(x)$ is non-constant and satisfies $u''+b(x)u' \geq 0$ in an interval, and $b(x)$ is bounded, then $u(x)$ attains its maximum on the boundaries of the interval.
   - If $u(x)$ is non-constant and satisfies $u'' + b(x)u' + h(x) > 0$ in an interval, and $b(x)$ and $h(x)$ are bounded, and $h \leq 0$, then a non-negative minimum of $u(x)$ can occur only on the boundaries of the interval.
   - If the elliptic operator $L[\cdot]$ has bounded coefficients and $u(\mathbf{x})$ satisfies the inequality

     $$L[u] = \sum_{i,j} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} \geq 0$$

     in some bounded domain D, then $u(\mathbf{x})$ cannot assume its maximum at an interior point of D unless $u(\mathbf{x})$ is identically constant.
   - If $L[\cdot]$ is a uniformly elliptic operator with bounded coefficients and $u(\mathbf{x}, t)$ satisfies the inequality

     $$L[u] - \frac{\partial u}{\partial t} = \sum_{i,j} a_{ij}(\mathbf{x}) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(\mathbf{x}) \frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} \geq 0$$

     in $D \times (0, T)$, where D is a bounded domain and $T < \infty$, then $u(\mathbf{x})$ can attain its maximum only for $t = 0$ or on ∂D.
3. A theorem in Durstine and Shaffer [3], applicable to ordinary differential equations and partial differential equations, states

   Let $L[\cdot]$ and $B[\cdot]$ be linear differential operators such that the equation

   $$\begin{aligned} L[u] + \phi(\mathbf{x}) &= 0, && \text{in a domain } D, \\ B_j[u] &= \lambda_j, && \text{for } j = 1, 2, \ldots, q \text{ on } \partial D, \end{aligned}$$

has a unique solution $u(\mathbf{x})$, and the Green's function does not change sign in D. If $w_1(\mathbf{x})$ and $w_2(\mathbf{x})$ satisfy

$$\begin{aligned} L[w_k] + \phi(\mathbf{x}) &= \epsilon_k(\mathbf{x}), && \text{in } D, \\ B_j[w_k] &= \lambda_j, && \text{for } j = 1, 2, \ldots, q \text{ on } \partial D, \end{aligned}$$

and

- ϵ_1/ϵ_2 is continuous,
- ϵ_1 does not change sign in D,
- either $1 > M \geq \frac{\epsilon_2}{\epsilon_1} \geq m$ or $M \geq \frac{\epsilon_2}{\epsilon_1} \geq m > 1$, then

$$w_1 + \frac{w_1 - w_2}{M - 1} < u(\mathbf{x}) < w_1 + \frac{w_1 - w_2}{m - 1}.$$

4. A theorem in Hille [4, pages 87–88], applicable to first order ordinary differential equations, states

   Let $F(x, y)$ and $G(x, y)$ be continuous in a region D (which contains the initial data) and suppose that $F(x, y) < G(x, y)$ everywhere in D. Let $y(x)$ and $z(x)$ be the solutions of

   $$\begin{aligned} y' &= F(x, y), && y(x_0) = y_0, \\ z' &= G(x, y), && z(x_0) = y_0. \end{aligned}$$

   Then, in the region where $y(x)$ and $z(x)$ are defined and continuous

   $$\begin{aligned} z(x) &< y(x), && \text{for } x < x_0, \\ y(x) &< z(x), && \text{for } x_0 < x. \end{aligned}$$

5. A theorem in Ding [2] states

   Consider the equation $\ddot{x} + g(x) = p(t)$ with
   - $p(t)$ is continuous and 2π periodic,
   - $g(x)$ is continuously differentiable and satisfies $\lim_{|x| \to \infty} \frac{g(x)}{x} = \infty$.

   If $p(t)$ is an even function, or if $p(t)$ is odd and $g(x)$ is an even function, then all solutions of this equation are bounded.

6. Other standard boundedness results include
   (a) **Theorem** If $p(x)$ is continuous, of period L, not identically zero, and satisfies $\int_0^L |p(x)|\,dx \leq 4/\pi$ and $\int_0^L p(x)\,dx \geq 0$, then all solutions of $u'' + p(x)u = 0$ are bounded as $x \to \pm\infty$.
   (b) **Theorem** If all solutions of $\mathbf{y}' = A(t)\mathbf{y}$ are bounded (where $\lim_{t \to \infty} \int^t \operatorname{tr}(A)\,dt > -\infty$) and if $\int^\infty |B(t)|\,dt < \infty$, then all solutions of $\mathbf{y}' = (A(t) + B(t))\mathbf{y}$ are bounded.

(c) **Theorem** If all solutions of $\mathbf{y}' = A(t)\mathbf{y}$ are bounded (where A is a periodic matrix) and if $\int^\infty |B(t)|\, dt < \infty$, then all solutions of $\mathbf{y}' = (A(t) + B(t))\mathbf{y}$ are bounded.

(d) **Theorem** If all solutions of $\mathbf{y}' = A\mathbf{y}$ are bounded as $t \to \infty$ (where A is a constant matrix) and if $\int^\infty |B(t)|\, dt < \infty$, then all solutions of $\mathbf{y}' = (A + B(t))\mathbf{y}$ are bounded.

(e) **Theorem** If all solutions to $y'' + f(x)y =$ are bounded and if $\int^\infty |g(x)|\, dx < \infty$, then all solutions of $y'' + (f(x) + g(x))y = 0$ are bounded.

(f) **Theorem** (Comparison of approximate solutions) Consider $x' = f(t, x)$ where f is continuous with Lipschitz constant k. Let u_1 and u_2 be approximate solutions with

$$|u_1'(t) - f(t, u_1(t))| \le \epsilon_1, \qquad |u_2'(t) - f(t, u_2(t))| \le \epsilon_2$$

except where the derivatives are discontinuous. Then, if $|u_1(t_0) - u_2(t_0)| \le \delta$, it follows that

$$|u_1(t) - u_2(t)| \le \delta e^{k|t-t_0|} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right)\left[e^{k|t-t_0|} - 1\right].$$

(g) **Theorem** Let $y(x)$ be any solution of $y'' - f(x)y = 0$ with $f(x)$ positive and continuous in $(0, \infty)$ and $xf(x) \in L(0, \infty)$. Then

$$C \exp\left(-\int_0^x [f(z) + 1]\, dz\right) \le [y(x)]^2 + [y'(x)]^2 \le C \exp\left(\int_0^x [f(z) + 1]\, dz\right),$$

where $C = [y(0)]^2 + [y'(0)]^2$.

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Ding, T. Boundedness of solutions of Duffing's equation. Tech. Rep. 58, University of Minnesota, Minneapolis, Minnesota, 1984. IMA Preprint Series.

[3] Durstine, R. M., and Shaffer, D. H. Determination of upper and lower bounds for solutions to linear differential equations. *Quart. Appl. Math 16*, 3 (1958), 315–317.

[4] Hille, E. *Lectures on Ordinary Differential Equations*. Addison–Wesley Publishing Co., Reading, MA, 1969.

[5] Protter, M. H., and Weinberger, H. F. *Maximum Principles in Differential Equations*. Springer–Verlag, New York, 1984.

[6] Sewell, M. J. *Maximum and Minimum Principles.* Cambridge University Press, New York, 1987.

[7] Sperb, R. *Maximum Principles and Their Applications.* Academic Press, New York, 1981.

[8] Varma, A., and Strieder, W. Approximate solutions of non-linear boundary-value problems. *IMA J. Appl. Mathematics 34* (1985), 165–171.

[9] Walter, W. *Differential and Integral Inequalities.* Springer–Verlag, New York, 1970.

133. McGarvey Iteration Technique

Applicable to First order ordinary differential equations.

Yields

A sequence of approximations to the solution.

Idea

The method consists of generating a sequence of functions by a recurrence relation. The initial function used is arbitrary.

Procedure

Given the first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y), \tag{133.1}$$

we choose $T_0(x, y) = T_0(y)$ to be an arbitrary function of y. Then we define the sequence of functions $\{T_n(x, y)\}$ by the recurrence relation

$$T_n(x, y) = -\int f(x, y) \left[\frac{\partial}{\partial y} T_{n-1}(x, y)\right] dx. \tag{133.2}$$

If we form $S_n(x, y) = \sum_{k=0}^{n} T_k(x, y)$, then $S_n(x, t) =$ constant is an approximate implicit solution to equation (133.1). As n increases, $S_n(x, y)$ will converge to the true solution of equation (133.1) if

$$\lim_{n\to\infty} \frac{\frac{\partial}{\partial y} T_n(x, y)}{\frac{\partial}{\partial y} S_n(x, y)} = 0.$$

Example

Suppose we wish to approximate the solution of the nonlinear ordinary differential equation

$$\begin{aligned} \frac{dy}{dx} &= x + \frac{1}{y}, \\ y(0) &= 4. \end{aligned}$$

In this case, equation (133.2) becomes

$$T_n(x, y) = -\int^x \left(x + \frac{1}{y}\right) \frac{\partial}{\partial y} T_{n-1}(x, y)\, dx. \tag{133.3}$$

We choose $T_0(y) = y$ (recall that T_0 is only a function of y). From equation (133.3), we can calculate

$$T_1(x,y) = -\frac{1}{2}x^2 - \frac{x}{y},$$
$$T_2(x,y) = -\frac{1}{2}\frac{x^2}{y^3} - \frac{1}{3}\frac{x^3}{y^2},$$
$$T_3(x,y) = -\frac{1}{2}\frac{x^3}{y^5} - \frac{13}{24}\frac{x^4}{y^4} - \frac{2}{15}\frac{x^5}{y^3}.$$

Note that we have not used any constants of integration in evaluating the $\{T_n\}$. This part of the analysis is independent of whether or not we choose such constants. We can now calculate $S_3(x,y)$ as

$$\begin{aligned}
&S_3(x,y)\\
&= \sum_{k=0}^{3} T_k(x,y)\\
&= y + \left(-\frac{1}{2}x^2 - \frac{x}{y}\right) + \left(-\frac{1}{2}\frac{x^2}{y^3} - \frac{1}{3}\frac{x^3}{y^2}\right) + \left(-\frac{1}{2}\frac{x^3}{y^5} - \frac{13}{24}\frac{x^4}{y^4} - \frac{2}{15}\frac{x^5}{y^3}\right),\\
&= \frac{120y^6 - 60x^2y^5 - 120xy^4 - 40x^3y^3 - 4x^2(4x^3+15)y^2 - 65x^4y - 60x^3}{120y^5}.
\end{aligned}$$

Now, for the first time, we use the initial condition: $y(0) = 4$. The implicit approximation to the solution of equation (133.1) is then given by

$$S_3(x,y) = S_3(x_0,y_0) = S_3(0,4),$$

or

$$\frac{120y^6 - 60x^2y^5 - 120xy^4 - 40x^3y^3 - 4x^2(4x^3+15)y^2 - 65x^4y - 60x^3}{120y^5} = 4. \tag{133.4}$$

For any value of x, equation (133.4) is a polynomial in y. Thus, for any x, we can solve for y. For this example, it turns out that the difference between the implicit solution given by equation (133.4) and the numerical solution is less than 2% for $0 \le x \le 20$.

Notes

1. The above example is from McGarvey [1].
2. This approximation technique may converge in cases where Picard approximations (see page 618) diverge.
3. For certain classes of equations, error estimates can be obtained for this technique.

Reference

[1] McGarvey, J. F. Approximating the general solution of a differential equation. *SIAM Review 24*, 3 (July 1982), 333–337.

134. Moment Equations: Closure

Applicable to A stochastic differential equation or a Fokker–Planck equation (which is a second order parabolic partial differential equation).

Yields

A system of ordinary differential equations from which different moments may be determined.

Idea

Interpreting the solution of the Fokker–Planck equation as a probability density, ordinary differential equations may sometimes be found for the moments of the random process.

Procedure

The solution of a Fokker–Planck equation is the probability density $P(\mathbf{x}, t)$ of a random process (see page 303). For an N-dimensional random process $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, the Fokker–Planck equation has the form

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(c_i P) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij} P), \tag{134.1}$$

where the coefficients $\{c_i\}$ and $\{a_{ij}\}$ are, in general, functions of t and $\mathbf{x}$. All of the coefficients are determined by the stochastic differential equation that created equation (134.1).

The expectation of a function of $\mathbf{x}$, say $f(\mathbf{x})$, is defined to be the integral of $f(\mathbf{x})$ times $P(\mathbf{x}, t)$, integrated over all values of $\mathbf{x}$. That is,

$$\mathrm{E}\left[f(\mathbf{x}(t))\right] = \int f(\mathbf{x}) P(\mathbf{x}, t)\, d\mathbf{x}.$$

Note that this expectation is a function of t. If equation (134.1) is multiplied by $f(\mathbf{x})$ and integrated over all values of $\mathbf{x}$, there results

$$\frac{d}{dt}\mathrm{E}\left[f(\mathbf{x}(t))\right] = -\sum_{i=1}^{N} \int f(\mathbf{x}) \frac{\partial}{\partial x_i}(c_i P)\, d\mathbf{x} + \sum_{i,j=1}^{N} \int f(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij} P)\, d\mathbf{x}. \tag{134.2}$$

Often, we may be able to integrate the right-hand side of equation (134.2) by parts to obtain an ordinary differential equation for $\mathrm{E}\left[f(\mathbf{x})\right]$.

Example

The system of stochastic differential equations

$$\begin{aligned} \frac{dx}{dt} + x &= z, \qquad & x(0) &= 0, \\ \frac{dz}{dt} + 2z &= N(t), & z(0) &= 1, \end{aligned} \tag{134.3.a-d}$$

where $N(t)$ is "white Gaussian noise" corresponds to the Fokker–Planck equation and initial condition

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{\partial}{\partial x}[(x-z)P] + 2\frac{\partial}{\partial z}[zP] + \frac{\partial^2}{\partial z^2}[P], \\ P(0,x,z) &= \delta(x)\delta(z-1), \end{aligned} \tag{134.4}$$

for the probability density $P(t,x,z)$ (see page 303). Suppose we desire the expected value of $x(t)$:

$$\mathrm{E}\,[x(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xP(t,x,z)\,dx\,dz.$$

Multiplying equation (134.4) by x and integrating from $-\infty$ to ∞ with respect to both x and z produces

$$\frac{d}{dt}\mathrm{E}\,[x(t)] = -\mathrm{E}\,[x(t)] + \mathrm{E}\,[z(t)], \tag{134.5}$$

where we have made the physically reasonable assumptions that $|x|P(t,x,z) \to 0$ as $|x| \to \infty$, and both $|z|P(t,x,z) \to 0$ and $|z|P_z(t,x,z) \to 0$ as $|z| \to \infty$. These assumptions were required to carry out the integrations by parts in the right-hand side of equation (134.2).

Note that equation (134.5) involves the expected value of z. To obtain an equation for $\mathrm{E}\,[z]$, equation (134.4) can be multiplied by z and then integrated to obtain

$$\frac{d}{dt}\mathrm{E}\,[z(t)] = -2\mathrm{E}\,[z(t)]. \tag{134.6}$$

From equation (134.3.b) and equation (134.3.d), the initial conditions for equation (134.5) and equation (134.6) are

$$\mathrm{E}\,[x(0)] = 0, \qquad \mathrm{E}\,[z(0)] = 1. \tag{134.7.a-b}$$

Alternatively, these initial conditions can be obtained directly from the initial conditions in equation (134.4) by taking expectations.

If equation (134.6) is solved with equation (134.7.b), then equation (134.5) can be solved with equation (134.7.a) to determine the expectation of both $x(t)$ and $z(t)$

$$\begin{aligned} \mathrm{E}\,[z(t)] &= e^{-2t}, \\ \mathrm{E}\,[x(t)] &= \frac{1}{3}\left(e^{-t} - e^{-2t}\right). \end{aligned}$$

If the second order moments (i.e., $\{\mathrm{E}\left[x^2(t)\right], \mathrm{E}\left[x(t)z(t)\right], \mathrm{E}\left[z^2(t)\right]\}$) are desired, the equations comparable to equation (134.5) and equation (134.6) are

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix}\mathrm{E}\left[x^2\right]\\ \mathrm{E}\left[xz\right]\\ \mathrm{E}\left[z^2\right]\end{bmatrix} &= \begin{bmatrix}-1 & 2 & 0\\ 0 & -3 & 1\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}\mathrm{E}\left[x^2\right]\\ \mathrm{E}\left[xz\right]\\ \mathrm{E}\left[z^2\right]\end{bmatrix} + \begin{bmatrix}0\\0\\2\end{bmatrix},\\
\begin{bmatrix}\mathrm{E}\left[x^2\right]\\ \mathrm{E}\left[xz\right]\\ \mathrm{E}\left[z^2\right]\end{bmatrix}_{t=0} &= \begin{bmatrix}0\\0\\1\end{bmatrix},
\end{aligned}
\tag{134.8}
$$

where we have dropped the explicit dependence on t for clarity. These equations were obtained by multiplying equation (134.4) by each of x^2, xz, and z^2, and then integrating with respect to x and z.

Notes

1. Another procedure for determining ordinary differential equations for the moments is described on page 572.
2. It is not always the case that the system of ordinary differential equations for the moments will close (i.e., there will be m equations for the m unknowns). For example, the stochastic differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + \epsilon x^3 = N(t),$$

   where $N(t)$ is white noise, corresponds to the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = -\dot{x}\frac{\partial P}{\partial x} + \frac{\partial}{\partial \dot{x}}[(\dot{x} + x + \epsilon x^3)P] + \frac{\partial^2 P}{\partial \dot{x}^2}.$$

   In this case, the equations for the first moments become

$$
\begin{aligned}
\frac{d}{dt}\mathrm{E}\left[x\right] &= \mathrm{E}\left[\dot{x}\right],\\
\frac{d}{dt}\mathrm{E}\left[\dot{x}\right] &= -\mathrm{E}\left[\dot{x}\right] + \mathrm{E}\left[x\right] - \epsilon\mathrm{E}\left[x^3\right].
\end{aligned}
$$

   Therefore, knowledge of $\mathrm{E}\left[x\right]$ requires knowledge of $\mathrm{E}\left[x^3\right]$. In this example, the system of ordinary differential equations that determine $\mathrm{E}\left[x^3\right]$ involves the quantity $\mathrm{E}\left[x^5\right]$, etc. However, if ϵ is small, then perturbation techniques may be used to approximately solve the moment equations.
3. For systems that do not close, two "closing" approximations that are commonly used are (see Boyce [4]):

   - Gaussian closure (also called "cumulant discard")
   - Correlation discard

- In the Gaussian closure technique, a high odd cumulant of the probability density is set to zero. This procedure yields an equation for $\mathrm{E}\left[x^k\right]$ in terms of $\{\mathrm{E}\left[x^j\right] \mid 0 < j < k\}$. Correlation discard is generally used for equations that have "colored noise" forcing terms. In this approximation technique, some high power of the dependent variable in the stochastic differential equation and the "colored noise" is assumed to be uncorrelated.
- Crandall [5] contains a review of non-Gaussian closure techniques. See also Ibrahim *et al.* [6].

4. For determining the moments of random functions defined by partial differential equations, see, for instance, Wan's paper [7].

References

[1] Assaf, S. A., and Zirkle, L. D. Approximate analysis of non-linear stochastic systems. *Int. J. Control 23*, 4 (1976), 477–492.

[2] Bobrik, R. V. Hierarchies of moment equations for the solution of the Schrodinger equation with random potential and their closure. *Teoreticheskaya i Matematicheskaya Fizika 68*, 2 (August 1986), 301–311.

[3] Bover, D. C. C. Moment equation methods for nonlinear stochastic systems. *J. Math. Anal. Appl. 65* (1978), 306–320.

[4] Boyce, W. E. Random eigenvalue problems. In *Probabilistic Methods in Applied Mathematics*, A. T. Bharucha-Reid, Ed. Academic Press, New York, 1968, pp. 1–73.

[5] Crandall, S. H. Non-Gaussian closure techniques for stationary random vibration. *Int. J. Non-Linear Mechanics 20*, 1 (1985), 1–8.

[6] Ibrahim, R. A., Soundararajan, A., and Heo, H. Stochastic response of nonlinear dynamic systems based on a non-Gaussian closure. *J. Appl. Mech. 52* (December 1985), 965–970.

[7] Wan, F. Y. M. Linear partial differential equations with random forcing. *Stud. Appl. Math. 51*, 2 (June 1972), 163–178.

135. Moment Equations: Itô Calculus

Applicable to A set of stochastic differential equations.

Yields

A system of ordinary differential equations from which different moments may be determined.

Idea

Using Itô calculus, a set of ordinary differential equations may be determined that will describe the moments of a random process.

Procedure

In the Itô calculus, there are two different types of differential elements. There are dt terms, which are small; and there are $d\beta$ terms (*Brownian motion* terms), which are random. Brownian motion is the integral of *white noise*; that is, $\beta(t) = \int_0^t n(s)\,ds$, when $n(s)$ is white noise.

We assume the standard scaling: $\mathrm{E}\left[(d\beta)^2\right] = dt$, where $\mathrm{E}[\cdot]$ is the expectation operator (taken over the random variables in the system). The Brownian motion terms also have mean zero: $\mathrm{E}[d\beta] = 0$.

Suppose that $x_1(t)$ and $x_2(t)$ are random processes described by the two stochastic differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_1(t) + b_1(t)n(t), \\ \frac{dx_2}{dt} &= a_2(t) + b_2(t)n(t), \end{aligned} \tag{135.1}$$

or

$$\begin{aligned} dx_1 &= a_1(t)\,dt + b_1(t)\,d\beta, \\ dx_2 &= a_2(t)\,dt + b_2(t)\,d\beta. \end{aligned}$$

Itô's lemma states that

$$d\left(x_1x_2\right) = x_1\,dx_2 + x_2\,dx_1 + b_1b_2\,dt. \tag{135.2}$$

This relation is different from the result in the classical calculus by the inclusion of the last term. This relationship may be used to determine moment equations for a random process.

Example

Given the stochastic differential equation

$$\begin{aligned} &\frac{d^2y}{dt^2} - n(t)y = 0, \\ &y(0) = 1, \quad y'(0) = 0, \end{aligned}$$

where $n(t)$ is white noise, we can define $z = \frac{dy}{dt}$ and so obtain the coupled system of stochastic differential equations

$$\begin{aligned} dy &= z\,dt, \quad y(0) = 1, \\ dz &= y\,d\beta, \quad z(0) = 0. \end{aligned} \tag{135.3}$$

Using Itô's lemma repeatedly on equation (135.3), we can derive the following relations

$$\begin{aligned} d\left(y^L\right) &= L y^{L-1} z\,dt, \\ d\left(z^K\right) &= \frac{K(K-1)}{2} y^2 z^{K-2}\,dt + K z^{K-1} y\,d\beta. \end{aligned} \tag{135.4}$$

If we define the $N+1$ different Nth order moments by

$$G_N^M(t) = \mathrm{E}\left[y^{N-M}(t) z^M(t)\right], \qquad M = 0, 1, \ldots, N,$$

then, from equation (135.4), we obtain the set of coupled ordinary differential equations

$$\frac{dG_N^M}{dt} = (N-M) G_N^{M+1} + \frac{M(M-1)}{2} G_N^{M-2}$$

for $\left\{G_N^M(t)\right\}$. For example, if we choose $N = 2$, then we obtain the system

$$\frac{d}{dt}\begin{bmatrix} G_2^0 \\ G_2^1 \\ G_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} G_2^0 \\ G_2^1 \\ G_2^2 \end{bmatrix}, \tag{135.5}$$

with the initial conditions

$$\begin{bmatrix} G_2^0 \\ G_2^1 \\ G_2^2 \end{bmatrix}_{t=0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of the matrix in equation (135.5) are the three cube roots of two. Hence, each of $\{G_2^0, G_2^1, G_2^2\}$ grows exponentially in time.

Notes

1. Note that the Fokker–Planck equation corresponding to equation (135.3) is

$$\begin{aligned} &\frac{1}{2} y^2 P_{zz} - z P_y = P_t, \\ &P(y, z, 0) = \delta(y-1)\delta(z), \end{aligned}$$

   where $P(y, z, t)$ represents the joint probability density of y and z at time t.
2. The coupled ordinary differential equations that are derived for the moments in this section are identical to the equations obtained by the method described on page 568.

References

[1] Kulkarny, V. A., and White, B. S. Focussing of waves in turbulent inhomogeneous media. *Phys. Fluids 251*, 10 (1982), 1770–1784.

[2] Schuss, Z. *Theory and Applications of Stochastic Differential Equations.* John Wiley & Sons, New York, 1980.

[3] Spigler, R. Monte Carlo-type simulation for solving stochastic ordinary differential equations. *Math. and Computers in Simulation 29* (1987), 243–251.

136. Monge's Method

Applicable to Some nonlinear second order partial differential equations with two independent variables.

Yields

An exact solution.

Idea

Application of some algebraic identities and then the use of equation splitting permits some nonlinear partial differential equations to be solved.

Procedure

Monge's method works for some differential equations of the form

$$R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} = V,$$

or

$$Rr + Ss + Tt = V, \tag{136.1}$$

for $z = z(x, y)$, where, as usual, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, $p = z_x$, $q = z_y$, and $\{R, S, T, V\}$ may be functions of $\{p, q, x, y, z\}$.

First, note that we can write

$$\begin{aligned} dp &= p_x\,dx + p_y\,dy = r\,dx + s\,dy, \\ dq &= q_x\,dx + q_y\,dy = s\,dx + t\,dy. \end{aligned} \tag{136.2.a-b}$$

Solving equation (136.2.a) for r and equation (136.2.b) for t and then using these values in equation (136.1), we obtain

$$[R\,dp\,dy + T\,dq\,dx - V\,dy\,dx] - s\left[R\,(dy)^2 - S\,dy\,dx + T\,(dx)^2\right] = 0. \tag{136.3}$$

By use of equation splitting (see page 520), we look for the simultaneous solutions to

$$\begin{aligned} R\,dp\,dy + T\,dq\,dx - V\,dy\,dx &= 0, \\ R\,(dy)^2 - S\,dy\,dx + T\,(dx)^2 &= 0. \end{aligned} \tag{136.4.a-b}$$

Any solution of equation (136.4) is also a solution of equation (136.3).

Such a solution is called an *intermediate integral* and will depend on an arbitrary constant or function. If we can find two such integrals, say

$$f(x, y, z, p, q) = A, \qquad g(x, y, z, p, q) = B,$$

where A and B are arbitrary constants, then we may be able to solve for $\{p = p(x, y, z),\ q = q(x, y, z)\}$. If we could, then we might be able to integrate the Pfaffian differential equation (see page 384) $dz = p\,dx + q\,dy$ to determine $z = z(x, y)$.

Example

Suppose we have the partial differential equation

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2y \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y} + 6y, \tag{136.5}$$

which can be written as: $y^2 r - 2ys + t = p + 6y$. Therefore, we have $\{R = y^2,\ S = -2y,\ T = 1,\ V = p + 6y\}$. The two equations in equation (136.4) then become

$$\begin{aligned} y^2\, dp\, dy + dq\, dx - (p + 6y)\, dy\, dx &= 0, \\ (y\, dy + dx)^2 &= 0. \end{aligned} \tag{136.6.a-b}$$

Equation (136.6.b) can be integrated to obtain

$$2x + y^2 = A, \tag{136.7}$$

where A is an arbitrary constant. Dividing equation (136.6.a) by dx (or, equivalently, by $(-y\, dy)$ from equation (136.6.b)), we obtain

$$-y\, dp + dq - (p + 6y)\, dy = 0,$$

which can be integrated to yield $-py + q - 3y^2 = \phi(A) = \phi(2x + y^2)$ or

$$y \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + 3y^2 = \phi(2x + y^2), \tag{136.8}$$

where ϕ is an arbitrary function. Equation (136.8) is an intermediate integral and the only one that equation (136.6) has (due to the double root appearing in equation (136.6.b)).

Because we do not have two intermediate integrals, we can not proceed with the derivation in the procedure. However, we can solve equation (136.8) directly to obtain a solution of equation (136.5). Because equation (136.8) is quasilinear, the method of characteristics (see page 432) may be used. The subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{-3y^2 + \phi(2x + y^2)}. \tag{136.9}$$

From the first equality in equation (136.9), we recover the integral in equation (136.7). Using equation (136.7) in the second equality in equation (136.9) yields

$$\frac{dy}{-1} = \frac{dz}{-3y^2 + \phi(A)},$$

with the solution $z - y^3 + y\phi(2x + y^2) = B$, where B is another arbitrary constant. Hence, a general integral of equation (136.5) is

$$\Phi\left(z - y^3 + y\phi(2x + y^2), 2x + y^2\right) = 0.$$

This leads to a general solution of equation (136.5)

$$z = y^3 - y\phi(2x + y^2) + \psi(2z + y^2), \tag{136.10}$$

where ϕ and ψ are arbitrary functions of their arguments.

Notes

1. Because equation splitting was used in going from equation (136.3) to equation (136.4), the solution obtained in equation (136.10) is not the most general solution.
2. See Ames [1, pages 60–65], Forsyth [2, Volume 6, pages 202–208], Piaggio [3, pages 181–187], and Sneddon [4, pages 131–135].

References

[1] Ames, W. F. Ad hoc exact techniques for nonlinear partial differential equations. In *Nonlinear Partial Differential Equations in Engineering*, W. F. Ames, Ed. Academic Press, New York, 1967.

[2] Forsyth, A. R. *Theory of Differential Equations*. Dover Publications, Inc., New York, 1959.

[3] Piaggio, H. T. H. *An Elementary Treatise on Differential Equations and Their Applications*. G. Bell & Sons, Ltd, London, England, 1926.

[4] Sneddon, I. N. *Elements of Partial Differential Equations*. McGraw–Hill Book Company, New York, 1957.

137. Newton's Method

Applicable to Ordinary and partial differential equations.

Yields

A sequence of approximations to the solution.

Idea

When a Newton iteration is applied to a nonlinear differential equation, each step of the iteration requires that a linear differential equation be solved.

Procedure

We illustrate the general procedure on an ordinary differential equation. Suppose we wish to approximate the solution to the first order ordinary differential equation

$$\begin{aligned} G(y', y, x) &= 0, \\ y(0) &= y_0, \end{aligned} \tag{137.1}$$

for $y(x)$ when $G(y', y, x)$ is a nonlinear function.

If an approximate solution of equation (137.1), say $y_k(x)$, is known, then $G(y', y, x)$ could be expanded about $y_k(x)$ to obtain

$$G(y', y, x) \simeq G(y'_k, y_k, x) + G_y(y'_k, y_k, x)(y - y_k) + G_{y'}(y'_k, y_k, x)(y' - y'_k) \tag{137.2}$$

to leading order. For the solution to equation (137.1), $G(y', y, x) = 0$, and so equation (137.2) becomes

$$(y' - y'_k)G_{y'}(y'_k, y_k, x) + (y - y_k)G_y(y'_k, y_k, x) \simeq -G(y'_k, y_k, x).$$

Therefore, if the linear ordinary differential equation

$$\begin{aligned} e'_k G_{y'} + e_k G_y &= -G, \\ e_k(0) &= 0, \end{aligned} \tag{137.3}$$

is solved for the "correction term" $e_k(x)$, then defining

$$y_{k+1}(x) = y_k(x) + e_k(x)$$

should yield a better approximation, $y_{k+1}(x)$, to $y(x)$.

Equation (137.3) can be solved exactly by the use of integrating factors (see page 356). However, it needs to be solved only approximately because the higher order approximations (i.e., y_{k+2}, $y_{k+3}, \ldots$) will correct errors made in solving equation (137.3).

Special Case

In the special case that the original equation is linear in y' and hence of the form

$$G(y', y, x) = y' - f(x, y) = 0,$$

then the definition of y_{k+1} may be succinctly represented as

$$\begin{aligned} y'_{k+1} - f_y(x, y_k(x)) y_{k+1} &= f(x, y_k(x)) - f_y(x, y_k(x)) y_k(x), \\ y_{k+1}(0) &= y_0. \end{aligned} \tag{137.4}$$

Example

Suppose we are looking for an approximation, near $x = 0$, of the solution to the nonlinear ordinary differential equation

$$\begin{aligned} y' + y^3 &= 0, \\ y(0) &= 1, \end{aligned}$$

which has the known exact solution

$$\begin{aligned} y(x) &= (1+2x)^{-1/2} \\ &= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \frac{35}{8}x^4 - \frac{63}{8}x^5 + \cdots . \end{aligned}$$

For this problem we recognize that $f(x, y) = -y^3$ and so equation (137.4) becomes

$$\begin{aligned} y'_{k+1} + 3y_k^2 y_{k+1} &= 2y_k^3, \\ y_{k+1}(0) &= 1. \end{aligned} \tag{137.5}$$

If we start with $y_0 = y(0) = 1$, then, from equation (137.5)

$$\begin{aligned} y'_1 + 3y_1 &= 2, \\ y_1(0) &= 1, \end{aligned}$$

with the solution

$$\begin{aligned} y_1(x) &= \frac{1}{3}\left(2 + e^{-3x}\right) \\ &= 1 - x + \frac{3}{2}x^2 - \frac{3}{2}x^3 + \cdots . \end{aligned}$$

If we use the approximation $y_1 \simeq 1 - x$, then the equation for y_2 (from equation (137.5), with $k = 1$) is

$$\begin{aligned} y'_2 + 3(1-x)^2 y_2 &= 2(1-x)^3, \\ y_2(0) &= 0, \end{aligned}$$

with the solution

$$\begin{aligned} y_2(x) &= -2e^{-x^3+3x^2-3x}\left(\int_0^x e^{x^3-3x^2+3x}(x-1)^3\,dx + 1\right) \\ &= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \frac{35}{8}x^4 - \frac{261}{40}x^5 + \cdots . \end{aligned}$$

We see then that $y_1(x)$ has the first 3 terms correct, whereas $y_2(x)$ (which used only the first order information in $y_1(x)$) has the first 5 terms correct.

Notes

1. For symbolic manipulation of the formulae appearing above, Geddes [4] discusses the number of correct terms at each step.
2. Most often, this iterative method will be implemented numerically and not performed analytically. This is because, by hand, it is often easier to find a Taylor series solution directly (see page 632) than to use Newton iterates. Rice and Boisvert [7, pages 101–111] have a numerical example of using Newton's method to solve an elliptic equation.
3. Error estimates for Newton's method (applied to first order equations) can be found in Mikhlin and Smolitskiy [6, pages 12–16].
4. When Newton's method is numerically applied to nonlinear boundary value problems, the method is often called *quasilinearization*. This is the same algorithm that is obtained when multiple shooting is used (see page 706), and the number of rays becomes very large. See Bellman and Kalaba [3] or Stoer and Bulirsch [8, pages 498–502] for details.
5. Geddes [4] showed that the number of correct coefficients in a power series solution obtained by this method, when applied to an explicit first order nonlinear ordinary differential equation, more than doubles at each step.
6. See also Ascher *al.* [1, pages 52–55].

References

[1] Ascher, U. M., Mattheij, R. M. M., and Russel, R. D. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1988.

[2] Barkatou, M. A. Rational Newton algorithm for computing formal solution of linear differential equations. No. 358 in Lecture Notes in Computer Science. Springer–Verlag, New York, 1989, pp. 183–195.

[3] Bellman, R. E., and Kalaba, R. *Quasilinearization and Nonlinear Boundary-Value Problems*. American Elsevier Publishing Company, New York, 1965.

[4] Geddes, K. O. Convergence behavior of the Newton iteration for first order differential equations. In *Symbolic and Algebraic Computation*, E. W. Ng, Ed., EUROSAM 79. Springer–Verlag, New York, 1979, pp. 189–199.

[5] Guenther, R. B., and Lee, J. W. Convergence of the Newton–Raphson method for boundary value problems of ordinary differential equations. In *Computation Solutions of Nonlinear Systems of Equations*. Amer. Math. Soc., Providence, RI, 1990, pp. 257–264.

[6] Mikhlin, S. G., and Smolitskiy, K. L. *Approximate Methods for Solutions of Differential and Integral Equations*. American Elsevier Publishing Company, New York, 1967.

[7] Rice, J. R., and Boisvert, R. F. *Solving Elliptic Problems Using ELLPACK*. Springer–Verlag, New York, 1985.

[8] Stoer, J., and Bulirsch, R. *Introduction to Numerical Analysis*. Springer–Verlag, New York, 1976. Translated by R. Bartels, W. Gautschi and C. Witzgall.

138. Padé Approximants

Applicable to Any type of function (whether or not it comes from a differential equation).

Yields

An approximation formula generally valid over an interval, and, often, information about whether singularities exist.

Idea

A Taylor series can be manipulated to produce information about the existence of singularities.

Procedure

When a power series representation of a function diverges, it indicates the inability of the power series to approximate the function in a certain region. A theorem of complex analysis states that if a Taylor series of a function diverges, then that function has singularities in the complex plane. A Padé approximant is a ratio of polynomials that contains the same information that a truncated power series does. Because the polynomial in the denominator may have roots in the region of interest, the Padé approximant may accurately indicate the presence of singularities.

Suppose we have found the kth order Taylor series solution to a differential equation (see page 632)

$$y_k(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k. \tag{138.1}$$

The (N, M) Padé approximant, $P_M^N(x)$, is a ratio of polynomials, with the polynomial in the numerator having degree N and the polynomial in the denominator having degree M:

$$P_M^N(x) = \frac{B_0 + B_1 x + \cdots + B_N x^N}{A_0 + A_1 x + \cdots + A_M x^M}, \tag{138.2}$$

with $N + M + 1 = k$. Without loss of generality, we take $A_0 = 1$. The remaining $N + M + 1$ coefficients $\{A_1, A_2, \dots, A_N, B_0, B_1, \dots, B_M\}$ are chosen so that the first $N + M + 1$ terms in the Taylor series expansion of $P_M^N(x)$ match the first $N + M + 1$ terms of the Taylor series in equation (138.1).

Usually we consider only the convergence of the Padé sequence $\{P_0^J(x), P_1^{J+1}(x), P_2^{J+2}(x), \dots\}$ having $N = M + J$ and J held constant while $M \to \infty$. The special sequence with $J = 0$ is called the *diagonal sequence*.

Example 1

Suppose we wish to approximate the solution of the ordinary differential equation

$$y' = y^2, \qquad y(0) = 1. \tag{138.3}$$

Because equation (138.3) is separable (see page 401), the solution to equation (138.3) can be found to be $y(x) = 1/(1-x)$. If we tried to find the Taylor series of $y(x)$ directly from equation (138.3), we would obtain

$$y(x) = 1 + x + x^2 + x^3 + x^4 + \cdots . \tag{138.4}$$

This geometric series is convergent, of course, only for $|x| < 1$. The solution has a singularity at $x = 1$, but this fact is not readily apparent from the expansion in equation (138.4).

The diagonal sequence of Padé approximants corresponding to equation (138.4) is

$$\begin{aligned} P_1^1(x) &= \frac{1}{1-x}, \\ P_2^2(x) &= \frac{1}{1-x}, \\ P_3^3(x) &= \frac{1}{1-x}. \end{aligned}$$

Therefore, the diagonal sequence of Padé approximants recovers the *exact* solution to the differential equation from only a few terms in the Taylor series. Of course, this is an exceptional example.

Example 2

Suppose we wish to approximate the solution of the ordinary differential equation

$$y' = 1 + y^2, \qquad y(0) = 0. \tag{138.5}$$

Because equation (138.5) is separable, the solution to equation (138.5) can be found to be $y(x) = \tan x$. If we tried to find the Taylor series of $y(x)$ directly from equation (138.5), we would find

$$y(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots . \tag{138.6}$$

Note that the exact solution has singularities at $x = \pm(2n+1)\pi/2$, whereas the Taylor series approximation does not appear to show this behavior. Using equation (138.6), we can compute the first few elements of the diagonal sequence

$$P_2^2(x) = \frac{3x}{3 - x^2},$$
$$P_3^3(x) = \frac{x(x^2 - 15)}{3(2x^2 - 5)},$$
$$P_4^4(x) = \frac{5x(21 - 2x^2)}{x^4 - 45x^2 + 105}.$$

Note that these Padé approximants have singularities where the denominator vanishes:

- For $P_2^2(x)$, these singularities are at $x \simeq \pm 1.7$.
- For $P_3^3(x)$, these singularities are at $x \simeq \pm 1.58$.
- For $P_4^4(x)$, these singularities are at $x \simeq \pm 1.5712$, and $x \simeq \pm 6.5$.

We observe that these Padé approximants are attempting to recover the singularities of the exact solution at $x = \pm\pi/2$ and $x = \pm 3\pi/2$. Because the Padé approximants have these singularities, they produce an accurate numerical approximation of the exact solution over a wide range of values.

Notes

1. Padé approximants are not always better than a Taylor series representation. In fact, it may happen that the Padé approximants diverge while the Taylor series converges. However, it often happens that $P_M^N(x)$ converges (as $N, M \to \infty$) to the true solution of the differential equation, even when the Taylor series solution diverges!
2. Padé approximants are also called *rational function approximations*.
3. Prendergast [7] proposes a technique to find the Padé approximants for the solution of a nonlinear differential equation without first finding the Taylor series. Martin and Zamudio–Cristi [6] address the same issue but for a smaller class of equations.
4. In Bender and Orszag [1, pages 383–410] is a discussion of computational techniques for computing Padé approximants numerically.
5. A *two-point Padé approximant* is one that utilizes Taylor series information about two different points. Often these points are chosen to be zero and infinity. For two-point Padé approximants the coefficients in equation (138.2) are chosen so that both Taylor series will be matched. See Bender and Orszag [1] or Magnus [5] for details.
6. Many symbolic computer languages have a function that finds Padé approximants analytically when a Taylor series is input. See Czapor and Geddes [3].
7. The Bulirsch–Stoer method is a numerical method for solving first order ordinary differential equations using Padé approximants, Richardson extrapolation, and the modified midpoint rule. See Press *et al.* [8, pages 563–568] for details.

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Cuyt, A. *Pade Approximants for Operators: Theory and Applications*. No. 1065. Springer–Verlag, New York, 1984.

[3] Czapor, S. R., and Geddes, K. O. A comparison of algorithms for the symbolic computation of Pade approximants. In *EUROSAM '84*, in J. Fitch, Ed. Springer–Verlag, New York, 1984, pp. 248–259.

[4] Geddes, K. O. Symbolic computation of Pade approximants. *ACM Trans. Math. Software 5*, 2 (June 1979), 218–233.

[5] Magnus, A. On the structure of the two-point Pade table. In *Analytic Theory of Continued Fractions*, W. B. Jones, W. J. Thron, and H. Waadeland, Eds., no. 932 in Lecture Notes in Mathematics. Springer–Verlag, New York, 1982, pp. 176–193.

[6] Martin, P., and Zamudio-Cristi, J. Fractional approximation for first-order differential equations with polynomial coefficients—application to zzzref31refzzz. *J. Math. Physics 23*, 12 (Dec 1982), 2276–2280.

[7] Prendergast, K. H. Rational approximation for non-linear ordinary differential equations. In *The Riemann Problem, Complete Integrability and Arithmetic Applications*, D. Chudnovsky and G. Chudnovsky, Eds., no. 925 in Lecture Notes in Mathematics. Springer–Verlag, New York, 1982.

[8] Press, W. H., Flannery, B. P., Teukolsky, S., and Vetterling, W. T. *Numerical Recipes*. Cambridge University Press, New York, 1986.

[9] Reusch, M. F., Ratzan, L., Pomphrey, N., , and Park, W. Diagonal Pade approximations for initial value problems. *SIAM J. Sci. Stat. Comput. 9*, 5 (September 1988), 829–838.

[10] Williamson, R. A. Pade approximations in the numerical solution of hyperbolic differential equations. In *Pade Approximation and Its Applications, Bad Honnef 1983*, H. Werner and H. J. Bunger, Eds., no. 1071 in Lecture Notes in Mathematics. Springer–Verlag, New York, 1984, pp. 252–264.

139. Perturbation Method: Method of Averaging

Applicable to Nonlinear differential equations that have a periodic solution and a small parameter.

Yields

An approximation to the solution, valid over an entire period.

Idea

Write the solution of a given differential equation as a function with slowly varying parts. Then, average those slowly varying parts over a complete cycle.

Procedure

We illustrate the method on a perturbed harmonic oscillator. Suppose we have the equation

$$\frac{d^2y}{dt^2} + y + \epsilon f\left(y, \frac{dy}{dt}\right) = 0. \tag{139.1}$$

Note that, when $\epsilon = 0$, equation (139.1) is a harmonic oscillator. The solution to equation (139.1), when $\epsilon = 0$, is therefore: $y(t) = A\cos(t + \Theta)$ (where A and Θ are constants). If ϵ is very small, we might expect a similar "looking" solution, so we assume that the solution to equation (139.1) is given by

$$y(t) = a\cos(t + \theta), \tag{139.2}$$

where $a(t)$ and $\theta(t)$ are "slowly varying" (another expression often used is "nearly constant"). Differentiating equation (139.2) with respect to t yields

$$\frac{dy}{dt} = -a\sin(t+\theta) + \frac{da}{dt}\cos(t+\theta) - a\frac{d\theta}{dt}\cos(t+\theta). \tag{139.3}$$

If a and θ are "slowly varying," then da/dt and $d\theta/dt$ will be "small" compared to a. Hence, we set the derivative of $y(t)$ to be

$$\frac{dy}{dt} = -a\sin(t+\theta). \tag{139.4}$$

Comparing equation (139.3) to equation (139.4), it is clear that we have made the assumption

$$\frac{da}{dt}\cos(t+\theta) - a\frac{d\theta}{dt}\cos(t+\theta) = 0. \tag{139.5}$$

This gives one equation relating the two unknowns, $a(t)$ and $\theta(t)$. Differentiating equation (139.4) and using equation (139.4) and equation (139.1) results in the expression

$$-\frac{da}{dt}\sin(t+\theta) - a\frac{d\theta}{dt}\cos(t+\theta) = \epsilon f\left(a\cos(t+\theta), -a\sin(t+\theta)\right). \tag{139.6}$$

The two equations in equation (139.5) and equation (139.6) can be solved to yield the relations

$$\begin{aligned} \frac{da}{dt} &= \epsilon f\left(a\cos(t+\theta), -a\sin(t+\theta)\right)\sin(t+\theta), \\ \frac{d\theta}{dt} &= \frac{\epsilon}{a} f\left(a\cos(t+\theta), -a\sin(t+\theta)\right)\cos(t+\theta). \end{aligned} \tag{139.7}$$

The equations in equation (139.2) and equation (139.7) are still exact. The change of variables from $\{y, y'\}$ to $\{a, \theta\}$ (by use of equation (139.2) and equation (139.5)) has been carried out without any approximation being made. The assumptions made have been motivated by the smallness of ϵ, but the system is still exact.

Now we use the "slowly varying" feature of a and θ to make the required approximation. If a and θ are "slowly varying," then the values of da/dt and $d\theta/dt$ should not change much over a single period of the solution. Hence, if we replace the right-hand sides of equation (139.7) by their averages over one period, then the solutions for $a(t)$ and $\theta(t)$ should not be changed very much. Therefore, we approximate the solution of equation (139.7) by the solution of

$$\frac{da}{dt} = \epsilon F(a), \qquad \frac{d\theta}{dt} = \frac{\epsilon}{a} G(a), \tag{139.8}$$

where

$$\begin{aligned} F(a) &= \frac{1}{2\pi}\int_0^{2\pi} f\left(a\cos(t+\theta), -a\sin(t+\theta)\right)\sin(t+\theta)\,d\theta, \\ G(a) &= \frac{1}{2\pi}\int_0^{2\pi} f\left(a\cos(t+\theta), -a\sin(t+\theta)\right)\cos(t+\theta)\,d\theta. \end{aligned} \tag{139.9}$$

The prescription is to evaluate equation (139.9) and then to solve equation (139.8). Knowing $a(t)$ and $\theta(t)$, we can evaluate equation (139.2) and so recover an approximation to $y(t)$.

Example 1

For the Van de Pol oscillator

$$\frac{d^2y}{dt^2} + y + \epsilon(y^2 - 1)\frac{dy}{dt} = 0,$$

we identify $f(y, y') = (y^2 - 1)y'$. Evaluating equation (139.9) with this f results in $F(a) = \frac{a}{2} - \frac{a^3}{8}$ and $G(a) = 0$. This, in turn, allows us to solve equation (139.8). We find

$$a^2(t) = \frac{4}{1 + \left(\frac{4}{a_0^2} - 1\right) e^{-\epsilon t}}, \qquad \theta(t) = \theta_0,$$

where $a_0 = a(0), \theta_0 = \theta(0)$. Note that as $t \to \infty$ the approximation in equation (139.2) tends to a sinusoidally varying function of magnitude two.

Example 2

For Duffing's equation

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0,$$

we identify $f(y, y') = y^3$. Evaluating equation (139.9) with this f results in $F(a) = 0$ and $G(a) = \frac{3}{4}a^3$. This, in turn, allows us to solve equation (139.8). We find

$$a(t) = a_0, \qquad \theta(t) = \theta_0 + \frac{3}{8}a_0^2 \epsilon t.$$

Notes

1. This method is also called the method of Krylov–Bogoliubov–Mitropolski.
2. The solution of $\dot{x} = \epsilon f(t, x, \epsilon)$ (when f has period 2π in t) can be approximated by averaging. The solution by kth order averaging is always valid with error $O(\epsilon^k)$ on time intervals of length $O(1/\epsilon)$. Accuracy is improved in two cases (see Murdock and Wang [5]):
   - If the average of f vanishes (i.e., $\frac{1}{2\pi}\int_0^{2\pi} f(t, x, \epsilon)\, dt = 0$), then the kth order averaging approximation is valid with error $O(\epsilon^{k-1})$ for intervals of length $O(1/\epsilon^2)$.
   - If the solutions approach a hyperbolic sink (exponentially attracting rest point), then the kth order averaging approxmimation has error $O(e^k)$ for all future time. This is known as the Sanchez–Palencia theorem.
3. There are many ways in which averaging techniques can be applied to differential equations; we have illustrated only one technique. Another useful technique is the method of averaged Lagrangians (see Whitham [9]). This technique is applied by finding the Lagrangian corresponding to a given differential equation (see page 61), assuming an expansion of the Lagrangian that contains slowly varying functions and a small parameter, and, at each order of the small parameter, solving the differential equation corresponding to that term of the Lagrangian.

4. Macsyma [4] has a package (`avg_pode`) that implements the method of averaging for ordinary differential equations.
5. See also Kevorkian and Cole [3, pages 279–287], Nayfeh [6, Chapter 5, pages 159–227], and Rand and Armbruster [7, Chapter 5, pages 107–131].

References

[1] Golec, J., and Ladde, G. Euler-type approximation for systems of stochastic differential equations. *J. Appl. Math. Simulation 2*, 4 (1989), 239–249.

[2] Gromyak, M. I. Justification of a scheme for averaging of hyperbolic systems with fast and slow variables. A mixed problem. *Ukrain. Mat. Zh. 38*, 5 (1986), 575–582.

[3] Kevorkian, J., and Cole, J. D. *Perturbation Methods in Applied Mathematics*. Springer–Verlag, New York, 1981.

[4] Macsyma. *Mathematics Reference Manual*. Macsyma, Arlington, MA, 1993.

[5] Murdock, J., and Wang, L.-C. Validity of the multiple scale method for very long intervals. *Z angew Math Phys 47* (1996), 760–789.

[6] Nayfeh, A. H. *Perturbation Methods*. John Wiley & Sons, New York, 1973.

[7] Rand, R. H., and Armbruster, D. *Perturbation Methods, Bifurcation Theory and Computer Algebra*. No. 65 in Applied Mathematical Sciences. Springer–Verlag, New York, 1987.

[8] Sanders, J. A., and Verhulst, F. *Averaging Methods in Nonlinear Dynamical Systems*. Springer–Verlag, New York, 1985.

[9] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

140. Perturbation Method: Boundary Layer Method

Applicable to Differential equations with a small parameter present for which regular perturbation series are inadequate.

Yields

This singular perturbation technique yields an expansion of the solution in terms of the small parameter.

Idea

If a regular perturbation series cannot match all the boundary conditions in a differential equation, there may be one or more regions where the solution is rapidly varying.

Procedure

Given a differential equation with a small parameter ϵ, attempt to find a solution in the form of a regular perturbation series (see page 610). Call this the "outer" solution. If the "outer" solution cannot match all of the initial conditions or boundary conditions, then attempt to place "boundary layers" (regions of rapid variation) near one or more of the boundaries.

Inside of each boundary layer, the solution will vary smoothly (in a stretched variable) from the value of a "outer" solution to the value on the boundary. If multiple "outer" solutions exist, then there may be internal boundary layers (called *shocks*). These internal boundary layers will change the solution smoothly from one "outer" solution to another.

Example

Consider the constant coefficient ordinary differential equation

$$\begin{aligned} &\epsilon \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0, \\ &y(0) = 4, \quad y(1) = 5, \end{aligned} \tag{140.1}$$

where ϵ is a number much smaller than one. Initially, we look for an "outer" solution in the form of a regular perturbation series (see page 610)

$$y = y_{\text{outer}} = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots . \tag{140.2}$$

Using equation (140.2) in equation (140.1) and setting the coefficients of different powers of ϵ to zero produces the sequence of equations

$$\begin{aligned} \frac{dy_0}{dx} + y_0 &= 0, \\ \frac{dy_1}{dx} + y_1 &= -\frac{d^2 y_0}{dx^2}, \\ &\vdots \end{aligned} \tag{140.3.a-b}$$

with the boundary conditions

$$\begin{aligned} y_0(0) &= 4, \qquad y_0(1) = 5, \\ y_i(0) &= 0, \qquad y_i(1) = 0, \quad \text{for } i = 1, 2, 3, \ldots. \end{aligned} \tag{140.4.a-b}$$

The most general solution of equation (140.3.a) is

$$y_0(x) = Ce^{-x} \tag{140.5}$$

for some constant C. This solution cannot satisfy both of the boundary conditions in equation (140.4.a); so, we suspect the existence of a boundary layer.

First, we search for a boundary layer near $x = 0$. If it is not possible to place one there, then we would attempt to place one near the other boundary, at $x = 1$. Because a change of order one is expected to take place in a thin x region, we scale x so that the width of the thin region becomes of order one (in the new variable $\widetilde{x}$)

$$\widetilde{x} = \frac{x}{\epsilon}. \tag{140.6}$$

(In other problems, the scaling may be different; it may be that $\widetilde{x} = x/\epsilon^\beta$, where β is an integer or a fraction.) Using the new independent variable as defined by equation (140.6), the equation (140.1) may be written as

$$\frac{d^2 y}{d\widetilde{x}^2} + \frac{dy}{d\widetilde{x}} + \epsilon y = 0. \tag{140.7}$$

The solution of this equation is called the "inner" solution. If we search for a regular perturbation series solution to equation (140.7), in the form of equation (140.2), then the sequence of equations begins

$$\begin{aligned} \frac{d^2 y_0}{d\widetilde{x}^2} + \frac{dy_0}{d\widetilde{x}} &= 0, \\ \frac{d^2 y_1}{d\widetilde{x}^2} + \frac{dy_1}{d\widetilde{x}} &= -y_0, \\ &\vdots \end{aligned} \tag{140.8.a-b}$$

Using the general solution to equation (140.8.a), we have

$$\begin{aligned} y_{\text{inner}}(\widetilde{x}) &= y_0(\widetilde{x}) + O(\epsilon) \\ &= D + Ee^{-\widetilde{x}} + O(\epsilon), \end{aligned} \tag{140.9}$$

where D and E are constants.

Because we have assumed that the boundary layer is at $x = 0$, the "inner" solution in equation (140.9) must satisfy the boundary condition at $x = 0$ (i.e., $y_{\text{inner}}(0) = 4$). The "outer" solution does not extend to $x = 0$ (because the boundary layer is present) but does extend to $x = 1$. Hence, the solution in equation (140.5) must satisfy the boundary condition at $x = 1$; that is, $y_{\text{outer}}(1) = 5$. Evaluating equation (140.5) and equation (140.9) at their respective boundaries results in

$$\begin{aligned} y_{\text{outer}}(x) &= 5e^{1-x} + O(\epsilon), \\ y_{\text{inner}}(\widetilde{x}) &= (4 - E) + Ee^{-\widetilde{x}} + O(\epsilon). \end{aligned} \tag{140.10}$$

To determine the constant E, we need a "matching principle." The "matching principle" is needed to ensure continuity of the solution as it changes from y_{inner} to y_{outer}. Because the transition occurs for x just larger than zero, we require

$$\lim_{x \to 0^+} y_{\text{inner}}(x) = \lim_{x \to 0^+} y_{\text{outer}}(x),$$

which we define to be y_{match}. Writing y_{inner} in terms of $\widetilde{x}$ and assuming that ϵ is arbitrarily small, this statement can be written as

$$\lim_{\widetilde{x} \to \infty} y_{\text{inner}}(\widetilde{x}) = \lim_{x \to 0^+} y_{\text{outer}}(x). \tag{140.11}$$

Sometimes this is called an *intermediate expansion* because the matching occurs on an intermediate scale. Using the solutions from equation (140.10) in equation (140.11), we determine that $E = 4 - 5e$.

Finally, we need to combine y_{inner} and y_{outer} together to obtain a uniformly valid approximation, y_{uniform}, over the entire interval: $x \in [0, 1]$. The uniform approximation is defined to be the sum of y_{inner} plus y_{outer}, minus the overlap value. That is,

$$\begin{aligned} y_{\text{uniform}} &= y_{\text{inner}} + y_{\text{outer}} - y_{\text{match}} \\ &= \left[5e + (4 - 5e)e^{-\widetilde{x}}\right] + \left[5e^{1-x}\right] - [5e] + O(\epsilon) \\ &= (4 - 5e)e^{-\widetilde{x}} + 5e^{1-x} + O(\epsilon) \\ &= (4 - 5e)e^{-x/\epsilon} + 5e^{1-x} + O(\epsilon). \end{aligned} \tag{140.12}$$

Figure 140.1 has graphs of the exact solution of equation (140.1) and the approximate solution given by equation (140.12) for $\epsilon = 0.1$.

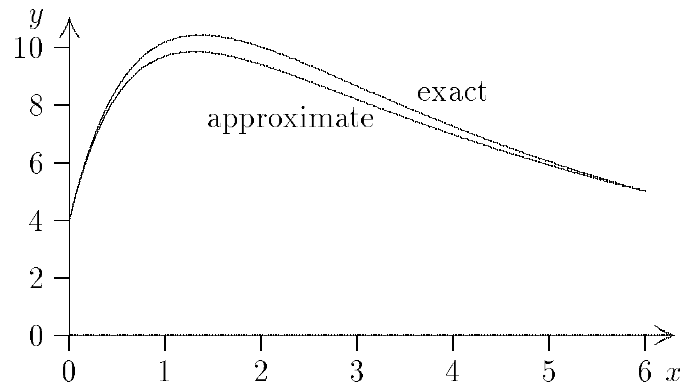

Figure 140.1: A comparison of the exact solution of equation (140.1) with the approximation in equation (140.12) for $\epsilon = 0.1$.

Notes

1. The exact solution to equation (140.1) is given by

$$y(x) = \frac{1}{e^{r_2} - e^{r_1}} \left[\left(4e^{r_2} - 5\right) e^{r_1 x} + \left(5 - 4e^{r_1}\right) e^{r_2 x} \right], \tag{140.13}$$

   where $r_1 = \left(-1 - \sqrt{1 - 4\epsilon}\right)/2\epsilon$ and $r_2 = \left(-1 + \sqrt{1 - 4\epsilon}\right)/2\epsilon$. For small values of ϵ, $r_1 \approx -1/\epsilon$ and $r_2 \approx -1$. Using these approximations in equation (140.13) and expanding everything to leading order, results in equation (140.12).

2. If the example were carried to second order in ϵ, then we would have found

$$\begin{aligned} y_{\text{outer}} &= \left[5e^{1-x}\right] + \epsilon \left[5(1-x)e^{1-x}\right] + O(\epsilon^2), \\ y_{\text{inner}} &= \left[5e + (4-5e)e^{-\widetilde{x}}\right] + \epsilon \left[5e\left(1 - e^{-\widetilde{x}}\right) - 5e\widetilde{x} + (4-5e)\widetilde{x}e^{-\widetilde{x}}\right] \\ &\quad + O(\epsilon^2), \\ y_{\text{uniform}} &= \left[5e^{1-x} + (4-5e)(1+x)e^{-x/\epsilon}\right] + \epsilon\left[5e^{1-x}(1-x) - e^{1-x/\epsilon}\right] \\ &\quad + O(\epsilon^2). \end{aligned}$$

3. In the example, we could have expected trouble initially. The original equation is of second order and so needs two boundary conditions. But the first order term in the regular perturbation series, equation (140.3.a), is a differential equation of first order, so it would be unlikely to match the two boundary conditions.

4. If it were not possible to match the "inner" and "outer" solutions in equation (140.11), then we would have tried to put a boundary layer

at $x = 1$. To do this, we scale x so that it has a large variation near $x = 1$, say $\widehat{x} = (1 - x)/\epsilon$. Using this new distance scale, the leading order terms in the "outer" and "inner" solutions would have the form of equation (140.5) and equation (140.9). Now, however, the outer solution would extend to $x = 0$ (so that $y_{\text{outer}} = 4e^{-x}$), whereas the inner solution would extend to $x = 1$ (so that $y_{\text{inner}} = (5-E)+Ee^{-\widehat{x}}$). At this point, we find that we cannot perform the necessary matching. We have $\lim_{x\to 1^-} y_{\text{outer}}(x) = 4e^{-1}$, but

$$\lim_{x\to 1^-} y_{\text{inner}}(x) = \lim_{\widehat{x}\to\infty} y_{\text{inner}}(\widehat{x}) = \begin{cases} 5, & \text{if } E = 0, \\ \infty, & \text{if } E > 0, \\ -\infty, & \text{if } E < 0. \end{cases}$$

We conclude that there is no boundary layer near $x = 1$, at least with the scaling $\widehat{x} = (1 - x)/\epsilon$.

5. Sometimes a boundary layer can appear in the middle of the region of interest. As an example of a "shock" or an "interior transition layer," consider the problem $\epsilon y'' + xy' = 0$ with the boundary values $y(-1) = 1$ and $y(1) = 2$. The solution to this problem is $y(x; \epsilon) = \frac{1}{2}\left(3 + \frac{\operatorname{erf}\left(x/2\sqrt{\epsilon}\right)}{\operatorname{erf}\left(1/2\sqrt{\epsilon}\right)}\right)$. Note the following limits, which indicate the non-uniformity of convergence:

$$\lim_{x\to 0^+} \lim_{\epsilon\to 0^+} y(x; \epsilon) = 2,$$
$$\lim_{x\to 0^-} \lim_{\epsilon\to 0^+} y(x; \epsilon) = 1,$$
$$\lim_{x\to 0} y(x; \epsilon) = \frac{3}{2}.$$

6. Kevorkian and Cole [2, pages 20–50 and 370–387] analyze the system:

$$\epsilon y'' - xy' + y = 0,$$
$$y(-1) = 1, \qquad y(1) = 2$$

and show that it has boundary layers at both $x = -1$ and $x = 1$.

7. For certain forms of simple equations, it is possible to predict the existence of boundary layers and other phenomena for generic boundary conditions. Table 140.1 shows the behavior that can be expected from the equation

$$\begin{aligned} &\epsilon y'' - p(x)y' - q(x)y = g(x), \qquad a \le x \le b \\ &y(a) = \alpha, \qquad y(b) = \beta, \end{aligned} \tag{140.14}$$

when ϵ is small and positive. For each case, there are simple examples that exhibit the predicted behavior. For example:

Conditions on $p(x)$	Type of solution
$p(x) \neq 0$ on $a \leq x \leq b$:	
(a) $p(x) < 0$	Boundary layer at $x = a$
(b) $p(x) > 0$	Boundary layer at $x = b$
$p(x) = 0$:	
(c) $q(x) > 0$	Boundary layers at $x = a$ and $x = b$
(d) $q(x) < 0$	Rapidly oscillating solution
(e) $q(x)$ changes sign	Classical turning point
$p' \neq q$, $p(0) = 0$ only at $x = 0$:	
(f) $p'(0) < 0$	No boundary layers, interior layer at $x = 0$
(g) $p'(0) > 0$	Boundary layers at $x = a$ and $x = b$, no interior layer at $x = 0$

Table 140.1: Possible behaviors for equation (140.14).

Equation	Boundary conditions	Solution
$\epsilon y'' + y' = 0$	$y(-1) = 0 \quad y(1) = 1$	$y(x) = \frac{e^{(1-x)/\epsilon} - e^{2/\epsilon}}{1 - e^{2/\epsilon}}$
$\epsilon y'' - y' = 0$	$y(-1) = 0 \quad y(1) = 1$	$y(x) = \frac{e^{(x-1)/\epsilon} - e^{-2/\epsilon}}{1 - e^{-2/\epsilon}}$
$\epsilon y'' - y = 0$	$y(-1) = 0 \quad y(1) = 1$	$y(x) = \frac{e^{\sqrt{\epsilon}(x-1)} - e^{\sqrt{\epsilon}(x+3)}}{1 - e^{4\sqrt{\epsilon}}}$
$\epsilon y'' + y = 0$	$y(-1) = 0 \quad y(1) = 1$	$y(x) = \frac{\sin\left((x+1)\sqrt{\epsilon}\right)}{\sin\left(2\sqrt{\epsilon}\right)}$

For non-generic boundary conditions, other solutions are possible. For example, equation (140.1) fits case (a) in table 140.1, which predicts the existence of a boundary layer near $x = 0$. However, if the boundary conditions for equation (140.1) had been $y(0) = y(1) = 4$, then the solution would have been $y(x) = 4$, with no boundary layers present.

8. A classic example showing the dependence of the solution on the boundary conditions is in Kevorkian and Cole [2, Section 2.5]. This nonlinear equation,

$$\begin{aligned} &\epsilon y'' + yy' - y = 0, \\ &y(0) = A, \qquad y(0) = B, \end{aligned} \tag{140.15}$$

has the solution behaviors shown in figure 140.2.

9. There are many matching principles that can be used to determine the unknown constants in the "inner" and "outer" solutions. One that is used in Van Dyke [9, page 64] is

> The n-term expansion of the inner solution (written in the outer variables) to m-terms is equal to the the m-term expansion of the outer solution (written in the inner variables) to n-terms.

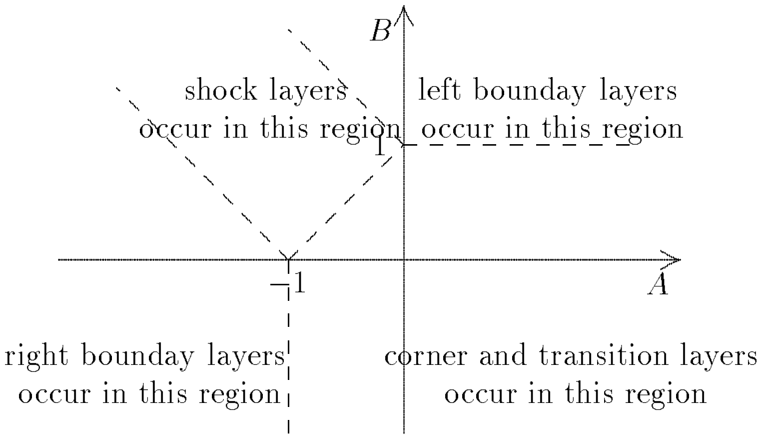

Figure 140.2: Different possible solutions to equation (140.15) for varying boundary conditions.

10. Sometimes there can be multiple boundary layers at a single boundary. That is, there are several layers of boundary layers (each with a different scaling) before the "outer" solution is matched to the value at the boundary.
11. There exist special numerical procedures that can be used for equations that have boundary layers. See, for instance, Miranker [5, Chapter 5, pages 88–108].
12. Lo [4] presents a technique for calculating many terms in an asymptotic expansion. The computer language Macsyma is used to perform the asymptotic matching at each stage.
13. This method is sometimes called the *method of matched asymptotic expansions*.
14. See also Bender and Orszag [1, Chapter 9, pages 417–483] Nayfeh [6, Chapter 4, pages 110–158], and Van Dyke [9, Chapter 5, pages 77–98].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Kevorkian, J., and Cole, J. D. *Perturbation Methods in Applied Mathematics*. Springer–Verlag, New York, 1981.

[3] Lagerstrom, P. A. *Matched Asymptotic Expansions*. Springer–Verlag, New York, 1988.

[4] Lo, L. L. Asymptotic matching by the symbolic manipulator MACSYMA. *J. Comput. Physics 61* (1985), 38–50.

[5] Miranker, W. L. *Numerical Methods for Stiff Equations*. D. Reidel Publishing Co., Boston, MA, 1981.

[6] Nayfeh, A. H. *Perturbation Methods.* John Wiley & Sons, New York, 1973.

[7] Pearson, C. E. On a differential equation of boundary layer type. *J. Math. and Physics 47* (1968), 134–154.

[8] Roberts, S. M. Further examples of the boundary value technique in singular perturbation problems. *J. Math. Anal. Appl. 133* (1988), 411–436.

[9] Van Dyke, M. *Perturbation Methods in Fluid Mechanics.* The Parabolic Press, Stanford, CA, 1975.

141. Perturbation Method: Functional Iteration

Applicable to Differential equations with a "small" term and homogeneous initial conditions or boundary conditions. Without the "small" term, the differential equation must be a linear and have a known Green's function.

Yields

A sequence of approximations.

Idea

If the given equation is only a "small" perturbation from a linear equation (with a known Green's function), then we may obtain an equivalent integral equation. This integral equation may be expanded methodically. Diagrams are often used to keep track of the terms.

Procedure

We will illustrate the general technique on a specific class of partial differential equations. Suppose we have the differential equation

$$\begin{gathered}\frac{\partial \phi}{\partial t} = H(t, x, \partial_x)\phi + V(x, \partial_x)\phi + A(x), \\ \phi(0, x) = 0, \qquad \phi(t, 0) = \phi(t, 1) = 0,\end{gathered} \tag{141.1}$$

for the unknown $\phi(t, x)$, where H and V are functionals. Let us presume that, in some sense, $||V\phi|| \ll ||H\phi||$. If the solution $G(t, x; y)$ of

$$\begin{gathered}\frac{\partial G}{\partial t} = H(t, x, \partial_x)G + \delta(x - y), \\ G(0, x; y) = 0, \qquad G(t, 0; y) = G(t, 1; y) = 0,\end{gathered} \tag{141.2}$$

is known, then the solution to equation (141.1) can be written as the equivalent integral equation

$$\begin{aligned}\phi(t, x) &= \int_0^1 G(t, x; y)[A(x) + V(x, \partial_x)\phi(t, x)]_{x=y} \, dy \\ &= \phi_0(t, x) + \int_0^1 G(t, x; y)V(y, \partial_y)\phi(t, y), \, dy,\end{aligned} \tag{141.3.a-b}$$

where $\phi_0(t, x) := \int_0^1 G(t, x; y)A(y) \, dy$. This is because $G(t, x; y)$ is a Green's function (see page 318) and superposition can be used (note that the boundary conditions in equation (141.1) and equation (141.2) are homogeneous).

If $\phi(t, y)$, as determined by the right-hand side of equation (141.3.b), is utilized in the integral in equation (141.3.b), then we obtain

$$\begin{aligned}\phi(t,x) &= \phi_0(t,x) + \int_0^1 G(t,x;y)\phi_0(t,y)\,dy \\ &\quad + \int_0^1 dy \int_0^1 du \;\left[G(t,x;y)V(y,\partial_y)\right]\left[G(t,y;u)V(u,\partial_u)\right]\phi(t,u).\end{aligned} \tag{141.4}$$

If $\phi(t, u)$, as determined by the right-hand side of equation (141.3.b), is utilized in the double integral in equation (141.4) and the process repeated, then we find

$$\begin{aligned}\phi(t,x) &= \phi_0(t,x) + \int_0^1 G(t,x;y)\phi_0(t,y)\,dy \\ &\quad + \int_0^1 dy \int_0^1 du \;\left[G(t,x;y)V(y,\partial_y)\right]\left[G(t,y;u)V(u,\partial_u)\right]\phi_0(t,u) \\ &\quad + \int_0^1 dy \int_0^1 du \int_0^1 dv \;\left[G(t,x;y)V(y,\partial_y)\right]\left[G(t,y;u)V(u,\partial_u)\right] \\ &\qquad\qquad \left[G(t,u;v)V(v,\partial_v)\right]\phi_0(t,v) + \cdots .\end{aligned} \tag{141.5}$$

Hence, we have produced a "natural" expansion of the solution to equation (141.1). Because writing the integrals in equation (141.5) becomes tedious, diagrams are often utilized. In a fairly obvious notation, we may write equation (141.5) as

$$\phi(t,x) = \phi_0(t,x) + F_1 + F_2 + F_3 + \cdots , \tag{141.6}$$

where each F_i is represented by a diagram in figure 141.1. The diagrams used in this method are never anything more than a shorthand notation for mathematical expressions. For each specific problem in which diagrammatic techniques are used, the diagrams must be appropriately defined. In this example, a node on a diagram corresponds to the operation $[G(t, \bullet; -)V(-, \partial_-)]$, and each line indicates an integral.

Example 1

We now show how functional iteration method can be used to approximate the solution of an ordinary differential equation, with a small parameter present. Given the differential equation with boundary conditions for $\phi(x)$

$$\begin{aligned}\frac{d^2\phi}{dx^2} &= \epsilon\left[1-\phi\right], \\ \phi(0) &= \phi(1) = 0,\end{aligned} \tag{141.7}$$

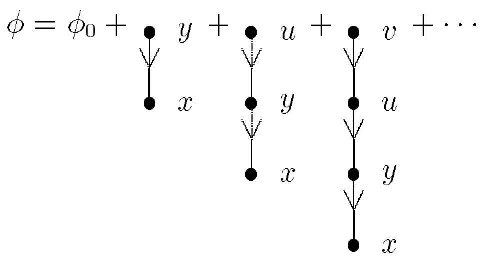

Figure 141.1: Diagrammatic representation of the solution in equation (141.6).

we first note that the exact solution is given by

$$\begin{aligned} \phi(x) &= 1 - \cos\sqrt{\epsilon}x + \frac{\cos\sqrt{\epsilon} - 1}{\sin\sqrt{\epsilon}} \sin\sqrt{\epsilon}x, \\ &= \epsilon\left(\frac{x^2 - x}{2}\right) - \epsilon^2\left(\frac{x^4 - 2x^3 + x}{24}\right) + O(\epsilon^3). \end{aligned} \qquad (141.8.a\text{-}b)$$

The Green's function that we need, $G(x;y)$, will satisfy the equation

$$\begin{aligned} \frac{d^2G}{dx^2} &= \delta(x - y), \\ G(0) &= G(1) = 0, \end{aligned}$$

and is given by (see the example for the Green's function method, on page 321)

$$G(x;y) = \begin{cases} x(y-1) & \text{for } 0 \le x \le y, \\ y(x-1) & \text{for } y < x \le 1. \end{cases}$$

The differential equation (141.7) can then be written as an integral equation, using this Green's function, as

$$\begin{aligned} \phi(x) &= \epsilon \int_0^1 G(x;y)\left[1 - \phi(y)\right] dy \\ &= \epsilon\phi_0(x) - \epsilon \int_0^1 G(x;y)\phi(y)\,dy, \end{aligned} \qquad (141.9.a\text{-}b)$$

where

$$\begin{aligned} \phi_0(x) &:= \int_0^1 G(x;y)\,dy \\ &= \int_x^1 x(y-1)\,dy + \int_0^x y(x-1)\,dy \\ &= \frac{x^2 - x}{2}. \end{aligned}$$

If the value of $\phi(x)$ (as defined by the right-hand side of equation (141.9.b)) is inserted for the function $\phi(y)$ in equation (141.9.b), the natural expansion arises

$$\begin{aligned}\phi(x) &= \epsilon\phi_0(x) - \epsilon^2 \int_0^1 G(x;y)\phi_0(y)\,dy \\ &\quad + \epsilon^3 \int_0^1 G(x;y) \int_0^1 G(y;z)\phi_0(z)\,dz\,dy - O(\epsilon^4),\end{aligned} \tag{141.10}$$

which can be represented by

$$\phi(x) = \phi_0(x) + F_1 + F_2 + F_3 + \cdots ,$$

where the $\{F_i\}$ are given in figure 141.1. In this example, a node on a diagram corresponds to multiplying by $G(\alpha;\beta)$ (for some specific α and β) and each line segment indicates an integration. It is easy to evaluate the first few diagrams, that is, to evaluate the first few terms in equation (141.10). The approximation obtained from equation (141.10) is identical to the expansion in equation (141.8.b).

Example 2

The Green's function is needed so that the solution of the original differential equation may be written in terms of an integral (as in equation (141.3.b) or equation (141.9.b)). For a first order equation, though, an integral representation can be found immediately. In this example, we analyze a nonlinear first order differential equation to indicate more fully how the diagrams may be used. Consider the nonlinear ordinary differential equation

$$\begin{aligned}\frac{dz}{dt} &= f(t) + g(t)z^2, \\ z(0) &= 0\end{aligned}$$

in which the nonlinear term (i.e., the $g(t)$ function) is "small." This equation may be integrated directly to obtain

$$z(t) = \int_0^t f(\tau)\,d\tau + \int_0^t g(\tau)z^2(\tau)\,d\tau. \tag{141.11}$$

If the value of $z(t)$ from the left-hand side of equation (141.11) is used in the right-hand side, then

$$\begin{aligned}z(t) &= \int_0^t f(\tau)\,d\tau + \int_0^t g(\tau)\left[\int_0^\tau f(\tau_1)\,d\tau_1\right]^2 d\tau \\ &\quad + 2\int_0^t g(\tau)\left[\int_0^\tau f(\tau_1)\,d\tau_1\right]\left[\int_0^\tau g(\tau_2)z^2(\tau_2)\,d\tau_2\right] d\tau \\ &\quad + \int_0^t g(\tau)\left[\int_0^\tau g(\tau_2)z^2(\tau_2)\,d\tau_2\right]^2 d\tau.\end{aligned} \tag{141.12}$$

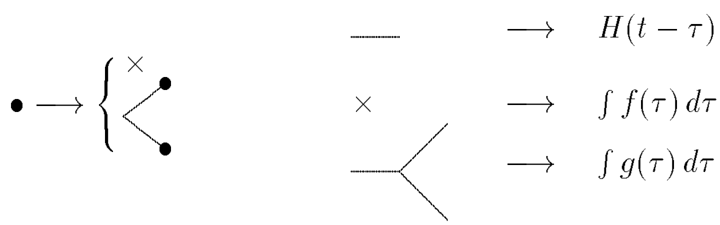

Figure 141.2: Rules for creating and interpreting diagrams.

A "natural" perturbation expansion would be to keep the first two terms in the right-hand side of equation (141.12) and to assume that the last two terms are "small." If $|z(t)| \ll 1$, then this may well be the case because the last two terms involve $|z|^2$ whereas the first two terms involve $|z|$.

The functional iteration technique can be used to derive equation (141.12) and the higher order extensions from diagrams. We need two sets of rules: One set of rules describes how the diagrams may be computed; the other set of rules describes how the diagrams are to be turned into mathematical expressions. If we use the rules in figure 141.2 (where $H(\cdot)$ denotes the Heaviside function), then the first two steps in the diagrammatic solution to $z(t)$ (from equation (141.11)) are given by the diagrams in figure 141.3.

Note that the third and fourth diagrams in figure 141.3 represent the same mathematical expression because they are topologically equivalent. The purpose of the Heaviside function is to restrict the range of integration. By careful inspection, the mathematical expressions associated with the last set of diagrams will be seen to be identical to equation (141.12).

Notes

1. In the physics literature, the Green's function is sometimes called the *propagator*. This is usually written in terms of a *path integral*, $G = \int e^{iS/\hbar}$, where S is the action, defined to be the integral of the Lagrangian. The diagrams produced in this context are sometimes called *Feynman diagrams*.
2. When nonlinear equations are approximated by this technique, as in example 2, keeping track of the terms in the expansion that are of the some order is greatly facilitated by some shorthand notation. The diagrams presented above perform such a task.
3. In more complicated problems, the diagrams will have several different types of line segments and several different types of nodes.
4. Often an "algebra of diagrams" is created, so that diagrams can be added, subtracted and multiplied without recourse to the mathematical expression that each diagram represents. This would require amplification of the rules that were used in example 2.

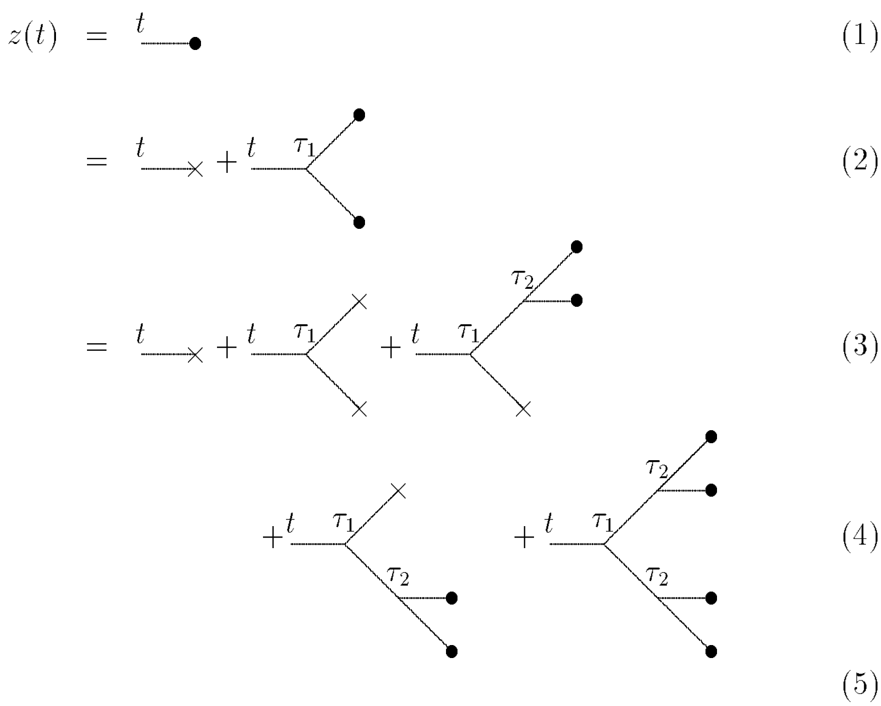

Figure 141.3: Two steps in the diagrammatic expansion of equation (141.11).

5. Presented in this section has been just one type of functional iteration; there are many others. For example, Picard iteration (see page 618) is a functional iteration method. Another method is a decomposition method frequently used by Adomian [2].
6. This technique is particularly important in problems in which there is no "small" parameter. In these cases, the *formally correct* diagrammatic expansion may be algebraically approximated by exactly summing certain classes of diagrams. See Mattuck [6] for details.

References

[1] Abrikosov, A. A., Gorkov, L. P., and Dzyaloshinski, I. E. *Methods of Quantum Field Theory in Statistical Physics*. Dover Publications, Inc., New York, 1963.

[2] Adomian, G. *Stochastic Systems*. Academic Press, New York, 1983.

[3] Drouffe, J.-M., and Saclay, C. Computer algebra as a research tool in physics. In *EUROCAL '85*, B. Buchberger and B. F. Caviness, Eds. Springer–Verlag, New York, 1985, pp. 58–67.

[4] Fishman, L., and McCoy, J. J. Factorization and path integration of the Helmholtz equation: Numerical algorithms. *J. Acoust. Soc. Am. 81*, 5 (May 1987), 1355–1376.

[5] Houard, J. C., and Irac-Astaud, M. A new approach to perturbation theory: Star diagrams. *J. Math. Physics 24*, 8 (Aug 1983), 1997–2005.

[6] Mattuck, R. D. *A Guide to Feynman Diagrams in the Many-Body Problem*. Academic Press, New York, 1976.

[7] Pascual, P., and Tarrach, R. *QCD: Renormalization for the Practitioner*. Springer–Verlag, New York, 1984.

[8] Schulman, L. S. *Techniques and Applications of Path Integration*. John Wiley & Sons, New York, 1981.

[9] Srinivasan, S. K., and Vasudevan, R. *Introduction to Random Differential Equations and Their Applications*. American Elsevier Publishing Company, New York, 1971.

142. Perturbation Method: Multiple Scales

Applicable to Nonlinear differential equations that have a small parameter present.

Yields

An approximation to the solution.

Idea

This is a singular perturbation technique, applicable to problems for which regular perturbation techniques fail. The assumption in this technique is that the solution depends on more than one "length" (or "time") scale.

Procedure

We presume that the solution depends on two (or more) different length (or time) scales. By trying different possibilities, we determine what these length scales are. These different length scales are treated as dependent variables when transforming the given ordinary differential equation into a partial differential equation, but then the length scales are treated as independent variables when solving the equations.

The dependent variable is then expanded in a regular perturbation series (see page 610), where each functions in the series depends on all of the different length scales. The different orders of ϵ are collected, and the sequential set of partial differential equations is solved.

As these equation are solved, the requirement is that each successive term must vanish no slower (as ϵ tends to zero) than the previous term.

Example

Suppose we have the ordinary differential equation

$$\begin{aligned} &\epsilon y'' + y' = 2, \\ &y(0) = 0, \quad y(1) = 1, \end{aligned} \tag{142.1}$$

for $y(x;\epsilon)$. We immediately recognize that equation (142.1) is likely to be a singular perturbation problem. This is because, when we set ϵ equal to zero, the equation becomes a first order differential equation, and it is very unlikely that the solution of this equation (which depends on a single constant) will match both boundary conditions.

We first need to determine what the proper length scales are for this problem. We guess that, for this problem, the proper length scales are $u := x$ and $v := x/\epsilon$. If we had guessed incorrectly, then we would not be able to carry out all of the calculations. First, equation (142.1) must be written in terms of these new variables. Writing $\frac{d}{dx}$ as

$$\frac{d}{dx} = \left(\frac{du}{dx}\frac{\partial}{\partial u} + \frac{dv}{dx}\frac{\partial}{\partial v}\right) = \left(\frac{\partial}{\partial u} + \frac{1}{\epsilon}\frac{\partial}{\partial v}\right),$$

the equation (142.1) becomes

$$\epsilon\left(\frac{\partial}{\partial u} + \frac{1}{\epsilon}\frac{\partial}{\partial v}\right)^2 y + \left(\frac{\partial}{\partial u} + \frac{1}{\epsilon}\frac{\partial}{\partial v}\right) y = 2. \tag{142.2}$$

We now propose the expansion of $y(x;\epsilon)$ as a regular perturbation series in the dependent variables u and v

$$y(x;\epsilon) = y_0(u,v) + \epsilon y_1(u,v) + \epsilon^2 y_2(u,v) + \cdots . \tag{142.3}$$

Using equation (142.3) in equation (142.2) and equating the different powers of ϵ results in an infinite sequence of equations, of which the first three are

$$\begin{aligned} O(\epsilon^{-1}): &\quad \frac{\partial^2 y_0}{\partial v^2} + \frac{\partial y_0}{\partial v} = 0, \\ O(\epsilon^{0}): &\quad \frac{\partial^2 y_1}{\partial v^2} + \frac{\partial y_1}{\partial v} = 2 - 2\frac{\partial^2 y_0}{\partial u \partial v} - \frac{\partial y_0}{\partial u}, \\ O(\epsilon^{1}): &\quad \frac{\partial^2 y_2}{\partial v^2} + \frac{\partial y_2}{\partial v} = -2\frac{\partial^2 y_1}{\partial u \partial v} - \frac{\partial y_1}{\partial u} - \frac{\partial^2 y_0}{\partial u^2}. \end{aligned} \tag{142.4.a-c}$$

The first partial differential equation can be solved to determine

$$y_0(u,v) = A(u) + B(u)e^{-v}, \tag{142.5}$$

where $A(u)$ and $B(u)$ are arbitrary functions of u. The second equation then becomes

$$\frac{\partial^2 y_1}{\partial v^2} + \frac{\partial y_1}{\partial v} = 2 - A'(u) + B'(u)e^{-v}, \tag{142.6}$$

which has the solution

$$y_1(u,v) = [2 - A'(u)]v + vB'(u)e^{-v} + D(u) + E(u)e^{-v}, \tag{142.7}$$

where $D(u)$ and $E(u)$ are arbitrary functions. Now, we use our solvability condition, which states that the higher order terms will vanish no slower than the lower order terms. For $y_1(u,v)$ (as given in equation (142.7)) to vanish no slower than $y_0(u,v)$ (as given in equation (142.5)), we require that $2 - A'(u) = 0$ and $B'(u) = 0$. Otherwise, for $x \neq 0$ and $\epsilon \ll 1$, the terms in y_1 would be larger than the terms in y_0 (because, in this case, $v \gg 1$).

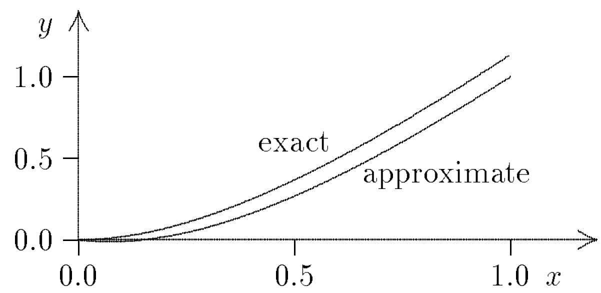

Figure 142.1: A comparison of the exact solution to equation (142.1) (given by equation (142.10)) and the approximate solution in equation (142.9), when $\epsilon = 0.5$.

Using these two constraints, we determine that $A(u) = 2u + A_0$ and $B(u) = B_0$, where A_0 and B_0 are constants. Hence, the first order solution becomes

$$y_0(u, v) = (2u + A_0) + B_0 e^{-v}. \tag{142.8}$$

Going back to the original variable (i.e., x), the leading term in the solution for $y(x; \epsilon)$ is (from equation (142.3) and equation (142.8))

$$y(x; \epsilon) \approx y_0(x) = (2x + A_0) + B_0 e^{-x/\epsilon}.$$

This expression can be matched to both of the boundary conditions in equation (142.1) to determine that

$$y(x; \epsilon) \approx 2x - \left(1 - e^{-x/\epsilon}\right). \tag{142.9}$$

The exact solution to equation (142.1) is given by

$$y(x; \epsilon) = 2x - \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}}. \tag{142.10}$$

Hence, we see that the approximate analysis has correctly obtained the first term in the expansion as ϵ tends to zero. Figure 142.1 has a comparison of equations (142.9) and (142.10) when $\epsilon = 0.5$.

Notes

1. It was not really necessary to solve equation (142.6) for y_1 to obtain the constraints on $A(u)$ and $B(u)$. By analysis of the equation for y_1, with an eye toward obtaining solutions that do not grow with v, the same conditions could have been obtained. This is an important procedure in more complicated problems for which explicit solutions are not easy to find. See the section on alternative theorems, beginning on page 15.

2. Any problem that can be solved by matched asymptotic expansions can also be solved by multiple scales, although the procedure may require more work.
3. Rubenfeld [8] gives an account of why the method of multiple scales sometimes gives *incorrect* results.
4. Fateman [3] describes a Macsyma program that will automatically utilize the method of multiple scales to approximate the solution of differential equations.
5. The method of multiple scales is often called *two timing*.
6. A Macsyma package to perform these computations is described in Len [5].
7. The choice of length scales depends on the particular problem. For some problems, three (or more) length scales may be appropriate. Each length scale may have a complicated dependence on the parameter ϵ.
8. The method of multiple scales does not result in an answer that is valid over an indefinitely long range. If, for instance, the two scales are x and ϵx, then the solution is valid, generally, for $x = O(\epsilon^{-1})$. The solution of $\dot{x} = \epsilon f(t, x, \epsilon)$ (when f has period 2π in t) can be approximated by the use of multiple scales. The kth order approximation using the two time scales t and $\tau = \epsilon t$ are valid with error $O(\epsilon^k)$ for intervals of length $O(1/\epsilon)$. It is often believed that adding a third scale $\sigma = \epsilon^2 t$ will result in solution valid for $O(1/\epsilon^2)$; this is incorrect (see Murdock and Wang [6]).
9. See also Bender and Orszag [1, Chapter 11, pages 544–568], Kevorkian and Cole [4, pages 115–151], and Nayfeh [7, Chapter 6, pages 228–307].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Brackbill, J. U., and Cohen, B. I. *Multiple Time Scales*. Academic Press, New York, 1985.

[3] Fateman, R. J. An approach to automatic asymptotic expansions. In *Proceedings of the 1976 ACM Symposium on Symbolic and Algebraic Computation (SYMSAC '76)*, R. D. Jenks, Ed. ACM, New York, 1976, pp. 365–371.

[4] Kevorkian, J., and Cole, J. D. *Perturbation Methods in Applied Mathematics*. Springer–Verlag, New York, 1981.

[5] Len, J. L. Analysis of ODEs using the multiple scales perturbation method. *Macsyma Newsletter* (April 1989), 4–10.

[6] Murdock, J., and Wang, L.-C. Validity of the multiple scale method for very long intervals. *Z angew Math Phys 47* (1996), 760–789.

[7] Nayfeh, A. H. *Perturbation Methods*. John Wiley & Sons, New York, 1973.

[8] Rubenfeld, L. A. On a derivative-expansion technique and some comments on multiple scaling in the asymptotic approximation of solutions of certain differential equations. *SIAM Review 20*, 1 (January 1978), 79–105.

[9] Sanders, J. A., and Verhulst, F. *Averaging Methods in Nonlinear Dynamical Systems*. Springer–Verlag, New York, 1985.

143. Perturbation Method: Regular Perturbation

Applicable to

Differential equations with a small parameter.

Yields

A series of terms of decreasing magnitude that approximate the solution of the original differential equation.

Idea

When an equation is changed by only a small amount, the solution will often only change by a small amount.

Procedure

Expand the dependent variables in a power series depending on the small parameter in the problem. Substitute this series into the original equation(s), the boundary condition(s), and the initial condition(s). Expand everything in a Taylor series, equate the terms corresponding to different powers of the small parameter, and solve the equations sequentially.

Example

Suppose we have the equation

$$\begin{aligned} &y'' + \epsilon y' + y = 0, \\ &y(0) = 1, \quad y'(0) = 0, \end{aligned} \tag{143.1}$$

where ϵ is a number whose magnitude is much smaller than 1. We suppose that the solution to equation (143.1), $y(x;\epsilon)$, can be expanded in a power series in ϵ as follows

$$y(x;\epsilon) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots . \tag{143.2}$$

Then, using equation (143.2) in equation (143.1), we obtain

$$\begin{aligned} (y_0'' + \epsilon y_1'' + \cdots) + \epsilon(y_0' + \epsilon y_1' + \cdots) + (y_0 + \epsilon y_1 + \cdots) &= 0, \\ y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \cdots &= 1, \\ y_0'(0) + \epsilon y_1'(0) + \epsilon^2 y_2'(0) + \cdots &= 0. \end{aligned} \tag{143.3}$$

Equating powers of ϵ in equation (143.3) to zero produces the sequence of equations

$$O(\epsilon^0): \qquad \begin{aligned} y_0'' + y_0 &= 0, \\ y_0(0) &= 1, \\ y_0'(0) &= 0, \end{aligned} \tag{143.4}$$

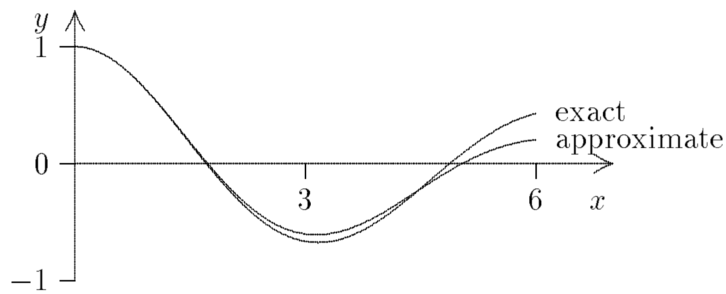

Figure 143.1: Comparison of the exact solution and the two term approximation to equation (143.1), when $\epsilon = 0.25$.

and

$$O(\epsilon^1): \qquad \begin{aligned} y_1'' + y_1 &= -y_0', \\ y_1(0) &= 1, \\ y_1'(0) &= 0. \end{aligned} \tag{143.5}$$

The solution to equation (143.4) is

$$y_0(x) = \cos x. \tag{143.6}$$

Using equation (143.6) in equation (143.5), we must now solve the equation

$$\begin{aligned} y_1'' + y_1 &= \sin x, \\ y_1(0) &= 1, \\ y_1'(0) &= 0. \end{aligned} \tag{143.7}$$

The solution to equation (143.7) is

$$y_1(x) = \frac{1}{2}(\sin x - x\cos x). \tag{143.8}$$

Therefore, the solution for $y(x;\epsilon)$ is approximately (using equations (143.6) and (143.8) in equation (143.2))

$$y(x;\epsilon) = \cos x + \frac{\epsilon}{2}(\sin x - x\cos x) + O(\epsilon^2). \tag{143.9}$$

We could continue this process indefinitely and calculate as many terms as were needed to obtain a desired accuracy. Figure 143.1 is a comparison of the first two terms of equation (143.9), when $\epsilon = 0.25$, with the exact solution.

Notes

1. The exact solution to equation (143.1) is given by

$$y(x;\epsilon) = \frac{\epsilon}{\sqrt{4-\epsilon^2}} e^{-\epsilon x/2} \sin\left(x\sqrt{1-\frac{\epsilon^2}{4}}\right) + e^{-\epsilon x/2} \cos\left(x\sqrt{1-\frac{\epsilon^2}{4}}\right),$$

which can be expanded for small ϵ to yield

$$y(x;\epsilon) = \cos x + \frac{\epsilon}{2}(\sin x - x\cos x) + O(\epsilon^2).$$

2. This method will *not* work on all equations that have a small parameter. As a simple example, consider

$$\epsilon y'' + y = 0, \qquad y(0) = 1, \quad y(1) = 2. \tag{143.10}$$

In this example, the first order equation (corresponding to equation (143.4)) is

$$y_0 = 0, \quad y(0) = 1, \quad y(1) = 2.$$

Clearly, this equation has no solution. Hence, the expansion in (143.3), must not be adequate to represent the solution of equation (143.10).

3. In deriving equations (143.4) and equation (143.5) from equation (143.3), it was implicitly assumed that each of $|y_1(x)|$, $|y_1'(x)|$, and $|y_1''(x)|$ are $O(1)$. Observe that this will *not* be the case when $x = O(1/\epsilon)$ (see equation (143.8)). Hence, we conclude that only when $x \ll 1/\epsilon$ can equation (143.9) be a good approximation to the solution of equation (143.1). If an approximation to the solution is desired over a larger range of x values, then the method of multiple scales might be used (see page 605). *Secular terms* is the name given to terms that become large and prevent a perturbation expansion from being valid.

4. If the solution to a differential equation is not analytic at $\epsilon = 0$, then the solution can *not* be expanded in the form of equation (143.2). Often, the best procedure is to utilize an expansion of the form

$$y(x;\epsilon) = y_0(x) + \mu_1(\epsilon)y_1(x) + \mu_2(\epsilon)y_2(x) + \dots,$$

and then determine the scaling functions $\{\mu_i\}$ as the $\{y_i\}$ are determined. It is frequently the case that the $\{\mu_i\}$ are given by terms of the form $\{\epsilon^n \log^m \epsilon\}$. Terms with $m \neq 0$ are sometimes called *switchback terms* (see Lagerstrom and Reinelt [4] or Van Dyke [7, pages 9–20 and 200–202]).

5. The functional iteration method (see page 598) produces the same terms that would be obtained by a regular perturbation expansion. The benefit of the diagrammatic method is that it allows easier manipulation of the terms.

6. See Bender and Orszag [1, pages 319–335], Farlow [2, Lesson 46, pages 370–378], Kevorkian and Cole [3, pages 17–20], and Lin and Segel [5, pages 45–55 and 225–241].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[3] Kevorkian, J., and Cole, J. D. *Perturbation Methods in Applied Mathematics*. Springer–Verlag, New York, 1981.

[4] Lagerstrom, P. A., and Reinelt, D. A. Note on logarithmic switchback terms in regular and singular perturbation expansion. *SIAM J. Appl. Math. 44*, 3 (June 1984), 451–462.

[5] Lin, C. C., and Segel, L. A. *Mathematics Applied to Deterministic Problems in the Natural Sciences*. The MacMillan Company, New York, 1974.

[6] Nayfeh, A. H. *Perturbation Methods*. John Wiley & Sons, New York, 1973.

[7] Van Dyke, M. *Perturbation Methods in Fluid Mechanics*. The Parabolic Press, Stanford, CA, 1975.

144. Perturbation Method: Strained Coordinates

Applicable to Differential equations that have a small parameter present.

Yields

An approximation to the solution, valid on a long time scale.

Idea

A regular perturbation expansion may give the correct answer but at the wrong location. By scaling the dependent variable and one or more of the independent variables by the small parameter, the solution may be approximated at the correct location.

Procedure

If the regular perturbation solution to a differential equation has secular terms but the original equation has bounded solutions, then the regular perturbation approximation is not valid for large values of the independent variables. One way to obtain a solution that is valid for longer scales is by "straining the coordinates"; that is, expanding the dependent variable *and* one or more of the independent variables in terms of the small parameter.

To completely specify the arbitrary functions and constants that arise, use the maxim: "Higher order approximation shall be no more singular than the first."

Example

Suppose we wish to approximate the solution to the nonlinear differential equation

$$\begin{aligned} &\frac{d^2y}{dt^2} + \omega^2 y = \epsilon y^3, \\ &y(0) = 1, \quad y'(0) = 0. \end{aligned} \tag{144.1}$$

This equation can be integrated once by first multiplying by y'. The resulting first order differential equation can be integrated in terms of elliptic functions. The explicit solution indicates that the solution is periodic.

If a regular perturbation technique is attempted, then the resulting equations can be solved in the usual manner (see page 610) to determine that

$$y(t;\epsilon) = \cos\omega t + \epsilon\left[\frac{3}{8}\frac{t}{\omega}\sin\omega t - \frac{1}{32\omega^2}\Big(\cos 3\omega t - \cos\omega t\Big)\right] + O(\epsilon^2).$$

Note that the second term in this solution becomes unbounded as t increases. Hence, secular terms are present.

In the method of straining, both the dependent variable and the independent variable are expanded in terms of ϵ. For this example, we presume the expansion has the form

$$\begin{aligned} t &= t(\tau;\epsilon) = \tau + \epsilon t_1(\tau) + O(\epsilon^2), \\ y &= y(\tau;\epsilon) = y_0(\tau) + \epsilon y_1(\tau) + O(\epsilon^2). \end{aligned} \tag{144.2.a-b}$$

Noting that the derivative with respect to t can be replaced with a derivative with respect to τ by

$$\frac{d}{dt} = (1 - \epsilon t_1' + \cdots)\frac{d}{d\tau},$$

(where a prime ($'$) denotes differentiation with respect to τ), we find that equation (144.1) can be turned into a sequence of equations, with each equation involving the next $y_k(\tau)$ term. The first two equations are

$$\begin{aligned} &\frac{d^2y_0}{d\tau^2} + \omega^2 y_0 = 0, \\ &\frac{d^2y_1}{d\tau^2} + \omega^2 y_1 = y_0^3 + 2t_1'\frac{d^2y_0}{d\tau^2} + t_1''\frac{dt_0}{d\tau}. \end{aligned} \tag{144.3}$$

The boundary conditions are similarly expanded. We find

$$\begin{aligned} &y_0(0) = 1, \qquad \frac{dy_0}{d\tau}(0) = 0, \\ &y_1(0) + t_1(0)\frac{dy_0}{d\tau}(0) = 0, \\ &\frac{dy_1}{d\tau}(0) - t_1'(0)\frac{dy_0}{d\tau}(0) + t_1(0) + \frac{dy_0}{d\tau}(0) = 0. \end{aligned} \tag{144.4}$$

Now we proceed to solve the equations sequentially, just as in the regular perturbation method. The first equation in (144.3) with the first pair of boundary conditions in equation (144.4) yields

$$y_0(\tau) = \cos\omega\tau. \tag{144.5}$$

Using this value for $y_0(\tau)$, the next equation (144.3) (which is for $y_1(\tau)$) becomes

$$\frac{d^2y_1}{d\tau^2} + \omega^2 y_1 = \frac{1}{4}\cos 3\omega\tau + \left(\frac{3}{4} - 2\omega^2 t_1'\right)\cos\omega\tau - \omega t_1''\sin\omega\tau. \tag{144.6}$$

To prevent $y_1(\tau)$ from having any secular terms, this equation cannot be forced at resonance. This means that the right-hand side of equation (144.6) cannot have any terms that are solutions of the homogeneous equation. To keep the right-hand side of equation (144.6) from having any $\cos\omega\tau$ or $\sin\omega\tau$ terms, we choose

$$\left(\frac{3}{4} - 2\omega^2 t_1'\right) = 0 \quad \text{or} \quad t_1 = \frac{3\tau}{8\omega^2}. \tag{144.7}$$

If we now solve equation (144.6), there will be no secular terms. Utilizing equation (144.7) in equation (144.2.a) results in

$$t = \tau + \frac{3\epsilon}{8\omega^2}\tau + \cdots ,$$

or

$$\tau = \left(1 - \frac{3\epsilon}{8\omega^2}\right) t + \cdots .$$

Using this last expression for τ in equation (144.5) results in our final form of the first order approximation

$$y_0(t) = \cos\left[\left(\omega - \frac{3\epsilon}{8\omega}\right) t\right].$$

Notes

1. Another common application of this method is to differential equations whose solutions are well behaved, but approximations by a regular perturbation scheme produce singular terms. For example, the differential equation

$$(x + \epsilon u)\frac{du}{dx} + u = 0, \qquad u(1) = 1, \tag{144.8}$$

   has a solution that is well behaved at $x = 0$, but the regular perturbation series $u(x;\epsilon) = u_0(x) + \epsilon u_1(x) + \ldots$ yields $u_0 = x^{-1}$, $u_1 = \frac{1}{2}\left(x^{-1} - x^{-3}\right)$, and higher order terms that are even more singular at $x = 0$. Applying strained coordinate techniques results in the exact solution of equation (144.8): $u(x) = \left(-x + \sqrt{x^2 + 2\epsilon + \epsilon^2}\right)/\epsilon$. This solution shows that $u(x)$ cannot be expanded in a power series in ϵ near $x = 0$.
2. The paper by Roberts and Shipman [7] concerns itself with equations of the form

$$[f(x) + \epsilon y]\frac{dy}{dx} + q(x)y = r(x)$$

   on the interval $0 < x < 1$, with $y(1) = c$, when the method of straining does *not* work.
3. This technique is a useful tool in many areas, including the theory of boundary layers and the structure and propagation of shock waves.
4. This technique is also known as the Lighthill method, the Lindstedt method, and the Poincaré–Lighthill method.

5. The computer language Macsyma [5] has a package (`lindst`) for automatically implementing this technique, see Len [3] for details.
6. See also Goldstein and Braun [2, pages 306–311], Nayfeh [6, Chapter 3, pages 56–109], and Van Dyke [8, Chapter 6, pages 99–120].

References

[1] Comstock, C. The Poincare–Lighthill perturbation technique and its generalizations. *SIAM Review 14*, 3 (July 1972), 433–446.

[2] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations.* U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[3] Len, J. L. Perturbation solution of ODEs in MACSYMA: Lindstedt's method. *MACSYMA Newsletter 5*, 2 (April 8), 6–9.

[4] Lighthill, M. J. A technique for rendering approximate solutions to physical problems uniformly valid. *Z. Flugwiss 9* (1961), 267–275.

[5] Macsyma. *Mathematics Reference Manual.* Macsyma, Arlington, MA, 1993.

[6] Nayfeh, A. H. *Perturbation Methods.* John Wiley & Sons, New York, 1973.

[7] Roberts, S. M., and Shipman, J. S. An iteration perturbation technique. *J. Comput. Physics 16* (1974), 285–297.

[8] Van Dyke, M. *Perturbation Methods in Fluid Mechanics.* The Parabolic Press, Stanford, CA, 1975.

[9] Whitham, G. The flow pattern of a supersonic projectile. *Comm. Pure Appl. Math 5* (1952), 301–348.

145. Picard Iteration

Applicable to Differential equations, a single equation, or a system.

Yields

A sequence of approximations to the solution.

Idea

We can write an ordinary differential equation as a fixed point formula. If we have a starting guess, we can iterate the equation to find an approximate solution to the original equation.

Procedure

Suppose we have the first order ordinary differential equation

$$\frac{dy}{dx} = f(y, x),$$

with the initial condition $y(x_0) = y_0$. This equation can be written as the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f\left(y(z), z\right)\, dz. \tag{145.1}$$

Note that equation (145.1) already incorporates the initial conditions. If we had a guess of $y(x)$, say $y_1(x)$, then we might be able to improve our guess by forming $y_2(x)$ as follows

$$y_2(x) = y_0 + \int_{x_0}^{x} f\left(y_1(z), z\right)\, dz.$$

Then, knowing $y_2(x)$, we could form $y_3(x)$ by the same technique. We can continue this process indefinitely, each time using the formula

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f\left(y_n(z), z\right)\, dz. \tag{145.2}$$

What we take for $y_1(x)$ is arbitrary; it is often easiest to take $y_1(x) = y_0$.

Example

Suppose we have the following ordinary differential equation

$$\frac{dy}{dx} = x^2 + y^2,$$

with $y(0) = 1$. In this case, the iteration formula, equation (145.2), becomes

$$y_{n+1}(x) = 1 + \int_{0}^{x} \left[z^2 + y_n(z)^2\right] dz.$$

If we take $y_1(x) = 1$, then we find

$$
\begin{aligned}
y_2(x) &= 1 + x + \frac{1}{3}x^3, \\
y_3(x) &= 1 + x + x^2 + \frac{2}{3}x^3 + \cdots, \\
y_4(x) &= 1 + x + x^2 + \frac{4}{3}x^3 + \frac{5}{6}x^4 + \cdots, \\
y_5(x) &= 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \frac{16}{15}x^8 + \cdots.
\end{aligned}
\tag{145.3}
$$

The Taylor series solution of this problem (see page 632) begins

$$y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \frac{6}{5}x^5 + \cdots.$$

Hence, each successive approximation in equation (145.3) appears to have one more correct term.

Notes

1. The successive approximations found by this method are not guaranteed to converge.
2. This method can also be used on systems of first order ordinary differential equations. For example, the scheme corresponding to the system

$$
\begin{aligned}
\frac{dy}{dt} &= f(y, z, t), \qquad y(0) = y_0, \\
\frac{dz}{dt} &= g(y, z, t), \qquad z(0) = z_0,
\end{aligned}
$$

is

$$
\begin{aligned}
y_{n+1}(t) &= y_0 + \int_0^t f(y_n(t), z_n(t), t)\, dt, \\
z_{n+1}(t) &= z_0 + \int_0^t g(y_n(t), z_n(t), t)\, dt.
\end{aligned}
$$

3. Picard iteration can be applied to ordinary differential equations of nth order without first writing the equation as a first order system. For example, the second order ordinary differential equation

$$
\begin{aligned}
&y'' = f\left(t, y(t), y'(t)\right), \\
&y(a) = A, \qquad y(b) = B,
\end{aligned}
$$

has the convenient iteration scheme

$$y_{n+1}(x) = A + (x-a)y_n'(a) + \int_a^x (x-t) f\left(t, y_n(t), y_n'(t)\right)\, dt$$

where $y_0(x) = A + (x-a)(B-A)/(b-a)$.

4. It is also possible to approximate partial differential equations by this technique. For example, the elliptic equation $\nabla^2 u = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ has the natural iteration formula $\nabla^2 u_n = f\left(x, y, u_{n-1}, \frac{\partial u_{n-1}}{\partial x}, \frac{\partial u_{n-1}}{\partial y}\right)$. Iyanaga and Kawada [2, page 998] have technical conditions for when this scheme will converge to the solution of the original equation. Rice and Boisvert [5, pages 79–82] illustrate this technique with the use of ELLPACK.
5. See also Boyce and DiPrima [1, pages 97–103] and Simmons [6, pages 418–422].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Iyanaga, S., and Kawada, Y. *Encyclopedic Dictionary of Mathematics*. MIT Press, Cambridge, MA, 1980.

[3] Lal, M., and Moffatt, D. Picard's successive approximation for non-linear two-point boundary value problems. *J. Comput. Appl. Math. 8*, 4 (1982), 233–236.

[4] Ozis, T. The extension of Picards's successive approximation for constructing two-side bounds for the solutions of diferential equations. *J. Comput. Appl. Math. 39* (1992), 7–14.

[5] Rice, J. R., and Boisvert, R. F. *Solving Elliptic Problems Using ELLPACK*. Springer–Verlag, New York, 1985.

[6] Simmons, G. F. *Differential Equations with Applications and Historical Notes*. McGraw–Hill Book Company, New York, 1972.

146. Reversion Method

Applicable to

Forced nonlinear ordinary differential equations.

Yields

A local approximation.

Idea

To derive the method, we assume a certain parameter is small and develop a perturbation expansion in that parameter. In practice, we use the formulae obtained by this method when the parameter is equal to 1.

Procedure

Suppose that the general nonlinear differential equation whose solution we wish to approximate near the initial value is given by

$$D_1 y + D_2 y^2 + \cdots + D_5 y^5 + \cdots = k\phi(x), \tag{146.1}$$

where the $\{D_i\}$ represent differential operators. We seek $y = y(x)$, where k is a constant and $\phi(x)$ is a known forcing function. For this method to work, we require that $D_1 \neq 0$.

We assume that $y(x)$ is analytic and k is sufficiently small so that the solution to equation (146.1) can be expanded in a power series in k. That is, we take

$$y(x) = a_1(x)k + a_2(x)k^2 + a_3(x)k^3 + \cdots . \tag{146.2}$$

Using equation (146.2) in equation (146.1) and equating powers of k results in an infinite sequence of equations for the $\{a_i(x)\}$. This sequence of equations begins

$$\begin{aligned} D_1 a_1 &= \phi(x), \\ D_1 a_2 &= -D_2 a_1^2, \\ D_1 a_3 &= -[2D_2 a_1 a_2 + D_3 a_1^3], \\ D_1 a_4 &= -[D_2(a_2^2 + 2a_1 a_3) + 3D_3 a_1^2 a_2 + D_4 a_1^4]. \end{aligned} \tag{146.3.a-d}$$

The reversion method is to assume the solution to equation (146.1) can be represented in the form of equation (146.2) when $k = 1$ and the coefficients are given by equation (146.3).

Example

Suppose we have the following nonlinear ordinary differential equation

$$\frac{dv}{dx} + \alpha v^2 = x, \qquad v(0) = v_0,$$

and we seek an approximation near $x = 0$. Changing variables to $y = v - v_0$ changes the equation into

$$\frac{dy}{dx} + 2v_0\alpha y + \alpha y^2 = x - \alpha v_0^2, \qquad y(0) = 0, \tag{146.4}$$

which simplifies the initial condition. Comparing equation (146.4) to equation (146.1), we make the identifications

$$\begin{aligned} D_1 &= \frac{d}{dx} + 2v_0\alpha, \qquad D_2 = \alpha, \\ k &= 1, \qquad \phi(x) = x - \alpha v_0^2. \end{aligned}$$

From equation (146.3.a), we obtain the following equation for a_1: $D_1 a_1 = \phi(x)$, or

$$\left(\frac{d}{dx} + 2v_0\alpha\right) a_1 = x - \alpha v_0^2. \tag{146.5}$$

Because $v(0) = 0$, we will take $a_1(0) = a_2(0) = \cdots = 0$. The solution to equation (146.5) with $a_1(0) = 0$ is

$$a_1 = \frac{x}{2v_0\alpha} + \frac{e^{-2v_0\alpha x} - 1}{4v_0^2\alpha^2} + \alpha v_0^2 \frac{e^{-2v_0\alpha x} - 1}{2v_0\alpha},$$

which was obtained by using a Laplace transform (see page 350).

The function a_2 can be determined from equation (146.3.b)

$$\left(\frac{d}{dx} + 2v_0\alpha\right) a_2 = \alpha a_1^2 = \alpha \left(\frac{x}{2v_0\alpha} + \frac{e^{-2v_0\alpha x} - 1}{4v_0^2\alpha^2} + \alpha v_0^2 \frac{e^{-2v_0\alpha x} - 1}{2v_0\alpha}\right)^2,$$

with $a_2(0) = 0$. This can also be solved by using Laplace transforms. Proceeding in this way, many terms in the series equation (146.2) can be evaluated.

Notes

1. The above example is from Pipes and Harvill [1, pages 653–665].
2. The extension of equation (146.3) can be found in Orstrand [2], which lists formulae for the first 13 terms.

References

[1] Pipes, L. A., and Harvill, L. R. *Applied Mathematics for Engineers and Physicists.* McGraw–Hill Book Company, New York, 1970.

[2] Van Orstrand, C. E. *Philosophical Magazine 19* (1910), 366.

147. Singular Solutions

Applicable to Nonlinear ordinary differential equations.

Yields

A singular solution.

Idea

Singular solutions may exist where the implicit function theorem does not hold in differential algebraic equations.

Procedure

The algebraic ordinary differential equation

$$F(x, y, y', \dots, y^{(n)}) = 0 \tag{147.1}$$

can often be explicitely solved for the $y^{(n)}$ term to determine that

$$\begin{aligned} y^{(n)} &= G_1(x, y, \dots, y^{(n-1)}) \\ y^{(n)} &= G_2(x, y, \dots, y^{(n-1)}) \\ &\vdots \\ y^{(n)} &= G_i(x, y, \dots, y^{(n-1)}). \end{aligned} \tag{147.2}$$

By the implicit function theorem, if $\frac{\partial F}{\partial y^{(n)}}(x, y, y', \dots, y^{(n)}) \neq 0$, then the solutions in equation (147.2) are the only solutions possible. However, at those points where $\frac{\partial F}{\partial y^{(n)}}(x, y, y', \dots, y^{(n)}) = 0$, there exists the possibility of singular solutions.

If the $y^{(n)}$ term is algebraically eliminated from the two equations

$$\begin{aligned} F(x, y, y', \dots, y^{(n)}) &= 0, \\ \frac{\partial F}{\partial y^{(n)}}(x, y, y', \dots, y^{(n)}) &= 0, \end{aligned}$$

then an equation of the form

$$H(x, y, y', \dots, y^{(n-1)}) = 0 \tag{147.3}$$

results. This is called the *p-discriminant equation*. Its solution(s) describe the *singular loci*.

After equation (147.3) is solved to determine possible singular solutions, it must be verified that they are, in fact, actual solutions to the original equation (147.1). Typically, the solution to equation (147.3), being a differential equation of $(n-1)$-st order, will involve only $n-1$ arbitrary constants.

Example

Given the nonlinear first order ordinary differential equation

$$F(x, y, y') = xy'^2 - 3yy' + 9x^2 = 0, \tag{147.4}$$

it is straightforward to compute

$$\frac{\partial F}{\partial y'} = 2xy' - 3y = 0. \tag{147.5}$$

Eliminating the y' term between equation (147.4) and equation (147.5) results in

$$y = \pm 2x^{3/2}. \tag{147.6}$$

In this case, both of the solutions in equation (147.6) satisfy equation (147.4). Note that the singular solutions in equation (147.6) do not depend on any constants, even though equation (147.4) was a first order differential equation.

Notes

1. The general nth order ordinary differential equation, linear in the nth derivative term,

   $$U(x, y, y', \ldots, y^{(n-1)})y^{(n)} + V(x, y, y', \ldots, y^{(n-1)}) = 0,$$

   has the singular solution $y = z(x)$ if $z(x)$ satisfies both of

   $$U(x, z, z', \ldots, z^{(n-1)}) = 0,$$
   $$V(x, z, z', \ldots, z^{(n-1)}) = 0.$$

2. Another way to determine singular solutions of the differential equation $f(x, y, y') = 0$ is to obtain the general solution $\phi(x, y, C) = 0$ (where C is an arbitrary constant) and then formally eliminate C between the two equations

   $$\phi(x, y, C) = 0,$$
   $$\frac{\partial \phi}{\partial x}(x, y, C) = 0.$$

   The resulting equation, which only involves x and y, is called the *c-discriminant equation*.
   For example, the differential equation $y'^2 + 4 - 4y = 0$ has the general solution $y(x) = 1 + (x - C)^2$; hence $\phi(x, y, C) = y - 1 - (x + C)^2$. Forming the c-discriminant results in the singular solution $y = 1$.
3. In general (see Piaggio [7, pages 65–79 and 192–201]), the p-discriminant equation will contain the envelope of the solutions, the cusp-locus and the tac-locus squared. The c-discriminant equation will contain the envelope of the solutions, the cusp-locus cubed and the node-locus squared. Of these, only the envelope is a solution to the original differential equation.

4. Some envelope solutions of differential equations may be found by use of Lie groups; see Bluman [1].
5. For polynomial functions, the algebraic elimination in the computation of the c-discriminant (or the p-discriminant) can be done by the use of resultants (see page 50).
6. See also El'sgol'ts [2, pages 81–88], Goldstein and Braun [3, pages 18–24], Ince [4, pages 83–91], and Murphy [6, pages 74–80].

References

[1] Bluman, G. Invariant solution for ordinary differential equations. *SIAM J. Appl. Math. 50*, 6 (December 1990), 1706–1715.

[2] El'sgol'ts, L. E. *Differential Equations and the Calculus of Variations*. MIR Publishers, Moscow, USSR, 1970.

[3] Goldstein, M. E., and Braun, W. H. *Advanced Methods for the Solution of Differential Equations*. U.S. Government Printing Office, Washington, D.C., 1973. NASA SP-316.

[4] Ince, E. L. *Ordinary Differential Equations*. Dover Publications, Inc., New York, 1964.

[5] Kaplan, W. *Ordinary Differential Equations*. Addison–Wesley Publishing Co., Reading, MA, 1958.

[6] Murphy, G. M. *Ordinary Differential Equations and Their Solution*. D. Van Nostrand Company, Inc., New York, 1960.

[7] Piaggio, H. T. H. *An Elementary Treatise on Differential Equations and Their Applications*. G. Bell & Sons, Ltd, London, England, 1926.

148. Soliton-Type Solutions

Applicable to Partial differential equations with wave-like solutions, often partial differential equations with only two independent variables.

Yields

Knowledge of whether solitons can be present.

Idea

See if there is a solitary wave solution to the partial differential equation. This indicates the possibility that the equation has solitons for solutions.

Procedure

A solitary wave is a localized, traveling wave; many nonlinear partial differential equations have solutions of this type. A soliton is a solitary wave that exhibits particle-like behavior. The particle-like properties include stability, localizability, and finite energy. A soliton is best described, however, in terms of its interaction with other solitary waves. We say that an equation possesses solitons when two or more colliding solitary waves do not break up and disperse but, instead, become more solitary waves.

In this technique, we change variables in such a way as to make such a solitary wave more apparent. If the original partial differential equation were in the independent variables x and t, we search for a solution of the form $u(x-ct)$. Here c represents the wave speed; if $c>0$ ($c<0$), then $u(x-ct)$ represents a wave traveling to the right (left). Note that many partial differential equations have solitary waves as solutions; most of these partial differential equations do *not* exhibit soliton behavior.

Example

One representation of the Korteweg–de Vries (KdV) equation is given by

$$u_t + \sigma u u_x + u_{xxx} = 0. \tag{148.1}$$

We change the independent variables from $\{x, t\}$ to $\{\eta, \zeta\}$ via (see page 168) $\{\eta = t,\ \zeta = x - ct.\}$. This change of variable turns equation (148.1) into

$$u_\eta - c u_\zeta + \sigma u u_\zeta + u_{\zeta\zeta\zeta} = 0. \tag{148.2}$$

If we now *presume* that equation (148.1) admits a wave-like solution, we can then take $u(\eta, \zeta) = v(\zeta) = v(x-ct)$. By assuming this functional form for $u(\eta, \zeta)$, equation (148.2) becomes

$$c v_\zeta + \sigma v v_\zeta + v_{\zeta\zeta\zeta} = 0. \tag{148.3}$$

Equation (148.3) is an autonomous ordinary differential equation. Hence, the order can be reduced by 1 (see page 230). In fact, for the equation (148.3), the exact solution can be obtained.

Equation (148.3) can be integrated with respect to ζ to obtain

$$-cv + \frac{1}{2}\sigma v^2 + v_{\zeta\zeta} = A,$$

where A is an arbitrary constant. This last equation, when multiplied by v_ζ, can be integrated again to obtain

$$-\frac{1}{2}cv^2 + \frac{1}{6}\sigma v^3 + (v_\zeta)^2 = Av + B, \tag{148.4}$$

where B is another arbitrary constant. Equation (148.4) can be solved algebraically for v_ζ and then this first order ordinary differential equation can be integrated in terms of elliptic functions (see Abramowitz and Stegun [2]).

Hence, we have shown that the KdV equation has solitary waves as solution. For a soliton type solution to exist for equation (148.1), it must be determined that a solution of equation (148.4) exists that is localized (i.e., differs appreciably from zero only in a bounded region). Finally, to actually show that the KdV has solitons, the interaction of these solitary waves must be investigated. From a much deeper analysis (see, for example, Whitham [9, Chapter 17, pages 577–620]) it is possible to show that the Korteweg–de Vries equation possesses solitons as solutions. In fact, the KdV equation can have, as its solutions, any number of solitons.

Notes

1. The technique that we have presented is no more than using similarity variables (see page 497) to obtain a solution of a specific form. Of course, the boundary conditions must admit a traveling wave solution, as well as the equations.
2. The wave speed (c in the Example) often must be determined as part of the solution. In the above example, it would be determined by the boundary conditions (as would A and B). Typically, in nonlinear problems, the velocity is amplitude dependent.
3. See Ablowitz and Segur [1, Chapter 17, pages 587–607].

References

[1] Ablowitz, M. J., and Segur, H. *Solitons and the Inverse Scattering Transform*. SIAM, Philadelphia, PA, 1981.

[2] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, D.C., 1964.

[3] Calogero, F., and Degasperis, A. *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations*. North–Holland Publishing Co., New York, 1982.

[4] Dodd, R. K., Eilbeck, J. C., and Morris, H. C. *Solitons and Nonlinear Wave Equations*. Academic Press, New York, 1982.

[5] Drazin, P. G., and Johnson, R. S. *Solitons: An Introduction*. Cambridge University Press, New York, 1989.

[6] Eckhaus, W., and Harten, A. V. *The Inverse Scattering Transformation and the Theory of Solitons*. North–Holland Publishing Co., New York, 1981.

[7] Lamb, G. L. *Elements of Soliton Theory*. John Wiley & Sons, New York, 1980.

[8] Newell, A. C. *Solitons in Mathematics and Physics*. SIAM, Philadelphia, PA, 1985.

[9] Whitham, G. B. *Linear and Nonlinear Waves*. Interscience Publishers, Inc., New York, 1974.

149. Stochastic Limit Theorems

Applicable to Linear differential equations that contain a small parameter and a random forcing term of a certain form.

Yields

A Fokker–Planck equation.

Idea

Some equations do not have a "white noise" forcing term and so a Fokker–Planck equation cannot be directly constructed (see page 303). However, it is often true that random forcing terms behave like "white noise" in some asymptotic limit. Hence, in this limit, a Fokker–Planck equation can be constructed.

Procedure

If $F(\mathbf{x}, t, \tau)$ is a "sufficiently random" mean zero function then, as ϵ tends to zero, the form

$$\frac{1}{\epsilon} F\left(\mathbf{x}, t, \frac{t}{\epsilon^2}\right) \tag{149.1}$$

behaves, in a certain sense, like a "white noise" term (see Papanicolaou and Kohler [4]). Using the "white noise" equivalent of equation (149.1), a Fokker–Planck equation can be obtained in the variables $\{\mathbf{x}, t\}$.

Hence, the prescription is to change a given equation so it has a term in the form of equation (149.1) and then obtain and analyze the corresponding Fokker–Planck equation.

Example

Using the geometric optics approximation to the wave equation, the scaled position and velocity of a ray in a weakly random medium satisfy

$$\begin{aligned}
\frac{dx}{dt} &= v, \\
\frac{dv}{dt} &= \frac{1}{\epsilon} F\left(x, \frac{t}{\epsilon^2}\right),
\end{aligned}$$

after a ray has traveled a long distance in the random medium. Here $F(\cdot)$ is a random function with mean zero (it represents the wave speed perturbation at any point). Assuming a "mixing condition" on F, which is a statement about how random $F(\cdot)$ is, the theorem in Papanicolaou and Kohler [4] can be used in the limit of ϵ going to zero.

Using this theorem, it can be shown that the probability density of the solution to equation (149.1) converges weakly to the solution of the following Fokker–Planck equation

$$\gamma \frac{\partial^2 P}{\partial v^2} - \frac{\partial P}{\partial x} = \frac{\partial P}{\partial t},$$

where the number γ is defined by $\gamma^2 = -\int_0^\infty \mathrm{E}\,[F(0,y)(F(0,0)]\,dy$, and $\mathrm{E}\,[\cdot]$ is the expectation operator. The details of the derivation are beyond the scope of this book. More details may be found in Kulkarny and White [3].

Notes

1. There are many different limit theorems that yield a "white noise" limit. For example, Keston and Papanicolaou's paper [1] is concerned with random differential equations of the form

$$\frac{dx}{dt} = \frac{1}{\epsilon^2} v,$$
$$\frac{dv}{dt} = \frac{1}{\epsilon} F(x, v).$$

2. The theorems in Keston and Papanicolaou [1] and in Papanicolaou and Kohler [4] have many technical requirements that must be satisfied. The "mixing condition" requirement has been verified for only a few physical process.
3. For some limit theorems, the Fokker–Planck formalism can be eliminated completely. For example, in Khas'minskii [2]m it is shown that the solution to the problem

$$\frac{dx}{dt} = \epsilon F(x, t, \omega, \epsilon), \qquad x(0) = x_0,$$

in an interval of order $O(1/\epsilon)$, can be uniformly approximated by the solution to the problem $\frac{d\overline{x}}{dt} = \epsilon \overline{F}(\overline{x})$, $\overline{x}(0) = x_0$, where

$$\overline{F}(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{E}\,[F(x, t, \omega, \epsilon)]\,dt,$$

if the stochastic process $F(x, t, \omega, \epsilon)$ satisfies the law of large numbers for fixed x.
4. Pardoux [5] finds a white noise limit of a partial differential equation.

References

[1] Keston, H., and Papanicolaou, G. A limit theorem for stochastic acceleration. *Comm. Math. Physics 78* (1980), 19–63.

[2] Khas'minskii, R. Z. A limit theorem for the solutions of differential equations with random right-hand sides. *Theory Prob. Appl. 11*, 3 (1966), 390–405.

[3] Kulkarny, V. A., and White, B. S. Focussing of waves in turbulent inhomogeneous media. *Phys. Fluids 251*, 10 (1982), 1770–1784.

[4] Papanicolaou, G., and Kohler, W. Asymptotic theory of mixing stochastic ordinary differential equations. *Comm. Pure Appl. Math 27* (1974), 641–668.

[5] Pardoux, E. Asymptotic analysis of a semi-linear PDE with wide-band noise disturbances. In *Stochastic Space–Time Models and Limit Theorems*, L. Arnold and P. Kotelenz, Eds. D. Reidel Publishing Co., Boston, MA, 1985, pp. 227–242.

[6] Van Den Broeck, C. Stochastic limit theorems: Some examples from nonequilibrium physics. In *Stochastic Space–Time Models and Limit Theorems*, L. Arnold and P. Kotelenz, Eds. D. Reidel Publishing Co., Boston, MA, 1985, pp. 179–189.

[7] White, B., and Franklin, J. A limit theorem for stochastic two-point boundary value problems of ordinary differential equations. *Comm. Pure Appl. Math 32* (1979), 253–276.

150. Taylor Series Solutions

Applicable to Initial value problems, both ordinary differential equations and partial differential equations.

Yields

An approximation to the solution near a point.

Idea

For an initial value problem, a Taylor series expansion can give an approximate solution.

Procedure

We will illustrate the general procedure on a first order linear ordinary differential equation. Suppose we have the differential equation

$$y'(x) = F(x, y), \tag{150.1}$$

(where $'$ indicates differentiation with respect to x) with the initial condition $y(a) = y_0$, where $F(x, y)$ is a known function. Evaluating equation (150.1) at $x = a$, we can determine $y'(a) = F(a, y_0)$. Differentiating equation (150.1) with respect to x, and using the chain rule, results in

$$y''(x) = F_x(x, y) + F_y(x, y)y_x. \tag{150.2}$$

Now equation (150.2) can be evaluated at $x = a$ to explicitly determine

$$\begin{aligned} y''(a) &= F_x(a, y(a)) + F_y(a, y(a))y_x(a) \\ &= F_x(a, y_0) + F_y(a, y_0)F(a, y_0), \end{aligned}$$

where we have used $y'(a) = F(a, y_0)$.

We can continue this process of differentiating equation (150.1) and evaluating the result to determine the nth derivative of $y(x)$ at the point $x = a$. The result will involve only the partial derivatives of $F(x, y)$ and the numerical values a and y_0. Knowing these values allows us to construct the Taylor series expansion of $y(x)$ about $x = a$ by use of

$$y(x) = y(a) + \frac{y'(a)}{1!}(x-a)^1 + \frac{y''(a)}{2!}(x-a)^2 + \frac{y'''(a)}{3!}(x-a)^3 + \cdots . \tag{150.3}$$

Example

Suppose we wish to approximate the solution of the nonlinear initial value problem

$$\begin{aligned} y' &= x^2 - y^2, \\ y(0) &= 1. \end{aligned} \tag{150.4.a-b}$$

From equation (150.4), it is straightforward to compute

$$\begin{aligned} y'' &= 2x - 2yy', \\ y''' &= 2 - 2(y')^2 - 2yy'', \\ y'''' &= -6y'y'' - 2yy''', \\ &\vdots \end{aligned} \tag{150.5}$$

Using equation (150.4.b), we evaluate equation (150.4.a) and then equation (150.5) sequentially, at $x = 0$, to determine

$$\begin{aligned} y'(0) &= -1, \\ y''(0) &= 2, \\ y'''(0) &= -4, \\ y''''(0) &= 20, \\ &\vdots \end{aligned} \tag{150.6}$$

Using the values from equation (150.6) in equation (150.3), with $a = 0$, the solution of equation (150.4) for $y(x)$ near $x = 0$ is given by

$$\begin{aligned} &= 1 - x + \frac{2}{2!}x^2 - \frac{4}{3!}x^3 + \frac{20}{4!}x^4 + \cdots \\ &= 1 - x + x^2 - \frac{2}{3}x^3 + \frac{5}{6}x^4 + \cdots . \end{aligned}$$

Notes

1. This method may be applied to higher order equations and systems of equations.
2. The method of series solution (see page 403), when used at an ordinary point, also yields a Taylor series solution.
3. The Taylor series worked out by this method can be used to compute Padé approximates to the solution. These Padé approximates may give information about singularities of the exact solution (see page 582). Fernández *et al.* [3] have developed a different technique for determining the location of singular points by postulating a form of the singularity.
4. A direct representation of the Taylor series may be obtained by implicit differentiation. We find that the solution to the differential equation $y' = f(t, y)$, with $y(0) = 0$, has the Lie series representation

$$y(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \left[\left(\frac{\partial}{\partial t} + f(t, z) \frac{\partial}{\partial z} \right)^n z \right] \Big|_{z=0} . \tag{150.7}$$

See Igumnov [6] for a computationally efficient way to determine $y(t)$ from equation (150.7) when $f(t, y)$ has a known Taylor series. Finizio and Ladas [4, pages 293–298] also describe a numerical scheme.

5. The numerical technique of analytical continuation (see page 698) combines Taylor series at several different points to approximate the solution of a differential equation in a large region.
6. Taylor's theorem has been generalized in a way in which the general term is a fractional derivative (see Osler [9] for details).
7. Corliss and Chang [2] describe a Fortran program for solving ordinary differential equations by the use of Taylor series.
8. Macsyma [8] has a package (`taylor_ode`) which computes Taylor series solutions of ordinary differential equations.

References

[1] Corliss, G., and Lowery, D. Choosing a stepsize for Taylor series methods for solving ODE's. *J. Comput. Appl. Math. 3*, 4 (1977), 251–256.

[2] Corliss, G. F., and Chang, Y. F. Solving ordinary differential equations using Taylor series. *ACM Trans. Math. Software 8* (1982), 114–144.

[3] Fernandez, F. M., Arteca, G. A., and Castro, E. A. Singular points from Taylor series. *J. Math. Physics 28*, 2 (Feb 1987), 323–329.

[4] Finizio, N., and Ladas, G. *Ordinary Differential Equations with Modern Applications*. Wadsworth Publishing Company, Belmont, CA, 1982.

[5] Hunter, C., and Guerrieri, B. Deducing the properties of singularities of functions from their Taylor series coefficients. *SIAM J. Appl. Math. 39*, 2 (October 1980), 248–263.

[6] Igumnov, V. P. Representation of solutions of differential equations by modified Lie series. *Differential Equations 20* (1984), 683–688.

[7] Kochavi, E., and Segev, R. Numerical solution of field problems by nonconforming Taylor discretization. *Appl. Math. Modeling 15* (March 1991), 152–157.

[8] Macsyma. *Mathematics Reference Manual*. Macsyma, Arlington, MA, 1993.

[9] Osler, T. J. Taylor's series generalized for fractional derivatives and applications. *SIAM Review 2*, 1 (February 1971), 37—48.

[10] Razzaghi, M., and Razzaghi, M. Solution of linear two-point boundary value problems via Taylor series. *J. Franklin Inst. 326*, 4 (1989), 511–521.

151. Variational Method: Eigenvalue Approximation

Applicable to Differential equations with eigenvalues to be determined.

Yields

Estimates for the eigenvalues.

Idea

If we guess approximate eigenfunctions, then we will obtain approximations to the eigenvalues. The "better" we guess the eigenfunctions, the better the estimates of the eigenvalues will be.

Procedure

Although the procedure is quite general, we will discuss it in the specific context of a Sturm–Liouville equation. Suppose we have the Sturm–Liouville equation on the interval $[a, b]$

$$L[y] = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - s(x)y = -\lambda r(x)y, \tag{151.1}$$

with $p(x) > 0$, $s(x) \geq 0$, and $y(a) = y(b) = 0$. If we expand $y(x)$ as

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{151.2}$$

where the $\{\phi_n(x)\}$ are an arbitrary set of complete functions that vanish at $x = a$ and $x = b$, and the $\{c_n\}$ are constants, then the $\{c_n\}$ must satisfy

$$\sum_{n=1}^{\infty} (A_{mn} - \lambda R_{mn})\, c_n = 0, \tag{151.3}$$

for $m = 1, 2, \ldots$, where

$$\begin{aligned} A_{mn} &= \int_a^b \left[p(x)\phi_m'(x)\phi_n'(x) + s(x)\phi_m(x)\phi_n(x)\right] dx, \\ R_{mn} &= \int_a^b r(x)\phi_m(x)\phi_n(x)\, dx. \end{aligned} \tag{151.4}$$

Equation (151.3) is obtained by substituting equation (151.2) into equation (151.1), multiplying the result by $\phi_m(x)$, integrating with respect to x from a to b, and using integration by parts. If the $\{\phi_n(x)\}$ are the eigenfunctions of the $L[y]$ operator in equation (151.1), then the matrices A and R are diagonal matrices and the eigenvalues $\{\lambda_i\}$ are easily obtained.

If, instead of equation (151.2), we use the finite sum

$$y(x) = \sum_{n=1}^{N} c_n \psi_n(x), \tag{151.5}$$

where the $\{\psi_n(x)\}$ are chosen to satisfy the boundary conditions, then equation (151.3) becomes

$$\sum_{n=1}^{N} \left(\overline{A}_{mn} - \bar{\lambda}\overline{R}_{mn}\right) c_n = 0, \tag{151.6}$$

for $m = 1, 2, \ldots, N$. In this equation, $\overline{A}$ and $\overline{R}$ are given by equation (151.4) with $\phi_k(x)$ replaced by $\psi_k(x)$. For equation (151.6) to have a non-trivial solution, $\bar{\lambda}$ must satisfy

$$|\mathcal{A} - \bar{\lambda}\mathcal{R}| = 0,$$

where $\mathcal{A}$ is the matrix formed out of the $\overline{A}_{mn}$ and $\mathcal{R}$ is the matrix formed out of the $\overline{R}_{mn}$. If the $\{\psi_k(x)\}$ that we have have chosen are "close" to the actual eigenfunctions of equation (151.1), then the $\{\bar{\lambda}_k\}$ obtained from equation (151) will be "close" to the eigenvalues $\{\lambda_k\}$ of equation (151.1).

It is always true that the smallest $\bar{\lambda}$ from equation (151) is larger than the smallest λ of equation (151.1).

Example

Suppose an approximation to the smallest eigenvalues of the Sturm–Liouville system

$$\begin{aligned} y'' &= -\lambda y, \\ y(-1) &= y(1) = 0 \end{aligned} \tag{151.7}$$

is desired. Equation (151.7) has the same form as equation (151.1), with $p(x) = 1$, $s(x) = 0$, $r(x) = 1$, $a = -1$, and $b = 1$. We guess that $y(x)$ can be well approximated by

$$y(x) = c_1(1 - x^2),$$

which is equation (151.5) with $N = 1$ and $\psi_1(x) = (1 - x^2)$. Using equation (151.4), we calculate

$$\begin{aligned} \overline{A}_{11} &= \int_{-1}^{1} (-2x)(-2x)\, dx = \frac{8}{3}, \\ \overline{R}_{11} &= \int_{-1}^{1} (1 - x^2)(1 - x^2)\, dx = \frac{16}{15}. \end{aligned} \tag{151.8}$$

Using equation (151.8) in equation (151.6) yields the eigenvalue equation for $\bar{\lambda}$, $\frac{8}{3} - \frac{16}{15}\bar{\lambda} = 0$, and therefore, $\bar{\lambda} = 2.5$. For this example, it turns out that the smallest eigenvalue is exactly $\lambda = \pi^2/4 \simeq 2.467$, which corresponds to the eigenfunction $\phi(x) = \cos(\pi x/2)$.

Notes

1. For the Sturm–Liouville equation (151.1), it can be shown that

$$\lambda = \frac{\left(-pyy_x\right)\big|_a^b + \int_a^b \left(p(y')^2 + sy^2\right)\,dx}{\int_a^b ry^2\,dx}.$$

   This is known as the *Rayleigh quotient*. This can be used to estimate the lowest eigenvalue because

$$\lambda_1 \leq \min_{u(x)} \left[\frac{\left(-puu_x\right)\big|_a^b + \int_a^b \left\{p(u')^2 + su^2\right\}\,dx}{\int_a^b ru^2\,dx}\right],$$

   where λ_1 represents the smallest eigenvalue, and the minimization is taken over all continuous functions that satisfy the boundary conditions associated with equation (151.1) (but not necessarily the differential equation itself). See Haberman [2, pages 172–176 and 224–226] for details.
2. There are similar relations for the eigenvalues of partial differential equations, which are also called the Rayleigh quotient. (See Butkov [1] for details.) For example, for the Helmholtz equation in a bounded region, $\nabla^2 u + \lambda u = 0$ there is the relation (see Haberman [2])

$$\lambda = \frac{-\oint u\nabla u\cdot\mathbf{n}\,ds + \iint\limits_R |\nabla u|^2\,dx\,dy}{\iint\limits_R u^2\,dx\,dy}.$$

3. This section's example is from Butkov [1, pages 573–586].
4. See also Zauderer [4, pages 450–483].

References

[1] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[2] Haberman, R. *Elementary Applied Partial Differential Equations.* Prentice–Hall, Inc., Englewood Cliffs, NJ, 1968.

[3] Weinberger, H. F. *Variational Methods for Eigenvalue Approximation.* SIAM, Philadelphia, PA, 1974.

[4] Zauderer, E. *Partial Differential Equations of Applied Mathematics.* John Wiley & Sons, New York, 1983.

152. Variational Method: Rayleigh–Ritz

Applicable to Differential equations that come from a variational principle.

Yields

An approximation valid over an interval.

Idea

The variational expression from which a differential equation is derived can be used to approximate the solution.

Procedure

Most equations of mathematical physics and engineering arise from a variational principle (see page 418). For example, the first variation of

$$J[u] = \iint_D \left(u_x^2 + u_y^2 + 2uf \right) \, dx \, dy \tag{152.1}$$

(also known as the Euler–Lagrange equation associated with equation (152.1)) is given by

$$\delta J = u_{xx} + u_{yy} - f = 0.$$

Hence, the solution to

$$\begin{aligned} u_{xx} + u_{yy} &= f, \qquad \text{in the region } D, \\ u &= g, \qquad \text{on the boundary of } D, \end{aligned}$$

is given by that function $u(x, y)$ that equals g on the boundary and minimizes equation (152.1).

The Rayleigh–Ritz method is to determine the functional that a differential equation comes from and then to find an approximate minimum. This is done by choosing a sequence of functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$ and then forming

$$u_N(x, y) = a_1\phi_1(x, y) + a_2\phi_2(x, y) + \cdots + a_n\phi_n(x, y), \tag{152.2}$$

where the $\{a_i\}$ are unknown. Of course, the $\{\phi_k\}$ must be chosen in such a way that the boundary conditions are satisfied. Now, the $\{a_i\}$ are chosen in such a way that the functional will be minimized. Specifically, using equation (152.2) in equation (152.1) (or the appropriate variational principal), the $\{a_i\}$ are chosen by solving the simultaneous system of equations given by

$$\frac{\partial}{\partial a_i} J[u_N] = 0, \qquad \text{for } i = 1, \ldots, N.$$

This will often be a simultaneous system of polynomial equations.

If the $\{\phi_i\}$ in equation (152.2) are chosen "well," then u_N will tend to u as $n \to \infty$.

Example 1

Suppose we wish to approximate the solution to the following Poisson equation in the unit square

$$\begin{aligned} u_{xx} + u_{yy} &= \sin \pi x, && \text{for } 0 < x < 1, \quad 0 < y < 1, \\ u &= 0, && \text{on } x = 0,\ x = 1,\ y = 0,\ y = 1. \end{aligned} \tag{152.3.a-b}$$

The above equation comes from the variational principle $\delta J = 0$, where

$$J[u] = \int_0^1 \int_0^1 \left(u_x^2 + u_y^2 + 2u \sin \pi x \right) \, dx \, dy. \tag{152.4}$$

We choose to approximate $u(x, y)$ by a linear combination of

$$\begin{aligned} \phi_1(x, y) &= x(1-x)y(1-y), \\ \phi_2(x, y) &= x^2(1-x)y(1-y), \\ \phi_3(x, y) &= x(1-x)y^2(1-y). \end{aligned}$$

Note that each of the $\{\phi_i\}$ vanish on the boundary of the square, and so u_3 will also (as equation (152.3.b) requires).

Using equation (152.2) (with $N = 3$) in equation (152.4) results in the minimization of the function

$$\Big[24\pi^3 a_3^2 + \left(35\pi^3 a_2 + 70\pi^3 a_1 + 2100 \right) + 24\pi^3 a_2^2 \\ + \left(70\pi^3 a_1 + 2100 \right) a_2 + 70\pi^3 a_1^2 + 4200 a_1 \Big] / 3150\pi^3. \tag{152.5}$$

Differentiating equation (152.5) with respect to each of a_1, a_2, and a_3 results in the linear system of equations

$$\begin{bmatrix} 140\pi^3 & 70\pi^3 & 70\pi^3 \\ 70\pi^3 & 48\pi^3 & 35\pi^3 \\ 70\pi^3 & 35\pi^3 & 48\pi^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -4200 \\ -2100 \\ -2100 \end{bmatrix},$$

with the solution $\{a_1 = -\frac{30}{\pi^3},\ a_2 = 0,\ a_3 = 0\}$. Using these values in equation (152.2) yields an approximation to the solution of equation (152.3).

Note that the exact solution to the problem in equation (152.3) can be found by finite Fourier transforms (see page 344) to be

$$u(x, y) = \frac{\sin \pi x}{\pi^2 \sinh \pi} \left[\sinh \pi y + \sinh(\pi(1 - y)) - \sinh \pi \right]. \tag{152.6}$$

Figure 152.1 has a comparison of the exact solution in (152.6) and the approximate solution found above. This figure compares the values of $u(0.1, y)$ and $u_3(0.1, y)$ as y varies from 0 to 1.

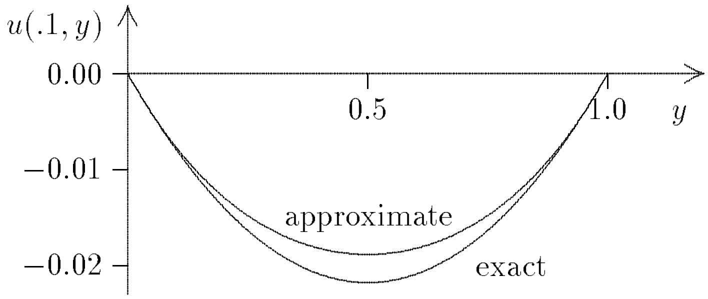

Figure 152.1: A comparison of the exact solution in equation (152.6) and the approximate solution in equation (152.2), when $x = 0.1$.

Example 2

A variation of this method, due to Kantorovich, is to choose the $\{\phi_k\}$ to depend only on y and to allow the $\{a_k\}$ to depend on x. For example, to approximate the solution of the Poisson equation

$$\begin{aligned} u_{xx} + u_{yy} &= -2, && \text{for } 0 < x < 1, \quad 0 < y < 1, \\ u &= 0, && \text{on } x = 0, x = 1, y = -1, y = 1, \end{aligned} \tag{152.7.a-b}$$

which corresponds to the first variation of

$$J[u] = \int_0^1 \int_{-1}^1 \left(u_x^2 + u_y^2 - 4u\right)\, dx\, dy, \tag{152.8}$$

we choose

$$u(x, y) \approx v(x, y) = f(x)(y^2 - 1). \tag{152.9}$$

where $f(x)$ is unknown. Using equation (152.9) in equation (152.8) results in

$$J[v] = \int_0^1 \left(\frac{16}{15} f'^2 + \frac{8}{3} f^2 + \frac{16}{3}\right) dx, \tag{152.10}$$

which must now be minimized. The first variation of equation (152.10) yields the following differential equation for $f(x)$

$$f'' - \frac{5}{2} f = \frac{5}{2}. \tag{152.11}$$

The function $f(x)$ must satisfy $f(0) = f(1) = 0$ for equation (152.7.b) to be satisfied. Solving equation (152.11) with these boundary conditions results in

$$f(x) = -1 + \cosh \alpha x + \left(\frac{1 - \cosh \alpha}{\sinh \alpha}\right) \sinh \alpha x, \tag{152.12}$$

where $\alpha = \sqrt{10}/2$. Combining equation (152.12) with equation (152.9) results in the final approximation to equation (152.7).

Notes

1. The Rayleigh–Ritz method also works for ordinary differential equations. For example, the variational principle corresponding to $J[u] = \int_0^1 [(y')^2 + y^2]\,dx$ is $\delta J = y'' + y = 0$.
2. This method is an example of a *weighted residual method* (see page 786).
3. This technique is often implemented numerically.
4. Example 2 is from Casti and Kalaba [2, pages 68–69].
5. See also Butkov [1, pages 573–586], Farlow [3, Lesson 45, pages 362–369], Kantorovich and Krylov [4, Chapter 4, pages 241–357], Mikhlin and Smolitskiy [5, Chapter 3, pages 147–269], Stakgold [6, pages 539–544], and Zauderer [7, pages 470–483].

References

[1] Butkov, E. *Mathematical Physics.* Addison–Wesley Publishing Co., Reading, MA, 1968.

[2] Casti, J., and Kalaba, R. *Imbedding Methods in Applied Mathematics.* Addison–Wesley Publishing Co., Reading, MA, 1973.

[3] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers.* John Wiley & Sons, New York, 1982.

[4] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis.* Interscience Publishers, Inc., New York, 1958.

[5] Mikhlin, S. G., and Smolitskiy, K. L. *Approximate Methods for Solutions of Differential and Integral Equations.* American Elsevier Publishing Company, New York, 1967.

[6] Stakgold, I. *Green's Functions and Boundary Value Problems.* John Wiley & Sons, New York, 1979.

[7] Zauderer, E. *Partial Differential Equations of Applied Mathematics.* John Wiley & Sons, New York, 1983.

153. WKB Method

Applicable to

Linear differential equations.

Yields

A global approximation.

Idea

The solution of an ordinary differential equation near an irregular singular point is often in the form of an exponential. Conversely, an exponential will often be a good approximation to an ordinary differential equation (even one without an irregular singular point.)

Procedure

If a given ordinary differential equation does not have a small parameter in it, multiply the highest order derivative term by a "small" parameter ϵ^2. This turns the equation into a singularly perturbed differential equation. Later, we will set ϵ equal to 1, and recover the original equation.

Given a singularly perturbed linear ordinary differential equation (of any order) $L[y] = 0$, look for a solution of the form

$$y(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right], \tag{153.1}$$

where we consider $\delta = \delta(\epsilon)$ to be a small number.

The technique is to use the approximation in (153.1) in the original equation and then apply dominant balance (see page 517) to determine a differential equation for $S_0(x)$. Solve this equation for $S_0(x)$. Then, using this solution for $S_0(x)$, apply dominate balance again to determine the next largest term. This will be a differential equation for the unknown $S_1(x)$. Solve this equation, and then iterate this procedure to determine several of the $\{S_i(x)\}$.

In order for the WKB approximation to be valid on an interval, we require that $\delta^n S_{n+1} \ll 1$ as $\delta \to 0$ and that $S_{n+1}(x)/S_n(x)$ be a bounded function of x on the given interval (for $n = 1, 2, \ldots$). If these do not hold, the expansion procedure is not valid. Note that if we have $\delta = 1$, the constraints on $\{S_i\}$ become constraints on the interval where the approximation is valid.

Special Case

For the singularly perturbed linear second order ordinary differential equation

$$\epsilon^2 y'' = Q(x)y, \tag{153.2}$$

with $Q(x) \neq 0$, we use equation (153.1) in equation (153.2) to determine

$$\frac{\epsilon^2}{\delta^2}\left(S_0'\right)^2 + \frac{2\epsilon^2}{\delta} S_0' S_1' + \frac{\epsilon^2}{\delta} S_0'' + \cdots = Q(x), \tag{153.3}$$

where the exponential term common to both sides has been factored out. The largest terms in equation (153.3) are $(S_0')^2\epsilon^2/\delta^2$ and $Q(x)$. Because $Q(x)$ is presumed to be of order one, we must have $\delta = \epsilon$ and $\left(S_0'\right)^2 = Q(x)$, or

$$S_0(x) = \pm \int^x \sqrt{Q(t)}\, dt. \tag{153.4}$$

Using $\delta = \epsilon$ and equation (153.4) in equation (153.3) and applying dominant balance again, yields a first order differential equation for $S_1(x)$

$$2S_0' S_1' + S_0'' = 0,$$

which can be integrated directly to yield

$$S_1(x) = -\frac{1}{4} \log Q(x). \tag{153.5}$$

Using equation (153.4) and equation (153.5) in equation (153.1), we determine the leading order approximation to the solution of equation (153.1) to be

$$\begin{aligned} y(x) \sim\, & C_1 \left[Q(x)\right]^{-1/4} \exp\left(\frac{1}{\epsilon} \int^x \sqrt{Q(t)} dt\right) \\ & + C_2 \left[Q(x)\right]^{-1/4} \exp\left(-\frac{1}{\epsilon} \int^x \sqrt{Q(t)} dt\right), \end{aligned} \tag{153.6}$$

for some constants C_1 and C_2. If a higher order approximation was desired, it is easy to derive that

$$\begin{aligned} S_2(x) &= \pm \int^x \left[\frac{Q''}{8Q^{3/2}} - \frac{5(Q')^2}{32Q^{5/2}}\right] dt, \\ S_3(x) &= \frac{Q''}{16Q^2} + \frac{5(Q')^2}{64Q^3}, \end{aligned}$$

because all of the equations for the higher order $\{S_i(x)\}$ are of first order.

Marić and Tomić [10] show that equation (153.6) is the correct asymptotic result if $\int^\infty \sqrt{Q}\, dt = \infty$ and $\int^\infty {Q'}^2 Q^{-5/2}\, dt < \infty$.

Example

Given the Airy equation

$$y'' = xy, \tag{153.7}$$

we introduce a small parameter ϵ^2 and write equation (153.7) as $\epsilon^2 y'' = xy$. This is now an equation of the same form as equation (153.2), with $Q(x) = x$. Hence, the approximation in equation (153.6) (with $\epsilon = 1$) yields

$$y(x) \sim C_1 x^{-1/4} \exp\left(\frac{2}{3}x^{3/2}\right) + C_2 x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right). \tag{153.8}$$

If we had included the $S_2(x)$ term, the approximation would be

$$\begin{aligned} y(x) \sim\ & C_1 x^{-1/4} \exp\left(\frac{2}{3}x^{3/2}\right)\left(1 + \frac{5}{48}x^{-3/2}\right) \\ & + C_2 x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right)\left(1 - \frac{5}{48}x^{-3/2}\right). \end{aligned} \tag{153.9}$$

In both equations (153.8) and (153.9), the approximations are valid only as $x \to \infty$.

Notes

1. WKB stands for Wentzel, Kramers, and Brillouin. This method is also sometimes called the WKBJ method or the Jeffreys method.
2. The eigenvalue problem $z'' + \lambda^2 V(x) z = 0$ with $z(0) = z(l) = 0$ can be analyzed by the WKB method. Using equation (153.6), the approximate solution is $z(x) = A(x) \sin\left(-\lambda^{-1} \int^x \sqrt{V(t)}\, dt + \phi(x)\right)$. The eigenvalues $\{\lambda_i\}$ are determined by where the oscillatory function vanishes. To leading order, as $n \to \infty$, the eigenvalues satisfy $\lambda_n = n\pi/L$, where $L = \int_0^l \sqrt{V(t)}\, dt$. A correction to this formula is in Lindblom and Robiscoe [7].
3. Ludwig [8] illustrates how the WKB method may be applied to partial differential equations.
4. The WKB approximation results in an asymptotic series. Hence, as more terms are taken in equation (153.1), the result may diverge.
5. WKB is a singular perturbation technique and boundary layer theory (see page 590) may be derived from it.
6. The approximation $y(x) \simeq \exp\left[\frac{S_0(x)}{\delta}\right]$ is often called the *geometrical optics approximation*. The approximation $y(x) \simeq \exp\left[\frac{S_0(x)}{\delta} + S_1(x)\right]$ is often called the *physical optics approximation*.
7. For the linear ODE of degree n, $\epsilon \frac{d^n y}{dx^n} = Q(x) y$, the physical optics approximation is $y(x) \simeq \exp\left[\frac{S_0(x)}{\delta} + S_1(x)\right]$ with $\delta = \epsilon^{1/n}$ and

$$S_0 = \omega \int^x [Q(x)]^{1/n}\, dt, \qquad S_1 = \frac{1-n}{2n} \log Q(x),$$

   where ω is any of the nth roots of unity (i.e., $\omega^n = 1$).

8. In regions where $Q(x)$ does not vanish, the classical WKB solutions of equation (153.2) in equation (153.6) are valid. Points where $Q(x)$ is equal to zero are called *turning points* or *transition points*; the solutions in (153.6) are not valid at these points. However, the *Langer connection formula* shows how the solution on each side of a turning point may be connected.

   Consider equation (153.2) when $Q(x)$ has a single, simple zero at $x = 0$ and is monotonically increasing everywhere. We presume the boundary condition $y(\infty) = 0$, to avoid the exponentially growing solution in equation (153.6) when $x \to \infty$. Consider a region that contains the turning point $x = 0$. Dividing this region into three smaller regions (with the turning point in the center region), asymptotic approximation may be obtained in each region. (Use WKB in the two outer regions, linearize $Q(x)$ in the center region, and write the answer in terms of Airy functions). By appropriate matching (see page 590), the arbitrary constants in these three solutions can be related. Hence, a uniformly valid approximation is given by:

$$y_{\text{unif}}(x) = C S_0^{1/6} Q(x)^{-1/4} \operatorname{Ai}\left[\left(\frac{3}{2\epsilon} S_0(x)\right)^{2/3}\right],$$

   where $S_0(x) = \int_0^x \sqrt{Q(t)}\,dt$ and C is an arbitrary constant.

   Many extensions to this simple formula have been found. The ordinary differential equations considered can be of higher order, there can be multiple turning points, and the turning point need not be simple. Wazwaz [14] considers a singular perturbation problem for a second order ordinary differential equation with two interior points of second order.

9. Note that WKB approximations to the two linearly independent solutions to $\epsilon y'' + a(x)y' + b(x)y = 0$ have the form

$$y_1(x) \approx c_1 \exp\left[-\int^x \frac{b(t)}{a(t)}\,dt\right],$$
$$y_2(x) \approx \frac{c_2}{a(x)} \exp\left[\int^x \frac{b(t)}{a(t)}\,dt - \frac{1}{\epsilon}\int^x a(t)\,dt\right],$$

   as $\epsilon \to 0^+$. See Bender and Orszag [1, Example 4 in Section 10.1].

10. Fedoryuk [3] considers the equation $\epsilon y'' + f(x, y) = 0$.
11. See Bender and Orszag [1, Chapter 10, pages 484–543].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw–Hill Book Company, New York, 1978.

[2] Farrell, P. A. Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem. *SIAM J. Numer. Anal. 25*, 3 (June 1988), 618–643.

[3] Fedoryuk, M. V. The WKB-method for a non-linear equation of the second order. *U.S.S.R. Comput. Maths. Math. Phys. 26*, 1 (1986), 121–128.

[4] Giler, S. Generalised WKBJ formulae. *J. Phys. A: Math. Gen. 21* (1988), 909–930.

[5] Kesarwani, R. N., and Varshni, Y. P. Five-term WKBJ approximation. *J. Math. Physics 21* (1980), 90–92.

[6] Langer, R. The asymptotic solutions of certain linear differential equations of the second order. *Trans. Amer. Math. Soc. 36* (1934), 90–106.

[7] Lindblom, L., and Robiscoe, R. T. Improving the accuracy of WKB eigenvalues. *J. Math. Physics 32*, 5 (May 1991), 1254–1258.

[8] Ludwig, D. Persistence of dynamical systems under random perturbations. *SIAM Review 17*, 4 (October 1975), 605–640.

[9] Lynn, R., and Keller, J. B. Uniform asymptotic solutions of second-order linear ordinary differential equations with turning point. *Comm. Pure Appl. Math 23* (1970), 379–408.

[10] Maric, V., and Tomic, M. On Liouville–Green (WKB) approximation for second order linear differential equations. *Differential Integral Equations 1*, 3 (1988), 299–304.

[11] McHugh, J. An historical survey of ordinary linear differential equations with a large parameter and turning points. *Arch. Hist. Exact. Sci. 7* (1971), 277–324.

[12] Sawi, M. E. On the WKBJ approximation. *J. Math. Physics 28*, 3 (March 1987), 556–558.

[13] Taylor, J. G. Improved error bounds for the Liouville–Green (or WKB) approximation. *J. Math. Anal. Appl. 85* (1982), 79–89.

[14] Wazwaz, A.-M. Two turning points of second order. *SIAM J. Appl. Math. 50*, 3 (June 1990), 883–892.

[15] Willner, B., and Rubenfeld, L. A. Uniform asymptotic solutions for a linear ordinary differential equation with one zzzref52refzzz-th order turning point: Analytic theory. *Comm. Pure Appl. Math 26* (1976), 343–367.

154. Introduction to Numerical Methods

Numerical analysis is a rapidly growing field, with new techniques being developed constantly. Presented in the last section of this book are some of the more commonly used methods. This section has been separated into three parts:

- Introductory material about numerical methods
- Methods that can be used for ordinary differential equations and, sometimes, also partial differential equations (When a method in this part can be used for a partial differential equation, there is a star (*) alongside the method number.)
- Methods that can be used only for partial differential equations

For some of the numerical methods presented in this section, a short C or Fortran computer program has been given. None of the codes have been optimized for performance. To economize on space, many of the comments that would normally appear in a well-documented computer code have been removed. When a C or Fortran computer code is given, the output is also indicated.

Below are some useful thoughts when solving differential equations numerically.

- Use prepared software packages whenever possible. Numerical codes are available for solving nearly any type of ordinary differential equation (see page 654).
- When writing a computer program, always test it on problems for which you know the solution, either analytically or from a different, reliable computer code.
- Perform numerical calculations with as many digits of precision as is reasonable for efficient execution. However, it is rarely useful to use less than "double precision."
- The standard way to determine if a numerical scheme is implemented correctly and the mesh sizes are small enough to justify the *a priori* error estimates is to reduce the size of the mesh and re-run the calculation. The resulting *a posteriori* error estimates should agree with the *a priori* error estimates.
- When choosing a numerical scheme to approximate the solution to a differential equation, the roundoff error should be balanced with the truncation error of the machine being used. A higher order method will not give more accurate answers if the major component of the error is due to roundoff. Likewise, performing calculations in "double

precision" will not give more accurate answers if the major component of the error is due to the discretization scheme.

As a rule of thumb, to calculate a first derivative by forward differences, the roundoff error and the truncation error will be approximately equal (and so accuracy will be high) if the difference in values used is the square root of the number of significant digits. For example, if your computer is working with 20 decimal digits of precision, then an accurate numerical approximation to the derivative of $y(t)$ will be obtained by $[y(t) - y(t + \Delta t)]/\Delta t$ for $\Delta t \simeq 10^{-10}$.

- Note that several of the methods described in earlier parts of this book may be readily implemented numerically. For some of those methods, references have been given that refer to numerical implementations. No mention of those methods is made in this section.
- Listed below are, in the author's opinion, the most useful methods appearing in this last section. These are the methods that might be tried first when a numerical approximation is required.
- In the numerical analysis of differential equations, there are many important topics that are not addressed in this book. These include
  1. Numerical boundary conditions for exterior problems (see Hagstrom and Hariharan [1])
  2. Efficiency of differential equation integration techniques (see Hosea and Shampine [2])
  3. Use of splines (see Sallam and Ameen [3])

Most Useful Methods for ODEs

- Boundary Value Problems: Box Method (page 701)
- Boundary Value Problems: Shooting Method* (page 706)
- Continuation Method* (page 710)
- Euler's Forward Method (page 730)
- Finite Element Method* (page 734)
- Predictor–Corrector Methods (page 759)
- Runge–Kutta Methods (page 763)
- Stiff Equations* (page 770)
- Weighted Residual Methods* (page 786)

Most Useful Methods for PDEs

- Continuation Method* (page 710)
- Finite Element Method* (page 734)
- Weighted Residual Methods* (page 786)
- Elliptic Equations: Finite Differences (page 805)
- Elliptic Equations: Relaxation (page 816)
- Hyperbolic Equations: Method of Characteristics (page 820)
- Hyperbolic Equations: Finite Differences (page 824)
- Method of Lines (page 831)
- Parabolic Equations: Implicit Method (page 839)

- Pseudospectral Method (page 851)

References

[1] Hagstrom, T., and Hariharan, S. I. Accurate boundary conditions for exterior problems in gas dynamics. *Math. of Comp. 51*, 184 (October 1988), 581–597.

[2] Hosea, M. E., and Shampine, L. F. Efficiency comparisons of methods for integrating ODEs. *Comp. & Maths. with Appls. 28*, 6 (1994), 45–55.

[3] Sallam, S., and Ameen, W. Numerical solution of general nth-order differential equations via splines. *Appl. Num. Math. 6* (1989/1990), 225–238.

155. Definition of Terms for Numerical Methods

A-stable A linear multistep method is A-stable if all solutions of the difference equation generated by the application of this method to the scalar test equation, $y' = \lambda y$, tend to zero as $x \to \infty$ for all complex λ with $\operatorname{Re} \lambda < 0$ and for all fixed step sizes h with $h > 0$. Note that an explicit multistep method cannot be A-stable.

Computational molecule A computational molecule is a pictorial representation of a finite difference scheme for a partial differential equation in two independent variables. In such a figure, the circles indicate which points are related by a difference scheme; the value being determined by the difference scheme is often shown shaded. For example, the computational molecule for the so-called "five-point star" approximation to the Laplacian, $\nabla^2 u_{i,j} \simeq \frac{1}{4}(u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1})$, is shown in figure 155.1.a. The computational molecule for the following explicit finite difference approximation to $u_t = u_{xx}$

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta t} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta x)^2}$$

is shown in figure 155.1.b.

Consistency of a finite difference scheme A method is consistent if the truncation errors tend to zero as the mesh is refined (i.e., as the characteristic scales in the mesh $\{\Delta x, \Delta t, \dots\}$ tend to zero). There are two types of consistency:

- **Conditionally consistent** If the truncation errors only tend to zero if $\{\Delta x, \Delta t, \dots\}$ tend to zero in a certain way. For example, it may be required that $(\Delta x)^2 < \Delta t$.
- **Unconditionally consistent** If the truncation errors tend to zero no matter how $\{\Delta x, \Delta t, \dots\}$, tend to zero.

Conservative scheme A conservative numerical scheme is one in which the "total energy" described by the differential system is conserved during the integration of the system.

Difference scheme A difference scheme is an approximation of a derivative term at a point by a collection of values near the point.

- **Centered scheme** A centered scheme is symmetric about the point at which the derivative is being approximated. For example, $y'(x) \simeq \frac{y(x+h)-y(x-h)}{2h}$, when $h \ll 1$.

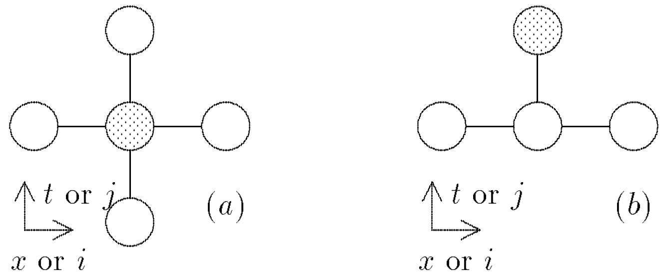

Figure 155.1: Computational molecules for two different approximations.

One-sided scheme A one-sided scheme uses values only from one side of the point at which a derivative is being approximated. Examples are forward and backward difference schemes.

Forward difference scheme A forward difference scheme is a one-sided difference scheme that uses points "ahead" of the point that is being approximated. For example, $y'(x) \simeq \frac{y(x+h)-y(x)}{h}$, when $h \ll 1$.

Backward difference scheme A backward difference scheme is a one-sided difference scheme that uses points "behind" the point that is being approximated. For example, $y'(x) \simeq \frac{y(x)-y(x-h)}{h}$, when $h \ll 1$.

Explicit method An explicit method is one for which there is an explicit formula, at a point, for the value of the unknown terms appearing in the differential equation.

Grid A grid is a set of points, called *mesh points*, on which the solution of a differential equation is approximated. If the points are uniformly spaced, then we have a *uniform grid*; otherwise we have a *non-uniform grid*. See page 675.

Implicit method An implicit method is one for which there is not an explicit formula, at a point, for the value of the unknown terms appearing in the differential equation. Generally a nonlinear algebraic equation must be solved to determine the value at a given point.

Mesh See Grid.

Order of a numerical method One less than the exponent in the error term of a method. See page 670.

Step size See page 670.

Stiff equations Stiff equations are differential equations that are ill posed in a computational sense. There are many different definitions of stiffness, two common ones are

- A system of differential equations is said to be stiff on the interval $[0, T]$ if there exists a component of a solution of the system that has a variation on $[0, T]$ that is large compared with $1/T$.
- A system is stiff if there exists more than one scale, with a great difference in size, on which the solution evolves. For instance, the system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ (where $\mathbf{A}$ is a constant matrix with eigenvalues $\lambda_i(\mathbf{A})$) is stiff if $\max_i |\lambda_i(\mathbf{A})| \gg \min_i |\lambda_i(\mathbf{A})|$.

Symplectic integration An integration method is said to be symplectic if the state of the (Hamiltonian) system following an integration step could have been reached from that before the step by some canonical transformation. The most straightforward way to test if a method is sympletic is to verify the Poisson-bracket relations between the before and after states. Given a method that determines $\mathbf{u}(\mathbf{x})$, where $\mathbf{u}$ and $\mathbf{x}$ are both s-dimensional, let J be the $s \times s$ Jacobian matrix that leads from "before" to "after": $J = \frac{\partial(\mathbf{u}_n, \mathbf{x}_n)}{\partial(\mathbf{u}_{n-1}, \mathbf{x}_{n-1})}$. Now define the matrix $K = \begin{bmatrix} 0_s & I_s \\ -I_s & 0_s \end{bmatrix}$. If $J^{\mathrm{T}} K J = K$, then the method is sympletic.

Truncation error The error when the exact solution is substituted into a finite difference scheme. See page 670.

156. Available Software

Applicable to Ordinary and partial differential equations that are to be approximated numerically.

Idea

When numerically approximating the solution to a differential equation, it is best to use commercially available software whenever possible. The routines commonly available for ordinary differential equations are adequate for nearly all types of problems. The routines commonly available for partial differential equations are not as well developed. For linear problems with no singularities, however, the available software is very good.

There are a multitude of commercially available computer libraries and isolated computer routines available. A taxonomy for differential equation software has been developed as part of the *Guide to Available Mathematical Software* (GAMS) project at the National Institute of Standards and Technology (NIST) [5], see table 156.1. GAMS [6] also has a listing of available software.

Because good software is readily available, we paraphrase the admonition that Byrne and Hindmarsh [8] give:

> ... if you are using a 10-line solver for differential equations ... you should consider using one of the programs referenced in this section. There is now commercially available "software" for differential equations with no error control, a user-specified step size, and no warning messages. We advise against using such programs, even on a small computer. The reasons are straightforward. For all but trivial problems, such programs cannot be sufficiently reliable for accurate computational results.

When using a prepared software package, it is always useful to test the package on problems similar to the one that you will use the package for. There are many collections of test problems for this purpose, see page 694.

Notes

1. Given a new problem to solve numerically, it is often attractive to design new software for this class of problem. However, it is usually more efficient to transform the problem and use well-tested codes. See, for example, Shampine and Zhang [25].
2. Addison *et al.* [2] present a decision tree to assist in the process of selecting an appropriate algorithm for the numerical solution of initial value ordinary differential equations. The decision tree can be used in an interactive manner. Where possible, the recommended software routines are in maintained libraries that have been extensively

I1	Ordinary differential equations (ODEs)
I1a	Initial value problems
I1a1	General, nonstiff, or mildly stiff
I1a1a	One-step methods (e.g., Runge–Kutta)
I1a1b	Multistep methods (e.g., Adams predictor-corrector)
I1a1c	Extrapolation methods (e.g., Bulirsch–Stoer)
I1a2	Stiff and mixed algebraic-differential equations
I1b	Multipoint boundary value problems
I1b1	Linear
I1b2	Nonlinear
I1b3	Eigenvalue (e.g., Sturm–Liouville)
I1c	Service routines (e.g., interpolation of solutions, error handling, test programs)
I2	Partial differential equations
I2a	Initial boundary value problems
I2a1	Parabolic
I2a1a	One spatial dimension
I2a1b	Two or more spatial dimensions
I2a2	Hyperbolic
I2b	Elliptic boundary value problems
I2b1	Linear
I2b1a	Second order
I2b1a1	Poisson (Laplace) or Helmholtz equation
I2b1a1a	Rectangular domain (or topologically rectangular in the coordinate system)
I2b1a1b	Nonrectangular domain
I2b1a2	Other separable problems
I2b1a3	Nonseparable problems
I2b1c	Higher order equations (e.g., biharmonic)
I2b2	Nonlinear
I2b3	Eigenvalue
I2b4	Service routines
I2b4a	Domain triangulation *(search also GAMS class P)*
I2b4b	Solution of discretized elliptic equations

Table 156.1: The GAMS taxonomy of differential equations software

tested. Addison *et al.* [3] contains a decision tree for boundary value problems.

3. Periodically, there are reviews in the literature of software applicable to a specific type of differential equation. See, for example, Machura and Sweet [17].

4. The books by Press *et al.* [22], contain collections of Fortran, PASCAL, and C codes for both ordinary differential equations and partial differential equations.
5. Many scientific software routines, including those for differential equations, may be obtained for free (via electronic mail) from a variety of computer networks. See the article by Dongarra and Grosse [12].
   - The ACM's *Transactions on Mathematical Software* (TOMS) is available at `http://gams.nist.gov/toms/Overview.html`.
   - Netlib is a collection of mathematical software, papers, and databases. It can be reached at `http://www.netlib.org`.
6. Numerical methods for first order PDEs may be found in Pennington and Berzins [20].
7. Even though it is possible to numerically approximate differential equations using spreadsheet programs, this is *not* recommended; see Enloe [14].
8. Software for small computers is summarized in Penn [19] and Teles *et al.* [26], [27].
9. Software is not listed for all of the GAMS taxonomy classes that have been established.
10. The following computer libraries are referred to in GAMS[1]. Their inclusion does not constitute an endorsement. Nor does it necessarily imply that unnamed packages are not worth trying. (All of the information in this note has been obtained from `http://gams.nist.gov`).
    - BIHAR

      A package of Fortran subprograms for the generalized biharmonic equation in rectangular geometry and polar coordinates subject to first kind boundary conditions. Distributed by netlib, see `http://www.netlib.org/bihar`.
    - CMLIB

      The NIST Core Math LIBrary (CMLIB) is a collection of high-quality, easily transportable Fortran subroutine sublibraries solving standard problems in many areas of mathematics and statistics (approximately 750 subroutines and functions). It is distributed by the Center for Computing and Applied Mathematics at NIST. The source for CMLIB has come from
      - BVSUP: see Scott and Watts [24]
      - CDRIV and SDRIV: see Kahaner *et al.* [16]
      - DEPAC: Code developed by Shampine and Watts.
      - FISHPAK: Code developed by Swarztrauber and Sweet.
      - SDASSL: see Petzold [21]
      - VHS3: Code developed by Sweet.

[1] Identification of commercial products does not imply recommendation or endorsement by NIST.

- CRAYFISHPAK

  A highly vectorized Fortran subroutine library for the solution of separable elliptic partial differential equations (e.g. Poisson's equation). Cartesian (2D and 3D), polar, cylindrical, spherical, surface spherical, and spherical cross-section geometries are supported, as well as both centered and staggered finite difference grids. Distributed by Green Mountain Software, Boulder, CO.

- DIFFPACK

  A set of object-oriented libraries for solving partial differential equations and several Unix utilities for general software management and numerical programming. Aimed at rapid prototyping of simulators based on PDEs while still offering high efficiency. Implemented in C++, the libraries are organized into several layers: Basic Tools, Linear Algebra Tools, Dp Kernel, Dp Utilities, and Dp Applications. Distributed by netlib, see `http://www.netlib.org/diffpack`. The Diffpack home page is `http://www.oslo.sintef.no/avd/33/3340/diffpack`.

- ELLPACK

  Solves linear elliptic boundary value problems in general 2D domains and in 3D boxes. Includes a problem-description language (a Fortran extension) allowing equations, domains, solution methods, and options to be specified at a very high level, but flexible enough to to do special processing (to solve nonlinear problems, for example). Incorporates over 50 problem solving modules for discretization, equation reordering, linear equation solution, etc. Distributed by Purdue Research Foundation, W. Lafayette, IN. This package is described in the book by Rice and Boisvert [23], see also `http;//www.cs,purdue/ellpack`.

- FISHPACK

  A package of Fortran subprograms for separable elliptic partial differential equations. Distributed by netlib, see `http://www.netlib.org/fishpack`.

- IMSLM

  The IMSL MATH/LIBRARY is a Fortran subprogram library for solving problems in applied mathematics (approximately 700 subroutines and functions.) Distributed by Visual Numerics of Houston, TX.

- MANPAK

  Utility programs for computations with submanifolds of R^n implicitly defined by a system of nonlinear equations. Includes subroutines for a wide variety algebraically explicit differential algebraic equations (DAEs); that is, DAEs in which either the algebraic equations and/or variables are explicitly specified. Distributed by netlib, see `http://www.netlib.org/contin/manpak`.

- NAG

  A Fortran subroutine library for solving standard problems in many areas of mathematics, statistics, and optimization (approximately 1000 subroutines.) Distributed by NAG, Downers Grove, IL.

- NMS

  A collection of high-quality, portable Fortran subroutines for solving common computational problems in mathematics, engineering, and statistics. From the book by Kahaner *et al.* [16].

- ODE

  A collection of software for solving initial and boundary value problems for ordinary differential equations. Distributed by netlib, see `http://www.netlib.org/ode`.

- ODEPACK and SODEPACK

  A collection of Fortran solvers for the initial value problem for ordinary differential equation systems. It currently includes six solvers, suitable for both stiff and nonstiff systems, and includes solvers for systems given in linearly implicit form as well as solvers for systems given in explicit form. (Available in single- and double-precision versions.) Distributed by netlib, see `http://www.netlib.org/odepack`.

- PDELIB

  A small collection of Fortran subroutines which solve general systems of nonlinear initial-boundary-value partial differential equations in one or two space dimensions. Each routine is based upon the method of lines.

- PDES

  Software to solve many types of partial differential equations collected from a variety of sources. Distributed by netlib, see `http://www.netlib.org/pdes`.

- PLTMG and DPLTMG

  A Fortran package for solving an elliptic partial differential equation in general regions of the plane. It features adaptive local mesh refinement, multigrid iteration, and a pseudo-arclength continuation option for parameter dependencies The package includes an initial mesh generator and several graphics packages. (Available in single- and double-precision versions.) Distributed by netlib, see `http://www.netlib.org/pltmg`.

- PORT

A Fortran subprogram library for solving a variety of mathematical problems. Distributed by Lucent Technologies, Liberty Corner, NJ.

- SLATEC

  The SLATEC Common Mathematical Library is a collection of Fortran subprograms for a wide variety of mathematical problems. A primary impetus for the library development was to provide portable, non-proprietary, mathematical software for supercomputers at a consortium of government-sponsored research laboratories. Distributed by the Energy Science and Technology Software Center, Oak Ridge, TN.

- TOMS

  The Collected Algorithms of the ACM, published by the journal ACM Transactions on Mathematical Software. Distributed by netlib, see `http://www.netlib.org/toms`.

References

[1] Adams, J. C. Mudpack — Multigrid portable Fortran software for the efficient solution of linear partial differential equations. *Appl. Math. and Comp. 34*, 2 (1989), 113–146.

[2] Addison, C. A., Enright, W. H., Gaffney, P. W., Gladwell, I., and Hanson, P. M. A decision tree for the numerical solution of initial value ordinary differential equations. *ACM Trans. Math. Software 17*, 1 (March 1991), 1–10.

[3] Addison, C. A., Enright, W. H., Gaffney, P. W., Gladwell, I., and Hanson, P. M. A decision tree for the numerical solution of boundary value ordinary differential equations, SMU Math Report 89-7, Southern Methodist University, Dallas, TX.

[4] Bank, R. E. *PLTMG: A Software Package for Solving Elliptic Partial Differential Equations.* SIAM, Philadelphia, PA, 1990.

[5] Boisvert, R. F., Howe, S. E., and Kahaner, D. K. GAMS: A framework for the management of scientific software. *ACM Trans. Math. Software 11*, 4 (December 1985), 313–355.

[6] Boisvert, R. F., Howe, S. E., Kahaner, D. K., and Springmann, J. L. Guide to available mathematical software. Tech. rep., National Institute of Standards and Technology, Gaithersburg, MD, March 1990. Center for Computing and Applied Mathematics NISTIR 90-4237.

[7] Boisvert, R. F., and Sweet, R. A. Mathematical software for elliptic boundary value problems. In *Sources and Development of Mathematical Software*, W. R. Cowell, Ed. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1984, pp. 200–263.

[8] Byrne, G. D., and Hindmarsh, A. C. Stiff ODE solvers: A review of current and coming attractions. *J. Comput. Physics 70* (1987), 1–62.

[9] Childs, B., Scott, M., Daniel, J. W., Denman, E., and Nelson, P., Eds. *Codes for Boundary-Value Problems in Ordinary Differential Equations*. Springer–Verlag, New York, 1979.

[10] Delves, L. M., McKerrell, A., and Peters, S. A. Performance of GEM2 on the ELLPACK problem population. *Internat. J. Numer. Methods Eng. 23* (1986), 229–238.

[11] Dew, P. M., and Walsh, J. E. A set of library routines for solving parabolic equations in one space variable. *ACM Trans. Math. Software 7*, 3 (Sept 1981), 295–314.

[12] Dongarra, J. J., and Grosse, E. Distribution of mathematical software via electronic mail. *Comm. of the ACM 30*, 5 (May 1987), 403–407.

[13] Dyksen, W. R., and Ribbens, C. J. Interactive ELLPACK: An interactive problem–solving environment for elliptic partial differential equations. *ACM Trans. Math. Software 13*, 2 (June 1987), 113–132.

[14] Enloe, C. L. Solving coupled, nonlinear differential equations with commercial spreadsheets. *Computers in Physics* (Jan/Feb 1989), 75–76.

[15] Gaffney, P. W. A performance evaluation of some FORTRAN subroutines for the solution of stiff oscillatory ordinary differential equations. *ACM Trans. Math. Software 10*, 1 (March 1984), 58–72.

[16] Kahaner, D., Moler, C., and Nash, S. *Numerical Methods and Software*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1989.

[17] Machura, M., and Sweet, R. A. A survey of software for partial differential equations. *ACM Trans. Math. Software 6*, 4 (Dec 1980), 461–488.

[18] Melgaard, D. K., and Sincovec, R. F. General software for two-dimensional nonlinear partial differential equations. *ACM Trans. Math. Software 7*, 1 (March 1981), 106–125.

[19] Penn, H. L. A review of differential equations software. *Collegiate Microcomputer 6* (1988), 33–42.

[20] Pennington, S. V., and Berzins, M. New NAG library software for first-order partial differential equations. *ACM Trans. Math. Software 20*, 1 (March 1994), 63–99.

[21] Petzold, L. R. Differential/algebraic equations are not ODE's. *SIAM J. Sci. Stat. Comput. 3*, 3 (1982), 367–384.

[22] Press, W. H., Flannery, B. P., Teukolsky, S., and Vetterling, W. T. *Numerical Recipes*. Cambridge University Press, New York, 1986.

[23] Rice, J. R., and Boisvert, R. F. *Solving Elliptic Problems Using ELLPACK*. Springer–Verlag, New York, 1985.

[24] Scott, M. R., and Watts, H. A. Computational solutions of linear two-point boundary value problems via orthonormalization. *SIAM J. Numer. Anal. 14* (1977), 40–70.

[25] Shampine, L. F., and Zhang, W. Efficient integration of ordinary differential equations by transformations. *Comp. & Maths. with Appls. 15* (3 1988), 213–220.

[26] Teles, E., Penn, H. L., and Wilkin, J. ODE software for the IBM PC. *College Math. J. 21*, 3 (May 1990), 242–245.

[27] Teles, E., Penn, H. L., and Wilkin, J. ODE software for the Macintosh. *College Math. J. 21*, 4 (September 1990), 330–332.

157. Finite Difference Formulas

Applicable to Differential equations that will be solved by the method of finite differences.

Idea

A table of finite difference formulas for some common grids and common equations can be useful.

Procedure

Given a differential equation to be approximated by finite differences and a grid (see page 675) on which the solution is desired, replace every derivative by a finite difference approximation to that derivative. Standard finite difference formulas presume that there is an underlying uniform grid with a spacing of h. (In two dimensions, the uniform grid spacing is commonly taken to be h in one direction and k in another direction).

In the formulas for ordinary differential equation systems $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$, we use the shorthand notation $x_n = x_0 + nh$, $\mathbf{y}_n = \mathbf{y}(x_n)$, $\mathbf{f}_n = \mathbf{f}(x_n, \mathbf{y}_n)$, and $\mathbf{v}_n \approx \mathbf{y}_n$.

In the formulas for partial differential equation systems $L[\mathbf{z}] = \mathbf{f}(x, y, \mathbf{z})$ (where $L[\,]$ is a two-dimensional differential operator), we use the shorthand notation $x_n = x_0 + nh$, $y_n = y_0 + nk$, $\mathbf{x}_{n,m} = (x_n, y_m)$, $\mathbf{z}_{n,m} = \mathbf{z}(x_n, y_m)$, $\mathbf{f}_{n,m} = \mathbf{f}(x_n, y_m, \mathbf{z}_{n,m})$, and $\mathbf{v}_{n,m} \approx \mathbf{z}_{n,m}$.

In this section we include tables of formulas for the following cases:

- One Dimension: Rectilinear Grid
- Two Dimensions: Rectilinear Grid
- Two Dimensions: Irregular Grid
- Two Dimensions: Triangular Grid
- Numerical Schemes for the ODE: $y' = f(x, y)$
- Explicit Numerical Schemes for the PDE: $au_x + u_t = 0$
- Implicit Numerical Schemes for the PDE: $au_x + u_t = S(x, t)$
- Numerical Schemes for the PDE: $F(u)_x + u_t = 0$
- Numerical Schemes for the PDE: $u_x = u_{tt}$

157.1 One Dimension: Rectilinear Grid

The following is a list of finite difference formulas of different accuracies for a grid with uniform spacing.

1. Formulas for the first derivative:

$$f'(x_0) = \frac{f_1 - f_0}{h} + O(h)$$
$$f'(x_0) = \frac{f_1 - f_{-1}}{2h} + O\big(h^2\big)$$
$$f'(x_0) = \frac{-f_2 + 4f_1 - 3f_0}{2h} + O\big(h^2\big)$$
$$f'(x_0) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + O\big(h^4\big)$$

2. Formulas for the second derivative:

$$f''(x_0) = \frac{f_2 - 2f_1 + f_0}{h^2} + O(h)$$
$$f''(x_0) = \frac{f_1 - 2f_0 + f_{-1}}{h^2} + O\big(h^2\big)$$
$$f''(x_0) = \frac{-f_3 + 4f_2 - 5f_1 + 2f_0}{h^2} + O\big(h^2\big)$$
$$f''(x_0) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + O\big(h^4\big)$$

3. Formulas for the third derivative:

$$f'''(x_0) = \frac{f_3 - 3f_2 + 3f_1 - f_0}{h^3} + O(h)$$
$$f'''(x_0) = \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3} + O\big(h^2\big)$$

4. Formulas for the fourth derivative:

$$f^{(4)}(x_0) = \frac{f_4 - 4f_3 + 6f_2 - 4f_1 + f_0}{h^4} + O(h)$$
$$f^{(4)}(x_0) = \frac{f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}}{h^4} + O\big(h^2\big)$$

157.2 Two Dimensions: Rectilinear Grid

The following is a list of finite difference formulas of different accuracies for rectangular grids with uniform spacing. Other formulas can be obtained from the last list simply by holding one variable constant.

1. Formulas for first order partial derivatives:

$$f_x(\mathbf{x}_{0,0}) = \frac{1}{2h}\left(f_{1,0} - f_{-1,0}\right) + O\big(h^2\big)$$
$$f_x(\mathbf{x}_{0,0}) = \frac{1}{4h}\left(f_{1,1} - f_{-1,1} + f_{1,-1} - f_{-1,-1}\right) + O\big(h^2\big)$$

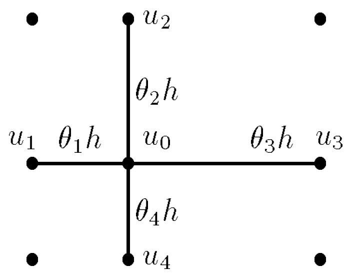

Figure 157.1: Spacing on an irregular domain.

2. Formulas for second order partial derivatives:

$$f_{xx}(\mathbf{x}_{0,0}) = \frac{1}{3h^2}\left(f_{1,1} - 2f_{0,1} + f_{-1,1} + f_{1,0} - 2f_{0,0} + f_{-1,0} + f_{1,-1} - 2f_{0,-1} + f_{-1,-1}\right) + O\big(h^2\big)$$

$$f_{xy}(\mathbf{x}_{0,0}) = \frac{1}{4h^2}\left(f_{1,1} - f_{1,-1} - f_{-1,1} + f_{-1,-1}\right) + O\big(h^2\big)$$

3. Formulas for the Laplacian:

$$\nabla^2 f(\mathbf{x}_{0,0}) = \frac{1}{h^2}\left(f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1} - 4f_{0,0}\right) + O\big(h^2\big)$$

$$\nabla^2 f(\mathbf{x}_{0,0}) = \frac{1}{12h^2}\Big(-60f_{0,0} + 16(f_{1,0} + f_{0,1} + f_{-1,0} + f_{0,-1}) - (f_{2,0} + f_{0,2} + f_{-2,0} + f_{0,-2})\Big) + O\big(h^4\big)$$

157.3 Two Dimensions: Irregular Grid

Nonuniform grids may be the only way to numerically solve some practical problems involving partial differential equations. For example, a nonuniform grid may be required near the boundaries of a domain. Also, adaptive grids and moving grids are sometimes more useful than a fixed grid (see page 675). The following finite difference formulas refer to the parameters defined in figure 157.1.

1. Formulas for first order partial derivatives:

$$\left.\frac{\partial u}{\partial x}\right|_{\mathbf{x}_{0,0}} = \frac{u_3 - u_1}{h(\theta_1 + \theta_3)} + O(h)$$

$$\left.\frac{\partial u}{\partial y}\right|_{\mathbf{x}_{0,0}} = \frac{u_2 - u_4}{h(\theta_2 + \theta_4)} + O(h)$$

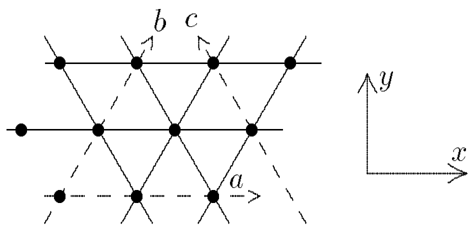

Figure 157.2: Definition of the coordinate system for a triangular domain.

2. Formulas for second order partial derivatives:

$$
\begin{aligned}
\left.\frac{\partial^2 u}{\partial x^2}\right|_{\mathbf{x}_{0,0}} &= \frac{2}{h^2}\left[\frac{u_1-u_0}{\theta_1(\theta_1+\theta_3)}+\frac{u_3-u_0}{\theta_3(\theta_1+\theta_3)}\right]+O(h)\\
\left.\frac{\partial^2 u}{\partial y^2}\right|_{\mathbf{x}_{0,0}} &= \frac{2}{h^2}\left[\frac{u_2-u_0}{\theta_2(\theta_2+\theta_4)}+\frac{u_4-u_0}{\theta_4(\theta_2+\theta_4)}\right]+O(h)\\
\nabla^2 u\,\Big|_{\mathbf{x}_{0,0}} &= \left(\frac{\partial^2 u}{\partial x^2}+\left.\frac{\partial^2 u}{\partial y^2}\right|_{\mathbf{x}_{0,0}}\right.\\
&= \frac{2}{h^2}\left[\frac{u_1}{\theta_1(\theta_1+\theta_3)}+\frac{u_2}{\theta_2(\theta_2+\theta_4)}+\frac{u_3}{\theta_3(\theta_1+\theta_3)}+\frac{u_4}{\theta_4(\theta_2+\theta_4)}\right.\\
&\qquad\left.-\left(\frac{1}{\theta_1\theta_3}+\frac{1}{\theta_2\theta_4}\right)u_0\right]+O(h)
\end{aligned}
$$

157.4 Two Dimensions: Triangular Grid

Sometimes it is easier to perform computations on a uniform triangular grid (see figure 157.2). If we represent the three directions on the triangular grid as $\{a, b, c\}$, then we can compute the partial derivatives:

$$
\begin{aligned}
\frac{\partial u}{\partial a} &= u_x, & \frac{\partial^2 u}{\partial a^2} &= u_{xx},\\
\frac{\partial u}{\partial b} &= \frac{1}{2}u_x+\frac{\sqrt{3}}{2}u_y, & \frac{\partial^2 u}{\partial b^2} &= \frac{1}{4}u_{xx}+\frac{\sqrt{3}}{2}u_{xy}+\frac{3}{4}u_{yy},\\
\frac{\partial u}{\partial c} &= -\frac{1}{2}u_x+\frac{\sqrt{3}}{2}u_y, & \frac{\partial^2 u}{\partial c^2} &= \frac{1}{4}u_{xx}-\frac{\sqrt{3}}{2}u_{xy}+\frac{3}{4}u_{yy}.
\end{aligned}
$$

These relations may be inverted to yield

Adams–Bashforth, order 2	$v_n - v_{n-1} = \frac{1}{2}h\,[3f_{n-1} - f_{n-2}]$
Adams–Bashforth, order 4	$v_n - v_{n-1} = \frac{1}{24}h\,[55f_{n-1} - 59f_{n-2} + 37f_{n-3} - 9f_{n-4}]$
Adams–Moulton, order 4	$v_n - v_{n-1} = \frac{1}{24}h\,[9f_n + 19f_{n-1} - 5f_{n-2} + f_{n-3}]$
backward Euler	$v_n - v_{n-1} = hf_n$
Euler's method	$v_n - v_{n-1} = hf_{n-1}$
explicit leapfrog	$v_{n+1} - v_{n-1} = hf_n$
implicit leapfrog	$v_n - v_{n-1} = \frac{1}{2}h(f_n + f_{n-1})$
Simpson's rule[a]	$v_n - v_{n-2} = \frac{1}{3}h(f_n + 4f_{n-1} + f_{n-2})$
trapezoidal rule[b]	$v_n - v_{n-1} = \frac{1}{2}h(f_n + f_{n-1})$

[a] Also known as Milne's method.
[b] Also known as Heun's method and as the Adams–Moulton method of order 2.

Table 157.1: Numerical schemes for the ODE: $y' = f(x, y)$

$$u_x = \frac{\partial u}{\partial a}, \qquad u_{yy} = \frac{1}{3}\left(2\frac{\partial^2 u}{\partial b^2} + 2\frac{\partial^2 u}{\partial c^2} - \frac{\partial^2 u}{\partial a^2}\right),$$

$$u_y = \frac{1}{\sqrt{3}}\left(\frac{\partial u}{\partial b} + \frac{\partial u}{\partial c}\right), \qquad u_{xy} = \frac{1}{\sqrt{3}}\left(\frac{\partial^2 u}{\partial b^2} - \frac{\partial^2 u}{\partial c^2}\right),$$

$$u_{xx} = \frac{\partial^2 u}{\partial a^2}, \qquad \nabla^2 u = u_{xx} + u_{yy} = \frac{2}{3}\left(\frac{\partial^2 u}{\partial a^2} + \frac{\partial^2 u}{\partial b^2} + \frac{\partial^2 u}{\partial c^2}\right).$$

See Gerald and Wheatley [6, Section 7.9] for a worked example using triangular coordinates.

157.5 Numerical Schemes for the ODE: $y' = f(x, y)$

Table 157.1 contains some common difference formulas for the ordinary differential equation $y' = f(x, y)$. Of these methods, Euler's method and the leapfrog method are explicit; all the others are implicit methods.

157.6 Explicit Numerical Schemes for the PDE: $au_x + u_t = 0$

Table 157.2 contains named explicit difference formulas for the partial differential equation $au_x + u_t = 0$. DuChateau and Zachmann [3, page 450] also list the local truncation error for each of these methods. In this listing, h is the uniform x spacing, and k is the uniform t spacing. The approximation to $u(x_n, t_j) = u(x_0 + nh, t_0 + jk)$ is represented by $u_{n,j}$.

Forward in time, forward in space (FTFS):
$a\frac{u_{n+1,j}-u_{n,j}}{h}+\frac{u_{n,j+1}-u_{n,j}}{k}=0$
Forward in time, centered in space (FTCS) (unstable):
$a\frac{u_{n+1,j}-u_{n-1,j}}{2h}+\frac{u_{n,j+1}-u_{n,j}}{k}=0$
Forward in time, backward in space (FTBS):
$a\frac{u_{n,j}-u_{n-1,j}}{h}+\frac{u_{n,j+1}-u_{n,j}}{k}=0$
Lax–Friedrichs method:
$a\frac{u_{n+1,j}-u_{n-1,j}}{2h}+\frac{u_{n,j+1}-\frac{1}{2}(u_{n-1,j}-u_{n+1,j})}{k}=0$
Lax–Wendroff method:
$u_{n,j+1}=u_{n,j}-\frac{ak}{2h}\left(u_{n+1,j}-u_{n-1,j}\right)$
$+\frac{a^2k^2}{2h^2}\left(u_{n-1,j}-2u_{n,j}+u_{n+1,j}\right)$

Table 157.2: Explicit numerical schemes for the PDE: $au_x+u_t=0$

Backward in time, backward in space (BTBS):
$a\frac{u_{n+1,j+1}-u_{n,j+1}}{h}+\frac{u_{n+1,j+1}-u_{n+1,j}}{k}=S_{n+1,j+1}$
Backward in time, centered in space (BTCS):
$a\frac{u_{n+1,j+1}-u_{n-1,j+1}}{2h}+\frac{u_{n,j+1}-u_{n,j}}{k}=S_{n,j+1}$
Crank–Nicolson:
$\frac{1}{2}\left(a\frac{u_{n+1,j+1}-u_{n-1,j+1}}{2h}+a\frac{u_{n+1,j}-u_{n-1,j}}{2h}\right)$
$+\frac{u_{n,j+1}-u_{n,j}}{k}=S_{n,j+1/2}$
Wendroff method:
$\frac{1}{2}\left(a\frac{u_{n+1,j+1}-u_{n,j+1}}{h}+a\frac{u_{n+1,j}-u_{n,j}}{h}\right)$
$+\frac{1}{2}\left(\frac{u_{n+1,j+1}-u_{n+1,j}}{k}+\frac{u_{n,j+1}-u_{n,j}}{k}\right)=S_{n+1/2,j+1/2}$

Table 157.3: Implicit numerical schemes for the PDE: $au_x+u_t=S(x,t)$

157.7 Implicit Numerical Schemes for the PDE: $au_x+u_t=S(x,t)$

Table 157.3 contains named implicit difference formulas for the partial differential equation $au_x+u_t=S(x,t)$. DuChateau and Zachmann [3, page 460] also list the local truncation error for each of these methods. In this listing, h is the uniform x spacing, and k is the uniform t spacing. The approximation to $u(x_n,t_j)=u(x_0+nh,t_0+jk)$ is represented by $u_{n,j}$, and $S_{n,j}$ is used to represent $S(x_n,t_j)$.

157.8 Numerical Schemes for the PDE: $F(u)_x+u_t=0$

Table 157.4 contains named difference formulas for the partial differential equation $F(u)_x+u_t=0$ (see DuChateau and Zachmann [3, page 475] for more details). In this listing, h is the uniform x spacing, k is the

Centered in time–centered in space (unstable):
$u_{n,j+1} = u_{n,j} - \frac{1}{2}s\left(F_{n+1,j} - F_{n-1,j}\right)$
Lax–Friedrichs method:
$u_{n,j+1} = \frac{1}{2}\left(u_{n+1,j} + u_{n-1,j}\right) - \frac{1}{2}s\left(F_{n+1,j} + F_{n-1,j}\right)$
Lax–Wendroff method:
$u_{n,j+1} = u_{n,j} - \frac{1}{2}s\left(F_{n+1,j} - F_{n-1,j}\right)$
$+\frac{1}{2}s^2\left[a_{n+1/2,j}\left(F_{n+1,j} - F_{n,j}\right) - a_{n-1/2,j}\left(F_{n,j} - F_{n-1,j}\right)\right]$
Richtmeyer method:
$u^*_{n+1/2} = \frac{1}{2}\left(u_{n+1,j} + u_{n,j}\right) - \frac{1}{2}\left(F_{n+1,j} - F_{n,j}\right)$
$u_{n,j+1} = u_{n,j} - s\left(F^*_{n+1/2} - F^*_{n-1}\right)$
MacCormack method:
$u^*_n = u_{n,j} - s\left(F_{n+1,j} - F_{n,j}\right)$
$u_{n,j+1} = \frac{1}{2}\left[u_{n,j} + u^*_n - s\left(F^*_n - F^*_{n-1}\right)\right]$
FTBS upwind method (use when $F'(u) > 0$):
$u_{n,j+1} = u_{n,j} + s\left(F_{n-1,j} - F_{n,j}\right)$
FTFS upwind method (use when $F'(u) < 0$):
$u_{n,j+1} = u_{n,j} - s\left(F_{n+1,j} - F_{n,j}\right)$

Table 157.4: Numerical schemes for the PDE: $F(u)_x + u_t = 0$

uniform t spacing, and the ratio of these is $s = k/h$. The approximation to $u(x_n, t_j) = u(x_0 + nh, t_0 + jk)$ is represented by $u_{n,j}$ and $F_{m,n} := F\left(u_{m,n}\right)$. A star superscript indicates an intermediate result (and $F^*_n := F(u^*_n)$). Finally, $a_n := F'_n = F'(u_n)$.

Note that some of the left-hand sides of the last listing can be obtained from this listing by taking $F(u) = au$.

157.9 Numerical Schemes for the PDE: $u_x = u_{tt}$

Table 157.5 contains named difference formulas for the partial differential equation $u_x = u_{tt}$. Lapidus and Pinder [9] discuss each of these methods in some detail. In this listing, h is the uniform x spacing, k is the uniform t spacing, and ρ is defined to be $\rho = h/k^2$. The approximation to $u(x_n, t_j) = u(x_0 + nh, t_0 + jk)$ is represented by $u_{n,j}$.

Notes

1. Fornberg [4] has a simple recursive technique for determining finite difference formula of high order.
2. For problems with periodic boundary conditions, it is possible to obtain finite differential formulas that are of infinite order; see page 851.
3. All of the discretization methods used should be of comparable order. That is, if one term in an equation has a discretization error of $O(h^2)$, then there is no reason for another term to have a discretization error of $O(h^4)$.

Classic explicit approximation:

$$u_{n+1,j} = (1-2\rho)u_{n,j} + \rho\left(u_{n,j+1} + u_{n,j-1}\right)$$

DuFort–Frankel explicit approximation:

$$(1+2\rho)u_{n+1,j} = 2\rho\left(u_{n,j+1} + u_{n,j-1}\right) + (1-2\rho)u_{n-1,j}$$

Richardson explicit approximation:

$$u_{n+1,j} - u_{n-1,j} - 2\rho\left(u_{n,j+1} + u_{n,j-1}\right) + 4\rho u_{n,j} = 0$$

Backward implicit approximation:

$$(1+2\rho)u_{n+1,j} - \rho\left(u_{n+1,j+1} + u_{n+1,j-1}\right) = u_{n,j}$$

Crank–Nicolson implicit approximation:

$$2(\rho+1)u_{n+1,j} - \rho\left(u_{n+1,j+1} + u_{n+1,j-1}\right) = 2(1-\rho)u_{n,j} + \rho\left(u_{n,j+1} + u_{n,j-1}\right)$$

Variable weighted implicit approximation (with $0 \le \theta \le 1$):

$$(1+2\rho\theta)u_{n+1,j} = \rho(1-\theta)\left(u_{n,j+1} + u_{n,j-1}\right) + \rho\theta\left(u_{n+1,j+1} + u_{n+1,j-1}\right) + \left[1 - 2\rho(1-\theta)\right]u_{n,j}$$

Table 157.5: Numerical schemes for the PDE: $u_x = u_{tt}$

4. Note that nonuniform grids may give rise to a number of consistency/stability phenomena that have no counterpart on uniform grids.

5. Macsyma [10] has a package (`fdif_pde`) which derives finite difference approximations for partial differential equations.
6. See also Abramowitz and Stegun [1, pages 883–885] and Lapidus and Pinder [9, section 4.3, pages 153–162].

References

[1] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington, D.C., 1964.

[2] Altas, I., and Stephenson, J. W. Finite difference schemes on irregular meshes. *Eighteenth Manitoba Conference on Numerical Mathematics and Computing Winnipeg, Canada 69* (1989), 21–32.

[3] DuChateau, P., and Zachmann, D. *Applied Partial Differential Equations.* Harper & Row Publishers, New York, 1989.

[4] Fornberg, B. Generation of finite difference formulas on arbitrarily spaced grids. *Math. of Comp. 51*, 184 (October 1988), 699–706.

[5] Ganzha, V. G., Mazurik, S. I., and Shapeev, V. P. Symbolic manipulations on a computer and their application to generation and investigation of difference schemes. In *EUROCAL '85*, B. Buchberger and B. F. Caviness, Eds. Springer–Verlag, New York, 1985, pp. 335–347.

[6] Gerald, C. F., and Wheatley, P. O. *Applied Numerical Analysis.* Addison–Wesley Publishing Co., Reading, MA, 1984.

[7] Heinrich, B. *Finite Difference Methods on Irregular Networks.* Birkhauser, Basel, Switzerland, 1987.

[8] Keller, H. B., and Pereyra, V. Symbolic generation of finite difference formulas. *Math. of Comp. 32* (1978), 955–971.

[9] Lapidus, L., and Pinder, G. F. *Numerical Solution of Partial Differential Equations in Science and Engineering.* John Wiley & Sons, New York, 1982.

[10] Macsyma. *Mathematics Reference Manual.* Macsyma, Arlington, MA, 1993.

[11] Voss, D. A fifth-order exponentially fitted formula. *SIAM J. Math. Anal. 25*, 3 (June 1988), 670–678.

158. Finite Difference Methodology

Applicable to Differential equations.

Yields

A finite difference scheme that can be used to numerically approximate a given differential equation.

Procedure

For the first order ordinary differential equation $y' = f(x, y)$, consider the general multistep (or k-step) method

$$N[v_n, v_{n+1}, \ldots, v_{n+k}] := \sum_{j=0}^{k} \alpha_j v_{n+j} - h \sum_{j=0}^{k} \beta_j f(x_{n+j}, v_{n+j}) = 0, \tag{158.1}$$

where $\alpha_0 \neq 0$, $n = k, k+1, \ldots$ and v_n is an approximation to $y(x_n)$ (where $x_n = nh$ and h is a small number called the *step size*). We presume the constants $\{\alpha_i\}$ and $\{\beta_i\}$ are known.

If $\beta_0 \neq 0$, then the scheme is an *implicit* difference method. If $\beta_0 = 0$, then the scheme is an *explicit* difference method. For explicit methods, equation (158.1) can be solved for v_n in terms of the other quantities in equation (158.1).

The exact solution to the equation $y' = f(x, y)$ will *not*, in general, satisfy $N[y_n, y_{n+1}, \ldots, y_{n+k}] = 0$ (here, $y_n = y(x_n)$). If $h \ll 1$, then a Taylor series can be employed to show that

$$y_{n+j} = y_n + jhy_n' + \frac{(jh)^2}{2} y_n'' + \cdots .$$

Using this expansion, a Taylor series can be taken of $N[y_n, y_{n+1}, \ldots, y_{n+k}]$ to obtain

$$\begin{aligned} N[y_n, y_{n+1}, \ldots, y_{n+k}] &= \sum_{j=0}^{k} \alpha_j y_{n-j} - h \sum_{j=0}^{k} \beta_j f(x_{n-j}, y_{n-j}) \\ &= h^{p+1} R_n + O(h^{p+2}), \end{aligned} \tag{158.2}$$

for some numbers p and R_n.

If $p \geq 1$, then the method is said to be *consistent*. If a method is consistent, then p is called the *order of the method*. We say that "the method is pth order accurate." The term $h^{p+1} R_n$ is called the *truncation error*. A theorem of numerical analysis states that there exist methods of order $p = 2k$.

The first and second *characteristic polynomials* of the method in equation (158.1) are defined as $\rho(x)$ and $\sigma(x)$, where

$$\rho(x) = \sum_{j=0}^{k} \alpha_j x^j, \qquad \sigma(x) = \sum_{j=0}^{k} \beta_j x^j.$$

If equation (158.1) is consistent, then it follows that $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$.

If $p > k + 2$, then the method will always be unstable (stability for the discretization of ordinary differential equations is defined on page 683). Specifically, if k is odd, then $p = k + 1$ is the largest p such that there is a stable method. Also, if k is even, then $p = k + 2$ is the largest p such that there is a stable method. If a difference method is stable and is of pth order accuracy, then $|v_n - y_n| = o(h^p)$ in any finite interval, $0 \le x \le L$.

Many finite difference formulas are tabulated on page 661. For example, for Euler's method and the trapezoidal rule, $k = 1$. For Simpson's rule, $k = 2$ and $p = 4$. To obtain a discretization for a differential equation, it is possible to obtain a finite difference formula for every term in the differential equation and then combine these formulas in the obvious manner. (Just replace each term in the differential equation with its finite difference approximation.) However, combining formulas in this way for partial differential equations—without understanding the underlying physics of the problem and the approximations—can quickly produce results that are unrelated to the true problem (see page 27).

Example

There are many procedures for generating finite difference formulas for the terms appearing in differential equations; we illustrate one straightforward method. Suppose we want to find an approximation to $f'(x_0)$, given the values $f(x_0 - h)$ and $f(x_0 + h)$. We write

$$f'(x_0) = \alpha f(x_0 - h) + \beta f(x_0 + h) + e(x_0; h), \tag{158.3}$$

where α and β are constants to be determined, and $e(x_0; h)$ represents the error term. Taking a Taylor series of the right-hand side of equation (158.3) (and using f_0 to represent $f(x_0)$, f_0' for $f'(x_0)$, etc.), we find

$$\begin{aligned} f_0' = \alpha &\left[f_0 - h f_0' + \frac{h^2}{2} f_0'' - \frac{h^3}{6} f_0''' + O(h^4) \right] \\ &+ \beta \left[f_0 + h f_0' + \frac{h^2}{2} f_0'' + \frac{h^3}{6} f_0''' + O(h^4) \right] + e(x_0; h). \end{aligned}$$

If we choose $\alpha = -\beta$, then this simplifies to

$$f_0' = \beta \left[2h f_0' + \frac{h^3}{3} f_0''' + O(h^4) \right] + e(x_0; h).$$

Finally, if we choose $\beta = 1/2h$, then we obtain $f_0' = f_0' + \frac{h^2}{6} f''' + O(h^3) + e(x_0; h)$. Hence, $e(x_0; h) = O(h^2)$. Putting all of this together, we have the finite difference approximation

$$f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + O(h^2).$$

This formula could be used to approximate the ordinary differential equation $y' = y^2$, on a uniform mesh, by

$$\frac{u(x_0+h) - u(x_0-h)}{2h} = u^2(x_0),$$

where $u(x) \approx y(x)$. Using $x_0 := nh$ and $u_n := u(nh)$ in this formula, we find $\frac{u_{n+1} - u_{n-1}}{2h} = u_n^2$. This can be manipulated into the explicit formula: $u_{n+1} = u_{n-1} + 2hu_n^2$.

Notes

1. Observe that a difference scheme can be stable and still not be consistent. Stability and accuracy are two entirely different concerns.
2. The Dahlquist relations are

$$\sum_{j=0}^{p} \alpha_j j^k = -k \sum_{j=0}^{p} \beta_j j^{k-1}. \qquad (158.4)$$

If they hold for $k = 0, 1, \ldots, p$, then we have (compare with equation (158.1))

$$\sum_{j=0}^{p} \alpha_j y(t - jh) = \sum_{j=0}^{p} \beta_j y'(t - jh) + O\left(h^{p+1}\right).$$

3. Finite difference schemes can be looked up (see page 661 or Isaacson and Keller [4, Chapter 8, pages 364–43)]) or they can be constructed as needed (see Lapidus and Pinder [8, pages 153–162] or Ganzha *et al.* [1]).
4. When approximating a differential equation on a bounded interval, the limit $h \to 0$, $n \to \infty$, nh fixed, is of interest. If the *local error* of a discretization scheme (as determined by equation (158.2)) is $O(h^{p+1})$, then the *global error* (the error at the end of the integration) will be $O(h^p)$.
5. Obrechkoff methods utilize derivatives of y in forming the finite difference scheme. The k-step Obrechkoff method using the first m derivatives of y may be written

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=1}^{m} h^i \sum_{j=0}^{k} \beta_{ij} y_{n+j}^{(i)}.$$

See Lambert [7] for details.

6. Often, a differential equation will have invariants that remain constant during the evolution of the differential equation. For example, in a conservative system the energy should remain constant. A numerical scheme should be used that ensures that these invariants remain constant. See symplectic methods (page 780) and Gear [2].
7. State-of-the-art software packages for ordinary differential equations do not use a single discretization scheme with a fixed step size. Rather, they vary their order (i.e., they choose from a collection of discretization formulas) and they vary the step size. Ideally, the optimal step size and order are determined at each step; this is an important aspect of the code's efficiency (see page 770).
8. To determine if a finite difference scheme for a partial differential equation is stable, see either the Courant consistency criterion (page 688) or the Von Neumann stability test (page 692).
9. There are other types of finite difference approximations that are not in the form of equation (158.1). See, for example, the cosine method (see page 716), the predictor–corrector method (see page 759), or the method of Runge–Kutta (see page 763).
10. There are many useful theorems in numerical analysis concerning methods for specific equations. For example; a method for $u_t = u_x$ with non-negative coefficients cannot have an accuracy of $p > 1$. See Iserles and Strang [5].
11. Energy propagation under dispersive partial differential equations travels with the *group velocity*. Even if an equation is non-dispersive, any finite difference approximation to it will be dispersive. Hence, study of the group velocity is an important part of the analysis of a finite difference scheme. See Trefethen [10] for details.

References

[1] Ganzha, V. G., Mazurik, S. I., and Shapeev, V. P. Symbolic manipulations on a computer and their application to generation and investigation of difference schemes. In *EUROCAL '85*, B. Buchberger and B. F. Caviness, Eds. Springer–Verlag, New York, 1985, pp. 335–347.

[2] Gear, C. W. Maintaining solution invariants in the numerical solution of ODEs. *SIAM J. Sci. Stat. Comput. 7*, 3 (July 1986), 734–743.

[3] Godunov, S. K., and Ryabenkii, V. S. *Difference Schemes: An Introduction to the Underlying Theory.* North–Holland Publishing Co., New York, 1987. Translates by E. M. Gelbard.

[4] Isaacson, E., and Keller, H. B. *Analysis of Numerical Methods.* John Wiley & Sons, New York, 1966.

[5] Iserles, A., and Strang, G. The optimal accuracy of difference schemes. *Trans. Amer. Math. Soc. 277*, 2 (June 1983), 779–803.

[6] Jackson, K. R. The convergence of integrand–approximation formulas for the numerical solution of IVPs for ODEs. *SIAM J. Numer. Anal. 25*, 1 (February 1988), 163–188.

[7] Lambert, J. D. *Computational Methods in Ordinary Differential Equations.* Cambridge University Press, New York, 1973.

[8] Lapidus, L., and Pinder, G. F. *Numerical Solution of Partial Differential Equations in Science and Engineering.* John Wiley & Sons, New York, 1982.

[9] Shampine, L. F. Implementation of implicit formulas for the solution of ODEs. *SIAM J. Sci. Stat. Comput. 1*, 1 (March 1980), 103–118.

[10] Trefethen, L. N. Group velocity in finite difference schemes. *SIAM Review 24*, 2 (April 1982), 113–136.

[11] Van Niekerk, F. D. Non-linear one step methods for initial value problems. *Comp. & Maths. with Appls. 13*, 4 (1987), 367–371.

159. Grid Generation

Applicable to Ordinary and partial differential equations.

Yields

A grid on which a differential equation may be numerically approximated.

Procedure

When a differential equation is going to be approximated numerically, the points at which the values of the dependent variable will be determined must be specified. This collection of points forms the *grid*, or *mesh*.

The most common computational grids are those in rectilinear coordinates or polar coordinates (see figure 159.1). These can be used when the domain of a problem naturally fits one of these geometries. For other domains, an appropriate computational grid must be determined. There are many ways in which to construct a grid for a specific equation on a specific domain.

There are many considerations that go into choosing a grid for a specific problem. The grid should be easy to generate, and the algebraic equations used on the grid (usually finite differences or finite elements) must be easy to generate. (On page 664 we have indicated how finite difference approximations may be found on triangular grids.) For finite element methods, it is common to use triangulated grids or grids composed of simple objects like triangles and rectangles. See example 3 in the section on finite element methods (on page 739) for an example.

Ideally, there should be many grid points where the solution (or its derivatives) are rapidly changing. Some grids naturally lend themselves to grid refinement in certain regions; this can be useful in adaptive techniques.

Example 1

For domains that can be described by combinations of simple geometric regions, a grid may be easy to find. See figure 159.2 for a simple computational grid for a domain that can be conveniently decomposed into a rectangle and a semicircle. In this figure we have also illustrated how the grid may be modified if it is found that the solution shows great variation in the upper left region of the domain.

Example 2

There are many ways in which a grid may be found for a domain. Figure 159.3, taken from Rice [8], shows six different grids for a single irregularly shaped domain. The first three grids (A, B, C) show different possibilities:

- Grid A is a simple triangulation of the domain.
- Grid B is a uniform rectilinear grid on the domain.

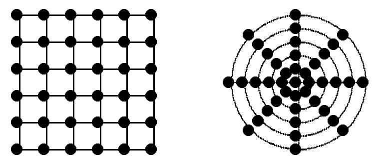

Figure 159.1: Two common computational grids, for rectilinear coordinates and for polar coordinates.

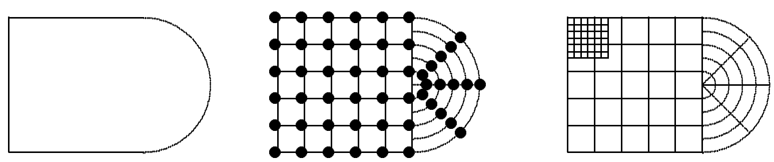

Figure 159.2: A domain, a possible grid on that domain, and a refined grid on that domain.

- Grid C is a uniform rectilinear mapping, logically mapped to the domain.

The second three grids (D, E, F) indicate how the the first three grids can adapt to some difficulties near the right boundary.

Notes

1. One of the greatest obstacles in generating numerical solution to fluid dynamics problems is the difficulty in geometrically describing complex configurations with computational grids.
2. Conformal mappings (see page 441) are frequently used to construct computational grids.
3. The multigrid method (see page 752) uses a sequence of grids, of varying coarseness, to approximate the solution of a differential equation.
4. Robert Schneiders maintains a comprehensive web site on mesh generation, see `http://www-users.informatik.rwth-aachen.de/~roberts/meshgeneration.html`. This site includes
   - Information on meshing research
   - A directory of people working on mesh generation,
   - Latest news on mesh generation
   - A list of programs (both public domain and commercial, more than 100 are mentioned – for most a URL is listed)
   - Information on conferences and short courses
   - Literature on mesh generation
   - Open positions

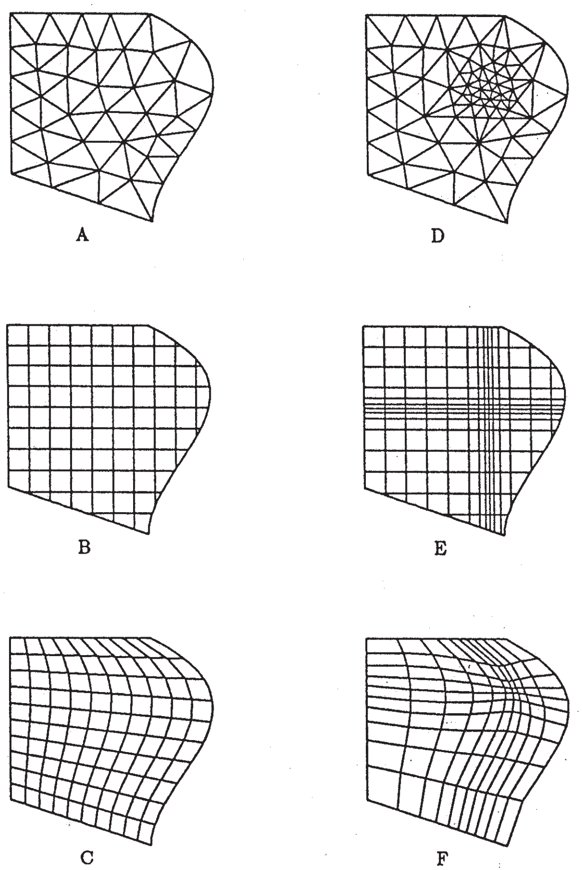

Figure 159.3: Six different grids for a domain (from Rice, J. R. Parallel Methods for Partial Differential Equations. In *The Characteristics of Parallel Algorithms*, L. H. Jamieson, D. B. Gannon, and R. J. Douglass, Eds. MIT Press, 1987.)

- Information on related topics (e.g., CFD, scientific computing, and computational geometry)

References

[1] Abrahamsson, L. Orthogonal grid generation for two dimensional ducts. *J. Comput. Appl. Math. 34* (1991), 305–314.

[2] Atlas, I., Manohar, R., and Stephenson, J. W. Adaptive mesh generation using quadratures. *Congr. Numer. 62* (1988), 37–45.

[3] Atlas, I., and Stephenson, J. W. A two-dimensional adaptive mesh generation method. *J. Comput. Physics 94* (1991), 201–224.

[4] Castillo, J. E., Ed. *Mathematical Aspects of Numerical Grid Generation*. SIAM, Philadelphia, PA, 1991.

[5] Eisman, P. R. Adaptive grid generation. *Comp. Meth. Appl. Mech. Eng. 64* (1987), 321–376.

[6] Mitchell, W. F. A comparison of adaptive refinement techniques for elliptic problems. *ACM Trans. Math. Software 15*, 4 (Dec 1989), 326–347.

[7] Pardhanani, A., and Carey, G. F. Optimization of computational grids. *Num. Meth. Part. Diff. Eqs. 4*, 2 (1988), 95–117.

[8] Rice, J. R. Parallel methods for partial differential equations. In *The Characteristics of Parallel Algorithms*, L. H. Jamieson, D. B. Gannon, and R. J. Douglass, Eds. MIT Press, Cambridge, MA, 1987, pp. 209–231.

[9] Sparis, P. D. A method for generating boundary-orthogonal curvilinear coordinate systems using the biharmonic equation. *J. Comput. Physics 61* (1985), 445–462.

[10] Thompson, J. F. Special issue on numerical grid generation. *Appl. Math. and Comp. 10–11* (1982).

[11] Thompson, J. F., Warsi, Z. U. A., and Mastin, C. W. *Numerical Grid Generation Foundations and Applications*. North–Holland Publishing Co., New York, 1985.

160. Richardson Extrapolation

Applicable to Approximation techniques for differential equations.

Yields

A procedure for increasing the accuracy.

Procedure

Suppose that a grid with a characteristic spacing h is used to numerically approximate the solution of a differential equation. Then the approximation $u(\mathbf{x}; h)$ at the point $\mathbf{x}$ in the domain will satisfy

$$u(\mathbf{x}; h) = y(\mathbf{x}) + R_m(\mathbf{x})h^m + O(h^{m+1}), \tag{160.1}$$

where $y(\mathbf{x})$ is the true solution to the differential equation, m is the order of the method, and the other terms represent the error (see page 670).

If the approximation scheme is kept the same, but the characteristic spacing of the grid is changed from h to k, then

$$u(\mathbf{x}; k) = y(\mathbf{x}) + R_m(\mathbf{x})k^m + O(k^{m+1}). \tag{160.2}$$

Equations (160.1) and (160.2) can be combined to yield the approximation

$$v(\mathbf{x}; h, k) := \frac{k^m u(\mathbf{x}; h) - h^m u(\mathbf{x}; k)}{k^m - h^m} = y(\mathbf{x}) + O(kh^m, hk^m).$$

Note that $v(\mathbf{x}; h, k)$ is one more order accurate than either $u(\mathbf{x}; h)$ or $u(\mathbf{x}; k)$. This process may be iterated to increase the accuracy even more.

In some cases, the order of the method, and hence m in equation (160.1), will be unknown. The Richardson extrapolation method may still be used, by either estimating m numerically, or by using the Shanks transformation. The Shanks transformation uses three successive terms of the form $A_n = A_\infty + \alpha h^n$ to estimate A_∞ via

$$A_\infty = \frac{A_{n+1}A_{n-1} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}.$$

This transformation may also be iterated; see Bender and Orszag [1, page 369] for details.

Example 1

Given the differential equation

$$\frac{dy}{dx} = y, \quad y(0) = 1,$$

we might choose to approximate the solution by Euler's method

$$u_{n+1;h} = (1 + h)u_{n;h}, \quad u_{0;h} = 1,$$

where $u_{n;h} \simeq y(nh)$, and the step size satisfies $h \ll 1$. Observe that our notation explicitly shows the dependence of the approximation on the grid size. Doing a detailed analysis, we can determine that

$$u_{n;h} = y(x) - \left(\frac{x}{2}\right) h + O(h^2), \tag{160.3}$$

where $x = nh$ and hence (here we choose $k = h/2$)

$$u_{2n;h/2} = y(x) - \left(\frac{x}{2}\right) \frac{h}{2} + O(h^2). \tag{160.4}$$

Combining equation (160.3) and equation (160.4) results in

$$w_{n;h} := 2u_{n;h} - u_{2n;h/2} = y(x) + O(h^2),$$

which is a numerical approximation that is second order accurate. Because h was reduced by a factor of 2 in going from equation (160.3) to equation (160.4), n had to be increased by a factor of 2 to maintain the same physical location, x.

Example 2

Suppose we have the differential equation

$$\frac{dy}{dx} = \frac{ty}{t^2+1}, \quad y(0) = 1. \tag{160.5}$$

The exact solution to equation (160.5) is $y(t) = \sqrt{1+t^2}$. Hence, $y(1) = \sqrt{2} \approx 1.41421$. Approximating equation (160.5) by use of Euler's method with a step size of h, we can obtain an approximation to the solution at $t = 1$, $u_h \approx y(1)$. As h decreases, this approximation should becomes better.

In table 160.1, we show the values of u_h that are obtained when the h's are made successively smaller by a factor of 2. Even though the last value is not very close to $\sqrt{2}$, we can improve the accuracy by using transformations. The first application of Richardson extrapolation is defined by (because Euler's method is first order accurate) $u_{h;R} := \frac{2u_h - u_{2h}}{2-1}$. The second application of Richardson extrapolation is defined by $u_{h;RR} := \frac{4u_{h;R} - u_{2h;R}}{4-1}$. The first application of the Shanks transformation is defined by

$$u_{h;S} := \frac{u_{2h}u_{h/2} - u_h^2}{u_{2h} + u_{h/2} - 2u_h}.$$

The second application then uses the numbers $u_{h;S}$ in the same formula to obtain $u_{h;SS}$. As expected, the transformed values are much closer to the true value of $y(1)$.

h	u_h	$u_{h;R}$	$u_{h;RR}$	$u_{h;S}$	$u_{h;SS}$
0.200	1.45847				
0.100	1.43792	1.41738		1.41198	
0.050	1.42646	1.41499	1.41420	1.41376	1.41420
0.025	1.42043	1.41441	1.41421	1.41411	
0.012	1.41735	1.41426	1.41421		

Table 160.1: Numerical approximations to the solution of equation (160.5) (More accurate results are obtained by applying Richardson extrapolation and the Shanks transformation to this data.)

Notes

1. In the example, the quantity $R_1(x)$ could be explicitly determined. However, to utilize this method, this value does not have to be known explicitly.
2. To numerically approximate the solution to $y' = f(x, y)$, the modified midpoint method determines $y(x + nh)$, given $y(x)$, by

$$\begin{aligned} z_0 &= y(x), \\ z_1 &= z_0 + hf'(x, z_0), \\ z_{m+1} &= z_{m-1} + 2hf'(x + mh, z_m), \qquad \text{for } m = 1, 2, \ldots, n-1, \\ y(x + nh) &\simeq \frac{1}{2}[z_n + z_{n-1} + hf'(x + nh, z_n)], \end{aligned}$$

   where h is a small step size. This method is of second order but has an error that only involves *even* powers of h. Hence, each Richardson extrapolation of this method increases the order by 2. See Press *et al.* [8, pages 83–86] for more details.
3. Richardson extrapolation is often referred to as *deferred approach to the limit*.
4. This method also works for non-uniform grids if every interval is subdivided.
5. Some functions are not well approximated by polynomials but are well approximated by rational functions (see the section on Padé approximants, page 582). Instead of using a polynomial fit for the error term (as in equation (160.1)), a rational function approximation could be made—this is the basis of the Bulirsch–Stoer method. See Press *et al.* [8, pages 563–568] for more details.
6. See also Isaacson and Keller [5, pages 372–374].

References

[1] Bender, C. M., and Orszag, S. A. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw–Hill Book Company, New York, 1978.

[2] Cash, J. R. On the numerical integration of nonlinear two-point boundary value problems using iterated deferred corrections: II. The development and analysis of highly stable deferred correction. *SIAM J. Numer. Anal. 25*, 4 (1988), 862–882.

[3] Christiansen, E., and Petersen, H. G. Estimation of convergence orders in repeated Richardson extrapolation. *BIT 29* (1989), 48–59.

[4] Deuflhard, P. Recent progress in extrapolation methods for ordinary differential equations. *SIAM Review 27*, 4 (December 1985), 505–535.

[5] Isaacson, E., and Keller, H. B. *Analysis of Numerical Methods*. John Wiley & Sons, New York, 1966.

[6] Lindberg, B. Compact deferred correction formulas. In *Numerical Integration of Differential Equations and Large Linear Systems*, J. Hinze, Ed. Springer–Verlag, New York, 1982, pp. 220–233.

[7] meier, R. F. On Richardson extrapolation for finite difference methods on regular grids. *Numer. Math. 55* (1989), 451–462.

[8] Press, W. H., Flannery, B. P., Teukolsky, S., and Vetterling, W. T. *Numerical Recipes*. Cambridge University Press, New York, 1986.

[9] Richardson, L. F. The approximate arithmetical solution by finite differences of physical problems involving differential equations. *Philos. Trans. Roy. Soc. London Ser. A 210* (1910), 307–357.

161. Stability: ODE Approximations

Applicable to Ordinary differential equations.

Yields

It is straightforward to determine if a finite difference scheme is stable.

Idea

If a finite difference scheme is stable, then a locally good approximation yields a globally good approximation (provided the differential equation is well posed).

Procedure 1

Difference schemes for ordinary differential equations may be stable or unstable. The definition closely parallels the definition for the stability and well-posedness of a differential equation. A stable difference scheme is one in which small changes in the initial and boundary data do not change the solution greatly. An unstable difference scheme is one that shows great sensitivity to the initial and boundary data.

To determine if the difference scheme for an ordinary differential equation is stable (or *zero-stable*), we apply the scheme to the equation $y' = 0$ (which has only a constant solution) and determine if the finite difference approximation stays bounded. Suppose we have the following difference scheme for the first order equation $y' = f(x, y)$:

$$\sum_{j=0}^{p} \alpha_j v_{n+j} - h \sum_{j=0}^{p} \beta_j f(x_{n+j}, v_{n+j}) = 0, \tag{161.1}$$

where v_n is an approximation to $y(x_n)$ (and $x_n = nh$ for $n = 1, 2, \dots$). Applying the above scheme to the test equation is equivalent to using $f(x, y) = 0$ in equation (161.1). This results in

$$\sum_{j=0}^{p} \alpha_j v_{n-j} = 0. \tag{161.2}$$

The method is said to be *stable* if all solutions of equation (161.2) are uniformly bounded for all n and all initial data $\{v_0, v_1, \dots, v_{p-1}\}$.

The difference equation (161.2) has solutions of the form $v_n = \lambda^n$. Using $v_n = \lambda^n$ in equation (161.2) results in the characteristic equation for λ

$$\lambda^n \rho(\lambda) = \sum_{j=0}^{p} \alpha_j \lambda^{n-j} = 0. \tag{161.3}$$

It is easily shown that the method is unstable if any of the roots to equation (161.3) have magnitudes greater than 1, or if there is a multiple root whose magnitude is equal to 1.

Procedure 2

Sometimes "stability" is defined in terms of how the approximate solution to the equation $y' = \lambda y$ behaves. Using $f(y, x) = \lambda y$ and then $v_n = \lambda^n$, we are led to the *stability polynomial*. The stability polynomial associated with equation (161.1) is defined to be $\pi(r; \overline{h}) = \rho(\lambda) - \overline{h}\sigma(\lambda)$, where $\overline{h}$ represents $h\lambda$ and $\rho(x)$ and $\sigma(x)$ represent the first and second characteristic polynomials (see page 671). Using the stability polynomial, we have the following definitions (see Lambert [10, pages 409–431]):

> The method in equation (161.1) is said to be *absolutely stable* for a given $\overline{h}$ if, for that $\overline{h}$, all the roots of $\pi(r; \overline{h})$ satisfy $|r_s| < 1$ for $s = 1, 2, \ldots, p$, and to be *absolutely unstable* otherwise. An interval (a, b) of the real line is said to be an *interval of absolute stability* if the method is absolutely stable for all $\overline{h} \in (a, b)$.

> The method in equation (161.1) is said to be *relatively stable* for a given $\overline{h}$ if, for that $\overline{h}$, the roots of $\pi(r; \overline{h})$ satisfy $|r_s| < |r_1|$ for $s = 2, 3, \ldots, p$, and to be *relatively unstable* otherwise. An interval (a, b) of the real line is said to be an *interval of relative stability* if the method is relatively stable for all $\overline{h} \in (a, b)$.

Using these definitions, we define the method in equation (161.1) to be absolutely/relatively stable in a region $\mathcal{R}$ of the complex plane if, for all $\overline{h} \in \mathcal{R}$, the roots of the stability polynomial $\pi(r; \overline{h})$ have the required associated property (defined above).

Using the notion of stability in a region, we define the following types of stability:

- A method has *A-stability* if $\{h\lambda \mid \Re(h\lambda) < 0\} \subset \mathcal{R}$.
- A method has *A(α)-stability* if $\{h\lambda \mid -\alpha < \pi - \arg(h\lambda) < \alpha\} \subset \mathcal{R}$.
- A method has *A_0-stability* if $\{h\lambda \mid \Im(h\lambda) = 0, \Re(h\lambda) < 0\} \subset \mathcal{R}$.

A picture of the region $\mathcal{R}$ is known as a *stability diagram*. When approximating a differential equation on a bounded interval, the limit $n \to \infty$, h fixed, is of interest. The stability diagram will indicate allowable values for h.

Example 1

Euler's method for the ordinary differential equation $y' = f(x, y)$ consists of the approximation: $v_{n+1} - v_n = hf(x_n, v_n)$. To determine if this method is stable, we apply this method to the equation $y' = 0$ to determine the difference scheme

$$v_n - v_{n-1} = 0. \qquad (161.4)$$

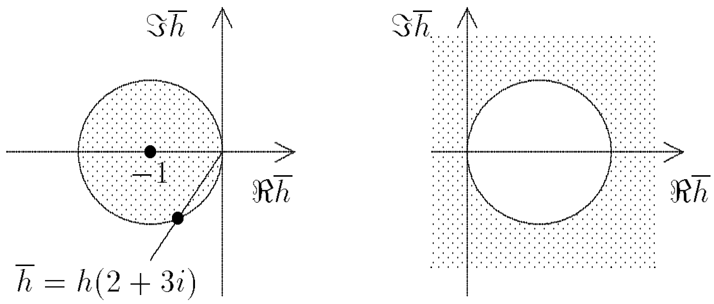

Figure 161.1: Stability diagrams for Euler's method (left) and Euler's backward method (right). Region of absolute stability is shown shaded.

Using $v_n = \lambda^n$ in equation (161.4) results in the characteristic equation

$$\rho(\lambda) = \lambda^n - \lambda^{n-1} = 0,$$

which has the roots $\lambda = 1$ and $\lambda = 0$ (with multiplicity $n-1$). Because the only root with magnitude 1, $\lambda = 1$, has multiplicity 1, and all the other roots have magnitudes less than 1, Euler's method is a stable method.

Example 2

Applying Euler's method to the equation $y' = f(x,y) = \lambda y$, we compute

$$\begin{aligned} v_{n+1} &= v_n + h f_n \\ &= v_n + h\lambda v_n \\ &= (1+\overline{h}) v_n. \end{aligned}$$

Hence, the region of absolute stability is given by $\mathcal{R} = \left\{ \overline{h} \mid \left|1+\overline{h}\right| \leq 1 \right\}$, see figure 161.1.a.

Applying Euler's backwards method to the equation $y' = f(x,y) = \lambda y$, we compute

$$\begin{aligned} y_{n+1} &= y_n + h f_{n+1} \\ &= y_n + h\lambda y_{n+1} \\ &= \frac{y_n}{1-\overline{h}}. \end{aligned}$$

Hence, the region of absolute stability is given by $\mathcal{R} = \left\{ \overline{h} \mid \frac{1}{\left|1-\overline{h}\right|} \leq 1 \right\}$, see figure 161.1.b.

Stability diagrams can be used to determine allowable step sizes. If we were to integrate the ordinary differential equation $y' = (2+3i)y$ using Euler's method, then the maximum allowable (real) step size that will produce an absolutely stable method is $h = \frac{4}{13}$; see figure 161.1.a. Stability diagrams are also used to qualitatively compare different difference schemes.

Notes

1. Observe that a difference scheme can be stable and still not be consistent. Stability and accuracy are two entirely different concerns.
2. For a stability analysis of second order ordinary differential equations, see Gear [6].
3. Generally, the sequence of methods, {one step methods, iteration methods, implicit methods}, demonstrate progressively better stability. That is, it is generally true that larger step sizes can be taken for implicit methods than for explicit methods.
4. Karim and Ismail [8] present five different ways in which to determine the stability of a difference scheme. They all lead to the same conclusion, but, on certain classes of equations, some methods are easier to apply than others.
5. To determine whether a finite difference scheme for a partial differential equation is stable, see either the Courant–Friedrichs–Lewy consistency criterion (page 688) or the Von Neumann stability test (page 692).
6. There are are many useful theorems in numerical analysis concerning the stability of methods for specific equations. For example, an A-stable method cannot have accuracy $p > 2$. See Dahlquist [3].
7. A consistent method is called *stiffly stable* if (1) for some constant $D < 0$, all solutions of the difference equation generated by the application of this method to the scalar test equation, $y' = \lambda y$, tend to zero as $n \to \infty$ for all complex λ with $\operatorname{Re} \lambda < D$ and for all fixed step sizes h with $h > 0$; and (2) there is an open set S whose closure contains the origin and the method is stable for $h\lambda \in S$. Here, h represents the grid spacing.
8. Mathematica has the package `OrderStar` which displays order stars for both absolute and relative stability.
9. There are many other types of stability that have been defined. A partial ordering of some common types of stability is given by the following list (see Butcher [2]):

$$\begin{aligned} &\text{algebraic stability} \\ &\qquad \Rightarrow \quad \text{Euclidean } AN\text{-stability} \quad \Rightarrow \quad \text{strong } AN\text{-stability} \\ &\qquad \Rightarrow \quad \text{weak } AN\text{-stability} \quad \Rightarrow \quad A\text{-stability} \end{aligned}$$

References

[1] Burrage, K. *(k,l)*-algebraic stability of Runge–Kutta methods. *IMA J. Num. Analysis 8*, 3 (1988), 385–400.

[2] Butcher, J. C. Linear and non-linear stability for general linear methods. *BIT 27* (1987), 182–189.

[3] Dahlquist, G. A special stability problem for linear multistep methods. *BIT 3* (1963), 27–43.

[4] Dekker, K., and Verwer, J. G. *Stability of Runge–Kutta Methods for Stiff Nonlinear Systems*. North–Holland Publishing Co., New York, 1984.

[5] Ganzha, V. G., and Liska, R. Application of the REDUCE computer algebra system to stability analysis of difference schemes. In *Computers and Mathematics*, E. Kaltofen and S. M. Watt, Eds. Springer–Verlag, New York, 1990, pp. 119–129.

[6] Gear, C. W. The stability of numerical methods for second order ordinary differential equations. *SIAM J. Numer. Anal. 15*, 1 (February 1978), 188–197.

[7] Iserles, A. Stability and dynamics of numerical methods for nonlinear ordinary differential equations. *IMA J. Num. Analysis 10* (1990), 1–30.

[8] Karim, A. I. A., and Ismail, G. A. The stability of multi-step formulae for solving differential equations. *Int. J. Comp. Math. 13* (1983), 53–67.

[9] Lambert, J. D. *Computational Methods in Ordinary Differential Equations*. Cambridge University Press, New York, 1973.

[10] Lambert, J. D. Developments in stability theory for ordinary differential equations. In *The State of the Art in Numerical Analysis*, A. Iserles and M. J. D. Powell, Eds. Clarendon Press, Oxford, England, 1987.

[11] Wanner, G. Order stars and stability. In *The State of the Art in Numerical Analysis*, A. Iserles and M. J. D. Powel, Eds. Clarendon Press, Oxford, England, 1987, pp. 451–471.

162. Stability: Courant Criterion

Applicable to Hyperbolic partial differential equations.

Yields

A statement about whether or not a difference scheme may converge to the exact solution of a hyperbolic equation.

Idea

The "numerical domain of dependence" for a hyperbolic equation must include the actual domain of dependence in order for the numerical approximation of the solution to converge to the true solution.

Procedure

A hyperbolic partial differential equation has characteristics (see page 432). Generally, the dependent variables will satisfy ordinary differential equations along the characteristics. These characteristics will propagate from the curves along which the initial data are given to every point in the domain. Given a specific point at which the solution is desired, the characteristics through that point must be determined.

If a numerical scheme for a hyperbolic equation attempts to compute a numerical approximation to the solution at a point, then all of the relevant characteristics must be present or the method may not converge to the correct solution.

Example

Suppose we have the wave equation

$$u_{tt} = c^2 u_{xx}, \tag{162.1}$$

for $u(x,t)$, where the constant c represents the wave speed. The initial conditions for equation (162.1) are assumed to be

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x). \end{aligned}$$

We define $v_{n,j} = u(t_n, x_j)$, where $t_n := n\Delta t$ and $x_j := j\Delta x$. If a second order centered difference scheme is used, then equation (162.1) might be approximated as

$$\frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{(\Delta t)^2} = c^2 \frac{u_{n,j+1} - 2u_{n,j} + u_{n,j-1}}{(\Delta x)^2},$$

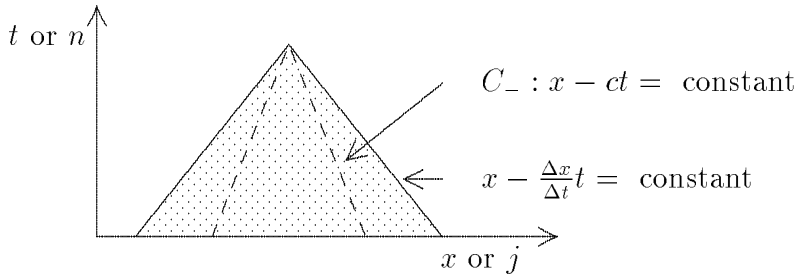

Figure 162.1: Characteristics (indicated by dashed lines) that are included in the numerical domain of dependence (shown shaded).

which can be manipulated into the explicit formula

$$u_{n+1,j} = 2\left[1 - \left(c\,\frac{\Delta t}{\Delta x}\right)^2\right]u_{n,j} + \left(c\,\frac{\Delta t}{\Delta x}\right)^2(u_{n,j+1} + u_{n,j-1}) - u_{n-1,j}. \tag{162.2}$$

Hence, the value of $u_{n+1,j}$ depends on $\{u_{n,j+k} \mid k = 0, \pm 1\}$ and $u_{n-1,j}$. Applying equation (162.2) to itself, we see that the value of $u_{n+1,j}$ depends on $\{u_{n-1,j+k} \mid k = 0, \pm 1, \pm 2\}$. Applying equation (162.2) again, we see that the value of $u_{n+1,j}$ depends on $\{u_{n-2,j+k} \mid k = 0, \pm 1, \pm 2, \pm 3\}$.

In general, the value of $u_{n+1,j}$ will depend on the points $\{u_{0,j+k} \mid k = 0, \pm 1, \ldots, \pm n\}$. These points along the initial curve (where the initial data are given) describe the *numerical domain of dependence*. See figure 162.1.

The characteristics of equation (162.1) are the two curves (shown dashed in the figures)

$$\begin{aligned} C_- : &\quad x - ct = x_i, \\ C_+ : &\quad x + ct = x_i, \end{aligned}$$

where x_i is any point on the initial curve. Hence, the value of $u(t_n, x_j)$ will depend on the values of $u(0, x_k)$ for $x_k = x_i - ct$ and $x_k = x_i + ct$.

If these values are not included in the numerical domain of dependence, then the numerical approximation will, generally, give the incorrect answer. This is simply because the numerical approximation does not use the data that are important in solving the problem.

The two different possible scenarios are shown in figures 162.1 and 162.2. In figure 162.1, the characteristics are included in the numerical domain of dependence (i.e., $\left(\frac{\Delta t}{\Delta x}\right)$ is less than 1). Because of this, the method *may* converge to the exact solution. In figure 162.2, the characteristics are not included in the numerical domain of dependence (i.e., $\left(\frac{\Delta t}{\Delta x}\right)$ is greater

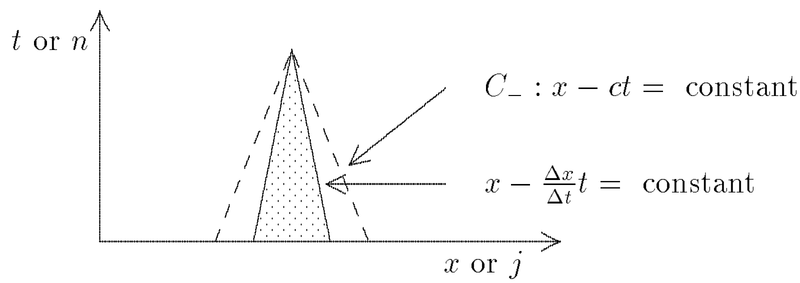

Figure 162.2: Characteristics (indicated by dashed lines) that are not included in the numerical domain of dependence (shown shaded).

than 1). Because of this, the method *cannot*, in general, converge to the exact solution of equation (162.1).

In summary, for this example, if Δx and Δt are chosen so that

- $c\frac{\Delta t}{\Delta x} > 1$, then the method *cannot* converge to the exact solution.
- $c\frac{\Delta t}{\Delta x} < 1$, then the method *may* converge to the exact solution.

Notes

1. This condition is also known as the Courant–Friedrichs–Lewy or CFL condition. The theorem proved by Courant *et al.* [1] is:

   > There are no explicit, unconditionally stable, consistant finite difference schemes for hyperbolic systems of partial differential equations.

2. Of course, more complicated hyperbolic problems will require a more detailed analysis.
3. Another test that can be used to determine the stability of a finite difference scheme for partial differential equations is the Von Neumann stability test (see page 692).
4. To determine if the difference scheme for an ordinary differential equation is stable, see page 670.
5. See also Davis [2, pages 45–47] and Isaacson and Keller [4, page 489]

References

[1] Courant, R., Friedrichs, K. O., and Lewy, H. Uber dir partiellen differenzengleichungen der mathematischen physik. *Mathematische Annalen 100* (1928), 32–74.

[2] Davis, J. L. *Finite Difference Methods in Dynamics of Continuous Media.* The MacMillan Company, New York, 1986.

[3] Gottlieb, D., and Tadmor, E. The CFL condition for spectral approximations to hyperbolic initial-boundary value problems. *Math. of Comp. 56*, 194 (April 1991), 565–588.

[4] Isaacson, E., and Keller, H. B. *Analysis of Numerical Methods*. John Wiley & Sons, New York, 1966.

163. Stability: Von Neumann Test

Applicable to Finite difference schemes for partial differential equations.

Yields

Knowledge of whether the difference scheme is stable.

Procedure

The Von Neumann test determines whether the difference scheme for a partial differential equation is stable. For difference schemes with constant coefficients, the test consists of examining all exponential solutions to determine whether they grow exponentially in the time variable even when the initial values are bounded functions of the space variable.

If any of them do increase without limit, then the method is *unstable*. Otherwise, it is *stable*.

This test can also be applied to equations with variable coefficients by introducing new, constant coefficients equal to the frozen values of the original ones at some specific point of interest.

Example

If the parabolic equation $u_t = u_{xx}$ is discretized via

$$u_t \simeq \frac{1}{k}\Big[u(x, t+k) - u(x,t)\Big],$$
$$u_{xx} \simeq \frac{1}{h^2}\Big[u(x+h,t) - 2u(x,t) + u(x-h,t)\Big],$$

and $v_{m,n}$ is used to represent $u(mh, nk)$, then the recurrence relation

$$u_{m,n+1} = u_{m,n} + \frac{k}{h^2}\left(u_{m+1,n} - 2u_{m,n} + u_{m-1,n}\right) \tag{163.1}$$

is obtained. To investigate all possible bounded exponential type solutions, we choose

$$u_{m,n} = e^{im\theta}e^{in\lambda}. \tag{163.2}$$

Substituting equation (163.2) into equation (163.1) results in the relation

$$e^{i\lambda} = 1 - 4\frac{k}{h^2}\sin^2\left(\frac{\theta}{2}\right), \tag{163.3}$$

which must be satisfied for λ and θ. It can be shown that the imaginary part of λ will be non-negative (and hence the method is stable) if

$$\frac{k}{h^2} \leq \frac{1}{2}. \tag{163.4}$$

Notes

1. A stability test for hyperbolic partial differential equations is the Courant–Friedrichs–Lewy consistency criterion (see page 688).
2. The Lax–Richtmyer equivalence theorem is the fundamental theorem in the theory of finite difference schemes for initial value problems:

   > A consistent finite difference scheme for a partial differential equation for which the initial value problem is well posed is convergent if and only if it is stable.

3. To determine whether the difference scheme for an ordinary differential equation is stable, see page 670.
4. See also Davis [1, pages 47–50], Garabedian [2, page 469 and page 477], Gottlieb and Orszag [3, pages 48–50], Isaacson and Keller [4, pages 523–529], and Lapidus and Pinder [5, pages 170–179].

References

[1] Davis, J. L. *Finite Difference Methods in Dynamics of Continuous Media*. The MacMillan Company, New York, 1986.

[2] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[3] Gottlieb, D., and Orszag, S. A. *Numerical Analysis of Spectral Methods: Theory and Applications*. SIAM, Philadelphia, PA, 1977.

[4] Isaacson, E., and Keller, H. B. *Analysis of Numerical Methods*. John Wiley & Sons, New York, 1966.

[5] Lapidus, L., and Pinder, G. F. *Numerical Solution of Partial Differential Equations in Science and Engineering*. John Wiley & Sons, New York, 1982.

164. Testing Differential Equation Routines

Applicable to Numerical approximations to differential equations.

Idea

Many differential equations have been used as examples to test differential equation solvers.

Procedure

As new differential equation integration techniques are developed, they are compared to existing techniques in terms of accuracy and efficiency. Many specific differential equation have been used as examples to indicate the performance of new algorithms and implementations. Tabulated below are some of those differential equations.

1. A test case that is often used to test computer codes for boundary value problems is *Troesch's problem* (see Roberts and Shipman [5]):

$$\begin{aligned} &\frac{d^2y}{dt^2} - n \sinh ny = 0, \\ &y(0) = 0, \quad y(1) = 1. \end{aligned}$$

2. Carroll [1] tests ODE system solvers with the equations (only some are listed below):

   (a) $y_1' = -6y_1 + 5y_2 + 2\sin x$, $y_2' = 94y_1 - 95y_2$, with $y_1(0) = y_2(0) = 0$.
   (b) $y' = -\beta y - y^2$ and $y(0) = 1$ for $\beta = \{1000, 800, 0.001, -10\}$.
   (c) $y_1' = -y_2 + (1 - y_1^2 - y_2^2)$, $y_2' = y_1 + (1 - y_1^2 - y_2^2)$, with $y_1(0) = 1$, $y_2(0) = 0$.
   (d) $y_1' = -y_1$, $y_2' = y_1^2 - 2y_2$, with $y_1(0) = y_2(0) = 5$.

3. Marletta [2] tests Sturm–Liouville problem solvers with the equations:

   (a) $-y'' + \left(\frac{2}{x^2} - \frac{1}{x}\right) y = \lambda y$ for $x \in (0, \infty)$.
   (b) $-y'' + \left(9e^{-2x} - 18e^{-x}\right) y = \lambda y$ for $x \in (-\infty, \infty)$.
   (c) $-\left((1 - x^2)y'\right)' = \lambda y$ for $x \in (-1, 1)$. (Legendre's equation)
   (d) $-y'' - \left(4000e^{-1.7(x-1.3)} - 2000e^{-3.4(x-1.3)} - \frac{2}{x^2}\right) y = \lambda y$ for $x \in (0, \infty)$. (Morse potential)
   (e) $-y'' + \left(x^2 + x^4\right) y = \lambda y$ for $x \in (-\infty, \infty)$.
   (f) $-(xy')' - \left(\frac{1}{4}\sec^2 x\right) y = \lambda y$ for $x \in (-\pi/2, \pi/2)$.
   (g) $-y'' - \frac{y}{x} = \lambda y$ for $x \in (0, \infty)$.

(h) $-\left(\frac{y'}{\sqrt{1-x^2}}\right)' = \frac{\lambda}{\sqrt{1-x^2}}y$ for $x \in (-1,1)$.

(i) $-y'' + \left(-2\beta\cos 2x + \beta^2 \sin^2 2x\right) y = \lambda y$, with $y(\pi/2) = y(-\pi/2) = 0$. (Coffey–Evans equation)

(j) $-y'' + y = \lambda w(x) y$ where $w(x) = \begin{cases} 0 & \text{for } x \in \left[0, \frac{1}{2}\right] \\ 1 & \text{for } x \in \left[\frac{1}{2}, 1\right] \end{cases}$, $y(0) = y(1) = 0$.

(k) $-y'' + x^\alpha y = \lambda y$

4. Shampine [6] tests stiff ODE solvers with the system

$$\begin{aligned} y_1' &= -0.04y_1 + 10^4 y_2 y_3, & y_1(0) &= 1 \\ y_2' &= 0.04y_1 - 10^4 y_2 y_3 - 3\cdot 10^7 y_2^2, & y_2(0) &= 0 \\ y_3' &= 3\cdot 10^7 y_2^2, & y_3(0) &= 0 \end{aligned}$$

5. Rice and Boisvert [4] have established a population of elliptic PDEs for testing purposes. It is divided into two groups, based on the domain geometry. There are 56 PDE problems defined on rectangular regions, most of which depend on parameters that control features of the problem. The problems themselves have differing

- Operator type (Poisson, Helmholtz, self-adjoint, constant coefficient, general)
- Boundary conditions (Dirichlet, Neumann, mixed)
- Solution features (entire, analytic, singular, peak, oscillatory, boundary layer, wave front, singularities, irregular, discontinuities, computationally complex)

Some of these problems are:

(a) $u_{xx} + u_{yy} = 1$ with $u = 0$ on the unit square ($x = 0, 1$ and $y = 0, 1$).

(b) $u_{xx} + u_{yy} = 6xye^{x+y}(xy + x + y - 3)$ with $u = 0$ on the unit square ($x = 0, 1$ and $y = 0, 1$).

(c) $u_{xx} + \frac{u_{yy}}{x^2} + \frac{2u_x}{x} + \frac{u_y}{x^2 \tan^3 y} = -100$ with $u = 0$ on $x = 0.1$ and $x = 1$, and $u = 0$ on $y = 0.1$ and $y = 1$.

(d) $\left(e^{xy}u_x\right)_x + \left(e^{-xy}u_y\right)_y - \frac{u}{1+x+y} = f$ with $u = 0$ on the unit square ($x = 0, 1$ and $y = 0, 1$). For this problem $f(x, y)$ is chosen so that the exact solution is $u = 3e^{xy}\sin\pi x \sin\pi y/4$.

(e) $u_{xx} + u_{yy} + \frac{3u_y}{5-y} = f$ with $u = 0$ on $x = \pm 0.5$ and $y = \mp 1$. For this problem $f(x, y)$ is chosen so that the exact solution has the form $u = (1 - y^2)(1 - 4x^2)(5 - y)^3(a + by)$.

(f) $u_{xx} + (1 + y^2)u_{yy} - u_x - (1 + y^2)u_y = f$ with

- $u + u_x = 0.27e^y$ on $x = 0$,
- $u - u_x = 0$ on $x = 1$,
- $u + u_y = 0.27e^x$ on $y = 0$,
- $u - u_y = 0.135(\log 2 - 1)(x^2 - x)^2$ on $y = 1$.

For this problem $f(x, y)$ is chosen so that the exact solution is $u = 0.135(e^{x+y} + (x^2 - x)^2 \log(1 + y^2)$.

References

[1] Carroll, J. A composite integration scheme for the nmerical solution of systems of ordinary differential equations. *J. Comput. Appl. Math. 25* (1989), 1–13.

[2] Marletta, M. Certification of algorithm 700. Numerical tests of the SLEIGN software for Sturm–Liouville problems. *ACM Trans. Math. Software 17*, 4 (December 1991), 481–490.

[3] Penn, H. L. A review of differential equations software. *Collegiate Microcomputer 6* (1988), 33–42.

[4] Rice, J. R., and Boisvert, R. F. *Solving Elliptic Problems Using ELLPACK.* Springer–Verlag, New York, 1985.

[5] Roberts, S. M., and Shipman, J. S. *Two-Point Boundary Value Problems: Shooting Methods.* American Elsevier Publishing Company, New York, 1972.

[6] Shampine, L. F. Ill-conditioned matrices and the integration of stiff ODEs. *J. Comput. Appl. Math. 48* (1993), 279–292.

165. Analytic Continuation*

Applicable to Initial value ordinary differential equations, a single equation, or a system.

Yields

A numerical approximation in the form of a Taylor series.

Idea

If the Taylor series of a function is known at a single point, then the Taylor series of that function may be found at another (nearby) point. This process may be repeated until a particular value is reached.

Procedure

Given a system of initial value ordinary differential equations, the method is to replace each dependent variable present by a Taylor series centered at a certain origin. The coefficients in each Taylor series are regarded as unknown quantities. The ordinary differential equations are used to obtain a set of recurrence relations from which the unknown coefficients may be calculated.

Thus, a formal power series solution may be determined to an initial value problem, and the series will be convergent in some region about the origin. Then, the truncated power series are evaluated at some point within the region of convergence. At this new point, initial values for the system are obtained from the already obtained Taylor series. Using these initial values, the recurrence relations then yield a second series solution valid in a region about the new origin.

This procedure can be iterated and the solution at a given point may be determined via a sequence of Taylor series. This algorithm is a numerical version of the process of analytic continuation.

Example

Suppose we have the system of ordinary differential equations

$$\begin{aligned} y' &= y^2 + z, \qquad && y(0) = 1, \\ z' &= z^2, && z(0) = 1. \end{aligned}$$

This system can be rewritten as the differential/algebraic system

$$\begin{aligned} a = y^2, \quad b = a + z, \quad c = z^2, \\ y' = b, \quad z' = c, \end{aligned} \tag{165.1}$$

with $b = 2$ and $a = c = y = z = 1$ when $t = 0$. If we define the Taylor series coefficients $\{a_k^{(j)}, b_k^{(j)}, c_k^{(j)}, y_k^{(j)}, z_k^{(j)}\}$ by the expansions

$$\begin{aligned} a(t) &= \sum_{k=0}^{\infty} a_k^{(j)}(t-t_j)^k, \quad & b(t) &= \sum_{k=0}^{\infty} b_k^{(j)}(t-t_j)^k, \\ c(t) &= \sum_{k=0}^{\infty} c_k^{(j)}(t-t_j)^k, \quad & y(t) &= \sum_{k=0}^{\infty} y_k^{(j)}(t-t_j)^k, \\ z(t) &= \sum_{k=0}^{\infty} z_k^{(j)}(t-t_j)^k, \end{aligned} \tag{165.2}$$

then, using equation (165.2) in equation (165.1), the following recurrence relations can be obtained

$$\begin{aligned} a_k^{(j)} &= \sum_{n=0}^{k} y_n^{(j)} y_{k-n}^{(j)}, \quad & b_k^{(j)} &= a_k^{(j)} + z_k^{(j)}, \\ c_k^{(j)} &= \sum_{n=0}^{k} z_n^{(j)} z_{k-n}^{(j)}, \quad & y_k^{(j)} &= b_k^{(j)}/(k+1), \\ z_k^{(j)} &= c_k^{(j)}/(k+1). \end{aligned} \tag{165.3}$$

The initial conditions give the starting values: $\{j = 0,\ t_0 = 0,\ a_0^{(0)} = c_0^{(0)} = y_0^{(0)} = z_0^{(0)} = 1,\ b_0^{(0)} = 2\}$. To determine the Taylor series about the point $t_0 = 0$, equation (165.3) is iterated for $k = 1, 2, \ldots, M$. The number of terms in each Taylor series required for a specified numerical accuracy M may be determined dynamically or fixed beforehand (if an appropriate analysis has been done).

Then a new point t_1 is chosen. A Taylor series for each of a, b, c, y, and z is then found about this new point by taking $j = 1$ and determining the initial conditions from.

$$a_0^{(1)} = \sum_{k=0}^{M} a_k^{(0)}(t_1 - t_0)^k, \quad b_0^{(1)} = \sum_{k=0}^{M} b_k^{(0)}(t_1 - t_0)^k, \quad \ldots .$$

The recurrence relations in equation (165.3) are then iterated again. This process can be repeated indefinitely.

Notes

1. Holubec and Stauffer [5] continue a Frobenius series instead of a Taylor series. This works particularly well on ordinary differential equations with regular singular points. They also discuss the appropriate step size to take at each stage in the calculation.
2. A Fortran computer program that generates the recurrence relations and then solves the system is described in Corliss and Chang [4].

3. Sometimes several hundred coefficients are required to obtain an accurate answer with this method. This is especially true when the expansion point for the Taylor series is near a singularity.
4. Interval bounds (see page 545) for the Taylor series coefficients are discussed in Moore [7, Chapter 11].
5. This technique has been extended to parabolic equations in Chang [1].

References

[1] Chang, Y. F. Solution of parabolic partial differential equations. In *Proceedings of the Sixth Manitoba Conference on Numerical Mathematics*, B. L. Hartnell and H. C. Williams, Eds. Utilitas Mathematics Publishing, Winnipeg, Canada, 1977, pp. 127–134.

[2] Chang, Y. F. Solving stiff systems by Taylor series. *Appl. Math. and Comp. 31* (1989), 251–269.

[3] Corliss, G., and Lowery, D. Choosing a stepsize for Taylor series methods for solving ODE's. *J. Comput. Appl. Math. 3*, 4 (1977), 251–256.

[4] Corliss, G. F., and Chang, Y. F. Solving ordinary differential equations using Taylor series. *ACM Trans. Math. Software 8* (1982), 114–144.

[5] Holubec, A., and Stauffer, A. D. Efficient solution of differential equations by analytic continuation. *J. Phys. A: Math. Gen. 18* (1985), 2141–2149.

[6] Holubec, A., Stauffer, A. D., Acacia, P., and Stauffer, J. A. Asymptotic shooting method for the solution of differential equations. *J. Phys. A: Math. Gen. 23* (1990), 4081–4095.

[7] Moore, R. E. *Interval Analysis*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1966.

166. Boundary Value Problems: Box Method

Applicable to Boundary value problems for ordinary differential equations.

Yields

A numerical approximation of the solution.

Idea

Using finite differences, the solution to a boundary value problem is determined (simultaneously) everywhere on the interval of interest.

Procedure

We will illustrate the procedure on the general second order linear ordinary differential equation. The same technique can be used, with only slight modifications, to systems of higher order ordinary differential equations, with the boundary data given virtually anywhere in the interval of interest.

Given the second order linear ordinary differential equation

$$\begin{aligned} a(x)y'' + b(x)y' + c(x)y = d(x), \\ y(x_L) = y_L, \quad y(x_U) = y_U, \end{aligned} \tag{166.1.a-b}$$

we introduce the variable $z(x) = y'(x)$ and write equation (166.1) as the system

$$\frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} z \\ \frac{d-cy-bz}{a} \end{bmatrix}. \tag{166.2}$$

Now, we choose a grid, not necessarily uniform, on the interval (x_L, x_U), say $x_L = x_1 < x_2 < \cdots < x_N = x_U$. At each one of the grid points, some finite difference scheme is chosen to approximate the equations in equation (166.2). The scheme used can vary from point to point. For instance, if Euler's method is used for every point, then

$$\begin{bmatrix} y \\ z \end{bmatrix}_{k+1} = \begin{bmatrix} y \\ z \end{bmatrix}_k + (x_{k+1} - x_k)\begin{bmatrix} z \\ \frac{d-cy-bz}{a} \end{bmatrix}_k \tag{166.3}$$

to first order, where $y_k = y(x_k)$, $z_k = z(x_k)$, and similarly for $\{a_k, b_k, c_k, d_k\}$. From equation (166.1.b). the values $y_1 = y_L$ and $y_N = y_U$ are known.

To determine all of the $\{z_k\}$, and the remaining $\{y_k\}$, all of the relations in equation (166.3) (i.e., for $k = 1, 2, \ldots, N$) should be combined into one large matrix equation. First, for ease of notation, define $h_k = x_{k+1} - x_k$, $e_k = d_k/a_k$, $f_k = c_k/a_k$ and $g_k = b_k/a_k$. In these new variables, equation (166.3) may be written as

$$\begin{aligned} y_{k+1} &= y_k + h_k z_k, \\ z_{k+1} &= z_k + h_k \left(e_k - f_k y_k - g_k z_k\right). \end{aligned} \tag{166.4}$$

Combining all of the equations in (166.4) results in

$$\begin{bmatrix} 1 & h_1 & -1 & 0 & 0 & 0 & \dots \\ h_1 f_1 & -1 + h_1 g_1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & h_2 & -1 & 0 & \dots \\ 0 & 0 & h_2 f_2 & -1 + h_2 g_2 & 0 & 1 & \\ \vdots & \vdots & & & & & \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ y_3 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} 0 \\ h_1 e_1 \\ 0 \\ h_2 e_2 \\ 0 \\ \vdots \\ h_N e_N \end{bmatrix}.$$

To this matrix equation should be added two more rows, one corresponding to $y_1 = y_L$ and one corresponding to $y_N = y_U$. With these two rows, there results an $2N \times 2N$ matrix equation. This equation can be solved to determine a numerical approximation to the solution at all of the grid points.

Example

The second order linear ordinary differential equation

$$\begin{aligned} & y'' + y = 3, \\ & y(0) = 3, \quad y\left(\frac{\pi}{2}\right) = 2, \end{aligned} \tag{166.5}$$

has the solution $y = 3 - \sin x$. We use the box method to numerically approximate this solution. Writing equation (166.5) as a system results in

$$\frac{d}{dx}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 3 - y \end{bmatrix}. \tag{166.6}$$

We choose a uniform grid: $x_n = (n-1)h$ for $n = 1, 2, 3, 4$ with $h = \pi/6$. Defining $y_n = y(x_n)$ and $z_n = z(x_n)$, and using Euler's method, equation (166.6) may be approximated as

$$\begin{aligned} y_{n+1} &= y_n + h z_n, \\ z_{n+1} &= z_n + h(3 - y_n). \end{aligned} \tag{166.7}$$

Combining all of the equations in equation (166.7) for $n = 1, 2, 3, 4$ results in

$$\begin{bmatrix} 1 & h & -1 & 0 & 0 & 0 & 0 & 0 \\ h & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & h & -1 & 0 & 0 & 0 \\ 0 & 0 & h & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & h & -1 & 0 \\ 0 & 0 & 0 & 0 & h & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ y_3 \\ z_3 \\ y_4 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3h \\ 0 \\ 3h \\ 0 \\ 3h \end{bmatrix}.$$

Then the following two rows are added, to incorporate the known values of $y(0)$ and $y(\pi/2)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ y_3 \\ z_3 \\ y_4 \\ z_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The Fortran program in program 166.1 numerically approximates the solution to the above equation. Note that this program uses a linear equation solver `LSOLVE`, whose source code is not listed. The output of the program is

```
 Here is the approximate solution:
    3.000   -0.701    2.633   -0.701    2.266   -0.509    2.000   -0.124
 Here is the exact solution
    3.000   -1.000    2.500   -0.866    2.134   -0.500    2.000    0.000
```

The values for y_n are only accurate to one decimal place in this example. Putting more points in the interval would decrease the error, as would using a higher order method in place of Euler's method.

Notes

1. In our example, if the two rows corresponding to the boundary terms were added to the matrix equation at the correct locations, the resulting matrix would be banded.
2. This technique is recommended for stiff boundary value problems because many points can be added where the solution undergoes large changes and different discretization schemes may be used in different regions.
3. For nonlinear equations or nonlinear boundary conditions, this method can be used iteratively by linearizing the nonlinear terms at each step.

```
      DIMENSION ARRAY(8,18),SOLN(8),RHS(8),NROW(100)
      PI=3.1415926
      NPOINT=8
      H=PI/2.* 2./FLOAT(NPOINT-2)
      DO 10 J=1,NPOINT
      DO 10 K=1,NPOINT
10    ARRAY(J,K)=0.0
C Create the matrix
      ARRAY(1,1)=1.0
        RHS(1  )=3.0
      ARRAY(NPOINT,NPOINT-1)=1.0
        RHS(NPOINT         )=2.0
      J=1
20    J=J+1
      IF( J .GE. NPOINT ) GOTO 30
C Here is the Y-equation
       ARRAY(J,J-1)=1
       ARRAY(J,J  )=H
       ARRAY(J,J+1)=-1
         RHS(J    )=0
      J=J+1
C Here is the Z-equation
       ARRAY(J,J-2)=H
       ARRAY(J,J-1)=-1
       ARRAY(J,J+1)=1
         RHS(J    )=3.0*H
       GOTO 20
C Solve the matrix system
30     CALL LSOLVE(NPOINT,ARRAY,SOLN,RHS,NROW,IFSING,NPOINT)
       WRITE(6,5) (SOLN(J),J=1,NPOINT)
5      FORMAT(' Here is the approximate solution:',/,8(1x,F8.3) )
C Compute the exact solution for comparison
       J=1
       DO 40 JJ=1,NPOINT/2
       SOLN(J  )=3.0-SIN( H*FLOAT(JJ-1) )
       SOLN(J+1)=   -COS( H*FLOAT(JJ-1) )
40     J=J+2
       WRITE(6,15) (SOLN(J),J=1,NPOINT)
15     FORMAT(' Here is the exact solution:',/,8(1x,F8.3) )
       END
```

Program 166.1: Fortran program for box method.

4. Other techniques for solving boundary value problems include collocation (see page 514), shooting (see page 706), and invariant imbedding (see page 747).
5. Ascher *et al.* [1], Daniel [2], and Mattheij [5] all have discussions of different techniques that can be applied to boundary value problems.
6. See also Isaacson and Keller [4, pages 427–432] and Roberts and Shipman [6, Chapter 8, pages 201–231].

References

[1] Ascher, U. M., Mattheij, R. M. M., and Russel, R. D. *Numerical*

Solution of Boundary Value Problems for Ordinary Differential Equations. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1988.

[2] Daniel, J. W. A road map of methods for approximating solutions of two-point boundary-value problems. In *Codes for Boundary-Value Problems in Ordinary Differential Equations*, B. Childs, M. Scott, J. W. Daniel, E. Denman, and P. Nelson, Eds. Springer–Verlag, New York, 1979, pp. 1–18.

[3] Gregory, J., and Zeman, M. Spline matrices and their applications to some higher order methods for boundary value problems. *SIAM J. Numer. Anal. 25*, 2 (April 1988), 399–410.

[4] Isaacson, E., and Keller, H. B. *Analysis of Numerical Methods.* John Wiley & Sons, New York, 1966.

[5] Mattheij, R. M. M. Decoupling and stability of algorithms for boundary value problems. *SIAM Review 27*, 1 (March 1985), 1–44.

[6] Roberts, S. M., and Shipman, J. S. *Two-Point Boundary Value Problems: Shooting Methods.* American Elsevier Publishing Company, New York, 1972.

167. Boundary Value Problems: Shooting Method*

Applicable to Nonlinear boundary value problems for ordinary differential equations.

Yields

A numerical approximation to the solution.

Idea

Using Newton's method, the correct initial conditions for a boundary value problem can be determined. Knowing the initial conditions, the differential equations can be numerically integrated in a straightforward manner.

Procedure

The general procedure can be illustrated by studying a second order ordinary differential equation. Suppose we wish to numerically approximate the solution $y(x)$ of the equation

$$\begin{aligned} &L[y'', y', y, x] = 0, \\ &y(0) = 0, \quad y(1) = A, \end{aligned} \tag{167.1}$$

where A is a given constant. The differential equation $L[\cdot] = 0$ may or may not be a linear differential equation. If $z(x; \alpha)$ is defined to be the solution of

$$\begin{aligned} &L[z'', z', z, x] = 0, \\ &z(0; \alpha) = 0, \quad z'(0; \alpha) = \alpha, \end{aligned} \tag{167.2}$$

then $y(x)$ will be equal to $z(x; \alpha)$ for one or more values of α. The parameter α in equation (167.2) must be determined so that

$$z(1; \alpha) = A.$$

Because equation (167.2) is an initial value problem, it is straightforward to integrate it numerically from $x = 0$ to $x = 1$. See, for instance, Euler's method (page 730). To use the shooting method, we integrate equation (167.2) numerically for some arbitrary initial guess for α, say α_0. If $z(1; \alpha_0) = A$, then $y(x) = z(x; \alpha_0)$ and we are done.

If $z(1; \alpha_0) \neq A$, then a new value of α must be chosen, say α_1. Equation (167.2) is then integrated for this new value of α. The process of choosing new values for α is repeated until the value of $z(1; \alpha)$ is sufficiently close to A. If the new α's are chosen well, then $z(1;\alpha)$ will converge to A and a numerical approximation to equation (167.1) will have been obtained. One way to choose the sequence of α's is by Newton's method

$$\alpha_{n+1} = \alpha_n - \frac{z(1;\alpha_n) - A}{\frac{\partial}{\partial\alpha} z(1;\alpha)\Big|_{\alpha=\alpha_n}}. \tag{167.3}$$

A numerical way to implement equation (167.3) might be

$$\alpha_{n+1} = \alpha_n - \frac{z(1;\alpha_n) - A}{[z(1;\alpha_n+\epsilon) - z(1;\alpha_n)]/\epsilon},$$

where ϵ is a small number.

Example

Suppose we have the nonlinear second order ordinary differential equation

$$\begin{aligned} &y'' + 2(y')^2 = 0, \\ &y(0) = 1, \quad y(1) = \frac{1}{2}. \end{aligned} \tag{167.4}$$

Because equation (167.4) has no explicit dependence on y, the "dependent variable missing" method (see page 260) can be used to solve this equation exactly. By this technique, the solution of equation (167.4) is found to be

$$y(x) = 1 + \frac{1}{2}\log\left(1 + \frac{1-e}{e}x\right).$$

Hence, $y'(0) = (1-e)/2e \simeq -0.31606$.

By use of the shooting method, a computer program should "discover" that $y'(0) \simeq$ `-0.31607`. The Fortran program in program 167.1 utilizes finite differences to determine $y'(0)$ for equation (167.4). The equation in equation (167.4) is turned into the two first order ordinary differential equations

$$\begin{aligned} \frac{dy}{dx} &= z, \\ \frac{dz}{dx} &= -2y^2, \end{aligned}$$

and then integrated by the use of Euler's method (see page 730).

An initial guess of $y'(0) = 0$ is used in the program. The successive approximations of $y'(0)$ appear below:

```
Iteration number   0  value of Y'(0)=   0.
Iteration number   1  value of Y'(0)=  -0.50000050
Iteration number   2  value of Y'(0)=  -0.49857452
Iteration number   3  value of Y'(0)=  -0.49102421
```

```
      Y0=1.D0
      Y1=0.5D0
      YP0=0.D0
C Perform a Newton iteration 9 times
      DO 10 NEWT=1,10
      WRITE(6,5) NEWT-1,YP0
5     FORMAT(' Iteration number',I4,' value of Y''(0)=',F13.8)
10    YP0=FNEWTON(Y0,Y1,YP0)
      END
C This function performs one Newton step
      FUNCTION FNEWTON(Y0,Y1,YP0)
      EPS=0.000001D0
      YP01=YP0
      YP02=YP0+EPS
      Z1=YAT1(Y0,YP01)
      Z2=YAT1(Y0,YP02)
      FNEWTON=YP0-(Z1-Y1)*EPS/(Z2-Z1)
      RETURN
      END
C This function determines Y(1); when Y(0) and Y'(0) are given
      FUNCTION YAT1(Y0,YP0)
      N=20000
      DX=1.D0/DFLOAT(N)
      Y=Y0
      YP=YP0
C This is the actual integration loop
      DO 10 J=1,N
      Y = Y  + DX * YP
10    YP= YP + DX * ( -2.D0*YP**2 )
      YAT1=Y
      RETURN
      END
```

Program 167.1: Fortran program for shooting method

```
Iteration number   4  value of Y'(0)=  -0.46366318
Iteration number   5  value of Y'(0)=  -0.40465858
Iteration number   6  value of Y'(0)=  -0.34199798
Iteration number   7  value of Y'(0)=  -0.31799014
Iteration number   8  value of Y'(0)=  -0.31608113
Iteration number   9  value of Y'(0)=  -0.31607109
```

Note that the computer program required a large number of steps in the interval $[0, 1]$ in order to achieve the accuracy shown (partly because we used Euler's method, which is of low order).

Notes

1. If this method is applied to a linear equation, the value of $y'(0)$ will converge to the correct value in a single step.
2. It is also possible to simultaneous integrate along several rays at once. This is called the *method of multiple shooting.* See Diekhoff *et al.* [1] or Stoer and Bulirsch [6] for details.

References

[1] Diekhoff, H.-J., Lory, P., Oberle, H. J., Pesch, H.-J., Rentrop, P., and Seydel, R. Comparing routines for the numerical solution of initial value problems in ordinary differential equations in multiple shooting. *Numer. Math. 27* (1977), 449–469.

[2] Keller, H. B., and Nelson, Jr., P. Hypercube implementations of parallel shooting. *Appl. Math. and Comp. 31* (1989), 574–603.

[3] Lemmert, R. The shooting method for some nonlinear Sturm–Liouville boundary value problems. *Z. Angew. Math. Phys. 40*, 5 (1989), 769–773.

[4] Marzulli, P., and Gheri, G. Estimation of the global discretization error in shooting methods for linear boundary value problems. *J. Comput. Appl. Math. 28* (1989), 309–314.

[5] Roberts, S. M., and Shipman, J. S. On the closed form solution of Troesch's problem. *J. Comput. Physics 21* (1976), 291–304.

[6] Stoer, J., and Bulirsch, R. *Introduction to Numerical Analysis.* Springer–Verlag, New York, 1976. Translated by R. Bartels, W. Gautschi and C. Witzgall.

168. Continuation Method*

Applicable to Any type of equation: algebraic or differential, a single equation, or a system.

Yields

A numerical approximation to the solution.

Idea

We embed a given problem into a problem with a continuation parameter σ in it. For one value of σ (say $\sigma = 1$), we obtain the original equations; whereas for a different value of σ (say $\sigma = 0$) we have an "easier" problem. We solve the simpler problem numerically and then slowly vary the continuation parameter from 0 to 1, obtaining a solution at each intermediate value.

Procedure

After setting up the problem as described above, we define a metric that tells how well a function satisfies the problem when the continuation parameter is between 0 and 1. First, we numerically solve the easier problem (at $\sigma = 0$). Then, the continuation parameter σ is increased by a small amount, and a solution is found by using Newton's method (this is accomplished by making the metric as small as possible). We increase σ some more and repeat this step until we have arrived at $\sigma = 1$.

Example

Suppose we wish to solve the following boundary value problem for $y = y(x)$,

$$y_{xx} + e^y = 0, \qquad y(0) = 1, \quad y(\pi/2) = 0. \tag{168.1}$$

We embed equation (168.1) into the problem for $v = v(x;\sigma)$,

$$v_{xx} + (1-\sigma)v + \sigma e^v = 0, \qquad v(0;\sigma) = 1, \quad v(\pi/2;\sigma) = 0. \tag{168.2}$$

Note that when $\sigma = 1$, we have $v(x;1) = y(x)$ and that, when $\sigma = 0$, the problem for $v(x;0)$ becomes

$$v(x;0)_{xx} + v(x;0) = 0, \qquad v(0;0) = 1, \quad v(\pi/2;0) = 1,$$

with the solution $v(x;0) = \cos x$.

The technique is to solve (168.2) numerically on a grid of values from 0 to $\pi/2$. We start with $\sigma = 0$ and $v(x;0) = \cos x$ and then increase σ by a small amount and allow $v(x;\sigma)$ to change accordingly.

We choose to solve equation (168.1) at the $N+1$ grid points: $\{x_n = hn \mid n = 0, 1, 2, \ldots, N\}$, where $h = \pi/2N$, and we define v_n^σ to be the numerical approximation to $v(x;\sigma)$ at the nth gridpoint. We take $v_0^\sigma = 1$ and $v_N^\sigma = 0$ so that the boundary conditions in equation (168.2) are always satisfied.

Now, we must define the metric. We choose

$$\epsilon_n^\sigma = \frac{v_{n+1}^\sigma - 2v_n^\sigma + v_{n-1}^\sigma}{h^2} + (1-\sigma)v_n^\sigma + \sigma e^{v_n^\sigma}. \tag{168.3}$$

We choose this metric because, when ϵ_n^σ is close to zero, equation (168.2) will be approximately satisfied. This metric was obtained by simply applying a centered second order difference formula to equation (168.2).

The procedure is now as follows (with $\sigma_0 = 0$, $k = 0$):

1. Increase σ by a small amount $\delta\sigma$ (i.e., $\sigma_{k+1} = \sigma_k + \delta\sigma$).
2. Find the $\{v_n^\sigma\}$ that make $\epsilon_n^{\sigma_k} \simeq 0$. This is best accomplished by Newton's method. That is, we keep iterating

$$\begin{bmatrix} v_2^{\sigma_k} \\ v_3^{\sigma_k} \\ \vdots \\ v_{N-1}^{\sigma_k} \end{bmatrix}_{m+1} = \begin{bmatrix} v_2^{\sigma_k} \\ v_3^{\sigma_k} \\ \vdots \\ v_{N-1}^{\sigma_k} \end{bmatrix}_{m} - J^{-1} \begin{bmatrix} \epsilon_2^{\sigma_k} \\ \epsilon_3^{\sigma_k} \\ \vdots \\ \epsilon_{N-1}^{\sigma_k} \end{bmatrix}_{m},$$

   where J is the Jacobian matrix defined by $J = \frac{\partial(\epsilon_2^{\sigma_k}, \epsilon_3^{\sigma_k}, \dots, \epsilon_{N-1}^{\sigma_k})}{\partial(v_2^{\sigma_k}, v_3^{\sigma_k}, \dots, v_{N-1}^{\sigma_k})}$, until the "difference" between $\begin{bmatrix} v_2^{\sigma_k} \\ v_3^{\sigma_k} \\ \vdots \\ v_{N-1}^{\sigma_k} \end{bmatrix}_{m+1}$ and $\begin{bmatrix} v_2^{\sigma_k} \\ v_3^{\sigma_k} \\ \vdots \\ v_{N-1}^{\sigma_k} \end{bmatrix}_{m}$ is smaller than some predefined constant (based on the machine's numerical capabilities).

   (a) Note that the Jacobian and the $\{\epsilon_n^\sigma\}$ all depend on the values of $\{v_n^{\sigma_k}\}_m$.
   (b) At each stage, when σ is increased, the values of $\{v_n^{\sigma_k}\}_0$ will be given by the last values of $\left\{v_n^{\sigma_{k-1}}\right\}$.
   (c) If $\delta\sigma$ is small enough, then Newton's method should converge.
3. If $\sigma_k \neq 1$, go back to the first step.
4. If $\sigma_k = 1$, then we have found a numerical approximation to the solution of equation (168.1).

Notes

1. There are computer codes available that can perform all of the above steps. The only input needed for them is the definition of the $\{\epsilon_n^\sigma\}$. For example, Rheinboldt [3] has the Fortran listing for a continuation package.

2. Continuation methods can be used to track different solution branches of a problem with bifurcations. If the Jacobian ever becomes singular (i.e., $\det J = 0$), a bifurcation point is likely. The null space of the Jacobian will indicate which directions are possible for the different solution branches.
3. It is not uncommon in practice to find that the iteration in equation (168.3) will not converge unless $\delta\sigma$ is *very* small (at least initially). The better continuation programs available will automatically determine $\delta\sigma$, making it as small as is needed but also increasing it when possible to speed up the calculation.
4. The method of invariant embedding (see page 747) is a specific type of continuation method.
5. Continuation methods are also known as *homotopy methods*,

References

[1] Allgower, R. L., and Georg, E. L. *Numerical Continuation Methods*. Springer–Verlag, New York, 1990.

[2] Bolstad, J. H., and Keller, H. B. A multigrid continuation method for elliptic problems with folds. *SIAM J. Sci. Stat. Comput. 7*, 4 (October 1986), 1081–1104.

[3] Rheinboldt, W. C. *Numerical Analysis of Parametrized Nonlinear Equations*, vol. 7. John Wiley & Sons, New York, 1986.

[4] Rheinboldt, W. C., and Burkardt, J. V. A locally parametrized continuation process. *ACM Trans. Math. Software 9*, 2 (June 1983), 215–235.

[5] Watson, L. T. Numerical linear algebra aspects of globally convergent homotopy methods. *SIAM Review 28*, 4 (December 1986), 529–545.

[6] Watson, L. T., Billups, S. C., and Morgan, A. P. Algorithm 652: HOMPACK: A suite of codes for globally convergent homotopy algorithms. *ACM Trans. Math. Software 13*, 3 (Sept 1987), 281–310.

169. Continued Fractions

Applicable to Linear second order ordinary differential equations.

Yields

A solution in terms of a continued fraction.

Idea

By finding a simple recurrence pattern, we can express the logarithmic derivative of the solution to an ordinary differential equation in terms of a continued fraction.

Procedure

Suppose we have a linear second order ordinary differential equation in the form

$$y = Q_0(x)y' + P_1(x)y''. \tag{169.1}$$

If equation (169.1) is differentiated with respect to x, then we obtain

$$y' = Q_1(x)y'' + P_2(x)y''', \tag{169.2}$$

where

$$Q_1 = \frac{Q_0 + P_1'}{1 - Q_0'}, \qquad P_2 = \frac{P_1}{1 - Q_0'}. \tag{169.3}$$

If equation (169.2) is differentiated with respect to x, then we obtain $y'' = Q_2(x)y''' + P_3(x)y''''$, where $Q_2 = \frac{Q_1+P_2'}{1-Q_1'}$, $P_3 = \frac{P_2}{1-Q_1'}$. This process can be repeated indefinitely to obtain

$$y^{(n)} = Q_n(x)y^{(n+1)} + P_{n+1}(x)y^{(n+2)}, \tag{169.4}$$

with $Q_n = \frac{Q_{n-1}+P_n'}{1-Q_{n-1}'}$, $P_{n+1} = \frac{P_n}{1-Q_{n-1}'}$.

Now, dividing equation (169.1) by y' produces

$$\begin{aligned}
\frac{y}{y'} &= Q_0 + P_1\frac{y''}{y'} \\
&= Q_0 + \frac{P_1}{y'/y''} \\
&= Q_0 + \cfrac{P_1}{Q_1 + P_2\cfrac{y'''}{y''}} \\
&= Q_0 + \cfrac{P_1}{Q_1 + \cfrac{P_2}{Q_2 + P_3\cfrac{y''''}{y'''}}},
\end{aligned} \tag{169.5}$$

where we have used equation (169.3) for the third equality and equation (169.4) (with $n = 3$) for the fourth equality.

We can extend the continued fraction in equation (169.5) indefinitely. If it terminates, then it represents the reciprocal of the logarithmic derivative of the solution to equation (169.1). If it does not terminate, then it will converge if the following three conditions are satisfied:

1. $P_n \to P$, $Q_n \to Q$ as $n \to \infty$.
2. The roots $\{\rho_1, \rho_2\}$ of $\rho^2 = Q\rho + P$ are of unequal modulus.
3. If $|\rho_2| < |\rho_1|$, then $\lim_{n\to\infty} |y^{(n)}|^{1/n} < \begin{cases} |\rho_2|^{-1} & \text{if } |\rho_2| \neq 0, \\ \infty & \text{if } |\rho_2| = 0. \end{cases}$

Example

Suppose we wish to find a continued fraction expansion for the reciprocal of the logarithmic derivative of the equation

$$xy'' - xy' - y = 0. \tag{169.6}$$

Comparing equation (169.6) with equation (169.1), we identify $Q_0(x) = -x$, $P_1(x) = x$. Using these values in equation (169.4), it is easy to show that $Q_n = 1 - x/(n+1)$ and $P_n = x/n$. Using these values, the partial sums for the continued fraction can be evaluated as

$$\begin{aligned} &\text{For 1 term:} && -\frac{x^2+2}{x}. \\ &\text{For 2 terms:} && -\frac{x^3+5x}{x^2+3}. \\ &\text{For 3 terms:} && -\frac{x^4+9x^2+8}{x^3+7x}, \\ &\text{For 4 terms:} && -\frac{x^5+14x^3+33x}{x^4+12x^2+15}. \end{aligned} \tag{169.7}$$

The information in equation (169.7) can be used to approximately evaluate y/y'.

Notes

1. This technique has rarely been extended, with any generality, to any types of differential equations other than linear second order ordinary differential equations. There has been a generalization to "matrix continued fractions" in Risken [8, Chapter 9]. In Bellman and Wing [2, page 19], continued fractions are used to represent the solution to a Riccati equation.
2. By taking partial sums of the continued fraction in equation (169.5), successively better approximations may be found. Rarely, though, can convergence be checked. See Field's paper [5].

3. Continued fractions have been used recently to obtain high accuracy approximations to eigenvalues and functions of mathematical physics; see Barnett [1] or Gerck and d'Oliveira [6].

References

[1] Barnett, A. R. High-precision evaluation of the regular and irregular Colounb wavefunctions. *J. Comput. Appl. Math. 8*, 1 (1982), 29–33.

[2] Bellman, R., and Wing, G. M. *An Introduction to Invariant Imbedding.* SIAM, Philadelphia, PA, 1992.

[3] Ditto, W. L., and Pickett, T. J. Nonperturbative solutions of nonlinear differential equations using continued fractions. *J. Math. Physics 29*, 8 (1988), 1761–1770.

[4] Ditto, W. L., and Pickett, T. J. Exact solution of nonlinear differential equations using continued fractions. *Nuovo Cimento B 105*, 4 (1990), 429–435.

[5] Field, D. A. Estimates of the speed of convergence of continued fraction expansion of functions. *Math. of Comp. 13*, 138 (April 1977), 495–502.

[6] Gerck, E., and d'Oliveira, A. B. Continued fraction calculation of the eigenvalues of tridiagonal matrices arising from the Schroedinger equation. *J. Comput. Appl. Math. 6*, 1 (1980), 81–82.

[7] Lentz, W. J. Continued fraction calculation of spherical Bessel functions. *Computers in Physics* (Jul/Aug 1990), 403–407.

[8] Risken, H. *The Fokker–Planck Equation.* Springer–Verlag, New York, 1984.

170. Cosine Method*

Applicable to Second order linear autonomous equations of a special form.

Yields

A finite difference scheme from which a numerical approximation to the solution may be obtained.

Idea

An exact representation of the solution is found. This exact representation is discretized to obtain an approximate numerical scheme.

Procedure

Suppose the following second order linear autonomous equation

$$\begin{aligned} &\mathbf{u}'' + A\mathbf{u} = \mathbf{0}, \\ &\mathbf{u}(0) = \mathbf{u}_0, \qquad \mathbf{u}'(0) = \mathbf{v}_0 \end{aligned} \tag{170.1}$$

is given for $\mathbf{u}(t)$, where A is a positive definite symmetric matrix. The solution to equation (170.1) has the *exact* representation

$$\mathbf{u}(t+k) + \mathbf{u}(t-k) = 2\cos\left(kA^{1/2}\right)\mathbf{u}(t),$$

where k represents a time step. Note that the cosine of a matrix is another matrix (see Moler and Van Loan [4] for how the exponential of a matrix may be computed).

The approximation scheme for (170.1) is based on the use of a rational function, $R(\cdot) = P(\cdot)/Q(\cdot)$, to approximate the cosine term:

$$\cos\left(kA^{1/2}\right) \simeq R\left(kA^{1/2}\right) = Q^{-1}\left(kA^{1/2}\right)P\left(kA^{1/2}\right).$$

Once a rational function has been chosen (i.e., P and Q have been picked), we define the approximation to $\mathbf{u}(t_j)$ to be $\mathbf{w}_j$ (where $t_j = jk$). The recurrence relation for $\mathbf{w}_j$ is then given by

$$Q\left(kA^{1/2}\right)(\mathbf{w}_{j+1} + \mathbf{w}_{j-1}) = 2P\left(kA^{1/2}\right)\mathbf{w}_j$$

or

$$\mathbf{w}_{j+1} = 2Q^{-1}\left(kA^{1/2}\right)P\left(kA^{1/2}\right)\mathbf{w}_j - \mathbf{w}_{j-1}.$$

Using Taylor series (see page 632), the first two values of $\mathbf{w}$ can be found to start the iteration

$$\begin{aligned}
\mathbf{w}_0 &= \mathbf{u}(0) = \mathbf{u}_0, \\
\mathbf{w}_1 = \mathbf{u}(k) &= \mathbf{u}(0) + k\mathbf{u}'(0) + \frac{k^2}{2!}\mathbf{u}''(0) + \frac{k^3}{3!}\mathbf{u}'''(0) + \dots \\
&= \mathbf{u}(0) + k\mathbf{u}'(0) - \frac{k^2}{2!}A\mathbf{u}(0) - \frac{k^3}{3!}A\mathbf{u}'(0) + \dots \qquad (170.2) \\
&= \mathbf{u}_0 + k\mathbf{v}_0 - \frac{k^2}{2!}A\mathbf{u}_0 - \frac{k^3}{3!}A\mathbf{v}_0 + \frac{k^4}{4!}A^2\mathbf{u}_0 + \dots,
\end{aligned}$$

where the differential equation itself has been used to compute the higher order derivatives of $\mathbf{u}$. The number of terms kept in this series should correspond to the accuracy of the rational approximation used for the cosine function.

Example

Suppose we have

$$\begin{aligned}
\mathbf{u}'' + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{u} = \mathbf{0}, \\
\mathbf{u}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{u}'(0) = \begin{bmatrix} 2\sqrt{3} \\ 2\sqrt{3} \end{bmatrix}.
\end{aligned} \qquad (170.3)$$

Here $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ is symmetric and positive definite (its eigenvalues are $\{1, 3\}$). The exact solution of the system in equation (170.3) can be found by converting it into the following first order system

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{v}_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2\sqrt{3} \\ 2\sqrt{3} \end{bmatrix},$$

where I is the 2×2 identity matrix and $\mathbf{v} = \mathbf{u}'$. The solution of this new system (see page 421) is

$$\begin{bmatrix} \mathbf{u}(t) \\ \mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} \cos t + 2\sin(\sqrt{3}t) \\ -\cos t + 2\sin(\sqrt{3}t) \\ -\sin t + 2\sqrt{3}\cos(\sqrt{3}t) \\ \sin t + 2\sqrt{3}\cos(\sqrt{3}t) \end{bmatrix}.$$

To use the cosine method, we need to approximate the cosine function. The (2,2) Padé approximant (see page 582) to the cosine function is

$$\cos(z) \simeq \frac{12 - 5z^2}{12 + z^2},$$

so that

$$P\left(kA^{1/2}\right) = 12I - 5k^2A,$$
$$Q\left(kA^{1/2}\right) = 12I + k^2A.$$

From this we obtain our discretization scheme

$$\begin{aligned}\mathbf{w}_{j+1} &= -\mathbf{w}_{j-1} + 2(12I + k^2A)^{-1}(12I - 5k^2A)\mathbf{w}_j \\ &= -\mathbf{w}_{j-1} + \alpha \begin{bmatrix} -15k^4 - 96k^2 + 144 & -72k^2 \\ -72k^2 & -15k^4 - 96k^2 + 144 \end{bmatrix} \mathbf{w}_j,\end{aligned} \tag{170.4}$$

where $\alpha = 2/3(k^2+4)(k^2+12)$.

The Fortran program in program 170.1 implements the above scheme with $k = 0.25$. To evaluate $\mathbf{w}_1$, we utilized the first five terms in equation (170.2). We chose to compare the output from the numerical approximation scheme to the exact solution when t is a multiple of 5. Even for t as large as 30, the results are accurate to two decimal places.

```
At time   5.00   W(J) =   1.6667   1.0993
                 EXACT=   1.6680   1.1007
At time  10.00   W(J) =  -2.8364  -1.1584
                 EXACT=  -2.8373  -1.1592
At time  15.00   W(J) =   0.7422   2.2617
                 EXACT=   0.7403   2.2596
At time  20.00   W(J) =   0.2361  -0.5798
                 EXACT=   0.2413  -0.5749
At time  25.00   W(J) =  -0.2625  -2.2450
                 EXACT=  -0.2680  -2.2504
At time  30.00   W(J) =   2.1371   1.8281
                 EXACT=   2.1386   1.8301
```

Notes

1. This method has been extended to apply to non-homogeneous problems, equations with time-dependent coefficients, and second order hyperbolic equations.
2. Since the iterates in equation (170.4) do not depend linearly on the step size k, the cosine method is not a multi-step method as defined on page 670.

References

[1] Bales, L. A., and Douglas, V. A. Cosine methods for nonlinear second-order hyperbolic equations. *Math. of Comp. 52*, 186 (1989), 299–319.

[2] Bales, L. A., Douglas, V. A., and Serbin, S. M. Cosine methods for second-order hyperbolic equations with time-dependent coefficients. *Math. of Comp. 45* (July 1985), 65–89.

[3] Coleman, J. P. Numerical methods for *y"=f(x,y)* via rational approximations for the cosine. *IMA J. Num. Analysis 9* (1989), 145–165.

```
       IMPLICIT DOUBLE PRECISION (A-H,O-Z)
       REAL*8 W(0:1000,2),MAT(2,2),K
       K=0.25D0
       TIME=K
       SQRT3=DSQRT(3.D0)
C Set up the initial conditions
       W(0,1)= 1.D0
       W(0,2)=-1.D0
       W(1,1)= 1 + K*2*SQRT3 - K**2/2.D0 - K**3*SQRT3 + K**4/24.D0
       W(1,2)=-1 + K*2*SQRT3 + K**2/2.D0 - K**3*SQRT3 - K**4/24.D0
C Set up the matrix for the recursion
       ALPHA = 2.D0/( 3.D0*(K**2+4)*(K**2+12) )
       MAT(1,1)= ALPHA  * ( - 15*K**4 - 96*K**2 + 144)
       MAT(1,2)= ALPHA  * (           - 72*K**2      )
       MAT(2,1)= MAT(1,2)
       MAT(2,2)= MAT(1,1)
C Loop in time
       DO 10 J=2,120
       TIME=TIME+K
       W(J,1)= -W(J-2,1) + MAT(1,1)*W(J-1,1) + MAT(1,2)*W(J-1,2)
       W(J,2)= -W(J-2,2) + MAT(2,1)*W(J-1,1) + MAT(2,2)*W(J-1,2)
C Compute the exact solution also
       IF( MOD(J,20) .NE. 0 ) GOTO 10
       EXACT1=   DCOS(TIME) + 2*DSIN(SQRT3*TIME)
       EXACT2= - DCOS(TIME) + 2*DSIN(SQRT3*TIME)
       WRITE(6,5) TIME,W(J,1),W(J,2),EXACT1,EXACT2
5      FORMAT('At time',F7.2,3x,'W(J) =',2F9.4,/,18X,'EXACT=',2F9.4)
10     CONTINUE
       END
```

Program 170.1: Fortran program for cosine method

[4] Moler, C., and Van Loan, C. Nineteen dubious ways to compute the exponential of a matrix. *SIAM Review 20*, 4 (October 1978), 801–836.

[5] Serbin, S. M. Some cosine schemes for second-order systems of ODE's with time-varying coefficients. *SIAM J. Sci. Stat. Comput. 6*, 1 (1985), 61–68.

[6] Serbin, S. M., and Fisher, A. L. A post-processor for the cosine method. *SIAM J. Sci. Stat. Comput. 9*, 1 (January 1988), 14–23.

171. Differential Algebraic Equations

Applicable to Differential algebraic equations, which are differential equations of the form

$$\mathbf{F}(x, \mathbf{y}, \mathbf{y}') = 0. \tag{171.1}$$

Often, $\mathbf{F}(\cdot)$ is nonlinear in the $\mathbf{y}'$ term, or $\mathbf{F}(\cdot)$ contains a collection of differential and algebraic equations. A special subcase of differential algebraic equations is standard ordinary differential equations, in the common form $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$.

Yields

A numerical approximation to the solution.

Idea

Differential algebraic equations are more difficult to solve than standard ordinary differential equations. These equations are invariably solved exclusively by numerical means. One common numerical technique is to use the backward Euler method. That is, equation (171.1) is approximated by

$$\mathbf{F}\left(x_{n+1}, \mathbf{y}_{n+1}, \frac{\mathbf{y}_{n+1} - \mathbf{y}_n}{x_{n+1} - x_n}\right) = 0,$$

and then the resulting system of nonlinear equations is solved for $\mathbf{y}_1$, then $\mathbf{y}_2$, etc.

Many special purpose codes have been written for these systems; see the references. There are, however, a few analytic solution techniques for differential algebraic equations, as the examples show.

Example 1

Algebraic differential equations arise, for instance, in the analysis of mechanical systems. Each component in a mechanical system will have equations of motion, as well as physical constraints (depending on how the given component is attached to other components in the system). It is these physical constraints that become algebraic constraints.

For example, consider a pendulum consisting of a point mass m, under the influence of gravity g, suspended by a massless rod of length l from an attachment point taken to be $x = 0$, $y = 0$. The equations of motion are

$$\begin{aligned} x' &= v_x, \\ y' &= v_y, \\ m v_x' &= -x\lambda, \\ m v_y' &= -y\lambda - g, \\ x^2 + y^2 &= l^2. \end{aligned} \tag{171.2}$$

Here, $\lambda(t)$ is the rod tension, and $v_x(t)$ and $v_y(t)$ are the x and y velocities.

Example 2

The differential equation

$$y = f(y') = (y')^5 + (y')^3 + y' + 5, \tag{171.3}$$

for $y(x)$, is an example of a differential algebraic equation. It is impossible for equation (171.3) to be analytically written in the form $y' = g(x, y)$.

However, it is possible to solve differential equations of the form $y = f(y')$ parametrically. The solution may be written as

$$y = f(t), \qquad x = \int t^{-1} f'(t)\,dt + C,$$

where C is an arbitrary constant. Hence, equation (171.3) has the solution

$$\begin{aligned} x &= \frac{5}{4}t^4 + \frac{3}{2}t^2 + \log t + C, \\ y &= t^5 + t^3 + t + 5. \end{aligned}$$

Example 3

If a differential algebraic equation is of the form $x = f(y')$, then the solution may be written parametrically as

$$x = f(t), \qquad y = \int t f'(t)\,dt + C,$$

where C is an arbitrary constant. Thus, the equation $x = (y')^3 - y' - 1$ has the parametric solution

$$\begin{aligned} x &= t^3 - t - 1, \\ y &= \frac{3}{4}t^4 - \frac{1}{2}t^2 + C. \end{aligned}$$

Example 4

If a differential algebraic equation is of the form $f(y') = 0$ and there exists at least one real root of $f(k) = 0$, then $y = kx + C$ is a solution (where C is an arbitrary constant). Thus, the equation $(y')^5 - 6(y')^2 - 8 = 0$ has the solution $y = 2x + C$.

Notes

1. If y is a solution to an algebraic differential equation, then y is called *differentially algebraic*. If u and v are differentially algebraic functions, then so are $u+v$, uv, u/v, $u \circ v$, u^{-1}, du/dt, and $\int_0^t u(s)\,ds$. Hence, all of the elementary functions (e.g., the rational functions, e^x, $\tan^{-1}$, Bessel functions) are differentially algebraic. Note that the Gamma function ($\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$) is *not* a differentially algebraic function. The Shannon–Pour-El–Lipshitz–Rubel theorem roughly states that the outputs of general purpose analog computers are differentially algebraic functions. See Rubel [18].
2. Differential algebraic equations of the form

$$\begin{aligned} \mathbf{u}' &= \mathbf{f}(\mathbf{u}, \mathbf{v}, t), \\ 0 &= \mathbf{g}(\mathbf{u}, \mathbf{v}, t), \end{aligned}$$

are said to be in *semi-explicit form*.
3. A class of algebraic differential equations that are often studied are systems of the form

$$\begin{aligned} E\mathbf{y}' &= A\mathbf{y} + \mathbf{g}(t), \\ \mathbf{y}(0) &= \mathbf{y}_0, \end{aligned} \tag{171.4}$$

where A and E are given matrices. In the cases of interest, A or E (or both) are singular, but $A - \lambda E$ is not identically zero. For example, the system

$$\begin{aligned} y_2' &= y_1 + g(x), \\ 0 &= y_2 + h(x), \end{aligned}$$

is an algebraic differential equation in the form of equation (171.4).
4. Consider equation (171.4) when $sE - A$ is a regular matrix pencil (i.e., $\det(sE - A)$ is not identically zero). (If $sE - A$ is not a regular matrix pencil then equation (171.4) is not well posed.) In this case, non-singular matrices P and Q can be found (see Gantmacher [6]) so that, with $\mathbf{y} = Q\mathbf{z} = \begin{bmatrix}\mathbf{z}_1 & \mathbf{z}_2\end{bmatrix}^{\mathrm{T}}$ and $\mathbf{h}(t) = P\mathbf{g}(t) = \begin{bmatrix}\mathbf{h}_1 & \mathbf{h}_2\end{bmatrix}^{\mathrm{T}}$, equation (171.4) takes the form

$$\begin{aligned} \mathbf{z}_1' + C\mathbf{z}_1 &= \mathbf{h}_1(t), \\ N\mathbf{z}_2' + \mathbf{z}_2 &= \mathbf{h}_2(t), \end{aligned}$$

where N is a nilpotent matrix of degree n (i.e., $N^n = 0$ and $N^{n-1} \neq 0$). This is known as Kronecker canonical form. The degree n defines the *index of the problem* in equation (171.4). The index is equal to the size of the largest Jordan block for the eigenvalue zero (i.e., $\lambda = 0$) of the matrix $E - \lambda A$. If the index is zero, then E is non-singular and the system is easily solved numerically. Systems with an index greater than 1 are algebraically incomplete, which means that the existence and the uniqueness of the solutions are not guaranteed. For example, the equations in equation (171.2) are of index 3.As another example, the differential algebraic equations (see Roche [15])

$$\begin{aligned} y' &= f(y, z) \\ 0 &= g(y, z) \end{aligned}$$

are of index 1 if $(\partial g/\partial z)^{-1}$ exists and is bounded in the neighborhood of the exact solution.

5. In Gear and Petzold [8] is the following algorithm in which the index of the problem in equation (171.4) can be reduced to zero by successive differentiations:

   (a) If E is non-singular, go to step (f).

   (b) Find non-singular matrices P and Q such that $PEQ = \begin{bmatrix} E_{11} & 0 \end{bmatrix}^{\mathrm{T}}$, with E_{11} having full rank.

   (c) Make the variable substitution $\mathbf{y} = Q\mathbf{z}$ and multiply the equations from the left by P giving

   $$\begin{bmatrix} E_{11} \\ 0 \end{bmatrix} \mathbf{z}' = \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} \mathbf{z} + \begin{bmatrix} \mathbf{h}_1(t) \\ \mathbf{h}_2(t) \end{bmatrix}.$$

   (d) Differentiate the lower part of the system to arrive at the new problem

   $$\begin{bmatrix} E_{11} \\ F_{21} \end{bmatrix} \mathbf{z}' = \begin{bmatrix} F_{11} \\ 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} \mathbf{h}_1(t) \\ -\mathbf{h}_2'(t) \end{bmatrix}.$$

   (e) If the "E" matrix for the new problem is singular, consider the new problem as the original problem and go to step (b).

   (f) Done.

   Note that the index of the original problem is equal to the number of times the above loop must be executed.

6. To indicate how much different the solution to algebraic differential equations can be from standard ordinary differential equations, consider the following amazing theorem in Rubel [16]:

   > Given any continuous function ϕ on $(-\infty, \infty)$ and any positive continuous function $\epsilon(t)$ on $(-\infty, \infty)$, there exists a C^∞ solution of the algebraic differential equation
   >
   > $$\begin{aligned} &3y'^4 y'' y''''^2 - 4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' \\ &\quad + 24y'^2 y''^4 y'''' - 12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 = 0, \end{aligned}$$
   >
   > with $|y(t) - \phi(t)| < \epsilon(t)$ for all $t \in (-\infty, \infty)$.

   Hence, *any* continuous function is a "valid" numerical approximation to a solution of the above equation!

7. A Fortran program for approximating the solution to differential algebraic equations of index 1, 2, and 3 is described in Hairer *et al.* [9]. This program is freely available via electronic mail.
8. See also Rheinboldt [14, Chapter 10, pages 183–202].

References

[1] Ascher, U. M., and Spiter, R. J. Collocation software for boundary value differential-algebraic equations. *SIAM J. Sci. Comput. 15*, 4 (July 1994), 938–952.

[2] Brenan, K. E., Campbell, S. L., and Petzold, L. R. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. No. 14 in Classics in Applied Mathematics. SIAM, Philadelphia, PA, 1995.

[3] Brenan, K. E., and Petzold, L. R. The numerical solution of higher index differential/algebraic equations by implicit methods. *SIAM J. Numer. Anal. 26*, 4 (August 1989), 976–996.

[4] Buck, R. C. The solutions to a smooth PDE can be dense in *C[I]*. *J. Differential Equations 41* (1981), 239–244.

[5] Burrage, K., and Petzold, L. On order reduction for Runge–Kutta methods applied to differential/algebraic systems and to stiff systems of ODES. *SIAM J. Numer. Anal. 27*, 2 (April 1990), 447–456.

[6] Gantmacher, F. R. *The Theory of Matrices*, vol. I and II. Chelsea Publishing Company, New York, 1959.

[7] Gear, C. W. Differential algebraic equations, indices, and integral algebraic equations. *SIAM J. Numer. Anal. 27*, 6 (December 1990), 1527–1534.

[8] Gear, C. W., and Petzold, L. R. ODE methods for the solution of differential/algebraic systems. *SIAM J. Numer. Anal. 21*, 4 (August 1984), 716–728.

[9] Hairer, E., Lubich, C., and Roche, M. *The Numerical Solution of Differential-Algebraic Systems by Runge–Kutta Methods*. Springer–Verlag, New York, 1989.

[10] Hairer, E., and Wanner, G. *Solving Ordinary Differential Equations, Volume II: Stiff and Differential-Algebraic Problems*. Springer–Verlag, New York, 1991.

[11] Hanke, M. On a least-squares collocation method for linear differential-algebraic equations. *Numer. Math. 54* (1988), 79–90.

[12] Leimkuhler, B., Petzold, L. R., and Gear, C. W. Approximation methods for the consistent initialization of differential algebraic equations. *SIAM J. Numer. Anal. 28*, 1 (February 1991), 205–226.

[13] Petzold, L., and Lotstedt, P. Numerical solution of nonlinear differential equations with algebraic constraints. II. Practical implications. *SIAM J. Sci. Stat. Comput. 7*, 3 (1986), 720–733.

[14] Rheinboldt, W. C. *Numerical Analysis of Parameterized Nonlinear Equations*. Wiley Interscience, New York, 1986.

[15] Roche, M. Rosenbrock methods for differential algebraic equations. *Numer. Math. 52* (1988), 45–6.

[16] Rubel, L. A. A universal differential equation. *Bull. Amer. Math. Soc. 4* (May 1981), 345–349.

[17] Rubel, L. A. Solutions of algebraic differential equations. *J. Differential Equations 49* (1983), 441–452.

[18] Rubel, L. A. Some mathematical limitations of the general-purpose analog computer. *Advances in Appl. Math. 9* (1988), 22–34.

172. Eigenvalue/Eigenfunction Problems

Applicable to Sturm–Liouville problems.

Yields

A numerical method for determining the eigenvalues and eigenfunctions of a regular Sturm–Liouville problem.

Idea

The Sturm–Liouville operator can be well approximated numerically by a simple discretization. This leads to a set of simultaneous equations, which can be represented as a matrix eigenvalue problem. The eigenvalues and eigenvectors of this matrix will approximate the eigenvalues and eigenfunctions of the Sturm–Liouville problem.

Procedure

Suppose we wish to approximate numerically the eigenvalues and eigenfunctions of the Sturm–Liouville system (see page 103)

$$\begin{aligned} &(p(x)y')' + q(x)y = \lambda y, \\ &y(0) = 0, \quad y(1) = 0, \end{aligned} \tag{172.1}$$

for $x \in [0, 1]$. We will illustrate how the method of finite differences can be used to approximate the eigenvalues and eigenvectors. Equation (172.1) can be approximated by

$$\begin{aligned} &D_-\big(p_{n+1/2} D_+ u_n\big) + q_n u_n = \lambda_h u_n, \\ &u_0 = 0, \quad u_N = 0, \end{aligned} \tag{172.2}$$

where $h = 1/N$, $u_n \simeq y(nh)$, $n = 1, 2, \ldots, N-1$, and a function with a subscript of n corresponds to an evaluation at $x = hn$. Also, the forward and backward differencing operators are defined by $D_- f_n := (f_n - f_{n-1})/h$ and $D_+ f_n := (f_{n+1} - f_n)/h$. It can be shown that (see Isaacson and Keller [9, pages 434–436] or Keller [10, Chapter 3, pages 39–48])

$$|\lambda - \lambda_h| \le Ch^2,$$

where C is some (unknown) constant. Therefore, for a sufficiently small h, the collection of $\{\lambda_h\}$ that satisfy equation (172.2) will closely approximate the collection of eigenvalues $\{\lambda\}$. The system in equation (172.2) is equivalent to the linear system of equations

$$A\mathbf{u}_h = h^2 \lambda_h \mathbf{u}_h, \tag{172.3}$$

where $\mathbf{u}_h = \begin{bmatrix} u_1 & \dots & u_{N-1} \end{bmatrix}^{\mathrm{T}}$ and A is the symmetric matrix

$$\begin{bmatrix} f_1 & p_{3/2} & 0 & 0 & \cdots & 0 & 0 \\ p_{3/2} & f_2 & p_{5/2} & 0 & \cdots & 0 & 0 \\ 0 & p_{5/2} & f_3 & p_{7/2} & & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & & p_{N-5/2} & f_{N-2} & p_{N-3/2} & 0 \\ 0 & 0 & \cdots & 0 & p_{N-3/2} & f_{N-1} & p_{N-1/2} \\ 0 & 0 & \cdots & 0 & 0 & p_{N-1/2} & f_N \end{bmatrix}, \tag{172.4}$$

where $f_m := h^2 q_m - (p_{m-1/2} + p_{m+1/2})$. Hence, the eigenvalues of (172.4), scaled by h^2 (see equation (172.3)), will approximate the eigenvalues of (172.1). Note that $\mathbf{u}_h$, the eigenvector of (172.4) corresponding to λ_h, is an approximation to the eigenfunction in (172.1). The eigenvalues and eigenvectors of equation (172.4) can be computed by standard numerical techniques. As N increases, more eigenvalues and eigenvectors are found and the accuracy of the lower order eigenvalues (and their associated eigenfunctions) increases.

Example

Consider the simple Sturm–Liouville system

$$\begin{aligned} y'' + y &= \lambda y, \\ y(0) = 0, &\quad y(1) = 0. \end{aligned} \tag{172.5}$$

For this system, the eigenfunctions and eigenvalues are given by

$$\begin{aligned} y_n(x) &= \sin n\pi x, \\ \lambda_n &= 1 - n^2\pi^2, \end{aligned} \tag{172.6}$$

for $n = 1, 2, \ldots$. Hence, the two eigenvalues with the least magnitude are $\lambda_1 = 1 - \pi^2 \simeq -8.86$ and $\lambda_2 = 1 - 4\pi^2 \simeq -38.47$. To utilize the numerical technique presented above, we compare equation (172.5) with equation (172.1) to determine that $p(x) = 1$ and $q(x) = 1$.

If $N = 3$ (so that $h = 1/3$), then the matrix in equation (172.4) is given by

$$\begin{bmatrix} -17/9 & 1 & 0 \\ 1 & -17/9 & 1 \\ 0 & 1 & -17/9 \end{bmatrix}. \tag{172.7}$$

The eigenvalues of the matrix in equation (172.7) are approximately `-1.9` and `-0.49`. When scaled by h^2, the estimates of the smallest eigenvalues of equation (172.5) become $\lambda_1 \simeq$`-4.3` and $\lambda_2 \simeq$`-17.0`.

For $N = 10$, the estimates are $\lambda_1 \simeq$`-7.1` and $\lambda_2 \simeq$`-30.7`, whereas for $N = 50$ the estimates are $\lambda_1 \simeq$`-8.5` and $\lambda_2 \simeq$`-36.9`. As N increases, the estimates become better. If a higher order scheme were used to discretize (172.2), then smaller values of N would be required to obtain a given accuracy.

Notes

1. Of course, Sturm–Liouville systems other than the one in equation (172.1) can be represented by a simple discretization such as in equation (172.2). More complicated boundary conditions may lead to a non-symmetric matrix in equation (172.3).
2. Many other techniques have been used to approximate the eigenvalues and eigenfunctions of differential systems. These methods include finite elements, Galerkin methods, invariant embedding, Prüfer substitution, shooting, and variational methods. See page 635 of this book, Cope [7], or Keller [10].
3. Many methods (such as the one illustrated here) approximate the kth eigenvalue of a regular Sturm–Liouville problem by the kth eigenvalue of a matrix problem of dimension n. Unfortunately, the accuracy deteriorates as k increases, and there is no approximation at all for $k > n$. However, it is possible to obtain approximations for all k that are uniformly accurate in k, see Shampine [17].
4. For the eigenvalues $\{\lambda_k\}$ of the Sturm–Liouville problem $-u'' + qu = \lambda u$, $u(0) = u(\pi) = 0$, when q has mean zero, Marti [13] gives the bounds $\left|\lambda_k - k^2\right| \leq P_{1,m}k^{-m} + P_{2,m}k^{-2m}$ for $k^2 \geq 3||q||_m$, where $||q||_m$ is the norm of q in a Sobolev space and the P's are homogeneous polynomials of degree at most 3 in $||q||_m$.

References

[1] Andrew, A. L. Correction of finite element eigenvalues for problems with natural or periodic boundary conditions. *BIT 28*, 2 (1988), 254–269.

[2] Babuska, I., and Osborn, J. E. Estimates of the errors in eigenvalue and eigenvector approximation by Galerkin methods with particular attention to the case of multiple eigenvalues. *SIAM J. Numer. Anal. 24*, 6 (December 1987), 1249–1276.

[3] Bailey, P. B., Garbow, B. S., Kaper, H. G., and Zettl, A. Algorithm 700: A FORTRAN software package for Sturm–Liouville problems. *ACM Trans. Math. Software 17*, 4 (December 1991), 500–501.

[4] Bailey, P. B., Garbow, B. S., Kaper, H. G., and Zettl, A. Eigenvalue and eigenfunction computations for Sturm–Liouville problems. *ACM Trans. Math. Software 17*, 4 (December 1991), 491–499.

[5] Bellman, R., and Wing, G. M. *An Introduction to Invariant Imbedding*. SIAM, Philadelphia, PA, 1992.

[6] Berghe, G. V., and Meyer, H. D. Accurate computation of higher Sturm–Liouville eigenvalues. *Numer. Math. 59* (1991), 243–254.

[7] Cope, D. A uniformly convergent series for Sturm–Liouville eigenvalues. *Quart. Appl. Math. 42*, 3 (October 1984), 373–380.

[8] Gartland, Jr., E. C. Accurate approximation of eigenvalues and zeros of selected eigenfunctions of regular Sturm–Liouville problems. *Math. of Comp. 42*, 166 (April 1984), 427–439.

[9] Isaacson, E., and Keller, H. B. *Analysis of Numerical Methods*. John Wiley & Sons, New York, 1966.

[10] Keller, H. B. *Numerical Solutions of Two Point Boundary Value Problems*. SIAM, Philadelphia, PA, 1976.

[11] Leroy, J. P., and Wallace, R. Extension of the renormalized Numerov method for second-order differential eigenvalue equations. *J. Comput. Physics 67* (1986), 239–252.

[12] Marletta, M. Certification of algorithm 700. Numerical tests of the SLEIGN software for Sturm–Liouville problems. *ACM Trans. Math. Software 17*, 4 (December 1991), 481–490.

[13] Marti, J. T. New upper and lower bounds for the eigenvalues of the Sturm–Liouville problem. *Computing 42* (1989), 239–243.

[14] Mikhailov, M. D., and Vulchanov, N. L. Computational procedures for Sturm–Liouville problems. *J. Comput. Physics 50* (1983), 323–336.

[15] Pruess, S. On shooting algorithms for calculating Sturm–Liouville eigenvalues. *J. Comput. Physics 75* (1988), 493–497.

[16] Pryce, J. D. *Numerical Solution of Sturm–Liouville Problems*. Oxford University Press, New York, 1993.

[17] Shampine, L. F. Uniformly accurate Sturm–Liouville eigenvalues. *Computing 47*, 3–4 (1992), 379–385.

173. Euler's Forward Method

Applicable to Initial value systems of first order ordinary differential equations.

Yields

A numerical marching scheme that is first order accurate.

Idea

A forward difference approximation to a derivative can be easily manipulated into a numerical scheme. The technique in this section is the most elementary finite difference approximation—other techniques are found on page 670.

Procedure

Given the first order system

$$\begin{aligned} \frac{d}{dt}\mathbf{y}(t) &= \mathbf{f}\,[t, \mathbf{y}(t)], \\ \mathbf{y}(t_0) &= \mathbf{y}_0, \end{aligned} \tag{173.1.a-b}$$

where $\mathbf{y}$ and $\mathbf{f}$ are vectors, we numerically approximate $d\mathbf{y}/dt$ by $[\mathbf{y}(t+\Delta t) - \mathbf{y}(t)]/\Delta t$, where Δt is a small *step size*. This numerical approximation is first order accurate. Using this approximation, equation (173.1.a) can be rewritten as

$$\mathbf{y}(t + \Delta t) \simeq \mathbf{y}(t) + \Delta t\ \mathbf{f}\,[t, \mathbf{y}(t)]. \tag{173.2}$$

Hence, to integrate equation (173.1), we iterate equation (173.2) and use the initial conditions from equation (173.1.b) for

$$\begin{aligned} \mathbf{y}(t_0) &= \mathbf{y}_0, \\ \mathbf{y}(t_0 + \Delta t) &\simeq \mathbf{y}(t_0) + \Delta t\ \mathbf{f}\,[t_0, \mathbf{y}_0(t_0)], \\ \mathbf{y}(t_0 + 2\Delta t) &\simeq \mathbf{y}(t_0 + \Delta t) + \Delta t\ \mathbf{f}\,[t_0 + \Delta t, \mathbf{y}_0(t_0 + \Delta t)], \\ \mathbf{y}(t_0 + 3\Delta t) &\simeq \mathbf{y}(t_0 + 2\Delta t) + \Delta t\ \mathbf{f}\,[t_0 + 2\Delta t, \mathbf{y}_0(t_0 + 2\Delta t)], \\ &\vdots \end{aligned}$$

Example

Suppose we want to approximate the value of $y(1)$ when $y(t)$ is defined by

$$\frac{dy}{dt} = \frac{ty}{t^2 + 1}, \qquad y(0) = 1. \tag{173.3}$$

Since this equation is separable, the exact solution is known to be $y(t) = \sqrt{1 + t^2}$. We can use this exact solution to compare the accuracy of the

```
void main(void) { RungeKutta(); }
void EulerForwardMethod(void) {
     int j, ndiv = 10;
     double t     = 0.0;
     double tinit = 0.0;
     double y     = 1.0;
     double tend  = 1.0;
     double deltat, exact;
     deltat = (tend - tinit) / ((double)ndiv);
     /* This is the integration loop */
     for (j = 1; j <= ndiv; j++)
     {
          t = t + deltat;
          y = y + (deltat * Yprime(t, y));
          exact = sqrt(1 + (t*t));
          printf("T= %6.3f  Y= %8.5f  Exact solution= %8.5f \n", t, y, exact);
     }
}
/* This function specifies the differential equation */
double Yprime( double t, double y) { return ( (t*y) / (t*t + 1)); }
```

Program 173.1: C program for Euler method.

```
       NDIV= 10
       TINIT= 0.D0
       TEND= 1.D0
       DELTAT=(TEND-TINIT)/DFLOAT(NDIV)
       T= 0.0
       Y= 1.0
C This is the integration loop
       DO 10 J=1,NDIV
       T= T + DELTAT
       Y= Y + DELTAT * YPRIME(T,Y)
       EXACT=DSQRT(1+T**2)
       WRITE(6,5) T,Y,EXACT
5      FORMAT(' T=', F6.3,'  Y=', F8.5,'  Exact solution=',F8.5)
10     CONTINUE
       END
C This function specifies the differential equation
       FUNCTION YPRIME(T,Y)
       YPRIME= T*Y / (T**2+1)
       RETURN
       END
```

Program 173.2: Fortran program for Euler method.

numerical approximation. The C (Fortran) code in program 173.1 (173.2) uses Euler's forward method to numerically approximate the solution of equation (173.3). The codes use a step size of $\Delta t = 0.1$. The output from the programs is listed below, with the exact solution alongside for comparison. The error in the calculated value for $y(1)$ is about 1.7%.

```
T= 0.100  Y= 1.00990  Exact solution= 1.00499
T= 0.200  Y= 1.02932  Exact solution= 1.01980
T= 0.300  Y= 1.05765  Exact solution= 1.04403
```

```
T= 0.400  Y= 1.09412  Exact solution= 1.07703
T= 0.500  Y= 1.13789  Exact solution= 1.11803
T= 0.600  Y= 1.18809  Exact solution= 1.16619
T= 0.700  Y= 1.24390  Exact solution= 1.22066
T= 0.800  Y= 1.30458  Exact solution= 1.28062
T= 0.900  Y= 1.36945  Exact solution= 1.34536
T= 1.000  Y= 1.43792  Exact solution= 1.41421
```

If the number of steps were increased (so the step size decreased), then the accuracy would improve. For example, if (in the above example) Δt was reduced to 0.01 (i.e., `NDIV=100`), then the calculated value of $y(1)$ would be `1.41672`. Hence, the error in the calculated value for $y(1)$ would decrease to about 0.17%.

Notes

1. This technique is the easiest to use and program of all the numerical methods presented in this book. A major drawback is that the step size Δt may have to be very small for accurate numerical values.
2. There is also a method known as *Euler's backward method*. For this implicit method, the difference scheme is given by

$$\mathbf{y}(t+\Delta t) \simeq \mathbf{y}(t) + \Delta t\ \mathbf{f}\left[t, \mathbf{y}(t+\Delta t)\right]. \tag{173.4}$$

   In general, equation (173.4) will be nonlinear in $\mathbf{y}(t+\Delta t)$. Hence, an iterative scheme (e.g., Newton's method) must be employed to find $\mathbf{y}(t+\Delta t)$ at each step.
3. The stability properties of Euler's forward and backward methods are completely different. Consider applying each method to the scalar differential equation $y' = -cy$, $y(0) = y_0$, where c is a positive constant. For Euler's forward method, we have

$$\begin{aligned} y(t+\Delta t) &\simeq y(t) + \Delta t\, y'(t), \\ &= y(t) - \Delta t\, (cy(t)), \\ &= (1 - c\Delta t) y(t), \\ &= y_0 (1 - c\Delta t)^{t/\Delta t}. \end{aligned} \tag{173.5}$$

   Whereas for Euler's backward method, we find

$$\begin{aligned} y(t+\Delta t) &\simeq y(t) + \Delta t\, y'(t+\Delta t), \\ &= y(t) - \Delta t\, (cy(t+\Delta t)), \\ &= \frac{y(t)}{1 + c\Delta t}, \\ &= \frac{y_0}{(1 + c\Delta t)^{t/\Delta t}}. \end{aligned} \tag{173.6}$$

   Note that the approximation in equation (173.5) diverges in an oscillatory fashion when $\Delta t > 2/c$, whereas the approximation in equation

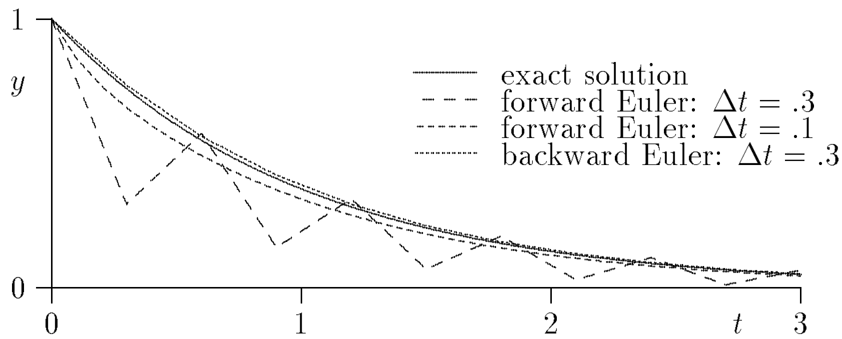

Figure 173.1: Different numerical techniques applied to $y' = -6y + 5e^{-t}$.

(173.6) is stable for any value of Δt. In particular, if $c \gg 1$ (so that the problem is stiff; see page 770), then Δt may have to be very small for Euler's forward method to be stable, whereas a larger value of Δt can be used with Euler's backward method.

4. As an indication of the different convergence properties of Euler's forward and backward methods, consider the equation: $\dot{y} = -6y + 5e^{-t}$. Figure 173.1 shows the exact solution ($y = e^{-t}$) and approximations obtained by using Euler's forward method (for $\Delta t = 0.3$ and $\Delta t = 0.1$) and Euler's backward method (for $\Delta t = 0.3$). On this problem, Euler's backward method is better than Euler's forward method for a fixed step size.
5. As always, ordinary differential equations of higher order can be written as a system of first order equations (see page 146).
6. See also Boyce and DiPrima [1, pages 399–406], Gear [2, pages 10–23], and Press *et al.* [3, pages 574–577].

References

[1] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[2] Gear, C. W. *Numerical Initial Value Problems in Ordinary Differential Equations*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1971.

[3] Press, W. H., Flannery, B. P., Teukolsky, S., and Vetterling, W. T. *Numerical Recipes*. Cambridge University Press, New York, 1986.

174. Finite Element Method*

Applicable to Differential equations that arise from variational principles. Principally ordinary differential equations and elliptic partial differential equations.

Yields

A numerical scheme for approximating the solution.

Procedure

The finite element method is one version of the method of weighted residuals (see page 786). The present method is characterized by having "local elements." The finite element method has a specialized vocabulary; several of the terms below are defined in the example.

Given a differential equation that comes from a variational principle and a domain in which the equation is to be solved the steps are as follows:

- Discretize the domain into simple shapes (these are the "finite elements"). Define a basis function $\phi_k(\mathbf{x})$ on each of the finite elements. These basis functions should have bounded support.
- Assemble the stiffness matrix and the load matrix. These depend only on the finite elements chosen and not on the differential equation to be approximated.
- Write the given differential equation as a variational principle. Approximate the unknown in the variational principle by a linear combination of the functions defined on the finite elements; that is, $u(\mathbf{x}) \simeq u_N(\mathbf{x}) := \sum_{k=1}^{N} c_k \phi_k(\mathbf{x})$. In this last expression, the $\{c_k\}$ are unknown and must be determined.
- Construct element stiffness matrices and load vectors element by element. Then assemble these together into the global stiffness matrix A and the global load vector $\mathbf{f}$.
- Relate the minimization in the variational principle to the minimization of the quadratic functional

$$I[u_N] = \mathbf{c}^{\mathrm{T}} A \mathbf{c} - 2\mathbf{c}^{\mathrm{T}} \mathbf{f}. \tag{174.1}$$

  When A is symmetric (as it frequently is), the minimization of equation (174.1) will occur when $\mathbf{c}$ is the solution of $A\mathbf{c} = \mathbf{f}$. In general, A will not be banded or tridiagonal, but it will be sparse. If the original differential equation was nonlinear, then $A = A(\mathbf{c})$ or $\mathbf{f} = \mathbf{f}(\mathbf{c})$.

There is a large literature on the finite element method. We choose to illustrate the basic ideas on simple examples: The first two examples are constant coefficient second order linear ordinary differential equations, the third example is for Laplace's equation. These examples show the major steps involved without the details that a sophisticated implementation requires.

Example 1

Suppose we have the constant coefficient second order linear ordinary differential equation

$$L[u] := -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x) \tag{174.2}$$

on the interval $0 \le x \le 1$. For simplicity, we take $p(x)$ and $q(x)$ to be constants. For this equation, we take the natural boundary conditions

$$u(0) = u(1) = 0. \tag{174.3}$$

If we use $I[v]$ to represent the "energy" of the system, then we may form

$$I[v] = \int_0^1 \left[p(v'(x))^2 + qv^2(x) - 2f(x)v(x)\right]\,dx. \tag{174.4}$$

It is straightforward to show that the first variation of $I[v]$ (see page 418) yields equations (174.2) and (174.3). Hence, $I[v]$ will be minimized when $v = u$.

Now we set up a uniform grid of $N+2$ points on the interval $0 \le x \le 1$ (i.e., $x_n = nh$ with $h = 1/(N+1)$ for $n = 0, 1, \dots, N+1$). We define the interval (x_k, x_{k+1}) to be "finite element number k." We choose as basis functions on the finite elements the linear functions $\phi_k(x)$ defined by

$$\phi_k(x) = \begin{cases} \frac{x-x_{k-1}}{h}, & \text{for } x_{k-1} \le x \le x_k, \\ \frac{x_{k+1}-x}{h}, & \text{for } x_k \le x \le x_{k+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{174.5}$$

These are the "hat functions" shown in figure 174.1. Note that

$$\phi_k'(x) = \begin{cases} \frac{1}{h}, & \text{for } x_{k-1} \le x \le x_k, \\ -\frac{1}{h}, & \text{for } x_k \le x \le x_{k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Now we approximate the function that minimizes equation (174.4), $u(x)$, by a linear combination of the $\phi_k(x)$. We take

$$u(x) \simeq u_N(x) := \sum_{k=1}^{N} c_k\phi_k(x), \tag{174.6}$$

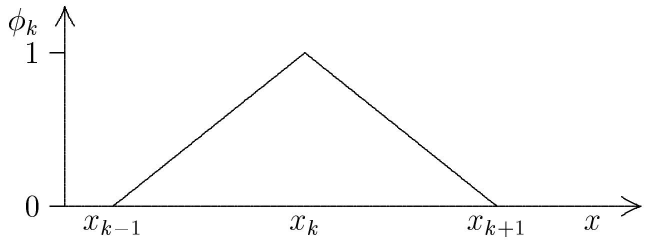

Figure 174.1: The "hat functions" in equation (174.5).

where the unknowns $\{c_k\}$ must be determined. Once the $\{c_k\}$ are known, then the approximation to $u(x)$ at any point can be found from equation (174.6). That is, on finite element k (i.e., for $x_k < x < x_{k+1}$)

$$\begin{aligned} u_N(x) &= c_k\phi_k(x) + c_{k+1}\phi_{k+1}(x), \\ u_N'(x) &= \frac{-c_k + c_{k+1}}{h}. \end{aligned} \tag{174.7}$$

Using $u_N(x)$ for $v(x)$ in equation (174.4) results in

$$\begin{aligned} I[u_N] &= \sum_{k=0}^{N} \int_{x_k}^{x_{k+1}} \left[p(u_N')^2 + q(u_N)^2 - 2fu_N\right] dx, \\ &= \sum_{k=0}^{N} \left[I_k^{\mathrm{s}} + I_k^{\mathrm{m}} + I_k^{\mathrm{l}}\right], \end{aligned} \tag{174.8}$$

where

$$\begin{aligned} I_k^{\mathrm{s}} &:= \int_{x_k}^{x_{k+1}} p(u_N')^2\, dx = \begin{bmatrix} c_k & c_{k+1} \end{bmatrix} K_k^{\mathrm{s}} \begin{bmatrix} c_k \\ c_{k+1} \end{bmatrix}, \\ I_k^{\mathrm{m}} &:= \int_{x_k}^{x_{k+1}} q(u_N)^2\, dx = \begin{bmatrix} c_k & c_{k+1} \end{bmatrix} K_k^{\mathrm{m}} \begin{bmatrix} c_k \\ c_{k+1} \end{bmatrix}, \\ I_k^{\mathrm{l}} &:= \int_{x_k}^{x_{k+1}} 2f(x)u_N(x)\, dx \end{aligned}$$

by virtue of equation (174.7). Here K_k^{s} is the *element stiffness matrix*, and K_k^{m} is the *element mass matrix*; they are defined by

$$K_k^{\mathrm{s}} = \frac{p}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad K_k^{\mathrm{m}} = \frac{qh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

If p and q were not taken to be constants, then these element matrices would not be so simple. A numerical integration would have been required to find the entries in these matrices.

A numerical integration is required to determine I_k^1. If, on finite element number k, $f(x)$ is approximated by $f(x) \simeq f_k\phi_k(x) + f_{k+1}\phi_{k+1}(x)$, then we find $I_k^1 = \left(\mathbf{f}_k^1\right)^{\mathrm{T}} \begin{bmatrix} c_k \\ c_{k+1} \end{bmatrix}$, where the *element load vector* is defined by $\mathbf{f}_k^1 = \dfrac{h}{3}\begin{bmatrix} 2f_k + \ \ f_{k+1} \\ f_k + 2f_{k+1} \end{bmatrix}$.

The system can now be *assembled* element by element. That is, we write a single matrix equation representing equation (174.8). For this example, we find that

$$I[u_N] = \mathbf{c}^{\mathrm{T}}(K+M)\mathbf{c} - 2\mathbf{f}^{\mathrm{T}}\mathbf{c}, \tag{174.9}$$

where $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_N \end{bmatrix}^{\mathrm{T}}$, $\mathbf{f} = \frac{h}{6}[(f_0+4f_1+f_2) \quad (f_1+4f_2+f_3) \quad \dots \quad (f_{N-2}+4f_{N-1}+f_N)]^{\mathrm{T}}$, and the *global stiffness matrix* K and the *global mass matrix* M are defined by

$$K = \frac{p}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & & -1 & 2 & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix},$$

$$M = \frac{qh}{6}\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & & 1 & 4 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 4 \end{bmatrix}.$$

To minimize the expression in equation (174.9), $\mathbf{c}$ should be chosen (because $K+M$ is a symmetric matrix in this example) to satisfy the matrix equation $(K+M)\mathbf{c} = \mathbf{f}$. This is a tridiagonal system of equations and may be solved by standard numerical linear algebra routines.

Example 2

This example shows more of the details for a specific application of the finite element method. Suppose that we wish to approximate the solution of the ordinary differential equation

$$\begin{aligned} &u'' - u' = e^x\left(e^{-x}u'\right)' = 0, \\ &u(0) = 2, \qquad u(4) = 1 + e^4, \end{aligned} \tag{174.10}$$

whose exact solution is $u(x) = 1 + e^x$. From page 418, we see that the variational principle associated with equation (174.10) is just $\delta J = 0$, where

$$J[v] = \int_0^4 e^{-x} \left(v'\right)^2 \, dx.$$

To use the finite element method on the problem in equation (174.10), we choose to use three elements: the intervals $[0,1]$, $[1,2]$, and $[2,4]$. We choose the polynomial basis functions

$$\begin{aligned}
&\text{on element } [0,1], \text{ the basis function is } f(x) = \alpha + \beta x + \gamma x^2, \\
&\text{on element } [1,2], \text{ the basis function is } g(x) = \delta + \epsilon x + \zeta x^2, \\
&\text{on element } [2,4], \text{ the basis function is } h(x) = \eta + \theta x + \iota x^2,
\end{aligned} \tag{174.11}$$

so that our approximation has the form

$$v = \begin{cases} f(x) & \text{on the interval } [0,1] \\ g(x) & \text{on the interval } [1,2] \\ h(x) & \text{on the interval } [2,4] \end{cases}$$

After $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota\}$ are determined, we will have found an approximate solution. The equations needed to satisfy the boundary conditions and for our approximation and its first derivative to be continuous on the interval $[0,1]$ are

$$\begin{aligned}
&\text{boundary conditions:} && f(0) = 2, && h(4) = 1 + e^4, \\
&\text{continuity conditions:} && f(1) = g(1), && f'(1) = g'(1), \\
& && g(2) = h(2), && g'(2) = h'(2).
\end{aligned} \tag{174.12}$$

Subject to the above constraints, we want to minimize $J[v]$. Using our chosen set of finite elements and basis functions, we have

$$\begin{aligned}
J[v] &= \int_0^1 e^{-x} \left(f'\right)^2 \, dx + \int_1^2 e^{-x} \left(g'\right)^2 \, dx + \int_2^4 e^{-x} \left(h'\right)^2 \, dx \\
&= (4e - 8)\gamma^2 + 4\beta\gamma + (8e^2 - 4e)\zeta^2 + 4e^2\epsilon\zeta + (e^2 - e)\theta^2 \\
&\quad + 4e^2\iota\theta + (8e^2 - 4e)\iota^2 + (e^2 - e)\epsilon^2 + (e - 1)\beta^2,
\end{aligned}$$

To minimize this last expression, subject to the constraints in equation (174.12), we use Lagrange multipliers. The expression obtained after Lagrange multipliers are introduced is differentiated with respect to each of the variables to obtain a linear system of 15 equations (9 equations for the variables in equation (174.11) and 6 equations for the Lagrange multipliers).

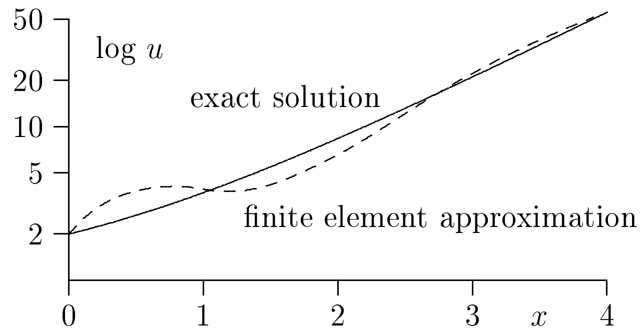

Figure 174.2: Exact solution and finite element approximation to (174.10).

This system can be solved to determine the basis function on each element:

$$\begin{aligned} f(x) &= -3.4508x^2 + 5.3673x + 2, \\ g(x) &= 4.1836x^2 - 9.9014x + 9.6343, \\ h(x) &= 8.8416x^2 - 28.5337x + 28.2666. \end{aligned}$$

Figure 174.2 has a comparison of the exact and approximate solutions. At points midway on the elements, we find

$$\begin{aligned} &u(0.5) = 2.65, \quad u(1.5) = 5.48, \quad u(3) = 21.09, \\ &f(0.5) = 3.82, \quad g(1.5) = 4.20, \quad h(3) = 22.24. \end{aligned}$$

A more accurate approximation could have been obtained by increasing the degree of the basis functions or by increasing the number of elements.

Example 3

Suppose that we want to approximate the solution to

$$\begin{aligned} &\nabla^2 u = 0 \quad \text{in the rectangle} \quad 0 \le x \le 2, \quad 0 \le y \le 1, \\ &u(x,0) = f(x), \qquad u(1,y) = j(y), \\ &u(x,1) = h(x), \qquad u(2,y) = g(y). \end{aligned} \tag{174.13}$$

For this problem, we choose we use three finite elements; two of these elements (I and II) are triangles and one (III) is a square (see figure 174.3). On the different elements, we choose to use the following polynomial functions to represent the solution:

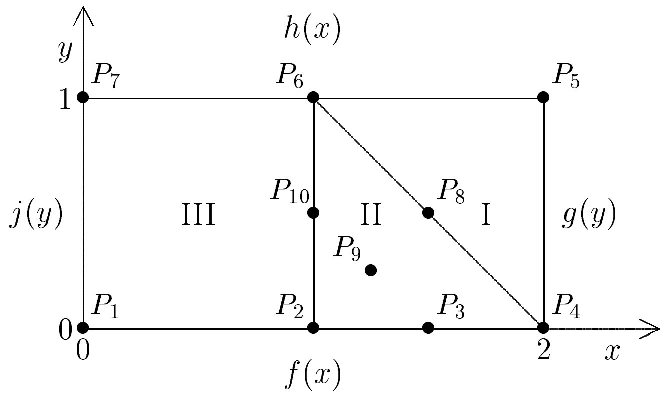

Figure 174.3: Finite elements used in example 3.

$$
\begin{aligned}
u_I &= a_{11} + a_{12}x + a_{13}y + a_{14}x^2 + a_{15}y^2 + a_{16}xy,\\
u_{II} &= a_{21} + a_{22}x + a_{23}y + a_{24}x^2 + a_{25}y^2 + a_{26}xy + a_{27}x^3 + a_{28}y^3,\\
u_{III} &= a_{31} + a_{32}x + a_{33}y.
\end{aligned}
$$

Now we must specify how the parameters in these approximate solutions are to be determined. Using a subscript on f, g, and h to denote evaluation at a node on figure 174.3, we choose to approximately satisfy the equation and boundary conditions on the individual elements as follows:

$$
\begin{aligned}
&\text{On element I:} && u_I\big|_{P_4} = g_4, && u_I\big|_{P_5} = g_5,\\
& && u_I\big|_{P_6} = h_6, && \nabla^2 u_I\big|_{P_5} = 0.\\
&\text{On element II:} && u_{II}\big|_{P_2} = f_2, && u_{II}\big|_{P_3} = f_3,\\
& && u_{II}\big|_{P_4} = f_4, && u_{II}\big|_{P_6} = h_6,\\
& && \nabla^2 u_{II}\big|_{P_6} = 0. && \\
&\text{On element III:} && u_{III}\big|_{P_1} = f_1, && u_{III}\big|_{P_2} = f_2,\\
& && u_{III}\big|_{P_6} = h_6, && u_{III}\big|_{P_7} = h_7.
\end{aligned}
\tag{174.14}
$$

To "connect" the elements, we choose the following conditions:

$$
\begin{aligned}
&\frac{\partial u_I}{\partial n}\Big|_{P_8} = \frac{\partial u_{II}}{\partial n}\Big|_{P_8}, && u_I\big|_{P_8} = u_{II}\big|_{P_8},\\
&\frac{\partial u_{II}}{\partial n}\Big|_{P_{10}} = \frac{\partial u_{III}}{\partial n}\Big|_{P_{10}}, && u_{II}\big|_{P_{10}} = u_{III}\big|_{P_{10}},\\
&\frac{\partial u_I}{\partial x}\Big|_{P_6} = \frac{\partial u_{III}}{\partial x}\Big|_{P_6}.
\end{aligned}
\tag{174.15}
$$

where n stands for the normal.

To actually carry out the solution technique, we choose the functions on the boundary to be $\{f(x) = x^2,\ g(y) = 4 + y - y^2,\ h(x) = x^2,\ j(y) = y - y^2\}$. For these functions, equation (174.13) has the exact solution $u(x, y) = x^2 + y - y^2$. Solving the linear equations in equations (174.14) and (174.15), we obtain the approximate solution

$$\begin{aligned} u_I &= -2y^2 + (10 - 4x)y + 2x^2 + x - 6, \\ u_{II} &= -8y^3 + 23y^2 + (8x - 23)y + 24x^3 - 107x^2 + 156x - 72, \\ u_{III} &= x. \end{aligned}$$

Comparing this approximate solution to the exact solution, we determine the maximum errors (and their locations) to be

Element	Maximum error	Location
I	1	$(1/2, 3/2)$
II	$\frac{16}{3\sqrt{3}}$	$(1, 1 - \frac{1}{\sqrt{3}})$
III	$1/4$	$(0, 1/2)$

Notes

1. Nearly every part of the finite element procedure that has been presented in example 1 can be generalized.

   - The basis functions do not have to be piecewise linear but could be piecewise quadratic, cubic, or higher order (they were chosen to be quadratic in example 2).
   - For physically two-dimensional structures, the "finite elements" can be triangles, quadrilaterals, or polygons with more sides (they can be tetrahedrons, cubes, or more complicated structures for three-dimensional structures). However, the smoothness conditions across the boundaries may be difficult to formulate.
   - Even in one dimension, the "finite elements" do not have to represent intervals of equal length (as in example 2).

2. The approximation to the solution in equation (174.6) will only be C^0, because the basis functions chosen in equation (174.5) are piecewise linear. The cubic Hermite approximation results in a C^1 approximation by choosing the following two basis functions per finite element:

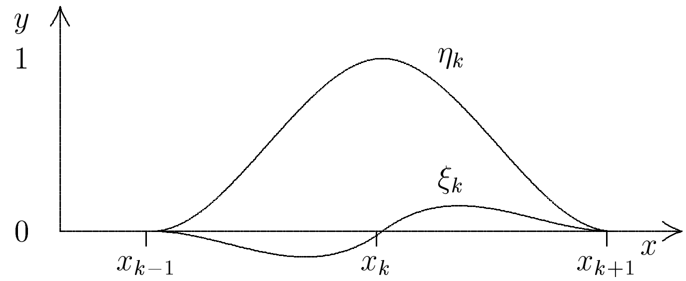

Figure 174.4: The functions for the cubic Hermite approximation.

$$\eta_k(x) = \begin{cases} 1 - 3\left(\frac{x-x_k}{h}\right)^2 - 2\left(\frac{x-x_k}{h}\right)^3, & \text{for } x_{k-1} \le x \le x_k, \\ 1 - 3\left(\frac{x-x_k}{h}\right)^2 + 2\left(\frac{x-x_k}{h}\right)^3, & \text{for } x_k \le x \le x_{k+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\zeta_k(x) = \begin{cases} (x - x_k)\left(1 + \frac{x-x_k}{h}\right)^2, & \text{for } x_{k-1} \le x \le x_k, \\ (x - x_k)\left(1 - \frac{x-x_k}{h}\right)^2, & \text{for } x_{k-1} \le x \le x_k, \\ 0, & \text{otherwise.} \end{cases}$$

These basis functions are continuous with their first derivatives at the nodes (endpoints of the intervals); see figure 174.4. Using these functions, an approximation of the form

$$u(x) \simeq u_N(x) := \sum_{k=1}^{N} d_k \eta_k(x) + e_k \zeta_k(x)$$

is supposed, where the constants $\{d_k, e_k\}$ must be determined.

3. In higher dimensions, smoother approximations are found analogously. Basis functions are chosen that are continuous (with several of their derivatives) at the nodes of the "finite elements." The nodes could be the vertices of a square (or cube), or some of the vertices and some points along the edges on the square (or cube).
4. Both Brebbia [3] and Mackerle and Fredriksson [7] have comprehensive listings of available software that numerically approximate the solutions of differential equations by finite elements.
5. Incidentally, by integrating by parts and using the boundary conditions in equation (174.3), it can be shown that equation (174.4) is equivalent to $I[v] = (v, L[v]) - 2\,(f, v)$, where $(g, h) := \int_0^1 g(x)h(x)\,dx$.
6. In some finite element programs, the discretization errors are controlled by letting the diameter of the largest element h approach zero. This is called the *h-version of the finite element method*. In the

p-version of the finite element method, the mesh is fixed while the degree of the polynomials on the elements is increased (this is also called the *global element method*). In the *hp-version*, both limits are considered simultaneously. See Babuška [2] for details.

7. Mackerle [6] contains a very large annotated bibliography.
8. A comprehensive listing of finite element resources is maintained by Roger Young and Ian MacPhedran; see `http://www.engr.usask.ca/~macphed/finite/fe_resources/fe_resources.html`.
9. See also Strang [10, pages 428–445].

References

[1] Allen, M. B., and Curran, M. C. Adaptive local grid refinement algorithms for finite-element collocations. *Num. Methods Part. Diff. Eqns. 5* (1989), 121–132.

[2] Babuska, I. The p and h-p versions of the finite element method. The state of the art. In *Finite Elements: Theory and Application*, D. L. Dwoyer, M. Y. Hussaini, and R. G. Voigt, Eds. Springer–Verlag, New York, 1988, pp. 199–239.

[3] Brebbia, C. A. *Finite Element Systems: A Handbook*. Springer–Verlag, New York, 1985.

[4] Delves, L. M., and Phillips, C. A fast implementation of the global element method. *J. Inst. Maths. Applics 25* (1980), 177–197.

[5] Lapidus, L., and Pinder, G. F. *Numerical Solution of Partial Differential Equations in Science and Engineering*. John Wiley & Sons, New York, 1982.

[6] Mackerle, J. Special volume—finite element methods: A guide to information sources. *Finite Elements in Analysis and Design 8*, 1–4 (December 1990).

[7] Mackerle, J., and Fredriksson, B. *Handbook of Finite Element Software: Supercomputers Mainframes, Minicomputers, Microcomputers*. Krieger Pub. Co., Melbourne, FL, 1988.

[8] Mitchell, A. R., and Wait, R. *The Finite Element Method in Differential Equations*. John Wiley & Sons, New York, 1977.

[9] Rockey, K. C., Evans, H. R., Griffiths, D. W., and Nethercot, D. A. *The Finite Element Method — A Basic Introduction for Engineers*, second ed. Halstead Press, New York, 1983.

[10] Strang, G. *Introduction to Appled Mathematics*. Wellesley–Cambridge Press, Wellesley, MA, 1986.

[11] Strang, G., and Fix, G. J. *An Analysis of the Finite Element Method*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1973.

[12] Wang, P. S. FINGER: A symbolic system for automatic generation of numerical programs in finite element analysis. *J. Symbolic. Comp. 2* (1986), 305–316.

175. Hybrid Computer Methods*

Applicable to Ordinary and partial differential equations.

Yields

A numerical approximation to the solution.

Idea

Sometimes the advantages of both digital and analog computers can be used simultaneously on a single differential equation.

Procedure

A hybrid computer is one that combines both digital and analog computing devices. Generally, in such a configuration, the analog computer is used to perform tasks that are very time consuming on a digital computer. The analog computer is constructed, generally by the user, out of capacitors, operational amplifiers, resistors, and other electronic components. The numbers in an analog computer are represented by electrical quantities such as voltage and amperage.

As an example of use, a partial differential equation can often be approximated by a large number of ordinary differential equations; for example, the method of lines (see page 831) or the Rayleigh–Ritz method (see page 638). Rather than introduce additional approximations in finding solutions of these ordinary differential equations, an analog computer may be used.

In other problems, the analog computer is used to evaluate integrals as they arise. These integrals are often multi-dimensional and would be computationally intensive on a digital computer.

The digital computer is nearly always used to control the solution procedure and to determine the discretization and the overall error.

Example

The block diagram in figure 175.1 shows how the differential equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx^2 = f(t)$$

might be solved by an analog computer. Each of the blocks in this figure is easily implemented by electronic components.

The blocks that perform the multiplications will generally have the numerical values of a and $-b$ specified by potentiometers. These values may be changed by adjusting the potentiometers by hand. Or, these values could be changed by a digital computer.

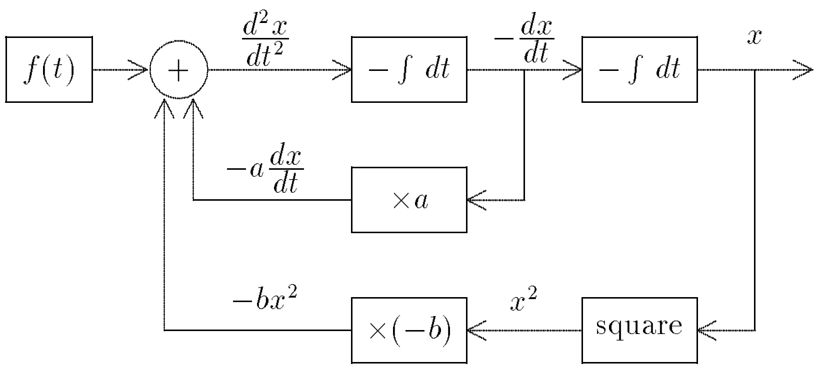

Figure 175.1: A block diagram for the analog solution of the differential equation $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx^2 = f(t)$.

Notes

1. For an example of a hybrid nonlinear parabolic equation solver, see El-Zorkany and Balasubramanian [3].
2. Recently, hybrid computers have been introduced that do not require the user to "plug" components together; the specification of the analog part of the machine is performed on the digital part of the machine.

References

[1] Allison, J. S., and Johnson, H. M. Stability of hybrid simulation of dynamic systems. *Mathematics and Computers in Simulation 21* (1979), 289–303.

[2] Amyot, J. R., and Camire, G. A. Stability of a class of hybrid computer models of dynamical systems. *Mathematics and Computers in Simulation 28* (1986), 57–64.

[3] El-Zorkany, H. I., and Balasubramanian, R. Hybrid solution of non-linear P.D.E.'s based on a special F.E. approximation I. One dimensional problem. In *Advances in Computer Methods For Partial Differential Equations-II*, R. Vichnevetsky, Ed., IMACS (AICA). North–Holland Publishing Co., New York, 1977, pp. 227–234.

[4] El-Zorkany, H. I., and Balasubramanian, R. Hybrid computer solution of PDE's using Laplace–modified Galerkin approximation. *Mathematics and Computers in Simulation 23* (1981), 304–311.

[5] El-Zorkany, H. I., and Balasubramanian, R. Hybrid simulation of multidimensional P.D.E.'s via solution of many one dimensional problems 1. Theoretical basis and hybrid implementation. *Mathematics and Computers in Simulation 25* (1983), 70–76.

[6] Lawson, P. A. Contraction mapping techniques applied to the hybrid computer solution of parabolic PDE's. *Mathematics and Computers in Simulation 23* (1981), 299–303.

[7] Neundorf, W. Iterative block methods for the hybrid computer solution of the method of lines. *Math. and Computers in Simulation 23*, 2 (1981), 142–148.

[8] Roubcek, T. Hybrid solution of weakly formulated boundary-value problems. *Mathematics and Computers in Simulation 26* (1984), 11–19.

[9] Shearer, J. L., Murphy, A. T., and Richardson, H. H. *Introduction to System Dynamics*. Addison–Wesley Publishing Co., Reading, MA.

176. Invariant Imbedding*

Applicable to Most often, two point boundary value problems for ordinary differential equations.

Yields

A new formulation as an initial value problem.

Idea

Invariant imbedding is a type of continuation method (see page 710). For the usual problems that are treated, the length of the interval of interest is considered to be the continuation parameter. Hence, the endpoint in a two point boundary value problem is treated as a variable. By differentiating with respect to this variable, an initial value problem can be created.

Procedure

The general technique involves some subtleties, so we choose to illustrate the technique on a class of two point boundary value problems. More details can be found in Casti and Kalaba [3]. Suppose we have the system of ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= a(t)x(t) + b(t)y(t), \\ \frac{dy}{dt} &= c(t)x(t) + d(t)y(t) + f(t), \end{aligned} \tag{176.1}$$

with

$$\begin{aligned} \alpha_1 x(0) + \alpha_2 y(0) &= 0, \\ \alpha_3 x(T) + \alpha_4 y(T) &= 1, \end{aligned} \tag{176.2}$$

on the interval $t \in [0, T]$, where the $\{\alpha_i\}$ are constants and $\{a, b, c, d\}$ are continuous functions. If we think of the endpoint T as being a variable, then the solution to equations (176.1) and (176.2) can be written, by use of superposition, as

$$\begin{aligned} x(t) = x(t, T) &= u(t, T) + p(t, T), \\ y(t) = y(t, T) &= v(t, T) + q(t, T), \end{aligned} \tag{176.3}$$

where the functions $\{u, v, p, q\}$ are defined by

$$\begin{aligned} \frac{du(t,T)}{dt} &= a(t)u + b(t)v, & \alpha_1 u(0,T) + \alpha_2 v(0,T) &= 0, \\ \frac{dv(t,T)}{dt} &= c(t)u + d(t)v + f(t), & \alpha_3 u(T,T) + \alpha_4 v(T,T) &= 0, \end{aligned} \tag{176.4}$$

and

$$\frac{dp(t,T)}{dt} = a(t)p + b(t)q, \qquad \alpha_1 p(0,T) + \alpha_2 q(0,T) = 0,$$
$$\frac{dq(t,T)}{dt} = c(t)p + d(t)q, \qquad \alpha_3 p(T,T) + \alpha_4 q(T,T) = 1. \quad (176.5)$$

Using algebraic manipulations, the systems in equations (176.4) and (176.5) can be written as initial value systems by the introduction of four new variables. Define the functions $\{r, s, m, n\}$ to be the solutions to the following nonlinear ordinary differential equations:

$$\begin{aligned}
r'(t) &= b(t)s(t) + [a(t) - \alpha_3 b(t)s(t) - \alpha_4 d(t)s(t)]r(t) - [\alpha_3 a(t) + \alpha_4 c(t)]r^2(t),\\
s'(t) &= c(t)r(t) + [d(t) - \alpha_3 a(t)r(t) - \alpha_4 c(t)r(t)]s(t) - [\alpha_3 b(t) + \alpha_4 d(t)]s^2(t),\\
m'(t) &= a(t)m(t) + b(t)n(t) - \Big\{[\alpha_3 a(t) + \alpha_4 c(t)]m(t)\\
&\qquad + [\alpha_3 b(t) + \alpha_4 d(t)]n(t) + f(t)\Big\}r(t),\\
n'(t) &= c(t)m(t) + d(t)n(t) + f(t) - \Big\{[\alpha_3 a(t) + \alpha_4 c(t)]m(t)\\
&\qquad + [\alpha_3 b(t) + \alpha_4 d(t)]n(t) + f(t)\Big\}s(t),
\end{aligned} \quad (176.6)$$

where $'$ denotes differentiation of a function with respect to its single argument (i.e., the variable t). The initial values for $\{r, s, m, n\}$ are given by

$$\begin{aligned}
\alpha_1 r(0) + \alpha_2 s(0) &= 0, \quad m(0) = 0,\\
\alpha_3 r(0) + \alpha_4 s(0) &= 1, \quad n(0) = 0.
\end{aligned} \quad (176.7)$$

Note that we must have $\alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$ if $r(0)$ and $s(0)$ are to be determined from equation (176.7). Using $\{r, s, m, n\}$, the equations for $\{p, q, u, v\}$ can now be written as

$$\begin{aligned}
\frac{dp(t,T)}{dT} &= -\Big\{r(T)[\alpha_3 a(T) + \alpha_4 c(T)] + s(T)[\alpha_3 b(T) + \alpha_4 d(T)]\Big\}p(t,T),\\
\frac{dq(t,T)}{dT} &= -\Big\{r(T)[\alpha_3 a(T) + \alpha_4 c(T)] + s(T)[\alpha_3 b(T) + \alpha_4 d(T)]\Big\}q(t,T),\\
\frac{du(t,T)}{dT} &= -\Big\{m(T)[\alpha_3 a(T) + \alpha_4 c(T)] + n(T)[\alpha_3 b(T) + \alpha_4 d(T)]\\
&\qquad + f(T)\Big\}p(t,T),\\
\frac{dv(t,T)}{dT} &= -\Big\{m(T)[\alpha_3 a(T) + \alpha_4 c(T)] + n(T)[\alpha_3 b(T) + \alpha_4 d(T)]\\
&\qquad + f(T)\Big\}q(t,T).
\end{aligned} \quad (176.8)$$

The initial conditions for $\{p, q, u, v\}$ may be written as

$$\begin{aligned}
p(t,t) &= r(t), \quad q(t,t) = s(t),\\
u(t,t) &= m(t), \quad v(t,t) = n(t).
\end{aligned} \quad (176.9)$$

Suppose that the solution of the original system, equations (176.1) and (176.2), is desired at the set of abscissas $\{t_1, t_2, t_3, \ldots, t_N\}$, where $t_N = T^*$, and T^* is the interval length of interest. The numerical technique is to numerically integrate the equations in equation (176.6) for $\{r, s, m, n\}$, from $t = 0$ to $t = T^*$. Hence, the values of $\{r, s, m, n\}$ will be known at the points $\{t_1, t_2, t_3, \ldots, t_N\}$.

Now, fix $t = t_1$ in equations (176.8) and (176.9). Integrate the resulting equations (with respect to T) from $T = t_1$ to $T = T^*$. This will yield $\{p(t_1, T^*),\ q(t_1, T^*),\ u(t_1, T^*),\ v(t_1, T^*)\}$. If these values are used in equation (176.3), then $x(t_1, T^*), y(t_1, T^*)$ will be determined. Of course, this is the same as $x(t_1), y(t_1)$. Hence, x and y have been determined at the first point of interest, t_1.

To obtain x and y at $t = t_2$, evaluate equations (176.8) and (176.9) at $t = t_2$ and integrate the resulting equations with respect to T (from t_2 to T^*). Repeat this for each of $t = t_3, t = t_4, \ldots, t = t_N$.

Example

Suppose we want to turn the boundary value problem

$$\begin{aligned} \frac{dx}{dt} &= 10y, \qquad && x(0) = 0, \\ \frac{dy}{dt} &= 10x, \qquad && y(10) = 1, \end{aligned}$$

into an initial value problem. Using the above notation, we find that $\alpha_1 = \alpha_4 = 1$, $\alpha_2 = \alpha_3 = 0$, $a(t) = d(t) = f(t) = 0$, $b(t) = c(t) = 10$, and $T^* = 10$. The system in equation (176.6) becomes

$$\begin{aligned} r' &= 10(s - r^2), \\ s' &= 10(s - 1), \\ m' &= 10(n - mr), \\ n' &= 10m(1 - s), \end{aligned} \tag{176.10}$$

with the initial conditions: $r(0) = 0$, $s(0) = 1$, $m(0) = 0$, $n(0) = 0$. It is easy to see that $n(t) = 0$, $m(t) = 0$, $s(t) = 1$, although these equations could have been integrated if this had not been observed. The system in (176.8) becomes

$$\begin{aligned} \frac{dp(t,T)}{dT} &= -10r(T)p(t,T), \\ \frac{dq(t,T)}{dT} &= -10r(T)q(t,T), \\ \frac{du(t,T)}{dT} &= -10m(T)p(t,T), \\ \frac{dv(t,T)}{dT} &= -10m(T)q(t,T), \end{aligned} \tag{176.11}$$

with the initial conditions: $p(t,t) = r(t)$, $q(t,t) = s(t)$, $u(t,t) = m(t)$, $v(t,t) = n(t)$. From the above observation, we conclude that $q(t,t) = 1$, $u(t,T) = 0$, $v(t,T) = 0$. Using equation (176.3), we find: $x(t) = x(t,10) = p(t,10)$ and $y(t) = y(t,10) = q(t,10)$. Let us suppose that we want to know the values of x and y for $t = 2, 4, 6, 8, 10$. The procedure to follow is

1. Integrate $r(t)$ from $t = 0$ up to $t = 10$ using equation (176.10). Hence, $r(2)$, $r(4)$, $r(6)$, $r(8)$, $r(10)$ will all be known.
2. Set $p(2,2) = r(2)$ and $q(2,2) = 1$. Integrate equation (176.11) for $p(t,T)$ and $q(t,T)$ from $T = 2$ to $T = 10$. Then, $\{p(2,10), q(2,10)\}$ will be known and hence, $\{x(2), y(2)\}$ will be known.
3. Set $p(4,4) = r(4)$ and $q(4,4) = 1$. Integrate equation (176.11) for $p(t,T)$ and $q(t,T)$ from $T = 4$ to $T = 10$. Then, $\{p(4,10), q(4,10)\}$ will be known and hence, $\{x(4), y(4)\}$ will be known.
4. Repeat steps (2) and (3) for $t = 6$, $t = 8$, and $t = 10$.

Notes

1. The paper by Scott [9] lists several different ways in which boundary value problems may be converted into stable initial value problems.
2. Imbedding methods can be used for more than just boundary value problems. This technique can also be applied to nonlinear variational problems, unconstrained nonlinear control processes, constrained control processes, and Fredholm integral equations. Imbedding methods have also been used in hyperbolic and parabolic partial differential equations.
3. Wasserstrom [11] discusses how imbedding methods can be analyzed as continuation methods (see page 710).
4. Other names for the invariant imbedding approach are "field method," "factorization method," "method of sweeps," "compound matrix method," and "Riccati transformation." In this last method, matrix Riccati equations (see page 395) are developed. See Ascher *et al.* [1] for details.

References

[1] Ascher, U. M., Mattheij, R. M. M., and Russel, R. D. *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1988.

[2] Bellman, R., and Wing, G. M. *An Introduction to Invariant Imbedding*. SIAM, Philadelphia, PA, 1992.

[3] Casti, J., and Kalaba, R. *Imbedding Methods in Applied Mathematics*. Addison–Wesley Publishing Co., Reading, MA, 1973.

[4] Dieci, L., Osborne, M. R., and Russell, R. D. A Riccati transformation method for solving linear BVPs. I: Theoretical aspects. *SIAM J. Numer. Anal. 25*, 5 (October 1988), 1055–1073.

[5] Lee, E. S. *Quasilinearization and Invariant Imbedding*. Academic Press, New York, 1968.

[6] Meyer, G. H. *Initial Value Methods for Boundary Value Problems: Theory and Application of Invariant Imbedding*. Academic Press, New York, 1973.

[7] Meyer, G. H. Invariant imbedding for fixed and free two point boundary value problems. In *Numerical Solutions of Boundary Value Problems for Ordinary Differential Equations*, A. K. Aziz, Ed. Academic Press, New York, 1975, pp. 249–275.

[8] Ng, B. S., and Reid, W. H. A numerical method for linear two-point boundary-value problems using compound matrices. *J. Comput. Physics 33*, 1 (Oct 1979), 70–85.

[9] Scott, M. R. On the conversion of boundary–value problems into stable initial–value problems via several invariant imbedding algorithms. In *Numerical Solutions of Boundary Value Problems for Ordinary Differential Equations*, A. K. Aziz, Ed. Academic Press, New York, 1975, pp. 89–146.

[10] Scott, M. R., and Vandevender, W. H. A comparison of several invariant imbedding algorithms for the solution of two-point boundary-value problems. *Appl. Math. and Comp. 1* (1975), 187–218.

[11] Wasserstrom, E. Numerical solutions by the continuation method. *SIAM Review 15*, 1 (January 1973), 89–119.

177. Multigrid Methods

Applicable to Ordinary and partial differential equations.

Yields

A numerical approximation technique.

Idea

After differential equations are discretized for the purpose of approximating the solution numerically, some linear algebraic operations must be performed. Frequently, a system of linear equations may need to be solved (e.g., see pages 701, 805, and 816). If the system of linear equations is large (e.g., when a fine discretization grid is used), then iterative methods are often used to solve the linear equations.

Multigrid methods are iterative methods for solving systems of linear equations arising from differential equations. Generally, different grids are used, with only a few iterations per grid. The last approximation on one grid becomes the first approximation on the next grid.

Procedure

We sketch the approximation process using the following ordinary differential equation as motivation:

$$\begin{aligned} &u''(x) - \sigma u(x) = -f(x) \\ &u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{177.1}$$

Consider approximating the solution of equation (177.1) on a uniform grid with a spacing of h (e.g., $x_j = jh$ and $v_j \approx u(x_j)$). Call this grid Ω^h. Using $\left(v_{j-1} - 2v_j + v_{j+1}\right)/h^2$ as an approximation to $u''(x_j)$, equation (177.1) can be written as

$$\frac{1}{h^2}\begin{bmatrix} 2+\sigma h^2 & -1 & 0 & & 0 \\ -1 & 2+\sigma h^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+\sigma h^2 & -1 \\ 0 & & 0 & -1 & 2+\sigma h^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{bmatrix} \tag{177.2}$$

or simply as $A^h \mathbf{v}^h = \mathbf{f}^h$. (Here a superscript indicates the spacing on the underlying grid.)

The solution of the linear system in equation (177.2) can be approximated by any of the standard iteration methods, such as Jacobi's method or the Gauss–Seidel method (see Golub and Van Loan [5]). Typically, these iterative methods begin to stall (i.e., the convergence rate decreases) when smooth error modes are present. Because a smooth mode on a fine grid looks less smooth on a coarser grid, it is advisable to move to a coarser grid. Iterating on this coarser grid will more effectively reduce the error term. The values on this coarse grid are then fed back to the fine grid.

To illustrate the process, let I_h^{2h} be the linear operator that performs restriction and maps a vector from Ω^h to Ω^{2h}. (For instance, every other value in the vector could be chosen.) Similarly, let I_{2h}^h be the linear operator that performs interpolation and maps a vector from Ω^{2h} to Ω^h. Let us use the term "Relax on" to mean "iterate some number of times using a standard technique such as Gauss–Seidel." Here, then, is how a multigrid method might be implemented:

Relax on $A^h\mathbf{u}^h = \mathbf{f}^h$ with an input initial guess $\mathbf{v}^h$.
Find the residual: $\mathbf{r}^h := A^h\mathbf{u}^h - \mathbf{f}^h$.
Move to a coarser grid: $\mathbf{f}^{2h} := I_h^{2h}\mathbf{r}^h$.

  Relax on $A^{2h}\mathbf{u}^{2h} = \mathbf{f}^{2h}$ with the initial guess $\mathbf{v}^{2h} = \mathbf{0}$.
  Find the residual: $\mathbf{r}^{2h} := A^{2h}\mathbf{u}^{2h} - \mathbf{f}^{2h}$.
  Move to a coarser grid: $\mathbf{f}^{4h} := I_{2h}^{4h}\mathbf{r}^{2h}$.

    Relax on $A^{4h}\mathbf{u}^{4h} = \mathbf{f}^{4h}$ with the initial guess $\mathbf{v}^{4h} = \mathbf{0}$.
    Find the residual: $\mathbf{r}^{4h} := A^{4h}\mathbf{u}^{4h} - \mathbf{f}^{4h}$.
    Move to a coarser grid: $\mathbf{f}^{8h} := I_{4h}^{8h}\mathbf{r}^{4h}$.

      $\vdots$
      Solve $A^{2^kh}\mathbf{u}^{2^kh} = \mathbf{f}^{2^kh}$ for $\mathbf{u}^{2^kh}$ (which we call $\mathbf{v}^{2^kh}$).
      $\vdots$

    Revise approximate solution on Ω^{4h}: $\mathbf{v}^{4h} \leftarrow \mathbf{v}^{4h} + I_{8h}^{4h}\mathbf{v}^{8h}$.
    Relax on $A^{4h}\mathbf{u}^{4h} = \mathbf{f}^{4h}$ with the initial guess $\mathbf{v}^{4h}$.

  Revise approximate solution on Ω^{2h}: $\mathbf{v}^{2h} \leftarrow \mathbf{v}^{2h} + I_{4h}^{2h}\mathbf{v}^{4h}$.
  Relax on $A^{2h}\mathbf{u}^{2h} = \mathbf{f}^{2h}$ with the initial guess $\mathbf{v}^{2h}$.

Revise approximate solution on Ω^h: $\mathbf{v}^h \leftarrow \mathbf{v}^h + I_{2h}^h\mathbf{v}^{2h}$.
Relax on $A^h\mathbf{u}^h = \mathbf{f}^h$ with the initial guess $\mathbf{v}^h$.

The overall effect is that an approximate solution to the system on the h-grid is input at the top, and a refined approximation to this solution is output at the bottom.

Note

1. The multigrid method is applicable to linear algebraic equations. Its importance for differential equations comes about because differential equations can be approximated by solving linear algebraic equations.

References

[1] Adams, J. Recent enhancements in MUDPACK A multigrid software package for elliptical partial differential equations. *Appl. Math. and Comp. 43* (May 1991), 79–94.

[2] Biggs, W. L. *A Multigrid Tutorial.* SIAM, Philadelphia, PA, 1988.

[3] Brandt, A., McCormick, S., and Ruge, J. Multigrid method for differential eigenproblems. *SIAM J. Sci. Stat. Comput. 4*, 2 (1983), 244–260.

[4] Goldstein, C. I. Multigrid analysis of finite element methods with numerical integration. *Math. of Comp. 56*, 194 (April 1991), 409–436.

[5] Golub, G. H., and Van Loan, C. F. *Matrix Computations*, second ed. The Johns Hopkins University Press, Baltimore, MD, 1989.

[6] Hackbusch, W., and Trottenberg, U. *Multigrid Methods.* Springer–Verlag, New York, 1982.

[7] Jespersen, D. *Multigrid Methods for Partial Differential Equations*, vol. 24 of *Studies in Mathematics.* Mathematical Association of America, Providence, RI, 1984.

[8] McCormick, S. *Multigrid Methods: Theory, Applications, and Supercomputing.* Marcel Dekker, New York, 1988.

[9] Wesseling, P. *An Introduction to Multigrid Methods.* John Wiley & Sons, New York, 1992.

178. Parallel Computer Methods

Applicable to All types of differential equations.

Yields

Numerical approximations to the solutions.

Idea

Parallel computers may be used to quickly obtain numerical approximations to differential equations.

Procedure

The physical basis for most differential equations is a local and asynchronous model. Hence, it should be possible to numerically approximate a partial differential equation by processors that are loosely coupled.

There are three major ways in which software for differential equations can exploit parallelism: in coding a method so that it can be performed simultaneously on several processors, in splitting variables (in a multi-variable system) between processors, and in using parallelism in performing the needed algebraic computations (i.e., solving algebraic systems of equations). We illustrate one parallel technique; it uses the first of these methods.

Example

This example for a two processor MIMD machine is from Iserles and Nørsett [8]. The Butcher-array is a convenient way in which to represent all of the information in a Runge–Kutta method for the equation $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$, $\mathbf{y}(x_0) = \mathbf{y}_0$ (see page 763). The Butcher array for a four-stage, fourth order Runge–Kutta method is

$$\begin{array}{c|cccc} 1/2 & 1/2 & 0 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 & 0 \\ 1/2 & -5/2 & 5/2 & 1/2 & 0 \\ 1/3 & -5/3 & 4/3 & 0 & 2/3 \\ \hline & -1 & 3/2 & -1 & 3/2 \end{array}$$

Because of the specific sparsity structure of this Butcher array, we can efficiently implement this technique on two processors. Given the value $\mathbf{y}_n$, to find the approximation at the next time step, $\mathbf{y}_{n+1}$, the steps are as follows:

- Use an iteration technique (perhaps Newton–Raphson) to solve the equations
  - $\boldsymbol{\xi}_1 = \mathbf{f}\left(t_n + \frac{1}{2}h, \mathbf{y}_n + \frac{1}{2}h\boldsymbol{\xi}_1\right)$ for $\boldsymbol{\xi}_1$ on processor 1,

  - $\boldsymbol{\xi}_2 = \mathbf{f}\left(t_n + \frac{2}{3}h, \mathbf{y}_n + \frac{2}{3}h\boldsymbol{\xi}_2\right)$ for $\boldsymbol{\xi}_2$ on processor 2.
- Copy the value of $\boldsymbol{\xi}_1$ to processor 2, and copy the value of $\boldsymbol{\xi}_2$ to processor 1.
- Use an iteration technique (perhaps Newton–Raphson) to solve the equations
  - $\boldsymbol{\xi}_3 = \mathbf{f}\left(t_n + \frac{1}{2}h, \mathbf{y}_n + h\left(-\frac{5}{2}\boldsymbol{\xi}_1 + \frac{5}{2}\boldsymbol{\xi}_2 + \frac{1}{2}\boldsymbol{\xi}_3\right)\right)$ for $\boldsymbol{\xi}_3$ on processor 1,
  - $\boldsymbol{\xi}_4 = \mathbf{f}\left(t_n + \frac{1}{3}h, \mathbf{y}_n + h\left(-\frac{5}{3}\boldsymbol{\xi}_1 + \frac{4}{3}\boldsymbol{\xi}_2 + \frac{2}{3}\boldsymbol{\xi}_4\right)\right)$ for $\boldsymbol{\xi}_4$ on processor 2.
- Copy $\boldsymbol{\xi}_4$ to processor 1 and then form the estimate at the next time value: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left(\frac{3}{2}(\boldsymbol{\xi}_2 + \boldsymbol{\xi}_4) - \boldsymbol{\xi}_1 - \boldsymbol{\xi}_3\right)$.

Notes

1. Many parallel computers can quickly perform matrix operations, such as solving a system of linear equations. Hence, these machines may be used to quickly approximate the solutions to differential equations by using methods (such as finite differences and finite elements) that produce large systems of linear algebraic equations.
   When solving differential equations numerically, it is not uncommon to have large computational needs. For example, a $50 \times 50 \times 50$ grid with 5 degrees of freedom per grid point, such as might be obtained from Euler's equation in fluid dynamics, will lead to matrices of size $N = 625,000$ and a bandwidth $m \approx 25000$. Even though sparse matrix techniques may be used, the complexity of the problem is very high. However, Rice [21] makes the point that linear algebra approaches are only tangentially relevant to solving partial differential equations and are, in fact, often misleading. Numerical analysis of differential equations begins with the equation itself, not with a discretized version of the equation.
2. Domain decompostion (see page 800) subdivides a large domain (on which an elliptic partial differential equation is defined) into many smaller domains. A separate processor can then be used on each smaller domain; see Quarteroni [19].
3. All types of processors have been used to numerically approximate the solutions to differential equations.
   - Hypercubes have been used by many, including Lustman *et al.* [12], Mu and Rice [15], and Murthy [16].
   - The use of neural networks for solving differential equations is considered in Dissanayake and Phan-Thien [5], Lee and Kang [10], and Meade and Fernandez [13].
   - By a simple replication of hardware, many Monte-Carlo simulations can be performed simultaneously (see pages 810 and 844). This is particularly useful for SIMD machines.

- Lattice gas methods (which use cellular automata) are a method of parallel computation; see page 828. Use of cellular automata to numerically approximate the solution of differential equations has also been considered in Boghosian and Levermore [3].
- It is also possible to build a specialized VLSI circuit to integrate a specific set of differential equations. A special purpose computer for high-speed, high-precision orbital mechanics computations has been built; see Applegate *et al.* [1]. This was used to demonstrate that the orbit of Pluto was chaotic; see Sussman and Wisdom [22].
- It is also possible to construct systolic arrays that solve a class of equations very quickly; see Megson and Evans [14].

4. The technique described in Garbey and Levine [7] numerically approximates hyperbolic equations by using both characteristics and cellular automata.
5. In the Kolmogorov theory of turbulence, computer memory usage scales as $R^{9/4}$ and computational work, including time integration, scales as R^3, where R is the Reynolds number. For engineering applications, Reynolds numbers of 10^3 are typical. For geophysical flows, Reynolds numbers of 10^8 are not unusual. See Jackson *et al.* [9] for details.
6. Lustman *et al.* [12] considers a parallel machine in which every processor computes the same problem but with a different step size. Extrapolation methods (see page 679) are then employed.

References

[1] Applegate, J. H., Douglas, M. R., Gursel, Y., Hunterm, P., Seitz, C. L., and Sussman, G. J. A digital orrery. *IEEE Trans. Computers C-34*, 9 (September 1985), 822–831.

[2] Bellen, A., Vermiglio, R., and Zennaro, M. Parallel ODE-solvers with stepsize control. *J. Comput. Appl. Math. 31* (1990), 277–293.

[3] Boghosian, B. M., and Levermore, C. D. A cellular automaton for Burgers' equation. *Complex Systems 1* (1987), 17–29.

[4] Burrage, K. *Parallel and Sequential Methods for Ordinary Differential Equations.* Numerical Mathematics and Science Publication Series. Clarendon Press, Oxford, England, 1995.

[5] Dissanayake, M. W. M. G., and Phan-Thien, N. Neural-network-based approximations for solving partial differential equations. *Comm. Num. Meth. Eng. 10*, 3 (March 1994), 195–201.

[6] Evans, D. J., and Sanugi, B. B. A parallel Runge–Kutta integration method. *Parallel Comput. 11*, 2 (1989), 245–251.

[7] Garbey, M., and Levine, D. Massively parallel computation of conservation laws. *Parallel Computing 16* (1990), 293–304.

[8] Iserles, A., and Nørsett, S. P. On the theory of parallel Runge–Kutta methods. *IMA J. Num. Analysis 10* (1990), 463–488.

[9] JACKSON, E., SHE, Z.-S., AND ORSZAG, S. A. A case study in parallel computing: I. Homogeneous turbulence on a hypercube. *J. Scientific Comput. 6*, 1 (March 1991), 27–45.

[10] LEE, H., AND KANG, I. S. Neural algorithm for solving differential equations. *J. Comput. Physics 91* (1990), 110–131.

[11] LIN, A. Parallel algorithms for boundary value problems. *J. Parallel & Distrib. Comput. 11* (1991), 284–290.

[12] LUSTMAN, L., NETA, B., AND KATTI, C. P. Solution of ordinary differential initial value problems on an INTEL hypercube. *Comp. & Maths. with Appls. 23*, 10 (1992), 65–72.

[13] MEADE, JR., A. J., AND FERNANDEZ, A. A. Solution of nonlinear ordinary differential equations by feedforward neural networks. *Math. Comput. Modelling 20*, 9 (1994), 19–44.

[14] MEGSON, G. M., AND EVANS, D. J. Systolic arrays for group explicit methods for solving first order hyperbolic equations. *Parallel Computing 16* (1990), 191–205.

[15] MU, M., AND RICE, J. R. A grid-based subtree-subcube assignment strategy for solving partial differential equations on hypercubes. *SIAM J. Sci. Comput. 13*, 3 (May 1992).

[16] MURTHY, C. S. R. Solving hyperbolic PDE's on hypercubes. *Comp. & Maths. with Appls. 21*, 21 (1991), 79–82.

[17] MURTHY, C. S. R., AND RAJARAMAN, V. A multiprocessor architecture for solving nonlinear partial differential equations. *Math. and Computers in Simulation 30* (1988), 453–464.

[18] ORETGA, J. M., AND VOIGT, R. G. Solution of partial differential equations on vector and parallel computers. *SIAM Review 27* (1985), 149–240.

[19] QUARTERONI, A. Domain decomposition and parallel processing for the numerical solution of partial differential equations. *Surv. Math. Ind. 1* (1991), 75–118.

[20] RIBBENS, C. J., WATSON, L. T., AND DESA, C. Trowards parallel mathematical software for elliptic partial differential equations. *ACM Trans. Math. Software 19*, 4 (December 1993), 457–473.

[21] RICE, J. R. Parallel methods for partial differential equations. In *The Characteristics of Parallel Algorithms*, L. H. Jamieson, D. B. Gannon, and R. J. Douglass, Eds. MIT Press, Cambridge, MA, 1987, pp. 209–231.

[22] SUSSMAN, G. J., AND WISDOM, J. Numerical evidence that the motion of Pluto is chaotic. *Science* (22 July 1988), 433–437.

[23] TAM, H. W. "one-stage parallel methods for the numerical solution of ordinary differential equations" and "two-stage parallel methods for the numerical solution of ordinary differential equations". *SIAM J. Sci. Stat. Comput. 13*, 5 (September 1992), 1039–1061 and 1062–1084.

[24] VAN DER HOUWEN, P. J., AND SOMMEIJER, B. P. Iterated Runge–Kutta methods on parallel computers. *SIAM J. Sci. Stat. Comput. 12*, 5 (September 1991), 1000–1028.

179. Predictor–Corrector Methods

Applicable to Ordinary differential equations of the form $y' = f(x, y)$.

Yields

A sequence of numerical approximations.

Idea

To integrate an ordinary differential equation from a point x_n to a new point $x_{n+1} = x_n + h$, a single formula may be used to predict y_{n+1}. Alternatively, the value of y_{n+1} could be predicted by one formula, and then that value could be refined by an iterative formula (the "corrector").

Procedure

For the first order ordinary differential equation $y' = f(x, y)$, suppose that the values of x and y are known at the sequence of $m + 1$ points $\{x_{n-m}, \ldots, x_{n-1}, x_n\}$. Then the values of y' are known at those same points (because y' is determined from x and y via $y' = f(x, y)$). An interpolatory polynomial of degree m can be fitted to $m + 1$ values of x and y'. This polynomial can be used to predict the value of y' in the interval (x_n, x_{n+1}). This, in turn, can be used to predict the value of y_{n+1} by a numerical approximation of the relation

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} y'(x)\, dx. \tag{179.1}$$

Such a formula is called an "predictor."

A modification of this step can be repeated. The values of x and y' are now known at the $m+1$ points $\{x_{n-m+1}, \ldots, x_n, x_{n+1}\}$. A polynomial can be fit through these points, and then the quantity in equation (179.1) can be recomputed. This formula, which furnishes a new estimate of y_{n+1}, is called a "corrector." The corrector may be used repeatedly.

Example

One set of predictor–corrector equations is the Adams–Bashforth predictor formula

$$y_{n+1} = y_n + \frac{h}{24}\left(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right), \tag{179.2}$$

and the Adams–Moulton corrector formula

$$y_{n+1} = y_n + \frac{h}{24}\left(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right), \tag{179.3}$$

where h is the difference between adjacent x points (The x points are assumed to be equally spaced). These equations are fourth order accurate.

Example

The Fortran program in program 179.1 uses the method in equation (179.2) and (179.3) to approximate the solution to the differential equation

$$\frac{dy}{dx} = 1 - x + \frac{y}{x}, \qquad y(1) = 0. \tag{179.4}$$

Because the solution of equation (179.4) is given by $y(x) = x(\log x - x + 1)$ (determined by integrating factors), it is easy to see that the values produced

```
Step number=  4  X= 1.60   Y= -0.2080
Step number=  5  X= 1.80   Y= -0.3820
Step number=  6  X= 2.00   Y= -0.6137
Step number=  7  X= 2.20   Y= -0.9054
Step number=  8  X= 2.40   Y= -1.2588
Step number=  9  X= 2.60   Y= -1.6756
Step number= 10  X= 2.80   Y= -2.1570
Step number= 11  X= 3.00   Y= -2.7041
Step number= 12  X= 3.20   Y= -3.3179
Step number= 13  X= 3.40   Y= -3.9991
Step number= 14  X= 3.60   Y= -4.7486
```

all are correct to the number of decimal places given.

Note that the program required that the values of y be given for $x = hj$ where $j = 1, 2, 3$. These "starting" values were obtained by using a Runge–Kutta method that was fourth order accurate (these calculations are not shown).

Notes

1. The corrector formula could be iterated as many times as is necessary to ensure convergence. This is called *correcting to convergence*. In general, however, if more than two iterations are required, then the step size h is probably too large.
2. Given the equation $y' = f(x, y)$, let P indicate an application of a predictor, C a single application of a corrector, and let E indicate an evaluation of the function f in terms of known values of its arguments. Correcting to convergence can then be represented by $P(EC)^\infty$. See Lambert [7] for an analysis of $P(EC)^m$ and $P(EC)^m E$, where m is a fixed number.
3. Note that the predictor–corrector method is a finite difference scheme that is not a linear multistep method as defined on page 670.
4. To obtain the starting values so that the predictor–corrector pair can be used, Runge–Kutta methods can be used first. This was done in the example above. When this is done, the Runge–Kutta method used should be at least as accurate as the predictor–corrector formula used. See Gear [4] for details.

```
      REAL*4 X(100),Y(100),YP(100)
C Define the initial values (found by Runge-Kutta)
        H=0.2
        X(1)= 1.0
        Y(1)= 0.0
       YP(1)= F(X(1),Y(1))
        X(2)= X(1) + H
        Y(2)=-0.02121
       YP(2)= F(X(2),Y(2))
        X(3)= X(2) + H
        Y(3)=-0.08894
       YP(3)= F(X(3),Y(3))
        X(4)= X(3) + H
        Y(4)=-0.20799
       YP(4)= F(X(4),Y(4))
C Here is the integration loop
       DO 10 N=4,14
       NP1=N+1
       X(NP1)=  X(N) + H
       Y(NP1)=  PREDIC(X,Y,YP,N,H)
       YP(NP1)= F(X(NP1),Y(NP1))
       Y(NP1)=  CORECT(X,Y,YP,N,H)
       Y(NP1)=  CORECT(X,Y,YP,N,H)
10     WRITE(6,5) N,X(N),Y(N)
5      FORMAT(' Step number=',I3,'  X=',F5.2,'   Y=',F8.4)
       END
C This function has the predictor
      FUNCTION PREDIC(X,Y,YP,N,H)
      REAL*4   X(100),Y(100),YP(100)
      PREDIC=Y(N) + H/24.*(55.*YP(N)-59.*YP(N-1)+37*YP(N-2)-9.*YP(N-3))
      RETURN
      END
C This function has the corrector
      FUNCTION CORECT(X,Y,YP,N,H)
      REAL*4   X(100),Y(100),YP(100)
      CORECT=Y(N) + H/24.*(9.*YP(N+1)+19.*YP(N)-5.*YP(N-1)+YP(N-2))
      RETURN
      END
C This function has the right hand side of the differential equation
      FUNCTION F(X,Y)
      F=1.0-X+Y/X
      RETURN
      END
```

Program 179.1: Fortran program for predictor–corrector method.

5. For the same accuracy, using a predictor–corrector pair to integrate a first order ordinary differential equation generally requires fewer evaluations of the function $f(x, y)$ than a Runge–Kutta method would.

6. One set of commonly used predictor–corrector equations is "Milne's method"

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y_n - y'_{n-1} + 2y'_{n-2}\right),$$
$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(y'_{n+1} + 4y'_n + y'_{n-1}\right).$$

   These equations are also fourth order accurate. Milne's method is *not* recommended because it is subject to an instability problem, in which the errors do *not* tend to zero as the step size h is made smaller. See Gerald and Wheatley [5, pages 314–323] for details.
7. The Adams–Bashforth formulas are a family of linear multistep methods that are often used as predictors for the equation $y' = f(x, y)$. The k-step fixed-stepsize Adams–Bashforth formula

$$y_n = y_{n-1} + h\sum_{j=1}^{k} \beta_j f(x_{n-j}, y_{n-j}),$$

   is equivalent to $y_n = y_{n-1} + \int_{x_{n-1}}^{x_n} p_n(s)\, ds$, where $p_n(x)$ is the unique polynomial of degree $k-1$ that interpolates $f(x_{n-j}, y_{n-j})$ at x_{n-j} for $j = 1, \ldots, k$.
8. See also Abramowitz and Stegun [1, formula 25.5.13–25.5.16, pages 896–897], Boyce and DiPrima [2, pages 431–438], and Bronson [3, 232–257].

References

[1] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, D.C., 1964.

[2] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[3] Bronson, R. *Modern Introductory Differential Equations*. Schaum's Outline Series. McGraw–Hill Book Company, New York, 1973.

[4] Gear, C. W. Runge–Kutta starters for multistep methods. *ACM Trans. Math. Software 6*, 3 (Sept 1980), 263–279.

[5] Gerald, C. F., and Wheatley, P. O. *Applied Numerical Analysis*. Addison–Wesley Publishing Co., Reading, MA, 1984.

[6] Karim, A. I. A., and Ismail, G. A. Nonequidistant modified predictor–corrector methods for solving systems of differential equations. *Int. J. Comp. Math. 17* (1985), 339–361.

[7] Lambert, J. D. *Computational Methods in Ordinary Differential Equations*. Cambridge University Press, New York, 1973.

[8] Van Der Houwen, P. J., and Sommeijer, B. P. Predictor–corrector methods for periodic second-order initial-value problems. *IMA J. Num. Analysis 7* (1987), 407–422.

180. Runge–Kutta Methods

Applicable to Initial value systems of first order ordinary differential equations.

Yields

A numerical approximation to the solution of an initial value system.

Idea

Given an ordinary differential equation and an initial value, the value of the dependent variable may be found at the next desired value of the independent variable by calculating several intermediate values.

Procedure

Given the first order ordinary differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{180.1}$$

the value of $y(x)$ at the point $x_0 + h$ may be approximated by a weighted average of values of $f(x, y)$ taken at different points in the interval $x_0 \leq x \leq x_0 + h$. The classical Runge–Kutta formula is given by

$$y(x_0 + h) = y(x_0) + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \tag{180.2}$$

where

$$\begin{aligned} k_1 &= f(x_0, y_0), \\ k_2 &= f(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1), \\ k_3 &= f(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2), \\ k_4 &= f(x_0 + h, y_0 + k_3). \end{aligned} \tag{180.3}$$

This approximation to $y(x_0+h)$ is fourth order accurate. After $y(x_0+h)$ has been determined, the same formula may be used to determine $y(x_0 + 2h)$. This process may be repeated.

The Butcher array is a convenient way in which to represent all of the information in a Runge–Kutta method for the equation $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ with $\mathbf{y}(x_0) = \mathbf{y}_0$. Specifically, the s-stage Runge–Kutta scheme (which uses s intermediate values)

$$\begin{aligned} \mathbf{y}_{n+1} &= \mathbf{y}_n + h\sum_{i=1}^{s} b_i \mathbf{k}_i, \\ \mathbf{k}_i &:= \mathbf{f}\left(x_n + c_i h, \mathbf{y}_n + h\sum_{j=1}^{s} a_{ij}\mathbf{k}_j\right), \end{aligned}$$

where $h := x_{n+1} - x_n$, $\sum_i b_i = 1$, and $c_i = \sum_{j=1}^{s} a_{ij}$ for each j, is represented in the tabular form

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^{\mathrm{T}} \end{array} \qquad \text{or} \qquad \begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_s \end{array}$$

Note that an explicit Runge–Kutta scheme has $a_{ij} = 0$ for $j \geq i$ (sometimes these zeros are omitted). See Butcher [5, page 163] or Dekker and Verwer [9, Chapter 3] for details. The explicit method in equations (180.2) and (180.3) has the Butcher array (with $s = 4$)

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Example 1

The C (Fortran) code in program 180.1 (180.2) calculates a numerical approximation to the solution of the equation

$$y' = 1 - x + \frac{y}{x}, \quad y(1) = 0, \tag{180.4}$$

using the method in equations (180.2) and (180.3). It uses a step size h of `0.1`. The exact solution of equation (180.4), determined by integrating factors, is $y(x) = x(\log x - x + 1)$. Hence, $y(2) = 2(\log 2 - 1) \simeq -0.6137$. This is the value returned by the programs.

Example 2

The derivation of a Runge–Kutta method is instructive because it indicates the arbitrary degrees of freedom that exist in Runge–Kutta methods. Given the equation $y' = f(t, y)$ and using $y_n = y(t_n)$ and $t_n = nh$ to find a 2-stage Runge–Kutta scheme, we assume a discrete approximation scheme of the form

$$\begin{aligned} y_{n+1} &= y_n + ak_1 + bk_2, \\ k_1 &= hf(t_n, y_n), \\ k_2 &= hf(t_n + \alpha h, y_n + \beta k_1). \end{aligned} \tag{180.5}$$

We want to find $\{a, b, \alpha, \beta\}$ to make the order of this scheme as high as possible. From equation (180.5) we can explicitly write y_{n+1} and then find a Taylor series expansion:

$$\begin{aligned} y_{n+1} &= y_n + ahf(t_n, y_n) + bhf(t_n + \alpha h, y_n + \beta hf(t_n, y_n)), \\ &= y_n + (a + b)hf_n + h^2 (\alpha b f_t + \beta b f_y f)_n + O(h^3), \end{aligned} \tag{180.6}$$

```
void main(void) { RungeKutta(); }
void RungeKutta(void) {
    int j;
    double h = 0.1;
    double x = 1.0;
    double y = 0.0;
    for(j=0; j<=9; j++) {
        y += Runge(x, y, h);
        x += h;
        printf("X= %6.2f  Y= %7.4f \n", x, y);
  }
}
/* This performs one integration step */
double Runge(double x, double y, double h) {
    double fk1, fk2, fk3, fk4;
    fk1 = F(x        , y              );
    fk2 = F(x + h/2.0, y + h*fk1 / 2.0);
    fk3 = F(x + h/2.0, y + h*fk2 / 2.0);
    fk4 = F(x + h    , y + h*fk3      );
    return(h * (fk1 + 2.0*fk2 + 2.0*fk3 + fk4) / 6.0);
}
/* This function has the right-hand side of the equation */
double F(double x, double y) { return(1.0 - x + y/x); }
```

Program 180.1: C program for Runge–Kutta method.

```
       H= 0.1
       X= 1.0
       Y= 0.0
       DO 10 J=1,9
       Y= Y+RUNGE(X,Y,H)
       X= X+H
10     WRITE(6,88) X,Y
88     FORMAT(' X=',F6.2,' Y=',F7.4)
       END
C This function performs one integration step
       FUNCTION RUNGE(X,Y,H)
       FK1= F(X,     Y)
       FK2= F(X+H/2.0,Y+H*FK1/2.0)
       FK3= F(X+H/2.0,Y+H*FK2/2.0)
       FK4= F(X+H,    Y+H*FK3)
       RUNGE= H*(FK1 + 2.0*FK2 + 2.0*FK3 + FK4)/6.0
       RETURN
       END
C This function has the right hand side of the equation
       FUNCTION F(X,Y)
       F= 1.0-X+Y/X
       RETURN
       END
```

Program 180.2: Fortran program for Runge–Kutta method

where a subscript of n denotes evaluation at the point (t_n, y_n). From $y' = f(t, y)$ we can directly construct a Taylor expansion in t to find:

$$y_{n+1} = y_n + h f_n + \frac{h^2}{2}\left(\frac{df}{dt}\right)_n + O\left(h^3\right), \tag{180.7}$$

$$= y_n + h f_n + \frac{h^2}{2}\left(f_t + f_y f\right)_n + O\left(h^3\right),$$

because $\frac{df}{dt} = f_t + f_y \frac{dy}{dt} = f_t + f_y f$. Comparing equations (180.6) and (180.7), we find the 3 equations

$$a + b = 1, \qquad \alpha b = \frac{1}{2}, \qquad \beta b = \frac{1}{2}, \tag{180.8}$$

for the 4 unknowns $\{a, b, \alpha, \beta\}$. Because these equations are undetermined, there are infinitely many second order Runge–Kutta schemes in the form of equation (180.5).

Fourth order Runge–Kutta methods result in 11 equations for 13 unknowns; 2 of the unknowns may be chosen arbitrarily to achieve some goal. For example, a fourth order Runge–Kutta method with a specific sparsity pattern is used in the section on parallel methods (see page 755) to allow a parallel implementation.

Example 3

To obtain accurate numerical results when using any method, an estimate of the local error must be obtained. This could be done by the standard technique of recomputing the answer with the step size halved; but this requires lots of additional computation. The Runge–Kutta–Fehlberg method is a fifth order method that uses 6 functional evaluations and allows an estimate of the error by re-using the same points:

$$\begin{aligned}
k_1 &= hf\left(x_n, y_n\right), \\
k_2 &= hf\left(x_n + \frac{1}{4}h, y_n + \frac{1}{4}k_1\right), \\
k_3 &= hf\left(x_n + \frac{3}{8}h, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2\right), \\
k_4 &= hf\left(x_n + \frac{12}{13}h, y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right), \\
k_5 &= hf\left(x_n + h, y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right), \qquad (180.9) \\
k_6 &= hf\left(x_n + \frac{1}{2}h, y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right), \\
y_{n+1} &= y_n + \left(\frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_k - \frac{1}{5}k_5\right) \\
\text{error} &\approx \frac{1}{360}k_1 - \frac{128}{4275}k_3 - \frac{2197}{75240}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6.
\end{aligned}$$

Notes

1. If $f(x, y)$ does not depend on y, then the solution of the initial value problem $y' = f(x)$, $y(x_0) = y_0$, is just the integral $y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$. The Runge–Kutta method in equation (180.2) then corresponds to the approximation of $y(x)$ by means of Simpson's rule.

2. There are several Runge–Kutta methods for first order equations. For example, the following scheme for equation (180.1)

$$\begin{aligned} y(x_0+h) &= y(x_0) + \frac{1}{2}(k_1+k_2), \\ k_1 &= hf(x_0, y_0), \\ k_2 &= hf(x_0+h, y_0+k_1), \end{aligned} \tag{180.10}$$

is of second order accuracy. A commonly used fourth order accurate method for first order ordinary differential equations (different from the one in equation (180.3)) is Gill's method; see Abramowitz and Stegun [1, formula 25.5.12].

3. There are also implicit Runge–Kutta methods, see Burrage and Butcher [4] or Butcher [5, Chapter 34]. There are also Runge–Kutta methods for ordinary differential equations of orders 2–10. See, for example, Abramowitz and Stegun [1, formulae 25.5.6–25.5.12] or Collatz [8, Section 2.4, pages 61–77]. For example, a Runge–Kutta scheme for the second order equation

$$y'' = g(x, y, y'), \qquad y(x_0) = y_0, \ y'(x_0) = v_0,$$

is given by

$$\begin{aligned} k_1 &= hg(x_0, y_0, v_0), \\ k_2 &= hg\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hv_0 + \frac{1}{8}hk_1, v_0 + \frac{1}{2}k_1\right), \\ k_3 &= hg\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hv_0 + \frac{1}{8}hk_1, v_0 + \frac{1}{2}k_2\right), \\ k_4 &= hg\left(x_0 + h, y_0 + hv_0 + \frac{1}{2}hk_3, v_0 + k_3\right), \end{aligned} \tag{180.11}$$

and

$$\begin{aligned} y(x_0+h) &= y_0 + hv_0 + \frac{1}{6}h(k_1+k_2+k_3), \\ y'(x_0+h) &= v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned} \tag{180.12}$$

This scheme is numerically fourth order accurate.

4. There are also Runge–Kutta methods for systems of first order ordinary differential equations. For example, the system

$$y' = m(x, y, z), \qquad z' = n(x, y, z)$$

of ordinary differential equations may be numerically approximated by first calculating

$$\begin{aligned} k_1 &= hm\,(x_0, y_0, z_0)\,, \\ l_1 &= hn\,(x_0, y_0, z_0)\,, \\ k_2 &= hm\,(x_0 + h, y_0 + k_1, z_0 + l_1)\,, \\ l_2 &= hn\,(x_0 + h, y_0 + k_1, z_0 + l_1)\,, \end{aligned} \tag{180.13}$$

and then the updated values are

$$\begin{aligned} y(x_0 + h) &= y(x_0) + \frac{1}{2}(k_1 + k_2), \\ z(x_0 + h) &= z(x_0) + \frac{1}{2}(l_1 + l_2). \end{aligned} \tag{180.14}$$

This formula is second order accurate. See Dekker and Verwer [9] for details.

5. The Butcher array can represent all multi-linear methods for approximating differential equations. For example
   - The backward Euler method $y_{n+1} = y_n + hf(t_n + h, y_{n+1})$ has the Butcher array $(s = 1)$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

   - The trapezoidal rule $y_{n+1} = y_n + \frac{h}{2}[f(t_n + y_n) + f(t_n + h, y_{n+1})]$ has the Butcher array $(s = 2)$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

6. Pseudo Runge–Kutta methods use not only the stages of the current step, but also the stages of the previous step. For example, for the equation $y' = f(x, y)$ the method has the form:

$$\begin{aligned} y_{n+1} &= y_n + \sum_{i=1}^{s} \alpha_i K_{i,n} \\ K_{i,n} &= hf\left(x_n + m_i h, y_n + \sum_{j=1}^{s} \bar{\lambda}_{i,j} K_{j,n-1} + \sum_{j=1}^{i-1} \lambda_{i,j} K_{j,n}\right). \end{aligned}$$

See Caira *et al.* [7] for details.

7. To obtain a Runge–Kutta method with a desired order, a minimum number of stages (i.e., function evaluations) are required. From Butcher [5] we have:

desired order:	1	2	3	4	5	6	7	8
minimal number of stages:	1	2	3	4	6	7	9	11

8. Mathematica has the package `Butcher` that sets up the equations to solve for a Runge–Kutta method, as in (180.8). The method can be choosen to be explicit, implicit, or diagonally implicit. The package can also create Butcher trees.
9. Runge–Kutta methods are always sympletic; see page 780.
10. RKSUITE is a suite of Fortran codes implementing Runge–Kutta methods. See `http://www.netlib.org/ode/rksuite/`.
11. The book by Butcher [5] has a very comprehensive account of Runge–Kutta methods (it includes 96 pages of references!). See also Boyce and DiPrima [3, pages 420–423].

References

[1] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, D.C., 1964.

[2] Bogacki, P., and Shampine, L. F. Interpolating high-order Runge–Kutta formulas. *Comp. & Maths. with Appls. 20*, 3 (1990), 15–24.

[3] Boyce, W. E., and DiPrima, R. C. *Elementary Differential Equations and Boundary Value Problems*, fourth ed. John Wiley & Sons, New York, 1986.

[4] Burrage, K., and Butcher, J. C. Stability criteria for implicit Runge–Kutta methods. *SIAM J. Numer. Anal. 16*, 1 (February 1979), 30–45.

[5] Butcher, J. C. *The Numerical Analysis of Ordinary Differential Equations*. John Wiley & Sons, New York, 1987.

[6] Butcher, J. C., and Cash, J. R. Towards efficient Runge–Kutta methods for stiff systems. *SIAM J. Numer. Anal. 27*, 3 (June 1990), 753–761.

[7] Caira, R., Costabile, C., and Costabile, F. A class of pseudo Runge–Kutta methods. *BIT 30* (1990), 642–649.

[8] Collatz, L. *The Numerical Treatment of Differential Equations*. Springer–Verlag, New York, 1966.

[9] Dekker, K., and Verwer, J. G. *Stability of Runge–Kutta Methods for Stiff Nonlinear Systems*. North–Holland Publishing Co., New York, 1984.

[10] Evans, D. J., and Sanugi, B. B. A nonlinear Runge–Kutta formula for initial value problems. *SIGNUM Newsletter 22*, 3 (July 1987), 27–30.

[11] Fi, J. M. Low order practical Runge–Kutta–Nystrom methods. *Computing 38* (1987), 281–297.

[12] Kutta, W. Beitrag zur naherungsweisen integration totaler differentialgleichungen. *Zeits. Math. Phys. 46* (1901), 435–453.

[13] Runge, C. Ueber die numerische Auflosung von differentialgleichungen. *Math. Ann. 46* (1895), 167–178.

[14] Shampine, L. F. Diagnosing stiffness for Runge–Kutta methods. *SIAM 12*, 2 (March 1991), 260–272.

181. Stiff Equations*

Applicable to Stiff differential equations (i.e., equations that evolve on more than one scale).

Yields

A numerical approximation technique.

Idea

Since stiff equations evolve on different scales, the techniques used to numerically approximate the solution should change as the different scales become important. This is because the stability aspects of a numerical technique often change as the equation changes (see page 683). Consider, for example, the definition of stiffly stable on page 686—as the eigenvalues of the problem change a method may no longer be stiffly stable.

Procedure

When trying to numerically approximate the solution to a stiff differential equation, the step size used in the discretization process should be variable, becoming very small when needed. The discretization formula should also change in different regions to reflect the different type of local solution (i.e., exponential growth, exponential decay, algebraic growth, etc.)

The step size should be made as small as is needed to obtain a desired accuracy, but it should be increased whenever possible to reduce the total number of computations. The step size should not be allowed to get so large, though, that the discretization technique becomes unstable.

A good choice of step size can be determined by monitoring the change in the solution of the differential equation. For any single step, the change in the function being approximated *and* all of its derivatives should not become too large.

Example

Suppose we have the problem

$$\begin{aligned} &\frac{d^2y}{dx^2} + (1-\epsilon)\frac{dy}{dx} - \epsilon y = 0, \\ &y(0) = 2, \qquad y'(0) = \epsilon - 1, \end{aligned} \tag{181.1}$$

where ϵ is a small positive number. The solution to equation (181.1) is

$$y(x) = e^{\epsilon x} + e^{-x}, \tag{181.2}$$

which has a steep decrease from $x = 0$ to $x \simeq -\log \epsilon$ and then has a gradual increase; see figure 181.1.

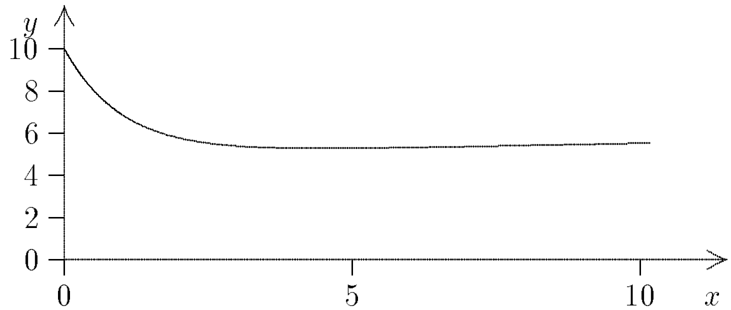

Figure 181.1: The solution to equation (181.1) is $y(x) = e^{\epsilon x} + e^{-x}$.

When using a simple discretization scheme (e.g., say, Euler's method), a small step size is required in the region from $x = 0$ to $x \simeq -\log \epsilon$ to resolve the exponential decay. After that region, however, the step size should be increased because the solution is no longer rapidly varying.

The Fortran program in program 181.1 implements this numerical idea for $\epsilon = 0.01$. It uses Euler's method and a variable step size. The parameter `TOL` determines how much the solution is allowed to change at any step. Note that the change in the solution is defined to also include the change in the value of the derivative. We have chosen `TOL` $= 0.01$.

A few lines of the output of the program are shown below

```
At T= 0.005  DELTAT= 0.0049  Y(T)= 1.9952  Exact value= 1.9952
At T= 0.317  DELTAT= 0.0049  Y(T)= 1.7307  Exact value= 1.7312
At T= 0.327  DELTAT= 0.0098  Y(T)= 1.7237  Exact value= 1.7243
At T= 1.001  DELTAT= 0.0098  Y(T)= 1.3761  Exact value= 1.3776
At T= 1.021  DELTAT= 0.0195  Y(T)= 1.3691  Exact value= 1.3707
At T= 1.685  DELTAT= 0.0195  Y(T)= 1.2005  Exact value= 1.2025
At T= 1.724  DELTAT= 0.0391  Y(T)= 1.1937  Exact value= 1.1958
At T= 2.349  DELTAT= 0.0391  Y(T)= 1.1170  Exact value= 1.1193
At T= 2.427  DELTAT= 0.0781  Y(T)= 1.1105  Exact value= 1.1129
At T= 2.974  DELTAT= 0.0781  Y(T)= 1.0788  Exact value= 1.0813
At T= 3.130  DELTAT= 0.1563  Y(T)= 1.0728  Exact value= 1.0755
At T= 3.599  DELTAT= 0.1563  Y(T)= 1.0613  Exact value= 1.0640
At T= 9.849  DELTAT= 0.3125  Y(T)= 1.1034  Exact value= 1.1036
At T=10.161  DELTAT= 0.3125  Y(T)= 1.1068  Exact value= 1.1070
```

During the program execution, the step size, `DELTAT`, has increased from 0.0049 to 0.3125. Hence, large steps were taken where the solution was not rapidly changing.

Notes

1. In the example shown, we can use the same discretization scheme throughout the region of interest—only the step size needs to be adjusted for efficient computation. In other problems, different discretization schemes will be needed in different regions.

```
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      TEND=10.0D0
      EPSLON=0.01D0
      TOL=0.01D0
      DELTAT=TEND
      OLDCHG=1.0D0
      T=0.0D0
      Y=2.0D0
      YP=EPSLON-1.D0
C Decrease the size of the time step
10    DELTAT=DELTAT/2.D0
20    IF ( DELTAT .GT. .5D0 ) GOTO 10
      CALL STEP(Y,YP,DELTAT,EPSLON,YN,YNP)
      CHANGE= DSQRT((Y-YN)**2 + (YP-YNP)**2)
      IF( CHANGE .GT. TOL        ) GOTO 10
      IF( CHANGE .GT. 2.D0*OLDCHG ) GOTO 10
C Store away the new values
      T = T + DELTAT
      Y = YN
      YP= YNP
      OLDCHG=CHANGE
      VAL=EXACT(T,EPSLON)
      WRITE(6,5) T, DELTAT, Y, VAL
5     FORMAT('  At T=',F6.3,'  DELTAT=',F7.4,
     1        '  Y(T)=',F7.4,'  Exact value=',F7.4)
C Increase the size of the time step
      DELTAT=2.D0*DELTAT
      IF( T .LT. TEND ) GOTO 20
      END
C This subroutine updates Y and Y' by Euler's method
      SUBROUTINE  STEP(Y,YP,DELTAT,EPSLON,YN,YNP)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      YN = Y  + DELTAT*( YP )
      YNP= YP + DELTAT*( EPSLON*Y - YP*(1.D0-EPSLON) )
      RETURN
      END
C This function computes the exact solution to compare against
      FUNCTION EXACT(T,EPSLON)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      EXACT=DEXP(EPSLON*T)+DEXP(-T)
      RETURN
      END
```

Program 181.1: Fortran program for stiff ODEs.

2. If the new independent variable $\tilde{x} = \epsilon x$ is introduced, then the solution in equation (181.2) may be written as $y(x) = e^{\tilde{x}} + e^{-\tilde{x}/\epsilon}$. In this representation of the solution, it is clear that there is a "boundary layer" near $\tilde{x} = 0$; see the section on boundary layers (page 590).
3. For an example of how the stability of a method may change as the solution of a differential equation evolves, see the stability analysis for Euler's method on page 732. In the example there, as the value of the positive constant c becomes smaller, the step size must also

become smaller to ensure stability.

4. Sometimes non-stiff methods can solve stiff problems, without any special difficulty except that they can be computationally expensive.
5. Changing the length of the step size leads to accurate solutions to stiff initial value ordinary differential equations and for partial differential equations that may be solved by a marching technique. For boundary value ordinary differential equations or for elliptic partial differential equations, the analogous technique is to numerically solve the equations on a non-uniform mesh. This mesh should be fine where the solution is rapidly changing, and coarse elsewhere.
6. It is *not* true that the eigenvalues of the matrix $A(t)$ in the system

$$\frac{d\mathbf{y}}{dt} = A(t)\mathbf{y} \tag{181.3}$$

will determine whether the system is stiff or not. For example, the matrix

$$A(t) = \begin{bmatrix} -1 - 9\cos^2 6t + 6\sin 12t & 12\cos^2 6t + \frac{9}{2}\sin 12t \\ -12\sin^2 6t + \frac{9}{2}\sin 12t & -1 - 9\sin^2 6t - 6\sin 12t \end{bmatrix} \tag{181.4}$$

has the constant eigenvalues -1 and -10, but the solution to equation (181.3) is

$$\mathbf{y} = C_1 e^{2t} \begin{bmatrix} \cos 6t + 2\sin 6t \\ 2\cos 6t - \sin 6t \end{bmatrix} + C_2 e^{-13t} \begin{bmatrix} \sin 6t - 2\cos 6t \\ 2\sin 6t + \cos 6t \end{bmatrix},$$

where C_1 and C_2 are arbitrary constants. Clearly the exponentials e^{-t} and e^{-10t} are not present in the solution. Also, the solution may blow up as t tends to infinity. Even so, the eigenvalues of the linearized problem are often the most useful piece of information available regarding the conditioning of the system. This example is from Dekker and Verwer [3, page 11].

7. If $\eta(t)$ is defined by $\eta = ||\mathbf{y}||^2 = \mathbf{y}^{\mathrm{H}}\mathbf{y}$, then, using equation (181.3), $\frac{d\eta}{dt} = \mathbf{y}^{\mathrm{H}}\left(A + A^{\mathrm{H}}\right)\mathbf{y}$. If $\lambda_{\max}$ represents the largest eigenvalue of $(A + A^{\mathrm{H}})$ then $\eta(t) \leq \eta_0 e^{\lambda_{\max} t}$. Hence, the eigenvalues of $(A + A^{\mathrm{H}})$ allow bounds to be determined for $\mathbf{y}(t)$. For the matrix in equation (181.4), the eigenvalues of $(A + A^{\mathrm{H}})$ are 4 and -26.
8. An equation is often realized to be stiff only after the differential equation has been numerically integrated. There are tests that can be performed during the integration procedure to determine whether the equation is stiff. See, for example, Gear [5] or Shampine [8].
9. For a recent review of software for stiff equations, see Aiken [1, Chapters 3–4, pages 70–202] or Byrne and Hindmars [2].
10. See also Gaffney [4], Miranker [6], and Petzold [7].

References

[1] Aiken, R. C. *Stiff Computation*. Oxford University Press, New York, 1985.

[2] Byrne, G. D., and Hindmarsh, A. C. Stiff ODE solvers: A review of current and coming attractions. *J. Comput. Physics 70* (1987), 1–62.

[3] Dekker, K., and Verwer, J. G. *Stability of Runge–Kutta Methods for Stiff Nonlinear Systems*. North–Holland Publishing Co., New York, 1984.

[4] Gaffney, P. W. A performance evaluation of some FORTRAN subroutines for the solution of stiff oscillatory ordinary differential equations. *ACM Trans. Math. Software 10*, 1 (March 1984), 58–72.

[5] Gear, C. W. Automatic detection and treatment of oscillatory and/or stiff ordinary differential equations. In *Numerical Integration of Differential Equations and Large Linear Systems*, J. Hinze, Ed. Springer–Verlag, New York, 1982, pp. 190–206.

[6] Miranker, W. L. *Numerical Methods for Stiff Equations*. D. Reidel Publishing Co., Boston, MA, 1981.

[7] Petzold, L. Automatic selection of methods for solving stiff and nonstiff systems of ordinary differential equations. *SIAM J. Sci. Stat. Comput. 4*, 1 (March 1983), 136–148.

[8] Shampine, L. F. Stiffness and nonstiff differential equation solvers, II: Detecting stiffness with Runge–Kutta methods. *ACM Trans. Math. Software 3*, 1 (March 1977), 44–53.

182. Integrating Stochastic Equations

Applicable to Stochastic differential equations.

Yields

A numerical approximation.

Idea

The "white Gaussian noise" term in a stochastic differential equation can be numerically approximated in many different ways.

Procedure

Suppose we have the stochastic differential equation

$$x' = b(x) + \sigma(x)n(t), \qquad x(0) = y, \tag{182.1}$$

where $n(t)$ represents white noise. There exist several numerical approximations for the quantity $x(T)$, where $T = mh$, h is a (small) time step, and T is a fixed time of order one. Three common numerical approximations of equation (182.1) are

$$\tilde{x}(t_{k+1}) = \tilde{x}(t_k) + b_k h + \sigma_k \sqrt{h}\alpha_k, \tag{182.2}$$

$$\widehat{x}(t_{k+1}) = \widehat{x}(t_k) + b_k h + \sigma_k \sqrt{h}\zeta_k, \tag{182.3}$$

$$\check{x}(t_{k+1}) = \check{x}(t_k) + \left(b - \frac{1}{2}\sigma\frac{\partial \sigma}{\partial x}\right)_k h + \sigma_k\sqrt{h}\zeta_k + \frac{1}{2}\left(\sigma\frac{\partial \sigma}{\partial x}\right)_k h\, \zeta_k^2, \tag{182.4}$$

with $\tilde{x}(0) = \widehat{x}(0) = \check{x}(0) = y$, where $t_k = kh$ and a subscript of k means evaluation at the kth point (e.g., $b_k = b(x(t_k))$). The $\{\alpha_k\}$ are independent random variables that take on the values $+1$ and -1 with probability $1/2$, while the $\{\zeta_k\}$ are independent Gaussian random variables with mean 0 and variance 1.

Each of the approximations in equations (182.2)–(182.4) have a different mean square error for a single step. If $\mathrm{E}[\cdot]$ represents the expectation operator, then

$$\begin{aligned} \mathrm{E}\left[(x(h) - \tilde{x}(h))^2\right] &= O(h), \\ \mathrm{E}\left[(x(h) - \widehat{x}(h))^2\right] &= O(h^2), \\ \mathrm{E}\left[(x(h) - \check{x}(h))^2\right] &= O(h^3). \end{aligned} \tag{182.5}$$

Hence, equation (182.4) is the most accurate if a sample of $x(T)$ is desired.

However, if the mean of a function of $x(T)$ is required, then each of the three approximations in equations (182.2)–(182.4) is first order accurate. That is, each of $\mathrm{E}[f(\tilde{x}(T))]$, $\mathrm{E}[f(\widehat{x}(T))]$, and $\mathrm{E}[f(\check{x}(T))]$ is equal to

$\mathrm{E}\left[f(x(T))\right] + O(h)$, for general functions f. This next approximation,

$$\begin{aligned}
z(t_{k+1}) &= z(t_k) + \left(b - \frac{1}{2}\sigma\frac{\partial\sigma}{\partial x}\right)_k h + \sigma_k\sqrt{h}\zeta_k + \frac{1}{2}\left(\sigma\frac{\partial\sigma}{\partial x}\right)_k h\ \zeta_k^2 \\
&\qquad + \left(\frac{1}{2}b\frac{\partial\sigma}{\partial x} + \frac{1}{2}\sigma\frac{\partial b}{\partial x} + \frac{1}{2}\frac{\partial\sigma}{\partial t} + \frac{1}{4}\sigma^2\frac{\partial^2\sigma}{\partial x^2}\right)_k h^{3/2}\zeta_k \\
&\qquad + \left(\frac{1}{2}b\frac{\partial b}{\partial x} + \frac{1}{2}\frac{\partial b}{\partial t} + \frac{1}{4}\sigma^2\frac{\partial^2 b}{\partial x^2}\right)_k h^2, \\
z(0) &= y,
\end{aligned} \tag{182.6}$$

has the better error estimate: $\mathrm{E}\left[f(z(T))\right] = \mathrm{E}\left[f(x(T))\right] + O(h^2)$. Note that, in equation (182.6), we have allowed b and σ to be functions of both t and x.

Example

Suppose we have the stochastic differential equation

$$x' = x + n(t), \qquad x(0) = 1, \tag{182.7}$$

where $n(t)$ is white noise, and we want to estimate $\mathrm{E}\left[x^2(1)\right]$. The Fokker–Planck equation corresponding to (182.7) is (see page 303)

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(xP) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(P),$$

with $P(0,x) = \delta(x-1)$. By using the method of moments (see page 568), the ordinary differential equation that describes $\mathrm{E}\left[x^2(t)\right]$ is given by

$$\frac{d}{dt}\mathrm{E}\left[x^2(t)\right] = 2\mathrm{E}\left[x^2(t)\right] + 1, \qquad \mathrm{E}\left[x^2(0)\right] = 1,$$

with the solution $\mathrm{E}\left[x^2(t)\right] = (3e^{2t}-1)/2$. Therefore, $\mathrm{E}\left[x^2(1)\right] = (3e^2 - 1)/2 \simeq 10.58$. This is the value that our numerical approximation should produce.

To implement the method in equation (182.3), the Fortran program in program 182.1 was constructed. The program takes the results of `NTRIAL` trials and averages these values together. Note that the program uses a routine called `RANDOM`, whose source code is not shown, which returns a random value uniformly distributed on the interval from 0 to 1.

A similar program was written which implemented the methods in equation (182.2) and equation (182.4). The results are indicated in table 182.1. It should be observed that the numerical results are increasingly accurate when the step size h is decreased.

Notes

1. Gaussian random variables may be generated from uniformly distributed random variables by the classical technique of Box and Muller [1]. This technique has been used in the function `ZETA` in program 182.1.

NTRIAL	h	Equation (182.2)	Equation (182.3)	Equation (182.4)
1000	0.25	8.14	8.40	11.19
1000	0.20	8.61	8.74	11.11
1000	0.10	9.62	9.30	10.59
1000	0.05	10.00	10.16	10.87
5000	0.25	8.14	8.40	11.19
5000	0.20	8.51	8.36	10.60
5000	0.10	9.46	9.35	10.59
5000	0.05	9.97	10.18	10.90

Table 182.1: Numerical comparison of different approximation techniques for equation (182.7)

```
C This program is a numerical implementation of equation (3)
      NTRIAL=1000
      H=0.05
      NTIME=20
      XINIT=1.0
      SUMX2=0.0
C Here is the integration loop
      DO 10 NSTEP=1,NTRIAL
      X=XINIT
      DO 20 K=1,NTIME
20    X=X + X*H + SQRT(H)*ZETA()
10    SUMX2=SUMX2 + X**2
      AVERAG=SUMX2/FLOAT(NTRIAL)
      WRITE(6,*) AVERAG
      END
C This function returns a gaussian random variable
      FUNCTION ZETA()
      DATA  TWOPI/6.2831853/
      Y1=RANDOM( DSEED )
      Y2=RANDOM( DSEED )
      ZETA= SQRT( -2.*ALOG(Y2) ) * COS( TWOPI*Y1 )
      RETURN
      END
```

Program 182.1: Fortran program for stochastic equation integration.

2. Because low numerical accuracy is obtained by this technique, a computer program does not need to work with extended precision arithmetic.
3. Milśhtein [9] and [10] describes equations (182.2)–(182.4) and presents a derivation of equation (182.6). He also includes a numerically fast implementation of equation (182.6) using Runge–Kutta methods.
4. Sun [16] presents a numerical method for approximating the solution to equations of the form $-(pu')' + (q + r\lambda)^2 u = f$, when p, q and r are all functions of the independent variable and both λ and f are random terms.

Differential equation	Solution
$d\mathbf{u} = A\mathbf{u}\,dt + d\mathbf{w}$	$\mathbf{u}(t) = e^{At}\mathbf{u}(0) + \int_0^t e^{A(t-\tau)}\,d\mathbf{w}(\tau)$
$dx = \beta x\,dt + \alpha x\,d\omega$	$x = e^{(\beta-(1/2)\alpha^2)t+\alpha\omega}$
$dx = \frac{1}{2}x\,dt + \sqrt{x^2-1}\,d\omega$	$x = \cosh\omega$
$dx = -(4ax^3 - 3x^2)\,dt - 2x\sqrt{x-ax^2}\,d\omega$	$x = a/(a+\omega^2)$

Table 182.2: Test problems for stochatic equation methods

5. Peterson [12] uses the test problems shown in table 182.2 to illustrate a numerical code for integrating stochastic differential equations.
6. Saito and Mitsui [14] describe 11 different numerical schemes for integrating stochastic differential equations and give stability diagrams based on the test equation $dx = \lambda x\,dt + \mu x\,d\omega$, $x(0) = 1$, whose solution is $x(t) = \exp\left\{\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu\omega(t)\right\}$.
7. Hofmann and Mathe [6] study the numerical phenomena when switching from (real) Monte-Carlo simulations to quasi-Monte-Carlo simulations (which is what computers carry out).

References

[1] Box, G. E. P., and Muller, M. E. A note on the generation of random normal deviates. *Ann. Math. Statistic 9* (1958), 610–611.

[2] Chang, C.-C. Numerical solution of stochastic differential equations with constant diffusion coefficients. *Math. of Comp. 49*, 180 (October 1987), 523–542.

[3] Drummond, I. T., Hoch, A., and Morgan, R. R. Numerical integration of stochastic differential equations with variable diffusivity. *J. Phys. A: Math. Gen. 19* (1986), 3871–3881.

[4] Golec, J., and Ladde, G. Euler-type approximation for systems of stochastic differential equations. *J. Appl. Math. Simulation 2*, 4 (1989), 239–249.

[5] Greenside, H. S., and Helfand, E. Numerical integration of stochastic differential equations - II. *The Bell System Technical Journal 60*, 8 (October 1981), 1927–1940.

[6] Hofmann, N., and Mathe, P. On quasi-Monte Carlo simulation of stochastic differential equations. *Math. of Comp. 66*, 218 (April 1997), 573–589.

[7] Janssen, R. Difference methods for stochastic differential equations with discontinuous coefficients. *Stochastics 13* (1984), 199–212.

[8] Janssen, R. Discretization of the Wiener-process in difference-methods for stochastic differential equations. *Stochastic Processes and Their Applications 18* (1984), 361–369.

[9] Milshtein, G. N. Approximate integration of stochastic differential equations. *Theory Prob. Appl. 19*, 4 (1974), 557–562.

[10] Milshtein, G. N. A method of second-order accuracy integration of stochastic differential equations. *Theory Prob. Appl. 23*, 2 (1978), 396–401.

[11] Newton, N. J. Asymptotically efficient Runge–Kutta methods for a class of Ito and Stratonovich equations. *SIAM J. Appl. Math. 51*, 2 (April 1991), 542–567.

[12] Peterson, W. P. Some experiments on numerical simulations of stochastic differential equations and a new algorithm. *J. Comput. Physics 113* (1994), 75–81.

[13] R umelin, W. Numerical treatment of stochastic differential equations. *SIAM J. Numer. Anal. 19*, 3 (June 1982), 604–613.

[14] Saito, Y., and Mitsui, T. Stability analysis of numerical schemes for stochastic differential equations. *SIAM J. Numer. Anal. 33*, 6 (December 1996), 254–2267.

[15] Spigler, R. Monte Carlo-type simulation for solving stochastic ordinary differential equations. *Math. and Computers in Simulation 29* (1987), 243–251.

[16] Sun, T.-C. A finite element method for random differential equations with random coefficients. *SIAM J. Numer. Anal. 16*, 6 (December 1979), 1019–1035.

183. Symplectic Integration

Applicable to
Hamiltonian systems.

Yields
An appropriate numerical approximation.

Idea
Hamiltonian systems have invariants that should be maintained during the numerical integration procedure.

Procedure
Consider an autonomous Hamiltonian system of the form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}. \tag{183.1}$$

The time evolution of these equations is area preserving or symplectic; equivalently, the flow conserves the two-form $d\mathbf{q} \wedge d\mathbf{p}$. A numerical method is called symplectic if, when applied to Hamiltonian problems, it generates numerical solutions that inherit the property of symplecticness. That is, the state of the system following an integration step could have been reached from that before the step by a canonical transformation.

There are two main groups of symplectic integrators. The first group consists of formulae that belong to standard families of numerical methods (e.g., Runge–Kutta methods) and just “happen” to be symplectic. These methods can be applied to general systems of differential equations. The second group consists of methods derived via generating functions. These methods cannot be applied to general systems of differential equations, not even small dissipative perturbations of Hamiltonian systems.

Procedure 1
The Runge–Kutta method with tableau

$$\begin{array}{|cccc}
a_{11} & a_{12} & \cdots & a_{1s} \\
a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
b_1 & b_2 & \ldots & b_s
\end{array}$$

(note that the usual $\{c_i\}$ do not appear because the system in equation (183.1) is autonomous) will be symplectic if the coefficients satisfy:

$$b_i a_{ij} + b_j a_{ji} - b_i b_j = 0, \qquad \text{for} \qquad 1 \leq i, j \leq s. \tag{183.2}$$

Procedure 2

We may choose to integrate the **p** equations with one Runge–Kutta scheme (using say $\{a_{ij}, b_i\}$), and the **q** equations with a different Runge–Kutta scheme (using say $\{A_{ij}, B_i\}$), with

$$\begin{array}{|cccc} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline b_1 & b_2 & \ldots & b_s \end{array} \qquad \begin{array}{|cccc} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \\ \hline B_1 & B_2 & \ldots & B_s \end{array}$$

This scheme will be symplectic if the coefficients satisfy

$$b_i A_{ij} + B_j a_{ji} - b_i B_j = 0, \qquad \text{for} \qquad 1 \le i, j \le s. \tag{183.3}$$

Example 1

A simple example of a first-order symplectic scheme for $H = p^2/2 + V(q)$ is $(q, p) \to (Q, P)$, where

$$\begin{aligned} Q &= q + (\Delta t)p, \\ P &= p - (\Delta t)\frac{\partial V}{\partial q}(q + (\Delta t)p). \end{aligned} \tag{183.4}$$

Example 2

For separable Hamiltonians (i.e., $H(\mathbf{p}, \mathbf{q}) = T(\mathbf{p}) + V(\mathbf{q})$), Candy and Rozmus [1] list the symplectic integration formulae in table 183.1. These formulae are to be used in the following fashion:

- Initial conditions: $(\mathbf{p}_0, \mathbf{q}_0)$ at $t = t_0$,
- Do for $i = 1$ to n;

  $\mathbf{p}_i = \mathbf{p}_{i-1} + b_i \mathbf{F}(\mathbf{q}_{i-1})\Delta t,$
  $\mathbf{q}_i = \mathbf{q}_{i-1} + a_i \mathbf{P}(\mathbf{p}_{i-1})\Delta t,$

- Integrated variables: $(\mathbf{p}_n, \mathbf{q}_n)$ at $t = t_0 + \Delta t$,

where $\mathbf{F}(\mathbf{q}) = -\nabla_{\mathbf{q}} V(\mathbf{q})$ and $\mathbf{P}(\mathbf{p}) = \nabla_{\mathbf{p}} T(\mathbf{p})$.

Example 3

This example illustrates what can happen if a non-symplectic method, such as forward Euler's method, is used. Consider the Hamiltoninan $H = p^2/2 + \Phi(q)$ for which the equations of motion are $\frac{dq}{dt} = p$ and $\frac{dp}{dt} = -\frac{\partial \Phi}{\partial q} = F(q)$. Integrating these equations using forward Euler results in

$$\begin{aligned} p_{n+1} &= p_n + hF(q_n) \\ q_{n+1} &= q_n + hp_n. \end{aligned} \tag{183.5}$$

There are are least three problems with this numerical scheme

Order(n)	Coefficients
1	$(a_1, b_1) = (1, 1)$
2	$(a_1, a_2, b_1, b_2) = (1/2, 1/2, 0, 1)$
3	$(a_1, a_2, a_3, b_1, b_2, b_3) = (2/3, -2/3, 1, 7/24, 3/4, -1/24)$
4	$a_1 = a_4 = (2 + 2^{1/3} + 2^{-1/3})/6$ $a_2 = a_3 = (1 - 2^{1/3} - 2^{-1/3})/6$ $b_1 = 0, b_2 = b_4 = (2 - 2^{1/3})^{-1}, b_3 = (1 - 2^{2/3})^{-1}$

Table 183.1: Symplectic integration schemes for separable Hamiltonians.

1. The Jacobian, defined by the determinant $J = \begin{vmatrix} \frac{\partial p_{n+1}}{\partial p_n} & \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} & \frac{\partial q_{n+1}}{\partial q_n} \end{vmatrix}$, is to leading order equal to $1 - h^2 F'(q_n)$. A value of $J < 1$ (or $J > 1$) leads to volume contraction (or expansion), neither of which is a property of a Hamiltonian systems.
2. The equations are not invariant to time reversal. That is, equation (183.5) can be inverted to yield

$$\begin{aligned} p_n &= p_{n+1} - hF(q_n) \\ q_n &= q_{n+1} - hp_n, \end{aligned} \tag{183.6}$$

   but this is not (183.5) with h replaced for $-h$ and n and $n+1$ interchanged.
3. The energy, defined by $E_n = p_n^2/2 + \Phi(q_n)$, is not independent of n. In fact,

$$E_{n+1} = E_n + \frac{h^2}{2}\left[F^2(q_n) - p_n^2 F'(q_n)\right] + O(h^3).$$

Notes

1. If $(\mathbf{p}^*, \mathbf{q}^*) = \psi(\mathbf{p}, \mathbf{q})$ is a variable transformation, then ψ will be area preserving if and only if the Jacobian determinant is identically unity: $\frac{\partial \mathbf{p}^*}{\partial \mathbf{p}}\frac{\partial \mathbf{q}^*}{\partial \mathbf{q}} - \frac{\partial \mathbf{p}^*}{\partial \mathbf{q}}\frac{\partial \mathbf{q}^*}{\partial \mathbf{p}} = 1$. This can be written as

$$\frac{\partial(\mathbf{p}^*, \mathbf{q}^*)}{\partial(\mathbf{p}, \mathbf{q})}^{\mathrm{T}} J \frac{\partial(\mathbf{p}^*, \mathbf{q}^*)}{\partial(\mathbf{p}, \mathbf{q})} = J \qquad \text{where} \qquad J = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

2. Symplecticness *characterizes* Hamiltonian flows; conservation of volume is a much weaker property shared by some non-Hamiltonian systems. Symplectic integrators do not in general conserve the energy (Hamiltonian) of a mechanical system.
3. It is impossible for an algorithm to simultaneously conserve the symplectic structure, the momentum map, and the Hamiltonian. Non-symplectic algorithms that conserve both momentum and energy have been studied by Simo and Wong [6].

4. The adjoint of a symplectic map, the inverse of a symplectic map, and the composition of two symplectic maps, all are symplectic.
5. A Hamiltonian system of the form $\{\dot{\mathbf{q}} = M^{-1}\mathbf{p},\ \dot{\mathbf{p}} = -\nabla F(\mathbf{q})\}$, with M a symmetric, positive definite matrix can, under the transformation $\{\mathbf{q} \mapsto M^{1/2}\mathbf{q},\ \mathbf{p} \mapsto M^{-1/2}\mathbf{p}\}$, be reduced to an equivalent system with $M = I$.
6. Zwillinger [8, pages 341–345] describes the exterior calculus in which two-forms are defined.
7. Ben Leimkuhler maintains a web page on symplectic methods; see `http://www.math.ukans.edu/~leimkuhl/symplectic.html`.

References

[1] Candy, J., and Rozmus, W. A sympletic integration algorithm for separable Hamiltonian functions. *J. Comput. Physics 92* (1991), 230–256.

[2] Channell, P. J., and Scovel, C. Symplectic integration of Hamiltonian systems. *Nonlinearity 3* (1990), 231–259.

[3] Greenspan, D. A counterexample of the usse of energy as a measure of computational accuracy. *J. Comput. Physics 91* (1990), 490–494.

[4] Qin, M., and Zhu, W. Construction of symplectic schemes for wave equations via hyperbolic functions zzzref37refzzz, zzzref38refzzz and zzzref39refzzz. *Computers Math. Applic. 26*, 8 (1993), 1–11.

[5] Sanz-Serna, J. M. Sympletic integrators for Hamiltonian problems: an overview. *Acta Numerica* (1991), 243–286.

[6] Simo, J. C., and Wong, K. K. Unconditionally stable algorithms for the orthogonal group that exactly conserve energy and momentum. *Internat. J. Numer. Methods Eng. 31* (1989), 19–52.

[7] Zhong, G., and Marsden, J. Lie–Poisson Hamilton–Jacobi theory and Lie–Poisson integrators. *Phys. Lett. A 133* (1988), 134–139.

[8] Zwillinger, D., Ed. *Standard Mathematical Tables and Formulae*, 30 ed. CRC, Boca Raton, FL, 1995.

184. Use of Wavelets

Applicable to Ordinary and partial differential equations.

Yields

A fast numerical scheme.

Idea

Using a weighted residual method with easily computed basis functions can lead to an efficient method.

Procedure

Wavelets are one set of functions that can be used with a Galerkin (weighted residual) method; see page 786. Orthogonal wavelets are defined (see Zwillinger [7, pages 663–667]) by specifying a set of parameters $\{h_k\}$ (with $h_k = 0$ if $k < 0$ or $k > n$) that satisfy

- Normalization: $\sum_{k=0}^{n} h_k = \sqrt{2}$
- Orthogonality: $\sum_k h_k h_{k-2j} = 2\delta_{0,j}$
- Accuracy p: $\sum_{k=0}^{n} (-1)^k k^j h_k = 0$ for $j = 0, \ldots, p-1$ with $p > 0$

Using these parameters, the solution to the equation

$$\phi(x) = \sqrt{2} \sum_{k=0}^{n} h_k \phi(2x - k),$$

called the *scaling function*, is guaranteed to exist. For each $j \geq 0$ and for $k = 0, 1, \ldots, 2^j$ set $\phi_{j,k} = 2^{j/2}\phi(2^j x - k)$. Define V^j to be the span of $\{\phi_{j,k}\}_{k=0}^{2^j}$. Then $V^m \supset V^{m-1} \cdots \supset V^1 \supset V^0$.

To use the Galerkin method, the dependent variable in the differential equation is projected into the space of trial functions belonging to V^m. That is, we make the approximation

$$y \approx \sum_k y_k \phi_{m,k}(x).$$

When the usual inner products are evaluated and orthogonality of the elements is used, linear algebraic equations can be obtained from a differential equation. If, at any time, a multiresolution is desired, this can be performed as a postprocessing step or as an adjunct calculation.

Notes

1. MathSoft maintains a web site containing wavelet reprints at `http://www.mathsoft.com/wavelets.html`. Specific collections of reprints are listed under “Wavelets and Ordinary Differential Equations” and “Wavelets and Partial Differential Equations.”

2. Jawerth and Sweldens [4] adapt wavelets so they become (bi)orthogonal with respect to the inner product defined by a differential operator. The stiffness matrix in the Galerkin method then becomes diagonal and can be trivially inverted. They also show how to construct an $O(N)$ algorithm for various constant and variable coefficient operators.
3. A reason to use wavelet expansions in numerical methods is that in wavelet coordinates differential operators may be preconditioned by a diagonal matrix. Moreover, a large class of operators, namely Calderón–Zygmund and pseudo-differential operators, are sparse in wavelet bases.
4. Wavelets are presently only capable of dealing with the simple boundary conditions. This is improving rapidly.
5. The wavelet corresponding to the scaling function $\phi(x)$ is the function $\psi(x) = \sqrt{2}\sum_{k=0}^{n}(-1)^k h_{n-k}\phi(2x-k)$. Using ψ we define the functions $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$; these are orthonormal and the entire collection $\{\psi_{j,k}\}_{j,k=-\infty}^{\infty}$ forms a basis for $L^2(R)$.

References

[1] Amaratunga, K., Williams, J. R., Qian, S., and Weiss, J. Wavelet–Galerkin solutions for one-dimensional partial differential equations. *Int. J. Num. Meth. Eng. 27* (1994), 2703–2716.

[2] Bacry, E., Mallat, S., and Papanicolaou, G. A wavelet based space–time adaptive numerical method for partial differential equations. *Mathematical Modelling and Numerical Analysis 26* (1992), 793–834.

[3] Enquist *et al.* Fast wavelet based algorithms for linear evolution equations. *SIAM J. Sci. Comput. 15*, 4 (July 1994).

[4] Jawerth, B., and Sweldens, W. Wavelet multiresolution analyses adapted for the fast solution of boundary value ordinary differential equations. In *Sixth Copper Mountain Conference on Multigrid Methods* (1993), N. D. Melson, T. A. Manteuffel, and S. F. McCormick, Eds., NASA Conference Publication 3224, pp. 259–273.

[5] Qian, Z., and Weiss, J. Wavelets and the numerical solution of boundary value problems. *Appl. Math. Lett. 6* (1993), 47–52.

[6] Xu, J. S., and Shann, W. C. Galerkin–wavelet methods for two-point boundary value problems. *Numer. Math. 63* (1992), 123–144.

[7] Zwillinger, D., Ed. *Standard Mathematical Tables and Formulae*, 30 ed. CRC, Boca Raton, FL, 1995.

185. Weighted Residual Methods*

Applicable to Ordinary and partial differential equations.

Yields

By introducing approximations, this method changes the numerical calculation of

- An ordinary differential equation to the numerical calculation of a set of algebraic equations
- A partial differential equation to the numerical calculation of a set of ordinary differential equations

Idea

We approximate the solution by taking a linear combination of an arbitrarily chosen set of functions. The coefficients of the functions, which may be constants or functions themselves, are unknown. We may use any of a number of schemes to find the numerical values for the unknown coefficients.

Procedure

We will illustrate the general technique via a specific example. Suppose we have the following partial differential equation to solve

$$\begin{aligned} u_t - N[u] &= 0, && \text{for } \mathbf{x} \in V, \quad t > 0, \\ u(0, \mathbf{x}) &= v(\mathbf{x}), && \text{for } \mathbf{x} \in V, \\ u(t, \mathbf{x}) &= f(t, \mathbf{x}), && \text{for } \mathbf{x} \in S, \end{aligned} \tag{185.1.a-c}$$

where $N[\cdot]$ is a differential operator in $\mathbf{x}$ and S is the boundary of V, the region in which we seek the solution.

We choose a $y(t, \mathbf{x})$ and some set of functions $\{u_i(t, \mathbf{x})\}$ with the properties

$$\begin{aligned} y(t, \mathbf{x}) &= f(t, \mathbf{x}), && \text{for } \mathbf{x} \in S, \\ u_j(t, \mathbf{x}) &= 0, && \text{for } \mathbf{x} \in S, \end{aligned}$$

and then form a trial solution by superposition

$$u_{\mathrm{T}}(t, \mathbf{x}) = y(t, \mathbf{x}) + \sum_{j=1}^{M} c_j(t) u_j(t, \mathbf{x}). \tag{185.2}$$

Note that the trial solution has been constructed in such a way that it automatically satisfies equation (185.1.c) but not equations (185.1.a) or

(185.1.b). If we use the trial solution in the original differential equation, (185.1.a), then the right-hand side will not be equal to zero but will be equal to some residual R_{E} given by

$$R_{\mathrm{E}}(u_{\mathrm{T}}) = (u_{\mathrm{T}})_t - N[u_{\mathrm{T}}]. \tag{185.3}$$

Instead of this definition of R_{E}, we might equally well have taken the square of equation (185.3). Likewise, the initial condition, equation (185.1.b), will not be satisfied, but there will be a residue R_{I} given by

$$R_{\mathrm{I}}(u_{\mathrm{T}}) = v(\mathbf{x}) - \sum_{j=1}^{M} c_j(0) u_j(0, \mathbf{x}).$$

Now, we choose M weighting functions $\{w_j(x)\}$. It is the choice of these weighting functions that defines the method. For example,

$$\begin{aligned}
\text{Galerkin:} \quad & w_j = u_j, \\
\text{Collocation:} \quad & w_j = \delta(\mathbf{x} - \mathbf{x}_j), \\
\text{least squares:} \quad & w_j = \frac{\partial R_{\mathrm{E}}(u_{\mathrm{T}})}{\partial c_j}, \\
\text{subdomain method:} \quad & w_j = \begin{cases} 1, & \text{for } x \in V_j, \\ 0, & \text{for } x \notin V_j, \end{cases}
\end{aligned} \tag{185.4}$$

where $\{\mathbf{x}_j \mid j = 1, 2, \ldots, M\}$ is a set of M points in V that must be chosen when collocation is used, and $\{V_j\}$ is a set of disjoint regions whose union is equal to V that must be chosen when the subdomain method is used.

Next, an inner product is defined by

$$(w, z) = \int_V w(\mathbf{x})\, z(\mathbf{x})\, dV, \tag{185.5}$$

or something similar. Then, finally, the unknown coefficients $\{c_j(t)\}$ will be determined from the two conditions

$$\begin{aligned}
(w_j,\, R_{\mathrm{E}}(u_{\mathrm{T}})) = 0, \qquad & \text{for } j = 1, 2, \ldots, M, \\
(w_j,\, R_{\mathrm{I}}(u_{\mathrm{T}})) = 0, \qquad & \text{for } j = 1, 2, \ldots, M.
\end{aligned} \tag{185.6.a-b}$$

The condition in equation (185.6.a) generates M simultaneous ordinary differential equations for the $\{c_j(t) \mid j = 1, 2, \ldots, M\}$, which will generally be nonlinear. The condition in equation (185.6.b) generates M simultaneous algebraic equations for $\{c_j(0) \mid j = 1, 2, \ldots, M\}$, which will generally be nonlinear.

The procedure is as follows. We solve equation (185.6.b) for the initial conditions for the $\{c_j(t)\}$. Using equation (185.6.a), we can then solve the ordinary differential equations to determine the $\{c_j(t)\}$ for all values of t. Using these values in equation (185.2), we have found an approximation to equation (185.1).

Example

Suppose we wish to approximate the solution to the equation

$$\begin{aligned} u_t &= N[u] = u^2 + u_{xx}, \qquad \text{for } 0 < x < 1, \quad t > 0 \\ u(0,x) &= \sin x = v(x), \\ u(t,0) &= 0, \\ u(t,1) &= 1. \end{aligned}$$

We choose $y(t,x) = x$ and $u_j(t,x) = \sin j\pi x$. Our trial solution then becomes the first M terms in a Fourier sine series

$$u_{\mathrm{T}}(t,x) = x + \sum_{j=1}^{M} c_j(t) \sin j\pi x.$$

Approximating $u(t,x)$ by $u_{\mathrm{T}}(t,x)$ the errors in the equation and the initial conditions are

$$\begin{aligned} R_{\mathrm{E}}(u_{\mathrm{T}}) &= \sum_{j=1}^{M} c_j'(t) \sin j\pi x - \left[x + \sum_{j=1}^{M} c_j(t) \sin j\pi x \right]^2 \\ &\qquad - \sum_{j=1}^{M} j^2 \pi^2 c_j(t) \sin(j\pi x), \\ R_{\mathrm{I}}(u_{\mathrm{T}}) &= \sin x - \sum_{j=1}^{M} c_j(0) \sin j\pi x. \end{aligned} \tag{185.7.a-b}$$

These two equations are in x and t. Ideally, we would like to have both expressions in equation (185.7) vanish identically. Because this is not possible (for all x and all t), we choose one of the four methods described in equation (185.4). Using the chosen method, we will obtain ordinary differential equations for the $\{c_j(t)\}$ and algebraic equations for the $\{c_j(0)\}$. When these equations are satisfied, the expressions in equation (185.7) will be "close" to zero.

Notes

1. It is also possible to choose the $\{u_i(t, \mathbf{x})\}$ to satisfy the differential equation (185.1) but not the boundary conditions. In this case, the integral in equation (185.5), which defines the inner product, becomes an integral over the boundary.
2. See the separate sections on collocation (page 514), least squares method (page 549), finite element method (page 734), Rayleigh–Ritz method (page 638), and wavelets (page 784).
3. Within the Galerkin framework, it is possible to generate finite elements, finite difference, and spectral methods.

4. This method can be used to change the calculation of an ordinary differential equation to the calculation of the solution of algebraic equations. The sequence of steps are the same as for partial differential equations, with the difference that both sets of equations in (185.6) will be algebraic equations. See the finite element method (page 734) for a worked example involving an ordinary differential equation.
5. See also Collatz [1, pages 408–418], Kantorovich and Krylov [5, pages 258–283], and Villadsen and Michelsen [6, Chapter 2, pages 67–95].

References

[1] Collatz, L. *The Numerical Treatment of Differential Equations.* Springer–Verlag, New York, 1966.

[2] Fletcher, C. A. J. *Computational Galerkin Methods.* Springer–Verlag, New York, 1984.

[3] Gottlieb, D., and Orszag, S. A. *Numerical Analysis of Spectral Methods: Theory and Applications.* SIAM, Philadelphia, PA, 1977.

[4] Haque, M., Baluch, M. H., and Mohsen, M. F. N. Solution of multiple point, nonlinear boundary value problems by method of weighted residuals. *Int. J. Comp. Math. 19* (1986), 69–84.

[5] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis.* Interscience Publishers, Inc., New York, 1958.

[6] Villadsen, J., and Michelsen, M. L. *Solution of Differential Equation Models by Polynomial Approximation.* Prentice–Hall, Inc., Englewood Cliffs, NJ, 1978.

186. Boundary Element Method

Applicable to Most often linear elliptic partial differential equations, often Laplace's equation. Sometimes parabolic, hyperbolic, or nonlinear elliptic equations.

Yields

An integral equation. The solution of the integral equation is used in an integral representation of the solution.

Idea

The problem of solving a partial differential equation *within* a given domain can be transformed into one solving an equivalent integral equation *on* the boundary of the domain. The unknown in the integral equation will be the "charge density" on the boundary of the domain.

Procedure

Suppose we have Laplace's equation (general linear elliptic equations have results analogous to those listed below)

$$\nabla^2 u(\mathbf{x}) = 0, \tag{186.1}$$

with the Dirichlet or Neumann data

$$u \Big|_S = f(\mathbf{x}) \quad \text{or} \quad \left.\frac{\partial u}{\partial n}\right|_S = g(\mathbf{x}), \tag{186.2.a-b}$$

where S is the boundary of the domain. Define $\psi(\mathbf{x};\mathbf{y})$ to be the free space Green's function of equation (186.1). That is, $\nabla^2\psi(\mathbf{x};\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$, where $\mathbf{y}$ is an arbitrary point inside the domain. Using Green's theorem, the solution to equation (186.1) and equation (186.2) can be represented in any of the following forms:

$$u(\mathbf{x}) = \int_S \sigma(\mathbf{z})\psi(\mathbf{x};\mathbf{z})\, d\mathbf{z}, \tag{186.3}$$

$$u(\mathbf{x}) = \int_S \mu(\mathbf{z})\frac{\partial\psi(\mathbf{x};\mathbf{z})}{\partial n}\, d\mathbf{z}, \tag{186.4}$$

$$u(\mathbf{x}) = \int_S \left[\eta(\mathbf{z})\psi(\mathbf{x};\mathbf{z}) + \zeta(\mathbf{z})\frac{\partial\psi(\mathbf{x};\mathbf{z})}{\partial n}\right]\, d\mathbf{z}. \tag{186.5}$$

In these equations, $\sigma(\mathbf{z})$ and $\eta(\mathbf{z})$ represent surface densities of the "single-layer" potential, $\mu(\mathbf{z})$ and $\zeta(\mathbf{z})$ represent the surface densities of the "double-layer" potential, $\mathbf{z}$ represents a point on the boundary, and n represents the outward pointing normal. If $\sigma(\mathbf{z})$, $\mu(\mathbf{z})$, or $\eta(\mathbf{z})$ and $\zeta(\mathbf{z})$ were known, then $u(\mathbf{x})$ could be computed via one of the above three equations. Note there is not a unique way to represent the solution by equation (186.5); there is a "degree of freedom" in this formulation that may be used for other purposes.

It turns out that the single-layer potential is continuous across the boundary S, whereas the double-layer potential has a jump of $\mu(\mathbf{y})$. This is because, as $\mathbf{x}$ tends to the boundary point $\mathbf{P}$ from the inside of the domain,

$$u(\mathbf{P}) = -\frac{1}{2}\mu(\mathbf{P}) + \int_S \mu(\mathbf{z}) \frac{\partial \psi(\mathbf{P};\mathbf{z})}{\partial n}\, d\mathbf{z}. \tag{186.6}$$

Using equation (186.6), a variety of boundary integral equations may be obtained.

For example, using equation (186.4) to represent the solution to the Dirichlet problem, if we allow the point $\mathbf{x}$ to approach the boundary, we determine from equation (186.6) that

$$f(\mathbf{y}) = -\frac{1}{2}\mu(\mathbf{y}) + \int_S \mu(\mathbf{z}) \frac{\partial \psi(\mathbf{z};\mathbf{y})}{\partial n}\, d\mathbf{z}.$$

This Fredholm integral equation of the second kind can, in principle, be solved for $\mu(\mathbf{y})$. After $\mu(\mathbf{y})$ is obtained, the value of $u(\mathbf{x})$ may be computed from equation (186.4).

If equation (186.3) had been used to represent the solution of the Neumann problem, then, after finding the normal derivative of equation (186.3), the following integral equation for $\sigma(\mathbf{y})$ results

$$g(\mathbf{y}) = -\frac{1}{2}\sigma(\mathbf{y}) + \int_S \sigma(\mathbf{z}) \frac{\partial \psi(\mathbf{z};\mathbf{y})}{\partial n}\, d\mathbf{z}.$$

After $\sigma(\mathbf{y})$ is obtained by solving the above integral equation, the value of $u(\mathbf{x})$ may be computed from equation (186.3).

Example

Consider Laplace's equation in the upper half plane, $\nabla^2 u = 0$ for $-\infty < x < \infty$ and $0 < y$, with the boundary conditions

$$\begin{aligned} u_y(x,0) &= 0 \qquad && -\infty < x < 0, \\ u_y(x,0) - k u(x,0) &= 0 \qquad && 0 < x < \infty, \end{aligned}$$

where k is a constant. The Green's function, $\nabla^2 \psi = \delta(x-\xi)\delta(y-\eta)$, in the upper half plane is

$$\psi(x,y;\xi,\eta) = -\frac{1}{2\pi}\log\sqrt{(x-\xi)^2 + (y-\eta)^2} - \frac{1}{2\pi}\log\sqrt{(x-\xi)^2 + (y+\eta)^2},$$

so that, on $y = 0$, we have $\psi(x, 0; \xi, \eta) = -\frac{1}{2\pi} \log\left((x-\xi)^2 + \eta^2\right)$. Now equation (186.3) can be simplified to $u(\mathbf{x}) = -\int_S \frac{\partial u(\mathbf{z})}{\partial n} \psi(\mathbf{x}; \mathbf{z})\, d\mathbf{z}$. Using the known values of u_n and ψ in this expression, we find

$$u(\xi, \eta) = \frac{k}{2\pi} \int_0^\infty u(x, 0) \log\left((x-\xi)^2 + \eta^2\right)\, dx. \tag{186.7}$$

If we define $\phi(x) = u(x, 0)$, then evaluation of equation (186.7) at $\eta = 0$ results in

$$\phi(\xi) = \frac{k}{\pi} \int_0^\infty \phi(x) \log|x - \xi|\, dx.$$

After this integral equation is solved for $\phi(x)$, the solution is given by equation (186.7).

Notes

1. Representing the solution in the form of equation (186.5) would be appropriate if the boundary conditions were mixed.
2. This technique has also been applied to the biharmonic equation in several applications. See Ingham and Kelmanson [7] for details.
3. After the boundary integral equation has been formulated, it is often solved numerically. Some numerical techniques for these equations can be found in Banerjee and Butterfield [1]. In practice one finds that the solution to the original elliptic equation could have been determined by solving a large sparse matrix system, while the boundary element method often requires that a smaller, dense, matrix system be solved to determine the potential. A worked example is shown in Lapidus and Pinder [8, pages 461–481].
4. The principle advantage of the reformulation in this section is that the dimensionality of the problem is reduced. As in the above example, a two-dimensional partial differential equation becomes a one-dimensional integral equation.
5. For problems in infinite domains, the behavior at infinity is (usually) automatically included in the boundary element formulation. Hence, there is no need for a "remote" boundary simulating an infinite distance. See Margulies [9].
6. The boundary element method has also been applied to parabolic equations; see Duran *et al.* [5] or Zamani [11]. It has also been applied to some hyperbolic equations; see Brebbia [4, Chapter 12, pages 191–199]. For an application to nonlinear elliptic equations, see Ingham and Kelmanson [7, Chapter 4].
7. The boundary element method and the finite element method have several features in common. See Brebbia [4, Chapter 9, pages 141–158] for a general account of the similarities and differences.
8. The presentation here has been for the *indirect* boundary element method. In this formulation, an integral equation for the potential

must be solved and then the solution to the original equation is given by an integral. It is also possible to directly determine an integral equation whose solution also satisfies the original equation. This is called the *direct* boundary element method. For example, given Laplace's equation, $\nabla^2\phi = 0$, if we define the Green's function $G(\mathbf{x}; \mathbf{y})$ by $\nabla^2 G = \delta(\mathbf{x} - \mathbf{y})$, then by Green's theorem

$$\begin{aligned}\frac{1}{2}\phi(\mathbf{y}) &= \int (G\nabla^2\phi - \phi\nabla^2 G)\, dV \\ &= \int \left(G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n} \right) dS.\end{aligned}$$

This integral equation can be solved directly for ϕ.

9. See Garabedian [6, Section 9.3, pages 334–348].

References

[1] Banerjee, R., and Butterfeld, P. K. *Boundary Element Methods in Engineering Science*. McGraw–Hill Book Company, New York, 1981.

[2] Brebbia, C. A. *Boundary Element Techniques in Computer Aided Engineering*. Martinus Nijhoff Publishers, Boston, 1984.

[3] Brebbia, C. A., Ed. *Topics in Boundary Element Research. Volume 1: Basic Principles and Applications*. Springer–Verlag, New York, 1984.

[4] Brebbia, C. A. *Topics in Boundary Element Research. Volume 2: Time-Dependent and Vibration Problems*. Springer–Verlag, New York, 1985.

[5] Duran, D., Cross, M., and Lewis, B. A. A preliminary analysis of boundary element methods applied to parabolic partial differential equations. In *New Developments in Boundary Element Methods*, C. A. Brebbia, Ed. Butterworth–Heinemann, 1981, pp. 179–190.

[6] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[7] Ingham, D. B., and Kelmanson, M. A. *Boundary Integral Equation Analyses of Singular, Potential, and Biharmonic Problems*. Springer–Verlag, New York, 1984.

[8] Lapidus, L., and Pinder, G. F. *Numerical Solution of Partial Differential Equations in Science and Engineering*. John Wiley & Sons, New York, 1982.

[9] Margulies, M. Exact treatment of the exterior problem in the combined FEM–BEM. In *New Developments in Boundary Element Methods*, C. A. Brebbia, Ed. Butterworths, London, England, 1980, pp. 43–64.

[10] Wardle, L. An introduction to the boundary element method. In *Computational Techniques for Differential Equations*, J. Noye, Ed. North–Holland Publishing Co., New York, 1984, pp. 525–549.

[11] Zamani, N. Some remarks on the use of the boundary element method in transient heat conduction problems. *Mathematics and Computers in Simulation 27* (1985), 61–64.

187. Differential Quadrature

Applicable to Nonlinear partial differential equations, a single equation, or a system. Most often, partial differential equations in two independent variables.

Yields

A system of ordinary differential equations whose solution approximates the solution of the original partial differential equation(s).

Idea

All of the derivatives with respect to one or more of the independent variables are replaced by a sum involving the dependent variable.

Procedure

To illustrate the general technique, we show how it works on a class of partial differential equations. Suppose we have the partial differential equation for $u(t,x)$

$$\begin{aligned} u_t &= g(t,x,u,u_x,u_{xx}), \\ u(0,x) &= h(x), \end{aligned} \tag{187.1}$$

on $t > 0$, $-\infty < x < \infty$. Instead of solving equation (187.1) for all values of x, we choose a finite set of x values at which the solution will be determined, say $\mathcal{S} = \{x_j \mid j = 1, \dots, N\}$. We now presume that the first derivatives with respect to x, at the points in $\mathcal{S}$, can be written as a linear combination of the values in $\mathcal{S}$. That is,

$$u_x(t,x_i) \simeq \sum_{j=1}^{N} a_{ij} u(t,x_j). \tag{187.2}$$

Viewing equation (187.2) as the linear transformation $u_x = Au$, it seems natural to approximate $u_{xx} = Au_x = A^2u$, or

$$u_{xx}(t,x_i) \simeq \sum_{k=1}^{N}\sum_{j=1}^{N} a_{ik}a_{kj} u(t,x_j). \tag{187.3}$$

Utilizing equations (187.2) and (187.3) in equation (187.1) results in the system of ordinary differential equations

$$\begin{aligned} u_t^i &= g\left(t, x_i, u^i, \sum_{j=1}^{N} a_{ij}u^j, \sum_{k=1}^{N}\sum_{j=1}^{N} a_{ik}a_{kj}u^j\right), \\ u^i(0) &= h(x_i), \end{aligned}$$

for $i = 1, \ldots, N$, where $u^i(t) := u(t, x_i)$. These initial value ordinary differential equations may be integrated numerically by any scheme.

Note that this method is similar to the method of lines (see page 831), except that the a_{ij} are *not* chosen in such a way that equation (187.2) represents a finite difference approximation to the derivative. The a_{ij} are instead chosen so that equation (187.2) is exact for all polynomials of degree less than or equal to $N-1$. That is, the a_{ij} satisfy the linear system

$$k\,(x_i)^{k-1} = \sum_{j=1}^{N} a_{ij}\,(x_j)^k\,. \tag{187.4}$$

for $k = 1, 2, \ldots, N$.

Example

We choose to numerically approximate the solution to the nonlinear partial differential equation

$$\begin{aligned} u_t &= u u_x, \\ u(0,x) &= 0.2x^2, \end{aligned}$$

which has the exact solution $u = 0.2(x+ut)^2$, or

$$u(t,x) = \frac{[1-(0.4)tx] - \sqrt{1-(0.8)tx}}{(0.4)t^2}.$$

The program shown in program 187.1 uses twenty x values in the interval from 0 to 1. Note that the source code for the linear equation solver (`LSOLVE`) is not shown. Some results of the program are shown next:

```
The time is now:    0.5000
Here is the approximate solution at this time value:
   0.0005    0.0020    0.0046    0.0083    0.0132    0.0192    0.0264
   0.0348    0.0446    0.0556    0.0681    0.0820    0.0974    0.1143
   0.1328    0.1530    0.1750    0.1985    0.2241    0.2620
Here is the exact solution at this time value:
   0.0005    0.0020    0.0046    0.0083    0.0132    0.0192    0.0264
   0.0349    0.0446    0.0557    0.0682    0.0822    0.0977    0.1147
   0.1334    0.1538    0.1760    0.2000    0.2260    0.2540

The time is now:    0.7500
Here is the approximate solution at this time value:
   0.0005    0.0021    0.0047    0.0085    0.0135    0.0198    0.0274
   0.0365    0.0470    0.0591    0.0729    0.0885    0.1060    0.1255
   0.1471    0.1712    0.1977    0.2255    0.2687    0.5368
Here is the exact solution at this time value:
   0.0005    0.0021    0.0047    0.0085    0.0135    0.0198    0.0275
   0.0365    0.0471    0.0593    0.0732    0.0889    0.1066    0.1263
   0.1484    0.1728    0.2000    0.2301    0.2634    0.3002
```

At $t = 0.75$, with the last value shown excluded, the relative error in the approximate solution is not more than 4%.

```
      DIMENSION X(50),U(50),UNEW(50),A(50,50),CORECT(50)
      DIMENSION SAVE(50,50),COEFF(50,50),RHS(50),NROW(50),SOLN(50)
C Set up the parameter values
      N=20
      TIME=0
      DELTAT=0.05
      NSTEP=15
C Set up the X points
      DO 10 J=1,N
10    X(J)=FLOAT(J)/FLOAT(N)
C Set up the coefficient matrix
      DO 20 K=1,N
      DO 20 J=1,N
20    SAVE(K,J)=X(J)**K
C For each I, determine A_[IJ]  by solving a system of equations
      DO 40 I=1,N
      DO 30 K=1,N
      RHS(K)=K*X(I)**(K-1)
      DO 30 J=1,N
30    COEFF(J,K)=SAVE(J,K)
      CALL LSOLVE(N,COEFF,SOLN,RHS,NROW,IFSING,50)
      IF( IFSING .NE. 1 ) STOP
      DO 40 J=1,N
40    A(I,J)=SOLN(J)
C Set up the initial conditions
      DO 50 J=1,N
50    U(J)=U0( X(J) )
C This is the loop in time
      DO 100 LOOPT=1,NSTEP
      TIME=TIME + DELTAT
      WRITE(6,5) TIME
C Iterate each one of the equations one time step
      DO 70 J=1,N
      SUM=0
      DO 60 K=1,N
60    SUM=SUM + A(J,K)*U(K)
70    UNEW(J)= U(J) + DELTAT * U(J) * SUM
      DO 80 J=1,N
80    U(J)=UNEW(J)
C Write out the approximate answer, and then the exact answer
      WRITE(6,*) ' Here is the approximate solution at this time value:'
      WRITE(6,15) (U(J), J=1,N)
      DO 90 J=1,N
90    CORECT(J)=EXACT(TIME, X(J) )
      WRITE(6,*) ' Here is the exact solution at this time value:'
100   WRITE(6,15) (CORECT(J), J=1,N)
5     FORMAT(' The time is now:',F10.4)
15    FORMAT( 30( 1X, 7(F9.4,1X) / ) )
      END
C This function has the initial conditions
      FUNCTION U0(X)
      U0=0.2*X**2
      RETURN
      END
C This function has the exact solution
      FUNCTION EXACT(T,X)
      TEMP=( 1.0 - (0.4)*T*X ) - SQRT( 1.0 - (0.8)*T*X )
      EXACT=TEMP / ( (0.4)*T**2 )
      RETURN
      END
```

Program 187.1: Fortran program for differential quadrature.

Notes

1. Note that the coefficient matrix in equation (187.4) is a Vandermonde matrix.
2. It is not clear that having the x values uniformly spaced produces the most accurate results. In Bellman *et al.* [1] the x values are chosen to be the roots of Legendre polynomials.
3. In Bellman *et al.* [1], a simple error analysis is performed. It is shown, for example, that the error in equation (187.2) is less than $Kh^{N-1}/(N-1)!$ if the mesh has a uniform spacing of h and if $|u^{(N)}(x)| \leq K$ in the domain of interest.
4. In Civan and Sliepcevich [3] a weighted sum of terms (similar to the approximation in equation (187.2)) is used to approximate the

second derivative terms (such as in equation (187.3)). This reduces the computational complexity of the coding.

References

[1] Bellman, R., Kashef, B. G., and Casti, J. Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations. *J. Comput. Physics 10* (1972), 40–52.

[2] Civan, F., and Sliepcevich, C. M. Solution of the Poisson equation by differential quadratures. *Internat. J. Numer. Methods Eng. 19* (1983), 711–724.

[3] Civan, F., and Sliepcevich, C. M. Differential quadrature for multidimensional problems. *J. Math. Anal. Appl. 101* (1984), 423–443.

[4] Civan, F., and Sliepcevich, C. M. On the solution of the Thomas–Fermi equation by differential quadrature. *J. Comput. Physics 56* (1984), 343–348.

[5] Naadimuthu, G., Bellman, R., Wang, K. M., and Lee, E. S. Differential quadrature and partial differential equations: Some numerical results. *J. Math. Anal. Appl. 98* (1984), 220–235.

188. Domain Decomposition

Applicable to Elliptic second order partial differential equations in non-regularly shaped domains.

Yields

An iterative solution procedure.

Idea

If the geometric domain in which a partial differential equation is to be solved can be written as the union of two (or more) regularly shaped domains, then it may be possible to write a recurrence relation for the solution.

Procedure

Suppose we wish to numerically approximate the solution to the elliptic equation

$$N[u] = F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = 0 \qquad (188.1)$$

in the domain $B = B_1 \cup B_2$ (see figure 188.1). We presume this is a Dirichlet problem, with the initial data, $f(x,y)$, given on the boundary of B.

Define the part of the boundary of B_1 (∂B_1) that is also a boundary of B to be α; the rest of the B_1 boundary of B_1 will be denoted by $\overline{\alpha}$. Likewise, define the part of the boundary of B_2 (∂B_2) that is also a boundary of B to be β; the rest of the B_2 boundary of B_2 will be denoted by $\overline{\beta}$.

The solution procedure is to first solve equation (188.1) only in B_1. Then, using this solution, we solve equation (188.1) only in the domain B_2. This is used to find a new solution of equation (188.1) in B_1, and then the process is repeated.

Initially, the data on the arc $\overline{\alpha}$ are chosen so that the data on ∂B_1 are piecewise continuous. That is, let $u_1(x,y)$ be the solution of equation (188.1) with the boundary conditions

$$u_1(x,y) = \begin{cases} f(x,y) & \text{on } \alpha, \\ \phi(x,y) & \text{on } \overline{\alpha}, \end{cases}$$

where $\phi(x,y)$ can be chosen in many different ways. After $u_1(x,y)$ is determined, let $v_1(x,y)$ be the solution of equation (188.1) with the boundary conditions

$$v_1(x,y) = \begin{cases} f(x,y) & \text{on } \beta, \\ u_1(x,y) & \text{on } \overline{\beta}. \end{cases}$$

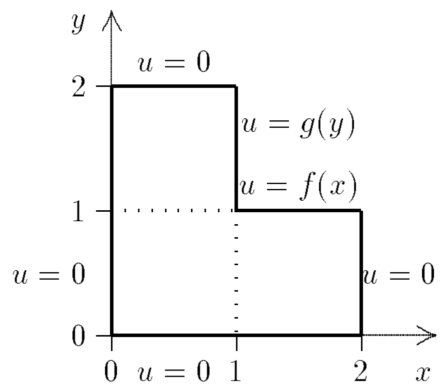

Figure 188.1: The domain for equation (188.1).

Then an iterative sequence of solutions to equation (188.1) is formed, $\{u_k(x,y),\, v_k(x,y) \mid k = 1, 2, \dots\}$ with

$$u_k(x,y) = \begin{cases} f(x,y) & \text{on } \alpha, \\ v_{k-1}(x,y) & \text{on } \overline{\alpha}, \end{cases}$$

$$v_k(x,y) = \begin{cases} f(x,y) & \text{on } \beta, \\ u_k(x,y) & \text{on } \overline{\beta}. \end{cases}$$

Under fairly general conditions, these functions will converge to the solution of equation (188.1). That is, the limiting $u_k(x,y)$ will be the solution to equation (188.1) in the region B_1, whereas the limiting $v_k(x,y)$ will be the solution to equation (188.1) in the region B_2.

In Kantorovich and Krylov [6, Chapter 7, pages 616–670], five assumptions are given that are required to assure the convergence of the above sequences. They are

1. Equation (188.1), with its boundary conditions, has a unique solution.
2. If $F[u^*] = F[u] = 0$, and $u^* > u$ on the boundary of the domain, then $u^* > u$ everywhere in the domain.
3. Within the domain, the solution to equation (188.1) is bounded by the values of u on the boundary of the domain.
4. A convergent sequence of uniformly bounded solutions to equation (188.1) converges to a solution to equation (188.1).
5. The boundary data are, at least, piecewise continuous.

Generally, non-pathological examples should satisfy these conditions.

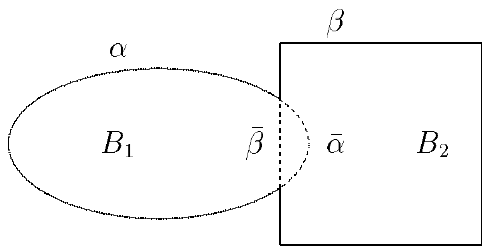

Figure 188.2: The domain for equation (188.2).

Example

Suppose we want to solve Laplace's equation in the L-shaped region shown in figure 188.2. For brevity, we define the following portions of the boundary

$$\Gamma_1 = \{x = 2, 0 \le y \le 1\} \cup \{0 \le x \le 2, y = 0\} \cup \{x = 0, 0 \le y \le 1\},$$
$$\Gamma_2 = \{x = 0, 0 \le y \le 2\} \cup \{0 \le x \le 1, y = 0\} \cup \{0 \le x \le 1, y = 2\}.$$

Then, the mathematical problem we wish to solve is

$$\begin{aligned} \nabla^2 u &= 0, & & \\ u &= 0, & &\text{on } \Gamma_1, \\ u &= 0, & &\text{on } \Gamma_2, \\ u &= f(x), & &\text{on } \{1 \le x \le 2, y = 1\}, \\ u &= g(y), & &\text{on } \{x = 1, 1 \le y \le 2\}. \end{aligned} \tag{188.2}$$

For this example, we break up the original domain into two rectangles, one vertical and one horizontal; the overlap region being the unit square. We start with

$$\begin{aligned} \nabla^2 u_1 &= 0, & & \\ u_1 &= 0, & &\text{on } \Gamma_1, \\ u_1 &= f(x), & &\text{on } \{1 \le x \le 2, y = 1\}, \\ u_1 &= \phi(x), & &\text{on } \{0 \le x \le 1, y = 1\}. \end{aligned} \tag{188.3}$$

Then, our iteration sequence becomes

$$\begin{aligned} \nabla^2 v_k &= 0, & & \\ v_k &= 0, & &\text{on } \Gamma_2, \\ v_k &= u_{k-1}(1, y), & &\text{on } \{x = 1, 0 \le y \le 1\}, \\ v_k &= g(y), & &\text{on } \{x = 1, 1 \le y \le 2\}, \end{aligned} \tag{188.4}$$

for $k = 1, 2, \dots$, whereas

$$\begin{aligned} \nabla^2 u_k &= 0, \\ u_k &= 0, && \text{on } \Gamma_1, \\ u_k &= f(x), && \text{on } \{1 \le x \le 2, y = 1\}, \\ u_k &= v_k(x,1), && \text{on } \{0 \le x \le 1, y = 1\}, \end{aligned} \tag{188.5}$$

for $k = 2, 3, \dots$.

In this case, because of the simple geometry, we can analytically write the solution to equation (188.4) and equation (188.5) by the use of Fourier transforms (see page 350). Note first, if we define $f_n(x) = u_n(x,1) = \sum_{k=1}^{\infty} f_{nk} \sin k\pi x$, then $u_n(x,y) = \displaystyle\sum_{k=1}^{\infty} \frac{f_{nk}}{\sinh(k\pi/2)} \sinh k\pi y \sin k\pi x$. Similarly, if we define the expansion $g_n(x) = v_n(1,y) = \sum_{k=1}^{\infty} g_{nk} \sin k\pi y$, then we obtain the result $v_n(x,y) = \displaystyle\sum_{k=1}^{\infty} \frac{g_{nk}}{\sinh(k\pi/2)} \sinh k\pi x \sin k\pi y$. Using these expansions in equations (188.4) and (188.5), we can readily determine that

$$\begin{aligned} f_{nk} &= B_k + \sum_{s=1}^{\infty} A_{ks} g_{n-1,s}, \\ g_{nk} &= C_k + \sum_{s=1}^{\infty} A_{ks} f_{n-1,s}, \end{aligned} \tag{188.6}$$

where

$$\begin{aligned} B_k &= \int_1^2 f(x) \sin(k\pi x/2)\, dx, \\ C_k &= \int_1^2 g(y) \sin(k\pi y/2)\, dy, \\ A_{ks} &= \frac{2}{\pi} \frac{1}{s^2 + k^2} \left[s \sin\left(\frac{k\pi}{2}\right) \cosh\left(\frac{s\pi}{2}\right) - k \cos\left(\frac{k\pi}{2}\right) \sinh\left(\frac{s\pi}{2}\right) \right]. \end{aligned}$$

In practice, the two recurrence relations in equation (188.6) would be iterated until a stationary value was obtained.

Notes

1. This method is usually implemented numerically, with little analysis done on the equations. For the above example, equations (188.3)–(188.5) would be approximated numerically by an elliptic equation package.
2. This method also works for coupled systems of elliptic equations. For two unknowns, a guess is made for one of the unknowns, and one of the equations is used to solve for the other unknown. Knowing this

second unknown, the first unknown is approximated numerically by the other equation, and the process is repeated. See Rice and Boisvert [9, pages 121–135] for some examples.

3. The procedure illustrated in this section is called *Schwarz's method*, it is only one of several different domain decomposition methods (see Glowinski *et al.* [5]).
4. In Chan *et al.* [3] it is shown that the convergence rate of the Schwarz alternating procedure, for general second-order elliptic equations, is independent of the aspect ratio for L-shaped, T-shaped, and C-shaped domains.
5. This technique works very well with parallel computers (see page 755), as the numerical problem on each domain can be solved by a separate processor; see Quarteroni [8].

References

[1] Canuto, C., and Funaro, D. The Schwarz algorithm for spectral methods. *SIAM J. Numer. Anal. 25*, 1 (February 1988), 24–40.

[2] Chan, T. F., Glowinski, R., Periaux, J., and Widlund, O. B., Eds. *Domain Decomposition Methods*. SIAM, Philadelphia, PA, 1989.

[3] Chan, T. F., Hou, T. Y., and Lions, P. L. Geometry related convergence results for domain decomposition algorithms. *SIAM J. Numer. Anal. 28*, 2 (April 1991), 378–391.

[4] Ehrlich, L. W. The numerical Schwarz alternating procedure and SOR. *SIAM J. Sci. Stat. Comput. 7*, 3 (July 1986), 989–993.

[5] Glowinski, R., Golub, G., Meurant, G., and Periaux, J., Eds. *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*. SIAM, Philadelphia, PA, 1988.

[6] Kantorovich, L. V., and Krylov, V. I. *Approximate Methods of Higher Analysis*. Interscience Publishers, Inc., New York, 1958.

[7] Meier, U. Two parallel SOR variants of the Schwarz alternating procedure. *Parallel Comp. 3*, 3 (1986), 205–215.

[8] Quarteroni, A. Domain decomposition and parallel processing for the numerical solution of partial differential equations. *Surv. Math. Ind. 1* (1991), 75–118.

[9] Rice, J. R., and Boisvert, R. F. *Solving Elliptic Problems Using ELLPACK*. Springer–Verlag, New York, 1985.

[10] Tang, W. P. Generalized Schwarz splittings. *SIAM J. Sci. Stat. Comput. 13*, 2 (1992), 573–595.

189. Elliptic Equations: Finite Differences

Applicable to Elliptic partial differential equations.

Yields

A numerical approximation of the solution.

Idea

By use of finite differences, a simultaneous system of equations may be determined. The solution of this algebraic system (which is often a linear system of equations) yields a numerical approximation to the differential equation.

Procedure

The method is simply to use finite differences everywhere and solve the resulting set of simultaneous equations. Because elliptic equations are boundary value problems, the solution at all points in the domain must be determined simultaneously.

We choose to illustrate the method on a second order elliptic equation of the form

$$\alpha u_{xx} + \beta u_{yy} = f(x, y, u, u_x, u_y), \tag{189.1}$$

where α and β are functions of x and y. We suppose that equation (189.1) applies inside a rectangle with $a \le x \le A$, $b \le y \le B$ and that the boundary conditions for equation (189.1) are

$$u(x, y) = \begin{cases} f(y), & \text{on } x = a, \\ g(y), & \text{on } x = A, \\ h(x), & \text{on } y = B, \end{cases} \tag{189.2}$$

$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} + u^3 = j(x), \quad \text{on} \quad y = b, \tag{189.3}$$

where $\{f, g, h, j\}$ are all known functions.

We first define a grid that fills the geometric domain (see page 675). For the rectangular geometry given, we choose a rectangular grid with an x spacing of h and a y spacing of k (where $h = (A - a)/(N - 1)$, and $k = (B - b)/(M - 1)$). Here, $N(M)$ is the number of grid points in the x (y) direction (see figure 189.1). Let the numerical approximation to $u(x, y)$ be given by v_{ij} (i.e., $v_{ij} \simeq u(a + ih, b + jk)$). We can then choose virtually any finite difference approximation to the derivatives appearing in equation (189.1). For instance, one second order approximation to equation (189.1)

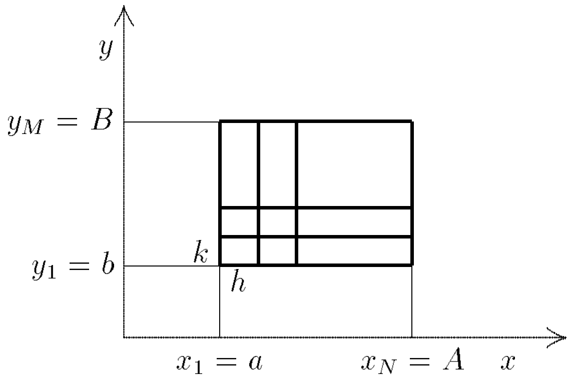

Figure 189.1: The numerical grid on which the problem is to be solved.

would be

$$\begin{aligned}\alpha_{ij}\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{h^2}&+\beta_{ij}\frac{v_{i,j+1}-2v_{i,j}+v_{i,j-1}}{k^2}\\ &= f\left(a+ih, b+jk, v_{ij}, \frac{v_{i+1,j}-v_{i-1,j}}{2h}, \frac{v_{i,j+1}-v_{i,j-1}}{2k}\right).\end{aligned} \tag{189.4}$$

For each i and j, equation (189.4) represents an algebraic equation among the $\{v_{ij}\}$. Now the boundary conditions must be incorporated. The boundary conditions in equation (189.2) can be written simply as

$$\begin{aligned} v_{0,j} &= f(b+jk), &\text{for}\quad j=1,2,\dots,M,\\ v_{N,j} &= g(b+jk), &\text{for}\quad j=1,2,\dots,M,\\ v_{i,m} &= h(a+ih), &\text{for}\quad i=1,2,\dots,N.\end{aligned} \tag{189.5}$$

The boundary condition in equation (189.3) can be written as

$$\frac{v_{i,1}-v_{i,0}}{k}+\frac{v_{i+1,0}-v_{i,0}}{h}+(v_{i,j})^3 = j(a+ih)\quad\text{for}\quad i=1,2,\dots,N. \tag{189.6}$$

If equation (189.4) is evaluated for $j = 1, 2, \dots, M$ and $i = 1, 2, \dots, N$, and equation (189.5) and equation (189.6) are included, there results a simultaneous system of equations for the $\{v_{ij}\}$. There are as many equations as there are unknowns. This system may then be solved numerically.

If the original elliptic equation (189.1) and the boundary conditions are linear in the independent variable, then the resulting system of equations will be linear. For this example, equation (189.6) is not linear (note the $(v_{i,j})^3$ term) because there is a u^3 term in equation (189.3). The most common type of elliptic systems have linear equations and linear boundary

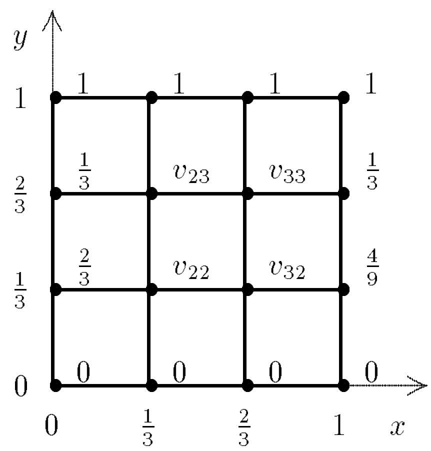

Figure 189.2: The grid on which equation (189.8) is solved.

conditions. For this type of elliptic system, a standard linear equation solver may be used. If the system of linear equations is too large to solve directly, an iterative method may be used (see page 816).

Example

Suppose we have the linear elliptic equation

$$(x+1)u_{xx} + (y+1)^2 u_{yy} = 1 + u, \tag{189.7}$$

on $0 \le x \le 1$, $0 \le y \le 1$ with

$$\begin{aligned} u(0,y) &= y, \quad u(1,y) = y^2, \\ u(x,0) &= 0, \quad u(x,1) = 1. \end{aligned} \tag{189.8}$$

If we choose $M = N = 4$ (so that $h = k = 1/3$), then there are 16 points $\{v_{ij} \mid 1 \le i \le 4, 1 \le j \le 4\}$ at which to determine an approximation to $u(x,y)$. The points $\{v_{ij} \mid i = 1 \text{ or } i = 4 \text{ or } j = 1 \text{ or } j = 4\}$ are determined directly by the boundary conditions in equation (189.8). Hence, the only unknowns that need to be determined are $\{v_{22}, v_{23}, v_{32}, v_{33}\}$; see figure 189.2. If equation (189.7) is discretized as

$$\begin{aligned} &(ih+1)\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} \\ &+ (jk+1)^2 \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{k^2} = 1 + v_{ij}, \end{aligned}$$

```
EQUATION.                  (X+1)*UXX+(Y+1)**2*UYY=1.0 + U
BOUNDARY.                  U=Y        ON   X=0.0
                           U=Y**2     ON   X=1.0
                           U=0.0      ON   Y=0.0
                           U=1.0      ON   Y=1.0
GRID.                      4 X POINTS
                           4 Y POINTS
DISCRETIZATION.            5 POINT STAR
SOLUTION.                  LINPACK BAND
OUTPUT.                    TABLE(U)
                           PLOT(U)
END.
```

Program 189.1: ELLPACK program for an elliptic problem.

then the equations for the unknown $\{v_{ij}\}$ may be written as

$$\begin{bmatrix} 57/9 & -16/9 & -4/3 & 0 \\ -25/9 & 25/3 & 0 & -4/3 \\ -5/3 & 0 & 7 & -16/9 \\ 0 & -5/3 & -25/9 & 9 \end{bmatrix} \begin{bmatrix} v_{22} \\ v_{23} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} -5/9 \\ 24/9 \\ -22/27 \\ 68/27 \end{bmatrix}. \tag{189.9}$$

The equations in equation (189.9) have the solution $v_{22} \simeq 0.0131$, $v_{23} \simeq 0.3791$, $v_{32} \simeq -0.0265$, $v_{33} \simeq 0.3419$.

Notes

1. The computer language ELLPACK (see Rice and Boisvert [4] is a high-level language that allows linear elliptic problems in two or three dimensions to be entered in an elementary way. The program generates a discretization scheme based on user preference. The geometry in two dimensions can be nearly arbitrary, with holes and other cutouts available. For example, to solve the problem in the example, the *entire* ELLPACK program is given in program 189.1. The use of ELLPACK for two and three-dimensional problems is highly recommended. There is also a version of ELLPACK available for parallel computation.
2. Picard iteration (see page 618), Newton's method (see page 578), and Monte-Carlo methods (see page 810) can also be used to numerically approximate the solution to elliptic problems.
3. Boisvert and Sweet [2] have a comprehensive listing of currently available software for solving elliptic problems.
4. See Twizell [5, pages 42–80].

References

[1] Birkoff, G., and Lynch, R. *Numerical Solution of Elliptic Problems.* SIAM, Philadelphia, PA, 1984.

[2] Boisvert, R. F., and Sweet, R. A. Mathematical software for elliptic boundary value problems. In *Sources and Development of Mathematical*

Software, W. R. Cowell, Ed. Prentice–Hall, Inc., Englewood Cliffs, NJ, 1984, pp. 200–263.

[3] Dyksen, W. R., and Ribbens, C. J. Interactive ELLPACK: An interactive problem–solving environment for elliptic partial differential equations. *ACM Trans. Math. Software 13*, 2 (June 1987), 113–132.

[4] Rice, J. R., and Boisvert, R. F. *Solving Elliptic Problems Using ELLPACK*. Springer–Verlag, New York, 1985.

[5] Twizell, E. H. *Computational Methods of Partial Differential Equations*. Ellis Horwood Limited, Chichester, England, 1984.

190. Elliptic Equations: Monte-Carlo Method

Applicable to Linear elliptic partial differential equations.

Yields

A numerical approximation to the solution of a linear elliptic partial differential equation at a single point.

Idea

Simulation of the motion of a random particle may be used to approximate the solution to linear elliptic equations.

Procedure

The steps for this method are straightforward. First, we give an overview; then, a more detailed presentation.

First, approximate the given elliptic partial differential equation by a finite difference method. Rewrite the finite difference formula as a recursive function for the value of the unknown at any given point. Then interpret this recursive formula as a set of transition probabilities that determine the motion of a random particle.

Now, write a computer program that will allow many (say K) particles to wander randomly around the domain of interest, based on the transition probabilities found from the difference formula. Simulate particles one at a time, with every particle starting off at the same point (say the point $\mathbf{z}$).

- If the boundary data are of the Dirichlet type (i.e., the value of the unknown is prescribed on the boundary), then, when a particle reaches the boundary, stop that particle and store away the value on the boundary. Begin another particle at the point $\mathbf{z}$.
- If the boundary data are not of the Dirichlet type (i.e., Neumann or mixed boundary conditions) then, when the particles reach the boundary, they will be given a finite probability to leave the boundary, and re-enter the domain of the problem. If the particle leaves the boundary, then continue the iteration process. If it does not leave the boundary, then the value at the boundary is stored away, and a new particle is started off at the point $\mathbf{z}$.

The simulation is finished after all K particles have been absorbed into the boundary. If the original elliptic equation was homogeneous, then an approximation to the solution, at the point $\mathbf{z}$, will be given by the average of all the values obtained (recall that when the particles stop at the boundary they obtain a value).

If the given elliptic equation was not homogeneous, then equation (190.4) shows how to obtain an approximation to the solution. In this latter case, the approximate value of the solution depends on the entire history of the particle.

In more detail, we now describe how the technique may be applied to the linear second order elliptical partial differential equation

$$L[u] = F(x, y), \tag{190.1}$$

with the operator $L[\cdot]$ defined by

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y,$$

where $\{A, B, C, D, E\}$ are all functions of $\{x, y\}$. The operator $L[\cdot]$ may be discretized to yield the approximation

$$\begin{aligned} L[u] \simeq \ & A_{i,j}\left[\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta x)^2}\right] \\ & + 2B_{i,j}\left[\frac{v_{i+1,j+1} - v_{i,j+1} - v_{i+1,j} + v_{i,j}}{(\Delta x)(\Delta y)}\right] \\ & + C_{i,j}\left[\frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{(\Delta y)^2}\right] \\ & + D_{i,j}\left[\frac{v_{i+1,j} - v_{i,j}}{\Delta x}\right] + E_{i,j}\left[\frac{v_{i,j+1} - v_{i,j}}{\Delta y}\right], \end{aligned} \tag{190.2}$$

where $x_i = x_0 + i(\Delta x)$, $y_j = y_0 + j(\Delta y)$, $v_{i,j} = u(x_i, y_j)$, and a subscript of i, j means an evaluation at the point (x_i, y_j). If the $\{\Gamma_{\cdot,\cdot}\}$ and $Q_{i,j}$ are defined by

$$\begin{aligned} \Gamma_{i+1,j+1} &= \left[\frac{2B_{i,j}}{(\Delta x)(\Delta y)}\right], \\ \Gamma_{i+1,j} &= \left[\frac{A_{i,j}}{(\Delta x)^2} - \frac{2B_{i,j}}{(\Delta x)(\Delta y)} + \frac{D_{i,j}}{\Delta x}\right], \\ \Gamma_{i,j+1} &= \left[\frac{C_{i,j}}{(\Delta y)^2} - \frac{2B_{i,j}}{(\Delta x)(\Delta y)} + \frac{E_{i,j}}{\Delta y}\right], \\ \Gamma_{i-1,j} &= \left[\frac{A_{i,j}}{(\Delta x)^2}\right], \\ \Gamma_{i,j-1} &= \left[\frac{C_{i,j}}{(\Delta x)^2}\right], \\ Q_{i,j} &= \left[\frac{2A_{i,j}}{(\Delta x)^2} - \frac{2B_{i,j}}{(\Delta x)(\Delta y)} + \frac{2C_{i,j}}{(\Delta y)^2} + \frac{D_{i,j}}{\Delta x} + \frac{E_{i,j}}{\Delta y}\right], \end{aligned}$$

then, using equation (190.2), equation (190.1) may approximated as

$$\begin{aligned} Q_{i,j}v_{i,j} = \Gamma_{i+1,j}v_{i+1,j} + \Gamma_{i+1,j+1}v_{i+1,j+1} + \Gamma_{i,j+1}v_{i,j+1} \\ + \Gamma_{i-1,j}v_{i-1,j} + \Gamma_{i,j-1}v_{i,j-1} - F_{i,j}, \end{aligned}$$

or dividing through by $Q_{i,j}$ and defining $p_{i,j} = \Gamma_{i,j}/Q_{i,j}$,

$$\begin{aligned} v_{i,j} = p_{i+1,j}v_{i+1,j} + p_{i+1,j+1}v_{i+1,j+1} + p_{i,j+1}v_{i,j+1} \\ + p_{i-1,j}v_{i-1,j} + p_{i,j-1}v_{i,j-1} - \frac{F_{i,j}}{Q_{i,j}}. \end{aligned} \tag{190.3}$$

Because the operator $L[\cdot]$ has been presumed to be elliptic, then Δx and Δy may be chosen small enough so that each of the p's are positive. The p's also add up to one, and we interpret them as probabilities of taking a step in a specified direction. Specifically, equation (190.3) is interpreted as follows: If a particle is at position (i, j) at step N, then,

- With probability $p_{i,j+1}$, the particle goes to $(i, j+1)$ at step $N+1$.
- With probability $p_{i,j-1}$, the particle goes to $(i, j-1)$ at step $N+1$.
- With probability $p_{i+1,j}$, the particle goes to $(i+1, j)$ at step $N+1$.
- With probability $p_{i-1,j}$, the particle goes to $(i-1, j)$ at step $N+1$.
- With probability $p_{i+1,j+1}$, the particle goes to $(i+1, j+1)$ at step $N+1$.

Now, suppose a particle starts at the point $P_0 = \mathbf{z}$ and undergoes a random walk according to the above prescription. After, say, m steps it will hit the boundary. Suppose that the sequence of points that this particle visits is $(P_0, P_1, P_2, \ldots, P_m)$. Then, an unbiased estimator of the value of $u(\mathbf{z})$ for the following elliptic problem

$$\begin{aligned} L[u] &= F(x, y), && \text{for all points } x, y \text{ in the domain } R, \\ u &= \phi(x, y), && \text{for all points } x, y \text{ on the boundary } \partial R, \end{aligned}$$

is given by

$$u(\mathbf{z}) \simeq \phi(P_m) - \sum_{j=0}^{m} \frac{F(P_j)}{Q(P_j)}. \tag{190.4}$$

In practice, several random paths will be taken, and the average taken to estimate $u(\mathbf{z})$. That is,

$$u(\mathbf{z}) \simeq \frac{1}{K} \sum_{k=1}^{K} \left\{ \phi(P^k_{m_k}) - \sum_{j=0}^{m_k} \frac{F(P^k_j)}{Q(P^k_j)} \right\}, \tag{190.5}$$

where $(P^k_0, P^k_1, \ldots, P^k_{m_k})$, represents the path taken by the kth random particle.

Example

Suppose we wish to numerically approximate the solution to Laplace's equation in an annulus. We have $\nabla^2 u = 0$ for $u(r, \theta)$ with the boundary

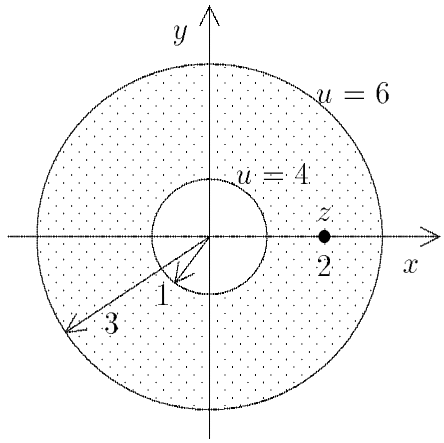

Figure 190.1: The domain in which Laplace's equation is solved.

conditions $u(1,\theta) = 4$ and $u(3,\theta) = 6$. (See figure 190.1.) We will approximate the value of $u(\mathbf{z})$, when $\mathbf{z} = (r = 2, \theta = 0)$. The exact solution for this problem is $u(r) = 4 + 2\log r/\log 3$, so that $u(\mathbf{z}) = 4 + \log 2/\log 3 \simeq 5.261$. To approximate the solution to this problem numerically, we will follow the steps outlined above. We will use the rectangular variables x and y, rather than the polar coordinate variables r and θ.

Using a standard second order approximation to the Laplacian, we find

$$\nabla^2 u \simeq \frac{v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{i,j}}{h^2} = 0, \tag{190.6}$$

where $v_{i,j} = u(hi, hj)$ and $h \ll 1$. Equation (190.6) can be manipulated into

$$v_{i,j} = \frac{v_{i+1,j}}{4} + \frac{v_{i-1,j}}{4} + \frac{v_{i,j+1}}{4} + \frac{v_{i,j-1}}{4}. \tag{190.7}$$

We interpret equation (190.7) probabilistically as follows: If a particle is at position (i,j) at step N, then,

- With probability 1/4, the particle goes to $(i, j+1)$ at step $N+1$.
- With probability 1/4, the particle goes to $(i, j-1)$ at step $N+1$.
- With probability 1/4, the particle goes to $(i+1, j)$ at step $N+1$.
- With probability 1/4, the particle goes to $(i-1, j)$ at step $N+1$.

Program 190.1 has Fortran code that was used to simulate the motion of the particles according to the above probability law. The output of that program is given below for $u(r = 2, \theta = 0)$. As more points are taken, the approximation becomes better.

```
      STEP=0.10
      SUM=0.0
      DO 10 IWALK=1,10000
      X=2.0
      Y=0.0
20    X=X + SIGN(STEP, RANDOM(DUMMY)-0.5 )
      Y=Y + SIGN(STEP, RANDOM(DUMMY)-0.5 )
      R=SQRT( X**2+Y**2 )
      IF( R.LT.3 .AND. R.GT.1 ) GOTO 20
C When a particle hits the boundary, sum the value
      IF( R .LE. 1) SUM=SUM+4
      IF( R .GE. 3) SUM=SUM+6
      IF( MOD(IWALK,1000) .NE. 0 ) GOTO 10
      APPROX=SUM/FLOAT(IWALK)
      WRITE(6,5) IWALK,APPROX
5     FORMAT(' Number of particles=',I5,'   Approximation=',F7.4)
10    CONTINUE
      END
```

Program 190.1: Fortran program for Monte-Carlo method applied to elliptic equations.

```
Number of particles= 1000   Approximation= 5.3440
Number of particles= 2000   Approximation= 5.3330
Number of particles= 3000   Approximation= 5.3200
Number of particles= 4000   Approximation= 5.3195
Number of particles= 6000   Approximation= 5.3030
Number of particles= 7000   Approximation= 5.3023
Number of particles= 8000   Approximation= 5.2958
Number of particles= 9000   Approximation= 5.2944
Number of particles=10000   Approximation= 5.2914
```

Note that the program uses a routine called `RANDOM`, whose source code is not given, which returns a random value uniformly distributed on the interval from zero to one.

Notes

1. If further accuracy is required, the options are
   (a) Increase the number of random particles.
   (b) Make the mesh discretization finer (i.e., reduce h).
   (c) Do both of the above.

   If the number of random particles is not very large, then (b) will not help much; and if the mesh is very coarse then, (a) will not help much. Generally, the variance of the answer (a measure of the "scatter") decreases as the number of trials to the minus one half power.
2. Because low numerical accuracy is obtained by this technique, a computer program does not need to work with extended precision arithmetic.
3. Sadeh and Franklin [8] present several worked examples. See also Farlow [5, pages 346–352] and Lattès [6, Chapter 8, pages 158–190].

References

[1] Bhavsar, V. C., and Gujar, U. G. VLSI algorithms for Monte Carlo solutions of partial differential equations. In *Advances in Computer Methods For Partial Differential Equations*, R. Vichnevetsky and R. S. Stepleman, Eds., IMACS. North–Holland Publishing Co., New York, 1984.

[2] Bhavsar, V. C., and Isaac, J. R. Design and analysis of parallel Monte Carlo algorithms. *SIAM J. Sci. Stat. Comput. 8*, 1 (1987), 573–595.

[3] Booth, T. E. Exact Monte Carlo solution of elliptic partial differential equations. *J. Comput. Physics 30* (1981), 396–404.

[4] Booth, T. E. Regional Monte Carlo solution of elliptic partial differential equations. *J. Comput. Physics 47* (1982), 281–290.

[5] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[6] Lattes, R. *Methods of Resolution for Selected Boundary Problems in Mathematical Physics*. Gordon and Breach, New York, 1969.

[7] Marshall, G. Monte Carlo methods for the solution of nonlinear partial differential equations. *Comput. Physics Comm. 56* (1989), 51–61.

[8] Sadeh, E., and Franklin, M. A. Monte Carlo solution of partial differential equations by special purpose digital computer. *IEEE Transactions on Computers C-23*, 4 (April 1974), 389–397.

[9] Vrbik, J. Monte Carlo simulation of the general elliptic operator. *J. Phys. A: Math. Gen. 20* (1987), 2693–2697.

```
      STEP=0.10
      SUM=0.0
      DO 10 IWALK=1,10000
      X=2.0
      Y=0.0
20    X=X + SIGN(STEP, RANDOM(DUMMY)-0.5 )
      Y=Y + SIGN(STEP, RANDOM(DUMMY)-0.5 )
      R=SQRT( X**2+Y**2 )
      IF( R.LT.3 .AND. R.GT.1 ) GOTO 20
C When a particle hits the boundary, sum the value
      IF( R .LE. 1) SUM=SUM+4
      IF( R .GE. 3) SUM=SUM+6
      IF( MOD(IWALK,1000) .NE. 0 ) GOTO 10
      APPROX=SUM/FLOAT(IWALK)
      WRITE(6,5) IWALK,APPROX
5     FORMAT(' Number of particles=',I5,'   Approximation=',F7.4)
10    CONTINUE
      END
```

Program 190.1: Fortran program for Monte-Carlo method applied to elliptic equations.

```
Number of particles= 1000   Approximation= 5.3440
Number of particles= 2000   Approximation= 5.3330
Number of particles= 3000   Approximation= 5.3200
Number of particles= 4000   Approximation= 5.3195
Number of particles= 6000   Approximation= 5.3030
Number of particles= 7000   Approximation= 5.3023
Number of particles= 8000   Approximation= 5.2958
Number of particles= 9000   Approximation= 5.2944
Number of particles=10000   Approximation= 5.2914
```

Note that the program uses a routine called `RANDOM`, whose source code is not given, which returns a random value uniformly distributed on the interval from zero to one.

Notes

1. If further accuracy is required, the options are
   (a) Increase the number of random particles.
   (b) Make the mesh discretization finer (i.e., reduce h).
   (c) Do both of the above.

   If the number of random particles is not very large, then (b) will not help much; and if the mesh is very coarse then, (a) will not help much. Generally, the variance of the answer (a measure of the "scatter") decreases as the number of trials to the minus one half power.
2. Because low numerical accuracy is obtained by this technique, a computer program does not need to work with extended precision arithmetic.
3. Sadeh and Franklin [8] present several worked examples. See also Farlow [5, pages 346–352] and Lattès [6, Chapter 8, pages 158–190].

References

[1] Bhavsar, V. C., and Gujar, U. G. VLSI algorithms for Monte Carlo solutions of partial differential equations. In *Advances in Computer Methods For Partial Differential Equations*, R. Vichnevetsky and R. S. Stepleman, Eds., IMACS. North–Holland Publishing Co., New York, 1984.

[2] Bhavsar, V. C., and Isaac, J. R. Design and analysis of parallel Monte Carlo algorithms. *SIAM J. Sci. Stat. Comput. 8*, 1 (1987), 573–595.

[3] Booth, T. E. Exact Monte Carlo solution of elliptic partial differential equations. *J. Comput. Physics 30* (1981), 396–404.

[4] Booth, T. E. Regional Monte Carlo solution of elliptic partial differential equations. *J. Comput. Physics 47* (1982), 281–290.

[5] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[6] Lattes, R. *Methods of Resolution for Selected Boundary Problems in Mathematical Physics*. Gordon and Breach, New York, 1969.

[7] Marshall, G. Monte Carlo methods for the solution of nonlinear partial differential equations. *Comput. Physics Comm. 56* (1989), 51–61.

[8] Sadeh, E., and Franklin, M. A. Monte Carlo solution of partial differential equations by special purpose digital computer. *IEEE Transactions on Computers C-23*, 4 (April 1974), 389–397.

[9] Vrbik, J. Monte Carlo simulation of the general elliptic operator. *J. Phys. A: Math. Gen. 20* (1987), 2693–2697.

191. Elliptic Equations: Relaxation

Applicable to Elliptic equations, most often Laplace's equations.

Yields

A numerical approximation to the solution.

Idea

The finite difference scheme for an elliptic equation can be interpreted as a local condition on the value of the solution. This local condition leads naturally to an iterative numerical procedure.

Procedure

Given an elliptic equation, choose a finite difference formula to approximate the equation on a grid in the domain of interest. This formula can be manipulated into a relation between the value of the unknown at a point and the values of the unknown at neighboring points. Hence, once values have been assigned to every point in the grid, this formula can be used iteratively to update the value at every point. When the values stops changing (to some specified precision), an approximate solution has been found.

Example

Suppose we want to approximate the solution to Laplace's equation on a square

$$\begin{aligned} \nabla^2 u &= 0, \\ u(0,y) = 0, \quad u(1,y) &= 0, \quad \text{for } 0 \le y \le 1, \\ u(x,0) = 0, \quad u(x,1) &= 1, \quad \text{for } 0 < x < 1. \end{aligned} \tag{191.1.a-c}$$

If we choose a grid with a uniform x spacing of Δx and a uniform y spacing of Δy, then equation (191.1.a) can be discretized as

$$\frac{1}{(\Delta x)^2}\left(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\right) + \frac{1}{(\Delta y)^2}\left(v_{i,j+1} - 2v_{i,j} + v_{i,j-1}\right) = 0, \tag{191.2}$$

where $v_{i,j} = u(i\,\Delta x, j\,\Delta y)$, for $i = 1, 2, \dots, 1/\Delta x$ and $j = 1, 2, \dots, 1/\Delta y$. Equation (191.2) can be manipulated to yield

$$v_{i,j} = \frac{1}{2(1+\lambda^2)}\left(\lambda^2\left(v_{i,j+1} + v_{i,j-1}\right) + v_{i+1,j} + v_{i-1,j}\right), \tag{191.3}$$

```
      REAL*8 V(6,6)
C Initializel the grid
      DO 10 I=2,5
      DO 10 J=2,5
10    V(I,J)=0.25D0
C Here is the boundary data
      DO 20 K=1,6
      V(K,1)=0.0D0
      V(K,6)=1.0D0
      V(1,K)=0.0D0
20    V(6,K)=0.0D0
C Perform the iterations
      EPS=0.0001D0
      NUM=0
40    NUM=NUM+1
      IFLAG=0
      DO 30 I=2,5
      DO 30 J=2,5
      VNEW= ( V(I+1,J) + V(I-1,J) + V(I,J+1) + V(I,J-1) ) / 4.D0
      IF( DABS(V(I,J)-VNEW) .GT. EPS )  IFLAG=1
30    V(I,J)=VNEW
C Determine if another iteration is required
      IF( IFLAG .EQ. 1 ) GOTO 40
      WRITE(6,5) NUM
5     FORMAT(' Number of iterations required:', I5)
      DO 50 J=1,6
50    WRITE(6,15) (V(I,7-J),I=1,6)
15    FORMAT( 7(1X,F9.4) )
      END
```

Program 191.1: Fortran program for relaxation method.

where $\lambda = \Delta y/\Delta x$. From equation (191.3), we see that $v_{i,j}$ can be replaced by a weighted average of the values at the neighboring points. Note that this is only true for points interior to the boundary.

The numerical technique is this: Initialize the values at all points in the grid (one common choice is to use the averaged value of the independent variable on the boundary); then systematically apply equation (191.3) to all the grid points until the solution converges. In theory, the points to be updated can be chosen in any order. In practice, some choices result in faster convergence.

The Fortran code in program 191.1 carries out this prescription for the problem in equation (191.1). In this program, $h = 0.2$, $k = 0.2$, and the number of iterative updates required before the approximation did not change more than `EPS` (set to 0.0001) was 16. The output from the computer program is given below

```
Number of iterations required:   16
   0.        1.0000    1.0000    1.0000    1.0000    0.
   0.        0.4545    0.5946    0.5946    0.4545    0.
   0.        0.2234    0.3294    0.3294    0.2234    0.
   0.        0.1097    0.1703    0.1703    0.1098    0.
   0.        0.0454    0.0718    0.0719    0.0454    0.
```

```
0.        0.        0.        0.        0.        0.
```

The symmetry of the solution was to be expected.

The exact solution to equation (191.1) can be determined by separation of variables (see page 487). The solution is

$$u(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty} \sin\left[(2n-1)\pi x\right] \frac{\sinh\left[(2n-1)\pi y\right]}{\sinh\left[(2n-1)\pi\right]}.$$

As can be verified, the numerical approximation is accurate to two decimal places.

Notes

1. The equations in (191.2) can be combined into one large system of linear equations, and then iterative methods can be applied to this system. Each different iterative method for a linear system can be interpreted as a relaxation method directly on the grid values.
2. Depending on the equation to which this method is applied and on the ordering in which the updated values are obtained, this technique is called

   - Alternating-direction-implicit (ADI) method
   - Gauss–Seidel or successive iteration scheme
   - Jacobi or simultaneous iteration scheme
   - Liebmann's method.
   - Successive over-relaxation (SOR) method

   In the ADI method, the finite difference approximation to Laplace's equation may be written

   $$\nabla^2 u \simeq \frac{u_{i,j-1}^{(2n)} - 2u_{i,j}^{(2n)} + u_{i,j+1}^{(2n)}}{(\Delta x)^2} + \frac{u_{i-1,j}^{(2n+1)} - 2u_{i,j}^{(2n+1)} + u_{i+1,j}^{(2n+1)}}{(\Delta y)^2} = 0.$$

   The superscripts indicate the iteration number. Hence, the updating is done alternately by rows and columns in the array of values.
3. This method, when applied to the elliptic equation $L[u] = 0$, can be interpreted as an approximation to the solution of the parabolic equation $u_t = L[u]$. By iterating until the solution stops changing, the steady-state solution of the parabolic equation is obtained. This interpretation allows error estimates to be obtained for this method (see Garabedian [4]).
4. See also Farlow [2, pages 304–305], Garabedian [3, pages 485–492], Gerald and Wheatley [4, pages 412–417], Isaacson and Keller [5, pages 463–478], and Smith [6, Chapter 5, pages 239–330].

References

[1] Chan, T. F., and Elman, H. C. Fourier analysis of iterative methods for elliptic problems. *SIAM Review 31*, 1 (March 1989), 20–49.

[2] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[3] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[4] Gerald, C. F., and Wheatley, P. O. *Applied Numerical Analysis*. Addison–Wesley Publishing Co., Reading, MA, 1984.

[5] Isaacson, E., and Keller, H. B. *Analysis of Numerical Methods*. John Wiley & Sons, New York, 1966.

[6] Smith, R. D. *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, third ed. Clarendon Press, Oxford, England, 1985.

[7] Vega-Fernandez, J. M., Duque-Carrillo, J. F., and Pe na-Bernal, J. J. A new way for solving Laplace's problem (the predictor jump method). *J. Math. Physics 26*, 3 (March 1985), 416–419.

192. Hyperbolic Equations: Method of Characteristics

Applicable to A single hyperbolic equation or a system of hyperbolic equations.

Yields

A numerical approximation scheme.

Idea

The method of characteristics (see page 432) can be used directly to create a numerical scheme to integrate hyperbolic equations.

Procedure

To simplify the analysis, we will illustrate the method on the second order hyperbolic partial differential equation

$$au_{xx} + bu_{xy} + cu_{yy} + d = 0. \tag{192.1}$$

In equation (192.1), the functions $\{a, b, c, d\}$ are assumed to depend on $\{x, y, u, u_x, u_y\}$. With the usual definitions of $p = u_x$ and $q = u_y$, equation (192.1) may be rewritten as the system of equations

$$\begin{aligned} E_1 &:= ap_x + bp_y + cq_y + d = 0, \\ E_2 &:= p_y - q_x = 0. \end{aligned}$$

If we define $E = E_1 + \lambda E_2$, then E may be written as

$$E = [ap_x + (\lambda + b)p_y] + (cq_y - \lambda q_x) + d = 0.$$

This, in turn, may be written as

$$E = \frac{d}{ds}(p + \mu q) + \left(d - q\frac{d\mu}{ds}\right) = 0, \tag{192.2}$$

along the curve defined parametrically by

$$\frac{dx}{ds} = a = -\frac{\lambda}{\mu}, \qquad \frac{dy}{ds} = \lambda + b = \frac{c}{\mu}, \tag{192.3}$$

if such a curve exists. For consistency in the equations in (192.3), we must choose μ to satisfy $a\mu^2 - b\mu + c = 0$; that is,

$$\mu_{1,2} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{192.4}$$

Define $\{\mu_1, \mu_2\}$ to be the distinct real roots given in equation (192.4) (if the roots are not distinct and real, then equation (192.1) is not hyperbolic), and define $\lambda_i = -a\mu_i$. Then equations (192.2) and (192.3) can be written as

$$\begin{aligned} \frac{d}{ds}\left(p+\mu_1 q\right) &= -\left(d - q\frac{d\mu_1}{ds}\right) \quad \text{on the curve } C_1, \\ \frac{d}{ds}\left(p+\mu_2 q\right) &= -\left(d - q\frac{d\mu_2}{ds}\right) \quad \text{on the curve } C_2, \end{aligned} \tag{192.5}$$

where the characteristic curves C_1 and C_2 are defined by

$$\begin{aligned} \text{on } C_1: \quad \frac{dx}{ds} &= a, \quad \frac{dy}{ds} = \lambda_1 + b, \\ \text{on } C_2: \quad \frac{dx}{ds} &= a, \quad \frac{dy}{ds} = \lambda_2 + b. \end{aligned} \tag{192.6.a-b}$$

These two characteristics curves have slopes that vary from point to point and are generally not orthogonal. Knowing $\{a, b, c, d\}$ allows us to determine $\{\mu_1, \mu_2\}$ and so $\{\lambda_1, \lambda_2\}$ can also be determined. Therefore, the characteristics curves can be calculated numerically.

Now, if $k_1 := p + \mu_1 q$ and $k_2 := p + \mu_2 q$ were known at some common point R (these values arise naturally from equation (192.5)), then $p(R)$ and $q(R)$ can be found by inverting these relations; that is

$$\begin{aligned} q(R) &= \frac{k_1 - k_2}{\mu_1 - \mu_2}, \\ p(R) &= \frac{\mu_1 k_1 - \mu_2 k_2}{\mu_1 - \mu_2}. \end{aligned} \tag{192.7}$$

The numerical procedure is now a straightforward application of the method of characteristics. First, the characteristic curves in equation (192.6) are identified, at some point, by determining μ_i and λ_i from equation (192.4). Then the equations for k_1 and k_2 (from equation (192.5)) are integrated a short distance along the characteristics. From the values of k_1 and k_2, values for p and q may be determined from equation (192.7). Finally, knowing p and q, the value of $u(x, y)$ can be determined. In more detail,

1. Given values at the points P and Q (see figure 192.1.a), we will determine the values of all the variables at the new point R.
2. Using equation (192.6), determine R by integrating along characteristic C_1 from P and along characteristic C_2 from Q until the curves intersect.
3. Using equation (192.5), integrate $k_1 = p + \mu_1 q$ from P to R and integrate $k_2 = p + \mu_2 q$ from Q to R. Knowing $\{k_1, k_2\}$ and $\{\mu_1, \mu_2\}$ at R allows $q(R)$ and $p(R)$ to be obtained from equation (192.7).

193. Hyperbolic Equations: Finite Differences

Applicable to Hyperbolic partial differential equations.

Yields

A numerical approximation scheme.

Idea

Finite differences can be used directly to numerically approximate the solution of a hyperbolic partial differential equation.

Procedure

The technique is to replace all of the derivatives appearing in the given hyperbolic partial differential equation by finite difference approximations. By rearranging the terms in this new equation, an explicit recurrence formula can generally be obtained.

A stability analysis can be performed on this recurrence relation to determine the step sizes that will ensure convergence of the numerical approximation to the true solution. A frequent problem encountered with this method is having enough starting values to begin iterating the recurrence relation. Starting values can generally be obtained by performing manipulations of the original equation.

Example

The hyperbolic equation

$$u_{tt} - \alpha^2 u_{xx} = 0, \tag{193.1}$$

on the interval $0 < x < L$, for $t > 0$, with the initial and boundary conditions

$$\begin{aligned} u(0,t) &= u(L,t) = 0, \\ u(x,0) &= f(x), \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \end{aligned} \tag{193.2}$$

can be numerically approximated directly by finite differences.

We choose a uniform grid of $M+1$ points in the x direction (i.e., $x_i = ih$ for $i = 0, 1, 2, \dots, M$ with $h = L/M$). We choose the step length in the t variable to be k and define $t_j = jk$. We also choose to use the following centered difference formulas for u_{xx} and u_{tt}

$$\begin{aligned} u_{tt}(x_i, t_j) &= \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2}, \\ u_{xx}(x_i, t_j) &= \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}. \end{aligned} \tag{193.3}$$

Each of these formulae is second order accurate. If we define $w_{i,j} = u(x_i, t_j)$, then using (193.3) in equation (193.1) results in

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2} - \alpha^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} = 0.$$

This last equation can be solved for $w_{i,j+1}$ to define the recurrence relation

$$w_{i,j+1} = 2(1 - \lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}, \tag{193.4}$$

for $i = 1, 2, \ldots, (M-1)$ and $j = 1, 2, \ldots$, where $\lambda = \alpha k/h$. The initial conditions and boundary conditions, from equation (193.2), can be represented as

$$\begin{aligned} w_{0,j} &= w_{M,j} = 0, && j = 1, 2, \ldots, \\ w_{i,0} &= f(x_i), && i = 1, 2, \ldots, M. \end{aligned} \tag{193.5}$$

Now comes the problem of starting the recurrence relation off. Suppose we wish to iterate equation (193.4). The values we first compute are the $\{w_{i,2}\}$, but these require knowledge of $\{w_{i,1}\}$, which is not given in equation (193.5). The procedure for obtaining these data is to perform a Taylor series expansion of $w_{i,1}$. We find that

$$\begin{aligned} w_{i,1} &= u(x_i, t_1) \\ &= u(x_i, k) \\ &\simeq u(x_i, 0) + k\frac{\partial u}{\partial t}(x_i, 0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, 0) + \ldots, \end{aligned} \tag{193.6}$$

where this last formula is second order accurate if we retain only the terms shown (higher order approximations can also be obtained). Now u_{tt} is known in terms of u_{xx} from equation (193.1), and $u(x, 0)$ is known in terms of $f(x)$ from (193.2). Therefore, (193.6) can be simplified to yield

$$w_{i,1} \simeq w_{i,0} + kg(x_1) + \frac{\alpha^2 k^2}{2} f''(x_i). \tag{193.7}$$

Special Case

The Fortran program in program 193.1 numerically approximates the solution of the hyperbolic equation

$$\begin{aligned} u_{xx} - 9u_{xx} &= 0, && \text{for } 0 < x < 1, \quad 0 < t, \\ u(0, t) = u(1, t) &= 0, && \text{for } 0 < t, \\ u(x, 0) &= \sin \pi x, && \text{for } 0 \le x \le 1, \\ u_t(x, 0) &= 0, && \text{for } 0 \le x \le 1. \end{aligned} \tag{193.8}$$

This system has the analytic solution $u(x, t) = \sin \pi x \cos 3\pi t$.

The program utilizes $M = 10$ and the value of k was chosen to be 0.02. The solution obtained for $t = 1$ at the points $x_i = 0.1i$ (for $i = 0, 1, \ldots, 10$) is

```
      REAL  W(100,100)
C Here are the initial values
      ALPHA=3.
      FL=1.
      M=10
      H=FL/FLOAT(M)
      FK=0.02
      N=1./FK
      FLAMBD=ALPHA*FK/H
      CONST=2.*(1.-FLAMBD**2)
C Set up the initial/boundary values in the matrix
      DO 10 J=1,N+1
      W(1,J)=0.
10    W(M+1,J)=0.
      DO 20 I=2,M
      XI=(I-1)*H
      W(I,1)=F(XI)
20    W(I,2)=W(I,1)+FK*G(XI)+FK**2*FPP(XI)/2.
C Here is the recurrence relation
      DO 30 J=2,N
      TT=J*FK
      DO 40 I=2,M
40    W(I,J+1)=CONST*W(I,J)+FLAMBD**2*(W(I+1,J)+W(I-1,J))-W(I,J-1)
30    WRITE(6,5) J,TT,(W(K,J+1), K=1,M+1)
5     FORMAT(' AT TIME STEP ',I4,' (T=',F7.3,')'/,4(1X,6(F9.4)/) )
      END
C These functions compute F(X), F''(X) and G(X)
      FUNCTION F(X)
      F=SIN(3.1415927*X)
      RETURN
      END
      FUNCTION G(X)
      G=0.
      RETURN
      END
      FUNCTION FPP(X)
      FPP=-(3.1415927)**2 * SIN(3.1415927*X)
      RETURN
      END
```

Program 193.1: Fortran: finite differences applied to hyperbolic equations.

```
    0.      -0.3082  -0.5862  -0.8069  -0.9485  -0.9973
   -0.9485  -0.8069  -0.5862  -0.3082   0.
```

By comparing these values to the exact solution, we observe that the numerical approximation is correct to two decimal places.

Notes

1. A stability analysis shows that equation (193.4) is stable if $\lambda < 1$.
2. If the k^2 term in equation (193.7) had been neglected, then the method would have been only a first order method.

3. See also Burden [1, pages 583–599], Davis [2, pages 42–44], and Garabedian [4, pages 463–475].

References

[1] Burden, R. L. *Numerical Analysis*. PWS Publishers, Boston, MA, 1985.

[2] Davis, J. L. *Finite Difference Methods in Dynamics of Continuous Media*. The MacMillan Company, New York, 1986.

[3] DuChateau, P., and Zachmann, D. *Applied Partial Differential Equations*. Harper & Row Publishers, New York, 1989.

[4] Garabedian, P. R. *Partial Differential Equations*. John Wiley & Sons, New York, 1964.

[5] Renaut-Williamson, R. A. Full discretisations of zzzref40refzzz and rational approximations to coshzzzref41refzzz. *SIAM J. Numer. Anal. 26*, 2 (April 1989), 338–347.

[6] Trefethen, L. N. Instability of difference models for hyperbolic initial boundary value problems. *Comm. Pure Appl. Math 37*, 3 (May 1984), 329–367.

194. Lattice Gas Dynamics

Applicable to Partial differential equations that physically arise from the motion of "particles."

Yields

A numerical approximation methodology.

Idea

Partial differential equations are usually derived from some microscopic dynamical system. It may be possible to simulate the dynamical system directly without first formulating differential equations.

Procedure

We illustrate the basic ideas behind this method for the case of a fluid. By considering the interacting particles that make up a fluid and using continuum theory, the usual Navier–Stokes equation can be derived (see, e.g., Hasslacher [9]). This equation describes the evolution of the fluid. To numerically approximate the solution to this equation, the equation is discretized, and the resulting algebraic equations are solved on a computer.

Because a computer will be used to solve a discrete problem, it may be easier (and faster) to directly simulate the motion of the original, discrete particles. The resulting simulation can mimic all of the effects that fluid systems have. By considering only local interaction laws in the simulation, we are led to use cellular automata to describe the dynamics of the particles. Methods have been found for constructing cellular automata that are microscopically reversible (and thus support a realistic thermodynamics), obey exact conservation laws, and model continuum phenomena.

Example

We will illustrate one possible set of interaction laws that can be used to simulate gas dynamics; this model goes by the name of HPP. We consider a rectilinear array in which a particle may be present in a cell (indicated by a dot), or it may be absent (indicated by a blank). At each "time step," the grid is considered in 2×2 blocks. The blocking alternates between even and odd time steps (see figure 194.1). At any time step, a particle in a cell is considered to be moving toward the center of the 2×2 block (see figure 194.1). Hence, a particle in the upper left corner will move to the lower right corner in one time step. On the next time step, because the blocking has changed, this particle will once again be in the upper left of its new block. Hence, it will continue moving on a diagonal path.

The particles travel straight, with one exception: When exactly two particles coming together from opposite directions collide, they bounce apart in the other two directions. These interactions are particle-conserving, deterministic, and invertible. In figure 194.2, we have indicated all possi-

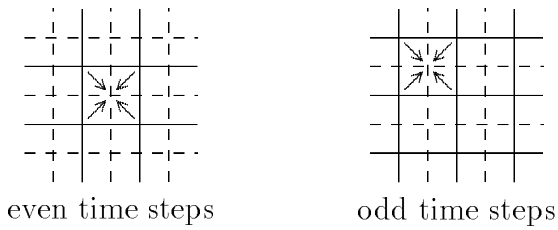

Figure 194.1: The 2×2 blocking of the rectilinear array at different time steps.

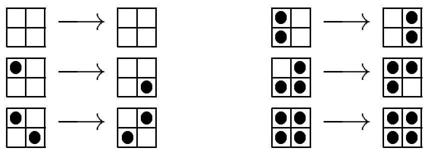

Figure 194.2: All possible motions and interactions on the rectilinear grid in one time step (up to rotations).

ble interaction possibilities (up to rotations). With just the information presented, it is possible to construct a full-scale simulation of a gas.

Notes

1. It should be noted that, for some regimes, a lattice gas may fail to well approximate the Navier–Stokes equation and yet be closer to the actual physics than the Navier–Stokes equation itself.
2. It is possible to amplify the simple example above by having many particles, interaction effects between the different particles, exclusion rules, etc.
3. The example above is for a rectilinear grid. The articles by Hasslacher [9] describe the use of hexagonal grids.
4. Papatheodorou and Fokas [12] have shown that "discrete soliton"–type behavior is possible in cellular automata.
5. Using special purpose hardware, simulation in lattice gas dynamics can be performed very quickly. See Margolus *et al.* [11].

References

[1] Chen, H., Matthaeus, W. H., and Klein, L. W. Theory of multicolor lattice gas: A cellular automaton poisson solver. *J. Comput. Physics 88*, 2 (1990), 433–466.

[2] Chen, S., Lee, M., Zhao, K. H., and Doolen, G. D. A lattice gas model with temperature. *Physica D 37*, 1–3 (1989), 42–59.

[3] Cottel, G. H., and Mas-Gallic, S. A particle method to solve the Navier–Stokes system. *Numer. Math. 57*, 8 (1990), 805–827.

[4] Doolen, G., Hasslacher, B., Frisch, U., Orszag, S., and Wolfram, S., Eds. *Lattice Gas Methods for Partial Differential Equations*. Addison–Wesley Publishing Co., Reading, MA, 1989.

[5] Elenin, G. G., and Krylov, V. V. Equilibrium equations for a multi-component nonideal lattice gas on sublattices. *Mat. Model. 2*, 1 (1990), 85–104.

[6] Enquist, B., and Hou, T. Y. Particle method approximation of oscillatory solutions to hyperbolic differential equations. *SIAM J. Numer. Anal. 26*, 2 (April 1989), 289–319.

[7] Frisch, U., d'Humieres, D., Hasslacher, B., Lallemand, P., Pomeau, Y., and Rivet, J. P. Lattice gas hydrodynamics in two and three dimensions. *Complex Systems 1*, 4 (1987), 649–707.

[8] Frisch, U., Hasslacher, B., and Pomeau, Y. Lattice gas automata for the Navier–Stokes equation. *Phys. Rev. Let. 56* (1986), 1505–1508.

[9] Hasslacher, B. "Background for lattice gas automata" and "The simple hexagonal model" and "The promise of lattice gas methods". *Los Alamos Science* (1987), 175–186, 187–200, and 211–217.

[10] Lawniczak, A. T., and Kapral, R., Eds. *Pattern Formation and Lattice Gas Automata*, vol. 6 of *Fields Institute Communications*. Amer. Math. Soc., Providence, RI, 1996.

[11] Margolus, N., Toffoli, T., and Vichiniac, G. Cellular-automata supercomputers for fluid-dynamics modeling. *PhysRevLet 56*, 16 (21April 1986), 1694–1696.

[12] Papatheodorou, T. A., and Fokas, A. S. Evolution theory, periodic particles, and solitons in cellular automata. *Stud. Appl. Math. 80* (1989), 165–182.

[13] Russo, G. A particle method for collisional kinetic equations. I. Basic theory and one-dimensional results. *J. Comput. Physics 87*, 2 (1990), 270–300.

[14] Toffoli, T. Cellular automata as an alternative to (rather than an approximation of) differential equations in modeling physics. *Physica D 10* (1984), 117–127.

[15] Tonegawa, T., Kaburagi, M., and Kanamori, J. Ground state analysis of the lattice gas model with two kinds of particles on the triangular lattice. *J. Phys. Soc. Japan 59*, 5 (1990), 1660–1675.

195. Method of Lines

Applicable to Elliptic, hyperbolic, and parabolic partial differential equations.

Yields

A system of partial differential equations with one fewer independent variables.

Idea

The basis of the method is substitution of finite differences for the derivatives with respect to one independent variable, and retention of the derivatives with respect to the remaining variables. This approach changes a given partial differential equation into a system of partial differential equations.

Procedure

We will illustrate the general method on a second order elliptic partial differential equation. Suppose the given equation is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{195.1}$$

in a domain Ω, where $\{A, B, C, D, E, F, G\}$ are functions of x and y. Because equation (195.1) is assumed to be elliptic, the necessary data for equation (195.1) are given on the boundary of Ω.

If we choose to discretize in the y variable, then we draw lines parallel to the x axis, with a constant distance h between adjacent lines. (See figure 195.1.) Suppose the lines are specified by

$$y = y_k = y_0 + kh, \qquad k = 0, 1, \ldots, N.$$

Then, we set $y = y_k$ in equation (195.1) and use finite differences for the derivatives with respect to y. For example, we can use

$$\begin{aligned}
\left.\frac{\partial u}{\partial y}\right|_{y=y_k} &\simeq \frac{1}{h}\left[u_{k+1}(x) - u_k(x)\right], \\
\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{y=y_k} &\simeq \frac{1}{h}\left[u'_{k+1}(x) - u'_k(x)\right], \\
\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=y_k} &\simeq \frac{1}{h^2}\left[u_{k+1}(x) - 2u_k(x) + u_{k-1}(x)\right],
\end{aligned} \tag{195.2}$$

where $u_k(x)$ is an approximation to $u(x, y_k)$. Using equation (195.2) in equation (195.1) (with $y = y_k$), we obtain a first order differential equation involving the unknown functions $\{u_{k-1}, u_k, u_{k+1}\}$. By taking $k =$

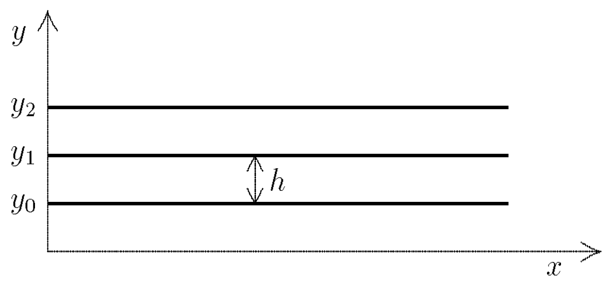

Figure 195.1: Subdivision of the domain to solve equation (195.1).

$0, 1, \ldots, N$, we obtain a system of first order ordinary differential equations for the $N+1$ unknown functions $\{u_0(x), u_1(x), \ldots, u_N(x)\}$.

If equation (195.1) is elliptic and Ω is convex, then the equations will constitute a two point boundary value system. Any standard (numerical) two point ordinary differential equation system solver can be used to solve this system.

Example

Suppose we have the following parabolic equation for $u(x, t)$

$$\begin{aligned} u_t &= u_{xx}, \\ u(0, x) &= \eta(x), \\ u(t, 0) &= \alpha(t), \\ u(t, 1) &= \beta(t). \end{aligned} \tag{195.3.a-d}$$

We discuss discretizing this equation in both x and t.

1. If we choose to discretize in the x variable, then we approximate $u(t, x_n)$ by $v_n(t)$, where $x_n = n/N = n\,\Delta x$. Then we can approximate the derivatives with respect to x in equation (195.3.a) by finite differences to obtain

$$\frac{d}{dt} v_n(t) \simeq \frac{v_{n+1}(t) - 2v_n(t) + v_{n-1}(t)}{(\Delta x)^2}, \tag{195.4}$$

for $n = 1, 2, \ldots, N-1$. The initial conditions and boundary conditions in (195.3) can be written as

$$\begin{aligned} v_n(0) &= \eta(n\,\Delta x), \qquad \text{for } n = 1, 2, \ldots, N-1, \\ v_0(t) &= \alpha(t), \\ v_N(t) &= \beta(t). \end{aligned} \tag{195.5}$$

(See figure 195.2.) If an explicit scheme (say forward Euler's method)

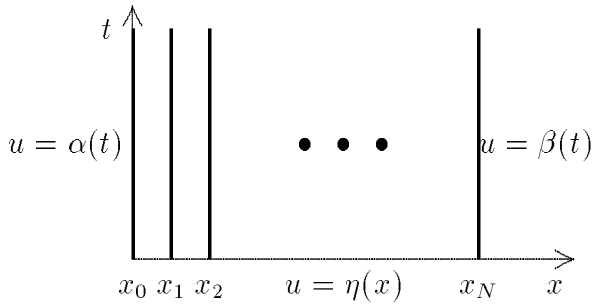

Figure 195.2: Subdivision of the domain.

is chosen to numerically approximate equation (195.4), then the simple formula

$$v_n(t+\Delta t) = v_n(t) + \frac{\Delta t}{(\Delta x)^2}\left[v_{n+1}(t) - 2v_n(t) + v_{n-1}(t)\right] \tag{195.6}$$

results. This formula can be iterated with equation (195.5) to find a numerical approximation to the solution of (195.3).

2. If, instead, we choose to discretize equation (195.3) in the t variable, then we would approximate $u(t_k, x)$ by $w_k(x)$, where $t_k = k\,\Delta t$. Approximating the t derivatives in equation (195.3) by finite differences, we obtain

$$\frac{w_k(x) - w_{k-1}(x)}{\Delta t} = \frac{d^2}{dx^2} w_k(x), \tag{195.7}$$

with the corresponding initial and boundary conditions

$$\begin{aligned} w_0(x) &= \eta(x) \\ w_m(0) &= \alpha(m\,\Delta t), \qquad \text{for } m = 0, 1, \ldots \\ w_m(1) &= \beta(m\,\Delta t), \qquad \text{for } m = 0, 1, \ldots . \end{aligned}$$

Note that equation (195.7) is a constant coefficient ordinary differential equation for the dependent variable $w_k(x)$. Hence, the explicit solution can be obtained and the differential system can be replaced by an algebraic system.

Notes

1. This method is sometimes called the *generalized Kantoravich method*.
2. Observe that the recurrence relation in equation (195.6) could have been obtained directly by applying finite differences to both the x

and t derivatives appearing in equation (195.3). This is not a clever use of the method of lines. A better approach would be to use a computer package to solve the initial value system in equations (195.4) and (195.5). This package could use an implicit method for the t derivative, and it could adjust the step size as necessary to reduce the error.

References

[1] Berzins, M. Global error estimation in the method of lines for parabolic equations. *SIAM J. Sci. Stat. Comput. 9*, 4 (July 1988), 687–703.

[2] Dew, P. M., and Walsh, J. E. A set of library routines for solving parabolic equations in one space variable. *ACM Trans. Math. Software 7*, 3 (Sept 1981), 295–314.

[3] Graney, I., and Richardson, A. A. The numerical solution of non-linear partial differential equations by the method of lines. *J. Comput. Appl. Math. 7*, 4 (1991), 229–236.

[4] Keast, P., and Muir, P. H. EPDCOL: A more efficient PDECOL code. *ACM Trans. Math. Software 17*, 2 (June 1991), 153–166.

[5] Kreiss, H.-O., and Scherer, G. Method of lines for hyperbolic differential equations. *SIAM J. Numer. Anal. 29*, 3 (June 1992).

[6] Melgaard, D. K., and Sincovec, R. F. General software for two-dimensional nonlinear partial differential equations. *ACM Trans. Math. Software 7*, 1 (March 1981), 106–125.

[7] Meyer, G. H. The method of lines for Poisson's equation with nonlinear or free boundary conditions. *Numer. Math. 29* (1978), 329–344.

[8] Mikhail, M. N. On the validity and stability of the method of lines for the solution of partial differential equations. *Appl. Math. and Comp. 22* (1987), 89–98.

[9] Schiesser, W. E. *The Numerical Method of Lines.* Academic Press, New York, 1991.

[10] Walter, W. *Differential and Integral Inequalities.* Springer–Verlag, New York, 1970.

196. Parabolic Equations: Explicit Method

Applicable to Parabolic partial differential equations.

Yields

An explicit numerical scheme.

Idea

Marching in time is the easiest way to solve a parabolic equation. For this explicit method, the time steps must be small.

Procedure

Suppose we have the parabolic differential equation

$$\begin{aligned} u_t &= L(u, \mathbf{x}, t), \\ u(t_0, \mathbf{x}) &= f(\mathbf{x}), \end{aligned} \tag{196.1}$$

for $u(\mathbf{x}, t)$, where $L(u, \mathbf{x}, t)$ is uniformly elliptic. The easiest way to solve equation (196.1) is by the use of "marching," which is an explicit method.

An explicit numerical approximation is determined by taking a forward difference in the t variable in equation (196.1) and having no other terms that involve future time values. For example, we can approximate $u(\mathbf{x}, t)$ by $v(\mathbf{x}, t)$ where $v(\mathbf{x}, t)$ satisfies

$$\begin{aligned} v(t + \Delta t, \mathbf{x}) &= v(t, \mathbf{x}) + \Delta t\, \widehat{L}(v(t, \mathbf{x}), \mathbf{x}, t), \\ v(t_0, \mathbf{x}) &= f(\mathbf{x}), \end{aligned} \tag{196.2}$$

and $\widehat{L}(\cdot)$ is any reasonable finite difference approximation to $L(v(t, \mathbf{x}), \mathbf{x}, t)$ that does not involve $v(t + \Delta t, \mathbf{x})$ (if it did involve this term, then the method would be implicit).

The main drawback of this method is that Δt must often be very small for the method to be stable. If $|\Delta \mathbf{x}|$ is the smallest discretization step in the evaluation of $\widehat{L}(v(t, \mathbf{x}), \mathbf{x}, t)$ then we require $\Delta t = O(|\Delta \mathbf{x}|^2)$ for equation (196.2) to be a numerically stable technique. More precise restrictions on Δt can be derived from the exact form of $L(v(t, \mathbf{x}), \mathbf{x}, t)$, and the numerical approximation used for the derivatives.

Example

Suppose we want to numerically approximate the solution to the diffusion problem

$$\begin{aligned} u_t &= u_{xx}, \\ u(t,0) &= 0, \\ u(t,1) &= 1, \\ u(0,x) &= 0, \end{aligned} \qquad \text{(196.3.a-d)}$$

for $t \geq 0$ with $0 \leq x \leq 1$. From the method of Fourier series or separation of variables (see pages 344 and 487), we find the analytic solution of equation (196.3) to be

$$u(t,x) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\pi^2 n^2 t} \sin \pi n x.$$

This exact solution will be used to ascertain the accuracy of the numerical solution.

To numerically approximate the solution to (196.3), we use a grid of N points between 0 and 1, $\{x_n \mid x_n = (n-1)\Delta x,\ n = 1, 2, \ldots, N\}$, where $\Delta x = 1/(N-1)$. We define $v_n(t)$ to be the approximation of $u(t,x)$ at the nth grid point: $v_n(t) \simeq u(t, x_n)$. The initial conditions in equation (196.3.d) can be represented as

$$v_m(0) = 0, \qquad m = 0, 1, 2, \ldots, N,$$

whereas the boundary conditions in equations (196.3.b,c) can be represented as

$$v_1(t) = 0, \qquad v_N(t) = 1.$$

Using a centered second order scheme for the u_{xx} term and a first order forward difference scheme for the u_t term, equation (196.3.a) can be discretized as

$$v_m(t + \Delta t) = v_m(t) + \Delta t \left(\frac{v_{m+1}(t) - 2v_m(t) + v_{m-1}(t)}{(\Delta x)^2} \right). \qquad (196.4)$$

The C (Fortran) code in program 196.1 (196.2) implements the above scheme for $N = 21$ and $\Delta t = 0.001$. We choose to compare the output from the program to the exact solution for $t = 0.1$ and $x = 0.5$. The exact solution is $u(0.1, 0.5) \simeq 0.2637$.

Table 196.1 shows the approximate value of $u(0.1, 0.5)$, for several different choices of N and Δt. From these values, we conclude

1. As N increases, the accuracy of the numerical solution increases.
2. As Δt deceases, the accuracy of the numerical solution increases.

The difference equation (196.4) was the example used to demonstrate the Von Neumann stability test (see page 692). It was determined there that the method will be stable if and only if $\Delta t/(\Delta x)^2$ is less than 1.

```
void  UPDATE(double *, double, double, int);
void main() {
    double X[1000], V[1000];
    double DELTAT=0.001, DELTAX, T=0;
    int    J, K, NTIME=100, N=21;
    DELTAX= (double) 1/(double)(N-1);
    /* Initialize the grid */
    for ( J=1; J<=N; J++ ) {
        X[J]= (double) (J-1)*DELTAX;
        V[J]= (double) 0;
    }
    V[N]= (double) 1;
    /* This is the loop for the number of time steps */
    for ( J=1; J<=NTIME; J++ ) {
        T += DELTAT;
        /* Update the grid */
        UPDATE(V,DELTAX,DELTAT,N);
        /* Output the answer */
        printf("The time is %8.4f\n",T);
        for ( K=1; K<=N; K++ ) { printf("(%8.4f,%8.4f)\n", X[K],V[K] ); }
    }
}
/* This subroutine increments the solution by one time step */
void  UPDATE(double *VOLD, double DELTAX, double DELTAT, int N) {
    double RATIO, VNEW[1000];
    int    J;
    RATIO=DELTAT/(DELTAX*DELTAX);
    for ( J=2; J<=N-1; J++ ) {
        VNEW[J]=VOLD[J] + RATIO*( VOLD[J+1] - 2*VOLD[J] + VOLD[J-1] );
    }
    for ( J=2; J<=N-1; J++ ) {  VOLD[J]=VNEW[J]; }
}
```

Program 196.1: C: explicit method applied to parabolic equations.

N	Δx	Δt	$\Delta t/(\Delta x)^2$	$v(0.1, 0.5)$
5	0.25	0.05	0.80	0.6400
5	0.25	0.01	0.16	0.2745
11	0.10	0.005	0.50	0.2628
11	0.10	0.001	0.10	0.2640
21	0.05	0.001	0.40	0.2639

Table 196.1: Approximate value of $u(0.1, 0.5)$ for different N and Δt. The exact value is $u(0.1, 0.5) \simeq 0.2637$

Note

1. See also Davis [1, Chapter 4, pages 167–193], Farlow [4, Lesson 38, pages 309–315], Press *et al.* [6, pages 635–640], Smith [7, Chapters 2 and 3, pages 11–174], and Twizell [8, pages 200–265].

```
      REAL*8 X(1000),V(1000)
      DELTAT=0.001D0
      NTIME=100
      N=21
      DELTAX=1.D0/DFLOAT(N-1)
C Initialize the grid
      DO 10 J=1,N
      X(J)=DFLOAT(J-1)*DELTAX
10    V(J)=0.D0
      V(N)=1.D0
      T=0.D0
C This is the loop for the number of time steps
      DO 20 J=1,NTIME
      T=T+DELTAT
C Update the grid
      CALL UPDATE(V,DELTAX,N,DELTAT)
C Output the answer
20    WRITE(6,5) T, (X(K),V(K),K=1,N)
5     FORMAT(' The time is=',F8.4,100(/10X,2F8.4) )
      END
C This subroutine increments the solution by one time step
      SUBROUTINE  UPDATE(VOLD,DELTAX,N,DELTAT)
      REAL*8  VOLD(1000),VNEW(1000)
      RATIO=DELTAT/DELTAX**2
      DO 100 J=2,N-1
100   VNEW(J)=VOLD(J) + RATIO*( VOLD(J+1) -2.D0 * VOLD(J) + VOLD(J-1) )
      DO 200 J=2,N-1
200   VOLD(J)=VNEW(J)
      RETURN
      END
```

Program 196.2: Fortran: explicit method applied to parabolic equations.

References

[1] Davis, J. L. *Finite Difference Methods in Dynamics of Continuous Media*. The MacMillan Company, New York, 1986.

[2] DuChateau, P., and Zachmann, D. *Applied Partial Differential Equations*. Harper & Row Publishers, New York, 1989.

[3] Evans, D. J., and Abdullah, A. R. B. A new explicit method for the solution of zzzref16refzzz. *Int. J. Comp. Math. 14* (1983), 325–353.

[4] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[5] Gerald, C. F., and Wheatley, P. O. *Applied Numerical Analysis*. Addison–Wesley Publishing Co., Reading, MA, 1984.

[6] Press, W. H., Flannery, B. P., Teukolsky, S., and Vetterling, W. T. *Numerical Recipes*. Cambridge University Press, New York, 1986.

[7] Smith, R. D. *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, third ed. Clarendon Press, Oxford, England, 1985.

[8] Twizell, E. H. *Computational Methods of Partial Differential Equations*. Ellis Horwood Limited, Chichester, England, 1984.

197. Parabolic Equations: Implicit Method

Applicable to Parabolic partial differential equations.

Yields

An implicit numerical scheme.

Idea

An implicit scheme will numerically approximate the solution of a parabolic equation and allow large time steps to be taken.

Procedure

Suppose we have the parabolic differential equation

$$\begin{aligned} u_t &= L(u, \mathbf{x}, t), \\ u(t_0, \mathbf{x}) &= f(\mathbf{x}), \end{aligned} \tag{197.1}$$

for $u(\mathbf{x}, t)$, where $L(u, \mathbf{x}, t)$ is uniformly elliptic. We desire an implicit difference scheme that will numerically approximate the solution to equation (197.1). An implicit method is one in which the value of $u(t + \Delta t, \mathbf{x})$ is not determined explicitly by the value of $u(t, \mathbf{x})$ but instead uses both $u(t + \Delta t, \mathbf{x})$ and $u(t, \mathbf{x})$.

For simplicity, we discuss only the case of a single space dimension. The difference scheme will utilize a uniform grid, with a spacing of Δx in the x direction and a spacing of Δt in the t direction. Define $v_{n,j}$ to be an approximation to $u(t_n, x_j)$, where $t_n = n\Delta t$ and $x_j = j\Delta x$.

To discretize equation (197.1) in t, we choose to use a forward difference in the t variable. That is,

$$u_t(t_n, x_j) = \frac{v_{n+1,j} - v_{n,j}}{\Delta t}.$$

Now the x derivatives will be approximated, at any point, by values at time t_n and at time t_{n+1}. That is,

$$\begin{aligned} u_x(t_n, x_j) &= (1 - \lambda_1)\frac{v_{n+1,j} - v_{n+1,j-1}}{\Delta x} + \lambda_1 \frac{v_{n,j} - v_{n,j-1}}{\Delta x}, \\ u_{xx}(t_n, x_j) &= (1 - \lambda_2)\frac{v_{n+1,j+1} - 2v_{n+1,j} + v_{n+1,j-1}}{(\Delta x)^2} \\ &\qquad + \lambda_2 \frac{v_{n,j+1} - 2v_{n,j} + v_{n,j-1}}{(\Delta x)^2}, \end{aligned} \tag{197.2}$$

where λ_1 and λ_2 are any real numbers between zero and one. For any such values, the scheme in equation (197.2) will be consistent. Note that if $\lambda_1 = \lambda_2 = 1$, there is only dependence on the values at a previous time step and an explicit method is recovered. If neither λ_1 nor λ_2 is equal to one, an implicit difference scheme results.

An implicit scheme often has the advantage that time steps can be taken that are much larger than the time steps that can be taken for an explicit method. More precise restrictions on Δt can be obtained from the form of $L(v, \mathbf{x}, t)$ and the values chosen for λ_1 and λ_2 in equation (197.2).

Example

Suppose we want to numerically approximate the solution to the diffusion problem

$$\begin{aligned} u_t &= u_{xx}, \\ u(t,0) &= 0, \\ u(t,1) &= 1, \\ u(0,x) &= 0, \end{aligned} \tag{197.3.a-d}$$

for $t \geq 0$ with $0 \leq x \leq 1$. From the method of Fourier series or separation of variables (see pages 344 and 487), we find the analytic solution to equation (197.3) is

$$u(t,x) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\pi^2 n^2 t} \sin \pi n x. \tag{197.4}$$

This exact solution will be used to determine the accuracy of the numerical solution.

To numerically approximate the solution to equation (197.3), we use a grid of N points between 0 and 1, $\{x_n \mid x_n = (n-1)\Delta x,\ n = 1, 2, \ldots, N\}$, where $\Delta x = 1/(N-1)$. The initial conditions in equation (197.3.d) can be represented as

$$v_{0,j} = 0, \qquad j = 0, 1, 2, \ldots, N, \tag{197.5}$$

whereas the boundary conditions in equation (197.3.b,c) can be represented as

$$v_{n,0} = 0, \qquad v_{n,N} = 1, \qquad \text{for } n = 1, 2, \ldots . \tag{197.6}$$

We choose to discretize the equation with $\lambda_1 = \lambda_2 = 1/2$; this produces the *Crank–Nicolson scheme*. The approximation to equation (197.3.a) is therefore

$$\frac{v_{n+1,j} - v_{n,j}}{\Delta t} = \frac{1}{2} \frac{v_{n+1,j+1} - 2v_{n+1,j} + v_{n+1,j-1}}{(\Delta x)^2} + \frac{1}{2} \frac{v_{n,j+1} - 2v_{n,j} + v_{n,j-1}}{(\Delta x)^2},$$

which can be manipulated into

$$-\rho v_{n+1,j+1} + (2+2\rho) v_{n+1,j} - \rho v_{n+1,j-1} = \rho v_{n,j+1} + (2-2\rho) v_{n,j} + \rho v_{n,j-1}, \tag{197.7}$$

N	Δx	Δt	$\Delta t/(\Delta x)^2$	$v(0.1, 0.5)$
5	0.25	0.01	0.80	0.2526
11	0.10	0.01	1.00	0.2508
11	0.10	0.005	0.50	0.2569
21	0.05	0.01	4.00	0.2507

Table 197.1: Approximate value of $u(0.1, 0.5)$ for different N and Δt. The exact value is $u(0.1, 0.5) \simeq 0.2637$

where we have defined $\rho = \Delta t/(\Delta x)^2$.

Note that for a given value of n, equation (197.7) is an algebraic equation for $v_{n+1,j}$ and two of its spatial neighbors. Hence, equation (197.7) cannot be used alone to determine $v_{n+1,j}$. Instead, a system of equations must be solved simultaneously. Utilizing equations (197.5) and (197.6), this system may be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\rho & 2+2\rho & -\rho & 0 & \cdots & 0 \\ 0 & -\rho & 2+2\rho & -\rho & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & 0 & & -\rho & 2+2\rho & -\rho \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{n+1,0} \\ v_{n+1,1} \\ v_{n+1,2} \\ \vdots \\ v_{n+1,N-1} \\ v_{n+1,N} \end{bmatrix} = \begin{bmatrix} 0 \\ \rho v_{n,1} + (2-2\rho) v_{n,2} + \rho v_{n,3} \\ \rho v_{n,2} + (2-2\rho) v_{n,3} + \rho v_{n,4} \\ \vdots \\ \rho v_{n,N-2} + (2-2\rho) v_{n,N-1} + \rho v_{n,N} \\ 1 \end{bmatrix}. \tag{197.8}$$

Because this system of linear equations has a banded matrix of width three, the system can be solved very efficiently.

The Fortran program in program 197.1 implements the above scheme with $N = 21$ and $\Delta t = 0.01$. Note that this program uses a matrix solver, `LSOLVE`, whose source code is not shown. We choose to compare the output from the program to the exact solution (given in equation (197.4)), for $t = 0.1$ and $x = 0.5$. The exact solution is $u(0.1, 0.5) \simeq 0.2637$.

Table 197.1 shows the approximate value of $u(0.1, 0.5)$, for several different choices of N and Δt. From these values, we conclude

1. As N increases, the accuracy of the numerical solution increases.
2. As Δt deceases, the accuracy of the numerical solution increases.

```
      DIMENSION  FMAT(100,100),RHS(100),V(100),X(100),NROW(200)
      N=21
      DELTAT=0.01
      NTIME=5
      DELTAX=1./DFLOAT(N-1)
      RHO=DELTAT/DELTAX**2
C Initialize the vector at T=0
      DO 10 J=1,N
      X(J)=DELTAX*(J-1)
10    V(J)=0
      T=0.
      DO 20 JTIME=1,NTIME
      T=T+DELTAT
C Set up the right hand side
      RHS(1)=0.
      RHS(N)=1.
      DO 30 J=2,N-1
30    RHS(J)= RHO*V(J-1)+(2.-2.*RHO)*V(J)+RHO*V(J+1)
C Set up the matrix
       DO 40 J=1,N
       DO 40 K=1,N
40     FMAT(J,K)=0.
       FMAT(1,1)=1.
       FMAT(N,N)=1.
       DO 50 J=2,N-1
       FMAT(J,J-1)=-RHO
       FMAT(J,J  )=2.+2.*RHO
50     FMAT(J,J+1)=-RHO
C Solve the matrix equation
     CALL LSOLVE(N,FMAT,V,RHS,NROW,IFSING,100)
C Print out the answer
20    WRITE(6,5) T, (X(K),V(K),K=1,N)
5     FORMAT(' Here is the solution at time=',F8.4,/,90(10X, 2F12.5/))
      END
```

Program 197.1: Fortran: implicit method applied to parabolic equations.

Notes

1. Observe from table 197.1 that the numerical method used resulted in reasonable approximations when $\Delta t/(\Delta x)^2$ was as large as 4. Using the Von Neumann test (see page 692), it can be shown that the Crank–Nicolson scheme is unconditionally stable for any value of $\Delta t/(\Delta x)^2$.
2. Another way to interpret this solution technique is as a sequence of elliptic problems, with one problem being solved at every time step. For example, given the parabolic system

$$\begin{aligned} u_t &= L[u] + f(\mathbf{x}, t), && \text{on } R, && t > 0, \\ u &= g(\mathbf{x}, t), && \text{on } \partial R, && t > 0, \\ u &= u_0(\mathbf{x}), && \text{on } R \cup \partial R, && t = 0, \end{aligned} \tag{197.9.a-c}$$

we can take a forward difference in t to obtain $u_t(t) \simeq \frac{u(t)-u(t-\Delta t)}{\Delta t}$, which allows equation (197.9) to be rewritten as $\frac{u(t)-u(t-\Delta t)}{\Delta t} \simeq L[u(t)]+f(\mathbf{x},t)$. This is an elliptic equation for $u(t)$ in which $u(\mathbf{x},t-\Delta t)$ plays the role of a nonhomogeneous forcing term. Hence, the successive time values of $u(\mathbf{x},t)$ may be determined by solving a sequence of elliptic problems. The boundary conditions for each elliptic problem come from equation (197.9.b), whereas the first value of $u(\mathbf{x},t)$ is given by $u_0(\mathbf{x})$. Rice and Boisvert [5, pages 111–120] present the template of an ELLPACK program that will numerically approximate the solution of parabolic equations by sequentially solving elliptic equations.

3. See also Davis [1, Chapter 4, pages 167–193], Farlow [3, Lesson 38, pages 309–315], and Smith [6, Chapters 2 and 3, pages 11–174].

References

[1] Davis, J. L. *Finite Difference Methods in Dynamics of Continuous Media.* The MacMillan Company, New York, 1986.

[2] DuChateau, P., and Zachmann, D. *Applied Partial Differential Equations.* Harper & Row Publishers, New York, 1989.

[3] Farlow, S. J. *Partial Differential Equations for Scientists and Engineers.* John Wiley & Sons, New York, 1982.

[4] Gerald, C. F., and Wheatley, P. O. *Applied Numerical Analysis.* Addison–Wesley Publishing Co., Reading, MA, 1984.

[5] Rice, J. R., and Boisvert, R. F. *Solving Elliptic Problems Using ELLPACK.* Springer–Verlag, New York, 1985.

[6] Smith, R. D. *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, third ed. Clarendon Press, Oxford, England, 1985.

198. Parabolic Equations: Monte-Carlo Method

Applicable to Linear parabolic partial differential equations.

Yields

A numerical approximation to the solution of a linear parabolic partial differential equation at a single point.

Idea

Simulation of the motion of a random particle may be used to approximate the solution to linear parabolic equations.

Procedure

The steps for this method are straightforward. First, we give an overview; then, a more detailed presentation.

First, approximate the elliptic part of the given parabolic partial differential equation by a finite difference method. Rewrite the finite difference formula as a recursive function for the value of the unknown at any given point. Then interpret this recursive formula as a set of transition probabilities that determine the motion of a random particle. By creating a finite difference scheme for the time derivative in the differential equation, a natural time scale will be associated with every step of the particle.

Now, write a computer program that will allow many (say K) particles to wander randomly around the domain of interest, based on the transition probabilities found from the difference formula. Simulate the particles one at a time, with every particle starting off at the same point (say the point $\mathbf{z}$). If the time step is Δt, and the solution is desired at $t = T$, then the particles will be allowed to wander randomly but for no more than $M = T/\Delta t$ steps.

- If the boundary data are of the Dirichlet type (i.e., the value of the unknown is prescribed on the boundary), then, when a particle reaches the boundary, stop that particle and store away the value on the boundary. Begin another particle at the point $\mathbf{z}$.
- If the boundary data are not of the Dirichlet type (say Neumann or mixed boundary conditions), then, when the particles reach the boundary, they will be given a finite probability to leave the boundary and re-enter the domain of the problem. If the particle leaves the boundary, continue the iteration process. If it does not leave the boundary, the value at the boundary is stored away, and a new particle is started off at the point $\mathbf{z}$.
- For parabolic equations there is also the possibility that the particle will not reach the boundary in M steps. If the particle has not

reached the boundary in M steps, then record the position that is finally reached. Using the initial conditions of the problem, there is a value associated with the point reached. Then, begin a new particle at the point $\mathbf{z}$.

If the parabolic equation was homogeneous, a numerical approximation to the solution at the point $\mathbf{z}$ will be given by an average of the K values stored away. If the given equation was not homogeneous, then equation (198.3) is used to obtain an estimate of the solution at the point $\mathbf{z}$. In this case, all points on the path that the particle traversed will be utilized.

In more detail, here is how the technique may be applied to the linear parabolic partial differential equation in the domain R

$$\begin{aligned} u_t &= L[u] + F(x,y,t), && x,y \in R \text{ and } t > 0, \\ u &= \phi(x,y,t), && x,y \in \partial R \text{ and } t > 0, \\ u(x,y,0) &= g(x,y), && x,y \in R, \end{aligned} \qquad (198.1.\text{a-c})$$

with the operator $L[\cdot]$ defined by

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y,$$

where $\{A,B,C,D,E\}$ are all functions of $\{x,y,t\}$. The operator $L[\cdot]$ may be discretized to yield the approximation

$$\begin{aligned} L[u] \simeq\ & A_{i,j}\left(\frac{v_{i+1,j,n} - 2v_{i,j,n} + v_{i-1,j,n}}{r}(\Delta x)^2\right)) \\ &+ 2B_{i,j}\left(\frac{v_{i+1,j+1,n} - v_{i,j+1,n} - v_{i+1,j,n} + v_{i,j,n}}{(\Delta x)(\Delta y)}\right) \\ &+ C_{i,j}\left(\frac{v_{i,j+1,n} - 2v_{i,j,n} + v_{i,j-1,n}}{(\Delta y)^2}\right) \\ &+ D_{i,j}\left(\frac{v_{i+1,j,n} - v_{i,j,n}}{\Delta x}\right) + E_{i,j}\left(\frac{v_{i,j+1,n} - v_{i,j,n}}{\Delta y}\right), \end{aligned}$$

where $x_i = x_0 + i(\Delta x)$, $y_j = y_0 + j(\Delta y)$, $t_n = n(\Delta t)$, $v_{i,j,n} = u(x_i, y_j, t_n)$, and a subscript of i,j,n means an evaluation at the point (x_i, y_j, t_n). If the $\{\Gamma_{\cdot,\cdot,\cdot}\}$ and $Q_{i,j,n}$ are defined by

$$\begin{aligned}
\Gamma_{i+1,j+1,n} &= \left[\frac{2B_{i,j,n}}{(\Delta x)(\Delta y)}\right], \\
\Gamma_{i+1,j,n} &= \left[\frac{A_{i,j,n}}{(\Delta x)^2} - \frac{2B_{i,j,n}}{(\Delta x)(\Delta y)} + \frac{D_{i,j,n}}{\Delta x}\right], \\
\Gamma_{i,j+1,n} &= \left[\frac{C_{i,j,n}}{(\Delta y)^2} - \frac{2B_{i,j,n}}{(\Delta x)(\Delta y)} + \frac{E_{i,j,n}}{\Delta y}\right], \\
\Gamma_{i-1,j,n} &= \left[\frac{A_{i,j,n}}{(\Delta x)^2}\right], \\
\Gamma_{i,j-1,n} &= \left[\frac{C_{i,j,n}}{(\Delta x)^2}\right], \\
Q_{i,j,n} &= \left[\frac{2A_{i,j,n}}{(\Delta x)^2} - \frac{2B_{i,j,n}}{(\Delta x)(\Delta y)} + \frac{2C_{i,j,n}}{(\Delta y)^2} + \frac{D_{i,j,n}}{\Delta x} + \frac{E_{i,j,n}}{\Delta y}\right],
\end{aligned}$$

and u_t is approximated by $\frac{u(x,y,t+\Delta t)-u(x,y,t)}{\Delta t}$, then equation (198.1.a) may be discretized as

$$\begin{aligned}
v_{i,j,n+1} = (\Delta t)\Big[&\Gamma_{i+1,j,n}v_{i+1,j,n} + \Gamma_{i+1,j+1,n}v_{i+1,j+1,n} + \Gamma_{i,j+1,n}v_{i,j+1,n} \\
&+ \Gamma_{i-1,j,n}v_{i-1,j,n} + \Gamma_{i,j-1,n}v_{i,j-1,n}\Big] \\
&+ \left[1 - Q_{i,j,n}(\Delta t)\right] v_{i,j,n} + (\Delta t)F_{i,j,n}.
\end{aligned} \tag{198.2}$$

If we now choose $\Delta t = 1/Q_{i,j,n}$ and define $p_{i,j,n} = \Gamma_{i,j,n}/Q_{i,j,n}$, then equation (198.2) can be written as

$$\begin{aligned}
v_{i,j,n+1} = p_{i+1,j,n}v_{i+1,j,n} &+ p_{i+1,j+1,n}v_{i+1,j+1,n} + p_{i,j+1,n}v_{i,j+1,n} \\
&+ p_{i-1,j,n}v_{i-1,j,n} + p_{i,j-1,n}v_{i,j-1,n} + \frac{F_{i,j,n}}{Q_{i,j,n}}.
\end{aligned}$$

Note that p's add up to 1. We interpret them as probabilities of taking a step in a specified direction. Specifically, if a particle is at position (i,j,n) at step n, then

- With probability $p_{i,j+1,n}$, the particle goes to $(i,j+1)$ at step $n+1$.
- With probability $p_{i,j-1,n}$, the particle goes to $(i,j-1)$ at step $n+1$.
- With probability $p_{i+1,j,n}$, the particle goes to $(i+1,j)$ at step $n+1$.
- With probability $p_{i-1,j,n}$, the particle goes to $(i-1,j)$ at step $n+1$.
- With probability $p_{i+1,j+1,n}$, the particle goes to $(i+1,j+1)$ at step $n+1$.

Now, suppose a particle starts at the point $P_0 = \mathbf{z}$ and undergoes a random walk according to the above prescription. We allow this particle to wander until a time of T has elapsed. If $Q_{i,j,n}$ is constant, then Δt is a constant, and we only need to count the number of steps taken. Either the particle will hit the boundary after, say, N steps, or it will not hit the boundary at all in M steps. Suppose that the sequence of points that this particle visits is $(P_0, P_1, P_2, \ldots, P_N)$, and $N = M$ if the boundary has not been reached. Then, an unbiased estimator of the value of $u(\mathbf{z})$ for the parabolic problem in (198.1) is given by

$$-\sum_{j=0}^{N} \frac{F(P_j)}{Q(P_j)} + \begin{cases} \phi(P_N, t_N), & \text{if the particle reached the boundary,} \\ g(P_N), & \text{if the particle did not reach the boundary.} \end{cases} \tag{198.3}$$

In practice, several random paths will be taken, and the average taken to estimate $u(x, y, t)$.

Example

Suppose we wish to numerically approximate the solution to the diffusion equation in the unit square, at a single point. Suppose we have the partial differential equation

$$u_t = \nabla^2 u, \tag{198.4}$$

for $u(t, x, y)$ with the boundary conditions

$$\begin{aligned} u(t, x, 0) &= u(t, x, 1) = 0, \\ u(t, 0, y) &= u(t, 1, y) = 0, \\ u(0, x, y) &= 10. \end{aligned} \tag{198.5}$$

The exact solution to equations (198.4) and (198.5) is

$$u(x, y, t) = \frac{16}{\pi^2} \sum_{n,m=1}^{\infty} \frac{e^{-\left[(2n-1)^2+(2m-1)^2\right]t}}{(2m-1)(2n-1)} \sin\left[(2m-1)\pi x\right] \sin\left[(2n-1)\pi y\right], \tag{198.6}$$

which was obtained by separation of variables (see page 487). Using equation (198.6) we determine that $u(0.6, 0.6, 0.5) \simeq 5.354$. We choose the point $\mathbf{z} = (0.6, 0.6)$ and try to numerically approximate the solution to equations (198.4) and (198.5) at the point $\mathbf{z}$ when $t = 0.5$. We follow the steps outlined above.

Using the standard second order approximation to the Laplacian, (see Abramowitz and Stegun [1, formula 25.3.30]), we find

$$\nabla^2 u \simeq \frac{u_{i+1,j,n} + u_{i-1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n}}{h^2} = 0,$$

where $u_{i,j,n} = u(hi, hj, n(\Delta t))$ and $h \ll 1$. Using our above approximation to the time derivative, we find that equation (198.4) may be approximated as

$$\frac{u_{i,j,n+1} - u_{i,j,n}}{\Delta t} = \frac{u_{i+1,j,n} + u_{i-1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n}}{h^2},$$

or (defining $\gamma = \Delta t/h^2$)

$$u_{i,j,n+1} = \gamma \left[u_{i+1,j,n} + u_{i-1,j,n} + u_{i,j+1,n} + u_{i,j-1,n}\right] + u_{i,j,n}(1 - 4\gamma). \tag{198.7}$$

If we choose $\gamma = 1/4$, then equation (198.7) simplifies to

$$u_{i,j,n+1} = \frac{u_{i+1,j}}{4} + \frac{u_{i-1,j}}{4} + \frac{u_{i,j+1}}{4} + \frac{u_{i,j-1}}{4}. \tag{198.8}$$

We interpret equation (198.8) probabilistically as follows: If a particle is at position (i, j) at step n, then

- With probability 1/4, the particle goes to $(i, j + 1)$ at step $n + 1$.
- With probability 1/4, the particle goes to $(i, j - 1)$ at step $n + 1$.
- With probability 1/4, the particle goes to $(i + 1, j)$ at step $n + 1$.
- With probability 1/4, the particle goes to $(i - 1, j)$ at step $n + 1$.

The Fortran program in program 198.1 was used to simulate the motion of the particles according to the above probability law. A total of `NSIM` random particles were started off. The outcome of that program is given below. As more paths are taken, the approximation becomes better. Obtaining many decimal places of accuracy requires a very large number of simulations.

```
STEP=0.03000   DT=0.00360  M=138
Average after 10000 particles is:  4.7320
Average after 20000 particles is:  4.7845
Average after 30000 particles is:  4.7847
```

Note that the program uses a routine called `RANDOM`, whose source code is not shown, that returns a random value uniformly distributed on the interval from zero to one.

Notes

1. If further accuracy is required, the options are
   (a) Increase the number of random particles.
   (b) Make the mesh discretization finer (i.e., decrease h).
   (c) Do both of the above.

   If the number of random particles is not very large, then (b) will not help much; and if the mesh is very coarse, then (a) will not help much. Generally, the variance of the answer (a measure of the "scatter") decreases as the number of trials to the minus one half power.
2. Because low numerical accuracy is obtained by this technique, a computer program does not need to work with extended precision arithmetic.

```
      NSIM=30000
      TIME=0.500
      XHOLD=0.60
      YHOLD=0.60
C Specify the step length
      STEP=0.02
C The step length determines the time step
      DT=4.*STEP**2
C Determine the number of time steps allowed
      M=TIME/DT
      SUM=0.
      DO 30 IWALK=1,NSIM
C Start off a new random walk
      X=XHOLD
      Y=YHOLD
      NSTEP=0
10    NSTEP=NSTEP+1
C Determine if M steps have been taken yet
      IF( NSTEP .GT. M ) GOTO 20
C Update the position
      X=X + SIGN(STEP,  RANDOM(DUMMY)-0.5 )
      Y=Y + SIGN(STEP,  RANDOM(DUMMY)-0.5 )
C If the particle escapes the box, start a new particle off
      IF( X.GT.1 .OR. X.LT.0 ) GOTO 40
      IF( Y.GT.1 .OR. Y.LT.0 ) GOTO 40
C Otherwise take another step
      GOTO 10
C Time has run out with the particle still in the grid
20    SUM=SUM+10
40    IF( MOD(IWALK,10000) .NE. 0 ) GOTO 30
      APPROX=SUM/FLOAT(IWALK)
      WRITE(6,5) IWALK,APPROX
30    APPROX=SUM/FLOAT(NSIM)
      WRITE(6,5) NSIM,APPROX
5     FORMAT(' Average after',I6,' particles is: ',F7.4)
      END
```

Program 198.1: Fortran program for Monte-Carlo method applied to parabolic equations.

3. If the time at which the solution is desired is so large that all of the particles end up at the boundaries, then the quantity really being calculated is the steady-state solution to the parabolic equation.
4. If a parabolic equation is interpreted as a Fokker–Planck equation (see page 303), then Itô equations can be associated with the parabolic equation. The Itô equations may be numerically integrated by the technique described on page 775.
5. Another type of Monte-Carlo approach for parabolic equations, using cellular automata, is described in Boghosian and Levermore [3].
6. Sadeh and Franklin [8] contain several worked examples. See also Farlow [4, pages 346–352].

References

[1] ABRAMOWITZ, M., AND STEGUN, I. A. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, D.C., 1964.

[2] BHAVSAR, V. C., AND GUJAR, U. G. VLSI algorithms for Monte Carlo solutions of partial differential equations. In *Advances in Computer Methods For Partial Differential Equations*, R. Vichnevetsky and R. S. Stepleman, Eds., IMACS. North–Holland Publishing Co., New York, 1984.

[3] BOGHOSIAN, B. M., AND LEVERMORE, C. D. A cellular automaton for Burgers' equation. *Complex Systems 1* (1987), 17–29.

[4] FARLOW, S. J. *Partial Differential Equations for Scientists and Engineers*. John Wiley & Sons, New York, 1982.

[5] MARSHALL, G. Monte Carlo methods for the solution of nonlinear partial differential equations. *Comput. Physics Comm. 56* (1989), 51–61.

[6] PUCKETT, E. G. Convergence of a random particle method to solutions of the Kolmogorov equation zzzref36refzzz. *Math. of Comp. 52*, 186 (April 1989), 615–645.

[7] ROBERTS, S. Convergence of a random walk method for the Burgers equation. *Math. of Comp. 52*, 186 (April 1989), 647–673.

[8] SADEH, E., AND FRANKLIN, M. A. Monte Carlo solution of partial differential equations by special purpose digital computer. *IEEE Transactions on Computers C-23*, 4 (April 1974), 389–397.

[9] SHERMAN, A. S., AND PESKIN, C. S. A Monte Carlo method for scalar reaction diffusion equations. *SIAM J. Sci. Stat. Comput. 7*, 4 (October 1986), 1360–1372.

[10] SHERMAN, A. S., AND PESKIN, C. S. Solving the Hodgkin–Huxley equations by a random walk method. *SIAM J. Sci. Stat. Comput. 9*, 1 (January 1988), 170–190.

199. Pseudospectral Method

Applicable to Most commonly, hyperbolic equations with periodic boundary conditions.

Yields

A numerical scheme for calculating the spatial derivatives.

Idea

A numerical finite Fourier transform can be used to obtain difference schemes that are of infinite order.

Procedure

On a uniformly spaced grid $\{x_1, x_2, \ldots, x_N\}$, with $x_{i+1} - x_i = h$, a numerical approximation to $\partial u/\partial x$ at the point x_k that is second order accurate is

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{2h}\left(u_{k+1} - u_{k-1}\right),$$

where $u_k = u(x_k)$. A numerical approximation that is fourth order accurate is given by

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{3h}\left(u_{k+1} - u_{k-1}\right) - \frac{1}{6h}\left(u_{k+2} - u_{k-2}\right).$$

A numerical approximation that is sixth order accurate is given by

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{2h}\left(u_{k+1} - u_{k-1}\right) - \frac{1}{3h}\left(u_{k+2} - u_{k-2}\right) + \frac{1}{30h}\left(u_{k+3} - u_{k-3}\right).$$

Methods of arbitrary high order may be constructed. For higher order methods, more points surrounding the point x_k will be utilized. In the limit, the following centered difference scheme of infinite order accuracy is obtained

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} = \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{jh}\left(u_{k+j} - u_{k-j}\right). \tag{199.1}$$

Eventually, when implementing methods of progressively higher order, the value of $u(x)$ at a point x_{k+j}, with $k + j > N$, will be required. If we assume that $u(x)$ is periodic, with period Nh, then $u(x_i) = u(x_{i+N})$. By periodicity, then the value at x_{j+k} is the same as the value at x_{j+k-N}. Hence, methods of arbitrarily high order may be constructed, and only the values $\{u_1, u_2, \ldots, u_N\}$ will be utilized.

Alternately, for given $u(x)$, a Fourier transform may be taken to determine

$$\widehat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{i\omega x}\, dx. \tag{199.2}$$

Once determined, $\widehat{u}(\omega)$ may be multiplied by $-i\omega$, and then an inverse transform taken to yield

$$\frac{\partial u}{\partial x} = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega \widehat{u}(\omega) e^{-i\omega x}\, d\omega. \tag{199.3}$$

An informal derivation of this statement is simple; consider differentiating the formula $u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(\omega) e^{-i\omega x}\, d\omega$ with respect to x.

Hence, the first derivative at every point in a domain may be computed by taking a Fourier transform, multiplying by $-i\omega$, and then taking an inverse Fourier transform. By discretizing equations (199.2) and (199.3), the Fourier transforms can be performed by "fast Fourier transforms" (FFTs). The FFT is a fast numerical technique for determining the finite Fourier transform of a function that is defined on a set of equally spaced grid points.

Hence, the derivative at every point in the grid can be computed by taking an FFT, multiplying by the discrete analogue of $i\omega$, and then taking an inverse FFT. This approach yields the same numerical scheme given in equation (199.1).

Using either technique, a highly accurate finite difference scheme is generated. This scheme may then be used to numerically approximate the u_x term appearing in a differential equation.

Example

Suppose we have the hyperbolic equation for $u(x,t)$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \tag{199.4}$$

for $t \geq 0$ on $0 \leq x \leq 1$ with the periodic boundary conditions

$$u(0,t) = u(1,t), \tag{199.5}$$

and the initial conditions

$$u(x,0) = \sin 2\pi x. \tag{199.6}$$

The solution of this system can be determined by the method of characteristics (see page 432) to be $u(x,t) = \sin 2\pi(x-t)$. We will compare the solution from our numerical scheme to this exact solution.

The pseudospectral method dictates that we take the derivatives of the periodic component (x in this example) by FFTs. We choose to use a one sided explicit difference scheme for the time derivative term. Of course, a more accurate derivative expression for the $\partial u/\partial t$ term would result in a more accurate numerical approximation (see Gottlieb and Turkel [8]).

A Fortran computer program is given in program 199.1 that finds a numerical approximation to the solution of equations (199.4)–(199.6). For

```
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL*8     V(100),X(100),EXACT(100)
      COMPLEX*16 VV(100),DERIV(100)
      N=8
      DELTAT=0.0001D0
      NTIME=10
      H=1.D0/DFLOAT(N)
      PI=3.141592653589D0
      W0=1.D0/DFLOAT(N/2-1)
C Initialize the vector with the initial conditions
      DO 10 J=1,N
      X(J)=DFLOAT(J-1)*H
10    V(J)=DSIN( 2.D0 * PI * X(J) )
C Here is the loop in time
      DO 20 LOOP=1,NTIME
      TIME=LOOP*DELTAT
C take the fourier transform of the V vector
      DO 30 J=1,N
30    VV(J)=V(J)
      CALL FFT(N,VV, 1.D0)
C multiply by (I W0)
      NBY2=N/2
      DO 40 J=1,N
40    DERIV(J)= VV(J) * DCMPLX(0.D0,1.D0) * DFLOAT(-NBY2-1+J) * W0
C Take the inverse Fourier transform
      CALL FFT(N,DERIV,-1.D0)
C Use the derivative values to update the mesh values
      DO 50 J=1,N
      V(J)=V(J) + DELTAT*DREAL( DERIV(J) )
50    EXACT(J)=DSIN( 2.D0*PI*( X(J)-TIME ) )
20    WRITE(6,5) TIME, (X(K),V(K),EXACT(K),K=1,N)
5     FORMAT(' Here is the solution at time',F6.3,/,
     1      8(2X,'X=',F7.4,'  Y(approx)=',F8.4,'  Y(exact)=',F8.4/))
      END
```

Program 199.1: Fortran program for spectral method.

comparison purposes, the exact solution is also printed out. Note that the program calls a subroutine (called `FFT(N,V,SIGNI)`), whose source code is not given, to perform the fast Fourier transform. This routine is input a complex-valued vector `V` and returns the same vector, where the values have been modified by

$$\mathtt{V}(k) = \frac{1}{\sqrt{\mathtt{N}}} \sum_{j=1}^{\mathtt{N}} \mathtt{V}(j) \exp\left[2\pi i (j-1)(k-1) \frac{\mathtt{SIGNI}}{\mathtt{N}}\right].$$

The last few lines of the program output are shown next:

```
Here is the solution at time 0.001
 X= 0.      Y(approx)=  0.0010  Y(exact)= -0.0063
 X= 0.1250  Y(approx)=  0.7078  Y(exact)=  0.7026
 X= 0.2500  Y(approx)=  1.0000  Y(exact)=  1.0000
 X= 0.3750  Y(approx)=  0.7064  Y(exact)=  0.7115
 X= 0.5000  Y(approx)= -0.0010  Y(exact)=  0.0063
```

```
X= 0.6250  Y(approx)= -0.7078  Y(exact)= -0.7026
X= 0.7500  Y(approx)= -1.0000  Y(exact)= -1.0000
X= 0.8750  Y(approx)= -0.7064  Y(exact)= -0.7115
```

Notes

1. To calculate higher order derivatives, higher powers of $(i\omega)$ should be used to multiply $\hat{u}(\omega)$. See any book on Fourier transforms (e.g., Butkov [2]).
2. Note that the method, when applied to partial differential equations, requires that the grid be uniform in every spatial variable in which a FFT is to be taken.
3. This scheme has also been applied to elliptic and parabolic equations, but the results are not much better than using a relatively low order finite difference scheme.
4. Comparing this method to finite differences, the pseudospectral method (the finite difference method) uses a global (local) interpolation of a function, then an approximation of a derivative is made from this interpolatory function.
5. Spectral methods are really more general than the limited exposition given here. Theoretically, spectral methods expand the unknown quantities in a series of orthogonal functions; these functions, in turn, result from the solution of a Sturm–Liouville problem. In practice, one considers either a Fourier expansion (as we have done here)—usually for periodic problems—or an expansion in terms of orthogonal polynomials. The Chebyshev polynomials are often used as they are amenable to the fast Fourier transform but also admit more general boundary values than those allowed in Fourier series. The use of Walsh series is discussed in Ohkita and Kobayashi [10].

References

[1] Ahner, H. F. Walsh functions and the solution of nonlinear differential equations. *Am. J. Phys. 56*, 7 (July 1988), 628–633.

[2] Butkov, E. *Mathematical Physics*. Addison–Wesley Publishing Co., Reading, MA, 1968.

[3] Canuto, C., Hussaini, M. Y., Quarteroni, A., and Zang, T. A. *Spectral Methods in Fluid Mechanics*. Springer–Verlag, New York, 1987.

[4] Cooley, J. W., Lewis, A. W., and Welch, P. D. The fast Fourier transform and its applications. *IEEE Transactions on Education E-12* (1969), 27–34.

[5] Fornberg, B. High-order finite differences and the pseudospectral method on staggered grids. *SIAM J. Numer. Anal. 12* (August 1990), 904–918.

[6] Fornberg, B. *A Practical Guide to Pseudospectral Methods*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, New York, 1996.

[7] Gottlieb, D., and Orszag, S. A. *Numerical Analysis of Spectral Methods: Theory and Applications*. SIAM, Philadelphia, PA, 1977.

[8] Gottlieb, D., and Turkel, E. On time discretizations for spectral methods. *Stud. Appl. Math. 63* (1980), 67–86.

[9] Mercier, B. *An Introduction to the Numerical Analysis of Spectral Methods.* Springer–Verlag, New York, 1989.

[10] Ohkita, M., and Kobayashi, Y. Piecewise-linear approximations of solutions of linear differential equations by Walsh functions. *Math. and Computers in Simulation 32* (1990), 297–308.

[11] Orszag, S. A. Comparison of pseudospectral and spectral approximation. *Stud. Appl. Math. 51* (1979), 253–259.

[12] Pickering, M. *An Introduction to Fast Fourier Transform Methods for Partial Differential Equations, with Applications.* John Wiley & Sons, New York, 1986.

[13] Tadmor, E. Stability analysis of finite-difference, pseudospectral and Fourier–Galerkin approximations for time-dependent problems. *SIAM Review 29*, 4 (1987), 525–555.

[14] Tal-Ezer, H. Spectral methods in time for parabolic problems. *SIAM J. Numer. Anal. 26*, 1 (February 1989), 1–11.

Mathematical Nomenclature

Applicable to The symbols used in this book.

Yields

Definitions of all special symbols.

Procedure

- $C^p[a,b]$: The class of functions that are continuous and have p continuous derivatives on the interval $[a,b]$.
- $\mathcal{F}$ Fourier transform operator.
- $H(x)$: The Heaviside function or step function; it is defined by

$$H(x) = \int_{-\infty}^{x} \delta(x)\,dx = \begin{cases} 0 & \text{if } x < 0, \\ 1/2 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

- $\Im$: The imaginary part of a quantity.
- $\mathcal{L}$: Laplace transform operator.
- O: We say that $f(x) = O(g(x))$ as $x \to x_0$ if there exists a positive constant C and a neighborhood U of x_0 such that $|f(x)| \leq C|g(x)|$ for all x in U.
- o: We say that $f(x) = o(g(x))$ as $x \to x_0$ if, given any $\mu > 0$, there exists a neighborhood U of x_0 such that $|f(x)| < \mu|g(x)|$ for all x in U.
- p: When $z = z(x,y)$, then $p = z_x$; when $y = y(x)$, then $p = y_x$.
- q: When $z = z(x,y)$, then $q = z_y$.
- r: When $z = z(x,y)$, then $r = z_{xx}$.
- s: When $z = z(x,y)$, then $s = z_{xy}$.
- t: When $z = z(x,y)$, then $t = z_{yy}$.
- $y_{x(n)}$: The nth derivative of y with respect to x.
- δ_{ij}: The Kronecker delta, it has the value 1 if $i = j$ and the value 0 if $i \neq j$.
- ϵ: This is often used to represent a small number assumed to be much less than one in magnitude.
- $\delta(x)$: The delta function; it has the properties that $\delta(x) = 0$ for $x \neq 0$, but $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.
- ∂S: If S is a region or volume, then ∂S denotes its boundary.
- $\widetilde{\nabla}$: The space-time gradient operator; it is defined by $\widetilde{\nabla} = [\nabla, \partial/\partial t]$.
- ∇^2: The Laplacian; it is defined by $\nabla^2(\phi) = \operatorname{div}(\operatorname{grad}\phi)$.
- $\mathbf{R}$: The real numbers.
- $\Re$: The real part of a quantity.
- ✡: The vector Laplacian; it is defined by ✡$\mathbf{v} = \operatorname{grad}(\operatorname{div}\mathbf{v}) - \operatorname{curl}\operatorname{curl}\mathbf{v}$.
- $\Box$: The d'Alembert operator; it is defined by $\Box = \partial^2/\partial t^2 - \nabla^2$.
- $\equiv$: A symmetric relation.
- $[L,H]$: The commutator of the two differential operators L and H (see method 7).
- $\{u,v\}$: The Lagrange bracket of the two independent variables u and v (see method 7).
- $[f,g]$: The Poisson bracket of the two functions f and g (see method 7).
- $\{y,x\}$: The Schwarzian derivative of y with respect to x (see method 7).

Errors in the Third Edition of

Handbook of Differential Equations

by Daniel Zwillinger

LAST UPDATED: November 22, 2000

(1) Section 11, **Fixed Point Existence Theorems**, pages 58 and 59
  (a) The name "Schrauder" should be "Schauder"
  (b) The following reference should be added:
    J. Schauder, "Der Fixpunktsatz in Funktionalraeumen," **Studia Math.**, 2, (1930), 171–180.
  (Thanks to G. Friesecke for these corrections.)

(2) Section 27, **Canonical Forms**, page 130, reference number 2 is now
  Bateman, H. *Partial Differential Equations of Mathematical Physics*, Dover Publications, New York, 1944.
  Which is incorrect. The reference should have been
  Bateman, H. *Differential Equations*, Longmans, Green and Co., New York, 1926, pages 75-79.
  (Thanks to Ali Nejadmalayeri for this correction.)

(3) Section 44.1.3, **Look-Up Technique**, page 189, last equation before section 44.2, presently has

$$y^{(m)} = axy^{-m/2}$$

This is incorrect, it should have been

$$y^{(m)} = ayx^{-m/2}$$

(Thanks to Flavio Noca for this correction.)

(4) Section 79, **Integrating Functions**, page 359, note number 10, the following should be added:
  The general solution to $u_x = yu_y$ is $u = f(x + \log y)$, where f is an arbitrary function.
  (Thanks to Alain Moussiaux for this observation.)

(5) Section 80, **Interchanging Dependent and Independent Variables**, page 361, note number 2, the reference to Bender and Orszag should be section 1.5, not 1.6.
  (Thanks to James Dare for this observation.)

(6) Section 85, **Reduction of order**, page 390, note number 2, presently contains
  More generally, if $\{z_1(x), \dots, z_p(x)\}$ are linearly independent solutions of equation (85.6), then the substitution

$$y(x) = \begin{bmatrix} z_1 & \dots & z_p & v \\ z_1' & \dots & z \dots\dots\vdots & \vdots \\ z_1^{(p)} & \dots & z_p^{(p)} & v^{(p)} \end{bmatrix}$$

  reduces equation (85.7) to a linear ordinary differential equation of order $n - p$ for $v(x)$.
  This should be changed to
  More generally, if $\{z_1(x), \dots, z_p(x)\}$ are linearly independent solutions of equation (85.6), then the substitution

$$y(x) = \begin{bmatrix} z_1 & \dots & z_p & z \\ z_1' & \dots & z ...... \vdots & \\ z_1^{(p)} & \dots & z_p^{(p)} & z^{(p)} \end{bmatrix} \phi(x)$$

where $\phi(x)$ need not be specified, reduces equation (85.6) to a linear ordinary differential equation of order $n-p$ for $y(x)$.

Here $y(x)$ can be written in the form

$$y(x) = A(x)z^{(p)} + B(x)z^{(p-1)} + \dots, \qquad A(x) \neq 0$$

and its derivatives have the form

$$y'(x) = A(x)z^{(p+1)} + \dots, \qquad y''(x) = A(x)z^{(p+2)} + \dots,$$

These equations can be used to eliminate $\{z^{(p)}, \dots, z^{(n)}\}$ and (85.6) will take the form

$$b_0 y^{(n-p)} + \dots + b_{n-p} y + V = 0$$

where V is linear in the $\{z, z', \dots, z^{(p-1)}\}$

(Thanks to Unal Goktas for this correction.)

(7) Section 93, **Inverse Scattering**, page 413, equation (93.1) is now

$$L[y] = y'' + a(x)y' + b(x) = f(x)$$

Which is incorrect. This should have been (note the missing y)

$$L[y] = y'' + a(x)y' + b(x)y = f(x)$$

(Thanks to Young Kim for this correction.)

(8) Section 106, **Inverse Scattering**, page 460, the Applicable to statement should have at the end

having the form of (106.2)

(Thanks to G. Friesecke for this observation.)

(9) Section 118, **Chaplygin's Method**, page 512, equations (118.5) and (118.6) and the surrounding text are now

Then define $u_1(x)$ to be the solution of

$$y' = M(x)y + N(x), \qquad y(x_0) = y_0. \tag{118.5}$$

and define $v_1(x)$ to be the solution of

$$y' = \widehat{M}(x)y + \widehat{N}(x), \qquad y(x_0) = y_0. \tag{118.6}$$

Which is incorrect. This should have been (note that the definitions have been switched):

Then define $v_1(x)$ to be the solution of

$$y' = M(x)y + N(x), \qquad y(x_0) = y_0. \tag{118.5}$$

and define $u_1(x)$ to be the solution of

$$y' = \widehat{M}(x)y + \widehat{N}(x), \qquad y(x_0) = y_0. \tag{118.6}$$

(Thanks to Bruno Van der Bossche for these corrections.)

(10) Section 145, **Picard Iteration**, page 619, note number one, the following should be added:

However, the successive approximations are guaranteed to converge to the true solution for all x sufficiently close to zero provided f is a continuously differentiable function.

(Thanks to G. Friesecke for this observation.)

(11) Section 148, **Soliton-Type Solutions**, pages 626–627

(a) In equation (148.3) the term cv_ζ should be $-cv_\zeta$.

(b) In equation (148.4) the term $(v_\zeta)^2$ should be $\frac{1}{2}(v_\zeta)^2$.

(c) An additional note should be added on page 627 to state

With the standard choice of $A = B = 0$, the solution to (148.4) can be solved in terms of elementary functions:

$$v(x) = \frac{3c}{\sigma}\left(\operatorname{sech}\left(\frac{\sqrt{c}x}{2}\right)\right)^2$$

(Thanks to G. Friesecke for these corrections.)

(12) Section 199, **Pseudospectral Method**, page 851 presently has:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{3h}(u_{k+1} - u_{k-1}) - \frac{1}{6h}(u_{k+2} - u_{k-2}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{1}{2h}(u_{k+1}-u_{k-1})-\frac{1}{3h}(u_{k+2}-u_{k-2})+\frac{1}{30h}(u_{k+3}-u_{k-3}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} = \sum_{j=1}^{\infty} \frac{2(-1)^{j+1}}{jh}(u_{k+j} - u_{k-j}).$$

Which are all incorrect. They should have been:

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{2}{3h}(u_{k+1} - u_{k-1}) - \frac{1}{12h}(u_{k+2} - u_{k-2}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} \simeq \frac{3}{5h}(u_{k+1}-u_{k-1})-\frac{3}{20h}(u_{k+2}-u_{k-2})+\frac{1}{60h}(u_{k+3}-u_{k-3}).$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_k} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{jh}(u_{k+j} - u_{k-j}).$$

(Thanks to Didier Clamond for these corrections.)