Students Solutions Manual

PARTIAL DIFFERENTIAL EQUATIONS

with FOURIER SERIES and BOUNDARY VALUE PROBLEMS

Second Edition

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Contents

Preface

This manual contains solutions with notes and comments to problems from the textbook

Partial Differential Equations
with Fourier Series and Boundary Value Problems
Second Edition

Most solutions are supplied with complete details and can be used to supplement examples from the text. Additional solutions will be posted on my website

www.math.missouri.edu/~ nakhle

as I complete them and will be included in future versions of this manual.

I would like to thank users of the first edition of my book for their valuable comments. Any comments, corrections, or suggestions would be greatly appreciated. My e-mail address is

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Errata

The following mistakes appeared in the first printing of the second edition (up-dates 24 March 2005).

Corrections in the text and figures

p. 224, Exercise #13 is better done after Section 4.4.
p. 268, Exercise #8(b), n should be even.
p.387, Exercise#12, use $y_2 = I_0(x)$ not $y_2 = J_1(x)$.
p.420, line 7, the integrals should be from $-\infty$ to ∞.
p. 425 Figures 5 and 6: Relabel the ticks on the x-axis as $-\pi$, $-\pi/2$, $\pi/2$, π, instead of -2π, $-\pi$, π, 2π.
p. 467, line (−3): Change reference (22) to (20).
p. 477, line 10: $(xt) \leftrightarrow (x, t)$.
p. 477, line 19: Change "interval" to "triangle"
p. 487, line 1: Change "is the equal" to "is equal"
p. 655, line 13: Change $\ln|\ln(x^2+y^2)|$ to $\ln(x^2+y^2)$.
p. A38, the last two lines of Example 10 should be:
$= (a_1 - 2a_0) + (2a_2 - a_1)x + \sum_{m=2}^{\infty} \ldots = \sum_{m=0}^{\infty} [(m+1)a_{m+1} + a_m(m^2-2)]x^m$.
Last page on inside back cover: Improper integrals, lines −3, the first integral should be from 0 to ∞ and not from $-\infty$ to ∞.

Corrections to Answers of Odd Exercises

Section 7.2, # 7: Change i to $-i$.
Section 7.8, # 13: $f(x) = 3$ for $1 < x < 3$ not $1 < x < 2$.
Section 7.8, # 35: $\sqrt{\frac{2}{\pi}} \frac{(e^{-iw}-1)}{w} \sum_{j=1}^{3} j \sin(jw)$. # 37: $i\sqrt{\frac{2}{\pi}} \frac{1}{w^3}$, # 51: $\frac{3}{\sqrt{2\pi}} [\delta_1 - \delta_0]$.
57: The given answer is the derivative of the real answer, which should be

$$\frac{1}{\sqrt{2\pi}}\Big((x+2)\big(\mathcal{U}_{-2}-\mathcal{U}_0\big)+(-x+2)\big(\mathcal{U}_0-\mathcal{U}_1\big)+\big(\mathcal{U}_1-\mathcal{U}_3\big)+(-x+4)\big(\mathcal{U}_3-\mathcal{U}_4\big)\Big)$$

59: The given answer is the derivative of the real answer, which should be
$\frac{1}{2}\frac{1}{\sqrt{2\pi}}\Big((x+3)\big(\mathcal{U}_{-3}-\mathcal{U}_{-2}\big)+(2x+5)\big(\mathcal{U}_{-2}-\mathcal{U}_{-1}\big)+(x+4)\big(\mathcal{U}_{-1}-\mathcal{U}_0\big)$
$+(-x+4)\big(\mathcal{U}_0-\mathcal{U}_1\big)+(-2x+5)\big(\mathcal{U}_1-\mathcal{U}_2\big)+(-x+3)\big(\mathcal{U}_2-\mathcal{U}_3\big)\Big)$
Section 7.10, # 9: $\frac{1}{2}\big[t\sin(x+t)+\frac{1}{2}\cos(x+t)-\frac{1}{2}\cos(x-t)\big]$.
Appendix A.2, # 43: $y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{18} x \cos 3x + \sum_{n=1,\ n\neq 3}^{6} \frac{\sin nx}{n(9-n^2)}$.
49: $y_p = \ldots \leftrightarrow y_p = x(\ldots$.
67: $y = -\frac{1}{8}e^x + \frac{1}{32}e^{3x} + (\frac{1}{8}x + \frac{3}{32})e^{-x}$.
Appendix A.3, # 9:$y = c_1 x + c_2\left[\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right]$.
25 $\ln(\cos x) \leftrightarrow \ln|\cos x|$. # 27 $y = c_1(1+x) + c_2 e^x - \frac{x^3}{2} - \frac{3}{2}x^2$.
Appendix A.4, # 13 $-1 + 4\sum_{n=0}^{\infty}(-1)^n x^n$
Appendix A.5, # 15 $y = 1 - 6x^2 + 3x^4 + \frac{4}{5}x^6 + \cdots$

Any suggestion or correction would be greatly appreciated. Please send them to my e-mail address

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Solutions to Exercises 1.1

1. If u_1 and u_2 are solutions of (1), then

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x} = 0.$$

Since taking derivatives is a linear operation, we have

$$\begin{aligned}\frac{\partial}{\partial t}(c_1u_1 + c_2u_2) + \frac{\partial}{\partial x}(c_1u_1 + c_2u_2) &= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} + c_1\frac{\partial u_1}{\partial x} + c_2\frac{\partial u_2}{\partial x} \\ &= c_1 \overbrace{\left(\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x}\right)}^{=0} + c_2 \overbrace{\left(\frac{\partial u_2}{\partial t} + + \frac{\partial u_2}{\partial x}\right)}^{=0} = 0,\end{aligned}$$

showing that $c_1u_1 + c_2u_2$ is a solution of (1).

5. Let $\alpha = ax + bt$, $\beta = cx + dt$, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = a\frac{\partial u}{\partial \alpha} + c\frac{\partial u}{\partial \beta} \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = b\frac{\partial u}{\partial \alpha} + d\frac{\partial u}{\partial \beta}.\end{aligned}$$

Recalling the equation, we obtain

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad (b-a)\frac{\partial u}{\partial \alpha} + (d-c)\frac{\partial u}{\partial \beta} = 0.$$

Let $a = 1,\ b = 2,\ c = 1,\ d = 1$. Then

$$\frac{\partial u}{\partial \alpha} = 0 \quad \Rightarrow \quad u = f(\beta) \quad \Rightarrow \quad u(x,\,t) = f(x+t),$$

where f is an arbitrary differentiable function (of one variable).

9. (a) The general solution in Exercise 5 is $u(x,\,t) = f(x+t)$. When $t = 0$, we get $u(x,\,0) = f(x) = 1/(x^2+1)$. Thus

$$u(x,\,t) = f(x+t) = \frac{1}{(x+t)^2+1}.$$

(c) As t increases, the wave $f(x) = \frac{1}{1+x^2}$ moves to the left.

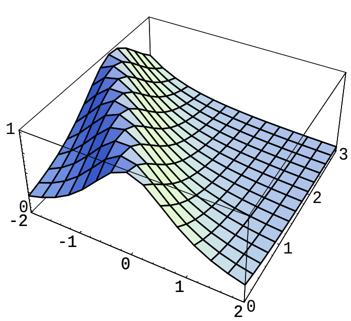


Figure for Exercise 9(b).

13. To find the characteristic curves, solve $\frac{dy}{dx} = \sin x$. Hence $y = -\cos x + C$ or $y + \cos x = C$. Thus the solution of the partial differential equation is $u(x,\,y) = f\,(y + \cos x)$. To verify the solution, we use the chain rule and get $u_x = -\sin x f'\,(y+\cos x)$ and $u_y = f'\,(y + \cos x)$. Thus $u_x + \sin x u_y = 0$, as desired.

Exercises 1.2

1. We have

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right).$$

So

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 v}{\partial t \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial t} = -\frac{\partial^2 u}{\partial x^2}.$$

Assuming that $\frac{\partial^2 v}{\partial t \partial x} = \frac{\partial^2 v}{\partial x \partial t}$, it follows that $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, which is the one dimensional wave equation with $c = 1$. A similar argument shows that v is a solution of the one dimensional wave equation.

5. (a) We have $u(x, t) = F(x + ct) + G(x - ct)$. To determine F and G, we use the initial data:

$$u(x, 0) = \frac{1}{1 + x^2} \quad \Rightarrow \quad F(x) + G(x) = \frac{1}{1 + x^2}; \quad (1)$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \quad \Rightarrow \quad cF'(x) - cG'(x) = 0$$

$$\Rightarrow \quad F'(x) = G'(x) \quad \Rightarrow \quad F(x) = G(x) + C, \qquad (2)$$

where C is an arbitrary constant. Plugging this into (1), we find

$$2G(x) + C = \frac{1}{1 + x^2} \quad \Rightarrow \quad G(x) = \frac{1}{2}\left[\frac{1}{1 + x^2} - C\right];$$

and from (2)

$$F(x) = \frac{1}{2}\left[\frac{1}{1 + x^2} + C\right].$$

Hence

$$u(x, t) = F(x + ct) + G(x - ct) = \frac{1}{2}\left[\frac{1}{1 + (x + ct)^2} + \frac{1}{1 + (x - ct)^2}\right].$$

9. As the hint suggests, we consider two separate problems: The problem in Exercise 5 and the one in Exercise 7. Let $u_1(x, t)$ denote the solution in Exercise 5 and $u_2(x, t)$ the solution in Exercise 7. It is straightforward to verify that $u = u_1 + u_2$ is the desired solution. Indeed, because of the linearity of derivatives, we have $u_{tt} = (u_1)_{tt} + (u_2)_{tt} = c^2(u_1)_{xx} + c^2(u_2)_{xx}$, because u_1 and u_2 are solutions of the wave equation. But $c^2(u_1)_{xx} + c^2(u_2)_{xx} = c^2(u_1 + u_2)_{xx} = u_{xx}$ and so $u_{tt} = c^2 u_{xx}$, showing that u is a solution of the wave equation. Now $u(x, 0) = u_1(x, 0) + u_2(x, 0) = 1/(1 + x^2) + 0$, because $u_1(x, 0) = 1/(1 + x^2)$ and $u_2(x, 0) = 0$. Similarly, $u_t(x, 0) = -2xe^{-x^2}$; thus u is the desired solution. The explicit formula for u is

$$u(x, t) = \frac{1}{2}\left[\frac{1}{1 + (x + ct)^2} + \frac{1}{1 + (x - ct)^2}\right] + \frac{1}{2c}\left[e^{-(x+ct)^2} - e^{-(x-ct)^2}\right].$$

13. The function being graphed is

$$u(x, t) = \sin \pi x \cos \pi t - \frac{1}{2}\sin 2\pi x \cos 2\pi t + \frac{1}{3}\sin 3\pi x \cos 3\pi t.$$

In frames 2, 4, 6, and 8, $t = \frac{m}{4}$, where $m = 1, 3, 5$, and 7. Plugging this into $u(x, t)$, we find

$$u(x, t) = \sin \pi x \cos \frac{m\pi}{4} - \frac{1}{2}\sin 2\pi x \cos \frac{m\pi}{2} + \frac{1}{3}\sin 3\pi x \cos \frac{3m\pi}{4}.$$

For $m = 1, 3, 5$, and 7, the second term is 0, because $\cos\frac{m\pi}{2} = 0$. Hence at these times, we have, for, $m = 1, 3, 5$, and 7,

$$u(x,\, \frac{m}{4}) = \sin \pi x \cos \pi t + \frac{1}{3} \sin 3\pi x \cos 3\pi t.$$

To say that the graph of this function is symmetric about $x = 1/2$ is equivalent to the assertion that, for $0 < x < 1/2$, $u(1/2 + x,\, \frac{m}{4}) = u(1/2 - x,\, \frac{m}{4})$. Does this equality hold? Let's check:

$$\begin{aligned} u(1/2 + x,\, \frac{m}{4}) &= \sin \pi(x + 1/2) \cos \frac{m\pi}{4} + \frac{1}{3} \sin 3\pi(x + 1/2) \cos \frac{3m\pi}{4} \\ &= \cos \pi x \cos \frac{m\pi}{4} - \frac{1}{3} \cos 3\pi x \cos \frac{3m\pi}{4}, \end{aligned}$$

where we have used the identities $\sin \pi(x + 1/2) = \cos \pi x$ and $\sin 2\pi(x + 1/2) = -\cos 3\pi x$. Similalry,

$$\begin{aligned} u(1/2 - x,\, \frac{m}{4}) &= \sin \pi(1/2 - x) \cos \frac{m\pi}{4} + \frac{1}{3} \sin 3\pi(1/2 - x) \cos \frac{3m\pi}{4} \\ &= \cos \pi x \cos \frac{m\pi}{4} - \frac{1}{3} \cos 3\pi x \cos \frac{3m\pi}{4}. \end{aligned}$$

So $u(1/2 + x,\, \frac{m}{4}) = u(1/2 - x,\, \frac{m}{4})$, as expected.

17. Same reasoning as in the previous exercise, we find the solution

$$u(x,\, t) = \frac{1}{2} \sin \frac{\pi x}{L} \cos \frac{c\pi t}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} \cos \frac{3c\pi t}{L} + \frac{2}{5} \sin \frac{7\pi x}{L} \cos \frac{7c\pi t}{L}.$$

21. (a) We have to show that $u(\frac{1}{2},\, t)$ is a constant for all $t > 0$. With $c = L = 1$, we have

$$u(x,\, t) = \sin 2\pi x \cos 2\pi t \Rightarrow u(1/2,\, t) = \sin \pi \cos 2\pi t = 0 \quad \text{for all } t > 0.$$

(b) One way for $x = 1/3$ not to move is to have $u(x,\, t) = \sin 3\pi x \cos 3\pi t$. This is the solution that corresponds to the initial condition $u(x,\, 0) = \sin 3\pi x$ and $\frac{\partial u}{\partial t}(x,\, 0) = 0$. For this solution, we also have that $x = 2/3$ does not move for all t.

25. The solution (2) is

$$u(x,\, t) = \sin \frac{\pi x}{L} \cos \frac{\pi c t}{L}.$$

Its initial conditions at time $t_0 = \frac{3L}{2c}$ are

$$u(x,\, \frac{3L}{2c}) = \sin \frac{\pi x}{L} \cos \left(\frac{\pi c}{L} \cdot \frac{3L}{2c} \right) = \sin \frac{\pi x}{L} \cos \frac{3\pi}{2} = 0;$$

and

$$\frac{\partial u}{\partial t}(x,\, \frac{3L}{2c}) = -\frac{\pi c}{L} \sin \frac{\pi x}{L} \sin \left(\frac{\pi c}{L} \cdot \frac{3L}{2c} \right) = -\frac{\pi c}{L} \sin \frac{\pi x}{L} \sin \frac{3\pi}{2} = \frac{\pi c}{L} \sin \frac{\pi x}{L}.$$

Solutions to Exercises 2.1

1. (a) $\cos x$ has period 2π. (b) $\cos \pi x$ has period $T = \frac{2\pi}{\pi} = 2$. (c) $\cos \frac{2}{3}x$ has period $T = \frac{2\pi}{2/3} = 3\pi$. (d) $\cos x$ has period 2π, $\cos 2x$ has period π, 2π, 3π,:. A common period of $\cos x$ and $\cos 2x$ is 2π. So $\cos x + \cos 2x$ has period 2π.

5. This is the special case $p = \pi$ of Exercise 6(b).

9. (a) Suppose that f and g are T-periodic. Then $f(x+T)\cdot g(x+T) = f(x)\cdot g(x)$, and so $f \cdot g$ is T periodic. Similarly,

$$\frac{f(x+T)}{g(x+T)} = \frac{f(x)}{g(x)},$$

and so f/g is T periodic.
(b) Suppose that f is T-periodic and let $h(x) = f(x/a)$. Then

$$\begin{aligned} h(x+aT) &= f\left(\frac{x+aT}{a}\right) = f\left(\frac{x}{a}+T\right) \\ &= f\left(\frac{x}{a}\right) \quad \text{(because } f \text{ is } T\text{-periodic)} \\ &= h(x). \end{aligned}$$

Thus h has period aT. Replacing a by $1/a$, we find that the function $f(ax)$ has period T/a.
(c) Suppose that f is T-periodic. Then $g(f(x+T)) = g(f(x))$, and so $g(f(x))$ is also T-periodic.

13.

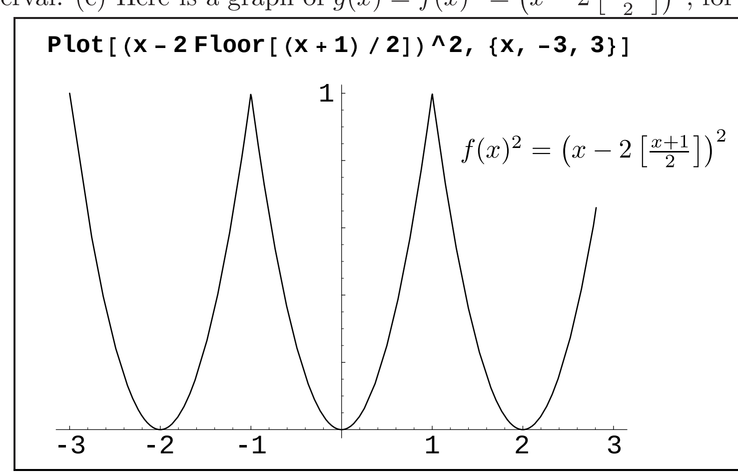
$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_0^{\pi/2} 1\,dx = \pi/2.$$

17. By Exercise 16, F is 2 periodic, because $\int_0^2 f(t)\,dt = 0$ (this is clear from the graph of f). So it is enough to describe F on any interval of length 2. For $0 < x < 2$, we have

$$F(x) = \int_0^x (1-t)\,dt = t - \frac{t^2}{2}\Big|_0^x = x - \frac{x^2}{2}.$$

For all other x, $F(x+2) = F(x)$. (b) The graph of F over the interval $[0, 2]$ consists of the arch of a parabola looking down, with zeros at 0 and 2. Since F is 2-periodic, the graph is repeated over and over.

21. (a) With $p = 1$, the function f becomes $f(x) = x - 2\left[\frac{x+1}{2}\right]$, and its graph is the first one in the group shown in Exercise 20. The function is 2-periodic and is equal to x on the interval $-1 < x < 1$. By Exercise 9(c), the function $g(x) = h(f(x)$ is 2-periodic for any function h; in particular, taking $h(x) = x^2$, we see that $g(x) = f(x)^2$ is 2-periodic. (b) $g(x) = x^2$ on the interval $-1 < x < 1$, because $f(x) = x$ on that interval. (c) Here is a graph of $g(x) = f(x)^2 = \left(x - 2\left[\frac{x+1}{2}\right]\right)^2$, for all x.



25. We have

$$\begin{aligned} |F(a+h)-F(a)| &= \left| \int_0^a f(x)\,dx - \int_0^{a+h} f(x)\,dx \right| \\ &= \left| \int_a^{a+h} f(x)\,dx \right| \leq M \cdot h, \end{aligned}$$

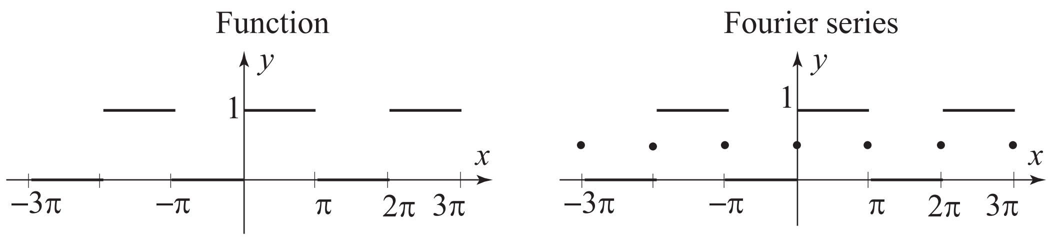
where M is a bound for $|f(x)|$, which exists by the previous exercise. (In deriving the last inequality, we used the following property of integrals:

$$\left| \int_a^b f(x)\,dx \right| \leq (b-a) \cdot M,$$

which is clear if you interpret the integral as an area.) As $h \to 0$, $M \cdot h \to 0$ and so $|F(a+h)-F(a)| \to 0$, showing that $F(a+h) \to F(a)$, showing that F is continuous at a.
(b) If f is continuous and $F(a) = \int_0^a f(x)\,dx$, the fundamental theorem of calculus implies that $F'(a) = f(a)$. If f is only piecewise continuous and a_0 is a point of continuity of f, let (x_{j-1}, x_j) denote the subinterval on which f is continuous and a_0 is in (x_{j-1}, x_j). Recall that $f = f_j$ on that subinterval, where f_j is a continuous component of f. For a in (x_{j-1}, x_j), consider the functions $F(a) = \int_0^a f(x)\,dx$ and $G(a) = \int_{x_{j-1}}^a f_j(x)\,dx$. Note that $F(a) = G(a) + \int_0^{x_{j-1}} f(x)\,dx = G(a) + c$. Since f_j is continuous on (x_{j-1}, x_j), the fundamental theorem of calculus implies that $G'(a) = f_j(a) = f(a)$. Hence $F'(a) = f(a)$, since F differs from G by a constant.

Solutions to Exercises 2.2

1. The graph of the Fourier series is identical to the graph of the function, except at the points of discontinuity where the Fourier series is equal to the average of the function at these points, which is $\frac{1}{2}$.



5. We compute the Fourier coefficients using he Euler formulas. Let us first note that since $f(x) = |x|$ is an even function on the interval $-\pi < x < \pi$, the product $f(x)\sin nx$ is an odd function. So

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{|x|\sin nx}^{\text{odd function}}\, dx = 0,$$

because the integral of an odd function over a symmetric interval is 0. For the other coefficients, we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\,dx \\ &= = \frac{1}{2\pi}\int_{-\pi}^{0}(-x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}x^2\Big|_0^{\pi} = \frac{\pi}{2}. \end{aligned}$$

In computing a_n $(n \geq 1)$, we will need the formula

$$\int x\cos ax\,dx = \frac{\cos(a\,x)}{a^2} + \frac{x\,\sin(a\,x)}{a} + C \quad (a \neq 0),$$

which can be derived using integration by parts. We have, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\ &= \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]\Big|_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier series is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x\,.$$

```
In[2]:=   s[n_, x_] := Pi / 2 - 4 / Pi Sum[1 / (2 k + 1) ^2 Cos[(2 k + 1) x], {k, 0, n}]

In[25]:=  partialsums = Table[s[n, x], {n, 1, 7}];
          f[x_] = x - 2 Pi Floor[(x + Pi) / (2 Pi)]
          g[x_] = Abs[f[x]]
          Plot[g[x], {x, -3 Pi, 3 Pi}]
          Plot[Evaluate[{g[x], partialsums}], {x, -2 Pi, 2 Pi}]
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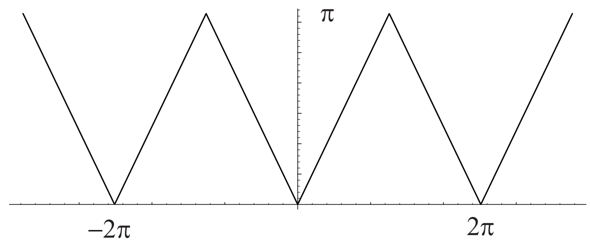
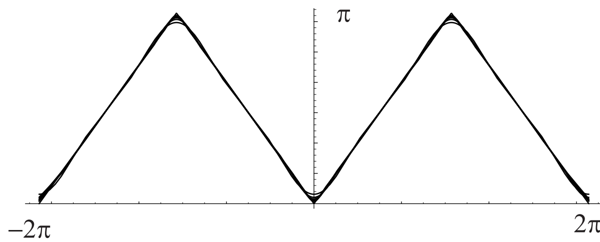


The function g(x) = | x | and its periodic extension



Partial sums of the Fourier series. Since we are summing over the odd integers, when n = 7, we are actually summing the 15th partial sum.

9. Just some hints:
(1) f is even, so all the b_n's are zero.
(2)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}.$$

(3) Establish the identity

$$\int x^2 \cos(ax)\,dx = \frac{2\,x\,\cos(a\,x)}{a^2} + \frac{\left(-2 + a^2\,x^2\right)\,\sin(a\,x)}{a^3} + C \qquad (a \neq 0),$$

using integration by parts.

13. You can compute directly as we did in Example 1, or you can use the result of Example 1 as follows. Rename the function in Example 1 $g(x)$. By comparing graphs, note that $f(x) = -2g(x+\pi)$. Now using the Fourier series of $g(x)$ from Example, we get

$$f(x) = -2\sum_{n=1}^{\infty} \frac{\sin n(\pi + x)}{n} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

17. Setting $x = \pi$ in the Fourier series expansion in Exercise 9 and using the fact that the Fourier series converges for all x to $f(x)$, we obtain

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2},$$

where we have used $\cos n\pi = (-1)^n$. Simplifying, we find

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

21. (a) Interpreting the integral as an area (see Exercise 16), we have

$$a_0 = \frac{1}{2\pi}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{1}{8}.$$

To compute a_n, we first determine the equation of the function for $\frac{\pi}{2} < x < \pi$. From Figure 16, we see that $f(x) = \frac{2}{\pi}(\pi - x)$ if $\frac{\pi}{2} < x < \pi$. Hence, for $n \geq 1$,

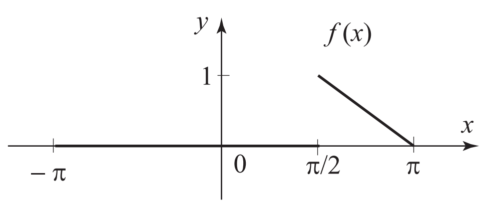
$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{\pi/2}^{\pi} \frac{2}{\pi}\overbrace{(\pi - x)}^{u}\overbrace{\cos nx}^{v'}\,dx \\
&= \frac{2}{\pi^2}(\pi - x)\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi} + \frac{2}{\pi^2}\int_{\pi/2}^{\pi}\frac{\sin nx}{n}\,dx \\
&= \frac{2}{\pi^2}\left[\frac{-\pi}{2n}\sin\frac{n\pi}{2}\right] - \frac{2}{\pi^2 n^2}\cos nx\Big|_{\pi/2}^{\pi} \\
&= -\frac{2}{\pi^2}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right].
\end{aligned}$$

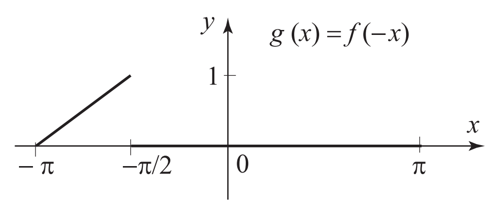
Also,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{\pi/2}^{\pi} \frac{2}{\pi}\overbrace{(\pi - x)}^{u}\overbrace{\sin nx}^{v'}\,dx \\
&= -\frac{2}{\pi^2}(\pi - x)\frac{\cos nx}{n}\Big|_{\pi/2}^{\pi} - \frac{2}{\pi^2}\int_{\pi/2}^{\pi}\frac{\cos nx}{n}\,dx \\
&= \frac{2}{\pi^2}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right].
\end{aligned}$$

Thus the Fourier series representation of f is

$$\begin{aligned}
f(x) &= \frac{1}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Bigg\{-\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]\cos nx \\
&\qquad + \left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]\sin nx\Bigg\}.
\end{aligned}$$



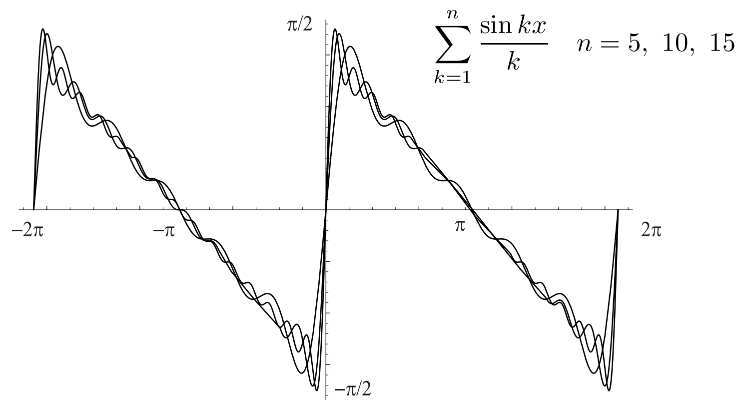


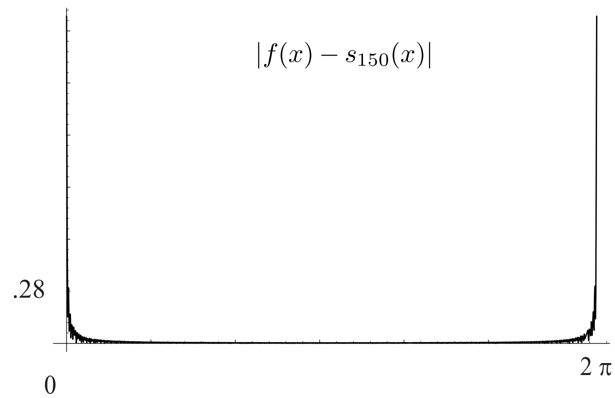
(b) Let $g(x) = f(-x)$. By performing a change of variables $x \leftrightarrow -x$ in the Fourier series of f, we obtain (see also Exercise 24 for related details) Thus the Fourier series representation of f is

$$\begin{aligned}
g(x) &= \frac{1}{8} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\Bigg\{-\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]\cos nx \\
&\qquad - \left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]\sin nx\Bigg\}.
\end{aligned}$$

25. For (a) and (b), see plots.
(c) We have $s_n(x) = \sum_{k=1}^{n}\frac{\sin kx}{k}$. So $s_n(0) = 0$ and $s_n(2\pi) = 0$ for all n. Also, $\lim_{x\to 0^+} f(x) = \frac{\pi}{2}$, so the difference between $s_n(x)$ and $f(x)$ is equal to $\pi/24$ at $x = 0$. But even we look near $x = 0$, where the Fourier series converges to $f(x)$, the difference $|s_n(x) - f(x)|$ remains larger than a positive number, that is about .28 and does not get smaller no matter how large n. In the figure, we plot $|f(x) - s_{150}(x)|$.

As you can see, this difference is 0 everywhere on the interval $(0,\, 2\pi)$, except near the points 0 and 2π, where this difference is approximately .28. The precise analysis of this phenomenon is done in the following exercise.





Solutions to Exercises 2.3

1. (a) and (b) Since f is odd, all the a_n's are zero and

$$\begin{aligned} b_n &= \frac{2}{p}\int_0^p \sin\frac{n\pi}{p}\,dx \\ &= \frac{-2}{n\pi}\cos\frac{n\pi}{p}\Big|_0^\pi = \frac{-2}{n\pi}\big[(-1)^n-1\big] \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.}\end{cases} \end{aligned}$$

Thus the Fourier series is $\displaystyle \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin\frac{(2k+1)\pi}{p}x$. At the points of discontinuity, the Fourier series converges to the average value of the function. In this case, the average value is 0 (as can be seen from the graph.

5. (a) and (b) The function is even. It is also continuous for all x. All the b_ns are 0. Also, by computing the area between the graph of f and the x-axis, from $x=0$ to $x=p$, we see that $a_0=0$. Now, using integration by parts, we obtain

$$\begin{aligned} a_n &= \frac{2}{p}\int_0^p -\left(\frac{2c}{p}\right)(x-p/2)\cos\frac{n\pi}{p}x\,dx = -\frac{4c}{p^2}\int_0^p \overbrace{(x-p/2)}^{u}\overbrace{\cos\frac{n\pi}{p}x}^{v'}\,dx \\ &= -\frac{4c}{p^2}\left[\overbrace{\frac{p}{n\pi}(x-p/2)\sin\frac{n\pi}{p}x\Big|_{x=0}^{p}}^{=0} - \frac{p}{n\pi}\int_0^p \sin\frac{n\pi}{p}x\,dx\right] \\ &= -\frac{4c}{p^2}\frac{p^2}{n^2\pi^2}\cos\frac{n\pi}{p}x\Big|_{x=0}^{p} = \frac{4c}{n^2\pi^2}(1-\cos n\pi) \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{8c}{n^2\pi^2} & \text{if } n \text{ is odd.}\end{cases} \end{aligned}$$

Thus the Fourier series is

$$f(x) = \frac{8c}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos\left[(2k+1)\frac{\pi}{p}x\right]}{(2k+1)^2}.$$

9. The function is even; so all the b_n's are 0,

$$a_0 = \frac{1}{p}\int_0^p e^{-cx}\,dx = -\frac{1}{cp}e^{-cx}\Big|_0^p = \frac{1-e^{-cp}}{cp};$$

and with the help of the integral formula from Exercise 15, Section 2.2, for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{p}\int_0^p e^{-cx}\cos\frac{n\pi x}{p}\,dx \\ &= \frac{2}{p}\frac{1}{n^2\pi^2+p^2c^2}\left[n\pi p e^{-cx}\sin\frac{n\pi x}{p} - p^2ce^{-cx}\cos\frac{n\pi x}{p}\right]\Big|_0^p \\ &= \frac{2pc}{n^2\pi^2+p^2c^2}\big[1-(-1)^n e^{-cp}\big]. \end{aligned}$$

Thus the Fourier series is

$$\frac{1}{pc}(1-e^{-cp}) + 2cp\sum_{n=1}^{\infty}\frac{1}{c^2p^2+(n\pi)^2}(1-e^{-cp}(-1)^n)\cos(\frac{n\pi}{p}x).$$

13. Take $p=1$ in Exercise 1, call the function in Exercise 1 $f(x)$ and the function in this exercise $g(x)$. By comparing graphs, we see that

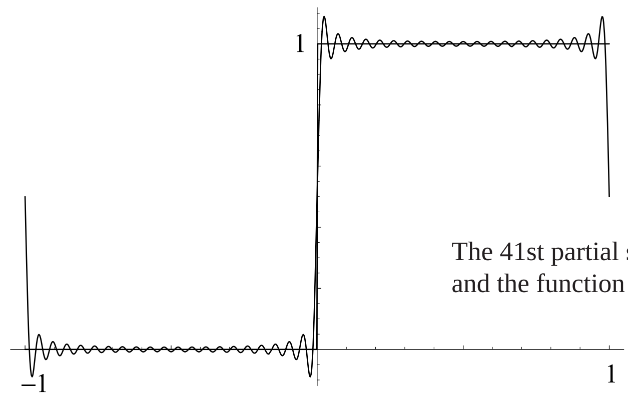
$$g(x) = \frac{1}{2}(1+f(x)).$$

Thus the Fourier series of g is

$$\frac{1}{2}\left(1+\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin(2k+1)\pi x\right)=\frac{1}{2}+\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\sin(2k+1)\pi x.$$

```
f[x_] = Which[x < 0, 0, 0 < x < 1, 1, x > 1, 0]
s[n_, x_] = 1/2 + 2/Pi Sum[1/(2 k + 1) Sin[(2 k + 1) Pi x], {k, 0, n}];
Plot[Evaluate[{f[x], s[20, x]}], {x, -1, 1}]

Which[x < 0, 0, 0 < x < 1, 1, x > 1, 0]
```



The 41st partial sum of the Fourier series and the function on the interval (-1, 1).

17. (a) Take $x=0$ in the Fourier series of Exercise 4 and get

$$0=\frac{p^2}{3}-\frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\quad\Rightarrow\quad\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$$

(b) Take $x=p$ in the Fourier series of Exercise 4 and get

$$p^2=\frac{p^2}{3}-\frac{4p^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n^2}\quad\Rightarrow\quad\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Summing over the even and odd integers separately, we get

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}+\sum_{k=1}^{\infty}\frac{1}{(2k)^2}.$$

But $\sum_{k=1}^{\infty}\frac{1}{(2k)^2}=\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{4}\frac{\pi^2}{6}$. So

$$\frac{\pi^2}{6}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}+\frac{\pi^2}{24}\quad\Rightarrow\quad\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}.$$

21. From the graph, we have

$$f(x)=\begin{cases}-1-x & \text{if } -1<x<0,\\ 1+x & \text{if } 0<x<1.\end{cases}$$

So

$$f(-x)=\begin{cases}1-x & \text{if } -1<x<0,\\ -1+x & \text{if } 0<x<1;\end{cases}$$

hence

$$f_e(x)=\frac{f(x)+f(-x)}{2}=\begin{cases}-x & \text{if } -1<x<0,\\ x & \text{if } 0<x<1,\end{cases}$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \begin{cases} -1 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Note that, $f_e(x) = |x|$ for $-1 < x < 1$. The Fourier series of f is the sum of the Fourier series of f_e and f_o. From Example 1 with $p = 1$,

$$f_e(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos[(2k+1)\pi x].$$

From Exercise 1 with $p = 1$,

$$f_o(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin[(2k+1)\pi x].$$

Hence

$$f(x) = \frac{1}{2} + \frac{4}{\pi} \sum_{k=0}^{\infty} \left[-\frac{\cos[(2k+1)\pi x]}{\pi(2k+1)^2} + \frac{\sin[(2k+1)\pi x]}{2k+1} \right].$$

25. Since f is $2p$-periodic and continuous, we have $f(-p) = f(-p+2p) = f(p)$. Now

$$a_0' = \frac{1}{2p} \int_{-p}^{p} f'(x)\, dx = \frac{1}{2p} f(x)\Big|_{-p}^{p} = \frac{1}{2p}(f(p) - f(-p)) = 0.$$

Integrating by parts, we get

$$\begin{aligned} a_n' &= \frac{1}{p} \int_{-p}^{p} f'(x) \cos \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \overbrace{f(x) \cos \frac{n\pi x}{p}\Big|_{-p}^{p}}^{=0} + \frac{n\pi}{p} \overbrace{\frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx}^{b_n} \\ &= \frac{n\pi}{p} b_n. \end{aligned}$$

Similarly,

$$\begin{aligned} b_n' &= \frac{1}{p} \int_{-p}^{p} f'(x) \sin \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \overbrace{f(x) \sin \frac{n\pi x}{p}\Big|_{-p}^{p}}^{=0} - \frac{n\pi}{p} \overbrace{\frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx}^{a_n} \\ &= -\frac{n\pi}{p} a_n. \end{aligned}$$

29. The function in Exercise 8 is piecewise smooth and continuous, with a piecewise smooth derivative. We have

$$f'(x) = \begin{cases} -\frac{c}{d} & \text{if } 0 < x < d, \\ 0 & \text{if } d < |x| < p, \\ \frac{c}{d} & \text{if } -d < x < 0. \end{cases}$$

The Fourier series of f' is obtained by differentiating term by term the Fourier series of f (by Exercise 26). Now the function in this exercise is obtained by multiplying $f'(x)$ by $-\frac{d}{c}$. So the desired Fourier series is

$$-\frac{d}{c} f'(x) = -\frac{d}{c} \frac{2cp}{d\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{dn\pi}{p}}{n^2} \left(-\frac{n\pi}{p} \right) \sin \frac{n\pi}{p} x = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{dn\pi}{p}}{n} \sin \frac{n\pi}{p} x.$$

33. The function $F(x)$ is continuous and piecewise smooth with $F'(x) = f(x)$ at all the points where f is continuous (see Exercise 25, Section 2.1). So, by Exercise 26, if we differentiate the Fourier series of F, we get the Fourier series of f. Write

$$F(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi}{p} x + B_n \sin \frac{n\pi}{p} x \right)$$

and

$$f(x) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right).$$

Note that the a_0 term of the Fourier series of f is 0 because by assumption $\int_0^{2p} f(x)\,dx = 0$. Differentiate the series for F and equate it to the series for f and get

$$\sum_{n=1}^{\infty} \left(-A_n \frac{n\pi}{p} \sin \frac{n\pi}{p} x + \frac{n\pi}{p} B_n \cos \frac{n\pi}{p} x \right) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right).$$

Equate the nth Fourier coefficients and get

$$-A_n \frac{n\pi}{p} = b_n \quad \Rightarrow \quad A_n = -\frac{p}{n\pi} b_n;$$
$$B_n \frac{n\pi}{p} = a_n \quad \Rightarrow \quad B_n = \frac{p}{n\pi} a_n.$$

This derives the nth Fourier coefficients of F for $n \geq 1$. To get A_0, note that $F(0) = 0$ because of the definition of $F(x) = \int_0^x f(t)\,dt$. So

$$0 = F(0) = A_0 + \sum_{n=1}^{\infty} A_n = A_0 + \sum_{n=1}^{\infty} -\frac{p}{n\pi} b_n;$$

and so $A_0 = \sum_{n=1}^{\infty} \frac{p}{n\pi} b_n$. We thus obtained the Fourier series of F in terms of the Fourier coefficients of f; more precisely,

$$F(x) = \frac{p}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left(-\frac{p}{n\pi} b_n \cos \frac{n\pi}{p} x + \frac{p}{n\pi} a_n \sin \frac{n\pi}{p} x \right).$$

The point of this result is to tell you that, in order to derive the Fourier series of F, you can integrate the Fourier series of f term by term. Furthermore, the only assumption on f is that it is piecewise smooth and integrates to 0 over one period (to guarantee the periodicity of F.) Indeed, if you start with the Fourier series of f,

$$f(t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} t + b_n \sin \frac{n\pi}{p} t \right),$$

and integrate term by term, you get

$$\begin{aligned} F(x) &= \int_0^x f(t)\,dt = \sum_{n=1}^{\infty} \left(a_n \int_0^x \cos \frac{n\pi}{p} t\,dt + b_n \int_0^x \sin \frac{n\pi}{p} t\,dt \right) \\ &= \sum_{n=1}^{\infty} \left(a_n \left(\frac{p}{n\pi} \right) \sin \frac{n\pi}{p} t \Big|_0^x dt + b_n \left(-\frac{p}{n\pi} \right) \cos \frac{n\pi}{p} t \Big|_0^x \right) \\ &= \frac{p}{\pi} \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left(-\frac{p}{n\pi} b_n \cos \frac{n\pi}{p} x + \frac{p}{n\pi} a_n \sin \frac{n\pi}{p} x \right), \end{aligned}$$

as derived earlier. See the following exercise for an illustration.

Solutions to Exercises 2.4

1. The even extension is the function that is identically 1. So the cosine Fourier series is just the constant 1. The odd extension yields the function in Exercise 1, Section 2.3, with $p = 1$. So the sine series is

$$\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{2k+1}.$$

This is also obtained by evaluating the integral in (4), which gives

$$b_n = 2\int_0^1 \sin(n\pi x)\,dx = -\frac{2}{n\pi}\cos n\pi x\Big|_0^1 = \frac{2}{n\pi}(1-(-1)^n).$$

9. We have

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx.$$

To evaluate this integral, we will use integration by parts to derive the following two formulas: for $a \neq 0$,

$$\int x\sin(ax)\,dx = -\frac{x\,\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} + C,$$

and

$$\int x^2\sin(ax)\,dx = \frac{2\,\cos(a\,x)}{a^3} - \frac{x^2\,\cos(a\,x)}{a} + \frac{2\,x\,\sin(a\,x)}{a^2} + C.$$

So

$$\begin{aligned}&\int x(1-x)\sin(ax)\,dx\\ &\quad= \frac{-2\,\cos(a\,x)}{a^3} - \frac{x\,\cos(a\,x)}{a} + \frac{x^2\,\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} - \frac{2\,x\,\sin(a\,x)}{a^2} + C.\end{aligned}$$

Applying the formula with $a = n\pi$, we get

$$\begin{aligned}&\int_0^1 x(1-x)\sin(n\pi x)\,dx\\ &= \frac{-2\,\cos(n\pi\,x)}{(n\pi)^3} - \frac{x\,\cos(n\pi\,x)}{n\pi} + \frac{x^2\,\cos(n\pi\,x)}{n\pi} + \frac{\sin(n\pi\,x)}{(n\pi)^2} - \frac{2\,x\,\sin(n\pi\,x)}{(n\pi)^2}\Big|_0^1\\ &= \frac{-2\,((-1)^n-1)}{(n\pi)^3} - \frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} = \frac{-2\,((-1)^n-1)}{(n\pi)^3}\\ &= \begin{cases}\frac{4}{(n\pi)^3} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

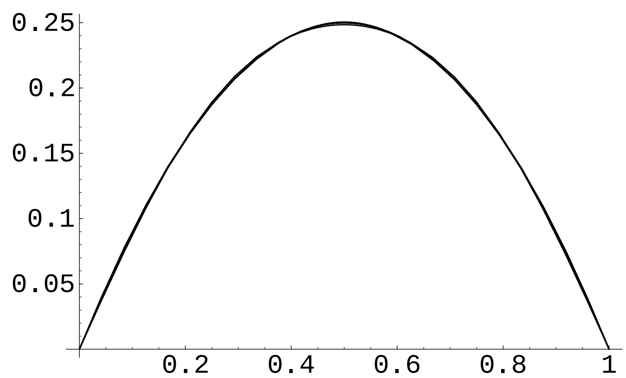
Thus

$$b_n = \begin{cases}\frac{8}{(n\pi)^3} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even,}\end{cases}$$

Hence the sine series in

$$\frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi x}{(2k+1)^3}.$$

```
b[k_] = 8 / Pi^3 1 / (2 k + 1) ^3;
ss[n_, x_] := Sum[b[k] Sin[(2 k + 1) Pi x ], {k, 0, n}];
partialsineseries = Table[ss[n, x], {n, 1, 5}];
f[x_] = x (1 - x)
Plot[Evaluate[{partialsineseries, f[x]}], {x, 0, 1}]
```



Perfect!

13. We have

$$\sin \pi x \cos \pi x = \frac{1}{2} \sin 2\pi x.$$

This yields the desired 2-periodic sine series expansion.

17. (b) Sine series expansion:

$$\begin{aligned}
b_n &= \frac{2}{p}\int_0^a \frac{h}{a} x \sin\frac{n\pi x}{p}\,dx + \frac{2}{p}\int_a^p \frac{h}{a-p}(x-p)\sin\frac{n\pi x}{p}\,dx \\
&= \frac{2h}{ap}\Big[-x\frac{p}{n\pi}\cos\frac{n\pi x}{p}\Big|_0^a + \frac{p}{n\pi}\int_0^a \cos\frac{n\pi x}{p}\,dx\Big] \\
&\quad + \frac{2h}{(a-p)p}\Big[(x-p)\frac{(-p)}{n\pi}\cos(\frac{n\,\pi\,x}{p})\Big|_a^p + \int_a^p \frac{p}{n\pi}\cos\frac{n\,\pi\,x}{p}\,dx\Big] \\
&= \frac{2h}{pa}\Big[\frac{-ap}{n\pi}\cos\frac{n\pi a}{p} + \frac{p^2}{(n\pi)^2}\sin\frac{n\pi a}{p}\Big] \\
&\quad + \frac{2h}{(a-p)p}\Big[\frac{p}{n\pi}(a-p)\cos\frac{n\pi a}{p} - \frac{p^2}{(n\pi)^2}\sin\frac{n\pi a}{p}\Big] \\
&= \frac{2hp}{(n\pi)^2}\sin\frac{n\pi a}{p}\Big[\frac{1}{a} - \frac{1}{a-p}\Big] \\
&= \frac{2hp^2}{(n\pi)^2(p-a)a}\sin\frac{n\pi a}{p}.
\end{aligned}$$

Hence, we obtain the given Fourier series.

Solutions to Exercises 2.5

1. We have

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ -1 & \text{if } -1 < x < 0; \end{cases}$$

The Fourier series representation is

$$f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin(2k+1)\pi x.$$

The mean square error (from (5)) is

$$E_N = \frac{1}{2} \int_{-1}^{1} f^2(x)\, dx - a_0^2 - \frac{1}{2} \sum_{n=1}^{N} (a_n^2 + b_n^2).$$

In this case, $a_n = 0$ for all n, $b_{2k} = 0$, $b_{2k+1} = \frac{4}{\pi(2k+1)}$, and

$$\frac{1}{2} \int_{-1}^{1} f^2(x)\, dx = \frac{1}{2} \int_{-1}^{1} dx = 1.$$

So

$$E_1 = 1 - \frac{1}{2}(b_1^2) = 1 - \frac{8}{\pi^2} \approx 0.189.$$

Since $b_2 = 0$, it follows that $E_2 = E_1$. Finally,

$$E_1 = 1 - \frac{1}{2}(b_1^2 + b_3^2) = 1 - \frac{8}{\pi^2} - \frac{8}{9\,\pi^2} \approx 0.099.$$

5. We have

$$\begin{aligned} E_N &= \frac{1}{2} \int_{-1}^{1} f^2(x)\, dx - a_0^2 - \frac{1}{2} \sum_{n=1}^{N} (a_n^2 + b_n^2) \\ &= 1 - \frac{1}{2} \sum_{n=1}^{N} b_n^2 = 1 - \frac{8}{\pi^2} \sum_{1 \le n \text{ odd} \le N} \frac{1}{n^2}. \end{aligned}$$

With the help of a calculator, we find that $E_{39} = .01013$ and $E_{41} = .0096$. So take $N = 41$.

9. We have $f(x) = \pi^2 x - x^3$ for $-\pi < x < \pi$ and, for $n \ge 1$, $b_n = \frac{12}{n^3}(-1)^{n+1}$. By Parseval's identity

$$\begin{aligned} \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{12}{n^3} \right)^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\pi^2 x - x^3\right)^2 \\ &= \frac{1}{\pi} \int_0^{\pi} \left(\pi^4 x^2 - 2\pi^2 x^4 + x^6\right) dx \\ &= \frac{1}{\pi} \left(\frac{\pi^4}{3} x^3 - \frac{2\pi^2}{5} x^5 + \frac{x^7}{7} \right) \Big|_0^{\pi} \\ &= \pi^6 \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right) = \frac{8}{105} \pi^6. \end{aligned}$$

Simplifying, we find that

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{(8)(2)}{(105)(144)} \pi^6 = \frac{\pi^6}{945}.$$

13. For the given function, we have $b_n = 0$ and $a_n = \frac{1}{n^2}$. By Parseval's identity, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} \quad \Rightarrow \quad \int_{-\pi}^{\pi} f^2(x)\,dx = \pi\sum_{n=1}^{\infty}\frac{1}{n^4} = \pi\zeta(4) = \frac{\pi^5}{90},$$

where we have used the table preceding Exercise 7 to compute $\zeta(4)$.

17. For the given function, we have

$$a_0 = 1,\ a_n = \frac{1}{3^n},\ b_n = \frac{1}{n} \text{ for } n \geq 1.$$

By Parseval's identity, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx &= a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \\
&= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(3^n)^2} + \frac{1}{n^2}\right) \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{9^n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
&= \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{9^n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}.
\end{aligned}$$

Using a geometric series, we find

$$\sum_{n=0}^{\infty}\frac{1}{9^n} = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}.$$

By Exercise 7(a),

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

So

$$\int_{-\pi}^{\pi} f^2(x)\,dx = 2\pi\left(\frac{1}{2} + \frac{1}{2}\frac{9}{8} + \frac{1}{2}\frac{\pi^2}{6}\right) = \frac{17\pi}{8} + \frac{\pi^3}{6}.$$

Solutions to Exercises 2.6

1. From Example 1, for $a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$,

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a - in} e^{inx} \quad (-\pi < x < \pi);$$

consequently,

$$e^{-ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a + in} e^{inx} \quad (-\pi < x < \pi),$$

and so, for $-\pi < x < \pi$,

$$\begin{aligned} \cosh ax &= \frac{e^{ax} + e^{-ax}}{2} \\ &= \frac{\sinh \pi a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1}{a + in} + \frac{1}{a - in} \right) e^{inx} \\ &= \frac{a \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} e^{inx}. \end{aligned}$$

2. From Example 1, for $a \neq 0, \pm i, \pm 2i, \pm 3i, \ldots$,

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a - in} e^{inx} \quad (-\pi < x < \pi);$$

consequently,

$$e^{-ax} = \frac{\sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a + in} e^{inx} \quad (-\pi < x < \pi),$$

and so, for $-\pi < x < \pi$,

$$\begin{aligned} \sinh ax &= \frac{e^{ax} - e^{-ax}}{2} \\ &= \frac{\sinh \pi a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1}{a - in} - \frac{1}{a + in} \right) e^{inx} \\ &= \frac{i \sinh \pi a}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{n}{n^2 + a^2} e^{inx}. \end{aligned}$$

5. Use identities (1); then

$$\begin{aligned} \cos 2x + 2\sin 3x &= \frac{e^{2ix} + e^{-2ix}}{2} + 2\frac{e^{3ix} - e^{-3ix}}{2i} \\ &= ie^{-3ix} + \frac{e^{-2ix}}{2} + \frac{e^{2ix}}{2} - ie^{3ix}. \end{aligned}$$

9. If $m = n$ then

$$\frac{1}{2p} \int_{-p}^{p} e^{i\frac{m\pi}{p}x} e^{-i\frac{n\pi}{p}x} dx = \frac{1}{2p} \int_{-p}^{p} e^{i\frac{m\pi}{p}x} e^{-i\frac{m\pi}{p}x} dx = \frac{1}{2p} \int_{-p}^{p} dx = 1.$$

If $m \neq n$, then

$$\begin{aligned}
\frac{1}{2p}\int_{-p}^{p} e^{i\frac{m\pi}{p}x}e^{-i\frac{n\pi}{p}x}dx &= \frac{1}{2p}\int_{-p}^{p} e^{i\frac{(m-n)\pi}{p}x}dx \\
&= \frac{-i}{2(m-n)\pi}e^{i\frac{(m-n)\pi}{p}x}\Big|_{-p}^{p} \\
&= \frac{-i}{2(m-n)\pi}\left(e^{i(m-n)\pi} - e^{-i(m-n)\pi}\right) \\
&= \frac{-i}{2(m-n)\pi}\left(\cos[(m-n)\pi] - \cos[-(m-n)\pi]\right) = 0.
\end{aligned}$$

13. (a) At points of discontinuity, the Fourier series in Example 1 converges to the average of the function. Consequently, at $x = \pi$ the Fourier series converges to $\frac{e^{a\pi}+e^{-a\pi}}{2} = \cosh(a\pi)$. Thus, plugging $x = \pi$ into the Fourier series, we get

$$\cosh(a\pi) = \frac{\sinh(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}(a+in)\overbrace{e^{in\pi}}^{=(-1)^n} = \frac{\sinh(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(a+in)}{a^2+n^2}.$$

The sum $\sum_{n=-\infty}^{\infty}\frac{in}{a^2+n^2}$ is the limit of the symmetric partial sums

$$i\sum_{n=-N}^{N}\frac{n}{a^2+n^2} = 0.$$

Hence $\sum_{n=-\infty}^{\infty}\frac{in}{a^2+n^2} = 0$ and so

$$\cosh(a\pi) = \frac{\sinh(\pi a)}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2} \quad\Rightarrow\quad \coth(a\pi) = \frac{a}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{a^2+n^2},$$

upon dividing both sides by $\sinh(a\pi)$. Setting $t = a\pi$, we get

$$\coth t = \frac{t}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(\frac{t}{\pi})^2+n^2} = \sum_{n=-\infty}^{\infty}\frac{t}{t^2+(\pi n)^2},$$

which is (b). Note that since a is not an integer, it follows that t is not of the form $k\pi i$, where $k$is an integer.

17. (a) In this exercise, we let a and b denote real numbers such that $a^2+b^2 \neq 0$. Using the linearity of the integral of complex-valued functions, we have

$$\begin{aligned}
I_1 + iI_2 &= \int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx \\
&= \int \left(e^{ax}\cos bx + ie^{ax}\sin bx\right)dx \\
&= \int e^{ax}\overbrace{(\cos bx + i\sin bx)}^{e^{ibx}}\,dx \\
&= \int e^{ax}e^{ibx}\,dx = \int e^{x(a+ib)}\,dx \\
&= \frac{1}{a+ib}e^{x(a+ib)} + C,
\end{aligned}$$

where in the last step we used the formula $\int e^{\alpha x}\,dx = \frac{1}{\alpha}e^{\alpha x} + C$ (with $\alpha = a+ib$), which is valid for all complex numbers $\alpha \neq 0$ (see Exercise 19 for a proof).
(b) Using properties of the complex exponential function (Euler's identity and the

fact that $e^{z+w} = e^z e^w$), we obtain

$$\begin{aligned}
I_1 + iI_2 &= \frac{1}{a+ib} e^{x(a+ib)} + C \\
&= \frac{\overline{(a+ib)}}{(a+ib)\cdot\overline{(a+ib)}} e^{ax} e^{ibx} + C \\
&= \frac{a-ib}{a^2+b^2} e^{ax}\big(\cos bx + i\sin bx\big) + C \\
&= \frac{e^{ax}}{a^2+b^2}\Big[\big(a\cos bx + b\sin bx\big) + i\big(-b\cos bx + a\sin bx\big)\Big] + C.
\end{aligned}$$

(c) Equating real and imaginary parts in (b), we obtain

$$I_1 = \frac{e^{ax}}{a^2+b^2}\big(a\cos bx + b\sin bx\big)$$

and

$$I_2 = \frac{e^{ax}}{a^2+b^2}\big(-b\cos bx + a\sin bx\big).$$

21. By Exercise 19,

$$\begin{aligned}
\int_0^{2\pi}\left(e^{it} + 2e^{-2it}\right) dt &= \frac{1}{i}e^{it} + \frac{2}{-2i}e^{-2it}\Big|_0^{2\pi} \\
&= -i\,\overbrace{e^{2\pi i}}^{=1} + i\,\overbrace{e^{-2\pi i}}^{=1} - (-i+i) = 0.
\end{aligned}$$

Of course, this result follows from the orthogonality relations of the complex exponential system (formula (11), with $p = \pi$).

25. First note that

$$\frac{1+it}{1-it} = \frac{(1+it)^2}{(1-it)(1+it)} = \frac{1-t^2+2it}{1+t^2} = \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}.$$

Hence

$$\begin{aligned}
\int \frac{1+it}{1-it} dt &= \int \frac{1-t^2}{1+t^2}\, dt + i\int \frac{2t}{1+t^2}\, dt \\
&= \int (-1 + \frac{2}{1+t^2}\, dt + i\int \frac{2t}{1+t^2}\, dt \\
&= -t + 2\tan^{-1} t + i\ln(1+t^2) + C.
\end{aligned}$$

Solutions to Exercises 2.7

1. (a) General solution of $y'' + 2y' + y = 0$. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ or $(\lambda + 1)^2 = 0$. It has one double characteristic root $\lambda = -1$. Thus the general solution of the homogeneous equation $y'' + 2y' + y = 0$ is

$$y = c_1 e^{-t} + c_2 t e^{-t}.$$

To find a particular solution of $y'' + 2y' + y = 25\cos 2t$, we apply Theorem 1 with $\mu = 1$, $c = 2$, and $k = 1$. The driving force is already given by its Fourier series: We have $b_n = a_n = 0$ for all n, except $a_2 = 25$. So $\alpha_n = \beta_n = 0$ for all n, except $\alpha_2 = \frac{A_2 a_2}{A_2^2 + B_2^2}$ and $\beta_2 = \frac{B_2 a_2}{A_2^2 + B_2^2}$, where $A_2 = 1 - 2^2 = -3$ and $B_2 = 4$. Thus $\alpha_2 = \frac{-75}{25} = -3$ and $\beta_2 = \frac{100}{25} = 4$, and hence a particular solution is $y_p = -3\cos 2t + 4\sin 2t$. Adding the general solution of the homogeneous equation to the particular solution, we obtain the general solution of the differential equation $y'' + 2y' + y = 25\cos 2t$

$$y = c_1 e^{-t} + c_2 t e^{-t} - 3\cos 2t + 4\sin 2t.$$

(b) Since $\lim_{t\to\infty} c_1 e^{-t} + c_2 t e^{-t} = 0$, it follows that the steady-state solution is

$$y_s = -3\cos 2t + 4\sin 2t.$$

5. (a) To find a particular solution (which is also the steady-state solution) of $y'' + 4y' + 5y = \sin t - \frac{1}{2}\sin 2t$, we apply Theorem 1 with $\mu = 1$, $c = 4$, and $k = 5$. The driving force is already given by its Fourier series: We have $b_n = a_n = 0$ for all n, except $b_1 = 1$ and $b_2 = -1/2$. So $\alpha_n = \beta_n = 0$ for all n, except, possibly, α_1, α_2, β_1, and $beta_2$. We have $A_1 = 4$, $A_2 = 1$, $B_1 = 4$, and $B_2 = 8$. So

$$\begin{aligned}
\alpha_1 &= \frac{-B_1 b_1}{A_1^2 + B_1^2} = \frac{-4}{32} = -\frac{1}{8},\\
\alpha_2 &= \frac{-B_2 b_2}{A_2^2 + B_2^2} = \frac{4}{65} = \frac{4}{65},\\
\beta_1 &= \frac{A_1 b_1}{A_1^2 + B_1^2} = \frac{4}{32} = \frac{1}{8},\\
\beta_2 &= \frac{A_2 b_2}{A_2^2 + B_2^2} = \frac{-1/2}{65} = -\frac{1}{130}.
\end{aligned}$$

Hence the steady-state solution is

$$y_p = -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{4}{65}\cos 2t - \frac{1}{130}\sin 2t.$$

(b) We have

$$\begin{aligned}
y_p &= -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{4}{65}\cos 2t - \frac{1}{130}\sin 2t,\\
(y_p)' &= \frac{1}{8}\sin t + \frac{1}{8}\cos t - \frac{8}{65}\sin 2t - \frac{1}{65}\cos 2t,\\
(y_p)'' &= \frac{1}{8}\cos t - \frac{1}{8}\sin t - \frac{16}{65}\cos 2t + \frac{2}{65}\sin 2t,\\
(y_p)'' + 4(y_p)' + 5y_p &= \left(\frac{1}{8} + \frac{4}{8} - \frac{5}{8}\right)\cos t + \left(-\frac{1}{8} + \frac{4}{8} + \frac{5}{8}\right)\sin t\\
&\quad + \left(\frac{2}{65} - \frac{32}{65} - \frac{5}{130}\right)\sin 2t + \left(-\frac{16}{65} - \frac{4}{65} + \frac{20}{65}\right)\cos 2t\\
&= \sin t + \left(\frac{2}{75} - \frac{32}{65} - \frac{5}{130}\right)\sin 2t - \frac{1}{2}\sin 2t,
\end{aligned}$$

which shows that y_p is a solution of the nonhomogeneous differential equation.

9. (a) Natural frequency of the spring is

$$\omega_0 = \sqrt{\frac{k}{\mu}} = \sqrt{10.1} \approx 3.164.$$

(b) The normal modes have the same frequency as the corresponding components of driving force, in the following sense. Write the driving force as a Fourier series $F(t) = a_0 + \sum_{n=1}^{\infty} f_n(t)$ (see (5). The normal mode, $y_n(t)$, is the steady-state response of the system to $f_n(t)$. The normal mode y_n has the same frequency as f_n. In our case, F is 2π-periodic, and the frequencies of the normal modes are computed in Example 2. We have $\omega_{2m+1} = 2m+1$ (the n even, the normal mode is 0). Hence the frequencies of the first six nonzero normal modes are 1, 3, 5, 7, 9, and 11. The closest one to the natural frequency of the spring is $\omega_3 = 3$. Hence, it is expected that y_3 will dominate the steady-state motion of the spring.

13. According to the result of Exercise 11, we have to compute $y_3(t)$ and for this purpose, we apply Theorem 1. Recall that y_3 is the response to $f_3 = \frac{4}{3\pi}\sin 3t$, the component of the Fourier series of $F(t)$ that corresponds to $n = 3$. We have $a_3 = 0$, $b_3 = \frac{4}{3\pi}$, $\mu = 1$, $c = .05$, $k = 10.01$, $A_3 = 10.01 - 9 = 1.01$, $B_3 = 3(.05) = .15$,

$$\alpha_3 = \frac{-B_3 b_3}{A_3^2 + B_3^2} = \frac{-(.15)(4)/(3\pi)}{(1.01)^2 + (.15)^2} \approx -.0611 \quad \text{and} \quad \beta_3 = \frac{A_3 b_3}{A_3 + B_3^2} \approx .4111.$$

So

$$y_3 = -.0611\cos 3t + .4111\sin 3t.$$

The amplitude of y_3 is $\sqrt{.0611^2 + .4111^2} \approx .4156$.

17. (a) In order to eliminate the 3rd normal mode, y_3, from the steady-state solution, we should cancel out the component of F that is causing it. That is, we must remove $f_3(t) = \frac{4\sin 3t}{3\pi}$. Thus subtract $\frac{4\sin 3t}{3\pi}$ from the input function. The modified input function is

$$F(t) - \frac{4\sin 3t}{3\pi}.$$

Its Fourier series is he same as the one of F, without the 3rd component, $f_3(t)$. So the Fourier series of the modified input function is

$$\frac{4}{\pi}\sin t + \frac{4}{\pi}\sum_{m=2}^{\infty} \frac{\sin(2m+1)t}{2m+1}.$$

(b) The modified steady-state solution does not have the y_3-component that we found in Exercise 13. We compute its normal modes by appealing to Theorem 1 and using as an input function $F(t) - f_3(t)$. The first nonzero mode is y_1; the second nonzero normal mode is y_5. We compute them with the help of Mathematica. Let us first enter the parameters of the problem and compute α_n and β_n, using the definitions from Theorem 1. The input/output from Mathematica is the following

```
Clear[a, mu, p, k, alph, bet, capa, capb, b, y]
mu = 1;
c = 5 / 100;
k = 1001 / 100;
p = Pi;
a0 = 0;
a[n_] = 0;
b[n_] = 2 / (Pi n) (1 - Cos[ n Pi]);
alph0 = a0 / k;
capa[n_] = k - mu (n Pi / p) ^2
capb[n_] = c n Pi / p
alph[n_] = (capa[n] a[n] - capb[n] b[n]) / (capa[n]^2 + capb[n]^2)
bet[n_] = (capa[n] b[n] + capb[n] a[n]) / (capa[n]^2 + capb[n]^2)
```

$$\frac{1001}{100} - n^2$$

$$\frac{n}{20}$$

$$-\frac{1 - \text{Cos}[n\,\pi]}{10\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi}$$

$$\frac{2\left(\frac{1001}{100} - n^2\right)(1 - \text{Cos}[n\,\pi])}{n\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi}$$

It appears that

$$\alpha_n = \frac{-(1-\cos(n\,\pi))}{10\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi} \quad \text{and} \quad \beta_n = \frac{2\left(\frac{1001}{100} - n^2\right)(1-\cos(n\,\pi))}{n\left(\frac{n^2}{400} + \left(\frac{1001}{100} - n^2\right)^2\right)\pi}$$

Note how these formulas yield 0 when n is even. The first two nonzero modes of the modified solution are

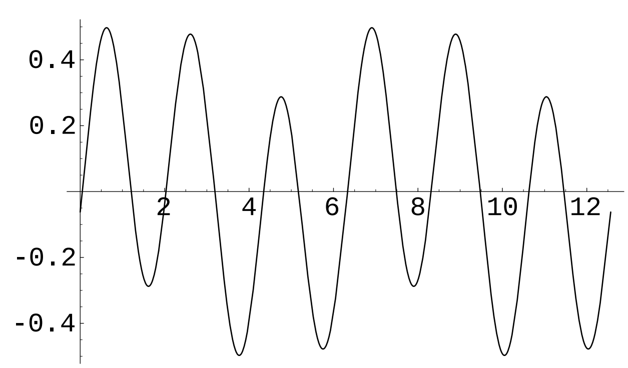
$$y_1(t) = \alpha_1 \cos t + \beta_1 \sin t = -.0007842 \cos t + .14131 \sin t$$

and

$$y_5(t) = \alpha_5 \cos 5t + \beta_5 \sin 5t - .00028 \cos 5t - .01698 \sin 5t.$$

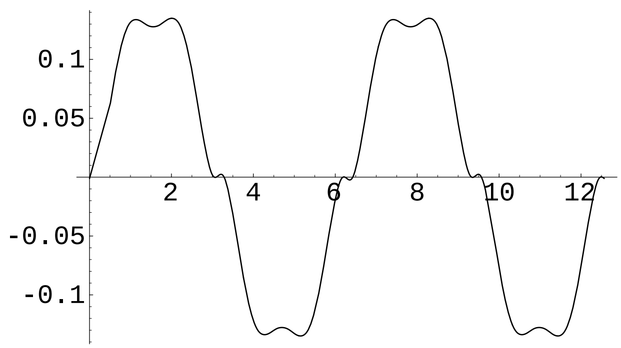
(c) In what follows, we use 10 nonzero terms of the original steady-state solution and compare it with 10 nonzero terms of the modified steady-state solution. The graph of the original steady-state solution looks like this:

```
steadystate[t_] = Sum[alph[n] Cos[n t] + bet[n] Sin[n t], {n, 1, 20}];
Plot[Evaluate[steadystate[t]], {t, 0, 4 Pi}]
```



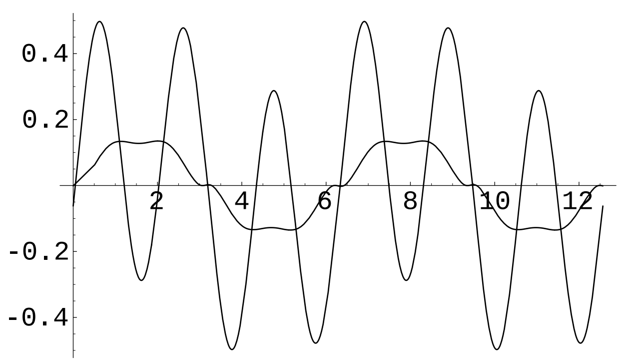
The modified steady-state is obtained by subtracting y_3 from the steady-state. Here is its graph.

```
modifiedsteadystate[t_] = steadystate[t] - (alph[3] Cos[3 t] + bet[3] Sin[3 t]);
Plot[Evaluate[modifiedsteadystate[t]], {t, 0, 4 Pi}]
```



In order to compare, we plot both functions on the same graph.

```
Plot[Evaluate[{steadystate[t], modifiedsteadystate[t]}], {t, 0, 4 Pi}]
```



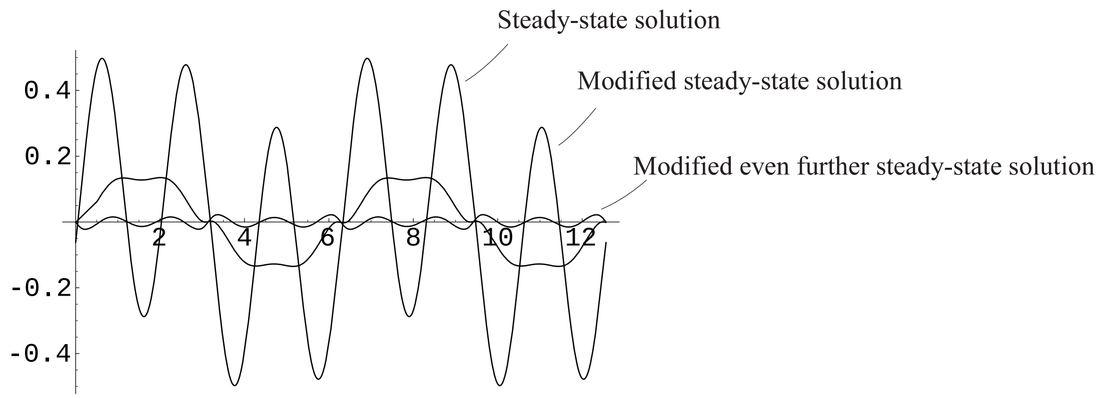
It seems like we were able to reduce the amplitude of the steady-state solution by a factor of 2 or 3 by removing the third normal mode. Can we do better? Let us analyze the amplitudes of the normal modes. These are equal to $\sqrt{\alpha_n^2 + \beta_n^2}$. We have the following numerical values:

```
amplitudes = N[Table[Sqrt[alph[n]^2 + bet[n]^2], {n, 1, 20}]]

{0.141312, 0., 0.415652, 0., 0.0169855, 0., 0.00466489, 0., 0.00199279, 0.,
 0.00104287, 0., 0.000616018, 0., 0.000394819, 0., 0.000268454, 0., 0.000190924, 0.}
```

It is clear from these values that y_3 has the largest amplitude (which is what we expect) but y_1 also has a relatively large amplitude. So, by removing the first component of F, we remove y_1, and this may reduce the oscillations even further. Let's see the results. We will plot the steady-state solution y_s, $y_s - y_3$, and $y_s - y_1 - y_3$.

```
modifiedfurther[t_] = modifiedsteadystate[t] - (alph[1] Cos[ t] + bet[1] Sin[ t]);
Plot[
 Evaluate[{modifiedfurther[t], steadystate[t], modifiedsteadystate[t]}], {t, 0, 4 Pi}]
```



21. (a) The input function $F(t)$ is already given by its Fourier series: $F(t) = 2\cos 2t + \sin 3t$. Since the frequency of the component $\sin 3t$ of the input function is 3 and is equal to the natural frequency of the spring, resonance will occur (because there is no damping in the system). The general solution of $y''+9y = 2\cos 2t+\sin 3t$ is $y = y_h + y_p$, where y_h is the general solution of $y''+9y = 0$ and y_p is a particular solution of the nonhomogeneous equation. We have $y_h = c_1 \sin 3t + c_2 \cos 3t$ and, to find y_p, we apply Exercise 20 and get

$$y_p = \left(\frac{a_2}{A_2}\cos 2t + \frac{b_2}{A_2}\sin 2t\right) + R(t),$$

where $a_2 = 2$, $b_2 = 0$, $A_2 = 9 - 2^2 = 5$, $a_{n_0} = 0$, $b_{n_0} = 1$, and

$$R(t) = -\frac{t}{6}\cos 3t.$$

Hence

$$y_p = \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t$$

and so the general solution is

$$y = c_1 \sin 3t + c_2 \cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t.$$

(b) To eliminate the resonance from the system we must remove the component of F that is causing resonance. Thus add to $F(t)$ the function $-\sin 3t$. The modified input function becomes $F_{\text{modified}}(t) = 2\cos 2t$.

25. The general solution is $y = c_1 \sin 3t + c_2 \cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t$. Applying the initial condition $y(0) = 0$ we get $c_2 + \frac{2}{5} = 0$ or $c_2 = -\frac{2}{5}$. Thus

$$y = c_1 \sin 3t - \frac{2}{5}\cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t.$$

Applying the initial condition $y'(0) = 0$, we obtain

$$\begin{aligned} y' &= 3c_1\cos 3t + \frac{6}{5}\sin 3t - \frac{6}{5}\sin 2t - \frac{1}{6}\cos 3t + \frac{t}{2}\sin 3t, \\ y'(0) &= 3c_1 - \frac{1}{6}, \\ y'(0) = 0 &\Rightarrow c_1 = \frac{1}{18}. \end{aligned}$$

Thus

$$y = \frac{1}{18}\sin 3t - \frac{2}{5}\cos 3t + \frac{2}{5}\cos 2t - \frac{t}{6}\cos 3t.$$

Solutions to Exercises 2.9

1.

$$|f_n(x)| = \left|\frac{\sin nx}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty.$$

The sequence converges uniformly to 0 for all real x, because $\frac{1}{\sqrt{n}}$ controls its size independently of x.

5. If $x = 0$ then $f_n(0) = 0$ for all n. If $x \neq 0$, then applying l'Hospital's rule, we find

$$\lim_{n\to\infty} |f_n(x)| = |x| \lim_{n\to\infty} \frac{n}{e^{-nx}} = |x| \lim_{n\to\infty} \frac{1}{|x|e^{-n}} = 0.$$

The sequence does not converge uniformly on any interval that contains 0 because $f_n(\frac{1}{n}) = e^{-1}$, which does not tend to 0.

9. $\left|\frac{\cos kx}{k^2}\right| \leq \frac{1}{k^2} = M_k$ for all x. Since $\sum M_k < \infty$ (p-series with $p > 1$), the series converges uniformly for all x.

17. $\left|\frac{(-1)^k}{|x|+k^2}\right| \leq \frac{1}{k^2} = M_k$ for all x. Since $\sum M_k < \infty$ (p-series with $p > 1$), the series converges uniformly for all x.

Solutions to Exercises 2.10

5. The cosine part converges uniformly for all x, by the Weierstrass M-test. The sine part converges for all x by Theorem 2(b). Hence the given series converges for all x.

9. (a) If $\lim_{k\to\infty} \sin kx = 0$, then

$$\lim_{k\to\infty} \sin^2 kx = 0 \quad \Rightarrow \quad \lim_{k\to\infty} (1 - \cos^2 kx) = 0 \quad \Rightarrow \quad \lim_{k\to\infty} \cos^2 kx = 1 \quad (*).$$

Also, if $\lim_{k\to\infty} \sin kx = 0$, then $\lim_{k\to\infty} \sin(k+1)x) = 0$. But $\sin(k+1)x = \sin kx \cos x + \cos kx \sin x$, so

$$\begin{aligned} 0 = \lim_{k\to\infty} (\overbrace{\sin kx \cos x}^{\to 0} + \cos kx \sin x) \quad &\Rightarrow \quad \lim_{k\to\infty} \cos kx \sin x = 0 \\ &\Rightarrow \quad \lim_{k\to\infty} \cos kx = 0 \text{ or } \sin x = 0. \end{aligned}$$

By (*), $\cos kx$ does not tend to 0, so $\sin x = 0$, implying that $x = m\pi$. Consequently, if $x \neq m\pi$, then $\lim k \to \infty \sin kx$ is not 0 and the series $\sum_{k=1}^{\infty} \sin kx$ does not converge by the nth term test, which proves (b).

Solutions to Exercises 3.1

1. $u_{xx} + u_{xy} = 2u$ is a second order, linear, and homogeneous partial differential equation. $u_x(0, y) = 0$ is linear and homogeneous.

5. $u_t u_x + u_{xt} = 2u$ is second order and nonlinear because of the term $u_t u_x$. $u(0, t) + u_x(0, t) = 0$ is linear and homogeneous.

9. (a) Let $u(x, y) = e^{ax}e^{by}$. Then

$$\begin{aligned} u_x &= ae^{ax}e^{by} \\ u_y &= be^{ax}e^{by} \\ u_{xx} &= a^2e^{ax}e^{by} \\ u_{yy} &= b^2e^{ax}e^{by} \\ u_{xy} &= abe^{ax}e^{by}. \end{aligned}$$

So

$$\begin{aligned} &Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \\ \Leftrightarrow\ & Aa^2e^{ax}e^{by} + 2Babe^{ax}e^{by} + Cb^2e^{ax}e^{by} \\ & +Dae^{ax}e^{by} + Ebe^{ax}e^{by} + Fe^{ax}e^{by} = 0 \\ \Leftrightarrow\ & e^{ax}e^{by}\big(Aa^2 + 2Bab + Cb^2 + Da + Eb + F\big) = 0 \\ \Leftrightarrow\ & Aa^2 + 2Bab + Cb^2 + Da + Eb + F = 0, \end{aligned}$$

because $e^{ax}e^{by} \neq 0$ for all x and y.
(b) By (a), in order to solve

$$u_{xx} + 2u_{xy} + u_{yy} + 2\,u_x + 2\,u_y + u = 0,$$

we can try $u(x, y) = e^{ax}e^{by}$, where a and b are solutions of

$$a^2 + 2ab + b^2 + 2a + 2b + 1 = 0.$$

But

$$a^2 + 2ab + b^2 + 2a + 2b + 1 = (a + b + 1)^2.$$

So $a + b + 1 = 0$. Clearly, this equation admits infinitely many pairs of solutions (a, b). Here are four possible solutions of the partial differential equation:

$$\begin{aligned} a = 1,\ b = -2 \quad &\Rightarrow \quad u(x, y) = e^x e^{-2y} \\ a = 0,\ b = -1 \quad &\Rightarrow \quad u(x, y) = e^{-y} \\ a = -1/2,\ b = -1/2 \quad &\Rightarrow \quad u(x, y) = e^{-x/2}e^{-y/2} \\ a = -3/2,\ b = 1/2 \quad &\Rightarrow \quad u(x, y) = e^{-3x/2}e^{y/2} \end{aligned}$$

13. We follow the outlined solution in Exercise 12. We have

$$A(u) = \ln(u),\ \phi(x) = e^x, \quad \Rightarrow \quad A(u(x(t)), t)) = A(\phi(x(0))) = \ln(e^{x(0)}) = x(0).$$

So the characteristic lines are

$$x = tx(0) + x(0) \quad \Rightarrow \quad x(0) = L(x\,t) = \frac{x}{t+1}.$$

So $u(x, t) = f(L(x, t)) = f\left(\frac{x}{t+1}\right)$. The condition $u(x, 0) = e^x$ implies that $f(x) = e^x$ and so

$$u(x, t) = e^{\frac{x}{t+1}}.$$

Check: $u_t = -e^{\frac{x}{t+1}}\frac{x}{(t+1)^2}$, $u_x = e^{\frac{x}{t+1}}\frac{1}{t+1}$,

$$u_t + \ln(u)u_x = -e^{\frac{x}{t+1}}\frac{x}{(t+1)^2} + \frac{x}{t+1}e^{\frac{x}{t+1}}\frac{1}{t+1} = 0.$$

17. We have

$$A(u) = u^2,\ \phi(x) = \sqrt{x}, \quad \Rightarrow \quad A(u(x(t)),\, t)) = A(\phi(x(0))) = x(0).$$

So the characteristic lines are

$$x = tx(0) + x(0) \quad \Rightarrow \quad x(0)(t+1) - x = 0.$$

Solving for $x(0)$, we find

$$x(0) = \frac{x}{t+1},$$

and so

$$u(x,\, t) = f\left(\frac{x}{t+1}\right).$$

Now

$$u(x,\, 0) = f(x) = \sqrt{x}.$$

So

$$u(x,\, t) = \sqrt{\frac{x}{t+1}}.$$

Solutions to Exercises 3.3

1. The solution is

$$u(x,\,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(b_n \cos c\frac{n\pi t}{L} + b_n^* \sin c\frac{n\pi t}{L}\right),$$

where b_n are the Fourier sine coefficients of f and b_n^* are $\frac{L}{cn\pi}$ times the Fourier coefficients of g. In this exercise, $b_n^* = 0$, since $g = 0$, $b_1 = 0.05$; and $b_n = 0$ for all $n > 1$, because f is already given by its Fourier sine series (period 2). So $u(x,\,t) = 0.05 \sin \pi x \cos t$.

5. (a) The solution is

$$u(x,\,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(b_n \cos(4n\pi t) + b_n^* \sin(4n\pi t)\right),$$

where b_n is the nth sine Fourier coefficient of f and b_n^* is $L/(cn)$ times the Fourier coefficient of g, where $L = 1$ and $c = 4$. Since $g = 0$, we have $b_n^* = 0$ for all n. As for the Fourier coefficients of f, we can get them by using Exercise 17, Section 2.4, with $p = 1$, $h = 1$, and $a = 1/2$. We get

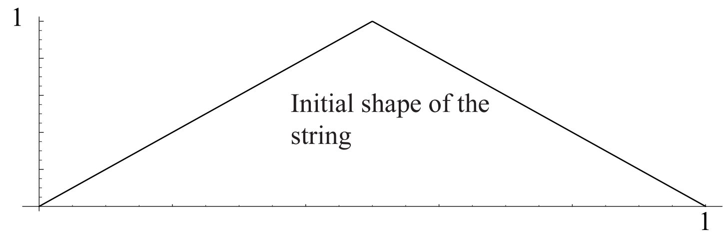
$$b_n = \frac{8}{\pi^2} \frac{\sin\frac{n\pi}{2}}{n^2}.$$

Thus

$$\begin{aligned} u(x,\,t) &= \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n^2} \sin(n\pi x)\cos(4n\pi t) \\ &= \frac{8}{\pi^2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x)\cos(4(2k+1)\pi t). \end{aligned}$$

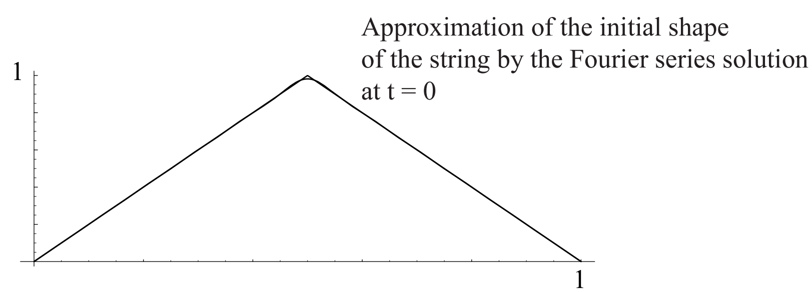
(b) Here is the initial shape of the string. Note the new Mathematica command that we used to define piecewise a function. (Previously, we used the If command.)

```
Clear[f]
f[x_] := 2 x   /; 0 < x < 1 / 2
f[x_] := 2 (1 - x)  /; 1 / 2 < x < 1
Plot[f[x], {x, 0, 1}]
```



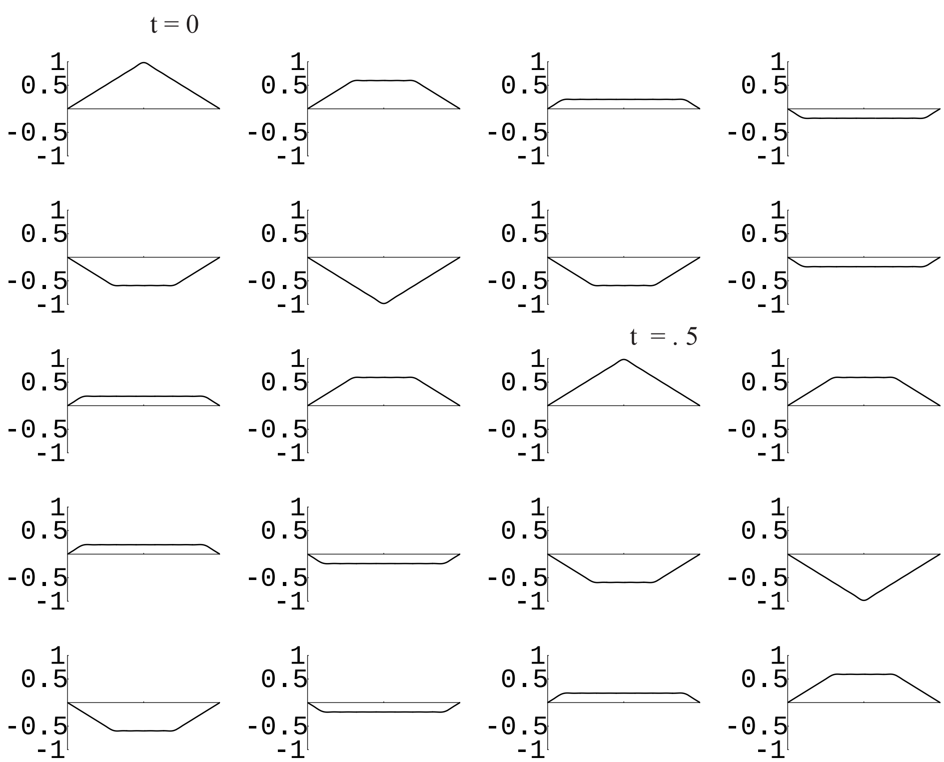
Because the period of $\cos(4(2k+1)\pi t)$ is $1/2$, the motion is periodic in t with period $1/2$. This is illustrated by the following graphs. We use two different ways to plot the graphs: The first uses simple Mathematica commands; the second one is more involved and is intended to display the graphs in a convenient array.

```
Clear[partsum]
partsum[x_, t_] :=
 8 / Pi^2 Sum[Sin[(-1)^k (2 k + 1) Pi x] Cos[4 (2 k + 1) Pi t] / (2 k + 1)^2, {k, 0, 10}]
Plot[Evaluate[{partsum[x, 0], f[x]}], {x, 0, 1}]
```



Here is the motion in an array.

```
tt = Table[
   Plot[Evaluate[ partsum[x, t] ], {x, 0, 1}, PlotRange → {{0, 1}, {-1, 1}},
    Ticks → {{.5}, {-1, -.5, .5, 1}}, DisplayFunction → Identity], {t, 0, 1, 1 / 20}];
Show[GraphicsArray[Partition[tt, 4]]]
```



The first frame is the initial shape at t = 0. Subsequent frames occur in icrements of time of size 1/20.

9. The solution is

$$u(x,\,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(b_n \cos(n\pi t) + b_n^* \sin(n\pi t)\right),$$

where $b_1^* = \frac{1}{\pi}$ and all other $b_n^* = 0$. The Fourier coefficients of f are

$$b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx.$$

To evaluate this integral, we will use integration by parts to derive first the formula: for $a \neq 0$,

$$\int x\sin(ax)\,dx = -\frac{x\,\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} + C,$$

and

$$\int x^2\sin(ax)\,dx = \frac{2\,\cos(a\,x)}{a^3} - \frac{x^2\,\cos(a\,x)}{a} + \frac{2\,x\,\sin(a\,x)}{a^2} + C;$$

thus

$$\begin{aligned}&\int x(1-x)\sin(ax)\,dx\\ &= \frac{-2\,\cos(a\,x)}{a^3} - \frac{x\,\cos(a\,x)}{a} + \frac{x^2\,\cos(a\,x)}{a} + \frac{\sin(a\,x)}{a^2} - \frac{2\,x\,\sin(a\,x)}{a^2} + C.\end{aligned}$$

Applying the formula with $a = n\pi$, we get

$$\begin{aligned}&\int_0^1 x(1-x)\sin(n\pi x)\,dx\\ &= \frac{-2\,\cos(n\pi\,x)}{(n\pi)^3} - \frac{x\,\cos(n\pi\,x)}{n\pi} + \frac{x^2\,\cos(n\pi\,x)}{n\pi} + \frac{\sin(n\pi\,x)}{(n\pi)^2} - \frac{2\,x\,\sin(n\pi\,x)}{(n\pi)^2}\Big|_0^1\\ &= \frac{-2\,((-1)^n-1)}{(n\pi)^3} - \frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi} = \frac{-2\,((-1)^n-1)}{(n\pi)^3}\\ &= \begin{cases}\frac{4}{(n\pi)^3} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

Thus

$$b_n = \begin{cases}\frac{8}{(n\pi)^3} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even,}\end{cases}$$

and so

$$u(x,\,t) = \frac{8}{\pi^3}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)\cos((2k+1)\pi t)}{(2k+1)^3} + \frac{1}{\pi}\sin(\pi x)\sin(\pi t).$$

13. To solve

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\\ u(0,t) = u(\pi,t) = 0,\\ u(x,0) = \sin x, \quad \frac{\partial u}{\partial t}(x,0) = 0,\end{gathered}$$

we follow the method of the previous exercise. We have $c = 1$, $k = .5$, $L = \pi$, $f(x) = \sin x$, and $g(x) = 0$. Thus the real number $\frac{Lk}{c\pi} = .5$ is not an integer and we have $n > \frac{kL}{\pi}$ for all n. So only Case III from the solution of Exercise 12 needs to be considered. Thus

$$u(x,\,t) = \sum_{n=1}^{\infty} e^{-.5t} \sin nx\big(a_n \cos \lambda_n t + b_n \sin \lambda_n t,$$

where

$$\lambda_n = \sqrt{(.5n)^2 - 1}.$$

Setting $t = 0$, we obtain

$$\sin x = \sum_{n=1}^{\infty} a_n \sin nx.$$

Hence $a_1 = 1$ and $a_n = 0$ for all $n > 1$. Now since

$$b_n = \frac{ka_n}{\lambda_n} + \frac{2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi}{L} x\, dx, \quad n = 1, 2, \ldots,$$

it follows that $b_n = 0$ for all $n > 1$ and and the solution takes the form

$$u(x,\,t) = e^{-.5t} \sin x\big(\cos \lambda_1 t + b_1 \sin \lambda_1 t\big),$$

where $\lambda_1 = \sqrt{(.5)^2 - 1} = \sqrt{.75} = \frac{\sqrt{3}}{2}$ and

$$b_1 = \frac{ka_1}{\lambda_1} = \frac{1}{\sqrt{3}}.$$

So

$$u(x,\,t) = e^{-.5t} \sin x\big(\cos(\frac{\sqrt{3}}{2} t) + \frac{1}{\sqrt{3}} \sin(\frac{\sqrt{3}}{2} t)\big).$$

Solutions to Exercises 3.4

1. We will use (5), since $g^* = 0$. The odd extension of period 2 of $f(x) = \sin \pi x$ is $f^*(x) = \sin \pi x$. So

$$u(x,\, t) = \frac{1}{2}\big[\sin(\pi(x + \frac{t}{\pi})) + \sin(\pi(x - \frac{t}{\pi}))\big] = \frac{1}{2}\big[\sin(\pi x + t) + \sin(\pi x - t)\big].$$

5. The solution is of the form

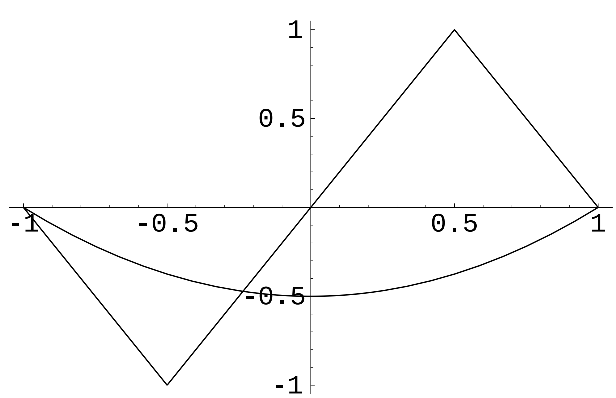
$$\begin{aligned} u(x,\, t) &= \frac{1}{2}\big[f^*(x-t) + f(x+t)\big] + \frac{1}{2}\big[G(x+t) - G(x-t)\big] \\ &= \frac{1}{2}\Big[\big(f^*(x-t) - G(x-t)\big) + \big(f^*(x+t) + G(x+t)\big)\Big], \end{aligned}$$

where f^* is the odd extension of f and G is as in Example 3. In the second equality, we expressed u as the average of two traveling waves: one wave traveling to the right and one to the left. Note that the waves are not the same, because of the G term. We enter the formulas in Mathematica and illustrate the motion of the string.

The difficult part in illustrating this example is to define periodic functions with *Mathematica*. This can be done by appealing to results from Section 2.1. We start by defining the odd extensions of f and G (called big g) on the interval [-1, 1].

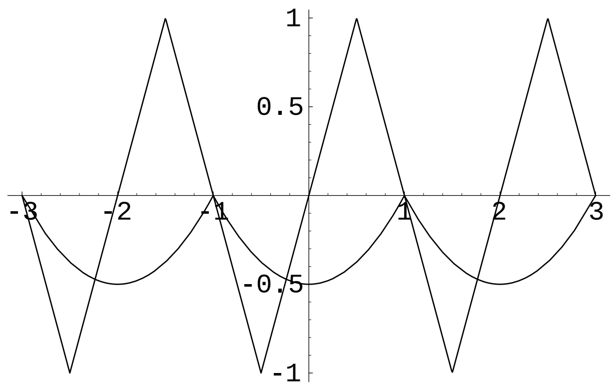
```
Clear[f, bigg]
f[x_] := 2 x   /; -1 / 2 < x < 1 / 2
f[x_] := 2 ( 1 - x)  /; 1 / 2 < x < 1
f[x_] := -2 ( 1 + x)  /; -1 < x < -1 / 2
bigg[x_] = 1 / 2 x^2 - 1 / 2

Plot[{f[x], bigg[x]}, {x, -1, 1}]
```



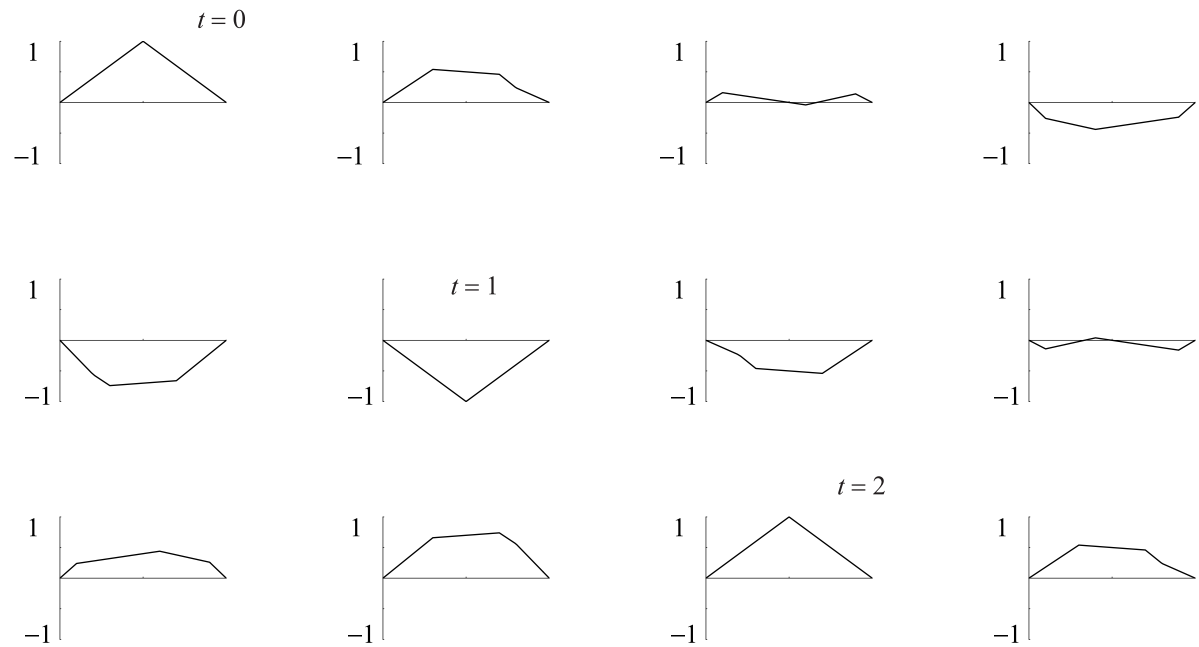
Here is a tricky Mathematica construction. (Review Section 2.1.)

```
extend[x_] := x - 2 Floor[(x + 1) / 2]
periodicf[x_] := f[extend[x]]
periodicbigg[x_] := bigg[extend[x]]
Plot[{periodicf[x], periodicbigg[x]}, {x, -3 , 3}]
```



Because f^* and G are 2-periodic, it follows immediately that $f^*(x \pm ct)$ and $G(x \pm ct)$ are $2/c$-periodic in t. Since $c = 1$, u is 2-periodic in t. The following is an array of snapshots of u. You can also illustrate the motion of the string using Mathematica (see the Mathematica notebooks). Note that in this array we have graphed the exact solution and not just an approximation using a Fourier series. This is a big advantage of the d'Alembert's solution over the Fourier series solution.

```
u[x_, t_] := 1/2 (periodicf[x - t] + periodicf[x + t]) +
  1/2 (periodicbigg[x - t] - periodicbigg[x + t])
tt = Table[
   Plot[Evaluate[ u[x, t] ], {x, 0, 1}, PlotRange → {{0, 1}, {-1, 1}},
    Ticks → {{.5}, {-1, -.5, .5, 1}}, DisplayFunction → Identity], {t, 0, 2.3, 1/5}];
Show[GraphicsArray[Partition[tt, 4]]]
```



9. You can use Exercise 11, Section 3.3, which tells us that the time period of motion is $T = \frac{2L}{c}$. So, in the case of Exercise 1, $T = 2\pi$, and in the case of Exercise 5, $T = 2$. You can also obtain these results directly by considering the formula for $u(x, t)$. In the case of Exercise 1, $u(x, t) = \frac{1}{2}\big[\sin(\pi x + t) + \sin(\pi x - t)\big]$ so $u(x, t + 2\pi) = \frac{1}{2}\big[\sin(\pi x + t2\pi) + \sin(\pi x - t2\pi)\big] = u(x, t)$. In the case of Exercise 5, use the fact that f^* and G are both 2-periodic.

13. We have

$$u(x, t) = \frac{1}{2}\left[f^*(x + ct) + f^*(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\, ds,$$

where f^* and g^* are odd and $2L$-periodic. So

$$u(x, t + \frac{L}{c}) = \frac{1}{2}\left[f^*(x + ct + L) + f^*(x - ct - L)\right] + \frac{1}{2c}\int_{x-ct-L}^{x+ct+L} g^*(s)\, ds.$$

Using the fact that f^* is odd, $2L$-period, and satisfies $f^*(L - x) = f^*(x)$ (this property is given for f but it extends to f^*), we obtain

$$\begin{aligned} f^*(x + ct + L) &= f^*(x + ct + L - 2L) = f^*(x + ct - L) \\ &= -f^*(L - x - ct) = -f^*(L - (x + ct)) = -f^*(x + ct). \end{aligned}$$

Similarly

$$\begin{aligned} f^*(x - ct - L) &= -f^*(L - x + ct) \\ &= -f^*(L - x + ct) = -f^*(L - (x - ct)) = -f^*(x - ct). \end{aligned}$$

Also $g^*(s+L) = -g^*(-s-L) = -g^*(-s-L+2L) = -g^*(L-s) = -g^*(s)$, by the given symmetry property of g. So, using a change of variables, we have

$$\frac{1}{2c}\int_{x-ct-L}^{x+ct+L} g^*(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s+L)\,ds = -\frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds.$$

Putting these identities together, it follows that $u(x, t+\frac{L}{c}) = -u(x, t)$.

17. (a) To prove that G is even, see Exercise 14(a). That G is $2L$-periodic follows from the fact that g is $2L$-periodic and its integral over one period is 0, because it is odd (see Section 2.1, Exercise 15).

Since G is an antiderivative of g^*, to obtain its Fourier series, we apply Exercise 33, Section 3.3, and get

$$G(x) = A_0 - \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n}\cos\frac{n\pi}{L}x,$$

where $b_n(g)$ is the nth Fourier sine coefficient of g^*,

$$b_n(g) = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

and

$$A_0 = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n}.$$

In terms of b_n^*, we have

$$\frac{L}{\pi}\frac{b_n(g)}{n} = \frac{2}{n\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx = cb_n^*,$$

and so

$$\begin{aligned} G(x) &= \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n} - \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n(g)}{n}\cos\frac{n\pi}{L}x \\ &= \sum_{n=1}^{\infty} cb_n^*\Big(1-\cos(\frac{n\pi}{L}x)\Big). \end{aligned}$$

(b) From (a), it follows that

$$\begin{aligned} G(x+ct)-G(x-ct) &= \sum_{n=1}^{\infty} cb_n^*\left[\Big(1-\cos(\frac{n\pi}{L}(x+ct))\Big) - \Big(1-\cos(\frac{n\pi}{L}(x-ct))\Big)\right] \\ &= \sum_{n=1}^{\infty} -cb_n^*\left[\cos(\frac{n\pi}{L}(x+ct)) - \cos(\frac{n\pi}{L}(x-ct))\right] \end{aligned}$$

(c) Continuing from (b) and using the notation in the text, we obtain

$$\begin{aligned} \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)\,ds &= \frac{1}{2c}\left[G(x+ct)-G(x-ct)\right] \\ &= \sum_{n=1}^{\infty} -b_n^*\frac{1}{2}\left[\cos(\frac{n\pi}{L}(x+ct))-\cos(\frac{n\pi}{L}(x-ct))\right] \\ &= \sum_{n=1}^{\infty} b_n^*\sin(\frac{n\pi}{L}x)\sin(\frac{n\pi}{L}ct) \\ &= \sum_{n=1}^{\infty} b_n^*\sin(\frac{n\pi}{L}x)\sin(\lambda_n t). \end{aligned}$$

(d) To derive d'Alembert's solution from (8), Section 3.3, proceed as follows:

$$\begin{aligned} u(x,\,t) &= \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi}{L}x)\cos(\lambda_n t)+\sum_{n=1}^{\infty} b_n^*\sin(\frac{n\pi}{L}x)\sin(\lambda_n t) \\ &= \frac{1}{2}\big(f^*(x-ct)+f^*(x+ct)\big)+\frac{1}{2c}\left[G(x+ct)-G(x-ct)\right], \end{aligned}$$

where in the last equality we used Exercise 16 and part (c).

21. Follow the labeling of Figure 8 in Section 3.4. Let $P_1=(x_0,\,t_0)$ be an arbitrary point in the region II. Form a characteristic parallelogram with vertices P_1, P_2, Q_1, Q_2, as shown in Figure 8 in Section 3.4. The vertices P_2 and Q_1 are on the characteristic line $x+2t=1$ and the vertex Q_2 is on the boundary line $x=1$. From Proposition 1, we have

$$u(P_1)=u(Q_1)+u(Q_2)-u(P_2)=u(Q_1)-u(P_2),$$

because $u(Q_2)=0$. We will find $u(P_2)$ and $u(Q_1)$ by using the formula $u(x,\,t)=-4t^2+x-x^2+8tx$ from Example 4, because P_2 and Q_1 are in the region I.

The point Q_1 is the intersection point of the characteristic lines $x-2t=x_0-2t_0$ and $x+2t=1$. Adding the equations and then solving for x, we get

$$x=\frac{x_0+1-2t_0}{2}.$$

The second coordinate of Q_1 is then

$$t=\frac{1-x_0+2t_0}{4}.$$

The point Q_2 is the intersection point of the characteristic line $x+2t=x_0+2t_0$ and $x=1$. Thus

$$t=\frac{x_0+2t_0-1}{2}.$$

The point P_2 is the intersection point of the characteristic lines $x+2t=1$ and $x-2t=1-(x_0+2t_0-1)$. Solving for x and t, we find the coordinates of P_2 to be

$$x=\frac{3-x_0-2t_0}{2}\quad\text{and}\quad t=\frac{-1+x_0+2t_0}{4}.$$

To simplify the notation, replace x_0 and t_0 by x and y in the coordinates of the points Q_1 and P_2 and let $\phi(x, t) = -4t^2 + x - x^2 + 8tx$. We have

$$\begin{aligned} u(x, t) &= u(Q_1) - u(P_2) \\ &= \phi\left(\frac{x+1-2t}{2}, \frac{1-x+2t}{4}\right) - \phi\left(\frac{3-x-2t}{2}, \frac{-1+x+2t}{4}\right) \\ &= 5 - 12t - 5x + 12tx, \end{aligned}$$

where the last expression was derived after a few simplifications that we omit. It is interesting to note that the formula satisfies the wave equation and the boundary condition $u(1, t) = 0$ for all $t > 0$. Its restriction to the line $x + 2t = 1$ (part of the boundary of region I) reduces to the formula for $u(x, t)$ for (x, t) in region I. This is to be expected since u is continuous in (x, t).

Solutions to Exercises 3.5

1. Multiply the solution in Example 1 by $\frac{78}{100}$ to obtain

$$u(x,\,t) = \frac{312}{\pi} \sum_{k=0}^{\infty} \frac{e^{-(2k+1)^2 t}}{2k+1} \sin(2k+1)x.$$

5. We have

$$u(x,\,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x),$$

where

$$\begin{aligned} b_n &= 2\int_0^1 x \sin(n\pi x)\,dx = 2\left[-\frac{x\,\cos(n\,\pi\,x)}{n\,\pi} + \frac{\sin(n\,\pi\,x)}{n^2\,\pi^2}\right]\Big|_0^1 \\ &= -2\frac{\cos n\,\pi}{n\,\pi} = 2\frac{(-1)^{n+1}}{n\pi}. \end{aligned}$$

So

$$u(x,\,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-(n\pi)^2 t} \sin(n\pi x)}{n}.$$

9. (a) The steady-state solution is a linear function that passes through the points (0, 0) and (1, 100). Thus, $u(x) = 100x$.
(b) The steady-state solution is a linear function that passes through the points (0, 100) and (1, 100). Thus, $u(x) = 100$. This is also obvious: If you keep both ends of the insulated bar at 100 degrees, the steady-state temperature will be 100 degrees.

13. We have $u_1(x) = -\frac{50}{\pi}x + 100$. We use (13) and the formula from Exercise 10, and get (recall the Fourier coefficients of f from Exercise 3)

$$\begin{aligned} u(x,\,t) &= -\frac{50}{\pi}x + 100 \\ &\quad + \sum_{n=1}^{\infty} \left[\frac{132}{\pi}\frac{\sin(n\frac{\pi}{2})}{n^2} - 100\left(\frac{2-(-1)^n}{n\pi}\right)\right] e^{-n^2 t} \sin nx. \end{aligned}$$

17. Fix $t_0 > 0$ and consider the solution at time $t = t_0$:

$$u(x,\,t_0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L} x e^{-\lambda_n^2 t_0}.$$

We will show that this series converges uniformly for all x (not just $0 \le x \le L$) by appealing to the Weierstrass M-test. For this purpose, it suffices to establish the following two inequalities:
(a) $\left|b_n \sin\frac{n\pi}{L} x e^{-\lambda_n^2 t_0}\right| \le M_n$ for all x; and
(b) $\sum_{n=1}^{\infty} M_n < \infty$.
To establish (a), note that

$$\begin{aligned} |b_n| &= \left|\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right| \le \frac{2}{L}\int_0^L \left|f(x)\sin\frac{n\pi}{L}x\right|\,dx \\ &\quad \text{(The absolute value of the integral is} \\ &\quad \le \text{ the integral of the absolute value.)} \\ &\le \frac{2}{L}\int_0^L |f(x)|\,dx = A \text{ (because } |\sin u| \le 1 \text{ for all } u). \end{aligned}$$

Note that A is a finite number because f is bounded, so its absolute value is bounded and hence its integral is finite on $[0, L]$. We have

$$\begin{aligned} \left|b_n \sin \frac{n\pi}{L} x e^{-\lambda_n^2 t_0}\right| &\leq A e^{-\lambda_n^2 t_0} = A e^{-\frac{c^2\pi^2 t_0}{L^2} n^2} \\ &\leq A \left(e^{-\frac{c^2\pi^2 t_0}{L^2}}\right)^n = A r^n, \end{aligned}$$

where $r = e^{-\frac{c^2\pi^2 t_0}{L^2}} < 1$. Take $M_n = A r^n$. Then a holds and $\sum M_n$ is convergent because it is a geometric series with ratio $r < 1$.

Solutions to Exercises 3.6

1. Since the bar is insulated and the temperature inside is constant, there is no exchange of heat, and so the temperature remains constant for all $t > 0$. Thus $u(x, t) = 100$ for all $t > 0$. This is also a consequence of (2), since in this case all the a_n's are 0 except $a_0 = 100$.

5. Apply the separation of variables method as in Example 1; you will arrive at the following equations in X and T:

$$X'' - kX = 0, \quad X(0) = 0, \; X'(L) = 0$$
$$T' - kc^2T = 0 \qquad .$$

We now show that the separation constant k has to be negative by ruling out the possibilities $k = 0$ and $k > 0$.

If $k = 0$ then $X'' = 0 \Rightarrow X = ax + b$. Use the initial conditions $X(0) = 0$ implies that $b = 0$, $X'(L) = 0$ implies that $a = 0$. So $X = 0$ if $k = 0$.

If $k > 0$, say $k = \mu^2$, where $\mu > 0$, then

$$\begin{aligned} X'' - \mu^2 X = 0 \quad &\Rightarrow \quad X = c_1 \cosh \mu x + c_2 \sinh \mu x; \\ X(0) = 0 \quad &\Rightarrow \quad 0 = c_1; X = c_2 \sinh \mu x; \\ X'(L) = 0 \quad &\Rightarrow \quad 0 = c_2 \mu \cosh(\mu L) \\ &\Rightarrow \quad c_2 = 0, \end{aligned}$$

because $\mu \neq 0$ and $\cosh(\mu L) \neq 0$. So $X = 0$ if $k > 0$. This leaves the case $k = -\mu^2$, where $\mu > 0$. In this case

$$\begin{aligned} X'' + \mu^2 X = 0 \quad &\Rightarrow \quad X = c_1 \cos \mu x + c_2 \sin \mu x; \\ X(0) = 0 \quad &\Rightarrow \quad 0 = c_1; X = c_2 \sin \mu x; \\ X'(L) = 0 \quad &\Rightarrow \quad 0 = c_2 \mu \cos(\mu L) \\ &\Rightarrow \quad c_2 = 0 \text{ or } \cos(\mu L) = 0. \end{aligned}$$

To avoid the trivial solution, we set $\cos(\mu L) = 0$, which implies that

$$\mu = (2k+1)\frac{\pi}{2L}, \quad k = 0, 1, \ldots .$$

Plugging this value of k in the equation for T, we find

$$T' + \mu^2 c^2 T = 0 \quad \Rightarrow \quad T(t) = B_k e^{-\mu^2 c^2 t} = B_k e^{-\left((2k+1)\frac{\pi}{2L}\right)^2 c^2 t}.$$

Forming the product solutions and superposing them, we find that

$$u(x, t) = \sum_{k=0}^{\infty} B_k e^{-\mu^2 c^2 t} = \sum_{k=0}^{\infty} B_k e^{-\left((2k+1)\frac{\pi}{2L}\right)^2 c^2 t} \sin\left[(2k+1)\frac{\pi}{2L}x\right].$$

To determine the coefficients B_k, we use the initial condition and proceed as in Example 1:

$$
\begin{aligned}
u(x,0)=f(x) \quad &\Rightarrow \quad f(x)=\sum_{k=0}^{\infty} B_k \sin\left[(2k+1)\frac{\pi}{2L}x\right];\\
&\Rightarrow \quad f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right]\\
&\qquad =\sum_{k=0}^{\infty} B_k \sin\left[(2k+1)\frac{\pi}{2L}x\right]\sin\left[(2n+1)\frac{\pi}{2L}x\right]\\
&\Rightarrow \quad \int_0^L f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx\\
&\qquad =\sum_{k=0}^{\infty} B_k \int_0^L \sin\left[(2k+1)\frac{\pi}{2L}x\right]\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx\\
&\Rightarrow \quad \int_0^L f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx\\
&\qquad = B_n \int_0^L \sin^2\left[(2n+1)\frac{\pi}{2L}x\right]\,dx,
\end{aligned}
$$

where we have integrated the series term by term and used the orthogonality of the functions $\sin\left[(2k+1)\frac{\pi}{2L}x\right]$ on the interval $[0, L]$. The orthogonality can be checked directly by verifying that

$$\int_0^L \sin\left[(2k+1)\frac{\pi}{2L}x\right]\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx=0$$

if $n\neq k$. Solving for B_n and using that

$$\int_0^L \sin^2\left[(2n+1)\frac{\pi}{2L}x\right]\,dx=\frac{L}{2}$$

(check this using a half-angle formula), we find that

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\left[(2n+1)\frac{\pi}{2L}x\right]dx.$$

9. This is a straightforward application of Exercise 7. For Exercise 1 the average is 100. For Exercise 2 the average is $a_0=0$.

13. The solution is given by (8), where c_n is given by (11). We have

$$
\begin{aligned}
\int_0^1 \sin^2\mu_n x\,dx &= \frac{1}{2}\int_0^1 (1-\cos(2\mu_n x)\,dx\\
&= \frac{1}{2}\left(x-\frac{1}{2\mu_n}\sin(2\mu_n x)\right)\Big|_0^1=\frac{1}{2}\left(1-\frac{1}{2\mu_n}\sin(2\mu_n)\right).
\end{aligned}
$$

Since μ_n is a solution of $\tan\mu=-\mu$, we have $\sin\mu_n=-\mu_n\cos\mu_n$, so

$$\sin 2\mu_n = 2\sin\mu_n\cos\mu_n=-2\mu_n\cos^2\mu_n,$$

and hence

$$\int_0^1 \sin^2\mu_n x\,dx=\frac{1}{2}\left(1+\cos^2\mu_n\right).$$

Also,

$$\int_0^{1/2} \sin \mu_n x\, dx = \frac{1}{\mu_n}\big(1 - \cos\frac{\mu_n}{2}\big).$$

Applying (11), we find

$$\begin{aligned} c_n &= \int_0^{1/2} 100 \sin \mu_n x\, dx \Big/ \int_0^1 \sin^2 \mu_n x\, dx \\ &= \frac{100}{\mu_n}\big(1 - \cos\frac{\mu_n}{2}\big) \Big/ \frac{1}{2}\left(1 + \cos^2 \mu_n\right) \\ &= \frac{200\big(1 - \cos\frac{\mu_n}{2}\big)}{\mu_n\left(1 + \cos^2 \mu_n\right)}. \end{aligned}$$

Thus the solution is

$$u(x,\, t) = \sum_{n=1}^{\infty} \frac{200\big(1 - \cos\frac{\mu_n}{2}\big)}{\mu_n\left(1 + \cos^2 \mu_n\right)} e^{-\mu_n^2 t} \sin \mu_n x.$$

17. Part (a) is straightforward as in Example 2. We omit the details that lead to the separated equations:

$$T' - kT = 0,$$
$$X'' - kX = 0, \quad X'(0) = -X(0),\ X'(1) = -X(1),$$

where k is a separation constant.
(b) If $k = 0$ then

$$\begin{aligned} X'' = 0 &\;\Rightarrow\; X = ax + b, \\ X'(0) = -X(0) &\;\Rightarrow\; a = -b \\ X'(1) = -X(1) &\;\Rightarrow\; a = -(a + b) \;\Rightarrow\; 2a = -b; \\ &\;\Rightarrow\; a = b = 0. \end{aligned}$$

So $k = 0$ leads to trivial solutions.
(c) If $k = \alpha^2 > 0$, then

$$\begin{aligned} X'' - \mu^2 X = 0 &\;\Rightarrow\; X = c_1 \cosh \mu x + c_2 \sinh \mu x; \\ X'(0) = -X(0) &\;\Rightarrow\; \mu c_2 = -c_1 \\ X'(1) = -X(1) &\;\Rightarrow\; \mu c_1 \sinh \mu + \mu c_2 \cosh \mu = -c_1 \cosh \mu - c_2 \sinh \mu \\ &\;\Rightarrow\; \mu c_1 \sinh \mu - c_1 \cosh \mu = -c_1 \cosh \mu - c_2 \sinh \mu \\ &\;\Rightarrow\; \mu c_1 \sinh \mu = -c_2 \sinh \mu \\ &\;\Rightarrow\; \mu c_1 \sinh \mu = \frac{c_1}{\mu} \sinh \mu. \end{aligned}$$

Since $\mu \neq 0$, $\sinh \mu \neq 0$. Take $c_1 \neq 0$ and divide by $\sinh \mu$ and get

$$\mu c_1 = \frac{c_1}{\mu} \;\Rightarrow\; \mu^2 = 1 \;\Rightarrow\; k = 1.$$

So $X = c_1 \cosh x + c_2 \sinh x$. But $c_1 = -c_2$, so

$$X = c_1 \cosh x + c_2 \sinh x = c_1 \cosh x - c_1 \sinh x = c_1 e^{-x}.$$

Solving the equation for T, we find $T(t) = e^t$; thus we have the product solution

$$c_0 e^{-x} e^t,$$

where, for convenience, we have used c_0 as an arbitrary constant.
(d) If $k = -\alpha^2 < 0$, then

$$\begin{aligned}
X'' + \mu^2 X = 0 \quad &\Rightarrow \quad X = c_1 \cos \mu x + c_2 \sin \mu x;\\
X'(0) = -X(0) \quad &\Rightarrow \quad \mu c_2 = -c_1\\
X'(1) = -X(1) \quad &\Rightarrow \quad -\mu c_1 \sin \mu + \mu c_2 \cos \mu = -c_1 \cos \mu - c_2 \sin \mu\\
&\Rightarrow \quad -\mu c_1 \sin \mu - c_1 \cos \mu = -c_1 \cos \mu - c_2 \sin \mu\\
&\Rightarrow \quad -\mu c_1 \sin \mu = -c_2 \sin \mu\\
&\Rightarrow \quad -\mu c_1 \sin \mu = \frac{c_1}{\mu} \sin \mu.
\end{aligned}$$

Since $\mu \neq 0$, take $c_1 \neq 0$ (otherwise you will get a trivial solution) and divide by c_1 and get

$$\mu^2 \sin \mu = -\sin \mu \;\Rightarrow\; \sin \mu = 0 \;\Rightarrow\; \mu = n\pi,$$

where n is an integer. So $X = c_1 \cos n\pi x + c_2 \sin n\pi x$. But $c_1 = -c_2\mu$, so

$$X = -c_1\bigl(n\pi \cos n\pi x - \sin n\pi x\bigr).$$

Call $X_n = n\pi \cos n\pi x - \sin n\pi x$.
(e) To establish the orthogonality of the X_n's, treat the case $k = 1$ separately. For $k = -\mu^2$, we refer to the boundary value problem

$$X'' + \mu_n^2 X = 0,\; X(0) = -X'(0),\; X(1) = -X'(1),$$

that is satisfied by the X_n's, where $\mu_n = n\pi$. We establish orthogonality using a trick from Sturm-Liouville theory (Chapter 6, Section 6.2). Since

$$X_m'' = \mu_m^2 X_m \text{ and } X_n'' = \mu_n^2 X_n,$$

multiplying the first equation by X_n and the second by X_m and then subtracting the resulting equations, we obtain

$$\begin{gathered}
X_n X_m'' = \mu_m^2 X_m X_n \text{ and } X_m X_n'' = \mu_n^2 X_n X_m\\
X_n X_m'' - X_m X_n'' = (\mu_n^2 - \mu_m^2) X_m X_n\\
\bigl(X_n X_m' - X_m X_n'\bigr)' = (\mu_n^2 - \mu_m^2) X_m X_n
\end{gathered}$$

where the last equation follows by simply checking the validity of the identity $X_n X_m'' - X_m X_n'' = \bigl(X_n X_m' - X_m X_n'\bigr)'$. So

$$\begin{aligned}
(\mu_n^2 - \mu_m^2) \int_0^1 X_m(x) X_n(x)\, dx &= \int_0^1 \bigl(X_n(x) X_m'(x) - X_m(x) X_n'(x)\bigr)'\, dx\\
&= X_n(x) X_m'(x) - X_m(x) X_n'(x)\Big|_0^1,
\end{aligned}$$

because the integral of the derivative of a function is the function itself. Now we use the boundary conditions to conclude that

$$\begin{aligned}
&X_n(x) X_m'(x) - X_m(x) X_n'(x)\Big|_0^1\\
&= X_n(1) X_m'(1) - X_m(1) X_n'(1) - X_n(0) X_m'(0) + X_m(0) X_n'(0)\\
&= -X_n(1) X_m(1) + X_m(1) X_n(1) + X_n(0) X_m(0) - X_m(0) X_n(0)\\
&= 0.
\end{aligned}$$

Thus the functions are orthogonal. We still have to verify the orthogonality when one of the X_n's is equal to e^{-x}. This can be done by modifying the argument that we just gave.
(f) Superposing the product solutions, we find that

$$u(x,\,t)=c_0e^{-x}e^t+\sum_{n=1}^{\infty}c_nT_n(t)X_n(x).$$

Using the initial condition, it follows that

$$u(x,\,0)=f(x)=c_0e^{-x}+\sum_{n=1}^{\infty}c_nX_n(x).$$

The coefficients in this series expansion are determined by using the orthogonality of the X_n's in the usual way. Let us determine c_0. Multiplying both sides by e^{-x} and integrating term by term, it follows from the orthogonality of the X_n that

$$\int_0^1 f(x)e^{-x}\,dx=c_0\int_0^1 e^{-2x}dx+\sum_{n=1}^{\infty}c_n\overbrace{\int_0^1 X_n(x)e^{-x}\,dx}^{=0}.$$

Hence

$$\int_0^1 f(x)e^{-x}\,dx=c_0\int_0^1 e^{-2x}dx=c_0\frac{1-e^{-2}}{2}.$$

Thus

$$c_0=\frac{2e^2}{e^2-1}\int_0^1 f(x)e^{-x}\,dx.$$

In a similar way, we prove that

$$c_n=\frac{1}{\kappa_n}\int_0^1 f(x)X_n(x)\,dx$$

where

$$\kappa_n=\int_0^1 X_n^2(x)\,dx\,.$$

This integral can be evaluated as we did in Exercise 15 or by straightforward computations, using the explicit formula for the X_n's, as follows:

$$\begin{aligned}\int_0^1 X_n^2(x)\,dx &= \int_0^1 \big(n\pi\cos n\pi x-\sin n\pi x\big)^2\,dx\\
&= \int_0^1 \big(n^2\pi^2\cos^2 n\pi x+\sin^2 n\pi x-2n\pi\cos(n\pi x)\sin(n\pi x)\big)\,dx\\
&= \overbrace{\int_0^1 n^2\pi^2\cos^2 n\pi x\,dx}^{=(n^2\pi^2)/2}+\overbrace{\int_0^1 \sin^2 n\pi x\,dx}^{=1/2}\\
&\quad -2n\pi\overbrace{\int_0^1 \cos(n\pi x)\sin(n\pi x)\,dx}^{=0}\\
&= \frac{n^2\pi^2+1}{2}.\end{aligned}$$

Solutions to Exercises 3.7

5. We proceed as in Exercise 3. We have

$$u(x,\, y,\, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin m\pi x \sin n\pi y,$$

where $\lambda_{mn} = \sqrt{m^2+n^2}$, $B_{mn} = 0$, and

$$\begin{aligned} B^*_{mn} &= \frac{4}{\sqrt{m^2+n^2}}\int_0^1\int_0^1 \sin m\pi x\, \sin n\pi y\, dx\, dy \\ &= \frac{4}{\sqrt{m^2+n^2}}\int_0^1 \sin m\pi x\, dx \int_0^1 \sin n\pi y\, dy \\ &= \begin{cases} \frac{16}{\sqrt{m^2+n^2}(mn)\pi^2} & \text{if } m \text{ and } n \text{ are both odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$u(x,\, y,\, t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{16\, \sin((2k+1)\pi x)\sin((2l+1)\pi y)}{\sqrt{(2k+1)^2+(2l+1)^2}\,(2k+1)(2l+1)\pi^2} \sin\sqrt{(2k+1)^2+(2l+1)^2}t$$

Solutions to Exercises 3.8

1. The solution is given by

$$u(x,\, y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh(n\pi y),$$

where

$$\begin{aligned} B_n &= \frac{2}{\sinh(2n\pi)} \int_0^1 x\sin(n\pi x)\,dx \\ &= \frac{2}{\sinh(2n\pi)} \left[-\frac{x\,\cos(n\,\pi\,x)}{n\,\pi} + \frac{\sin(n\,\pi\,x)}{n^2\,\pi^2} \right] \Big|_0^1 \\ &= \frac{2}{\sinh(2n\pi)} \frac{-(-1)^n}{n\,\pi} = \frac{2}{\sinh(2n\pi)} \frac{(-1)^{n+1}}{n\,\pi}. \end{aligned}$$

Thus,

$$u(x,\, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sinh(2n\pi)} \sin(n\pi x)\sinh(n\pi y).$$

5. Start by decomposing the problem into four subproblems as described by Figure 3. Let $u_j(x,\, y)$ denote the solution to problem j $(j = 1,\, 2,\, 3,\, 4)$. Each u_j consists of only one term of the series solutions, because of the orthogonality of the sine functions. For example, to compute u_1, we have

$$u_1(x,\, y) = \sum_{n=1}^{\infty} A_n \sin n\pi x \sinh[n\pi(1-y)],$$

where

$$A_n = \frac{2}{\sinh n\pi} \int_0^1 \sin 7\pi x \sin n\pi x\, dx.$$

Since the integral is 0 unless $n = 7$ and, when $n = 7$,

$$A_7 = \frac{2}{\sinh 7\pi} \int_0^1 \sin^2 7\pi x\, dx = \frac{1}{\sinh 7\pi}.$$

Thus

$$u_1(x,\, y) = \frac{1}{\sinh 7\pi} \sin 7\pi x \sinh[7\pi(1-y)].$$

In a similar way, appealing to the formulas in the text, we find

$$\begin{aligned} u_2(x,\, y) &= \frac{1}{\sinh \pi} \sin \pi x \sinh(\pi y) \\ u_3(x,\, y) &= \frac{1}{\sinh 3\pi} \sinh[3\pi(1-x)] \sin(3\pi y) \\ u_4(x,\, y) &= \frac{1}{\sinh 6\pi} \sinh 6\pi x \sin(6\pi y); \\ u(x,\, y) &= u_1(x,\, y) + u_2(x,\, y) + u_3(x,\, y) + u_4(x,\, y) \\ &= \frac{1}{\sinh 7\pi} \sin 7\pi x \sinh[7\pi(1-y)] + \frac{1}{\sinh \pi} \sin(\pi x)\sinh(\pi y) \\ &\quad + \frac{1}{\sinh 3\pi} \sinh[3\pi(1-x)]\sin(3\pi y) + \frac{1}{\sinh 6\pi} \sinh(6\pi x)\sin(6\pi y) \end{aligned}$$

Solutions to Exercises 3.9

1. We apply (2), with $a = b = 1$:

$$u(x,\, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y,$$

where

$$\begin{aligned}
E_{mn} &= \frac{-4}{\pi^2(m^2+n^2)} \int_0^1\int_0^1 x \sin m\pi x \sin n\pi y \, dx\, dy \\
&= \frac{-4}{\pi^2(m^2+n^2)} \int_0^1 x \sin m\pi x \, dx \overbrace{\int_0^1 \sin n\pi y\, dy}^{=\frac{1-(-1)^n}{n\pi}} \\
&= \frac{-4}{\pi^4(m^2+n^2)} \frac{1-(-1)^n}{n} \left(-\frac{x\,\cos(m\,\pi\,x)}{m} + \frac{\sin(m\,x)}{m^2\pi} \right)\Big|_0^1 \\
&= \frac{4}{\pi^4(m^2+n^2)} \frac{1-(-1)^n}{n} \frac{(-1)^m}{m}.
\end{aligned}$$

Thus

$$u(x,\, y) = \frac{8}{\pi^4} \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^m}{(m^2+(2k+1)^2)m(2k+1)} \sin m\pi x \sin((2k+1)\pi y).$$

5. We will use an eigenfunction expansion based on the eigenfunctions $\phi(x,\, y) = \sin m\pi x \sin n\pi y$, where $\Delta\pi(x,\, y) = -\pi^2\big(m^2+n^2\big) \sin m\pi x \sin n\pi y$. So plug

$$u(x,\, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y$$

into the equation $\Delta u = 3u - 1$, proceed formally, and get

$$\begin{gathered}
\Delta\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y\right) = 3\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1 \\
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn}\Delta\left(\sin m\pi x \sin n\pi y\right) = 3\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1 \\
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -E_{mn}\pi^2\big(m^2+n^2\big) \sin m\pi x \sin n\pi y \\
= 3\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{mn} \sin m\pi x \sin n\pi y - 1 \\
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \big(3+\pi^2\big(m^2+n^2\big)\big)E_{mn} \sin m\pi x \sin n\pi y = 1.
\end{gathered}$$

Thinking of this as the double sine series expansion of the function identically 1, it follows that $\big(3+\pi^2\big(m^2+n^2\big)\big)E_{mn}$ are double Fourier sine coefficients, given by (see (8), Section 3.7)

$$\begin{aligned}
\big(3+\pi^2\big(m^2+n^2\big)\big)E_{mn} &= 4\int_0^1\int_0^1 \sin m\pi x \sin n\pi y\, dx\, dy \\
&= 4\,\frac{1-(-1)^m}{m\pi}\,\frac{1-(-1)^n}{n\pi} \\
&= \begin{cases} 0 & \text{if either } m \text{ or } n \text{ is even,} \\ \frac{16}{\pi^2 m\, n} & \text{if both } m \text{ and } n \text{ are even.} \end{cases}
\end{aligned}$$

Thus

$$E_{mn} = \begin{cases} 0 & \text{if either } m \text{ or } n \text{ is even,} \\ \frac{16}{\pi^2 m\, n\big(3+\pi^2\big(m^2+n^2\big)\big)} & \text{if both } m \text{ and } n \text{ are even,} \end{cases}$$

and so

$$u(x, y) = \frac{16}{\pi^2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{\sin((2k+1)\pi x) \sin((2l+1)\pi y)}{(2k+1)\,(2l+1)\big(3 + \pi^2\big((2k+1)^2 + (2l+1)^2\big)\big)}.$$

Solutions to Exercises 3.10

1. We use a combination of solutions from (2) and (3) and try a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} \sin mx \big[A_m \cosh m(1-y) + B_m \sinh my\big].$$

(If you have tried a different form of the solution, you can still do the problem, but your answer may look different from the one derived here. The reason for our choice is to simplify the computations that follow.) The boundary conditions on the vertical sides are clearly satisfied. We now determine A_m and B_m so as to satisfy the conditions on the other sides. Starting with $u(1, 0) = 100$, we find that

$$100 = \sum_{m=1}^{\infty} A_m \cosh m \sin mx.$$

Thus $A_m \cosh m$ is the sine Fourier coefficient of the function $f(x) = 100$. Hence

$$A_m \cosh m = \frac{2}{\pi}\int_0^{\pi} 100 \sin mx\, dx \quad \Rightarrow \quad A_m = \frac{200}{\pi m \cosh m}\left[1-(-1)^m\right].$$

Using the boundary condition $u_y(x, 1) = 0$, we find

$$0 = \sum_{m=1}^{\infty} \sin mx \big[A_m(-m)\sinh[m(1-y)] + mB_m \cosh my\big]\Big|_{y=1}.$$

Thus

$$0 = \sum_{m=1}^{\infty} mB_m \sin mx \cosh m.$$

By the uniqueness of Fourier series, we conclude that $mB_m \cosh m = 0$ for all m. Since $m \cosh m \neq 0$, we conclude that $B_m = 0$ and hence

$$\begin{aligned} u(x, y) &= \frac{200}{\pi}\sum_{m=1}^{\infty} \frac{[1-(-1)^m]}{m \cosh m} \sin mx \cosh[m(1-y)] \\ &= \frac{400}{\pi}\sum_{k=0}^{\infty} \frac{\sin[(2k+1)x]}{(2k+1)\cosh(2k+1)} \cosh[(2k+1)(1-y)]. \end{aligned}$$

5. We combine solutions of different types from Exercise 4 and try a solution of the form

$$u(x, y) = A_0 + B_0 y + \sum_{m=1}^{\infty} \cos\frac{m\pi}{a}x\big[A_m \cosh[\frac{m\pi}{a}(b-y)] + B_m \sinh[\frac{m\pi}{a}y]\big].$$

Using the boundary conditions on the horizontal sides, starting with $u_y(x, b) = 0$, we find that

$$0 = B_0 + \sum_{m=1}^{\infty} \frac{m\pi}{a} B_m \cos\frac{m\pi}{a}x \cosh\big[\frac{m\pi}{a}b\big].$$

Thus $B_0 = 0$ and $B_m = 0$ for all $m \geq 1$ and so

$$A_0 + \sum_{m=1}^{\infty} A_m \cos\frac{m\pi}{a}x \cosh[\frac{m\pi}{a}(b-y)].$$

Now, using $u(x,\,0) = g(x)$, we find

$$g(x) = A_0 + \sum_{m=1}^{\infty} A_m \cosh[\frac{m\pi}{a}b] \cos\frac{m\pi}{a}x.$$

Recognizing this as a cosine series, we conclude that

$$A_0 = \frac{1}{a}\int_0^a g(x)\,dx$$

and

$$A_m \cosh[\frac{m\pi}{a}b] = \frac{2}{a}\int_0^a g(x)\cos\frac{m\pi}{a}x\,dx;$$

equivalently, for $m \geq 1$,

$$A_m = \frac{2}{a\cosh[\frac{m\pi}{a}b]}\int_0^a g(x)\cos\frac{m\pi}{a}x\,dx.$$

9. We follow the solution in Example 3. We have

$$u(x,\,y) = u_1(x,\,y) + u_2(x,\,y),$$

where

$$u_1(x,\,y) = \sum_{m=1}^{\infty} B_m \sin mx \sinh my,$$

with

$$B_m = \frac{2}{\pi m\cosh(m\pi)}\int_0^\pi \sin mx\,dx = \frac{2}{\pi m^2\cosh(m\pi)}(1-(-1)^m);$$

and

$$u_2(x,\,y) = \sum_{m=1}^{\infty} A_m \sin mx \cosh[m(\pi-y)],$$

with

$$A_m = \frac{2}{\pi\cosh(m\pi)}\int_0^\pi \sin mx\,dx = \frac{2}{\pi m\cosh(m\pi)}(1-(-1)^m).$$

Hence

$$u(x,\,y) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(1-(-1)^m)}{m\cosh(m\pi)}\sin mx\left[\frac{\sinh my}{m} + \cosh[m(\pi-y)]\right]$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{(2k+1)\cosh[(2k+1)\pi]}\left[\frac{\sinh[(2k+1)y]}{(2k+1)} + \cosh[(2k+1)(\pi-y)]\right].$$

Solutions to Exercises 4.1

1. We could use Cartesian coordinates and compute u_x, u_y, u_{xx}, and u_{yy} directly from the definition of u. Instead, we will use polar coordinates, because the expression $x^2+y^2=r^2$, simplifies the denominator, and thus it is easier to take derivatives. In polar coordinates,

$$u(x,y)=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}=r^{-1}\cos\theta\,.$$

So

$$u_r=-r^{-2}\cos\theta,\quad u_{rr}=2r^{-3}\cos\theta,\quad u_\theta=-r^{-1}\sin\theta,\quad u_{\theta\theta}=-r^{-1}\cos\theta.$$

Plugging into (1), we find

$$\nabla^2 u=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=\frac{2\cos\theta}{r^3}-\frac{\cos\theta}{r^3}-\frac{\cos\theta}{r^3}=0\quad(\text{if } r\neq 0).$$

If you used Cartesian coordinates, you should get

$$u_{xx}=\frac{2x(x^2-3y^2)}{(x^2+y^2)^2}\quad\text{and}\quad u_{yy}=-\frac{2x(x^2-3y^2)}{(x^2+y^2)^2}.$$

5. In spherical coordinates:

$$u(r,\,\theta,\,\phi)=r^3\Rightarrow u_{rr}=6r,\ u_\theta=0,u_{\theta\theta}=0,\ u_{\phi\phi}=0.$$

Plugging into (3), we find

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial\theta^2}+\cot\theta\,\frac{\partial u}{\partial\theta}+\csc^2\theta\,\frac{\partial^2 u}{\partial\phi^2}\Big)=6r+6r=12r.$$

9. (a) If $u(r,\theta,\phi)$ depends only on r, then all partial derivatives of u with respect to θ and ϕ are 0. So (3) becomes

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial\theta^2}+\cot\theta\,\frac{\partial u}{\partial\theta}+\csc^2\theta\,\frac{\partial^2 u}{\partial\phi^2}\Big)=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}.$$

(b) If $u(r,\theta,\phi)$ depends only on r and θ, then all partial derivatives of u with respect to ϕ are 0. So (3) becomes

$$\nabla^2 u=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial\theta^2}+\cot\theta\,\frac{\partial u}{\partial\theta}\Big).$$

Solutions to Exercises 4.2

1. We appeal to the solution (5) with the coefficients (6). Since $f(r)=0$, then $A_n=0$ for all n. We have

$$\begin{aligned} B_n &= \frac{1}{\alpha_n J_1(\alpha_n)^2}\int_0^2 J_0(\frac{\alpha_n r}{2}) r\,dr \\ &= \frac{4}{\alpha_n^3 J_1(\alpha_n)^2}\int_0^{\alpha_n} J_0(s) s\,ds \quad (\text{let } s=\frac{\alpha_n}{2}r) \\ &= \frac{4}{\alpha_n^3 J_1(\alpha_n)^2}[sJ_1(s)]\Big|_0^{\alpha_n} \\ &= \frac{4}{\alpha_n^2 J_1(\alpha_n)} \quad \text{for all } n\geq 1. \end{aligned}$$

Thus

$$u(r,\,t)=4\sum_{n=1}^{\infty}\frac{J_0(\frac{\alpha_n r}{2})}{\alpha_n^2 J_1(\alpha_n)}\sin(\frac{\alpha_n t}{2}).$$

5. Since $g(r)=0$, we have $B_n=0$ for all n. We have

$$A_n=\frac{2}{J_1(\alpha_n)^2}\int_0^1 J_0(\alpha_1 r)J_0(\alpha_n r) r\,dr=0 \quad \text{for } n\neq 1 \text{ by orthogonality.}$$

For $n=1$,

$$A_1=\frac{2}{J_1(\alpha_1)^2}\int_0^1 J_0(\alpha_1 r)^2 r\,dr=1,$$

where we have used the orthogonality relation (12), Section 4.8, with $p=0$. Thus

$$u(r,\,t)=J_0(\alpha_1 r)\cos(\alpha_1 t).$$

9. (a) Modifying the solution of Exercise 3, we obtain

$$u(r,\,t)=\sum_{n=1}^{\infty}\frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2}J_0(\alpha_n r)\sin(\alpha_n ct).$$

(b) Under suitable conditions that allow us to interchange the limit and the summation sign (for example, if the series is absolutely convergent), we have, for a given $(r,\,t)$,

$$\begin{aligned} \lim_{c\to\infty} u(r,\,t) &= \lim_{c\to\infty}\sum_{n=1}^{\infty}\frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2}J_0(\alpha_n r)\sin(\alpha_n ct) \\ &= \sum_{n=1}^{\infty}\lim_{c\to\infty}\frac{J_1(\alpha_n/2)}{\alpha_n^2 c J_1(\alpha_n)^2}J_0(\alpha_n r)\sin(\alpha_n ct) \\ &= 0, \end{aligned}$$

because $\lim_{c\to\infty}\frac{J_1(\alpha_n/2)}{\alpha_n^2 cJ_1(\alpha_n)^2}=0$ and $\sin(\alpha_n ct)$ is bounded. If we let $u_1(r,\,t)$ denote the solution corresponding to $c=1$ and $u_c(r,\,t)$ denote the solution for arbitrary $c>0$. Then, it is easy t check that

$$u_c(r,\,t)=\frac{1}{c}u_1(r,\,ct).$$

This shows that if c increases, the time scale speeds proportionally to c, while the displacement decreases by a factor of $\frac{1}{c}$.

Solutions to Exercises 4.3

1. The condition $g(r,\theta) = 0$ implies that $a^*_{mn} = 0 = b^*_{mn}$. Since $f(r,\theta)$ is proportional to $\sin 2\theta$, only $b_{2,n}$ will be nonzero, among all the a_{mn} and b_{mn}. This is similar to the situation in Example 2. For $n = 1, 2, \ldots$, we have

$$\begin{aligned} b_{2,n} &= \frac{2}{\pi J_3(\alpha_{2,n})^2}\int_0^1\int_0^{2\pi}(1-r^2)r^2\sin 2\theta J_2(\alpha_{2,n}r)\sin 2\theta r\,d\theta\,dr \\ &= \frac{2}{\pi J_3(\alpha_{2,n})^2}\int_0^1\overbrace{\int_0^{2\pi}\sin^2 2\theta\,d\theta}^{=\pi}(1-r^2)r^3 J_2(\alpha_{2,n}r)\,dr \\ &= \frac{2}{J_3(\alpha_{2,n})^2}\int_0^1(1-r^2)r^3 J_2(\alpha_{2,n}r)\,dr \\ &= \frac{2}{J_3(\alpha_{2,n})^2}\frac{2}{\alpha_{2,n}^2}J_4(\alpha_{2,n}) = \frac{4J_4(\alpha_{2,n})}{\alpha_{2,n}^2 J_3(\alpha_{2,n})^2}, \end{aligned}$$

where the last integral is evaluated with the help of formula (15), Section 4.3. We can get rid of the expression involving J_4 by using the identity

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x).$$

With $p = 3$ and $x = \alpha_{2,n}$, we get

$$\overbrace{J_2(\alpha_{2,n})}^{=0} + J_4(\alpha_{2,n}) = \frac{6}{\alpha_{2,n}}J_3(\alpha_{2,n}) \quad\Rightarrow\quad J_4(\alpha_{2,n}) = \frac{6}{\alpha_{2,n}}J_3(\alpha_{2,n}).$$

So

$$b_{2,n} = \frac{24}{\alpha_{2,n}^3 J_3(\alpha_{2,n})}.$$

Thus

$$u(r,\,\theta,\,t) = 24\sin 2\theta\sum_{n=1}^{\infty}\frac{J_2(\alpha_{2,n}r)}{\alpha_{2,n}^3 J_3(\alpha_{2,n})}\cos(\alpha_{2,n}t).$$

5. We have $a_{mn} = b_{mn} = 0$. Also, all a^*_{mn} and b^*_{mn} are zero except $b^*_{2,n}$. We have

$$b^*_{2,n} = \frac{2}{\pi\alpha_{2,n}J_3(\alpha_{2,n})^2}\int_0^1\int_0^{2\pi}(1-r^2)r^2\sin 2\theta J_2(\alpha_{2,n}r)\sin 2\theta r\,d\theta\,dr.$$

The integral was computed in Exercise 1. Using the computations of Exercise 1, we find

$$b^*_{2,n} = \frac{24}{\alpha_{2,n}^4 J_3(\alpha_{2,n})}.$$

hus

$$u(r,\,\theta,\,t) = 24\sin 2\theta\sum_{n=1}^{\infty}\frac{J_2(\alpha_{2,n}r)}{\alpha_{2,n}^4 J_3(\alpha_{2,n})}\sin(\alpha_{2,n}t).$$

9. (a) For $l = 0$ and all $k \geq 0$, the formula follows from (7), Section 4.8, with $p = k$:

$$\int r^{k+1}J_k(r)\,dr = r^{k+1}J_{k+1}(r) + C.$$

(b) Assume that the formula is true for l (and all $k \geq 0$). Integrate by parts, using

$u = r^{2l}$, $dv = r^{k+1}J_{k+1}(r)\,dr$, and hence $du = 2lr^{2l-1}dr$ and $v = r^{k+1}J_{k+1}(r)$:

$$\begin{aligned}\int r^{k+1+2l}J_k(r)\,dr &= \int r^{2l}[r^{k+1}J_k(r)]\,dr \\ &= r^{2l}\,r^{k+1}J_{k+1}(r) - 2l\int r^{2l-1}r^{k+1}J_{k+1}(r)\,dr \\ &= r^{k+1+2l}J_{k+1}(r) - 2l\int r^{k+2l}J_{k+1}(r)\,dr \\ &= r^{k+1+2l}J_{k+1}(r) - 2l\int r^{(k+1)+1+2(l-1)}J_{k+1}(r)\,dr\end{aligned}$$

and so, by the induction hypothesis, we get

$$\begin{aligned}\int r^{k+1+2l}J_k(r)\,dr &= r^{k+1+2l}J_{k+1}(r) - 2l\sum_{n=0}^{l-1}\frac{(-1)^n 2^n (l-1)!}{(l-1-n)!}r^{k+2l-n}J_{k+n+2}(r) + C \\ &= r^{k+1+2l}J_{k+1}(r) \\ &\quad + \sum_{n=0}^{l-1}\frac{(-1)^{n+1}2^{n+1}l!}{(l-(n+1))!}r^{k+1+2l-(n+1)}J_{k+(n+1)+1}(r) + C \\ &= r^{k+1+2l}J_{k+1}(r) + \sum_{m=1}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}r^{k+1+2l-m}J_{k+m+1}(r) + C \\ &= \sum_{m=0}^{l}\frac{(-1)^m 2^m l!}{(l-m)!}r^{k+1+2l-m}J_{k+m+1}(r) + C,\end{aligned}$$

which completes the proof by induction for all integers $k \geq 0$ and all $l \geq 0$.

13. The proper place for this problem is in the next section, since its solution invovles solving a Dirichlet problem on the unit disk. The initial steps are similar to the solution of the heat problem on a rectangle with nonzero boundary data (Exercise 11, Section 3.8). In order to solve the problem, we consider the following two subproblems: Subproblem #1 (Dirichlet problem)

$$\begin{aligned}(u_1)_{rr} + \frac{1}{r}(u_1)_r + \frac{1}{r^2}(u_1)_{\theta\theta} &= 0, \qquad 0 < r < 1,\ 0 \leq \theta < 2\pi, \\ u_1(1,\,\theta) &= \sin 3\theta, \qquad 0 \leq \theta < 2\pi.\end{aligned}$$

Subproblem #2 (to be solved after finding $u_1(r,\,\theta)$ from Subproblem #1)

$$\begin{aligned}(u_2)_t &= (u_2)_{rr} + \tfrac{1}{r}(u_2)_r + \tfrac{1}{r^2}(u_2)_{\theta\theta}, \quad 0 < r < 1,\ 0 \leq \theta < 2\pi,\ t > 0, \\ u_2(1,\,\theta,\,t) &= 0, \qquad 0 \leq \theta < 2\pi,\ t > 0, \\ u_2(r,\,\theta,\,0) &= -u_1(r,\,\theta), \qquad 0 < r < 1,\ 0 \leq \theta < 2\pi.\end{aligned}$$

You can check, using linearity (or superposition), that

$$u(r,\,\theta,\,t) = u_1(r,\,\theta) + u_2(r,\,\theta,\,t)$$

is a solution of the given problem.

The solution of subproblem #1 follows immediately from the method of Section 4.5. We have

$$u_2(r,\,\theta) = r^3 \sin 3\theta.$$

We now solve subproblem #2, which is a heat problem with 0 boundary data and initial temperature distribution given by $-u_2(r,\,\theta) = -r^3 \sin 3\theta$. reasoning as in Exercise 10, we find that the solution is

$$u_2(r,\,\theta,\,t) = \sum_{n=1}^{\infty} b_{3n} J_3(\alpha_{3n} r) \sin(3\theta) e^{-\alpha_{3n}^2 t},$$

where

$$\begin{aligned}
b_{3n} &= \frac{-2}{\pi J_4(\alpha_{3n})^2} \int_0^1 \int_0^{2\pi} r^3 \sin^2 3\theta J_3(\alpha_{3n} r) r\, d\theta\, dr \\
&= \frac{-2}{J_4(\alpha_{3n})^2} \int_0^1 r^4 J_3(\alpha_{3n} r)\, dr \\
&= \frac{-2}{J_4(\alpha_{3n})^2} \frac{1}{\alpha_{3n}^5} \int_0^{\alpha_{3n}} s^4 J_3(s)\, ds \quad \text{(where } \alpha_{3n} r = s\text{)} \\
&= \frac{-2}{J_4(\alpha_{3n})^2} \frac{1}{\alpha_{3n}^5} s^4 J_4(s)\Big|_0^{\alpha_{3n}} \\
&= \frac{-2}{\alpha_{3n} J_4(\alpha_{3n})}.
\end{aligned}$$

Hence

$$u(r,\,\theta,\,t) = r^3 \sin 3\theta - 2\sin(3\theta) \sum_{n=1}^{\infty} \frac{J_3(\alpha_{3n} r)}{\alpha_{3n} J_4(\alpha_{3n})} e^{-\alpha_{3n}^2 t}.$$

Exercises 4.4

1. Since f is already given by its Fourier series, we have from (4)

$$u(r, \theta) = r\cos\theta = x.$$

5. Let us compute the Fourier coefficients of f. We have

$$a_0 = \frac{50}{\pi}\int_0^{\pi/4} d\theta = \frac{25}{2};$$

$$a_n = \frac{100}{\pi}\int_0^{\pi/4}\cos n\theta\, d\theta = \frac{100}{n\pi}\sin n\theta\Big|_0^{\pi} = \frac{100}{n\pi}\sin\frac{n\pi}{4};$$

$$b_n = \frac{100}{\pi}\int_0^{\pi/4}\sin n\theta\, d\theta = -\frac{100}{n\pi}\cos n\theta\Big|_0^{\pi} = \frac{100}{n\pi}(1-\cos\frac{n\pi}{4}).$$

Hence

$$f(\theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{4}\cos n\theta + (1-\cos\frac{n\pi}{4})\sin n\theta\right);$$

and

$$u(r, \theta) = \frac{25}{2} + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\frac{n\pi}{4}\cos n\theta + (1-\cos\frac{n\pi}{4})\sin n\theta\right) r^n.$$

9. $u(r\,\theta) = 2r^2\sin\theta\cos\theta = 2xy$. So $u(x\,y) = T$ if and only if $2xy = T$ if and only if $y = \frac{T}{2x}$, which shows that the isotherms lie on hyperbolas centered at the origin.

13. We follow the steps in Example 4 (with $\alpha = \frac{\pi}{4}$) and arrive at the same equation in Θ and R. The solution in Θ is

$$\Theta_n(\theta) = \sin(4n\theta), \quad n = 1, 2, \ldots,$$

and the equation in R is

$$r^2R'' + rR' - (4n)^2R = 0.$$

The indicial equation for this Euler equation is

$$\rho^2 - (4n)^2 = 0 \quad\Rightarrow\quad \rho = \pm 4n.$$

Taking the bounded solutions only, we get

$$R_n(r) = r^{4n}.$$

Thus the product solutions are $r^{4n}\sin 4\theta$ and the series solution of the problem is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^{4n}\sin 4n\theta.$$

To determine b_n, we use the boundary condition:

$$u_r(r, \theta)\big|_{r=1} = \sin\theta \quad\Rightarrow\quad \sum_{n=1}^{\infty} b_n 4nr^{4n-1}\sin 4n\theta\big|_{r=1} = \sin\theta$$

$$\Rightarrow\quad \sum_{n=1}^{\infty} b_n 4n\sin 4n\theta = \sin\theta$$

$$\Rightarrow\quad 4nb_n = \frac{2}{\pi/4}\int_0^{\pi/4}\sin\theta\sin 4n\theta\, d\theta.$$

Thus

$$\begin{aligned} b_n &= \frac{2}{\pi n}\int_0^{\pi/4} \sin\theta \sin 4n\theta \, d\theta \\ &= \frac{1}{\pi n}\int_0^{\pi/4} [-\cos[(4n+1)\theta] + \cos[(4n-1)\theta]] \, d\theta \\ &= \frac{1}{\pi n}\left[-\frac{\sin[(4n+1)\theta]}{4n+1} + \frac{\sin[(4n-1)\theta]}{4n-1}\right]\Big|_0^{\pi/4} \\ &= \frac{1}{\pi n}\left[-\frac{\sin[(4n+1)\frac{\pi}{4}}{4n+1} + \frac{\sin[(4n-1)\frac{\pi}{4}}{4n-1}\right] \\ &= \frac{1}{\pi n}\left[-\frac{\cos(n\pi)\sin\frac{\pi}{4}]}{4n+1} - \frac{\cos(n\pi)\sin\frac{\pi}{4}}{4n-1}\right] \\ &= \frac{(-1)^n}{\pi n}\frac{\sqrt{2}}{2}\left[\frac{-1}{4n+1} - \frac{1}{4n-1}\right] \\ &= \frac{(-1)^{n+1}\sqrt{2}}{\pi}\frac{4}{16n^2-1}. \end{aligned}$$

Hence

$$u(r,\,\theta) = \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{16n^2-1} r^{4n}\sin 4n\theta.$$

17. Since u satsifies Laplace's equation in the disk, the separation of variables method and the fact that u is 2π-periodic in θ imply that u is given by the series (4), where the coefficients are to be determined from the Neumann boundary condition. From

$$u(r,\,\theta) = a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \big[a_n\cos n\theta + b_n \sin n\theta\big],$$

it follows that

$$u_r(r,\,\theta) = \sum_{n=1}^{\infty}\left(n\frac{r^{n-1}}{a^n}\right)\big[a_n\cos n\theta + b_n \sin n\theta\big].$$

Using the boundary condition $u_r(a,\,\theta) = f(\theta)$, we obtain

$$f(\theta) = \sum_{n=1}^{\infty}\frac{n}{a}\big[a_n\cos n\theta + b_n \sin n\theta\big].$$

In this Fourier series expansion, the $n = 0$ term must be 0. But the $n = 0$ term is given by

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta,$$

thus the compatibility condition

$$\int_0^{2\pi} f(\theta)\,d\theta = 0$$

must hold. Once this condition is satsified, we determine the coefficients a_n and b_n by using the Euler formulas, as follows:

$$\frac{n}{a}a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

and

$$\frac{n}{a}b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

Hence

$$a_n = \frac{a}{n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \quad \text{and} \quad b_n = \frac{a}{n\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta.$$

Note that a_0 is still arbitray. Indeed, the solution of a Neumann problem is not unique. It can be determined only up to an additive constant (which does not affect the value of the normal derivative at the boundary).

21. Using the fact that the solutions must be bounded as $r \to \infty$, we see that $c_1 = 0$ in the first of the two equations in (3), and $c_2 = 0$ in the second of the two equations in (3). Thus

$$R(r) = R_n(r) = c_n r^{-n} = \left(\frac{r}{a}\right)^n \quad \text{for } n = 0,\ 1,\ 2,\ \ldots.$$

The general solution becomes

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta)\,, \quad r > a.$$

Setting $r = a$ and using the boundary condition, we obtain

$$f(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\,,$$

which implies that the a_n and b_n are the Fourier coefficients of f and hence are given by (5).

25. The hint does it.

29. (a) Recalling the Euler formulas for the Fourier coefficients, we have

$$\begin{aligned}
u(r,\,\theta) &= a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n [a_n \cos n\theta + b_n \sin n\theta] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi\, d\phi \cos n\theta + \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi\, d\phi \sin n\theta\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[\frac{1}{\pi}\int_0^{2\pi} f(\phi)\big[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\big]\, d\phi\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, d\phi + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n(\theta - \phi)\, d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - \phi)\right] d\phi.
\end{aligned}$$

(b) Continuing from (a) and using Exercise 28, we obtain

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{1-\left(\frac{r}{a}\right)^2}{1-2\left(\frac{r}{a}\right)\cos\theta+\left(\frac{r}{a}\right)^2} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)P\big(r/a,\,\theta-\phi\big)\,d\phi. \end{aligned}$$

Solutions to Exercises 4.5

1. Using (2) and (3), we have that

$$u(\rho,\, z) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho)\sinh(\lambda_n z), \quad \lambda_n = \frac{\alpha_n}{a},$$

where $\alpha_n = \alpha_{0,n}$ is the nth positive zero of J_0, and

$$\begin{aligned}
A_n &= \frac{2}{\sinh(\lambda_n h) a^2 J_1(\alpha_n)^2} \int_0^a f(\rho) J_0(\lambda_n \rho)\rho\, d\rho \\
&= \frac{200}{\sinh(2\alpha_n) J_1(\alpha_n)^2} \int_0^1 J_0(\alpha_n \rho)\rho\, d\rho \\
&= \frac{200}{\sinh(2\alpha_n)\alpha_n^2 J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0(s) s\, ds \quad (\text{let } s = \alpha_n \rho) \\
&= \frac{200}{\sinh(2\alpha_n)\alpha_n^2 J_1(\alpha_n)^2} [J_1(s)s]\Big|_0^{\alpha_n} \\
&= \frac{200}{\sinh(2\alpha_n)\alpha_n J_1(\alpha_n)}.
\end{aligned}$$

So

$$u(\rho,\, z) = 200 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \rho)\sinh(\alpha_n z)}{\sinh(2\alpha_n)\alpha_n J_1(\alpha_n)}.$$

5. (a) We proceed exactly as in the text and arrive at the condition $Z(h) = 0$ which leads us to the solutions

$$Z(z) = Z_n(z) = \sinh(\lambda_n (h - z)), \quad \text{where } \lambda_n = \frac{\alpha_n}{a}.$$

So the solution of the problem is

$$u(\rho,\, z) = \sum_{n=1}^{\infty} C_n J_0(\lambda_n \rho)\sinh(\lambda_n (h - z)),$$

where

$$C_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f(\rho) J_0(\lambda_n \rho)\rho\, d\rho.$$

(b) The problem can be decomposed into the sum of two subproblems, one treated in the text and one treated in part (a). The solution of the problem is the sum of the solutions of the subproblems:

$$u(\rho,\, z) = \sum_{n=1}^{\infty} \Big(A_n J_0(\lambda_n \rho)\sinh(\lambda_n z) + C_n J_0(\lambda_n \rho)\sinh(\lambda_n (h - z)) \Big),$$

where

$$A_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f_2(\rho) J_0(\lambda_n \rho)\rho\, d\rho,$$

and

$$C_n = \frac{2}{a^2 J_1(\alpha_n)^2 \sinh(\lambda_n h)} \int_0^a f_1(\rho) J_0(\lambda_n \rho)\rho\, d\rho.$$

9. We use the solution in Exercise 8 with $a = 1$, $h = 2$, $f(z) = 10z$. Then

$$\begin{aligned} B_n &= \frac{1}{I_0\left(\frac{n\pi}{2}\right)} \int_0^2 0z \sin \frac{n\pi z}{2}\, dz \\ &= \frac{40}{n\pi\, I_0\left(\frac{n\pi}{2}\right)} (-1)^{n+1}. \end{aligned}$$

Thus

$$u(\rho, z) = \frac{40}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, I_0\left(\frac{n\pi}{2}\right)} I_0\left(\frac{n\pi}{2}\rho\right) \sin \frac{n\pi z}{2}.$$

Solutions to Exercises 4.6

1. Write (1) in polar coordinates:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = -k\phi \qquad \phi(a, \theta) = 0.$$

Consider a product solution $\phi(r, \theta) = R(r)\Theta(\theta)$. Since θ is a polar angle, it follows that

$$\Theta(\theta + 2\pi) = \Theta(\theta);$$

in other words, Θ is 2π-periodic. Plugging the product solution into the equation and simplifying, we find

$$\begin{aligned} R''\Theta + \tfrac{1}{r}R'\Theta + \tfrac{1}{r^2}R\Theta'' &= -kR\Theta; \\ \left(R'' + \tfrac{1}{r}R' + kR\right)\Theta &= -\tfrac{1}{r^2}R\Theta''; \\ r^2\tfrac{R''}{R} + r\tfrac{R'}{R} + kr^2 &= -\tfrac{\Theta''}{\Theta}; \end{aligned}$$

hence

$$r^2\frac{R''}{R} + r\frac{R'}{R} + kr^2 = \lambda,$$

and

$$-\frac{\Theta''}{\Theta} = \lambda \quad \Rightarrow \quad \Theta'' + \lambda\Theta = 0,$$

where λ is a separation constant. Our knowledge of solutions of second order linear ode's tells us that the last equation has 2π-periodic solutions if and only if

$$\lambda = m^2, \quad m = 0, \pm 1, \pm 2, \ldots.$$

This leads to the equations

$$\Theta'' + m^2\Theta = 0,$$

and

$$r^2\frac{R''}{R} + r\frac{R'}{R} + kr^2 = m^2 \quad \Rightarrow \quad r^2R'' + rR' + (kr^2 - m^2)R = 0.$$

These are equations (3) and (4). Note that the condition $R(a) = 0$ follows from $\phi(a, \theta) = 0 \Rightarrow R(a)\Theta(\theta) = 0 \Rightarrow R(a) = 0$ in order to avoid the constant 0 solution.

5. We proceed as in Example 1 and try

$$u(r, \theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r, \theta),$$

where $\phi_{mn}(r, \theta) = J_m(\lambda_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2\phi_{mn}$, and get

$$\begin{aligned} \nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r, \theta)\right) &= 1 - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r, \theta) \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r, \theta)\right) = 1 - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r, \theta) \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\lambda_{mn}^2\phi_{mn}(r, \theta) = 1 - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r, \theta) \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(1 - \alpha_{mn}^2)\phi_{mn}(r, \theta) = 1. \end{aligned}$$

We recognize this expansion as the expansion of the function 1 in terms of the functions ϕ_{mn}. Because the right side is independent of θ, it follows that all A_{mn} and B_{mn} are zero, except $A_{0,n}$. So

$$\sum_{n=1}^{\infty}(1 - \alpha_{mn}^2)A_{0,n}J_0(\alpha_{0,n})r) = 1,$$

which shows that $(1-\alpha_{mn}^2)A_{0,n} = a_{0,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 0 of the function 1. This series is computed in Example 1, Section 4.8. We have

$$1 = \sum_{n=1}^{\infty} \frac{2}{\alpha_{0,n} J_1(\alpha_{0,n})} J_0(\alpha_{0,n} r) \quad 0 < r < 1.$$

Hence

$$(1-\alpha_{mn}^2)A_{0,n} = \frac{2}{\alpha_{0,n} J_1(\alpha_{0,n})} \quad \Rightarrow \quad A_{0,n} = \frac{2}{(1-\alpha_{mn}^2)\alpha_{0,n} J_1(\alpha_{0,n})};$$

and so

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{2}{(1-\alpha_{mn}^2)\alpha_{0,n} J_1(\alpha_{0,n})} J_0(\alpha_{0,n} r).$$

9. Let

$$h(r) = \begin{cases} r & \text{if } 0 < r < 1/2, \\ 0 & \text{if } 1/2 < r < 1. \end{cases}$$

Then the equation becomes $\nabla^2 u = f(r,\theta)$, where $f(r,\theta) = h(r)\sin\theta$. We proceed as in the previous exercise and try

$$u(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \phi_{mn}(r,\theta),$$

where $\phi_{mn}(r,\theta) = J_m(\lambda_{mn} r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$. We plug this solution into the equation, use the fact that $\nabla^2(\phi_{mn}) = -\lambda_{mn}^2\phi_{mn} = -\alpha_{mn}^2\phi_{mn}$, and get

$$\begin{aligned} \nabla^2\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}(r,\theta)\right) &= h(r)\sin\theta \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\nabla^2\left(\phi_{mn}(r,\theta)\right) = h(r)\sin\theta \\ &\Rightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\alpha_{mn}^2\phi_{mn}(r,\theta) = h(r)\sin\theta. \end{aligned}$$

We recognize this expansion as the expansion of the function $h(r)\sin\theta$ in terms of the functions ϕ_{mn}. Because the right side is proportional to $\sin\theta$, it follows that all A_{mn} and B_{mn} are zero, except $B_{1,n}$. So

$$\sin\theta \sum_{n=1}^{\infty} -\alpha_{1n}^2 B_{1,n} J_1(\alpha_{1n} r) = h(r)\sin\theta,$$

which shows that $-\alpha_{1n}^2 B_{1,n}$ is the nth Bessel coefficient of the Bessel series expansion of order 1 of the function $h(r)$:

$$\begin{aligned} -\alpha_{1n}^2 B_{1,n} &= \frac{2}{J_2(\alpha_{1,n})^2}\int_0^{1/2} r^2 J_1(\alpha_{1,n} r)\,dr \\ &= \frac{2}{\alpha_{1,n}^3 J_2(\alpha_{1,n})^2}\int_0^{\alpha_{1,n}/2} s^2 J_1(s)\,ds \\ &= \frac{2}{\alpha_{1,n}^3 J_2(\alpha_{1,n})^2} s^2 J_2(s)\Big|_0^{\alpha_{1,n}/2} \\ &= \frac{J_2(\alpha_{1,n}/2)}{2\alpha_{1,n} J_2(\alpha_{1,n})^2}. \end{aligned}$$

Thus

$$u(r, \theta) = \sin\theta \sum_{n=1}^{\infty} -\frac{J_2(\alpha_{1,n}/2)}{2\alpha_{1,n}^3 J_2(\alpha_{1,n})^2} J_1(\alpha_{1,n} r).$$

Solutions to Exercises 4.7

1. Bessel equation of order 3. Using (7), the first series solution is

$$J_3(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+3)!}\left(\frac{x}{2}\right)^{2k+3} = \frac{1}{1\cdot 6}\frac{x^3}{8} - \frac{1}{1\cdot 24}\frac{x^5}{32} + \frac{1}{2\cdot 120}\frac{x^7}{128} + \cdots.$$

5. Bessel equation of order $\frac{3}{2}$. The general solution is

$$\begin{aligned} y(x) &= c_1 J_{\frac{3}{2}} + c_2 J_{-\frac{3}{2}} \\ &= c_1\left(\frac{1}{1\cdot\Gamma(\frac{5}{2})}\left(\frac{x}{2}\right)^{\frac{3}{2}} - \frac{1}{1\cdot\Gamma(\frac{7}{2})}\left(\frac{x}{2}\right)^{\frac{7}{2}} + \cdots\right) \\ &\quad + c_2\left(\frac{1}{1\cdot\Gamma(\frac{-1}{2})}\left(\frac{x}{2}\right)^{-\frac{3}{2}} - \frac{1}{1\cdot\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{\frac{1}{2}} + \cdots\right). \end{aligned}$$

Using the basic property of the gamma function and (15), we have

$$\begin{gathered} \Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{3}{4}\sqrt{\pi} \\ \Gamma(\tfrac{7}{2}) = \tfrac{5}{2}\Gamma(\tfrac{5}{2}) = \tfrac{15}{8}\sqrt{\pi} \\ -\tfrac{1}{2}\Gamma(-\tfrac{1}{2}) = \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \;\Rightarrow\; \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}. \end{gathered}$$

So

$$\begin{aligned} y(x) &= c_1\sqrt{\frac{2}{\pi x}}\left(\frac{4}{3}\frac{x^2}{4} - \frac{8}{15}\frac{x^4}{16} + \cdots\right) \\ &\quad c_2\sqrt{\frac{2}{\pi x}}(-1)\left(-\frac{1}{2}\frac{2}{x} - \frac{x}{2} - \cdots\right) \\ &= c_1\sqrt{\frac{2}{\pi x}}\left(\frac{x^2}{3} - \frac{x^4}{30} + \cdots\right) + c_2\sqrt{\frac{2}{\pi x}}\left(\frac{1}{x} + \frac{x}{2} - \cdots\right) \end{aligned}$$

9. Divide the equation through by x^2 and put it in the form

$$y'' + \frac{1}{x}y' + \frac{x^2-9}{x^2}y = 0 \quad \text{for } x > 0.$$

Now refer to Appendix A.6 for terminology and for the method of Frobenius that we are about to use in this exercise. Let

$$p(x) = \frac{1}{x} \quad \text{for} \quad q(x) = \frac{x^2-9}{x^2}.$$

The point $x = 0$ is a singular point of the equation. But since $x\,p(x) = 1$ and $x^2\,q(x) = x^2 - 9$ have power series expansions about 0 (in fact, they are already given by their power series expansions), it follows that $x = 0$ is a regular singular point. Hence we may apply the Frobenius method. We have already found one series solution in Exercise 1. To determine the second series solution, we consider the indicial equation

$$r(r-1) + p_0 r + q_0 = 0,$$

where $p_0 = 1$ and $q_0 = -9$ (respectively, these are the constant terms in the series expansions of $xp(x)$ and $x^2q(x)$). Thus the indicial equation is

$$r - 9 = 0 \quad \Rightarrow \quad r_1 = 3,\; r_2 = -3.$$

The indicial roots differ by an integer. So, according to Theorem 2, Appendix A.6, the second solution y_2 may or may not contain a logarithmic term. We have, for $x > 0$,

$$y_2 = ky_1 \ln x + x^{-3}\sum_{m=0}^{\infty} b_m x^m = ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3},$$

where $a_0 \neq 0$ and $b_0 \neq 0$, and k may or may not be 0. Plugging this into the differential equation

$$x^2y'' + xy' + (x^2 - 9)y = 0$$

and using the fact that y_1 is a solution, we have

$$\begin{aligned}
y_2 &= ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3} \\
y_2' &= ky_1' \ln x + k\frac{y_1}{x} + \sum_{m=0}^{\infty}(m-3)b_m x^{m-4}; \\
y_2'' &= ky_1'' \ln x + k\frac{y_1'}{x} + k\frac{xy_1' - y_1}{x^2} + \sum_{m=0}^{\infty}(m-3)(m-4)b_m x^{m-5} \\
&= ky_1'' \ln x + 2k\frac{y_1'}{x} - k\frac{y_1}{x^2} + \sum_{m=0}^{\infty}(m-3)(m-4)b_m x^{m-5};
\end{aligned}$$

$$\begin{aligned}
&x^2y_2'' + xy_2' + (x^2-9)y_2 \\
&= kx^2y_1'' \ln x + 2kxy_1' - ky_1 + \sum_{m=0}^{\infty}(m-3)(m-4)b_m x^{m-3} \\
&\quad +kxy_1' \ln x + ky_1 + \sum_{m=0}^{\infty}(m-3)b_m x^{m-3} \\
&\quad +(x^2-9)\left[ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-3}\right] \\
&= k \ln x \big[\overbrace{x^2y_1'' + xy_1' + (x^2-9)y_1}^{=0}\big] \\
&\quad +2kxy_1' + \sum_{m=0}^{\infty}\big[(m-3)(m-4)b_m + (m-3)b_m - 9b_m\big]x^{m-3} \\
&\quad +x^2\sum_{m=0}^{\infty} b_m x^{m-3} \\
&= 2kxy_1' + \sum_{m=0}^{\infty}(m-6)mb_m x^{m-3} + \sum_{m=0}^{\infty} b_m x^{m-1}.
\end{aligned}$$

To combine the last two series, we use reindexing as follows

$$\begin{aligned}
&\sum_{m=0}^{\infty}(m-6)mb_m x^{m-3} + \sum_{m=0}^{\infty} b_m x^{m-1} \\
&\quad = -5b_1x^{-2} + \sum_{m=2}^{\infty}(m-6)mb_m x^{m-3} + \sum_{m=2}^{\infty} b_{m-2}x^{m-3} \\
&\quad = -5b_1x^{-2} + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}.
\end{aligned}$$

Thus the equation

$$x^2y_2'' + xy_2' + (x^2-9)y_2 = 0$$

implies that

$$2kxy_1' - 5b_1x^{-2} + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3} = 0.$$

This equation determines the coefficients b_m $(m \geq 1)$ in terms of the coefficients of y_1. Furthermore, it will become apparent that k cannot be 0. Also, b_0 is arbitrary but by assumption $b_0 \neq 0$. Let's take $b_0 = 1$ and determine the the first five b_m's.

Recall from Exercise 1

$$y_1 = \frac{1}{1\cdot 6}\frac{x^3}{8} - \frac{1}{1\cdot 24}\frac{x^5}{32} + \frac{1}{2\cdot 120}\frac{x^7}{128} + \cdots .$$

So

$$y_1' = \frac{3}{1\cdot 6}\frac{x^2}{8} - \frac{5}{1\cdot 24}\frac{x^4}{32} + \frac{7}{2\cdot 120}\frac{x^6}{128} + \cdots$$

and hence (taking $k = 1$)

$$2kxy_1' = \frac{6k}{1\cdot 6}\frac{x^3}{8} - \frac{10k}{1\cdot 24}\frac{x^5}{32} + \frac{14k}{2\cdot 120}\frac{x^7}{128} + \cdots .$$

The lowest exponent of x in

$$2kxy_1' - 5b_1x^{-2} + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}$$

is x^{-2}. Since its coefficient is $-5b_1$, we get $b_1 = 0$ and the equation becomes

$$2xy_1' + \sum_{m=2}^{\infty}\big[(m-6)mb_m + b_{m-2}\big]x^{m-3}.$$

Next, we consider the coefficient of x^{-1}. It is $(-4)2b_2 + b_0$. Setting it equal to 0, we find

$$b_2 = \frac{b_0}{8} = \frac{1}{8}.$$

Next, we consider the constant term, which is the $m = 3$ term in the series. Setting its coefficient equal to 0, we obtain

$$(-3)3b_3 + b_1 = 0 \quad \Rightarrow \quad b_3 = 0$$

because $b_1 = 0$. Next, we consider the term in x, which is the $m = 4$ term in the series. Setting its coefficient equal to 0, we obtain

$$(-2)4b_4 + b_2 = 0 \quad \Rightarrow \quad b_4 = \frac{1}{8}b_2 = \frac{1}{64}.$$

Next, we consider the term in x^2, which is the $m = 5$ term in the series. Setting its coefficient equal to 0, we obtain $b_5 = 0$. Next, we consider the term in x^3, which is the $m = 6$ term in the series plus the first term in $2kxy_1'$. Setting its coefficient equal to 0, we obtain

$$0 + b_4 + \frac{k}{8} = 0 \quad \Rightarrow \quad k = -8b_4 = -\frac{1}{8}.$$

Next, we consider the term in x^4, which is the $m = 7$ term in the series. Setting its coefficient equal to 0, we find that $b_7 = 0$. It is clear that $b_{2m+1} = 0$ and that

$$y_2 \approx -\frac{1}{8}y_1 \ln x + \frac{1}{x^3} + \frac{1}{8\,x} + \frac{1}{64}x + \cdots$$

Any nonzero constant multiple of y_2 is also a second linearly independent solution of y_1. In particular, $384\,y_2$ is an alternative answer (which is the answer given in the text).

13. The equation is of the form given in Exercise 10 with $p = 3/2$. Thus its general solution is

$$y = c_1x^{3/2}J_{3/2}(x) + c_2x^{3/2}Y_{3/2}(x).$$

Using Exercise 22 and (1), you can also write this general solution in the form

$$\begin{aligned} y &= c_1 x\left[\frac{\sin x}{x} - \cos x\right] + c_2 x\left[-\frac{\cos x}{x} - \sin x\right] \\ &= c_1\left[\sin x - x\cos x\right] + c_2\left[-\cos x - x\sin x\right]. \end{aligned}$$

In particular, two linearly independent solution are

$$y_1 = \sin x - x\cos x \quad \text{and} \quad y_2 = \cos x + x\sin x.$$

This can be verified directly by using the differential equation (try it!).

17. We have

$$\begin{aligned} y &= x^{-p}u, \\ y' &= -px^{-p-1}u + x^{-p}u', \\ y'' &= p(p+1)x^{-p-2}u + 2(-p)x^{-p-1}u' + x^{-p}u'', \\ xy'' + (1+2p)y' + xy &= x\left[p(p+1)x^{-p-2}u - 2px^{-p-1}u' + x^{-p}u''\right] \\ &\quad +(1+2p)\left[-px^{-p-1}u + x^{-p}u'\right] + x\,x^{-p}u \\ &= x^{-p-1}\left[x^2u'' + [-2px + (1+2p)x]u'\right. \\ &\quad \left.+[p(p+1) - (1+2p)p + x^2]u\right] \\ &= x^{-p-1}\left[x^2u'' + xu' + (x^2 - p^2)u\right]. \end{aligned}$$

Thus, by letting $y = x^{-p}u$, we transform the equation

$$xy'' + (1+2p)y' + xy = 0$$

into the equation

$$x^{-p-1}\left[x^2u'' + xu' + (x^2 - p^2)u\right] = 0,$$

which, for $x > 0$, is equivalent to

$$x^2u'' + xu' + (x^2 - p^2)u = 0,$$

a Bessel equation of ordr $p > 0$ in u. The general solution of the last equation is

$$u = c_1 J_p(x) + c_2 Y_p(x).$$

Thus the general solution of the original equation is

$$Y = c_1 x^{-p} J_p(x) + c_2 x^{-p} Y_p(x).$$

21. Using (7),

$$\begin{aligned} J_{-\frac{1}{2}}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k - \frac{1}{2} + 1)}\left(\frac{x}{2}\right)^{2k-\frac{1}{2}} \\ &= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k + \frac{1}{2})} \frac{x^{2k}}{2^{2k}} \\ &= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{2^{2k}k!}{(2k)!\sqrt{\pi}} \frac{x^{2k}}{2^{2k}} \quad \text{(by Exercise 44(a))} \\ &= \sqrt{\frac{2}{\pi\, x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k}}{(2k)!} = \sqrt{\frac{2}{\pi\, x}} \cos x. \end{aligned}$$

22. (a) Using (7),

$$
\begin{aligned}
J_{\frac{3}{2}}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{3}{2}+1)}\left(\frac{x}{2}\right)^{2k+\frac{3}{2}} \\
&= \sqrt{\frac{2}{x}}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+2+\frac{1}{2})}\frac{x^{2k+2}}{2^{2k+2}} \\
&= \sqrt{\frac{2}{\pi x}}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{2\,2^{2k+1}k!}{(2k+3)(2k+1)!}\frac{x^{2k+2}}{2^{2k+2}} \\
&\quad (\Gamma(k+2+\frac{1}{2}) = \Gamma(k+1+\frac{1}{2})\Gamma(k+1+\frac{1}{2}) \text{ then use Exercise 44(b)}) \\
&= \sqrt{\frac{2}{\pi\, x}}\sum_{k=0}^{\infty} \frac{(-1)^k(2k+2)}{(2k+3)!}\,x^{2k+2} \quad \text{(multiply and divide by } (2k+2)) \\
&= \sqrt{\frac{2}{\pi\, x}}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k)}{(2k+1)!}\,x^{2k} \quad \text{(change } k \text{ to } k-1) \\
&= \sqrt{\frac{2}{\pi\, x}}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}[(2k+1)-1]}{(2k+1)!}\,x^{2k} \\
&= \sqrt{\frac{2}{\pi\, x}}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!}\,x^{2k} - \sqrt{\frac{2}{\pi\, x}}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k+1)!}\,x^{2k} \\
&= \sqrt{\frac{2}{\pi\, x}}\left(-\cos x + \frac{\sin x}{x}\right).
\end{aligned}
$$

25. (a) Let $u = \frac{2}{a}\, e^{-\frac{1}{2}(at-b)}$, $Y(u) = y(t)$, $e^{-at+b} = \frac{a^2}{4}\, u^2$; then

$$
\frac{dy}{dt} = \frac{dY}{du}\frac{du}{dt} = Y'\,(-e^{-\frac{1}{2}(at-b)});\ \frac{d^2y}{dt^2} = \frac{d}{du}\left(Y'\,(-e^{-\frac{1}{2}(at-b)})\right) = Y''e^{-at+b} + Y'\frac{a}{2}e^{-\frac{1}{2}(at-b)}.
$$

So

$$
Y''e^{-at+b} + Y'\frac{a}{2}e^{-\frac{1}{2}(at-b)} + Ye^{-at+b} = 0 \quad \Rightarrow \quad Y'' + \frac{a}{2}Y'e^{-\frac{1}{2}(at-b)} + Y = 0,
$$

upon multiplying by e^{at-b}. Using $u = \frac{2}{a}e^{-\frac{1}{2}(at-b)}$, we get

$$
Y'' + \frac{1}{u}Y' + Y = 0 \quad \Rightarrow \quad u^2Y'' + uY' + u^2Y = 0,
$$

which is Bessel's equation of order 0.
(b) The general solution of $u^2Y'' + uY' + u^2Y = 0$ is $Y(u) = c_1J_0(u) + c_2Y_0(u)$. But $Y(u) = y(t)$ and $u = \frac{2}{a}e^{-\frac{1}{2}(at-b)}$, so

$$
y(t) = c_1J_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}) + c_2Y_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).
$$

(c) (i) If $c_1 = 0$ and $c_2 \neq 0$, then

$$
y(t) = c_2Y_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).
$$

As $t \to \infty$, $u \to 0$, and $Y_0(u) \to -\infty$. In this case, $y(t)$ could approach either $+\infty$ or $-\infty$ depending on the sign of c_2. $y(t)$ would approach infinity linearly as near 0, $Y_0(x) \approx \ln x$ so $y(t) \approx \ln\left(\frac{2}{a}e^{-\frac{1}{2}(at-b)}\right) \approx At$.
(ii) If $c_1 \neq 0$ and $c_2 = 0$, then

$$
y(t) = c_1J_0(\frac{2}{a}e^{-\frac{1}{2}(at-b)}).
$$

As $t \to \infty$, $u(t) \to 0$, $J_0(u) \to 1$, and $y(t) \to c_1$. In this case the solution is bounded.
(ii) If $c_1 \neq 0$ and $c_2 \neq 0$, as $t \to \infty$, $u(t) \to 0$, $J_0(u) \to 1$, $Y_0(u) \to -\infty$. Since Y_0 will dominate, the solution will behave like case (i).

It makes sense to have unbounded solutions because eventually the spring wears out and does not affect the motion. Newton's laws tell us the mass will continue with unperturbed momentum, i.e., as $t \to \infty$, $y'' = 0$ and so $y(t) = c_1 t + c_2$, a linear function, which is unbounded if $c_1 \neq 0$.

33. (a) In (13), let $u^2 = t$, $2u\,du = dt$, then

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt = \int_0^\infty u^{2(x-1)} e^{-u^2}(2u)du = 2\int_0^\infty u^{2x-1} e^{-u^2}\,du.$$

(b) Using (a)

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 2\int_0^\infty u^{2x-1}e^{-u^2}du\, 2\int_0^\infty v^{2y-1}e^{-v^2}dv \\ &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}u^{2x-1}v^{2y-1}du\,dv.\end{aligned}$$

(c) Switching to polar coordinates: $u = r\cos\theta$, $v = r\sin\theta$, $u^2+v^2 = r^2$, $dudv = rdrd\theta$; for (u, v) varying in the first quadrant ($0 \le u < \infty$ and $0 \le v < \infty$), we have $0 \le \theta \le \frac{\pi}{2}$, and $0 \le r < \infty$, and the double integral in (b) becomes

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}(r\cos\theta)^{2x-1}(r\sin\theta)^{2y-1}rdrd\theta \\ &= 2\int_0^{\frac{\pi}{2}}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta\,\overbrace{2\int_0^\infty r^{2(x+y)-1}e^{-r^2}\,dr}^{=\Gamma(x+y)} \\ &\qquad \text{(use (a) with } x+y \text{ in place of } x) \\ &= 2\Gamma(x+y)\int_0^{\frac{\pi}{2}}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}d\theta,\end{aligned}$$

implying (c).

41. Let $I = \int_0^{\pi/2}\sin^{2k+1}\theta\,d\theta$. Applying Exercise 33, we take $2x-1 = 0$ and $2y-1 = 2k+1$, so $x = \frac{1}{2}$ and $y = k+1$. Then

$$2I = \frac{\Gamma(\frac{1}{2})\,\Gamma(k+1)}{\Gamma(k+1+\frac{1}{2})} = \frac{\sqrt{\pi}\,k!}{(k+\frac{1}{2})\Gamma(k+\frac{1}{2})} = \frac{2\sqrt{\pi}\,k!}{(2k+1)\Gamma(k+\frac{1}{2})}.$$

As in (a), we now use $\Gamma(k+\frac{1}{2}) = \frac{(2k)!}{2^{2k}k!}\sqrt{\pi}$, simplify, and get

$$I = \frac{2^{2k}(k!)^2}{(2k+1)!}.$$

Solutions to Exercises 4.8

1. (a) Using the series definition of the Bessel function, (7), Section 4.7, we have

$$\begin{aligned}
\frac{d}{dx}[x^{-p}J_p(x)] &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^p k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^p k!\Gamma(k+p+1)}\frac{d}{dx}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k 2k}{2^p k!\Gamma(k+p+1)}\frac{1}{2}\left(\frac{x}{2}\right)^{2k-1} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^p (k-1)!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k-1} \\
&= -\sum_{m=0}^{\infty}\frac{(-1)^m}{2^p m!\Gamma(m+p+2)}\left(\frac{x}{2}\right)^{2m+1} \qquad \text{(set } m=k-1\text{)} \\
&= -x^{-p}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+p+2)}\left(\frac{x}{2}\right)^{2m+p+1} = -x^{-p}J_{p+1}(x).
\end{aligned}$$

To prove (7), use (1):

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x) \quad \Rightarrow \quad \int x^p J_{p-1}(x)\,dx = x^p J_p(x) + C.$$

Now replace p by $p+1$ and get

$$\int x^{p+1} J_p(x)\,dx = x^{p+1} J_{p+1}(x) + C,$$

which is (7). Similarly, starting with (2),

$$\begin{aligned}
\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x) \quad &\Rightarrow \quad -\int x^{-p}J_{p+1}(x)\,dx = x^{-p}J_p(x) + C \\
&\Rightarrow \quad \int x^{-p}J_{p+1}(x)\,dx = -x^{-p}J_p(x) + C.
\end{aligned}$$

Now replace p by $p-1$ and get

$$\int x^{-p+1}J_p(x)\,dx = -x^{-p+1}J_{p-1}(x) + C,$$

which is (8).
(b) To prove (4), carry out the differentiation in (2) to obtain

$$x^{-p}J_p'(x) - px^{-p-1}J_p(x) = -x^{-p}J_{p+1}(x) \quad \Rightarrow \quad xJ_p'(x) - pJ_p(x) = -xJ_{p+1}(x),$$

upon multiplying through by x^{p+1}. To prove (5), add (3) and (4) and then divide by x to obtain

$$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x).$$

To prove (6), subtract (4) from (3) then divide by x.

5. $\int J_1(x)\,dx = -J_0(x) + C$, by (8) with $p = 1$.

9.

$$\begin{aligned}
\int J_3(x)\,dx &= \int x^2[x^{-2}J_3(x)]\,dx \\
&\quad x^2 = u,\, x^{-2}J_3(x)\,dx = dv,\ 2x\,dx = du,\, v = -x^{-2}J_2(x) \\
&= -J_2(x) + 2\int x^{-1}J_2(x)\,dx = -J_2(x) - 2x^{-1}J_1(x) + C \\
&= J_0(x) - \frac{2}{x}J_1(x) - \frac{2}{x}J_1(x) + C \text{(use (6) with } p=1\text{)} \\
&= J_0(x) - \frac{4}{x}J_1(x) + C.
\end{aligned}$$

13. Use (6) with $p = 4$. Then

$$
\begin{aligned}
J_5(x) &= \frac{8}{x} J_4(x) - J_3(x) \\
&= \frac{8}{x}\left[\frac{6}{x} J_3(x) - J_2(x)\right] - J_3(x) \quad \text{(by (6) with } p = 3) \\
&= \left(\frac{48}{x^2} - 1\right) J_3(x) - \frac{8}{x} J_2(x) \\
&= \left(\frac{48}{x^2} - 1\right)\left(\frac{4}{x} J_2(x) - J_1(x)\right) - \frac{8}{x} J_2(x) \quad \text{(by (6) with } p = 2) \\
&= \left(\frac{192}{x^3} - \frac{12}{x}\right) J_2(x) - \left(\frac{48}{x^2} - 1\right) J_1(x) \\
&= \frac{12}{x}\left(\frac{16}{x^2} - 1\right)\left[\frac{2}{x} J_1(x) - J_0(x)\right] - \left(\frac{48}{x^2} - 1\right) J_1(x) \\
&\quad \text{(by (6) with } p = 1) \\
&= -\frac{12}{x}\left(\frac{16}{x^2} - 1\right) J_0(x) + \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1(x).
\end{aligned}
$$

17. (a) From (17),

$$
\begin{aligned}
A_j &= \frac{2}{J_1(\alpha_j)^2} \int_0^1 f(x) J_0(\alpha_j x) x\, dx = \frac{2}{J_1(\alpha_j)^2} \int_0^c J_0(\alpha_j x) x\, dx \\
&= \frac{2}{\alpha_j^2 J_1(\alpha_j)^2} \int_0^{c\alpha_j} J_0(s) s\, ds \quad (\text{let } \alpha_j x = s) \\
&= \frac{2}{\alpha_j^2 J_1(\alpha_j)^2} J_1(s) s \Big|_0^{c\alpha_j} = \frac{2c J_1(\alpha_j)}{\alpha_j J_1(\alpha_j)^2}.
\end{aligned}
$$

Thus, for $0 < x < 1$,

$$
f(x) = \sum_{j=1}^{\infty} \frac{2c J_1(\alpha_j)}{\alpha_j J_1(\alpha_j)^2} J_0(\alpha_j x).
$$

(b) The function f is piecewise smooth, so by Theorem 2 the series in (a) converges to $f(x)$ for all $0 < x < 1$, except at $x = c$, where the series converges to the average value $\frac{f(c+)+f(c-)}{2} = \frac{1}{2}$.

21. (a) Take $m = 1/2$ in the series expansion of Exercise 20 and you'll get

$$
\sqrt{x} = 2 \sum_{j=1}^{\infty} \frac{J_{1/2}(\alpha_j x)}{\alpha_j\, J_{3/2}(\alpha_j)} \quad \text{for } 0 < x < 1,
$$

where α_j is the jth positive zero of $J_{1/2}(x)$. By Example 1, Section 4.7, we have

$$
J_{1/2}(x) = \sqrt{\frac{2}{\pi\, x}} \sin x.
$$

So

$$
\alpha_j = j\pi \quad \text{for } j = 1, 2, \ldots.
$$

(b) We recall from Exercise 11 that

$$
J_{3/2}(x) = \sqrt{\frac{2}{\pi\, x}} \left(\frac{\sin x}{x} - \cos x\right).
$$

So the coefficients are

$$\begin{aligned} A_j &= \frac{2}{\alpha_j\, J_{3/2}(\alpha_j)} = \frac{2}{j\pi\, J_{3/2}(j\pi)} \\ &= \frac{2}{j\pi\,\sqrt{\frac{2}{\pi\, j\pi}}\left(\frac{\sin j\pi}{j\pi} - \cos j\pi\right)} \\ &= (-1)^{j-1}\sqrt{\frac{2}{j}} \end{aligned}$$

and the Bessel series expansion becomes, or $0 < x < 1$,

$$\sqrt{x} = \sum_{j=1}^{\infty}(-1)^{j-1}\sqrt{\frac{2}{j}}J_{1/2}(\alpha_j x).$$

(c) Writing $J_{1/2}(x)$ in terms of $\sin x$ and simplifying, this expansion becomes

$$\begin{aligned} \sqrt{x} &= \sum_{j=1}^{\infty}(-1)^{j-1}\sqrt{\frac{2}{j}}J_{1/2}(\alpha_j x) \\ &= \sum_{j=1}^{\infty}(-1)^{j-1}\sqrt{\frac{2}{j}}\sqrt{\frac{2}{\pi\,\alpha_j}}\sin\alpha_j \\ &= \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\frac{\sin(j\pi x)}{\sqrt{x}}. \end{aligned}$$

Upon multiplying both sides by $\sqrt{x}$, we obtain

$$x = \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\sin(j\pi x) \quad \text{for } 0 < x < 1,$$

which is the familiar Fourier sine series (half-range expansion) of the function $f(x) = x$.

25. By Theorem 2 with $p = 1$, we have

$$\begin{aligned} A_j &= \frac{2}{J_2(\alpha_{1,j})^2}\int_{\frac{1}{2}}^{1} J_1(\alpha_{1,j}x)\,dx \\ &= \frac{2}{\alpha_{1,j}J_2(\alpha_{1,j})^2}\int_{\frac{\alpha_{1,j}}{2}}^{\alpha_{1,j}} J_1(s)\,ds \quad (\text{let } \alpha_{1,j}x = s) \\ &= \frac{2}{\alpha_{1,j}J_2(\alpha_{1,j})^2}[-J_0(s)\Big|_{\frac{\alpha_{1,j}}{2}}^{\alpha_{1,j}} \quad (\text{by (8) with } p = 1) \\ &= \frac{-2\left[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\right]}{\alpha_{1,j}J_2(\alpha_{1,j})^2} \\ &= \frac{-2\left[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\right]}{\alpha_{1,j}J_0(\alpha_{1,j})^2}, \end{aligned}$$

where in the last equality we used (6) with $p = 1$ at $x = \alpha_{1,j}$ (so $J_0(\alpha_{1,j})+J_2(\alpha_{1,j}) = 0$ or $J_0(\alpha_{1,j}) = -J_2(\alpha_{1,j})$). Thus, for $0 < x < 1$,

$$f(x) = -2\sum_{j=1}^{\infty}\frac{-2\left[J_0(\alpha_{1,j}) - J_0(\frac{\alpha_{1,j}}{2})\right]}{\alpha_{1,j}J_0(\alpha_{1,j})^2}J_1(\alpha_{1,j}x).$$

29. By Theorem 2 with $p = 1$, we have

$$\begin{aligned} A_j &= \frac{1}{2\,J_2(\alpha_{1,j})^2}\int_0^2 J_1(\alpha_{2,j}x/2)x\,dx \\ &= \frac{2}{\alpha_{1,j}^2 J_2(\alpha_{1,j})^2}\int_0^{\alpha_{1,j}} J_1(s)s\,ds \quad (\text{let } \alpha_{1,j}x/2 = s). \end{aligned}$$

Since we cannot evaluate the definite integral in a simpler form, just leave it as it is and write the Bessel series expansion as

$$1 = \sum_{j=1}^{\infty} \frac{2}{\alpha_{1,j}^2 J_2(\alpha_{1,j})^2}\left[\int_0^{\alpha_{1,j}} J_1(s)s\,ds\right] J_1(\alpha_{1,j}x/2) \quad \text{for } 0 < x < 2.$$

33. $p = \frac{1}{2}$, $y = c_1 J_{\frac{1}{2}}(\lambda x) + c_2 Y_{\frac{1}{2}}(\lambda x)$. For y to be bounded near 0, we must take $c_2 = 0$. For $y(\pi) = 0$, we must take $\lambda = \lambda_j = \frac{\alpha_{\frac{1}{2},j}}{\pi} = j$, $j = 1, 2, \ldots$ (see Exercises 21); and so

$$y = y_i = c_{1,j} J_1(\frac{\alpha_{\frac{1}{2},j}}{\pi}x) = c_{1,j}\sqrt{\frac{2}{\pi x}}\sin(jx)$$

(see Example 1, Section 4.7).

One more formula. To complement the integral formulas from this section, consider the following interesting formula. Let a, b, c, and p be positive real numbers with $a \neq b$. Then

$$\int_0^c J_p(ax)\,J_p(bx)x\,dx = \frac{c}{b^2 - a^2}\big[aJ_p(bc)J_{p-1}(ac) - bJ_p(ac)J_{p-1}(bc)\big].$$

To prove this formula, we note that $y_1 = J_p(ax)$ satisfies

$$x^2 y_1'' + xy_1' + (a^2x^2 - p^2)y_1 = 0$$

and $y_2 = J_p(bx)$ satisfies

$$x^2 y_2'' + xy_2' + (b^2x^2 - p^2)y_2 = 0.$$

Write these equations in the form

$$\big(xy_1'\big)' + y_1' + \frac{a^2x^2 - p^2}{x}y_1 = 0$$

and

$$\big(xy_2'\big)' + y_2' + \frac{b^2x^2 - p^2}{x}y_1 = 0.$$

Multiply the first by y_2 and the second by $y - 1$, subtract, simplify, and get

$$y_2\big(xy_1'\big)' - y_1\big(xy_2'\big)' = y_1y_2(b^2 - a^2)x.$$

Note that

$$y_2\big(xy_1'\big)' - y_1\big(xy_2'\big)' = \frac{d}{dx}\big[y_2(xy_1') - y_1(xy_2')\big].$$

So

$$(b^2 - a^2)y_1y_2x = \frac{d}{dx}\big[y_2(xy_1') - y_1(xy_2')\big],$$

and, after integrating,

$$(b^2 - a^2)\int_0^c y_1(x)y_2(x)x\,dx = \big[y_2(xy_1') - y_1(xy_2')\big]\Big|_0^c = x\big[y_2y_1' - y_1y_2'\big]\Big|_0^c.$$

On the left, we have the desired integral times $(b^2 - a^2)$ and, on the right, we have

$$c\big[J_p(bc)aJ_p'(ac) - bJ_p(ac)J_p'(bc)\big] - c\big[aJ_p(0)J_p'(0) - bJ_p(0)J_p'(0)\big].$$

Since $J_p(0) = 0$ if $p > 0$ and $J_0'(x) = -J_1(x)$, it follows that $J_p(0)J_p'(0) - J_p(0)J_p'(0) = 0$ for all $p > 0$. Hence the integral is equal to

$$I = \int_0^c J_p(ax)\, J_p(bx)x\, dx = \frac{c}{b^2 - a^2}\big[aJ_p(bc)J_p'(ac) - bJ_p(ac)J_p'(bc)\big].$$

Now using the formula

$$J_p'(x) = \frac{1}{2}\big[J_{p-1}(x) - J_{p+1}(x)\big],$$

we obtain

$$I = \frac{c}{2(b^2 - a^2)}\big[aJ_p(bc)\big(J_{p-1}(ac) - J_{p+1}(ac)\big) - bJ_p(ac)\big(J_{p-1}(bc) - J_{p+1}(bc)\big)\big].$$

Simplify with the help of the formula

$$J_{p+1}(x) = \frac{2p}{x}J_p(x) - J_{p-1}(x)$$

and you get

$$\begin{aligned} I &= \frac{c}{2(b^2 - a^2)}\Big[aJ_p(bc)\big(J_{p-1}(ac) - (\frac{2p}{ac}J_p(ac) - J_{p-1}(ac))\big) \\ &\qquad -bJ_p(ac)\big(J_{p-1}(bc) - (\frac{2p}{bc}J_p(bc) - J_{p-1}(bc))\big)\Big] \\ &= \frac{c}{b^2 - a^2}\big[aJ_p(bc)J_{p-1}(ac) - bJ_p(ac)J_{p-1}(bc)\big], \end{aligned}$$

as claimed.

Note that this formula implies the orthogonality of Bessel functions. In fact its proof mirrors the proof of orthogonality from Section 4.8.

Solutions to Exercises 4.9

1. We have

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos\big(-x\sin\theta\big)\,d\theta = \frac{1}{\pi}\int_0^{\pi} \cos\big(x\sin\theta\big)\,d\theta.$$

So

$$J_0(0) = \frac{1}{\pi}\int_0^{\pi} d\theta = 1.$$

For $n \neq 0$,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos\big(n\theta - x\sin\theta\big)\,d\theta;$$

so

$$J_n(0) = \frac{1}{\pi}\int_0^{\pi} \cos n\theta\,d\theta = 0.$$

5. All the terms in the series

$$1 = J_0(x)^2 + 2\sum_{n=1}^{\infty} J_n(x)^2$$

are nonnegative. Since they all add-up to 1, each must be less than or equal to 1. Hence

$$J_0(x)^2 \leq 1 \Rightarrow |J_0(x)| \leq 1$$

and, for $n \geq 2$,

$$2J_n(x)^2 \leq 1 \Rightarrow |J_n(x)| \leq \frac{1}{\sqrt{2}}.$$

Solutions to Exercises 5.1

1. Start with Laplace's equation in spherical coordinates

$$(1) \qquad \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial \theta^2} + \cot\theta \frac{\partial u}{\partial \theta} + \csc^2\theta\,\frac{\partial^2 u}{\partial \phi^2}\Big) = 0,$$

where $0 < r < a$, $0 < \phi < 2\pi$, and $0 < \theta < \pi$. To separate variable, take a product solution of the form

$$u(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) = R\Theta\Phi,$$

and plug it into (1). We get

$$R''\Theta\Phi + \frac{2}{r}R'\Theta\Phi + \frac{1}{r^2}\Big(R\Theta''\Phi + \cot\theta R\Theta'\Phi + \csc^2\theta\, R\Theta\Phi''\Big) = 0.$$

Divide by $R\Theta\Phi$ and multiply by r^2:

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \csc^2\theta\,\frac{\Phi''}{\Phi} = 0.$$

Now proceed to separate the variables:

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} = -\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \csc^2\theta\,\frac{\Phi''}{\Phi}\right).$$

Since the left side is a function of r and the right side is a function of ϕ and θ, each side must be constant and the constants must be equal. So

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} = \mu$$

and

$$\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \csc^2\theta\,\frac{\Phi''}{\Phi} = -\mu.$$

The equation in R is equivalent to (3). Write the second equation in the form

$$\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \mu = -\csc^2\theta\,\frac{\Phi''}{\Phi};$$

$$\sin^2\theta\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \mu\right) = -\frac{\Phi''}{\Phi}.$$

This separates the variables θ and ϕ, so each side must be constant and the constant must be equal. Hence

$$\sin^2\theta\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \mu\right) = \nu$$

and

$$\nu = -\frac{\Phi''}{\Phi} \quad\Rightarrow\quad \Phi'' + \nu\Phi = 0.$$

We expect 2π-periodic solutions in Φ, because ϕ is and azimuthal angle. The only way for the last equation to have 2π-periodic solutions that are essentially different is to set $\nu = m^2$, where $m = 0, 1, 2, \ldots$. This gives the two equations

$$\Phi'' + m^2\Phi = 0$$

(equation (5)) and

$$\sin^2\theta\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \mu\right) = m^2,$$

which is equivalent to (6).

Solutions to Exercises 5.2

1. This problem is similar to Example 2. Note that f is its own Legendre series:

$$f(\theta) = 20\,(P_1(\cos\theta) + P_0(\cos\theta)).$$

So really there is no need to compute the Legendre coefficients using integrals. We simply have $A_0 = 20$ and $A_1 = 20$, and the solution is

$$u(r,\,\theta) = 20 + 20\,r\,\cos\theta.$$

3. We have

$$u(r,\,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta),$$

with

$$\begin{aligned} A_n &= \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\,\sin\theta\,d\theta \\ &= \frac{2n+1}{2}\int_0^{\frac{\pi}{2}} 100\,P_n(\cos\theta)\,\sin\theta\,d\theta + \frac{2n+1}{2}\int_{\frac{\pi}{2}}^{\pi} 20\,P_n(\cos\theta)\,\sin\theta\,d\theta. \end{aligned}$$

Let $x = \cos\theta$, $dx = -\sin\theta\,d\theta$. Then

$$A_n = 50(2n+1)\int_0^1 P_n(x)\,dx + 10\,(2n+1)\int_{-1}^0 P_n(x)\,dx.$$

The case $n = 0$ is immediate by using $P_0(x) = 1$,

$$A_0 = 50\int_0^1 dx + 10\int_{-1}^0 dx = 60.$$

For $n > 0$, the integrals are not straightforward and you need to refer to Exercise 10, Section 5.6, where they are evaluated. Quoting from this exercise, we have

$$\int_0^1 P_{2n}(x)dx = 0, \quad n = 1, 2, \ldots,$$

and

$$\int_0^1 P_{2n+1}(x)\,dx = \frac{(-1)^n(2n)!}{2^{2n+1}(n!)^2(n+1)}, \quad n = 0, 1, 2, \ldots.$$

Since $P_{2n}(x)$ is an even function, then, for $n > 0$,

$$\int_{-1}^0 P_{2n}(x)\,dx = \int_0^1 P_{2n}(x)\,dx = 0.$$

Hence for $n > 0$,

$$A_{2n} = 0.$$

Now $P_{2n+1}(x)$ is an odd function, so

$$\int_{-1}^0 P_{2n+1}(x)\,dx = -\int_0^1 P_{2n+1}(x)\,dx = -\frac{(-1)^n(2n)!}{2^{2n+1}(n!)^2(n+1)}.$$

Hence for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} A_{2n+1} &= 50(4n+3)\int_0^1 P_{2n+1}(x)\,dx + 10\,(4n+3)\int_{-1}^0 P_{2n+1}(x)\,dx \\ &= 50(4n+3)\frac{(-1)^n(2n)!}{2^{2n+1}(n!)^2(n+1)} - 10\,(4n+3)\frac{(-1)^n(2n)!}{2^{2n+1}(n!)^2(n+1)} \\ &= 40\,(4n+3)\frac{(-1)^n(2n)!}{2^{2n+1}(n!)^2(n+1)} \\ &= 20\,(4n+3)\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(n+1)}. \end{aligned}$$

So
$$u(r,\theta)=60+20\sum_{n=0}^{\infty}(4n+3)\frac{(-1)^n(2n)!}{2^{2n}(n!)^2(n+1)}r^{2n+1}P_{2n+1}(\cos\theta).$$

5. Solution We have
$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta),$$
with
$$\begin{aligned}A_n &= \frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\,\sin\theta\,d\theta\\ &= \frac{2n+1}{2}\int_0^{\frac{\pi}{2}}\cos\theta\,P_n(\cos\theta)\,\sin\theta\,d\theta\\ &= \frac{2n+1}{2}\int_0^{1} x\,P_n(x)\,dx,\end{aligned}$$
where, as in Exercise 3, we made the change of variables $x=\cos\theta$. At this point, we have to appeal to Exercise 11, Section 5.6, for the evaluation of this integral. (The cases $n=0$ and 1 can be done by referring to the explicit formulas for the P_n, but we may as well at this point use the full result of Exercise 11, Section 5.6.) We have
$$\int_0^1 xP_0(x)\,dx=\frac{1}{2};\ \int_0^1 xP_1(x)\,dx=\frac{1}{3};$$
$$\int_0^1 x\,P_{2n}(x)\,dx=\frac{(-1)^{n+1}(2n-2)!}{2^{2n}((n-1)!)^2n(n+1)};\quad n=1,2,\ldots;$$
and
$$\int_0^1 x\,P_{2n+1}(x)\,dx=0;\quad n=1,2,\ldots.$$
Thus,
$$A_0=\frac{1}{2}\frac{1}{2}=\frac{1}{4};\quad A_1=\frac{3}{2}\frac{1}{3}=\frac{1}{2};\quad A_{2n+1}=0,\quad n=1,\,2,\,3,\ldots;$$
and for $n=1,\,2,\,\ldots$,
$$A_{2n}=\frac{2(2n)+1}{2}\frac{(-1)^{n+1}(2n-2)!}{2^{2n}((n-1)!)^2n(n+1)}=(4n+1)\frac{(-1)^{n+1}(2n-2)!}{2^{2n+1}((n-1)!)^2n(n+1)}.$$
So
$$u(r,\theta)=\frac{1}{4}+\frac{1}{2}r\cos\theta+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(4n+1)(2n-2)!}{2^{2n+1}((n-1)!)^2n(n+1)}r^{2n}P_{2n}(\cos\theta).$$

Solutions to Exercises 5.3

1. (c) Starting with (4) with $n = 2$, we have

$$Y_{2,m}(\theta,\phi) = \sqrt{\frac{5}{4\pi}\frac{(2-m)!}{(2+m)!}}\, P_2^m(\cos\theta)e^{im\phi},$$

where $m = -2, -1, 0, 1, 2$. To compute the spherical harmonics explicitely, we will need the explicit formula for the associated Legendre functions from Example 1, Section 5.7. We have

$$P_2^{-2}(x) = \tfrac{1}{8}(1-x^2);\quad P_2^{-1}(x) = \tfrac{1}{2}x\sqrt{1-x^2};\quad P_2^0(x) = P_2(x) = \tfrac{3x^2-1}{2};$$
$$P_2^1(x) = -3x\sqrt{1-x^2};\qquad P_2^2(x) = 3(1-x^2).$$

So

$$\begin{aligned} Y_{2,-2}(\theta,\phi) &= \sqrt{\frac{5}{4\pi}\frac{(2+2)!}{(2-2)!}}\, P_2^{-2}(\cos\theta)e^{-2i\phi} \\ &= \sqrt{\frac{5}{4\pi}\frac{4!}{1}}\,\frac{1}{8}\overbrace{(1-\cos^2\theta)}^{=\sin^2\theta}e^{-2i\phi} \\ &= \sqrt{\frac{30}{\pi}}\,\frac{1}{8}\sin^2\theta e^{-2i\phi} = \frac{3}{4}\sqrt{\frac{5}{6\,\pi}}\,\sin^2\theta e^{-2i\phi}; \end{aligned}$$

$$\begin{aligned} Y_{2,-1}(\theta,\phi) &= \sqrt{\frac{5}{4\pi}\frac{(2+1)!}{(2-1)!}}\, P_2^{-1}(\cos\theta)e^{-i\phi} \\ &= \sqrt{\frac{5}{4\pi}\frac{3!}{1!}}\,\frac{1}{2}\cos\theta\,\overbrace{\sqrt{1-\cos^2\theta}}^{=\sin\theta}\,e^{-i\phi} \\ &= \sqrt{\frac{15}{2\pi}}\,\frac{1}{2}\cos\theta\sin\theta e^{-i\phi} = \frac{3}{2}\sqrt{\frac{5}{6\,\pi}}\,\cos\theta\sin\theta e^{-i\phi}. \end{aligned}$$

Note that since $0 \le \theta \le \pi$, we have $\sin\theta \ge 0$, and so the equality $\sqrt{1-\cos^2\theta} = \sin\theta$ that we used above does hold. Continuing the list of spherical harmonics, we have

$$\begin{aligned} Y_{2,0}(\theta,\phi) &= \sqrt{\frac{5}{4\pi}\frac{(2+0)!}{(2-0)!}}\, P_2(\cos\theta)e^{-i\phi} \\ &= \sqrt{\frac{5}{4\pi}}\,\frac{3\cos^2\theta-1}{2} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta-1). \end{aligned}$$

The other spherical harmonics are computed similarly; or you can use the identity in Exercise 4. We have

$$\begin{aligned} Y_{2,2} &= (-1)^2\,\overline{Y_{2,-2}} = \overline{Y_{2,-2}} = \frac{3}{4}\sqrt{\frac{5}{6\,\pi}}\,\sin^2\theta\overline{e^{-2i\phi}} \\ &= \frac{3}{4}\sqrt{\frac{5}{6\,\pi}}\,\sin^2\theta e^{2i\phi}. \end{aligned}$$

In the preceding computation, we used two basic properties of the operation of complex conjugation:

$$\overline{az} = a\overline{z}\quad \text{if } a \text{ is a real number;}$$

and

$$\overline{e^{ia}} = e^{-ia}\quad \text{if } a \text{ is a real number.}$$

Finally,

$$\begin{aligned} Y_{2,1} &= (-1)^1\,\overline{Y_{2,-1}} = -\overline{Y_{2,-1}} = -\frac{3}{2}\sqrt{\frac{5}{6\,\pi}}\,\cos\theta\sin\overline{e^{-i\phi}} \\ &= -\frac{3}{2}\sqrt{\frac{5}{6\,\pi}}\,\cos\theta\sin e^{i\phi}. \end{aligned}$$

■

5. (a) If $m = 0$, the integral becomes

$$\int_0^{2\pi} \phi\,d\phi = \frac{1}{2}\phi^2\Big|_0^{2\pi} = 2\pi^2.$$

Now suppose that $m \neq 0$. Using integration by parts, with $u = \phi,\ du = d\phi,\ dv = e^{-im\phi},\ v = \frac{1}{-im}e^{-im\phi}$, we obtain:

$$\int_0^{2\pi} \overbrace{\phi}^{=u}\ \overbrace{e^{-im\phi}}^{=dv}\,d\phi = \left[\phi\,\frac{1}{-im}e^{-im\phi}\right|_0^{2\pi} + \frac{1}{im}\int_0^{2\pi} e^{-im\phi}\,d\phi$$

We have

$$e^{-im\phi}\Big|_{\phi=2\pi} = \Big[\cos(m\phi) - i\sin(m\phi)\Big|_{\phi=2\pi} = 1,$$

and

$$\begin{aligned} \int_0^{2\pi} e^{-im\phi}\,d\phi &= \int_0^{2\pi}\big(\cos m\phi - i\sin m\phi\big)\,d\phi \\ &= \begin{cases} 0 & \text{if } m \neq 0, \\ 2\pi & \text{if } m = 0 \end{cases} \end{aligned}$$

So if $m \neq 0$,

$$\int_0^{2\pi} \phi\,e^{-im\phi}\,d\phi = \frac{2\pi}{-im} = \frac{2\pi}{m}i.$$

Putting both results together, we obtain

$$\int_0^{2\pi} \phi\,e^{-im\phi}\,d\phi = \begin{cases} \frac{2\pi}{m}i & \text{if } m \neq 0, \\ 2\pi^2 & \text{if } m = 0. \end{cases}$$

(b) Using $n = 0$ and $m = 0$ in (9), we get

$$\begin{aligned} A_{0,0} &= \frac{1}{2\pi}\sqrt{\frac{1}{4\pi}\frac{0!}{0!}}\overbrace{\int_0^{2\pi}\phi\,d\phi}^{=2\pi^2}\int_0^{\pi} P_0(\cos\theta)\sin\theta\,d\theta \\ &= \pi\frac{1}{2\sqrt{\pi}}\int_0^{\pi}\frac{1}{\sqrt{2}}\sin\theta\,d\theta \\ &= \pi\frac{1}{2\sqrt{\pi}}\overbrace{\int_0^{\pi}\sin\theta\,d\theta}^{=2} = \sqrt{\pi} \end{aligned}$$

Using $n = 1$ and $m = 0$ in (9), we get

$$\begin{aligned} A_{1,0} &= \frac{1}{2\pi}\sqrt{\frac{3}{4\pi}\frac{1}{1}}\int_0^{2\pi}\phi\,d\phi\int_0^{\pi} P_1(\cos\theta)\sin\theta\,d\theta \\ &= \frac{1}{4\pi}\sqrt{\frac{3}{\pi}}(2\pi^2)\overbrace{\int_{-1}^{1} P_1(x)\,dx}^{=0} \\ &= 0, \end{aligned}$$

where we used $\int_{-1}^{1} P_1(x)\,dx = 0$, because $P_1(x) = x$ is odd. Using $n = 1$ and $m = -1$ in (9), and appealing to the formulas for the associated Legendre functions from Section 5.7, we get

$$\begin{aligned} A_{1,-1} &= \frac{1}{2\pi}\sqrt{\frac{3}{4\pi}\frac{2!}{0!}}\overbrace{\int_0^{2\pi} \phi\, e^{i\phi}\,d\phi}^{=\frac{2\pi}{(-1)}i}\int_0^{\pi} P_1^{-1}(\cos\theta)\sin\theta\,d\theta \\ &= -i\sqrt{\frac{3}{2\pi}}\frac{1}{2}\overbrace{\int_0^{\pi}\sin^2\theta\,d\theta}^{=\frac{\pi}{2}} \quad (P_1^{-1}(\cos\theta) = \frac{1}{2}\sin\theta) \\ &= -\frac{i}{4}\sqrt{\frac{3\pi}{2}}. \end{aligned}$$

Using $n = 1$ and $m = 1$ in (9), and appealing to the formulas for the associated Legendre functions from Section 5.7, we get

$$\begin{aligned} A_{1,1} &= \frac{1}{2\pi}\sqrt{\frac{3}{4\pi}\frac{0!}{2!}}\overbrace{\int_0^{2\pi} \phi\, e^{-i\phi}\,d\phi}^{=\frac{2\pi}{1}i}\int_0^{\pi} P_1^{1}(\cos\theta)\sin\theta\,d\theta \\ &= i\sqrt{\frac{3}{8\pi}}\overbrace{\int_0^{\pi} -\sin^2\theta\,d\theta}^{=-\frac{\pi}{2}} \quad (P_1^{1}(\cos\theta) = -\sin\theta) \\ &= -\frac{i}{4}\sqrt{\frac{3\pi}{2}}. \end{aligned}$$

(c) The formula for $A_{n,0}$ contains the integral $\int_0^{\pi} P_n^0(\cos\theta)\sin\theta\,d\theta$. But $P_n^0 = P_n$, the nth Legendre polynomial; so

$$\begin{aligned} \int_0^{\pi} P_n^0(\cos\theta)\sin\theta\,d\theta &= \int_0^{\pi} P_n(\cos\theta)\sin\theta\,d\theta \\ &= \int_{-1}^{1} P_n(x)\,dx \\ &= 0 \quad (n = 1, 2, \ldots), \end{aligned}$$

where the last equality follows from the orthogonality of Legendre polynomials (take $m = 0$ in Theorem 1, Section 5.6, and note that $P_0(x) = 1$, so $\int_{-1}^{1}(1)P_n(x)\,dx = 0$, as desired.)

9. We apply (11). Since f is its own spherical harmonics series, we have

$$u(r, \theta, \phi) = Y_{0,0}(\theta, \phi) = \frac{1}{2\sqrt{\pi}}. \quad \blacksquare$$

Solutions to Exercises 5.4

5. We apply Theorem 3 and note that since f depends only on r and not on θ or ϕ, the series expansion should also not depend on θ or ϕ. So all the coefficients in the series are 0 except for the coefficients $A_{j,0,0}$, which we will write as A_j for simplicity. Using (16) with $m = n = 0$, $a = 1$, $f(r, \theta, \phi) = 1$, and $Y_{0,0}(\theta, \phi) = \frac{1}{2\sqrt{\pi}}$, we get

$$\begin{aligned} A_j &= \frac{2}{j_1^2(\alpha_{\frac{1}{2},j})} \int_0^1 \int_0^{2\pi} \int_0^{\pi} j_0(\lambda_{0,j}\, r) \frac{1}{2\sqrt{\pi}} r^2 \sin\theta \, d\theta \, d\phi \, dr \\ &= \frac{1}{\sqrt{\pi}\, j_1^2(\alpha_{\frac{1}{2},j})} \overbrace{\int_0^{2\pi} d\phi}^{=2\pi} \overbrace{\int_0^{\pi} \sin\theta \, d\theta}^{=2} \int_0^1 j_0(\lambda_{0,j}\, r) r^2 \, dr \\ &= \frac{4\sqrt{\pi}}{j_1^2(\alpha_{\frac{1}{2},j})} \int_0^1 j_0(\lambda_{0,j}\, r) r^2 \, dr, \end{aligned}$$

where $\lambda_{0,j} = \alpha_{\frac{1}{2},j}$, the jth zero of the Bessel function of order $\frac{1}{2}$. Now

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

(see Example 1, Section 4.7), so the zeros of $J_{1/2}$ are precisely the zeros of $\sin x$, which are $j\pi$. Hence

$$\lambda_{0,j} = \alpha_{\frac{1}{2},j} = j\pi.$$

Also, recall that

$$j_0(x) = \frac{\sin x}{x}$$

(Exercises 38, Section 4.8), so

$$\begin{aligned} \int_0^1 j_0(\lambda_{0,j}\, r) r^2 \, dr &= \int_0^1 j_0(j\pi\, r) r^2 \, dr = \int_0^1 \frac{\sin(j\pi r)}{j\pi\, r} r^2 \, dr \\ &= \frac{1}{j\pi} \overbrace{\int_0^1 \sin(j\pi\, r) r \, dr}^{=\frac{(-1)^{j+1}}{j\pi}} \\ &= \frac{(-1)^{j+1}}{(j\pi)^2}, \end{aligned}$$

where the last integral follows by integration by parts. So,

$$A_j = \frac{4\sqrt{\pi}}{j_1^2(j\pi)} \frac{(-1)^{j+1}}{(j\pi)^2}.$$

This can be simplified by using a formula for j_1. Recall from Exercise 38, Section 4.7,

$$j_1(x) = \frac{\sin x - x\cos x}{x^2}.$$

Hence

$$j_1^2(j\pi) = \left[\frac{\sin(j\pi) - j\pi\cos(j\pi)}{(j\pi)^2}\right]^2 = \left[\frac{-\cos(j\pi)}{j\pi}\right]^2 = \left[\frac{(-1)^{j+1}}{j\pi}\right]^2 = \frac{1}{(j\pi)^2},$$

and

$$A_j = 4(-1)^{j+1}\sqrt{\pi},$$

and so the series expansion becomes: for $0 < r < 1$,

$$\begin{aligned} 1 &= \sum_{j=1}^{\infty} 4(-1)^{j+1}\sqrt{\pi}\frac{\sin j\pi r}{j\pi r} Y_{0,0}(\theta,\, \phi) \\ &= \sum_{j=1}^{\infty} 4(-1)^{j+1}\sqrt{\pi}\frac{\sin j\pi r}{j\pi r}\frac{1}{2\sqrt{\pi}} \\ &= \sum_{j=1}^{\infty} 2(-1)^{j+1}\frac{\sin j\pi r}{j\pi r}. \end{aligned}$$

It is interesting to note that this series is in fact a half range sine series expansion. Indeed, multiplying both sides by r, we get

$$r = \frac{2}{\pi}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{\sin j\pi r}{j} \qquad (0 < r < 1),$$

which is a familiar sines series expansion (compare with Example 1, Section 2.4).■

Solutions to Exercises 5.5

1. Putting $n = 0$ in (9), we obtain

$$P_0(x) = \frac{1}{2^0} \sum_{m=0}^{0} (-1)^m \frac{(0-2m)!}{m!(0-m)!(0-2m)!} x^{0-2m}.$$

The sum contains only one term corresponding to $m = 0$. Thus

$$P_0(x) = (-1)^0 \frac{0!}{0!0!0!} x^0 = 1,$$

because $0! = 1$. For $n = 1$, formula (9) becomes

$$P_1(x) = \frac{1}{2^1} \sum_{m=0}^{M} (-1)^m \frac{(2-2m)!}{m!(1-m)!(1-2m)!} x^{1-2m},$$

where $M = \frac{1-1}{2} = 0$. Thus the sum contains only one term corresponding to $m = 0$ and so

$$P_1(x) = \frac{1}{2^1}(-1)^0 \frac{2!}{0!1!1!} x^1 = x.$$

For $n = 2$, we have $M = \frac{2}{2} = 1$ and (9) becomes

$$\begin{aligned} P_2(x) &= \frac{1}{2^2} \sum_{m=0}^{1} (-1)^m \frac{(4-2m)!}{m!(2-m)!(2-2m)!} x^{2-2m} \\ &= \frac{1}{2^2} \overbrace{(-1)^0 \frac{4!}{0!2!2!} x^2}^{m=0} + \frac{1}{2^2} \overbrace{(-1)^1 \frac{(4-2)!}{1!1!0!} x^0}^{m=1} \\ &= \frac{1}{4} 6x^2 + \frac{1}{4}(-1)2 = \frac{3}{2}x^2 - \frac{1}{2}. \end{aligned}$$

For $n = 3$, we have $M = \frac{3-1}{2} = 1$ and (9) becomes

$$\begin{aligned} P_3(x) &= \frac{1}{2^3} \sum_{m=0}^{1} (-1)^m \frac{(6-2m)!}{m!(3-m)!(3-2m)!} x^{3-2m} \\ &= \frac{1}{2^3}(-1)^0 \frac{6!}{0!3!3!} x^3 + \frac{1}{2^3}(-1)^1 \frac{4!}{1!2!1!} x^1 \\ &= \frac{5}{2}x^3 - \frac{3}{2}x. \end{aligned}$$

For $n = 4$, we have $M = \frac{4}{2} = 2$ and (9) becomes

$$\begin{aligned} P_4(x) &= \frac{1}{2^4} \sum_{m=0}^{2} (-1)^m \frac{(8-2m)!}{m!(4-m)!(4-2m)!} x^{4-2m} \\ &= \frac{1}{2^4} \frac{8!}{0!4!4!} x^4 - \frac{1}{2^4} \frac{6!}{1!3!2!} x^2 + \frac{1}{2^4} \frac{4!}{2!2!0!} x^0 \\ &= \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}$$

5. Using the explicit formulas for the Legendre polynomials, we find

$$\begin{aligned} \int_{-1}^{1} P_3(x)\, dx &= \int_{-1}^{1} (\frac{5}{2}x^3 - \frac{3}{2}x)\, dx \\ &= (\frac{5}{8}x^4 - \frac{3}{4}x^2)\Big|_{-1}^{1} = 0 \end{aligned}$$

Another faster way to see the answer is to simply note that P_3 is an odd function, so its integral over any symmetric interval is 0. There is yet another more important reason for this integral to equal 0. In fact,

$$\int_{-1}^{1} P_n(x)\,dx = 0 \qquad \text{for all } n \neq 0.$$

This is a consequence of orthogonality that you will study in Section 5.6.

9. This is Legendre's equation with $n(n+1) = 30$ so $n = 5$. Its general solution is of the form

$$\begin{aligned} y &= c_1 P_5(x) + c_2 Q_5(x) \\ &= c_1 \frac{1}{8}(63x^5 - 70x^3 + 15x) + c_2\left(1 - 15x^2 + 30x^4 + \cdots\right) \\ &= c_1(63x^5 - 70x^3 + 15x) + c_2\left(1 - 15x^2 + 30x^4 + \cdots\right) \end{aligned}$$

In finding $P_5(x)$, we used the given formulas in the text. In finding the first few terms of $Q_5(x)$, we used (3) with $n = 5$. (If you are comparing with the answers in your textbook, just remember that c_1 and c_2 are arbitrary constants.)

13. This is Legendre's equation with $n(n+1) = 6$ or $n = 2$. Its general solution is $y = c_1 P_2(x) + c_2 Q_2(x)$. The solution will be bounded on $[-1, 1]$ if and only if $c_2 = 0$; that's because P_2 is bounded in $[-1, 1]$ but Q_2 is not. Now, using $P_2(x) = \frac{1}{2}(3x^2 - 1)$, we find

$$y(0) = c_1 P_2(0) + c_2 Q_2(0) = -\frac{c_1}{2} + c_2 Q_2(0)$$

If $c_2 = 0$, then $c_1 = 0$ and we obtain the zero solution, which is not possible (since we are given $y'(0) = 1$, the solution is not identically 0). Hence $c_2 \neq 0$ and the solutions is not bounded.

17. (To do this problem we can use the recurrence relation for the coefficients, as we have done below in the solution of Exercise 19. Instead, we offer a different solution based on an interesting observation.) This is Legendre's equation with $n(n+1) = \frac{3}{4}$ or $n = \frac{1}{2}$. Its general solution is still given by (3) and (4), with $n = \frac{1}{2}$:

$$y = c_1 y_1 + c_2 y_2,$$

where

$$\begin{aligned} y_1(x) &= 1 - \frac{\frac{1}{2}(\frac{1}{2}+1)}{2!}x^2 + \frac{(\frac{1}{2}-2)\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+3)}{4!}x^4 + \cdots \\ &= 1 - \frac{3}{8}x^2 - \frac{21}{128}x^4 + \cdots \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= x - \frac{(\frac{1}{2}-1)(\frac{1}{2}+2)}{3!}x^3 + \frac{(\frac{1}{2}-3)(\frac{1}{2}-1)(\frac{1}{2}+2)(\frac{1}{2}+4)}{5!}x^5 + \cdots \\ &= x + \frac{5}{24}x^3 + \frac{15}{128}x^5 + \cdots \end{aligned}$$

Since $y_1(0) = 1$ and $y_2(0) = 0$, $y_1'(0) = 0$ and $y_2'(0) = 1$ (differentiate the series term by term, then evaluate at $x = 0$), it follows that the solution is $y = y_1(x) + y_2(x)$, where y_1 and y_2 are as describe above.

29. (a) Since

$$\left|(x + i\sqrt{1-x^2}\cos\theta)^n\right| = \left|x + i\sqrt{1-x^2}\cos\theta\right|^n,$$

it suffices to prove the inequality

$$\left|x + i\sqrt{1-x^2}\cos\theta\right| \le 1,$$

which in turn will follow from

$$\left|x + i\sqrt{1-x^2}\cos\theta\right|^2 \le 1.$$

For any complex number $\alpha + i\beta$, we have $|\alpha + i\beta|^2 = \alpha^2 + \beta^2$. So

$$\begin{aligned}\left|x + i\sqrt{1-x^2}\cos\theta\right|^2 &= x^2 + (\sqrt{1-x^2}\cos\theta)^2 \\ &= x^2 + (1-x^2)\cos^2\theta \\ &\le x^2 + (1-x^2) = 1,\end{aligned}$$

which proves the desired inequality.
(b) Using Laplace's formula, we have, for $-1 \le x \le 1$,

$$\begin{aligned}|P_n(x)| &= \frac{1}{\pi}\left|\int_0^\pi (x + i\sqrt{1-x^2}\cos\theta)^n\,d\theta\right| \\ &\le \frac{1}{\pi}\int_0^\pi \left|(x + i\sqrt{1-x^2}\cos\theta)^n\right| d\theta \\ &\le \frac{1}{\pi}\int_0^\pi d\theta \qquad \text{(by (a))} \\ &= 1\end{aligned}$$

Solutions to Exercises 5.6

1. Bonnet's relation says: For $n = 1, 2, \ldots,$

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x).$$

We have $P_0(x) = 1$ and $P_1(x) = x$. Take $n = 1$, then

$$\begin{aligned} 2P_2(x) + P_0(x) &= 3xP_1(x), \\ 2P_2(x) &= 3x \cdot x - 1, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1). \end{aligned}$$

Take $n = 2$ in Bonnet's relation, then

$$\begin{aligned} 3P_3(x) + 2P_1(x) &= 5xP_2(x), \\ 3P_3(x) &= 5x\big(\frac{1}{2}(3x^2 - 1)\big) - 2x, \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x. \end{aligned}$$

Take $n = 3$ in Bonnet's relation, then

$$\begin{aligned} 4P_4(x) + 3P_2(x) &= 7xP_3(x), \\ 4P_4(x) &= 7x\big(\frac{5}{2}x^3 - \frac{3}{2}x\big) - \frac{3}{2}\big(x^2 - 1\big), \\ P_4(x) &= \frac{1}{4}\left[\frac{35}{2}x^4 - 15x^2 + \frac{3}{2}\right]. \end{aligned}$$

5. By Bonnet's relation with $n = 3$,

$$\begin{aligned} 7xP_3(x) &= 4P_4(x) + 3P_2(x), \\ xP_3(x) &= \frac{4}{7}P_4(x) + \frac{3}{7}P_2(x). \end{aligned}$$

So

$$\begin{aligned} \int_{-1}^{1} x\, P_2(x)P_3(x)\, dx &= \int_{-1}^{1} \left(\frac{4}{7}P_4(x) + \frac{3}{7}P_2(x)\right) P_2(x)\, dx \\ &= \frac{4}{7}\int_{-1}^{1} P_4(x)\, P_2(x)\, dx + \frac{3}{7}\int_{-1}^{1} [P_2(x)]^2\, dx \\ &= 0 + \frac{3}{7}\,\frac{2}{5} = \frac{6}{35}, \end{aligned}$$

where we have used Theorem 1(i) and (ii) to evaluate the last two integrals.
9. (a) Write (4) in the form

$$(2n+1)P_n(t) = P'_{n+1}(t) - P'_{n-1}(t).$$

Integrate from x to 1,

$$\begin{aligned} (2n+1)\int_x^1 P_n(t)\, dt &= \int_x^1 P'_{n+1}(t)\, dt - \int_x^1 P'_{n-1}(t)\, dt \\ &= P_{n+1}(t)\Big|_x^1 - P_{n-1}(t)\Big|_x^1 \\ &= \big(P_{n-1}(x) - P_{n+1}(x)\big) + \big(P_{n+1}(1) - P_{n-1}(1)\big). \end{aligned}$$

By Example 1, we have $P_{n+1}(1) - P_{n-1}(1) = 0$. So for $n = 1, 2, \ldots$

$$\int_x^1 P_n(t)\,dt = \frac{1}{2n+1}\left[P_{n-1}(x) - P_{n+1}(x)\right].$$

(b) First let us note that because P_n is even when is even and odd when is odd, it follows that $P_n(-1) = (-1)^n P_n(1) = (-1)^n$. Taking $x = -1$ in (a), we get

$$\int_{-1}^1 P_n(t)\,dt = \frac{1}{2n+1}[P_{n-1}(-1) - P_{n+1}(-1)] = 0,$$

because $n-1$ and $n+1$ are either both even or both odd, so $P_{n-1}(-1) = P_{n+1}(-1)$.
(c) We have

$$0 = \int_{-1}^1 P_n(t)\,dt = \int_{-1}^x P_n(t)\,dt + \int_x^1 P_n(t)\,dt.$$

So

$$\begin{aligned}\int_{-1}^x P_n(t)\,dt &= -\int_x^1 P_n(t)\,dt\\ &= -\frac{1}{2n+1}[P_{n-1}(x) - P_{n+1}(x)]\\ &= \frac{1}{2n+1}[P_{n+1}(x) - P_{n-1}(x)]\end{aligned}$$

13. We will use $D^n f$ to denote the nth derivative of f. Using Exercise 12,

$$\int_{-1}^1 (1-x^2)\,P_{13}(x)\,dx = \frac{(-1)^{13}}{2^{13}(13)!}\int_{-1}^1 D^{13}[(1-x^2)]\,(x^2-1)^{13}\,dx = 0$$

because $D^{13}[(1-x^2)] = 0$.

17. Using Exercise 12,

$$\begin{aligned}\int_{-1}^1 \ln(1-x)\,P_2(x)\,dx &= \frac{(-1)^2}{2^2 2!}\int_{-1}^1 D^2[\ln(1-x)]\,(x^2-1)^2\,dx\\ &= \frac{1}{8}\int_{-1}^1 \frac{-1}{(1-x)^2}\,(x-1)^2(x+1)^2\,dx\\ &= \frac{-1}{8}\int_{-1}^1 (x+1)^2\,dx = \frac{-1}{24}(x+1)^3\Big|_{-1}^1 = \frac{-1}{3}.\end{aligned}$$

21. For $n > 0$, we have

$$D^n[\ln(1-x)] = \frac{-(n-1)!}{(1-x)^n}$$

. So

$$\begin{aligned}\int_{-1}^1 \ln(1-x)\,P_n(x)\,dx &= \frac{(-1)^n}{2^n n!}\int_{-1}^1 D^n[\ln(1-x)]\,(x^2-1)^n\,dx\\ &= \frac{(-1)^{n+1}(n-1)!}{2^n n!}\int_{-1}^1 \frac{(x+1)^n(x-1)^n}{(1-x)^n}\,dx\\ &= \frac{-1}{2^n n}\int_{-1}^1 (x+1)^n\,dx = \frac{-1}{2^n n(n+1)}(x+1)^{n+1}\Big|_{-1}^1\\ &= \frac{-2}{n(n+1)}\end{aligned}$$

For $n = 0$, we use integration by parts. The integral is a convergent improper integral (the integrand has a problem at 1)

$$\begin{aligned} \int_{-1}^{1} \ln(1-x)\, P_0(x)\, dx &= \int_{-1}^{1} \ln(1-x)\, dx \\ &= -(1-x)\ln(1-x) - x\Big|_{-1}^{1} = -2 + 2\ln 2. \end{aligned}$$

To evaluate the integral at $x = 1$, we used $\lim_{x\to 1}(1-x)\ln(1-x) = 0$.

29. Call the function in Exercise 28 $g(x)$. Then

$$g(x) = \frac{1}{2}\left(|x| + x\right) = \frac{1}{2}\left(|x| + P_1(x)\right).$$

Let B_k denote the Legendre coefficient of g and A_k denote the Legendre coefficient of $f(x) = |x|$, for $-1 < x < 1$. Then, because $P_1(x)$ is its own Legendre series, we have

$$B_k = \begin{cases} \frac{1}{2} A_k & \text{if } k \neq 1 \\ \frac{1}{2}\left(A_k + 1\right) & \text{if } k = 1 \end{cases}$$

Using Exercise 27 to compute A_k, we find

$$B_0 = \frac{1}{2}A_0 = \frac{1}{4},\ B_1 = \frac{1}{2} + \frac{1}{2}A_1 = \frac{1}{2} + 0 = \frac{1}{2}, B_{2n+1} = 0,\ n = 1, 2, \ldots,$$

and

$$B_{2n} = \frac{1}{2}A_{2n} = \frac{(-1)^{n+1}(2n-2)!}{2^{2n+1}((n-1)!)^2 n}\left(\frac{4n+1}{n+1}\right).$$

Solutions to Exercises 6.1

1. Let $f_j(x) = \cos(j\pi x)$, $j = 0, 1, 2, 3$, and $g_j(x) = \sin(j\pi x)$, $j = 1, 2, 3$. We have to show that $\int_0^2 f_j(x)g_k(x)\,dx = 0$ for all possible choices of j and k. If $j = 0$, then

$$\int_0^2 f_j(x)g_k(x)\,dx = \int_0^2 \sin k\pi x\,dx = \frac{-1}{k\pi}\cos(k\pi x)\Big|_0^2 = 0.$$

If $j \neq 0$, and $j = k$, then using the identity $\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$,

$$\begin{aligned}\int_0^2 f_j(x)g_j(x)\,dx &= \int_0^2 \cos(j\pi x)\sin(j\pi x)\,dx\\ &= \frac{1}{2}\int_0^2 \sin(2j\pi x)\,dx\\ &= \frac{-1}{4j\pi}\cos(2j\pi x)\Big|_0^2 = 0.\end{aligned}$$

If $j \neq 0$, and $j \neq k$, then using the identity

$$\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big),$$

we obtain

$$\begin{aligned}\int_0^2 f_j(x)g_k(x)\,dx &= \int_0^2 \sin(k\pi x)\cos(j\pi x)\,dx\\ &= \frac{1}{2}\int_0^2 \big(\sin(k+j)\pi x + \sin(k-j)\pi x\big)\,dx\\ &= \frac{-1}{2\pi}\Big(\frac{1}{k+j}\cos(k+j)\pi x + \frac{1}{k-j}\cos(k-j)\pi x\Big)\Big|_0^2 = 0.\end{aligned}$$

5. Let $f(x) = 1$, $g(x) = 2x$, and $h(x) = -1 + 4x$. We have to show that

$$\int_{-1}^1 f(x)g(x)w(x)\,dx = 0, \quad \int_{-1}^1 f(x)h(x)w(x)\,dx = 0, \quad \int_{-1}^1 g(x)h(x)w(x)\,dx = 0.$$

Let's compute:

$$\int_{-1}^1 f(x)g(x)w(x)\,dx = \int_{-1}^1 2x\sqrt{1-x^2}\,dx = 0,$$

because we are integrating an odd function over a symmetric interval. For the second integral, we have

$$\begin{aligned}\int_{-1}^1 f(x)h(x)w(x)\,dx &= \int_{-1}^1 (-1+4x^2)\sqrt{1-x^2}\,dx\\ &= \int_0^\pi (-1+4\cos^2\theta)\sin^2\theta\,d\theta\\ &\qquad (x=\cos\theta,\ dx=-\sin\theta\,d\theta, \sin\theta\geq 0 \text{ for } 0\leq\theta\leq\pi.)\\ &= -\int_0^\pi \sin^2\theta\,d\theta + 4\int_0^\pi (\cos\theta\sin\theta)^2\,d\theta\\ &= -\overbrace{\int_0^\pi \frac{1-\cos 2\theta}{2}\,d\theta}^{=\frac{\pi}{2}} + 4\int_0^\pi \Big(\frac{1}{2}\sin(2\theta)\Big)^2\,d\theta\\ &= -\frac{\pi}{2} + \overbrace{\int_0^\pi \frac{1-\cos(4\theta)}{2}\,d\theta}^{=\frac{\pi}{2}} = 0\end{aligned}$$

For the third integral, we have

$$\int_{-1}^{1} g(x)h(x)w(x)\,dx = \int_{-1}^{1} 2x(-1+4x^2)\sqrt{1-x^2}\,dx = 0,$$

because we are integrating an odd function over a symmetric interval.

9. In order for the functions 1 and $a+bx+x^2$ to be orthogonal, we must have

$$\int_{-1}^{1} 1\cdot(a+bx+x^2)\,dx = 0$$

Evaluating the integral, we find

$$\begin{aligned} ax+\frac{b}{2}x^2+\frac{1}{3}x^3\Big|_{-1}^{1} &= 2a+\frac{2}{3}=0 \\ a &= -\frac{1}{3}. \end{aligned}$$

In order for the functions x and $\frac{1}{3}+bx+x^2$ to be orthogonal, we must have

$$\int_{-1}^{1} 1\cdot(\frac{1}{3}+bx+x^2)x\,dx = 0$$

Evaluating the integral, we find

$$\begin{aligned} \frac{1}{6}x^2+\frac{b}{3}x^3+\frac{1}{4}x^4\Big|_{-1}^{1} &= \frac{b}{3}=0 \\ b &= 0. \end{aligned}$$

13. Using Theorem 1, Section 5.6, we find the norm of $P_n(x)$ to be

$$\|P_n\| = \left(\int_{-1}^{1} P_n(x)^2\,dx\right)^{\frac{1}{2}} = \left(\frac{2}{2n+1}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{2n+1}}.$$

Thus the orthonormal set of functions obtained from the Legendre polynomials is

$$\frac{\sqrt{2}}{\sqrt{2n+1}}P_n(x),\ n=0,\,2,\,\ldots.$$

17. For Legendre series expansions, the inner product is defined in terms of integration against the Legendre polynomials. That is,

$$(f,\,P_j) = \int_{-1}^{1} f(x)P_j(x)\,dx = \frac{2}{2j+1}A_j$$

where A_j is the Legendre coefficient of f (see (7), Section 5.6). According to the generalized Parseval's identity, we have

$$\begin{aligned} \int_{-1}^{1} f^2(x)\,dx &= \sum_{j=0}^{\infty}\frac{|(f,\,P_j)|^2}{\|P_j\|^2} \\ &= \sum_{j=0}^{\infty}\left(\frac{2}{2j+1}A_j\right)^2\frac{2}{2j+1} \\ &= \sum_{j=0}^{\infty}\frac{2}{2j+1}A_j^2. \end{aligned}$$

(The norm $\|P_j\|$ is computed in Exercise 13.)

Solutions to Exercises 6.2

1. Sturm-Liouville form: $\big(xy'\big)' + \lambda y = 0$, $p(x) = x$, $q(x) = 0$, $r(x) = 1$. Singular problem because $p(x) = 0$ at $x = 0$.

5. Divide the equation through by x^2 and get $\frac{y''}{x} - \frac{y'}{x^2} + \lambda \frac{y}{x} = 0$. Sturm-Liouville form: $\big(\frac{1}{x}y'\big)' + \lambda \frac{y}{x} = 0$, $p(x) = \frac{1}{x}$, $q(x) = 0$, $r(x) = \frac{1}{x}$. Singular problem because $p(x)$ and $r(x)$ are not continuous at $x = 0$.

9. Sturm-Liouville form: $\big((1-x^2)\,y'\big)' + \lambda y = 0$, $p(x) = 1-x^2$, $q(x) = 0$, $r(x) = 1$. Singular problem because $p(\pm 1) = 0$.

13. Before we proceed with the solution, we can use our knowledge of Fourier series to guess a family of orthogonal functions that satisfy the Sturm-Liouville problem: $y_k(x) = \sin\frac{2k+1}{2}x$, $k = 0, 1, 2, \ldots$. It is straightforward to check the validity of our guess. Let us instead proceed to derive these solutions. We organize our solution after Example 2. The differential equation fits the form of (1) with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. In the boundary conditions, $a = 0$ and $b = \pi$, with $c_1 = d_2 = 1$ and $c_2 = d_1 = 0$, so this is a regular Sturm-Liouville problem.

We consider three cases.

CASE 1: $\lambda < 0$. Let us write $\lambda = -\alpha^2$, where $\alpha > 0$. Then the equation becomes $y'' - \alpha^2 y = 0$, and its general solution is $y = c_1 \sinh \alpha x + c_2 \cosh \alpha x$. We need $y(0) = 0$, so substituting into the general solution gives $c_2 = 0$. Now using the condition $y'(\pi) = 0$, we get $0 = c_1\alpha \cosh \alpha\pi$, and since $\cosh x \neq 0$ for all x, we infer that $c_1 = 0$. Thus there are no nonzero solutions in this case.

CASE 2: $\lambda = 0$. Here the general solution of the differential equation is $y = c_1 x + c_2$, and as in Case 1 the boundary conditions force c_1 and c_2 to be 0. Thus again there is no nonzero solution.

CASE 3: $\lambda > 0$. In this case we can write $\lambda = \alpha^2$ with $\alpha > 0$, and so the equation becomes $y'' + \alpha^2 y = 0$. The general solution is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. From $y(0) = 0$ we get $0 = c_1 \cos 0 + c_2 \sin 0$ or $0 = c_1$. Thus $y = c_2 \sin \alpha x$. Now we substitute the other boundary condition to get $0 = c_2 \alpha \cos \alpha\pi$. Since we are seeking nonzero solutions, we take $c_2 \neq 0$. Thus we must have $\cos \alpha\pi = 0$, and hence $\alpha = \frac{2k+1}{2}$. Since $\lambda = \alpha^2$, the problem has eigenvalues

$$\lambda_k = \left(\frac{2k+1}{2}\right)^2,$$

and corresponding eigenfunctions

$$y_k = \sin\frac{2k+1}{2}x,\ k = 0, 1, 2, \ldots.$$

17. **Case I** If $\lambda = 0$, the general solution of the differential equation is $X = ax + b$. As in Exercise 13, check that the only way to satisfy the boundary conditions is to take $a = b = 0$. Thus $\lambda = 0$ is not an eigenvalue since no nontrivial solutions exist.

Case II If $\lambda = -\alpha^2 < 0$, then the general solution of the differential equation is $X = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. We have $X' = c_1 \alpha \sinh x + c_2 \alpha \cosh \alpha x$. In order to have nonzero solutions, we suppose throughout the solution that c_1 or c_2 is nonzero. The first boundary condition implies

$$c_1 + \alpha c_2 = 0 \quad c_1 = -\alpha c_2.$$

Hence both c_1 and c_2 are nonzero. The second boundary condition implies that

$$c_1(\cosh \alpha + \alpha \sinh \alpha) + c_2(\sinh \alpha + \alpha \cosh \alpha) = 0.$$

Using $c_1 = -\alpha c_2$, we obtain

$$\begin{aligned} -\alpha c_2(\cosh \alpha + \alpha \sinh \alpha) + c_2(\sinh \alpha + \alpha \cosh \alpha) &= 0 \quad (\text{divide by} c_2 \neq 0) \\ \sinh \alpha (1 - \alpha^2) &= 0 \\ \sinh \alpha = 0 \text{ or } 1 - \alpha^2 &= 0 \end{aligned}$$

Since $\alpha \neq 0$, it follows that $\sinh \alpha \neq 0$ and this implies that $1-\alpha^2 = 0$ or $\alpha = \pm 1$. We take $\alpha = 1$, because the value -1 does not yield any new eigenfunctions. For $\alpha = 1$, the corresponding solution is

$$X = c_1 \cosh x + c_2 \sinh x = -c_2 \cosh x + c_2 \sinh x,$$

because $c_1 = -\alpha c_2 = -c_2$. So in this case we have one negative eigenvalue $\lambda = -\alpha^2 = -1$ with corresponding eigenfunction $X = \cosh x - \sinh x$.

Case III If $\lambda = \alpha^2 > 0$, then the general solution of the differential equation is

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x\,.$$

We have $X' = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$. In order to have nonzero solutions, one of the coefficients c_1 or c_2 must be $\neq 0$. Using the boundary conditions, we obtain

$$\begin{aligned} c_1 + \alpha c_2 &= 0 \\ c_1(\cos\alpha - \alpha \sin\alpha) + c_2(\sin\alpha + \alpha\cos\alpha) &= 0 \end{aligned}$$

The first equation implies that $c_1 = -\alpha c_2$ and so both c_1 and c_2 are $neq0$. From the second equation, we obtain

$$\begin{aligned} -\alpha c_2(\cos\alpha - \alpha \sin\alpha) + c_2(\sin\alpha + \alpha\cos\alpha) &= 0 \\ -\alpha(\cos\alpha - \alpha \sin\alpha) + (\sin\alpha + \alpha\cos\alpha) &= 0 \\ \sin\alpha(\alpha^2+1) &= 0 \end{aligned}$$

Since $\alpha^2 + 1 \neq 0$, then $\sin\alpha = 0$, and so $\alpha = n\pi$, where $n = 1, 2, \ldots$. Thus the eigenvalues are

$$\lambda_n = (n\pi)^2$$

with corresponding eigenfunctions

$$y_n = -n\pi \cos n\pi x + \sin n\pi x, \qquad n = 1, 2, \ldots.$$

21. If $\lambda = \alpha^2$, then the solutions are of the form $c_1 \cosh \alpha x + c_2 \sinh \alpha x$. Using the boundary conditions, we find

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 = 0 \\ y(1) = 0 \quad &\Rightarrow \quad c_2 \sinh\alpha = 0. \end{aligned}$$

But $\alpha \neq 0$, hence $\sinh\alpha \neq 0$, and so $c_2 = 0$. There are no nonzero solutions if $\lambda > 0$ and so the problem as no positive eigenvalues. This does not contradict Theorem 1 because if we consider the equation $y'' - \lambda y = 0$ as being in the form (1), then $r(x) = -1 < 0$ and so the problem is a singular Sturm-Liouville problem to which Theorem 1 does not apply.

25. The eigenfunctions in Example 2 are $y_j(x) = \sin jx$, $j = 1, 2, \ldots$. Since f is one of these eigenfunctions, it is equal to its own eigenfunction expansion.

29. You can verify the orthogonality directly by checking that

$$\int_0^{2\pi} \sin\frac{nx}{2} \sin\frac{mx}{2}\, dx = 0 \quad \text{if } m \neq n \ (m,\, n \text{ integers}).$$

You can also quote Theorem 2(a) because the problem is a regular Sturm-Liouville problem.

33. (a) From Exercise 36(b), Section 4.8, with $y = J_0(\lambda r)$ and $p = 0$, we have

$$2\lambda^2 \int_0^a [y(r)]^2 r\, dr = [ay'(a)]^2 + \lambda^2 a^2 [y(a)]^2.$$

But y satisfies the boundary condition $y'(a) = -\kappa y(a)$, so

$$\begin{aligned} 2\lambda^2 \int_0^a [y(r)]^2 r\, dr &= a^2\kappa^2[y(a)]^2 + \lambda^2 a^2 [y(a)]^2; \\ \int_0^a [y(r)]^2 r\, dr &= \frac{a^2}{2}\left[\kappa^2 \frac{[J_0(\lambda_k a)]^2}{\lambda_k^2} + [J_0(\lambda_k a)]^2\right] \\ &= \frac{a^2}{2}\left[[J_0(\lambda_k a)]^2 + [J_1(\lambda_k a)]^2\right], \end{aligned}$$

because, by (7), $[J_0(\lambda_k a)]^2 = \left[\frac{\lambda_k}{\kappa} J_1(\lambda_k a)\right]^2$.
(b) Reproduce the sketch of proof of Theorem 1. The given formula for the coefficients is precisely formula (5) in this case.

Solutions to Exercises 6.3

1. (a) The initial shape of the chain is given by the function

$$f(x) = -.01(x - .5), \quad 0 < x < .5,$$

and the initial velocity of the chain is zero. So the solution is given by (10), with $L = .5$ and $B_j = 0$ for all j. Thus

$$u(x, t) = \sum_{j=1}^{\infty} A_j J_0\left(\alpha_j \sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\,t\right).$$

To compute A_j, we use (11), and get

$$\begin{aligned} A_j &= \frac{2}{J_1^2(\alpha_j)} \int_0^{.5} (-.01)(x - .5) J_0\left(\alpha_j\sqrt{2x}\right)\,dx \\ &= \frac{-.02}{J_1^2(\alpha_j)} \int_0^{.5} (x - .5) J_0\left(\alpha_j\sqrt{2x}\right)\,dx \end{aligned}$$

Make the change of variables $s = \alpha_j\sqrt{2x}$, or $s^2 = 2\alpha_j^2 x$, so $2s\,ds = 2\alpha_j^2\,dx$ or $dx = \frac{s}{\alpha_j^2}\,ds$. Thus

$$\begin{aligned} A_j &= \frac{-.02}{J_1^2(\alpha_j)} \int_0^{\alpha_j} (\frac{.5}{\alpha_j^2} s^2 - .5) J_0(s) \frac{s}{\alpha_j^2}\,ds \\ &= \frac{-.01}{\alpha_j^4 J_1^2(\alpha_j)} \int_0^{\alpha_j} (s^2 - \alpha_j^2) J_0(s) s\,ds \\ &= \frac{.01}{\alpha_j^4 J_1^2(\alpha_j)} \left[2\frac{\alpha_j^4}{\alpha_j^2} J_2(\alpha_j)\right] \\ &= \frac{.02}{\alpha_j^2 J_1^2(\alpha_j)} J_2(\alpha_j), \end{aligned}$$

where we have used the integral formula (15), Section 4.3, with $a = \alpha = \alpha_j$. We can give our answer in terms of J_1 by using formula (6), Section 4.8, with $p = 1$, and $x = \alpha_j$. Since α_j is a zero of J_0, we obtain

$$\frac{2}{\alpha_j} J_1(\alpha_j) = J_0(\alpha_j) + J_2(\alpha_j) = J_2(\alpha_j).$$

So

$$A_j = \frac{.02}{\alpha_j^2 J_1^2(\alpha_j)} \frac{2}{\alpha_j} J_1(\alpha_j) = \frac{.04}{\alpha_j^3 J_1(\alpha_j)}.$$

Thus the solution is

$$u(x, t) = \sum_{j=1}^{\infty} \frac{.04}{\alpha_j^3 J_1(\alpha_j)} J_0\left(\alpha_j\sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\,t\right),$$

where $g \approx 9.8\ m/sec^2$.

Going back to the questions, to answer (a), we have the normal modes

$$u_j(x, t) = \frac{.04}{\alpha_j^3 J_1(\alpha_j)} J_0\left(\alpha_j\sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\,t\right).$$

The frequency of the jth normal mode is

$$\nu_j = \sqrt{2g}\,\frac{\alpha_j}{4\pi}.$$

A six-term approximation of the solution is

$$u(x,\,t) \approx \sum_{j=1}^{6} \frac{.04}{\alpha_j^3 J_1(\alpha_j)} J_0\left(\alpha_j \sqrt{2x}\right) \cos\left(\sqrt{2g}\,\frac{\alpha_j}{2}\,t\right).$$

At this point, we use Mathematica (or your favorite computer system) to approximate the numerical values of the coefficients. Here is a table of relevant numerical data.

j	1	2	3	4	5	6
α_j	2.40483	5.52008	8.65373	11.7915	14.9309	18.0711
ν_j	.847231	1.94475	3.04875	4.15421	5.26023	6.36652
A_j	.005540	−.000699	.000227	−.000105	.000058	−.000036

Table 1 Numerical data for Exercise 1.

Exercises 6.4

1. This is a special case of Example 1 with $L = 2$ and $\lambda = \alpha^4$. The values of α are the positive roots of the equation

$$\cos 2\alpha = \frac{1}{\cosh 2\alpha}.$$

There are infinitely many roots, α_n ($n = 1, 2, \ldots$), that can be approximated with the help of a computer (see Figure 1). To each α_n corresponds one eigenfunction

$$X_n(x) = \cosh \alpha_n x - \cos \alpha_n x - \frac{\cosh 2\alpha_n - \cos 2\alpha_n}{\sinh 2\alpha_n - \sin 2\alpha_n} (\sinh \alpha_n x - \sin \alpha_n x) .$$

5. There are infinitely many eigenvalues $\lambda = \alpha^4$, where α is a positive root of the equation

$$\cos \alpha = \frac{1}{\cosh \alpha}.$$

As in Example 1, the roots of this equation, α_n ($n = 1, 2, \ldots$), can be approximated with the help of a computer (see Figure 1). To each α_n corresponds one eigenfunction

$$X_n(x) = \cosh \alpha_n x - \cos \alpha_n x - \frac{\cosh \alpha_n - \cos \alpha_n}{\sinh \alpha_n - \sin \alpha_n} (\sinh \alpha_n x - \sin \alpha_n x) .$$

The eigenfunction expansion of $f(x) = x(1 - x)$, $0 < x < 1$, is

$$f(x) = \sum_{n=1}^{\infty} A_n X_n(x),$$

where

$$A_n == \frac{\int_0^1 x(1 - x) X_n(x)\, dx}{\int_0^1 X_n^2(x)\, dx} .$$

After computing several of these coefficients, it was observed that:

$$\int_0^1 X_n^2(x)\, dx = 1 \quad \text{for all } n = 1, 2, \ldots,$$

$$A_{2n} = 0 \quad \text{for all } n = 1, 2, \ldots.$$

The first three nonzero coefficients are

$$A_1 = .1788, \quad A_3 = .0331, \quad A_5 = .0134.$$

So

$$f(x) \approx .1788\, X_1(x) + .0331\, X_3(x) + .0134\, X_5(x),$$

where X_n described explicitely in Example 1. We have

$$\begin{aligned}
X_1(x) &= \cosh(4.7300\, x) - \cos(4.7300\, x) + .9825\, (\sin(4.7300\, x) - \sinh(4.7300\, x)), \\
X_2(x) &= \cosh(1.0008\, x) - \cos(1.0008\, x) + 1.0008\, (\sin(1.0008\, x) - \sinh(1.0008\, x)), \\
X_3(x) &= \cosh(10.9956\, x) - \cos(10.9956\, x) + \sin(10.9956\, x) - \sinh(10.9956\, x), \\
X_4(x) &= \cosh(14.1372\, x) - \cos(14.1372\, x) + \sin(14.1372\, x) - \sinh(14.1372\, x), \\
X_5(x) &= \cosh(17.2788\, x) - \cos(17.2788\, x) + \sin(17.2788\, x) - \sinh(17.2788\, x).
\end{aligned}$$

9. Assume that μ and X are an eigenvalue and a corresponding eigenfunction of the Sturm-Liouville problem

$$X'' + \mu X = 0, \quad X(0) = 0,\ X(L) = 0 .$$

Differentiate twice to see that X also satisfies the fourth order Sturm-Liouville problem

$$X^{(4)} - \lambda X = 0,$$
$$X(0) = 0,\ X''(0) = 0,\ X(L) = 0,\ X''(L) = 0\,.$$

If α and X are an eigenvalue and a corresponding eigenfunction of

$$X'' + \mu X = 0, \quad X(0) = 0,\ X(L) = 0,$$

then differentiating twice the equation, we find

$$X^{(4)} + \mu X'' = 0, \quad X(0) = 0,\ X(L) = 0.$$

But $X'' = -\mu X$, so $X^{(4)} - \mu^2 X = 0$ and hence X satisfies the equation $X^{(4)} - \lambda X = 0$ with $\lambda = \mu^2$. Also, from $X(0) = 0,\ X(L) = 0$ and the fact that $X'' = -\mu X$, it follows that $X''(0) = 0$ and $X''(L) = 0$.

Exercises 6.6

1. $u_{xxyy} = 0$, $u_{xxxx} = 4!$, $u_{yyyy} = -4!$, $\nabla^4 u = 0$.

5. Express v in Cartesian coordinates as follows:

$$\begin{aligned} v &= r^2 \cos(2\theta)(1 - r^2) \\ &= r^2[\cos^2\theta - \sin^2\theta](1 - r^2) \\ &= (x^2 - y^2)\big(1 - (x^2 + y^2)\big). \end{aligned}$$

Let $u = x^2 - y^2$. Then u is harmonic and so v is biharmonic by Example 1, with $A = 1$, $D = 1$, $B = C = 0$.

7. Write $v = r^2 \cdot r^n \cos n\theta$ and let $u = r^n \cos n\theta$. Then u is harmonic (use the Laplacian in polar coordinates to check this last assertion) and so v is biharmonic, by Example 1 with $A = 1$ and $B = C = D = 0$.

9. Write $v = ar^2 \ln r + br^2 + c \ln r + d = \phi + \psi$, where $\phi = [ar^2 + c] \ln r$ and $\psi = br^2 + d$. From Example 1, it follows that ψ is biharmonic. Also, $\ln r$ is harmonic (check the Laplacian in polar coordinates) and so, by Example 1, ϕ is biharmonic. Consequently, v is biharmonic, being the sum of two biharmonic functions.

13. We follow the method of Theorem 1, as illustrated by Example 2. First, solve the Dirichlet problem $\nabla^2 w = 0$, $w(1, \theta) = \cos 2\theta$, for $0 \le r < 1$, $0 \le \theta \le 2\pi$. The solution in this case is $w(r, \theta) = r^2 \cos 2\theta$. (This is a simple application of the method of Section 4.4, since the boundary function is already given by its Fourier series.) We now consider a second Dirichlet problem on the unit disk with boundary values $v(1, \theta) = \frac{1}{2}\big(w_r(1, \theta) - g(\theta)\big)$. Since $g(\theta) = 0$ and $w_r(r, \theta) = 2r \cos 2\theta$, it follows that $v(1, \theta) = \cos 2\theta$. The solution of th Dirichlet problem in v is $v(r, \theta) = r^2 \cos 2\theta$. Thus the solution of biharmonic problem is

$$u(r, \theta) = (1 - r^2)r^2 \cos 2\theta + r^2 \cos 2\theta = 2r^2 \cos 2\theta - r^4 \cos 2\theta.$$

This can be verified directly by plugging into the biharmonic equation and the boundary conditions.

17. $u(1, 0) = 0$ implies that $w = 0$ and so $v(1, \theta) = -\frac{g(\theta)}{2}$. So

$$v(r, \theta) = -\frac{1}{2}\big[a_0 + \sum_{n=1}^{\infty} r^n(\cos n\theta + b_n \sin n\theta)\big],$$

where a_n and b_n are the Fourier coefficients of g. Finally,

$$u(r, \theta) = (1 - r^2)v(r, \theta) = -\frac{1}{2}(1 - r^2)\Big[a_0 + \sum_{n=1}^{\infty} r^n(\cos n\theta + b_n \sin n\theta)\Big].$$

Exercises 6.7

12. Correction to the suggested proof: $y_2 = I_0$ and not J_1.

17. Let $u_1(r, t)$ denote the solution of the problem in Example 3 and let $u_2(r, t)$ denote the solution in Example 3. Then, by linearity or superposition, $u(r, t) = u_1(r, t) + u_2(r, t)$ is the desired solution.

Solutions to Exercises 7.1

1. We have

$$f(x) = \begin{cases} 1 & \text{if } -a < x < a, \quad (a > 0) \\ 0 & \text{otherwise,} \end{cases}$$

This problem is very similar to Example 1. From (3), if $\omega \neq 0$, then

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt = \frac{1}{\pi}\int_{-a}^{a}\cos\omega t\,dt = \left[\frac{\sin\omega t}{\pi\omega}\right]_{-a}^{a} = \frac{2\sin a\omega}{\pi\omega}.$$

If $\omega = 0$, then

$$A(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt = \frac{1}{\pi}\int_{-a}^{a} dt = \frac{2a}{\pi}.$$

Since $f(x)$ is even, $B(\omega) = 0$. For $|x| \neq a$ the function is continuous and Theorem 1 gives

$$f(x) = \frac{2a}{\pi}\int_0^{\infty}\frac{\sin a\omega\cos\omega x}{\omega}\,d\omega.$$

For $x = \pm a$, points of discontinuity of f, Theorem 1 yields the value $1/2$ for the last integral. Thus we have the Fourier integral representation of f

$$\frac{2a}{\pi}\int_0^{\infty}\frac{\sin a\omega\cos\omega x}{\omega}\,d\omega = \begin{cases} 1 & \text{if } |x| < a, \\ 1/2 & \text{if } |x| = a, \\ 0 & \text{if } |x| > a. \end{cases}$$

■

5. since $f(x) = e^{-|x|}$ is even, $B(w) = 0$ for all w, and

$$\begin{aligned} A(w) &= \frac{2}{\pi}\int_0^{\infty} e^{-t}\cos wt\,dt \\ &= \frac{2}{\pi}\frac{e^{-t}}{1+w^2}\left[-\cos wt + w\sin wt\right]\Big|_0^{\infty} \\ &= \frac{2}{\pi}\frac{1}{1+w^2}, \end{aligned}$$

where we have used the result of Exercise 17, Sec. 2.6, to evaluate the integral. Applying the Fourier integral representation, we obtain:

$$e^{-|x|} = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+w^2}\cos wx\,dw.$$

9. The function

$$f(x) = \begin{cases} x & \text{if } -1 < x < 1, \\ 2 - x & \text{if } 1 < x < 2, \\ -2 - x & \text{if } -2 < x < -1, \\ 0 & \text{otherwise,} \end{cases}$$

is odd, as can be seen from its graph. Hence $A(w) = 0$ and

$$\begin{aligned} B(w) &= \frac{2}{\pi}\int_0^{\infty} f(t)\sin wt\,dt \\ &= \frac{2}{\pi}\int_0^1 t\sin wt\,dt + \frac{2}{\pi}\int_1^2 (2-t)\sin wt\,dt \\ &= \frac{2}{\pi}\left[\frac{-t}{w}\cos wt\Big|_0^1 + \int_0^1\frac{\cos wt}{w}\,dt\right] \\ &\quad + \frac{2}{\pi}\left[-\frac{2-t}{w}\cos wt\Big|_1^2 + \int_1^2\frac{\cos wt}{w}\,dt\right] \\ &= \frac{2}{\pi w^2}\left[2\sin w - \sin 2w\right]. \end{aligned}$$

Since f is continuous, we obtain the Fourier integral representation of

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{w^2}\left[2\sin w - \sin 2w\right]\sin wx, dw.$$

13. (a) Take $x = 1$ in the Fourier integral representation of Example 1:

$$\frac{2}{\pi}\int_0^\infty \frac{\sin w \cos w}{w}\, dw = \frac{1}{2} \quad \Rightarrow \quad \int_0^\infty \frac{\sin w \cos w}{w}\, dw = \frac{\pi}{4}.$$

(b) Integrate by parts: $u = \sin^2 w$, $du = 2\sin w \cos w\, dw$, $dv = \frac{1}{w^2} dw$, $v = -\frac{1}{w}$:

$$\int_0^\infty \frac{\sin^2 \omega}{\omega^2}\, d\omega = \overbrace{\frac{\sin^2 w}{w}\bigg|_0^\infty}^{=0} + 2\int_0^\infty \frac{\sin w \cos w}{w^2} dw = \frac{\pi}{2},$$

by (a).

17.

$$\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2}\, d\omega = \begin{cases} 0 & \text{if } x < 0, \\ \pi/2 & \text{if } x = 0, \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

Solution. Define

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \pi/2 & \text{if } x = 0, \\ \pi e^{-x} & \text{if } x > 0. \end{cases}$$

Let us find the Fourier integral representation of f:

$$A(w) = \frac{1}{\pi}\int_0^\infty \pi e^{-x} \cos wx\, dx = \frac{1}{1+w^2}$$

(see Exercise 5);

$$B(w) = \frac{1}{\pi}\int_0^\infty \pi e^{-x} \sin wx\, dx = \frac{w}{1+w^2},$$

(see Exercise 17, Sec. 2.6). So

$$f(x) = \int_0^\infty \frac{\cos wx + w \sin wx}{1+w^2}\, dw,$$

which yields the desired formula.

25. We have

$$\operatorname{Si}(a) = \int_0^a \frac{\sin t}{t}\, dt, \qquad \operatorname{Si}(b) = \int_0^b \frac{\sin t}{t}\, dt.$$

So

$$\operatorname{Si}(b) - \operatorname{Si}(a) = \int_0^b \frac{\sin t}{t}\, dt - \int_0^a \frac{\sin t}{t}\, dt = \int_a^b \frac{\sin t}{t}\, dt.$$

Solutions to Exercises 7.2

1. In computing $\widehat{f}$, the integral depends on the values of f on the interval $(-1,\,1)$. Since on this interval f is odd, it follows that $f(x)\cos wx$ is odd and $f(x)\sin wx$ is even on the interval $(-1,1)$. Thus

$$\begin{aligned}\widehat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\overbrace{\int_{-1}^{1} f(x)\cos wx\,dx}^{=0} - \frac{i}{\sqrt{2\pi}}\int_{-1}^{1} f(x)\sin wx\,dx\\ &= -\frac{2i}{\sqrt{2\pi}}\int_0^1 \sin wx\,dx\\ &= \frac{2i}{\sqrt{2\pi}}\frac{\cos wx}{w}\Big|_0^1\\ &= i\sqrt{\frac{2}{\pi}}\frac{\cos w-1}{w}.\end{aligned}$$

5. Use integration by parts to evaluate the integrals:

$$\begin{aligned}\widehat{f}(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos wx - i\sin wx)\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\cos wx\,dx - \frac{i}{\sqrt{2\pi}}\overbrace{\int_{-1}^{1}(1-|x|)\sin wx\,dx}^{=0}\\ &= \frac{2}{\sqrt{2\pi}}\int_0^1 \overbrace{(1-x)}^{u}\overbrace{\cos wx\,dx}^{dv}\\ &= \frac{2}{\sqrt{2\pi}}\Big((1-x)\frac{\sin wx}{w}\Big)\Big|_0^1 + \frac{2}{\sqrt{2\pi}w}\int_0^1 \sin wx\,dx\\ &= -\sqrt{\frac{2}{\pi}}\frac{\cos wx}{w^2}\Big|_0^1\\ &= \sqrt{\frac{2}{\pi}}\frac{1-\cos w}{w^2}.\end{aligned}$$

9. In Exercise 1,

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \times (\text{area between graph of } f(x) \text{ and the } x\text{axis}) = 0.$$

In Exercise 7,

$$\widehat{f}(0) = \frac{1}{\sqrt{2\pi}} \times (\text{area between graph of } f(x) \text{ and the } x\text{axis}) = \frac{100}{\sqrt{2\pi}}.$$

13. We argue as in Exercise 11. Consider $g(x) = \frac{\sin ax}{x}$ where we assume $a > 0$. For the case $a < 0$, use $\sin(-ax) = -\sin ax$ and linearity of the Fourier transform. Let $f(x) = 1$ if $|x| < a$ and 0 otherwise. Recall from Example 1 that

$$\mathcal{F}(f(x))(w) = \sqrt{\frac{2}{\pi}}\frac{\sin aw}{w} = \sqrt{\frac{2}{\pi}}g(w).$$

Multiplying both sides by $\sqrt{\frac{\pi}{2}}$ and using the linearity of the Fourier transform, it follows that

$$\mathcal{F}(\sqrt{\frac{\pi}{2}}f(x))(w) = g(w).$$

So

$$\mathcal{F}g = \mathcal{F}\mathcal{F}\left(\sqrt{\frac{\pi}{2}}f(x)\right) = \sqrt{\frac{\pi}{2}}f(x),$$

by the reciprocity relation. Using the symbol w as a variable, we get

$$\mathcal{F}\left(\frac{\sin ax}{x}\right) = \sqrt{\frac{\pi}{2}}f(w) = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |w| < a, \\ 0 & \text{otherwise.} \end{cases}$$

17. (a) Consider first the case $a > 0$. Using the definition of the Fourier transform and a change of variables

$$\begin{aligned} \mathcal{F}(f(ax))(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}\,dx \\ &= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\frac{\omega}{a}x}\,dx \qquad (ax = X,\ dx = \frac{1}{a}dX) \\ &= \frac{1}{a}\mathcal{F}(f)(\frac{w}{a}). \end{aligned}$$

If $a < 0$, then

$$\begin{aligned} \mathcal{F}(f(ax))(w) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)e^{-i\omega x}\,dx \\ &= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(x)e^{-i\frac{\omega}{a}x}\,dx \\ &= -\frac{1}{a}\mathcal{F}(f)(\frac{w}{a}). \end{aligned}$$

Hence for all $a \neq 0$, we can write

$$= \frac{1}{|a|}\mathcal{F}(f)\left(\frac{\omega}{a}\right).$$

(b) We have

$$\mathcal{F}(e^{-|x|})(w) = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}.$$

By (a),

$$\mathcal{F}(e^{-2|x|})(w) = \frac{1}{2}\sqrt{\frac{2}{\pi}}\frac{1}{1+(w/2)^2} = \sqrt{\frac{2}{\pi}}\frac{2}{4+w^2}.$$

(c) Let $f(x)$ denote the function in Example 2. Then $g(x) = f(-x)$. So

$$\widehat{g}(w) = \widehat{f}(-w) = \frac{1+iw}{\sqrt{2\pi}(1+w^2)}.$$

Let $h(x) = e^{-|x|}$. Then $h(x) = f(x) + g(x)$. So

$$\widehat{h}(w) = \widehat{f}(w) + \widehat{g}(w) = \frac{1-iw}{\sqrt{2\pi}(1+w^2)} + \frac{1+iw}{\sqrt{2\pi}(1+w^2)} = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}.$$

21. We have $\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}}e^{-w^2/4}$, by Theorem 5. Using Exercise 20, we have

$$\begin{aligned} \mathcal{F}(\frac{\cos x}{e^{x^2}}) &= \mathcal{F}(\cos x e^{-x^2}) \\ &= \frac{1}{2\sqrt{2}}\left(e^{-\frac{(w-a)^2}{4}} + e^{-\frac{(w+a)^2}{4}}\right) \end{aligned}$$

25 Let

$$g(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and note that $f(x) = \cos(x)\, g(x)$. Now $\mathcal{F}(g(x)) = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$. Using Exercise 20, we have

$$\begin{aligned} \mathcal{F}(f(x)) &= \mathcal{F}(\cos x g(x)) \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\Big(\frac{\sin(w-1)}{w-1} + \frac{\sin(w+1)}{w+1}\Big). \end{aligned}$$

29. Take $a = 0$ and relabel $b = a$ in Exercise 27, you will get the function $f(x) = h$ if $0 < x < a$. Its Fourier transform is

$$\widehat{f}(w) == \sqrt{\frac{2}{\pi}} h e^{-i\frac{a}{2}w} \frac{\sin\left(\frac{a\,w}{2}\right)}{w}.$$

Let $g(x)$ denote the function in the figure. Then $g(x) = \frac{1}{a} x\, f(x)$ and so, by Theorem 3,

$$\begin{aligned} \widehat{g}(w) &= \frac{1}{a} i \frac{d}{dw} \widehat{f}(w) \\ &= i\sqrt{\frac{2}{\pi}} \frac{h}{a} \frac{d}{dw}\left[e^{-i\frac{a}{2}w} \frac{\sin\left(\frac{a\,w}{2}\right)}{w}\right] \\ &= i\sqrt{\frac{2}{\pi}} \frac{h}{a} \frac{e^{-i\frac{a}{2}w}}{w}\left[a\frac{\cos\left(\frac{a\,w}{2}\right) - i\sin\left(\frac{a\,w}{2}\right)}{2} - \frac{\sin\left(\frac{a\,w}{2}\right)}{w}\right] \\ &= i\sqrt{\frac{2}{\pi}} \frac{h}{a} e^{-i\frac{a}{2}w}\left[\frac{-2\sin\left(\frac{a\,w}{2}\right) + a\,w\,e^{-i\frac{a}{2}w}}{2w^2}\right]. \end{aligned}$$

33. Let $g(x)$ denote the function in this exercise. By the reciprocity relation, since the function is even, we have $\mathcal{F}(\mathcal{F}(g)) = g(-x) = g(x)$. Taking inverse Fourier transforms, we obtain $\mathcal{F}^{-1}(g) = \mathcal{F}(g)$. Hence it is enough to compute the Fourier transform. We use the notation and the result of Exercise 34. We have

$$g(x) = 2f_{2a}(x) - f_a(x).$$

Verify this identity by drawing the graphs of f_{2a} and f_a and then drawing the graph of $f_{2a}(x) - f_a(x)$. With the help of this identity and the result of Exercise 34, we have

$$\begin{aligned} \widehat{g}(w) &= 2\widehat{f_{2a}}(w) - \widehat{f_a}(w) \\ &= 2\frac{8a}{\sqrt{2\pi}} \frac{\sin^2(aw)}{4(aw)^2} - \frac{4a}{\sqrt{2\pi}} \frac{\sin^2\left(\frac{aw}{2}\right)}{(aw)^2} \\ &= \frac{4}{a\sqrt{2\pi}}\left[\frac{\sin^2(aw)}{w^2} - \frac{\sin^2\left(\frac{aw}{2}\right)}{w^2}\right]. \end{aligned}$$

37. By Exercise 27,

$$\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-w^2/4}.$$

By Theorem 3(i)

$$\begin{aligned}\mathcal{F}(xe^{-x^2}) &= i\frac{d}{dw}\Big(\frac{1}{\sqrt{2}}e^{-w^2/4}\Big)\\ &= \frac{i}{2\sqrt{2}}e^{-w^2/4}.\end{aligned}$$

41. We have $\mathcal{F}(\frac{1}{1+x^2}) = \sqrt{\frac{\pi}{2}}e^{-|w|}$. So if $w > 0$

$$\mathcal{F}(\frac{x}{1+x^2})(w) = i\sqrt{\frac{\pi}{2}}\frac{d}{dw}e^{-w} = -i\sqrt{\frac{\pi}{2}}e^{-w}.$$

If $w < 0$

$$\mathcal{F}(\frac{x}{1+x^2})(w) = i\sqrt{\frac{\pi}{2}}\frac{d}{dw}e^{w} = i\sqrt{\frac{\pi}{2}}e^{w}.$$

If $w = 0$,

$$\mathcal{F}(\frac{x}{1+x^2})(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx = 0$$

(odd integrand). We can combine these answers into one formula

$$\mathcal{F}(\frac{x}{1+x^2})(w) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}\,(w)e^{-|w|}.$$

45. Theorem 3 (i) and Exercise 19:

$$\begin{aligned}\mathcal{F}(xe^{-\frac{1}{2}(x-1)^2}) &= i\frac{d}{dw}\Big(\mathcal{F}(e^{-\frac{1}{2}(x-1)^2})\Big)\\ &= i\frac{d}{dw}\Big(e^{-iw}\mathcal{F}(e^{-\frac{1}{2}x^2})\Big)\\ &= i\frac{d}{dw}\Big(e^{-iw}e^{-\frac{1}{2}w^2})\Big) = i\frac{d}{dw}\Big(e^{-\frac{1}{2}w^2-iw})\Big)\\ &= i(-w-i)e^{-\frac{1}{2}w^2-iw}\\ &= (1-iw)e^{-\frac{1}{2}w^2-iw}.\end{aligned}$$

49.

$$\widehat{h}(\omega) = e^{-\omega^2}\cdot\frac{1}{1+\omega^2} = \mathcal{F}\big(\frac{1}{\sqrt{2}}e^{-x^2/4}\big)\cdot\mathcal{F}\big(\sqrt{\frac{\pi}{2}}e^{-|x|}\big).$$

Hence

$$\begin{aligned}h(x) &= f*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2}}e^{-\frac{(x-t)^2}{4}}\sqrt{\frac{\pi}{2}}e^{-|t|}dt\\ &= \frac{1}{2\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{(x-t)^2}{4}}e^{-|t|}dt.\end{aligned}$$

53. Let $f(x) = xe^{-x^2/2}$ and $g(x) = e^{-x^2}$.
(a) $\mathcal{F}(f)(w) = -iwe^{-\frac{w^2}{2}}$, and $\mathcal{F}(g)(w) = \frac{1}{\sqrt{2}}e^{-\frac{w^2}{4}}$.
(b)

$$\begin{aligned}\{*\} &= \{\cdot\}\\ &= -i\frac{w}{\sqrt{2}}e^{-\frac{w^2}{4}}e^{-\frac{w^2}{2}}\\ &= -i\frac{w}{\sqrt{2}}e^{-3\frac{w^2}{4}}.\end{aligned}$$

(c) With the help of Theorem 3

$$\begin{aligned}
f * g &= \mathcal{F}^{-1}\Big(-i\frac{w}{\sqrt{2}}e^{-3\frac{w^2}{4}}\Big)\\
&= \mathcal{F}^{-1}\Big(i\frac{1}{\sqrt{2}}\frac{4}{6}\frac{d}{dw}e^{-3\frac{w^2}{4}}\Big)\\
&= \frac{1}{\sqrt{2}}\frac{2}{3}\mathcal{F}^{-1}\Big(i\frac{d}{dw}e^{-3\frac{w^2}{4}}\Big) = \frac{1}{\sqrt{2}}\frac{2}{3}x\mathcal{F}^{-1}\Big(e^{-3\frac{w^2}{4}}\Big)\\
&= \frac{2}{3\sqrt{3}}xe^{-\frac{1}{3}x^2}.
\end{aligned}$$

In computing $\mathcal{F}^{-1}\Big(e^{-3\frac{w^2}{4}}\Big)$, use Exercise 10(a) and (5) to obtain

$$\mathcal{F}^{-1}\Big(e^{-aw^2}\Big) = \frac{1}{\sqrt{2a}}e^{-\frac{(-x)^2}{4a}} = \frac{1}{\sqrt{2a}}e^{-\frac{x^2}{4a}}.$$

57. Recall that f is integrable means that

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

If f and g are integrable, then

$$f * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)\,dt.$$

So, using properties of the integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} |f * g(x)|dx &= \int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)\,dt\right|dx\\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x-t)g(t)|\,dx\,dt\\
&\quad \text{(Interchange order of integration.}\\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overbrace{\int_{-\infty}^{\infty}|f(x-t)|\,dx}^{=\int_{-\infty}^{\infty}|f(x)|dx}|g(t)|\,dt\\
&\quad \text{(Change variables in the inner integral } X = x-t.\text{)}\\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(x)|\,dx\int_{-\infty}^{\infty}|g(t)|\,dt < \infty;
\end{aligned}$$

thus $f * g$ is integrable.

Solutions to Exercises 7.3

1.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = \frac{1}{1+x^2}, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Follow the solution of Example 1. Fix t and Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w,\,t) = -w^2\widehat{u}(w,\,t),$$
$$\widehat{u}(w,\,0) = \mathcal{F}\Big(\frac{1}{1+x^2}\Big) = \sqrt{\frac{\pi}{2}}e^{-|w|}, \qquad \frac{d}{dt}\widehat{u}(w,\,0) = 0.$$

Solve the second order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)\cos wt + B(w)\sin wt.$$

Using $\frac{d}{dt}\widehat{u}(w,\,0) = 0$, we get

$$-A(w)w\sin wt + B(w)w\cos wt\Big|_{t=0} = 0 \;\Rightarrow\; B(w)w = 0 \;\Rightarrow\; B(w) = 0.$$

Hence

$$\widehat{u}(w,\,t) = A(w)\cos wt.$$

Using $\widehat{u}(w,\,0) = \sqrt{\frac{\pi}{2}}e^{-|w|}$, we see that $A(w) = \sqrt{\frac{\pi}{2}}e^{-|w|}$ and so

$$\widehat{u}(w,\,t) = \sqrt{\frac{\pi}{2}}e^{-|w|}\cos wt.$$

Taking inverse Fourier transforms, we get

$$u(x,\,t) = \int_{-\infty}^{\infty} e^{-|w|}\cos wt\, e^{ixw}\,dw.$$

5.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2},$$
$$u(x,0) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{x}, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

Fix t and Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w,\,t) = -c^2w^2\widehat{u}(w,\,t),$$
$$\widehat{u}(w,\,0) = \mathcal{F}\Big(\sqrt{\frac{\sin x}{x}}\Big)(w) = \widehat{f}(w) = \begin{cases} 1 & \text{if } |w| < 1 \\ 0 & \text{if } |w| > 1, \end{cases}$$
$$\frac{d}{dt}\widehat{u}(w,\,0) = 0.$$

Solve the second order differential equation in $\widehat{u}(w,\,t)$:

$$\widehat{u}(w,\,t) = A(w)\cos cwt + B(w)\sin cwt.$$

Using $\frac{d}{dt}\widehat{u}(w,\,0) = 0$, we get

$$\widehat{u}(w,\,t) = A(w)\cos cwt.$$

Using $\widehat{u}(w, 0) = \widehat{f}(w)$, we see that

$$\widehat{u}(w, t) = \widehat{f}(w) \cos wt.$$

Taking inverse Fourier transforms, we get

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w) \cos cwt\, e^{ixw}\, dw = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos cwt\, e^{ixw}\, dw.$$

17.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^4 u}{\partial x^4}$$

$$u(x,0) = \begin{cases} 100 & \text{if } |x| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Fourier transform the problem with respect to the variable x:

$$\frac{d^2}{dt^2}\widehat{u}(w, t) = w^4 \widehat{u}(w, t),$$

$$\widehat{u}(w, 0) = \widehat{f}(w) = 100\sqrt{\frac{2}{\pi}}\frac{\sin 2w}{w}.$$

Solve the second order differential equation in $\widehat{u}(w, t)$:

$$\widehat{u}(w, t) = A(w)e^{-w^2 t} + B(w)e^{w^2 t}.$$

Because a Fourier transform is expected to tend to 0 as $w \to \pm\infty$, if we fix $t > 0$ and let $w \to \infty$ or $w \to -\infty$, we see that one way to make $\widehat{u}(w, t) \to 0$ is to take $B(w) = 0$. Then $\widehat{u}(w, t) = A(w)e^{-w^2 t}$, and from the initial condition we obtain $B(w) = \widehat{f}(w)$. So

$$\widehat{u}(w, t) = \widehat{f}(w)e^{-w^2 t} = 100\sqrt{\frac{2}{\pi}}\frac{\sin 2w}{w}e^{-w^2 t}.$$

Taking inverse Fourier transforms, we get

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 100\sqrt{\frac{2}{\pi}}\frac{\sin 2w}{w}e^{-w^2 t}\, e^{ixw}\, dw \\ &= \frac{100}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2w}{w}e^{-w^2 t}\, e^{ixw}\, dw. \end{aligned}$$

21. (a) To verify that

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

is a solution of the boundary value problem of Example 1 is straightforward. You just have to plug the solution into the equation and the initial and boundary conditions and see that the equations are verified. The details are sketched in Section 3.4, following Example 1 of that section.
(b) In Example 1, we derived the solution as an inverse Fourier transform:

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\widehat{f}(w)\, \cos cwt + \frac{1}{cw}\widehat{g}(w)\, \sin cwt\right] e^{iwx}\, dx.$$

Using properties of the Fourier transform, we will show that

$$(1) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)\, \cos cwt e^{iwx}\, dw = \frac{1}{2}[f(x-ct) + f(x+ct)];$$

$$(2) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{w}\widehat{g}(w)\, \sin cwt\, e^{iwx}\, dw = \frac{1}{2}\int_{x-ct}^{x+ct} g(s)\, ds.$$

To prove (1), note that

$$\cos cwt = \frac{e^{icwt} + e^{-icwt}}{2},$$

so

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)\, \cos cwt e^{iwx}\, dw \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)\, (\frac{e^{icwt} + e^{-icwt}}{2}) e^{iwx}\, dw \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)\, e^{iw(x+ct)}\, dw \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w)\, e^{iw(x-ct)}\, dw\right] \\
&= \frac{1}{2}[f(x+ct) + f(x-ct)];
\end{aligned}$$

because the first integral is simply the inverse Fourier transform of $\widehat{f}$ evaluated at $x + ct$, and the second integral is the inverse Fourier transform of $\widehat{f}$ evaluated at $x - ct$. This proves (1). To prove (2), we note that the left side of (2) is an inverse Fourier transform. So (2) will follow if we can show that

$$\text{(3)} \qquad \mathcal{F}\left\{\int_{x-ct}^{x+ct} g(s)\, ds\right\} = \frac{2}{w}\widehat{g}(w)\, \sin cwt.$$

Let G denote an antiderivative of g. Then (3) is equivalent to

$$\mathcal{F}\left(G(x+ct) - G(x-ct)\right)(w) = \frac{2}{w}\widehat{G'}(w)\, \sin cwt.$$

Since $\widehat{G'} = iw\widehat{G}$, the last equation is equivalent to

$$\text{(4)} \qquad \mathcal{F}\left(G(x+ct)\right)(w) - \mathcal{F}\left(G(x-ct)\right)(w) = 2i\widehat{G}(w)\, \sin cwt.$$

Using Exercise 19, Sec. 7.2, we have

$$\begin{aligned}
\mathcal{F}\left(G(x+ct)\right)(w) - \mathcal{F}\left(G(x-ct)\right)(w) &= e^{ictw}\mathcal{F}(G)(w) - e^{-ictw}\mathcal{F}(G)(w) \\
&= \mathcal{F}(G)(w)\left(e^{ictw} - e^{-ictw}\right) \\
&= 2i\widehat{G}(w)\, \sin cwt,
\end{aligned}$$

where we have applied the formula

$$\sin ctw = \frac{e^{ictw} - e^{-ictw}}{2i}.$$

This proves (4) and completes the solution.

25. Using the Fourier transform, we obtain

$$\begin{aligned}
\frac{d^2}{dt^2}\widehat{u}(w,\, t) &= c^2\, w^4 \widehat{u}(w,\, t), \\
\widehat{u}(w,\, 0) &= \widehat{f}(w), \qquad \frac{d}{dt}\widehat{u}(w,\, 0) = \widehat{g}(w)\,.
\end{aligned}$$

Thus

$$\widehat{u}(w,\, t) = A(w)e^{-cw^2 t} + B(w)e^{cw^2 t}.$$

Using the initial conditions:

$$\widehat{u}(w,\, 0) = \widehat{f}(w) \;\Rightarrow\; A(w) + B(w) = \widehat{f}(w) \;\Rightarrow\; A(w) = \widehat{f}(w) - B(w);$$

and

$$\begin{aligned}
\frac{d}{dt}\widehat{u}(w,\,0) = \widehat{g}(w) \quad &\Rightarrow \quad -cw^2 A(w) + cw^2 B(w) = \widehat{g}(w) \\
&\Rightarrow \quad -A(w) + B(w) = \frac{\widehat{g}(w)}{cw^2} \\
&\Rightarrow \quad -\widehat{f}(w) + 2B(w) = \frac{\widehat{g}(w)}{cw^2} \\
&\Rightarrow \quad B(w) = \frac{1}{2}\left(\widehat{f}(w) + \frac{\widehat{g}(w)}{cw^2}\right); \\
&\Rightarrow \quad A(w) = \frac{1}{2}\left(\widehat{f}(w) - \frac{\widehat{g}(w)}{cw^2}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{u}(w,\,t) &= \frac{1}{2}\left(\widehat{f}(w) - \frac{\widehat{g}(w)}{cw^2}\right) e^{-cw^2 t} + \frac{1}{2}\left(\widehat{f}(w) + \frac{\widehat{g}(w)}{cw^2}\right) e^{cw^2 t}. \\
&= \widehat{f}(w)\frac{(e^{cw^2 t} + e^{-cw^2 t})}{2} + \frac{\widehat{g}(w)}{cw^2}\frac{(e^{cw^2 t} - e^{-cw^2 t})}{2} \\
&= \widehat{f}(w)\cosh(cw^2 t) + \frac{\widehat{g}(w)}{cw^2}\sinh(cw^2 t)
\end{aligned}$$

Taking inverse Fourier transforms, we get

$$u(x,\,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\widehat{f}(w)\cosh(cw^2 t) + \frac{\widehat{g}(w)}{cw^2}\sinh(cw^2 t)\right) e^{ixw}\,dw.$$

Solutions to Exercises 7.4

1. Repeat the solution of Example 1 making some adjustments: $c = \frac{1}{2}$, $g_t(x) = \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{x^2}{t}}$,

$$\begin{aligned} u(x,t) &= f * g_t(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) \frac{\sqrt{2}}{\sqrt{t}} e^{-\frac{(x-s)^2}{t}}\, ds \\ &= \frac{20}{\sqrt{t\pi}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{t}}\, ds \quad (v = \frac{x-s}{\sqrt{t}},\ dv = -\frac{1}{\sqrt{t}}\, ds) \\ &= \frac{20}{\sqrt{\pi}} \int_{\frac{x-1}{\sqrt{t}}}^{\frac{x+1}{\sqrt{t}}} e^{-v^2}\, ds \\ &= 10 \left(\operatorname{erf}(\frac{x+1}{\sqrt{t}}) - \operatorname{erf}(\frac{x-1}{\sqrt{t}}) \right). \end{aligned}$$

5. Apply (4) with $f(s) = s^2$:

$$\begin{aligned} u(x,t) &= f * g_t(x) \\ &= \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^2 e^{-\frac{(x-s)^2}{t}}\, ds. \end{aligned}$$

You can evaluate this integral by using integration by parts twice and then appealing to Theorem 5, Section 7.2. However, we will use a different technique based on the operational properties of the Fourier transform that enables us to evaluate a much more general integral. Let n be a nonnegative integer and suppose that f and $s^n f(s)$ are integrable and tend to 0 at $\pm\infty$. Then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^n f(s)\, ds = (i)^n \left[\frac{d^n}{dw^n} \mathcal{F}(f)(w) \right]_{w=0}.$$

This formula is immediate if we recall Theorem 3(ii), Section 7.2, and that

$$\widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s)\, ds.$$

We will apply this formula with

$$f(s) = \frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}}.$$

We have

$$\begin{aligned} \mathcal{F}\left(\frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}} \right)(w) &= e^{-iwx} \mathcal{F}\left(\frac{1}{\sqrt{2t}} e^{-\frac{s^2}{t}} \right)(w) \quad \text{(by Exercise 19, Sec. 7.2)} \\ &= e^{-iwx} e^{-w^2 t} = e^{-(iwx + w^2 t)} \quad \text{(See the proof of Th. 1.)} \end{aligned}$$

So

$$\begin{aligned}
u(x, t) &= \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} s^2 e^{-\frac{(x-s)^2}{t}} \, ds \\
&= -\left[\frac{d^2}{dw^2}\mathcal{F}\left(\frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}}\right)(w)\right]_{w=0} \\
&= -\left[\frac{d^2}{dw^2} e^{-(iwx+w^2t)}\right]_{w=0} \\
&= -\left[\frac{d}{dw} - e^{-(iwx+w^2t)}(ix+2wt)\right]_{w=0} \\
&= \left[-e^{-(iwx+w^2t)}(ix+2wt)^2 + 2te^{-(iwx+w^2t)}\right]_{w=0} \\
&= x^2 + 2t.
\end{aligned}$$

You can check the validity of this answer by plugging it back into the heat equation. The initial condition is also obviously met: $u(x, 0) = x^2$.

The approach that we took can be used to solve the boundary value problem with $f(x)x^n$ as initial temperature distribution. See the end of this section for interesting applications.

9. Fourier transform the problem:

$$\frac{du}{dt}\widehat{u}(w, t) = -e^{-t} w^2 \widehat{u}(w, t), \quad \widehat{u}(w, 0) = \widehat{f}(w).$$

Solve for $\widehat{u}(w, t)$:

$$\widehat{u}(w, t) = \widehat{f}(w) e^{-w^2(1-e^{-t})}.$$

Inverse Fourier transform and note that

$$u(x, t) = f * \mathcal{F}^{-1}\left(e^{-w^2(1-e^{-t})}\right).$$

With the help of Theorem 5, Sec. 7.2 (take $a = 1 - e^{-t}$), we find

$$\mathcal{F}^{-1}\left(e^{-w^2(1-e^{-t})}\right) = \frac{1}{\sqrt{2}\sqrt{1-e^{-t}}} e^{-\frac{x^2}{4(1-e^{-t})}}.$$

Thus

$$u(x, t) = \frac{1}{2\sqrt{\pi}\sqrt{1-e^{-t}}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4(1-e^{-t})}} \, ds.$$

13. If in Exercise 9 we take

$$f(x) = \begin{cases} 100 & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

then the solution becomes

$$u(x, t) = \frac{50}{\sqrt{\pi}\sqrt{1-e^{-t}}} \int_{-1}^{1} e^{-\frac{(x-s)^2}{4(1-e^{-t})}} \, ds.$$

Let $z = \frac{x-s}{2\sqrt{1-e^{-t}}}$, $dz = \frac{-ds}{2\sqrt{1-e^{-t}}}$. Then

$$\begin{aligned}
u(x, t) &= \frac{50}{\sqrt{\pi}\sqrt{1-e^{-t}}} 2\sqrt{1-e^{-t}} \int_{\frac{x-1}{2\sqrt{1-e^{-t}}}}^{\frac{x+1}{2\sqrt{1-e^{-t}}}} e^{-z^2} \, dz \\
&= \frac{100}{\sqrt{\pi}} \int_{\frac{x-1}{2\sqrt{1-e^{-t}}}}^{\frac{x+1}{2\sqrt{1-e^{-t}}}} e^{-z^2} \, dz \\
&= 50\left[\operatorname{erf}\left(\frac{x+1}{2\sqrt{1-e^{-t}}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{1-e^{-t}}}\right)\right].
\end{aligned}$$

As t increases, the expression $\operatorname{erf}\left(\frac{x+1}{2\sqrt{1-e^{-t}}}\right)-\operatorname{erf}\left(\frac{x-1}{2\sqrt{1-e^{-t}}}\right)$ approaches very quickly $\operatorname{erf}\left(\frac{x+1}{2}\right)-\operatorname{erf}\left(\frac{x-1}{2}\right)$, which tells us that the temperature approaches the limiting distribution

$$50\left[\operatorname{erf}\left(\frac{x+1}{2}\right)-\operatorname{erf}\left(\frac{x-1}{2}\right)\right].$$

You can verify this assertion using graphs.

17. (a) If

$$f(x)=\begin{cases} T_0 & \text{if } a<x<b,\\ 0 & \text{otherwise,}\end{cases}$$

then

$$u(x,\,t)=\frac{T_0}{2c\sqrt{\pi t}}\int_a^b e^{-\frac{(x-s)^2}{4c^2t}}\,ds.$$

(b) Let $z=\frac{x-s}{2c\sqrt{t}}$, $dz=\frac{-ds}{2c\sqrt{t}}$. Then

$$\begin{aligned} u(x,\,t) &= \frac{T_0}{2c\sqrt{\pi t}}2c\sqrt{t}\int_{\frac{x-b}{2c\sqrt{t}}}^{\frac{x-a}{2c\sqrt{t}}} e^{-z^2}\,dz\\ &= \frac{T_0}{2}\left[\operatorname{erf}\left(\frac{x-a}{2c\sqrt{t}}\right)-\operatorname{erf}\left(\frac{x-b}{2c\sqrt{t}}\right)\right].\end{aligned}$$

25. Let $u_2(x,\,t)$ denote the solution of the heat problem with initial temperature distribution $f(x)=e^{-(x-1)^2}$. Let $u(x,\,t)$ denote the solution of the problem with initial distribution e^{-x^2}. Then, by Exercise 23, $u_2(x,\,t)=u(x-1,\,t)$

By (4), we have

$$u(x,\,t)=\frac{1}{c\sqrt{2t}}e^{-x^2/(4c^2t)}*e^{-x^2}.$$

We will apply Exercise 24 with $a=\frac{1}{4c^2t}$ and $b=1$. We have

$$\begin{aligned} \frac{ab}{a+b} &= \frac{1}{4c^2t}\times\frac{1}{\frac{1}{4c^2t}+1}\\ &= \frac{1}{1+4c^2t}\\ \frac{1}{\sqrt{2(a+b)}} &= \frac{1}{\sqrt{2(\frac{1}{4c^2t}+1)}}\\ &= \frac{c\sqrt{2t}}{\sqrt{4c^2t+1}}.\end{aligned}$$

So

$$\begin{aligned} u(x,\,t) &= \frac{1}{c\sqrt{2t}}e^{-x^2/(4c^2t)}*e^{-x^2}\\ &= \frac{1}{c\sqrt{2t}}\cdot\frac{c\sqrt{2t}}{\sqrt{4c^2t+1}}e^{-\frac{x^2}{1+4c^2t}}\\ &= \frac{1}{\sqrt{4c^2t+1}}e^{-\frac{x^2}{1+4c^2t}},\end{aligned}$$

and hence

$$u_2(x,\,t)=\frac{1}{\sqrt{4c^2t+1}}e^{-\frac{(x-1)^2}{1+4c^2t}}.$$

29. Parts (a)-(c) are obvious from the definition of $g_t(x)$.
(d) The total area under the graph of $g_t(x)$ and above the x-axis is

$$\begin{aligned}\int_{-\infty}^{\infty} g_t(x)\,dx &= \frac{1}{c\sqrt{2t}}\int_{-\infty}^{\infty} e^{-x^2/(4c^2t)}\,dx\\ &= \frac{2c\sqrt{t}}{c\sqrt{2t}}\int_{-\infty}^{\infty} e^{-z^2}\,dz \quad \left(z=\frac{x}{2c\sqrt{t}},\ dx = 2c\sqrt{t}\,dz\right)\end{aligned}$$

$$\sqrt{2}\int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{2\pi},$$

by (4), Sec. 7.2.
(e) To find the Fourier transform of $g_t(x)$, apply (5), Sec. 7.2, with

$$a = \frac{1}{4c^2t}, \quad \frac{1}{\sqrt{2a}} = 2c\sqrt{2t}, \quad \frac{1}{4a} = c^2t.$$

We get

$$\begin{aligned}\widehat{g}_t(w) &= \frac{1}{c\sqrt{2t}}\mathcal{F}\left(e^{-x^2/(4c^2t)}\right)\,dx\\ &= \frac{1}{c\sqrt{2t}} \times 2c\sqrt{2t}e^{-c^2t\omega^2}\\ &= e^{-c^2t\omega^2}.\end{aligned}$$

(f) If f is an integrable and piecewise smooth function, then at its points of continuity, we have

$$\lim_{t\to 0} g_t * f(x) = f(x).$$

This is a true fact that can be proved by using properties of Gauss's kernel. If we interpret $f(x)$ as an initial temperature distribution in a heat problem, then the solution of this heat problem is given by

$$u(x, t) = g_t * f(x).$$

If $t \to 0$, the temperature $u(x, t)$ should approach the initial temperature distribution $f(x)$. Thus $\lim_{t\to 0} g_t * f(x) = f(x)$.

Alternatively, we can use part (e) and argue as follows. Since

$$\lim_{t\to 0}\mathcal{F}(g_t)(\omega) = \lim_{t\to 0} e^{-c^2t\omega^2} = 1,$$

So

$$\lim_{t\to 0}\mathcal{F}(g_t * f) = \lim_{t\to 0}\mathcal{F}(g_t)\mathcal{F}(f) = \mathcal{F}(f).$$

You would expect that the limit of the Fourier transform be the transform of the limit function. So taking inverse Fourier transforms, we get $\lim_{t\to 0} g_t * f(x) = f(x)$. (Neither one of the arguments that we gave is rigorous.)

A generalization of Exercise 5 Suppose that you want to solve the heat equation $u_t = u_{xx}$ subject to the initial condition $u(x, 0) = x^n$ where n is a nonnegative integer. We have already done the case $n = 0$ (in Exercise 19) and $n = 2$ (in Exercise 5). For the general case, proceed as in Exercise 5 and apply (4) with $f(s) = s^n$:

$$\begin{aligned}u(x, t) &= f * g_t(x)\\ &= \frac{1}{\sqrt{2t}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} s^n e^{-\frac{(x-s)^2}{t}}\,ds.\end{aligned}$$

Use the formula

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} s^n f(s)\,ds = (i)^n\left[\frac{d^n}{dw^n}\mathcal{F}(f)(w)\right]_{w=0},$$

with

$$f(s) = \frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}}$$

and

$$\mathcal{F}\left(\frac{1}{\sqrt{2t}} e^{-\frac{(x-s)^2}{t}}\right)(w) = e^{-(iwx+w^2t)}$$

(see the solution of Exercise 5). So

$$u(x,\,t) = (i)^n \left[\frac{d^n}{dw^n} e^{-(iwx+w^2t)}\right]_{w=0}.$$

To compute this last derivative, recall the Taylor series formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

So knowledge of the Taylor series gives immediately the values of the derivatives at a. Since

$$e^{aw} = \sum_{n=0}^{\infty} \frac{(aw)^n}{n!},$$

we get

$$\left[\frac{d^j}{dw^j} e^{aw}\right]_{w=0} = a^j.$$

Similarly,

$$\left[\frac{d^k}{dw^k} e^{bw^2}\right]_{w=0} = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{b^j (2j)!}{j!} & \text{if } k = 2j. \end{cases}$$

Returning to $u(x,\,t)$, we compute the nth derivative of $e^{-(iwx+w^2t)}$ using the Leibniz rule and use the what we just found and get

$$\begin{aligned}
u(x,\,t) &= (i)^n \left[\frac{d^n}{dw^n} e^{-w^2t} e^{-iwx}\right]_{w=0} \\
&= (i)^n \sum_{j=0}^{n} \binom{n}{j} \frac{d^j}{dw^j} e^{-w^2t} \cdot \frac{d^{n-j}}{dw^{n-j}} e^{-iwx}\Big|_{w=0} \\
&= (i)^n \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j} \frac{d^{2j}}{dw^{2j}} e^{-w^2t} \cdot \frac{d^{n-2j}}{dw^{n-2j}} e^{-iwx}\Big|_{w=0} \\
&= (i)^n \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j} \frac{(-t)^j (2j)!}{j!} \cdot (-ix)^{n-2j} \\
&= \sum_{j=0}^{[\frac{n}{2}]} \binom{n}{2j} \frac{t^j (2j)!}{j!} \cdot x^{n-2j}.
\end{aligned}$$

For example, if $n = 2$,

$$u(x,\,t) = \sum_{j=0}^{1} \binom{2}{2j} \frac{t^j (2j)!}{j!} \cdot x^{2-2j} = x^2 + 2t,$$

which agrees with the result of Exercise 5. If $n = 3$,

$$u(x,\,t) = \sum_{j=0}^{1} \binom{3}{2j} \frac{t^j (2j)!}{j!} \cdot x^{3-2j} = x^3 + 6tx.$$

You can easily check that this solution verifies the heat equation and $u(x,\,0)=x^3$. If $n=4$,

$$u(x,\,t) \;=\; \sum_{j=0}^{2}\begin{pmatrix}4\\2j\end{pmatrix}\frac{t^j(2j)!}{j!}\cdot x^{4-2j}=x^4+12tx^2+12t^2.$$

Here too, you can check that this solution verifies the heat equation and $u(x,\,0)=x^4$.

We now derive a recurrence relation that relates the solutions corresponding to $n-1$, n, and $n+1$. Let $u_n=u_n(x,\,t)$ denote the solution with initial temperature distribution $u_n(x,\,0)=x^n$. We have the following recurrence relation

$$u_{n+1}=xu_n+2ntu_{n-1}.$$

The proof of this formula is very much like the proof of Bonnet's recurrence formula for the Legendre polynomials (Section 5.6). Before we give the proof, let us verify the formula with $n=3$. The formula states that $u_4=4u_3+6tu_2$. Sine $u_4=x^4+12tx^2+12t^2$, $u_3=x^3+6tx$, and $u_2=x^2+2t$, we see that the formula is true for $n=3$. We now prove the formula using Leibniz rule of differentiation. As in Section 5.6, let us use the symbol D^n to denote the nth derivative, We have

$$\begin{aligned}
u_{n+1}(x,\,t) &= (i)^{n+1}\left[\frac{d^{n+1}}{dw^{n+1}}e^{-(iwx+w^2t)}\right]_{w=0}\\
&= (i)^{n+1}\left[D^{n+1}e^{-(iwx+w^2t)}\right]_{w=0}\\
&= (i)^{n+1}\left[D^{n}\left(De^{-(iwx+w^2t)}\right)\right]_{w=0}\\
&= (i)^{n+1}\left[D^{n}\left(-(ix+2wt)e^{-(iwx+w^2t)}\right)\right]_{w=0}\\
&= (i)^{n+1}\left[D^{n}\left(-(ix+2wt)e^{-(iwx+w^2t)}\right)\right]_{w=0}\\
&= (i)^{n+1}\left[-D^{n}\left(e^{-(iwx+w^2t)}\right)(ix+2wt)-2ntD^{n-1}\left(e^{-(iwx+w^2t)}\right)\right]_{w=0}\\
&= x(i)^n\,D^n\left(e^{-(iwx+w^2t)}\right)\Big|_{w=0}+2nt\,(i)^{n-1}D^{n-1}\left(e^{-(iwx+w^2t)}\right)\Big|_{w=0}\\
&= xu_n+2ntu_{n-1}.
\end{aligned}$$

Solutions to Exercises 7.5

1. To solve the Dirichlet problem in the upper half-plane with the given boundary function, we use formula (5). The solution is given by

$$\begin{aligned} u(x, y) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}\,ds \\ &= \frac{50y}{\pi}\int_{-1}^{1} \frac{ds}{(x-s)^2+y^2} \\ &= \frac{50}{\pi}\left\{\tan^{-1}\left(\frac{1+x}{y}\right)+\tan^{-1}\left(\frac{1-x}{y}\right)\right\}, \end{aligned}$$

where we have used Example 1 to evaluate the definite integral.

5. Appealing to (4) in Section 7.5, with $y = y_1,\ y_2, y_1 + y_2$, we find

$$\mathcal{F}(P_{y_1})(w) = e^{-y_1|w|}, \quad \mathcal{F}(P_{y_2})(w) = e^{-y_2|w|}, \quad \mathcal{F}(P_{y_1+y_2})(w) = e^{-(y_1+y_2)|w|}.$$

Hence

$$\mathcal{F}(P_{y_1})(w)\cdot\mathcal{F}(P_{y_2})(w) = e^{-y_1|w|}e^{-y_2|w|} = e^{-(y_1+y_2)|w|} = \mathcal{F}(P_{y_1+y_2})(w).$$

But

$$\mathcal{F}(P_{y_1})(w)\cdot\mathcal{F}(P_{y_2})(w) = \mathcal{F}(P_{y_1} * P_{y_2})(w),$$

Hence

$$\mathcal{F}(P_{y_1+y_2})(w) = \mathcal{F}(P_{y_1} * P_{y_2})(w);$$

and so $P_{y_1+y_2} = P_{y_1} * P_{y_2}$.

9. Modify the solution of Example 1(a) to obtain that, in the present case, the solution is

$$u(x, y) = \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{a+x}{y}\right)+\tan^{-1}\left(\frac{a-x}{y}\right)\right].$$

To find the isotherms, we must determine the points (x, y) such that $u(x, y) = T$. As in the solution of Example 1(b), these points satisfy

$$x^2+\left(y-a\cot(\frac{\pi T}{T_0})\right)^2 = \left(a\csc(\frac{\pi T}{T_0})\right)^2.$$

Hence the points belong to the arc in the upper half-plane of the circle with center $(0, a\cot(\frac{\pi T}{T_0}))$ and radius $a\csc(\frac{\pi T}{T_0})$. The isotherm corresponding to $T = \frac{T_0}{2}$ is the arc of the circle

$$x^2+\left(y-a\cot(\frac{\pi}{2})\right)^2 = \left(a\csc(\frac{\pi}{2})\right)^2,$$

or

$$x^2+y^2 = a^2.$$

Thus the isotherm in this case is the upper semi-circle of radius a and center at the origin.

13. Parts (a)-(c) are clear. Part (e) fellows from a table. For (d), you can use (e) and the fact that the total area under the graph of $P_y(x)$ and above the x-axis is $\sqrt{2\pi}\widehat{P_y}(0) = \sqrt{2\pi}e^{-0} = \sqrt{2\pi}$.

(f) If f is an integrable function and piecewise smooth, consider the Dirichlet problem with boundary values $f(x)$. Then we know that the solution is $u(x, y) = P_y * f(x)$. In particular, the solution tends to the boundary function as $y \to 0$. But this means that $\lim_{y\to 0} P_y * f(x) = f(x)$.

The proof of this fact is beyond the level of the text. Another way to justify the convergence is to take Fourier transforms. We have

$$\mathcal{F}(P_y * f)(w) = \mathcal{F}(P_y)(w)\cdot\mathcal{F}(f)(w) = e^{-y|w|}\mathcal{F}(f)(w).$$

Since $\lim_{y\to 0} e^{-y|w|} = 1$, it follows that

$$\lim_{y\to 0} \mathcal{F}(P_y * f)(w) = \lim_{y\to 0} e^{-y|w|}\mathcal{F}(f)(w) = \mathcal{F}(f)(w).$$

Taking inverse Fourier transforms, we see that $\lim_{y\to 0} P_y * f(x) = f(x)$.

The argument that we gave is not rigorous, since we did not justify that the inverse Fourier transform of a limit of functions is the limit of the inverse Fourier transforms.

Solutions to Exercises 7.6

1. The even extension of $f(x)$ is

$$f_e(x) = \begin{cases} 1 & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier transform of $f_e(x)$ is computed in Example 1, Sec. 7.2 (with $a = 1$). We have, for $w \geq 0$,

$$\mathcal{F}_c(f)(w) = \mathcal{F}(f_e)(w) = \sqrt{\frac{2}{\pi}}\frac{\sin w}{w}.$$

To write f as an inverse Fourier cosine transform, we appeal to (6). We have, for $x > 0$,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(f)(w)\cos wx\,dw,$$

or

$$\frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wx\,dw = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{if } x > 1, \\ \frac{1}{2} & \text{if } x = 1. \end{cases}$$

Note that at the point $x = 1$, a point of discontinuity of f, the inverse Fourier transform is equal to $(f(x+) + f(x-))/2$.

5. The even extension of $f(x)$ is

$$f_e(x) = \begin{cases} \cos x & \text{if } -2\pi < x < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

Let's compute the Fourier cosine transform using definition (5), Sec. 7.6:

$$\begin{aligned}
\mathcal{F}_c(f)(w) &= = \sqrt{\frac{2}{\pi}}\int_0^{2\pi} \cos x \cos wx\,dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^{2\pi} \frac{1}{2}\left[\cos(w+1)x + \cos(w-1)x\right]\,dx \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin(w+1)x}{w+1} + \frac{\sin(w-1)x}{w-1}\right]\Bigg|_0^{2\pi} \qquad (w \neq 1) \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin 2(w+1)\pi}{w+1} + \frac{\sin 2(w-1)\pi}{w-1}\right] \qquad (w \neq 1) \\
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{\sin 2\pi w}{w+1} + \frac{\sin 2\pi w}{w-1}\right] \qquad (w \neq 1) \\
&= \sqrt{\frac{2}{\pi}}\sin 2\pi w \frac{w}{w^2-1} \qquad (w \neq 1).
\end{aligned}$$

Also, by l'Hospital's rule, we have

$$\lim_{w \to 0}\sqrt{\frac{2}{\pi}}\sin 2\pi w \frac{w}{w^2-1} = \sqrt{2\pi},$$

which is the value of the cosine transform at $w = 1$.

To write f as an inverse Fourier cosine transform, we appeal to (6). We have, for $x > 0$,

$$\frac{2}{\pi}\int_0^\infty \frac{w}{w^2-1}\sin 2\pi w\,\cos wx\,dw = \begin{cases} \cos x & \text{if } 0 < x < 2\pi, \\ 0 & \text{if } x > 2\pi. \end{cases}$$

For $x = 2\pi$, the integral converges to 1/2. So

$$\frac{2}{\pi}\int_0^\infty \frac{w}{w^2-1}\sin 2\pi w\,\cos 2\pi w\,dw = \frac{1}{2}.$$

9. Applying the definition of the transform and using Exercise 17, Sec. 2.6 to evaluate the integral,

$$\begin{aligned}\mathcal{F}_s(e^{-2x})(w) &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\sin wx\,dx \\ &= \sqrt{\frac{2}{\pi}}\,\frac{e^{-2x}}{4+w^2}\left[-w\cos wx - 2\sin wx\right]\Big|_{x=0}^{\infty} \\ &= \sqrt{\frac{2}{\pi}}\frac{w}{4+w^2}.\end{aligned}$$

The inverse sine transform becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{w}{4+w^2}\sin wx\,dw.$$

13. We have $f_e(x) = \frac{1}{1+x^2}$. So

$$\mathcal{F}_c\left(\frac{1}{1+x^2}\right) = \mathcal{F}\left(\frac{1}{1+x^2}\right) = \sqrt{\frac{\pi}{2}}e^{-w} \quad (w>0),$$

by Exercise 11, Sec. 7.2.

17. We have $f_e(x) = \frac{\cos x}{1+x^2}$. So

$$\mathcal{F}_c\left(\frac{\cos x}{1+x^2}\right) = \mathcal{F}\left(\frac{\cos x}{1+x^2}\right) = \sqrt{\frac{\pi}{2}}\left(e^{-|w-1|} + e^{-(w+1)}\right) \quad (w>0),$$

by Exercises 11 and 20(b), Sec. 7.2.

21. From the definition of the inverse transform, we have $\mathcal{F}_c f = \mathcal{F}_c^{-1} f$. So $\mathcal{F}_c\mathcal{F}_c f = \mathcal{F}_c\mathcal{F}_c^{-1} f = f$. Similarly, $\mathcal{F}_s\mathcal{F}_s f = \mathcal{F}_s\mathcal{F}_s^{-1} f = f$.

Solutions to Exercises 7.7

1. Fourier sine transform with respect to x:

$$\tfrac{d}{dt}\widehat{u}_s(w,\,t) = -w^2\widehat{u}_s(w,\,t) + \sqrt{\tfrac{2}{\pi}}w\,\overbrace{u(0,\,t)}^{=0}$$

$$\tfrac{d}{dt}\widehat{u}_s(w,\,t) = -w^2\widehat{u}_s(w,\,t).$$

Solve the first-order differential equation in $\widehat{u}_s(w,\,t)$ and get

$$\widehat{u}_s(w,\,t) = A(w)e^{-w^2t}.$$

Fourier sine transform the initial condition

$$\widehat{u}_s(w,\,0) = A(w) = \mathcal{F}_s(f(x))(w) = T_0\sqrt{\frac{2}{\pi}}\frac{1-\cos bw}{w}.$$

Hence

$$\widehat{u}_s(w,\,t) = \sqrt{\frac{2}{\pi}}\frac{1-\cos bw}{w}e^{-w^2t}.$$

Taking inverse Fourier sine transform:

$$u(x,\,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos bw}{w}e^{-w^2t}\sin wx\,dw.$$

5. If you Fourier cosine the equations (1) and (2), using the Neumann type condition

$$\frac{\partial u}{\partial x}(0,t) = 0,$$

you will get

$$\tfrac{d}{dt}\widehat{u}_c(w,\,t) = c^2\Big[-w^2\widehat{u}_c(w,\,t) - \sqrt{\tfrac{2}{\pi}}\,\overbrace{\frac{d}{dx}u(0,\,t)}^{=0}\Big]$$

$$\tfrac{d}{dt}\widehat{u}_c(w,\,t) = -c^2w^2\widehat{u}_c(w,\,t).$$

Solve the first-order differential equation in $\widehat{u}_c(w,\,t)$ and get

$$\widehat{u}_c(w,\,t) = A(w)e^{-c^2w^2t}.$$

Fourier cosine transform the initial condition

$$\widehat{u}_c(w,\,0) = A(w) = \mathcal{F}_c(f)(w).$$

Hence

$$\widehat{u}_s(w,\,t) = \mathcal{F}_c(f)(w)e^{-c^2w^2t}.$$

Taking inverse Fourier cosine transform:

$$u(x,\,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \mathcal{F}_c(f)(w)e^{-c^2w^2t}\cos wx\,dw.$$

9. (a) Taking the sine transform of the heat equation (1) and using $u(0,t) = T_0$ for $t > 0$, we get

$$\frac{d}{dt}\,\widehat{u}_s(w,t) = c^2\Big[-w^2\widehat{u}_s(w,t) + \sqrt{\frac{2}{\pi}}wu(0,\,t)\Big];$$

or

$$\frac{d}{dt}\,\widehat{u}_s(w,t) + c^2\omega^2\widehat{u}_s(w,t) = c^2\sqrt{\frac{2}{\pi}}wT_0.$$

Taking the Fourier sine transform of the boundary condition $u(x,0) = 0$ for $x > 0$, we get $\widehat{u}_s(w,0) = 0$.
(b) A particular solution of the differential equation can be guessed easily: $\widehat{u}_s(w,\,t) = \sqrt{\frac{2}{\pi}}\,\frac{T_0}{w}$. The general solution of the homogeneous differential equation:

$$\frac{d}{dt}\,\widehat{u}_s(w,t) + c^2\omega^2\widehat{u}_s(w,t) = 0$$

is $\widehat{u}_s(w,\,t) = A(w)e^{-c^2w^2t}$. So the general solution of the nonhomogeneous differential equation is

$$\widehat{u}_s(w,\,t) = A(w)e^{-c^2w^2t}\sqrt{\frac{2}{\pi}}\,\frac{T_0}{w}.$$

Using $\widehat{u}_s(w,\,0) = A(w)\sqrt{\frac{2}{\pi}}\,\frac{T_0}{w} = 0$, we find $A(w) = -\sqrt{\frac{2}{\pi}}\frac{T_0}{w}$. So

$$\widehat{u}_s(w\,t) = \sqrt{\frac{2}{\pi}}\frac{T_0}{w} - \sqrt{\frac{2}{\pi}}\frac{T_0}{w}e^{-c^2w^2t}.$$

Taking inverse sine transforms, we find

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}\int_0^\infty\left(\frac{T_0}{w} - \frac{T_0}{w}e^{-c^2w^2t}\right)\sin wx\,dw \\ &= T_0\,\overbrace{\frac{2}{\pi}\int_0^\infty \frac{\sin wx}{w}\,dw}^{=\operatorname{sgn}(x)=1} - \frac{2T_0}{\pi}\int_0^\infty \frac{\sin wx}{w}e^{-c^2w^2t}\,dw \\ &= T_0 - \frac{2T_0}{\pi}\int_0^\infty \frac{\sin wx}{w}e^{-c^2w^2t}\,dw \end{aligned}$$

13. Proceed as in Exercise 11 using the Fourier sine transform instead of the cosine transform and the condition $u(x\,0) = 0$ instead of $u_y(x,\,0) = 0$. This yields

$$\tfrac{d^2}{dx^2}\widehat{u}_s(x,\,w) - w^2\widehat{u}_s(x,\,w) + \sqrt{\tfrac{2}{\pi}}\,\overbrace{u(x,\,0)}^{=0} = 0$$

$$\tfrac{d^2}{dx^2}\widehat{u}_s(x,\,w) = w^2\widehat{u}_s(x,\,w).$$

The general solution is

$$\widehat{u}_s(x,\,w) = A(w)\cosh wx + B(w)\sinh wx.$$

Using

$$\widehat{u}_s(0,\,w) = 0 \quad\text{and}\quad \widehat{u}_s(1,\,w) = \mathcal{F}_s(e^{-y}) = \sqrt{\frac{2}{\pi}}\frac{w}{1+w^2},$$

we get

$$A(w) = 0 \quad\text{and}\quad B(w) = \sqrt{\frac{2}{\pi}}\frac{w}{1+w^2}\cdot\frac{1}{\sinh w}.$$

Hence

$$\widehat{u}_s(x,\,w) = \sqrt{\frac{2}{\pi}}\frac{w}{1+w^2}\frac{\sinh wx}{\sinh w}.$$

Taking inverse sine transforms:

$$u(x,\,y) = \frac{2}{\pi}\int_0^\infty \frac{w}{1+w^2}\frac{\sinh wx}{\sinh w}\sin wy\,dw.$$

Solutions to Exercises 7.8

1. As we move from left to right at a point x_0, if the graph jumps by c units, then we must add the scaled Dirac delta function by $c\delta_{x_0}(x)$. If the jump is upward, c is positive; and if the jump is downward, c is negative. With this in mind, by looking at the graph, we see that

$$f'(x) = \frac{1}{2}\,\delta_{-2}(x) + \frac{1}{2}\,\delta_{-1}(x) - \frac{1}{2}\,\delta_1(x) - \frac{1}{2}\,\delta_2(x) = \frac{1}{2}\left(\delta_{-2}(x) + \delta_{-1}(x) - \delta_1(x) - \delta_2(x)\right).$$

13. We do this problem by reversing the steps in the solutions of the previous exercises. Since $f(x)$ has zero derivative for $x < -2$ or $x > 3$, it is therefore constant on these intervals. But since $f(x)$ tends to zero as $x \to \pm\infty$, we conclude that $f(x) = 0$ for $x < -2$ or $x > 3$. At $x = -2$, we have a jump upward by one unit, then the function stays constant for $-2 < x < -1$. At $x = -1$, we have another jump upward by one unit, then the function stays constant for $-1 < x < -1$. At $x = 1$, we have another jump upward by one unit, then the function stays constant for $1 < x < 3$. At $x = 3$, we have a jump downward by three units, then the function stays constant for $x > 3$. Summing up, we have

$$f(x) = \begin{cases} 0 & \text{if } x < -2, \\ 1 & \text{if } -2 < x < -1, \\ 2 & \text{if } -1 < x < 1, \\ 3 & \text{if } 1 < x < 3, \\ 0 & \text{if } 3 < x. \end{cases}$$

17. We use the definition (7) of the derivative of a generalized function and the fact that the integral against a delta function δ_a picks up the value of the function at a. Thus

$$\begin{aligned} \langle \phi'(x),\, f(x)\rangle &= \langle \phi(x),\, -f'(x)\rangle = -\langle \phi(x),\, f'(x)\rangle \\ &= -\langle \delta_0(x) - \delta_1(x),\, f'(x)\rangle = -f'(0) + f'(1). \end{aligned}$$

21. From Exercise 7, we have $\phi'(x) = \frac{1}{a}\big(\mathcal{U}_{-2a}(x) - \mathcal{U}_{-a}(x)\big) - \frac{1}{a}\big(\mathcal{U}_a(x) - \mathcal{U}_{2a}(x)\big)$. Using (9) (or arguing using jumps on the graph), we find

$$\phi''(x) = \frac{1}{a}\big(\delta_{-2a}(x) - \delta_{-a}(x)\big) - \frac{1}{a}\big(\delta_a(x) - \delta_{2a}(x)\big) = \frac{1}{a}\big(\delta_{-2a}(x) - \delta_{-a}(x) - \delta_a(x) + \delta_{2a}(x)\big).$$

25. Using the definition of ϕ and the definition of a derivative of a generalized function, and integrating by parts, we find

$$\begin{aligned}
\langle \phi'(x),\, f(x)\rangle &= -\langle \phi(x),\, f'(x)\rangle = -\int_{-\infty}^{\infty} \phi(x) f'(x)\,dx \\
&= -\int_{-1}^{0} 2(x+1) f'(x)\,dx - \int_{0}^{1} -2(x-1) f'(x)\,dx \\
&= -2(x+1)f(x)\Big|_{-1}^{0} + 2\int_{-1}^{0} f(x)\,dx + 2(x-1)f(x)\Big|_{0}^{1} - 2\int_{0}^{1} f(x)\,dx \\
&= -2f(0) + 2\int_{-1}^{0} f(x)\,dx + 2f(0) - 2\int_{0}^{1} f(x)\,dx \\
&= \langle 2\big(\mathcal{U}_{-1}(x) - \mathcal{U}_0(x)\big),\, f(x)\rangle - \langle 2\big(\mathcal{U}_0(x) - \mathcal{U}_1(x)\big),\, f(x)\rangle. \\
&= \langle 2\big(\mathcal{U}_{-1}(x) - \mathcal{U}_0(x)\big) - 2\big(\mathcal{U}_0(x) - \mathcal{U}_1(x)\big),\, f(x)\rangle.
\end{aligned}$$

Thus

$$\phi'(x) = 2\big(\mathcal{U}_{-1}(x) - \mathcal{U}_0(x)\big) - 2\big(\mathcal{U}_0(x) - \mathcal{U}_1(x)\big).$$

Reasoning similarly, we find

$$\begin{aligned}
\langle \phi''(x),\, f(x)\rangle &= -\langle \phi'(x),\, f'(x)\rangle = -\int_{-\infty}^{\infty} \phi'(x) f'(x)\,dx \\
&= -2\int_{-1}^{0} f'(x)\,dx + 2\int_{0}^{1} f'(x)\,dx \\
&= -2\big(f(0) - f(-1)\big) + 2\big(f(1) - f(0)\big) = 2f(-1) - 4f(0) + 2f(1) \\
&= \langle 2\delta_{-1} - 4\delta_0 + 2\delta_1,\, f(x)\rangle.
\end{aligned}$$

Thus

$$\phi''(x) = 2\delta_{-1} - 4\delta_0 + 2\delta_1.$$

29. We use (13) and the linearity of the Fourier transform:

$$\mathcal{F}\big(3\delta_0 - 2\delta_{-2}\big) = \frac{1}{\sqrt{2\pi}}\big(3 - 2e^{2iw}\big).$$

33. Using the operational property in Theorem 3(i), Section 7.2, we find

$$\begin{aligned}
\mathcal{F}\big(x\,(\mathcal{U}_{-1} - \mathcal{U}_1)\big) &= i\frac{d}{dw}\mathcal{F}\big(\mathcal{U}_{-1} - \mathcal{U}_1\big) \\
&= i\frac{d}{dw}\left[-\frac{i}{\sqrt{2\pi}\,w}e^{iw} + \frac{i}{\sqrt{2\pi}\,w}e^{-iw}\right] \\
&= \frac{-i(i)}{\sqrt{2\pi}}\frac{d}{dw}\left[\frac{e^{iw} - e^{-iw}}{w}\right] \\
&= \frac{1}{\sqrt{2\pi}}\frac{d}{dw}\left[\frac{2i\sin w}{w}\right] \qquad (\text{Recall } e^{iu} - e^{-iu} = 2i\sin u) \\
&= \frac{2i}{\sqrt{2\pi}}\left[\frac{w\,\cos w - \sin w}{w^2}\right] \\
&= i\sqrt{\frac{2}{\pi}}\left[\frac{w\,\cos w - \sin w}{w^2}\right].
\end{aligned}$$

The formula is good at $w=0$ if we take the limit as $w\to 0$. You will get

$$\mathcal{F}\big(x\,(\mathcal{U}_{-1}-\mathcal{U}_1)\big)=\lim_{w\to 0} i\sqrt{\frac{2}{\pi}}\left[\frac{w\,\cos w-\sin w}{w^2}\right]=i\sqrt{\frac{2}{\pi}}\lim_{w\to 0}\frac{-w\,\sin w}{2w}=0.$$

(Use l'Hospital's rue.) Unlike the Fourier transform in Exercise 31, the transform here is a nice continuous function. There is a major difference between the transforms of the two exercises. In Exercise 31, the function is not integrable and its Fourier transform exists only as a generalized function. In Exercise 33, the function is integrable and its Fourier transform exists in the usual sense of Section 7.2. In fact, look at the transform in Exercise 31, it is not even defined at $w=0$.

An alternative way to do this problem is to realize that

$$\phi'(x)=-\delta_{-1}-\delta_1+\mathcal{U}_{-1}-\mathcal{U}_1.$$

So

$$\begin{aligned}\mathcal{F}\big(\phi'(x)\big) &= \mathcal{F}\big(-\delta_{-1}-\delta_1+\mathcal{U}_{-1}-\mathcal{U}_1\big)\\ &= \frac{1}{\sqrt{2\pi}}\left(-e^{iw}-e^{-iw}-i\frac{e^{iw}}{w}+i\frac{e^{-iw}}{w}\right).\end{aligned}$$

But

$$\mathcal{F}\big(\phi'(x)\big)=i\,w\,\mathcal{F}\big(\phi(x)\big).$$

So

$$\begin{aligned}\mathcal{F}\big(\phi(x)\big) &= \frac{i}{w}\mathcal{F}\big(e^{iw}+e^{-iw}+i\frac{e^{iw}}{w}-i\frac{e^{-iw}}{w}\big)\\ &= \frac{i}{\sqrt{2\pi}\,w}\left[2\cos w+\frac{i}{w}(2i\sin w)\right]\\ &= i\sqrt{\frac{2}{\pi}}\left[\frac{w\cos w-\sin w}{w^2}\right].\end{aligned}$$

37. Write $\tau^2(x)=x^2\,\mathcal{U}_0(x)$, then use the operational properties

$$\begin{aligned}\mathcal{F}\left(\tau^2(x)\right) &= \mathcal{F}\left(x^2\,\mathcal{U}_0(x)\right)\\ &= -\frac{d^2}{dw^2}\mathcal{F}\left(\mathcal{U}_0(x)\right)\\ &= -\frac{-i}{\sqrt{2\pi}}\frac{d^2}{dw^2}\left[\frac{1}{w}\right]\\ &= \frac{2i}{\sqrt{2\pi}}\frac{1}{w^3}=i\sqrt{\frac{2}{\pi}}\frac{1}{w^3}.\end{aligned}$$

41. We have

$$f'(x)=-\mathcal{U}_{-2}(x)+2\mathcal{U}_{-1}(x)-\mathcal{U}_1(x)+\delta_{-1}(x)-2\,\delta_2(x).$$

So

$$\begin{aligned}\mathcal{F}\left(f'(x)\right) &= \mathcal{F}\left(-\mathcal{U}_{-2}(x)+2\mathcal{U}_{-1}(x)-\mathcal{U}_1(x)+\delta_{-1}(x)-2\,\delta_2(x)\right)\\ &= -\frac{-i}{\sqrt{2\pi}}\frac{e^{2iw}}{w}+2\frac{-i}{\sqrt{2\pi}}\frac{e^{iw}}{w}-\frac{-i}{\sqrt{2\pi}}\frac{e^{-iw}}{w}+\frac{1}{\sqrt{2\pi}}e^{iw}-\frac{2}{\sqrt{2\pi}}e^{-2iw}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{F}(f(x)) &= \frac{-i}{w}\mathcal{F}(f'(x)) \\ &= \frac{1}{\sqrt{2\pi}\,w}\left[\frac{e^{2iw}}{w} - 2\frac{e^{iw}}{w} + \frac{e^{-iw}}{w} - ie^{iw} + 2ie^{-2iw}\right].\end{aligned}$$

45. You may want to draw a graph to help you visualize the derivatives. For $f(x) = \sin x$ if $|x| < \pi$ and 0 otherwise, we have $f"(x) = \cos x$ if $|x| < \pi$ and 0 otherwise. Note that since f is continuous, we do not add delta functions at the endpoints $x = \pm\pi$ when computing f'. For f'', the graph is discontinuous at $x = \pm\pi$ and we have

$$f''(x) = -\delta_{-\pi} + \delta_\pi - \sin x$$

if $|x| \leq \pi$ and 0 otherwise. Thus

$$f''(x) = -\delta_{-\pi} + \delta_\pi - f(x) \quad \text{for all } x.$$

Taking the Fourier transform, we obtain

$$\begin{aligned}\mathcal{F}(f''(x)) &= \mathcal{F}(-\delta_{-\pi} + \delta_\pi - f(x))\,; \\ -w^2\mathcal{F}(f(x)) &= -\frac{1}{\sqrt{2\pi}}e^{i\pi w} + \frac{1}{\sqrt{2\pi}}e^{-i\pi w} - \mathcal{F}(f(x))\,;\end{aligned}$$

$$\Rightarrow\ (1+w^2)\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\overbrace{\left(e^{i\pi w} + e^{-i\pi w}\right)}^{=2\cos(\pi w)}$$

$$\Rightarrow\ \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\frac{\cos(\pi w)}{1+w^2}.$$

49. From Example 9, we have

$$f' = \delta_{-1} - \delta_1.$$

So from

$$\frac{d}{dx}\big(f * f\big) = \frac{df}{dx} * f$$

we have

$$\begin{aligned}\frac{d}{dx}\big(f * f\big) &= \frac{df}{dx} * f = \big(\delta_{-1} - \delta_1\big) * f; \\ &= \delta_{-1} * f - \delta_1 * f; = \frac{1}{\sqrt{2\pi}}\big(f(x+1) - f(x-1)\big).\end{aligned}$$

Using the explicit formula for f, we find

$$\frac{d}{dx}\big(f * f\big) = \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{if } -2 < x < 0, \\ -\frac{1}{\sqrt{2\pi}} & \text{if } 0 < x < 2, \\ 0 & \text{otherwise,}\end{cases}$$

as can be verified directly from the graph of $f * f$ in Figure 18.

53. Using (20), we have

$$\begin{aligned}\phi * \psi &= (\delta_{-1} + 2\delta_2) * (\delta_{-1} + 2\delta_2) \\ &= \delta_{-1} * \delta_{-1} + 2\delta_{-1} * \delta_2 + 2\delta_2 * \delta_{-1} + 4\delta_2 * \delta_2 \\ &= \frac{1}{\sqrt{2\pi}}\left[\delta_{-2} + 4\delta_1 + 4\delta_4\right].\end{aligned}$$

57. Following the method of Example 9, we have

$$\begin{aligned}
\frac{d}{dx}(\phi * \psi) &= \frac{d\psi}{dx} * \phi \\
&= (\delta_{-1} - \delta_1) * (\mathcal{U}_{-1} - \mathcal{U}_1 + \mathcal{U}_2 - \mathcal{U}_3) \\
&= \delta_{-1} * \mathcal{U}_{-1} - \delta_{-1} * \mathcal{U}_1 + \delta_{-1} * \mathcal{U}_2 - \delta_{-1} * \mathcal{U}_3 - \delta_1 * \mathcal{U}_{-1} \\
&\quad + \delta_1 * \mathcal{U}_1 - \delta_1 * \mathcal{U}_2 + \delta_1 * \mathcal{U}_3 \\
&= \frac{1}{\sqrt{2\pi}} (\mathcal{U}_{-2} - \mathcal{U}_0 + \mathcal{U}_1 - \mathcal{U}_2 - \mathcal{U}_0 + \mathcal{U}_2 - \mathcal{U}_3 + \mathcal{U}_4) \\
&= \frac{1}{\sqrt{2\pi}} ((\mathcal{U}_{-2} - \mathcal{U}_0) - (\mathcal{U}_0 - \mathcal{U}_1) - (\mathcal{U}_3 - \mathcal{U}_4)).
\end{aligned}$$

Integrating $\frac{d}{dx}(\phi * \psi)$ and using the fact that $\phi * \psi$ equal 0 for large $|x|$ and that there are no discontinuities on the graph, we find

$$\begin{aligned}
\phi * \psi(x) &= \begin{cases} \frac{1}{\sqrt{2\pi}}(x+2) & \text{if } -2 < x < 0 \\ \frac{1}{\sqrt{2\pi}}(-x+2) & \text{if } 0 < x < 1 \\ \frac{1}{\sqrt{2\pi}} & \text{if } 1 < x < 3 \\ \frac{1}{\sqrt{2\pi}}(-x+4) & \text{if } 3 < x < 4 \\ 0 & \text{otherwise} \end{cases} \\
&= \frac{1}{\sqrt{2\pi}}\Big((x+2)\big(\mathcal{U}_{-2} - \mathcal{U}_0\big) + (-x+2)\big(\mathcal{U}_0 - \mathcal{U}_1\big) \\
&\quad + \big(\mathcal{U}_1 - \mathcal{U}_3\big) + (-x+4)\big(\mathcal{U}_3 - \mathcal{U}_4\big)\Big).
\end{aligned}$$

Solutions to Exercises 7.9

1. Proceed as in Example 1 with $c = 1/2$. Equation (3) becomes in this case

$$\begin{aligned} u(x, t) &= \frac{2}{\sqrt{2t}} e^{-x^2/t} * \delta_1(x) \\ &= \frac{1}{\sqrt{\pi t}} e^{-(x-1)^2/t}, \end{aligned}$$

since the effect of convolution by δ_1 is to shift the function by 1 unit to the right and multiply by $\frac{1}{\sqrt{2\pi}}$.

5. We use the superposition principle (see the discussion preceeding Example 4). If ϕ is the solution of $u_t = \frac{1}{4} u_{xx} + \delta_0$, $u(x, 0) = 0$ and ψ is the solution of $u_t = \frac{1}{4} u_{xx}$, $u(x, 0) = \mathcal{U}_0(x)$, then you can check that $\phi + \psi$ is the solution of $u_t = \frac{1}{4} u_{xx} + \delta_0$, $u(x, 0) = \mathcal{U}_0(x)$. By Examples 1,

$$\phi(x, t) = \frac{2\sqrt{t}}{\sqrt{\pi}} e^{-x^2/t} - \frac{2|x|}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, \frac{x^2}{t}\right)$$

and by Exercise 20, Section 7.4,

$$\psi(x, t) = \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{t}}\right).$$

9. Apply Theorem 2 with $c = 1$ and $f(x, t) = \cos ax$; then

$$\begin{aligned} u(x, t) &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-s)}} e^{-(x-y)^2/(4(t-s))} \cos(ay)\, dyds \\ &= \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2(t-s)}} e^{-y^2/(4(t-s))} \cos(a(x-y))\, dyds \\ &\qquad \text{(Change variables } x - y \leftrightarrow y) \\ &= \cos(ax) \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2(t-s)}} e^{-y^2/(4(t-s))} \cos(ay)\, dyds \\ &\qquad \text{(Integral of odd function is 0.} \\ &= \cos(ax) \int_0^t e^{-a^2(t-s)} ds \quad \text{(Fourier transform of a Guaussian.)} \\ &= \frac{1}{a^2} \cos(ax)\big(1 - e^{-a^2 t}\big). \end{aligned}$$

Solutions to Exercises 7.10

1. Apply Proposition 1 with $f(x,\,s) = e^{-(x+s)^2}$, then

$$\begin{aligned}\frac{dU}{dx} &= f(x,\,x) + \int_0^x \frac{\partial}{\partial x} f(x,\,s)\,ds\\ &= e^{-4x^2} - \int_0^x 2(x+s)e^{-(x+s)^2}\,ds\\ &= e^{-4x^2} - \int_x^{2x} 2ve^{-v^2}\,dv \quad ((v = x+s)\\ &= e^{-4x^2} + e^{-v^2}\Big|_x^{2x} = 2e^{-4x^2} - e^{-x^2}.\end{aligned}$$

5. Following Theorem 1, we first solve $\phi_t = \phi_{xx}$, $\phi(x,\,0,\,s) = e^{-s}x^2$, where $s > 0$ is fixed. The solution is $\phi(x,\,t,\,s) = e^{-s}(2t + x^2)$ (see the solution of Exercise 5, Section 7.4). The the desired solution is given by

$$\begin{aligned}u(x,\,t) &= \int_0^t \phi(x,\,t-s,\,s)ds\\ &= \int_0^t e^{-s}(2(t-s) + x^2)ds\\ &= -2te^{-s} + 2se^{-s} + 2e^{-s} - x^2e^{-s}\Big|_0^t\\ &= -2 + 2t + x^2 + e^{-t}(2 - x^2).\end{aligned}$$

9. Following Theorem 2, we first solve $\phi_{tt} = \phi_{xx}$, $\phi(x,\,0,\,s) = 0$, $\phi_t(x,\,0,\,s) = \cos(s + x)$ where $s > 0$ is fixed. By d'Alembert's method, the solution is

$$\phi(x,\,t,\,s) = \frac{1}{2}\int_{x-t}^{x+t} \cos(s+y)\,dy = \frac{1}{2}\big[\sin(s + x + t) - \sin(s + x - t)\big].$$

The the desired solution is given by

$$\begin{aligned}u(x,\,t) &= \int_0^t \phi(x,\,t-s,\,s)ds\\ &= \frac{1}{2}\int_0^t \big[\sin(x+t) - \sin(x - t + 2s)\big]ds\\ &= \frac{1}{2}\Big[s\sin(x+t) + \frac{1}{2}\cos(x - t + 2s)\Big]\Big|_0^t\\ &= \frac{1}{2}\Big[t\sin(x+t) + \frac{1}{2}\cos(x+t) - \frac{1}{2}\cos(x-t)\Big].\end{aligned}$$

13. start by solving $\phi_{tt} = \phi_{xx}$, $\phi(x,\,0,\,s) = 0$, $\phi_t(x,\,0,\,s) = \delta_0(x)$ where $s > 0$ is fixed. By d'Alembert's method, the solution is

$$\phi(x,\,t,\,s) = \frac{1}{2}\int_{x-t}^{x+t} \delta_0(y)\,dy = \frac{1}{2}\big[\mathcal{U}_0(x+t) - \mathcal{U}_0(x-t)\big].$$

By Theorem 2, the solution of $u_{tt} = u_{xx} + \delta_0(x)$, $u(x,\,0) = 0$, $\phi_t(x,\,0) = 0$ is given by

$$
\begin{aligned}
\phi(x,\,t) &= \int_0^t \phi(x,\,t-s,\,s)ds \\
&= \frac{1}{2}\int_0^t \big[\mathcal{U}_0(x+t-s)-\mathcal{U}_0(x-t+s)\big]ds \\
&= \frac{1}{2}\big[-\tau(x+t-s)-\tau(x-t+s)\big]\Big|_0^t \\
&= -\tau(x)+\frac{1}{2}\big[\tau(x+t)+\tau(x-t)\big],
\end{aligned}
$$

where $\tau = \tau_0$ is the antiderivative of $\mathcal{U}_0$ described in Example 2, Section 7.8,

Solutions to Exercises 8.1

1. $|11\cos 3t| \leq 11$, so (2) holds if you take $M = 11$ and a any positive number, say $a = 1$. Note that (2) also holds with $a = 0$.

5. $|\sinh 3t| = |(e^{3t} - e^{-3t})/2| \leq (e^{3t} + e^{3t})/2 = e^{3t}$. So (2) holds with $M = 1$ and $a = 3$.

9. Using linearity of the Laplace transform and results from Examples 1 and 2, we have

$$\begin{aligned}\mathcal{L}(\sqrt{t} + \frac{1}{\sqrt{t}})(s) &= \mathcal{L}(t^{1/2}) + \mathcal{L}(t^{-1/2}) \\ &= \frac{\Gamma(3/2)}{s^{3/2}} + \frac{\Gamma(1/2)}{s^{1/2}}\end{aligned}$$

Now $\Gamma(1/2) = \sqrt{\pi}$, so $\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2$. Thus

$$\mathcal{L}(\sqrt{t} + \frac{1}{\sqrt{t}})(s) = \frac{\sqrt{\pi}}{2s^{3/2}} + \sqrt{\frac{\pi}{s}}.$$

13. Use Example 3 and Theorem 4:

$$\begin{aligned}\mathcal{L}(t\,\sin 4t)(s) &= -\frac{d}{ds}\mathcal{L}(\sin(4t)) = -\frac{d}{ds}\frac{4}{s^2 + 4^2} \\ &= \frac{8s}{(s^2 + 4^2)^2}\end{aligned}$$

17. We have

$$\mathcal{L}(e^{2t}\sin 3t)(s) = \mathcal{L}(\sin 3t)(s-2) = \frac{3}{(s-2)^2 + 3^2}\frac{3}{(s-2)^2 + 9}$$

21. We have

$$\begin{aligned}\mathcal{L}((t+2)^2\cos t)(s) &= \mathcal{L}((t^2 + 4t + 4)\cos t)(s) \\ &= \mathcal{L}(t^2\cos t)(s) + 4\mathcal{L}(t\cos t)(s) + 4\mathcal{L}(\cos t)(s) \\ &= \frac{d^2}{ds^2}\mathcal{L}(\cos t)(s) - 4\frac{d}{ds}\left[\mathcal{L}(\cos t)(s)\right] + 4\mathcal{L}(\cos t)(s) \\ &= \frac{d^2}{ds^2}\frac{s}{s^2+1} - 4\frac{d}{ds}\left[\frac{s}{s^2+1}\right] + \frac{4s}{s^2+1} \\ &= \frac{d}{ds}\frac{-s^2+1}{(s^2+1)^2} - 4\frac{-s^2+1}{(s^2+1)^2} + \frac{4s}{s^2+1} \\ &= \frac{2s(s^2-3)}{(s^2+1)^3} + 4\frac{s^2-1}{(s^2+1)^2} + \frac{4s}{s^2+1}\end{aligned}$$

25. Since

$$\mathcal{L}(t) = \frac{1}{s^2};$$

then

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

29. Using

$$\begin{aligned}\mathcal{L}\left(e^{at}t^n\right) &= \frac{n!}{(s-a)^{n+1}},\\ \mathcal{L}\left(\frac{e^{at}t^n}{n!}\right) &= \frac{1}{(s-a)^{n+1}},\\ \mathcal{L}\left(e^{at}\cos bt\right) &= \frac{s-a}{(s-a)^2+b^2},\end{aligned}$$

we find that

$$f(t) = \frac{e^{3t}t^4}{4!} + e^{3t}\cos t.$$

33. Partial fractions:

$$\begin{aligned}\frac{2s-1}{s^2-s-2} &= \frac{2s-1}{(s-2)(s+1)}\\ &= \frac{A}{s-2} + \frac{B}{s+1};\\ 2s-1 &= A(s+1)+B(s-2)\end{aligned}$$

Take particular values of s:

$$\begin{aligned}s=-1 \quad &\Rightarrow \quad -3=-3B \;\Rightarrow\; B=1\\ s=2 \quad &\Rightarrow \quad 3=3A \;\Rightarrow\; A=1\end{aligned}$$

So

$$\begin{aligned}F(s) &= = \frac{1}{s-2} + \frac{1}{s+1};\\ f(t) &= e^{2t}+e^{-t}\end{aligned}$$

37. Partial fractions:

$$\begin{aligned}\frac{1}{s^2+3s+2)} &= \frac{A}{s+2} + \frac{B}{s+1};\\ 1 &= A(s+1)+B(s+2)\\ F(s) &= \frac{-1}{s+2} + \frac{1}{s+1};\\ f(t) &= -e^{-2t}+e^{-t}\end{aligned}$$

41. The change of variables $\tau = t-\pi$ transforms the initial value problem $y''+y = \cos t, \quad y(\pi)=0,\ y'(\pi)=0$, into

$$y''+y=\cos(\tau+\pi) = -\cos\tau, \quad y(0)=0,\ y'(0)=0$$

Laplace transformt the equation and use the given initial conditions:

$$\begin{aligned}
\mathcal{L}(y'') + \mathcal{L}(y) &= \mathcal{L}(-\cos\tau) \\
s^2Y - sy(0) - y'(0) + Y &= -\frac{s}{s^2+1} \\
Y(s^2+1) &= = -\frac{s}{s^2+1} \\
Y &= -\frac{s}{(s^2+1)^2} = \frac{1}{2}\frac{d}{ds}\frac{1}{s^2+1} \\
y &= -\frac{1}{2}\tau\sin\tau = -\frac{1}{2}(t-\pi)\sin(t-\pi) \\
&= \frac{1}{2}(t-\pi)\sin t
\end{aligned}$$

45. Laplace transform the equation $y'' - y' - 6y = e^t \cos t$ and use the initial conditions $y(0) = 0,\ y'(0) = 1$:

$$\begin{aligned}
s^2Y - sy(0) - y'(0) - sY + y(0) - 6Y &= \frac{s-1}{(s-1)^2+1}; \\
Y(s^2 - s - 6) &= 1 + \frac{s-1}{(s-1)^2+1};
\end{aligned}$$

So

$$\begin{aligned}
Y &= \frac{1}{(s-3)(s+2)} + \frac{1}{(s-3)(s+2)}\frac{s-1}{(s-1)^2+1}; \\
Y &= \frac{-1}{5(s+2)} + \frac{1}{5(s-3)} + \frac{1}{(s-3)(s+2)}\frac{s-1}{s^2-2s+2} \\
&= \frac{-1}{5(s+2)} + \frac{1}{5(s-3)} + \frac{1}{(s+2)(s^2-2s+2)} + \frac{2}{(s-3)(s+2)(s^2-2s+2)}
\end{aligned}$$

We now find the partial frctions decomposition of the last term on the right. Write

$$\begin{aligned}
\frac{1}{(s+2)(s^2-2s+2)} &= \frac{A}{s+2} + + \frac{Bs+C}{s^2-2s+2}; \\
1 &= A(s^2-2s+2) + (Bs+C)(s+2); \\
s = -2 &\Rightarrow 1 = 10A \Rightarrow A = 1/10 \\
\text{constant term} &\Rightarrow 1 = \frac{1}{5} + 2C \Rightarrow C = 2/5; \\
\text{coefficient of } s^2 &\Rightarrow 0 = \frac{1}{10} + B \Rightarrow B = -1/10; \\
\frac{1}{(s+2)(s^2-2s+2)} &= \frac{1}{10(s+2)} + + \frac{-s+4}{10(s^2-2s+2)} \\
&= Y_1.
\end{aligned}$$

Also

$$
\begin{aligned}
\frac{2}{(s-3)(s+2)(s^2-2s+2)} &= \frac{A}{s-3} + + \frac{B}{s+2} + \frac{CS+D}{(s-3)(s+2)(s^2-2s+2)}; \\
2 &= A(s+2)(s^2-2s+2) + B(s-3)(s^2-2s+2) \\
&\quad +(CS+D)(s-3)(s+2); \\
s=-2 &\Rightarrow B = -1/25 \\
s=-3 &\Rightarrow A = 2/25 \\
\text{constant term} &\Rightarrow 2 = \frac{8}{25} + \frac{6}{25} - 6D \;\Rightarrow\; D = -6/25; \\
\text{coefficient of } s^3 &\Rightarrow C = -1/25; \\
\frac{2}{(s-3)(s+2)(s^2-2s+2)} &= \frac{2}{25(s-3)} - \frac{1}{25(s+2)} + \frac{-s-6}{25(s^2-2s+2)} \\
&= Y_2;
\end{aligned}
$$

Now

$$
\begin{aligned}
Y &= \frac{-1}{5(s+2)} + \frac{1}{5(s-3)} + Y_1 + Y_2 \\
y &= -\frac{7}{50}e^{-2t} + \frac{7}{25}e^{3t} - \frac{7}{50}e^t \cos t + \frac{1}{50}e^t \sin t.
\end{aligned}
$$

Solutions to Exercises 8.2

1. To compute the Laplace transform of $f(t) = \mathcal{U}_0(t-1) - t + 1$, use

$$\mathcal{L}\left[\mathcal{U}_0(t-a)\right](s) = \frac{e^{-as}}{s};$$

so

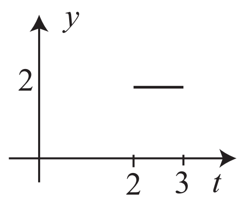
$$\begin{aligned} \mathcal{L}\left[\mathcal{U}_0(t-1) - t + 1\right](s) &= \mathcal{L}\left[\mathcal{U}_0(t-1)\right] - \mathcal{L}\left[t\right] + \mathcal{L}\left[1\right] \\ &= \frac{e^{-s}}{s} - \frac{1}{s^2} + \frac{1}{s}. \end{aligned}$$

5. Use the identity $\sin t = -\sin(t-\pi)$. Then

$$\begin{aligned} \mathcal{L}\left[\sin t\, \mathcal{U}_0(t-\pi)\right](s) &= -\mathcal{L}\left[\sin(t-\pi)\,\mathcal{U}_0(t-\pi)\right](s) \\ &= -e^{-\pi s}\mathcal{L}\left[\sin t\right](s) = \frac{-e^{-\pi s}}{s^2+1}. \end{aligned}$$

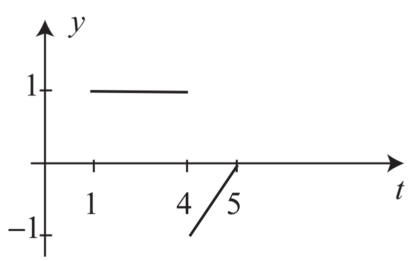
9.

$$\begin{aligned} y &= 2\left(\mathcal{U}_0(t-2) - \mathcal{U}_0(t-3)\right); \\ Y &= 2\frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} \end{aligned}$$



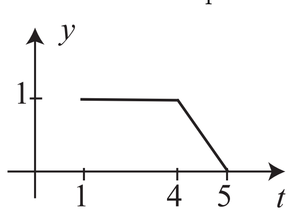
13.

$$\begin{aligned} y &= \left(\mathcal{U}_0(t-1) - \mathcal{U}_0(t-4)\right) + (t-5)\left(\mathcal{U}_0(t-4) - \mathcal{U}_0(t-5)\right) \\ &= \mathcal{U}_0(t-1) - \mathcal{U}_0(t-4) + (t-5)\mathcal{U}_0(t-5) + (t-4)\mathcal{U}_0(t-4) - \mathcal{U}_0(t-4); \\ Y &= \frac{e^{-s}}{s} - 2\frac{e^{-4s}}{s} + \frac{e^{-4s}}{s^2} - \frac{e^{-5s}}{s^2} \end{aligned}$$



The following is a variation on Exercise 13.

13 bis. Find the Laplace transform of the function in the picture



$$\begin{aligned} y &= (\mathcal{U}_0(t-1) - \mathcal{U}_0(t-4)) + (5-t)\left(\mathcal{U}_0(t-4) - \mathcal{U}_0(t-5)\right) \\ &= \mathcal{U}_0(t-1) - \mathcal{U}_0(t-4) + (t-5)\mathcal{U}_0(t-5) - (t-5)\mathcal{U}_0(t-4) \\ &= \mathcal{U}_0(t-1) - \mathcal{U}_0(t-4) + (t-5)\mathcal{U}_0(t-5) - (t-4)\mathcal{U}_0(t-4) + \mathcal{U}_0(t-4); \\ Y &= \frac{e^{-s}}{s} - \frac{e^{-4s}}{s} + \frac{e^{-5s}}{s^2} - \frac{e^{-4s}}{s^2} + \frac{e^{-4s}}{s} \\ &= \frac{e^{-s}}{s} + \frac{1}{s^2}\left(e^{-5s} - e^{-4s}\right) \end{aligned}$$

17. Let $y(t) = \sin t$, then $Y(s) = \frac{1}{s^2+1}$; so if

$$f(t) = \mathcal{U}_0(t-1)\sin(t-1),$$

then $F(s) = \dfrac{e^{-s}}{s^2+1}$.

21. Let $y(t) = \sqrt{t}$, then $Y(s) = \frac{\Gamma(3/2)}{s^{3/2}}$, where $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\pi}{2}$. If

$$\phi(t) = \frac{2}{\pi}\sqrt{t} \Rightarrow \Phi(s) = \frac{1}{s^{3/2}};$$

and so if

$$f(t) = \frac{2}{\pi}\sqrt{t-1}\,\mathcal{U}_0(t-1) \text{ then } F(s) = \frac{e^{-s}}{s^{3/2}}.$$

25. We will compute $t * t$ in two different ways. First method: We have

$$\begin{aligned} t &\xrightarrow{\mathcal{L}} \frac{1}{s^2}; \\ t &\xrightarrow{\mathcal{L}} \frac{1}{s^2}; \\ t * t &\xrightarrow{\mathcal{L}} \frac{1}{s^2}\cdot\frac{1}{s^2} = \frac{1}{s^4} \\ \frac{1}{s^4} &\xrightarrow{\mathcal{L}^{-1}} \frac{t^3}{6} = t * t. \end{aligned}$$

Alternatively,

$$\begin{aligned} t * t &= \int_0^t (t-\tau)\tau\, d\tau \\ &= t\frac{1}{2}\tau^2 - \frac{1}{3}\tau^3\Big|_0^t = \frac{1}{2}t^3 - \frac{1}{3}t^3 = \frac{1}{6}t^3. \end{aligned}$$

29. We have $F(s) = \frac{1}{s(s^2+1)} = \frac{1}{s}\cdot\frac{1}{s^2+1}$, and

$$\begin{aligned} \frac{1}{s} &\xrightarrow{\mathcal{L}^{-1}} 1; \\ \frac{1}{s^2+1} &\xrightarrow{\mathcal{L}^{-1}} \sin t. \end{aligned}$$

So

$$f(t) = 1 * \sin t = \int_0^t \sin\tau\, d\tau = 1 - \cos t.$$

33. Passing the equations $y'' + y = \delta_0(t-1), y(0) = 0,\ y'(0) = 0$, through the Laplace transform, we get:

$$\begin{aligned} s^2Y - sy(0) - y'(0) + Y &= e^{-s} \\ (s^2+1)Y &= e^{-s} \\ Y &= \frac{e^{-s}}{s^2+1}. \end{aligned}$$

Thus the solution is

$$y = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+1}\right).$$

To compute this inverse transform, we observe

$$\begin{aligned} \sin t &\xrightarrow{\mathcal{L}} \frac{1}{s^2+1}; \\ \mathcal{U}_0(t-1)\sin(t-1) &\xrightarrow{\mathcal{L}} \frac{e^{-s}}{s^2+1}; \end{aligned}$$

so $y = \mathcal{U}_0(t-1)\sin(t-1)$.

37. Take the Laplace transform on both sides of $y'' + 4y = \mathcal{U}_0(t-1)e^{t-1}$ and use the initial conditions $y(0) = 1,\ y'(0) = 0$, and you will get

$$\begin{aligned} \mathcal{L}(y'' + 4y) &= \mathcal{L}\left(\mathcal{U}_0(t-1)e^{t-1}\right) \\ \mathcal{L}(y'') + \mathcal{L}(4y) &= \mathcal{L}\left(\mathcal{U}_0(t-1)e^{t-1}\right); \\ s^2Y - sy(0) - y'(0) + 4Y &= e^{-s}\frac{1}{s-1} \quad (s>1) \\ s^2Y + 4Y &= \frac{e^{-s}}{s-1} \\ Y(4+s^2) &= \frac{e^{-s}}{s-1}; \\ Y &= \frac{e^{-s}}{(s-1)(s^2+4)}, \end{aligned}$$

where we have used Theorem 1, Sec. 8.2. Thus the solution is the inverse Laplace transform of

$$\frac{e^{-s}}{(s-1)(s^2+4)}.$$

Use partial fractions

$$\begin{aligned} \frac{1}{(s-1)(s^2+4)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+4} \\ &= \frac{A(s^2+4) + (s-1)(Bs+C)}{(s-1)(s^2+4)} \\ 1 &= A(s^2+4) + (s-1)(Bs+C); \\ \text{Set } s=1 &\Rightarrow 1 = 5A,\ A = \frac{1}{5}; \\ \text{Set } s=2i &\Rightarrow 1 = (2i-1)(B(2i)+C); \\ \text{Set } s=-2i &\Rightarrow 1 = (-2i-1)(B(-2i)+C); \\ &\Rightarrow B = -\frac{1}{5},\ C = -\frac{1}{5}. \end{aligned}$$

Hence

$$\frac{1}{(s-1)(s^2+4)} = \frac{1}{5}\left[\frac{1}{s-1} - \frac{s+1}{s^2+4}\right]$$

and so

$$\frac{e^{-s}}{(s-1)(s^2+4)} = \frac{e^{-s}}{5}\left[\frac{1}{s-1} - \frac{s+1}{s^2+4}\right]$$

You can use the Table of Laplace transforms at the end of the book to verify the following computations:

$$\begin{aligned}
e^t &\xrightarrow{\mathcal{L}} \frac{1}{s-1}\\
\sin(2t) &\xrightarrow{\mathcal{L}} \frac{2}{s^2+4};\\
\frac{1}{2}\sin(2t) &\xrightarrow{\mathcal{L}} \frac{1}{4+s^2};\\
\cos(2t) &\xrightarrow{\mathcal{L}} \frac{s}{s^2+4};\\
\frac{1}{5}\left[e^t - \frac{1}{2}\sin(2t) - \cos(2t)\right] &\xrightarrow{\mathcal{L}} \frac{1}{5}\left[\frac{1}{s-1} - \frac{s+1}{s^2+4}\right]\\
\mathcal{U}_0(t-a)f(t-a) &\xrightarrow{\mathcal{L}} e^{-as}F(s);\\
\frac{1}{5}\left[e^{t-1} - \frac{1}{2}\sin(2(t-1)) - \cos(2(t-1))\right]\mathcal{U}_0(t-1) &\xrightarrow{\mathcal{L}} \frac{e^{-s}}{5}\left[\frac{1}{s-1} - \frac{s+1}{s^2+4}\right]
\end{aligned}$$

From this we derive the solution

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{5}\left[\frac{1}{s-1} - \frac{s+1}{s^2+4}\right]\right) = \frac{1}{5}\left[e^{t-1} - \frac{1}{2}\sin(2(t-1)) - \cos(2(t-1))\right]\mathcal{U}_0(t-1).$$

41. Taking the Laplace transform of the equations $y'' + 4y = \cos t,\ y(0) = 0,\ y'(0) = 0$, we obtain

$$\begin{aligned}
s^2Y + 4Y &= \frac{s}{s^2+1}\\
Y &= \frac{1}{2}\frac{s}{s^2+1}\cdot\frac{2}{s^2+2^2}.
\end{aligned}$$

So

$$y = \frac{1}{2}\cos t * \sin(2t).$$

45. $T = 2$, for $0 < t < 1$, $f(t) = t$ and for $1 < t < 2$, $f(t) = 2 - t$; so, by the previous exercise,

$$\begin{aligned}
F(s) &= \frac{1}{1-e^{-2s}}\left[\int_0^1 te^{-st}\,dt + \int_0^1 (2-t)e^{-st}\,dt\right]\\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-st}}{s}(2-t)\Big|_1^2 - \frac{1}{s}\int_1^2 e^{-st}dt\right]\\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - 2\frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right]\\
&= \frac{e^{2s}}{(e^{2s}-1)s^2}\left[1 - 2e^{-s} + e^{-2s}\right]\\
&= \frac{(e^s-1)^2}{(e^s-1)(e^s+1)s^2} = \frac{e^s-1}{(e^s+1)s^2}
\end{aligned}$$

49. We have

$$\sin t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} \qquad \text{for all } t.$$

So for $t \neq 0$, we divide by t both sides and get

$$\frac{\sin t}{t} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k+1)!} \qquad \text{for all } t \neq 0.$$

As $t \to 0$, the left side approaches 1. The right side is continuous, and so as $t \to 0$, it approaches the value at 0, which is 1. Hence both sides of the equality approach 1 as $t \to 0$, and so we may take the expansion to be valid for all t. Apply the result of the previous exercise, then

$$\begin{aligned}
\mathcal{L}(f(t))(s) &= \mathcal{L}\left(\sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k+1)!}\right)(s) \\
&= \sum_{k=0}^{\infty} (-1)^k (2k)! \frac{1}{s^{2k+1}(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{s^{2k+1}(2k+1)}.
\end{aligned}$$

Recall the expansion of the inverse tangent:

$$\tan^{-1} u = \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k+1}}{2k+1} \qquad |u| < 1.$$

So

$$\tan^{-1} \frac{1}{s} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{s^{2k+1}(2k+1)} \qquad |s| > 1.$$

Comparing series, we find that, for $s > 1$,

$$\mathcal{L}(\frac{\sin t}{t}) = \tan^{-1}(\frac{1}{s}).$$

The formula is in fact valid valid for all $s > 0$. See Exercise 56.

53. We have

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{a\sqrt{\pi}} e^{-\frac{t^2}{4a^2}}\right)(s) &= \frac{1}{a\sqrt{\pi}} \int_0^{\infty} e^{-st} e^{-\frac{t^2}{4a^2}}\, dt \\
&= \frac{1}{a\sqrt{\pi}} \int_0^{\infty} e^{-\left(\frac{t}{2a}+as\right)^2 + a^2 s^2}\, dt \\
&= \frac{1}{a\sqrt{\pi}} e^{a^2 s^2} \int_0^{\infty} e^{-\left(\frac{t}{2a}+as\right)^2}\, dt \\
&\quad (\text{Let } T = \frac{t}{2a} + as,\ dT = \frac{1}{2a} dt.) \\
&= \frac{1}{\sqrt{\pi}} e^{a^2 s^2} \int_{as}^{\infty} e^{-T^2}\, 2\, dT \\
&= e^{a^2 s^2} \frac{2}{\sqrt{\pi}} \int_{as}^{\infty} e^{-t^2}\, dt \\
&= e^{a^2 s^2} \operatorname{erfc}(as).
\end{aligned}$$

54. Note that, for $a > 0$,

$$\begin{aligned}
\mathcal{L}(f(at))(s) &= \int_0^{\infty} e^{-st} f(at)\, dt \\
&= \frac{1}{a} \int_0^{\infty} e^{-\frac{s}{a} t} f(t)\, dt = \frac{1}{a} \mathcal{L}(f(t))\left(\frac{s}{a}\right).
\end{aligned}$$

Using this and Exercise 52, we find

$$\mathcal{L}\left(\operatorname{erf}(at)\right)(s) = \frac{1}{a}\frac{1}{\frac{s}{a}}e^{\frac{s^2}{4a^2}}\operatorname{erfc}\left(\frac{s}{2a}\right) = \frac{1}{s}e^{\frac{s^2}{4a^2}}\operatorname{erfc}\left(\frac{s}{2a}\right).$$

57. Bessel's equation of order 0 is

$$xy'' + y' + xy = 0.$$

Applying the Laplace transform, we obtain

$$\begin{aligned}
\mathcal{L}(xy'') + \mathcal{L}(y') + \mathcal{L}(xy) &= 0; \\
-\frac{d}{ds}\mathcal{L}(y'') + \mathcal{L}(y') - \frac{d}{ds}\mathcal{L}(y) &= 0 \\
-\frac{d}{ds}\Big[s^2Y - sy(0) - y'(0)\Big] + sY - y(0) - Y' &= 0 \\
\Big[-2sY - s^2Y' + sy(0)\Big] + sY - y(0) - Y' &= 0 \\
-Y'(1+s^2) - sY &= 0 \\
Y' + \frac{s}{1+s^2}Y &= 0.
\end{aligned}$$

An integating factor for this first order linear differential equation is

$$e^{\int \frac{s}{1+s^2}\,ds} = e^{\frac{1}{2}\ln(1+s^2)} = \sqrt{1+s^2}$$

After multiplyig by the integrating factor, the equation becmes

$$\frac{d}{ds}[\sqrt{1+s^2}Y] = 0\,.$$

Integrating both sides, we get

$$\sqrt{1+s^2}Y = K$$

or

$$Y = \frac{K}{\sqrt{1+s^2}},$$

where K is a constant.
From Exercise 50, $y = KJ_0(t)$.

Solutions to Exercises 8.3

1. The solution is the same as Example 2. Simply take $T_0 = 70$ in that example.

5. Using the formula from Example 3, we get

$$\begin{aligned}
u(x, t) &= \int_0^t (t-\tau)[\tau - (t-x)\,\mathcal{U}_0(\tau - x)]\,d\tau \\
&= \int_0^t (t-\tau)\tau\,d\tau - \int_0^t (t-\tau)(\tau - x)(\mathcal{U}_0(\tau - x)\,d\tau \\
&= \frac{t}{2}\tau^2 - \frac{1}{3}\tau^3\Big|_0^t - \int_0^t (t-\tau)(\tau - x)\,\mathcal{U}_0(\tau - x)\,d\tau \\
&= \frac{t^3}{3!} - \int_0^t (t-\tau)(\tau - x)\,\mathcal{U}_0(\tau - x)\,d\tau
\end{aligned}$$

Note that $\mathcal{U}_0(\tau - x) = 1$ if $\tau > x$ and 0 if $\tau < x$. So the integral is 0 if $t < x$ (since in this case $\tau \le t < x$). If $x < \tau < t$, then

$$\begin{aligned}
&\int_0^t (t-\tau)(\tau - x)\,\mathcal{U}_0(\tau - x)\,d\tau \\
&= \int_x^t (t-\tau)(\tau - x)\,d\tau \\
&= \int_x^t (t\tau - tx - \tau^2 + \tau x)\,d\tau \\
&= \frac{1}{2}t\tau^2 - tx\tau - \frac{1}{3}\tau^3 + \frac{1}{2}\tau^2 x\Big|_x^t \\
&= \frac{1}{2}t^3 - t^2 x - \frac{1}{3}t^3 + \frac{1}{2}t^2 x - \frac{1}{2}tx^2 + tx^2 + \frac{1}{3}x^3 - \frac{1}{2}x^3 \\
&= \frac{1}{6}t^3 - \frac{1}{2}t^2 x + \frac{1}{2}tx^2 - \frac{1}{6}x^3 \\
&= \frac{1}{6}(t-x)^3
\end{aligned}$$

Hence

$$u(x, t) = \begin{cases} \frac{1}{6}t^3 & \text{if } t < x \\ \frac{1}{6}t^3 - \frac{1}{6}(t-x)^3 & \text{if } t > x, \end{cases}$$

or

$$u(x, t) = \frac{1}{6}t^3 - \frac{1}{6}(t-x)^3\,\mathcal{U}_0(t - x).$$

9. Transforming the problem, we find (see Exercise 7 for similar details)

$$\begin{aligned}
s^2U(x,\, s) - su(x,\, 0) - u_t(x,\, 0) &= U_{xx}(x,\, 0);\\
s^2U(x,\, s) - 1 &= U_{xx}(x,\, 0);\\
U_{xx}(x,\, 0) - s^2U(x,\, s) &= -1;\\
U(x,\, s) &= A(s)e^{-sx} + \frac{1}{s^2};\\
U(0,\, s) &= \mathcal{L}(\sin t) = \frac{1}{1+s^2};\\
&\Rightarrow \quad A(s) + \frac{1}{s^2} = \frac{1}{1+s^2}\\
&\Rightarrow \quad A(s) = \frac{1}{1+s^2} - \frac{1}{s^2}\\
U(x,\, s) &= \left[\frac{1}{1+s^2} - \frac{1}{s^2}\right] e^{-sx} + \frac{1}{s^2}\\
u(x,\, t) &= t - (t-x)\,\mathcal{U}_0(t-x) + \sin(t-x)\,\mathcal{U}_0(t-x).
\end{aligned}$$

13. Verify that

$$u(x,\, t) = u_1(x,\, t) + u_2(x,\, t),$$

where u_1 is a solution of

$$\begin{aligned}
u_t &= u_{xx};\\
u(0,\, t) &= 70;\\
u(x,\, 0) &= 70;
\end{aligned}$$

and u_2 is a solution of

$$\begin{aligned}
u_t &= u_{xx};\\
u(0,\, t) &= 30;\\
u(x,\, 0) &= 0.
\end{aligned}$$

It is immediate that the solution of the first problem is $u_1 = 70$. The solution of the second problem is similar to Example 2:

$$u_2(x,\, t) = 30\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

Solutions to Exercises 8.4

1. (a) Let $z^2 = x$, $\tilde{u}(z, t) = u(z^2, t) = u(x, t)$. Then

$$2z\frac{dz}{dx} = 1 \quad \text{or} \quad \frac{dz}{dx} = \frac{1}{2z}.$$

So

$$\frac{\partial}{\partial x}u(x, t) = \frac{\partial}{\partial x}\tilde{u}(z, t) = \frac{\partial \tilde{u}}{\partial z}\frac{dz}{dx} = \frac{\partial \tilde{u}}{\partial z}\frac{1}{2z}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x}\left[\frac{\partial \tilde{u}}{\partial z}\frac{1}{2z}\right] \\
&= \frac{d}{dx}\left(\frac{1}{2z}\right)\frac{\partial \tilde{u}}{\partial z} + \frac{1}{2z}\frac{\partial}{\partial x}\left(\frac{\partial \tilde{u}}{\partial z}\right) \\
&= \frac{-1}{2z^2}\frac{dz}{dx}\frac{\partial \tilde{u}}{\partial z} + \frac{1}{2z}\frac{\partial^2 \tilde{u}}{\partial z^2}\frac{dz}{dx} \\
&= \frac{-1}{4z^3}\frac{\partial \tilde{u}}{\partial z} + \frac{1}{4z^2}\frac{\partial^2 \tilde{u}}{\partial z^2}
\end{aligned}$$

(b) Substituting what we found in (a) into (6) and using u in place of $\tilde{u}$ to simplify notation, we get

$$\begin{aligned}
u_{tt} &= g\left[z^2\left[-\frac{1}{4z^3}u_z + \frac{1}{4z^2}u_{zz}\right] + \frac{1}{2z}u_z\right] \\
&= \frac{g}{4}\left[u_{zz} + \frac{1}{z}u_z\right]
\end{aligned}$$

5. Using Exercise 9 of Section 4.3, we find

$$\begin{aligned}
\mathcal{H}_0(x^{2N}\,\mathcal{U}_0(a-x))(s) &= \int_0^\infty x^{2N}\,\mathcal{U}_0(a-x)J_0(sx)x\,dx \\
&= \int_0^a J_0(sx)x^{2N+1}\,dx \\
&\quad \text{(change variables } sx \leftrightarrow x\text{)} \\
&= \frac{1}{s^{2N+2}}\int_0^{as} x^{2N+1}J_0(x)\,dx \\
&= \frac{1}{s^{2N+2}}\sum_{n=0}^{N}(-1)^n 2^N\frac{N!}{(N-n)!}x^{2N+1-n}J_{n+1}(x)\Big|_0^{as} \\
&= \frac{1}{s^{2N+2}}\sum_{n=0}^{N}(-1)^n 2^N\frac{N!}{(N-n)!}(as)^{2N+1-n}J_{n+1}(as) \\
&= \sum_{n=0}^{N}(-1)^n 2^N\frac{N!}{(N-n)!}\frac{a^{2N+1-n}}{s^{1+n}}J_{n+1}(as)
\end{aligned}$$

9. (a) The chain starts to move from rest with an initial velocity of $v(x) = \sqrt{x}$.
(b) We have $A \equiv 0$ and

$$B(s) = \frac{2}{\sqrt{g}s}\mathcal{H}_0(1/z)(s) = \frac{2}{\sqrt{g}s^2}.$$

So

$$u(x,\, t) = \frac{2}{\sqrt{g}} \int_0^\infty \sin\left(\frac{\sqrt{g}}{2} st\right) J_0\left(\sqrt{x}s\right) \frac{1}{s}\, ds.$$

13. Similar to Example 1.

Solutions to Exercises 12.1

1. We have $M(x, y) = xy$, $N(x, y) = y$, $M_y(x, y) = x$, $N_x(x, y) = y$. The right side of (4) is equal to

$$\int_0^1 \int_0^1 -x\, dx\, dy = -\frac{1}{2}.$$

Starting with the side of the square on the x-axis and moving counterclockwise, label the sides of the square by 1, 2, 3, and 4. We have

$$\int_C \big(M(x, y)\, dx + N(x, y)\, dy\big) = \sum_{j=1}^4 \int_{\text{side } j} \big(M(x, y)\, dx + N(x, y)\, dy\big) = \sum_{j=1}^4 I_j.$$

On side 1, $y = 0$, hence $M = N = 0$ and so $I_1 = 0$. On side 2, $x = 1$ and y varies from 0 to 1; hence $M = y$, $N = y$, and $dx = 0$, and so

$$I_2 = \int_0^1 y\, dy = \frac{1}{2}.$$

On side 3, $y = 1$ and x varies form 1 to 0; hence $M = x$, $N = 1$, and $dy = 0$, and so

$$I_3 = \int_1^0 x\, dx = -\frac{1}{2}.$$

On side 4, $x = 0$ and y varies form 1 to 0; hence $M = 0$, $N = y$, and $dx = 0$, and so

$$I_4 = \int_1^0 y\, dy = -\frac{1}{2}.$$

Consequently,

$$\int_C \big(M(x, y)\, dx + N(x, y)\, dy\big) = 0 + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2},$$

which verifies Green's theorem in this case.

5. We have $M(x, y) = 0$, $N(x, y) = x$, $M_y(x, y) = 0$, $N_x(x, y) = 1$. The right side of (6) is equal to

$$\iint_D dx\, dy = \text{ (area of annular region)} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

We have

$$\int_\Gamma \big(M(x, y)\, dx + N(x, y)\, dy\big) = \int_{C_2} + \int_{C_1} \big(M(x, y)\, dx + N(x, y)\, dy\big) = I_1 + I_2.$$

Parametrize C_1 by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, $dx = -\sin t\, dt$, $dy = \cos t$. Hence

$$I_1 = \int_0^{2\pi} \cos^2 t\, dt = \pi.$$

Parametrize C_2 by $x = \frac{1}{2}\cos t$, $y = \frac{1}{2}\sin t$, t varies from 2π to 0, $dx = -\frac{1}{2}\sin t\, dt$, $dy = \frac{1}{2}\cos t$. Hence

$$I_2 = \frac{1}{4}\int_{2\pi}^0 dt == -\int_0^{2\pi} \frac{1}{4}\cos^2 t\, dt = -\frac{\pi}{4}.$$

Consequently,

$$\int_\Gamma \big(M(x, y)\, dx + N(x, y)\, dy\big) = \pi - \frac{\pi}{4} = \frac{3\pi}{4},$$

which verifies Green's theorem in this case.

9. Take $u(x, y) = y$ and $v(x, y) = x$. Then $\nabla^2 v = 0$, $\nabla u = (0, 1)$, $\nabla v = (1, 0)$, so $\nabla u \cdot \nabla v = 0$ and, by (9),

$$\int_C y \frac{\partial x}{\partial n}\, ds = \int_C u \frac{\partial v}{\partial n}\, ds = \int_\Gamma 0\, ds = 0.$$

13. Same solution as in Example 1. Use Theorem 2 instead of Theorem 1.

17. We use the 2nd integral in Example 1. Let us parametrize the ellipse by $x(t) = a\,\cos t$, $y(t) = b\,\sin t$, $dy = b\cos t\, dt$, $0 \leq t \leq 2\pi$.

$$\begin{aligned}
\text{Area} &= \int_C x\, dy = \int_0^{2\pi} a\,\cos t\, b\,\sin t\, dt \\
&= a\,b \int_0^{2\pi} \cos^2 t\, dt = a\,b \int_0^{2\pi} \frac{1 + \cos(2t)}{2}\, dt \\
&= \pi ab.
\end{aligned}$$

Solutions to Exercises 12.2

1. The function $u(x, y) = e^x \cos y$ is harmonic for all (x, y) (check that $\nabla^2 u = 0$ for all (x, y)). Applying (1) at $(x_0, y_0) = (0, 0)$ with $r = 1$, we obtain

$$1 = u(0, 0) = \frac{1}{2\pi}\int_0^{2\pi} e^{\cos t}\cos(\sin t)\,dt.$$

5. $u(x, y) = x^2 - y^2$, $u_{xx} = 2$, $u_{yy} = -2$, $u_{xx} + u_{yy} = 0$ for all (x, y). Since u is harmonic for all (x, y) it is harmonic on the given square region and continuus on its boundary. Since the region is bounded, u attains its maximum and minimum values on the boundary, by Corollary 1. Starting with the side of the square on the x-axis and moving counterclockwise, label the sides of the square by 1, 2, 3, and 4.

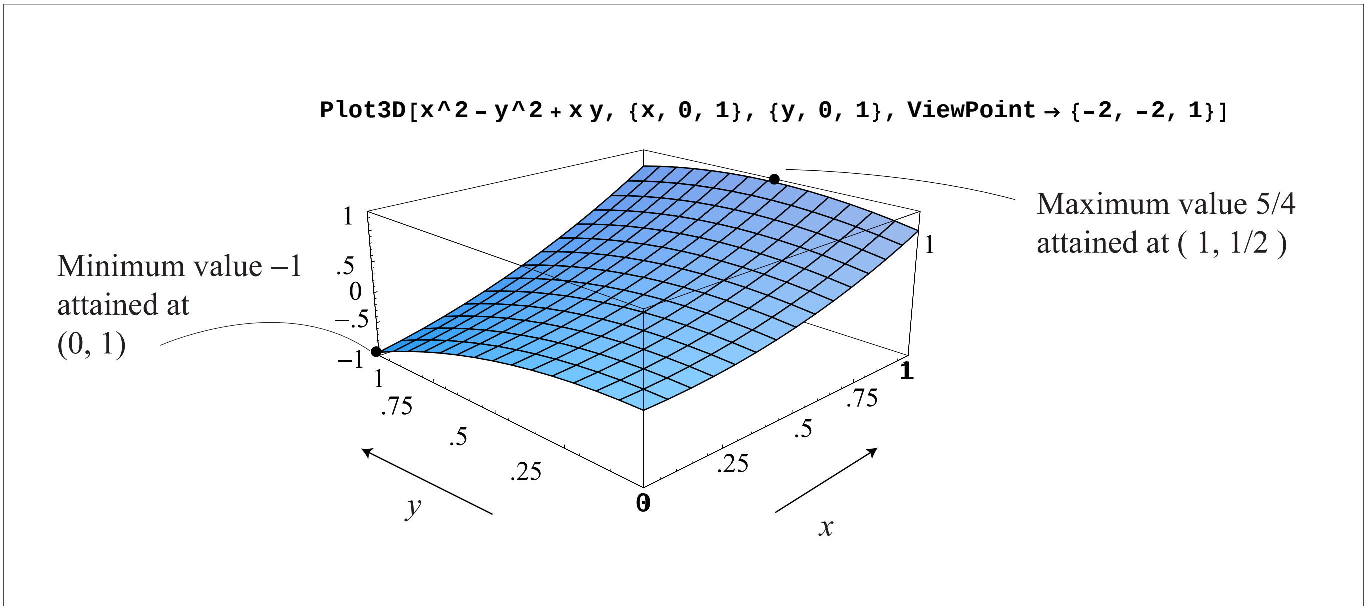
On side 1, $0 \le x \le 1$, $y = 0$, and $u(x, y) = u(x, 0) = x^2$. On this side, the maximum value is 1 and is attained at the point (1, 0), and the minimum value is 0 and is attained at the point (0, 0).

On side 2, $0 \le y \le 1$, $x = 1$, and $u(x, y) = u(1, y) = 1 + y - y^2 = f(y)$. On this side, $f'(y) = -2y + 1$, $f'(y) = 0 \Rightarrow y = 1/2$. Minimum value $f(0) = f(1) = 1$, attained at the points (1, 0) and (1, 1). Maximum value $f(1/2) = 1 - 1/4 + 1/2 = 5/4$, attained at the point (1, 1/2).

On side 3, $0 \le x \le 1$, $y = 1$, and $u(x, y) = u(x, 1) = x^2 + x - 1 = f(x)$. On this side, $f'(x) = 2x + 1$, $f'(x) = 0 \Rightarrow x = -1/2$. Extremum values occur at the endpoints: $f(0) = -1$, $f(1) = 1$. Thus the minimum value is -1 and is attained at the point (0, 1). Maximum value is 1 and is attained at the point (1, 1).

On side 4, $0 \le y \le 1$, $x = 0$, and $u(x, y) = u(0, y) = -y^2$. On this side, the maximum value is 0 and is attained at the point (0, 0), and the minimum value is -1 and is attained at the point (0, 1).

Consequently, the maximum value of u on the square is 5/4 and is attained at the point (1, 1/2); and the minimum value of u on the square is -1 and is attained at the point (0, 1) (see figure).



Solutions to Exercises 12.3

1. We have

$$\int_{\Gamma} G(x,\, y,\, x_0,\, y_0)\, ds = 0,$$

because $G(x,\, y,\, x_0,\, y_0) = 0$ for all $(x,\, y)$ on Γ (Theorem 3(ii)).

5. Let $u(x,\, y) = x$, then u is harmonic for all $(x,\, y)$ and so, by Theorem 2,

$$\int_{\Gamma} x \frac{\partial}{\partial n} G(x,\, y,\, \frac{1}{2},\, \frac{1}{3})\, ds = 2\pi \frac{1}{2} = \pi.$$

9. By Theorem 4, with $f(x,\, y) = x^2 y^3$,

$$\nabla^2 \left(\iint_{\Omega} x^2 y^3\, G(x,\, y,\, x_0,\, y_0)\, dx\, dy \right) = 2\pi x_0^2 y_0^3.$$

Solutions to Exercises 12.4

1. The function $u(r, \theta) = 1$ is harmonic in the unit disk and has boundary values $f(\theta) = 1$. Use (10) with $f(\theta) = 1$ and you get, for all $0 \le r < 1$,

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \phi)} d\theta.$$

5. For $n = 1, 2, \ldots$, the function $u(r, \theta) = r^n \cos n\theta$ is harmonic in the disk of radius $R > 0$ and has boundary values (when $r = R$) $f(\theta) = R^n \cos n\theta$. Use (10) with $f(\theta) = R^n \cos n\theta$ and you get, for all $0 \le r < R$,

$$r^n \cos n\theta = \frac{R^n (R^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{\cos n\theta}{R^2 + r^2 - 2rR\cos(\theta - \phi)} d\theta.$$

Solutions to Exercises 12.5

1. (a) $(3+2i)(2-i) = 6 - 3i + 4i - 2i^2 = 8 + i$.
(b) $(3-i)\overline{(2-i)} = (3-i)(2+i) = 7 + i$.
(c)

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{\overline{1-i}}{\overline{1-i}} = \frac{(1+i)(1+i)}{1^2+1^2} = i.$$

5. (a) $\text{Arg}\,(i) = \frac{\pi}{2}$. (b) $|i| = 1$. (c) $i = e^{i\frac{\pi}{2}}$.

9. (a) $\text{Arg}\,(1+i) = \frac{\pi}{4}$. (b) $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$. (c) $1+i = \sqrt{2}\,e^{i\cdot\frac{\pi}{4}}$.

> In computing the values of $\text{Arg}\, z$, just remember that $\text{Arg}\, z$ takes its values in the interval $(-\pi, \pi]$. Consequently, $\text{Arg}\, z$ is not always equal to $\tan^{-1}(y/x)$ (see Section 12.5, (8), for the formula that relates $\text{Arg}\, z$ to $\tan^{-1}(y/x)$). You can use Mathematica to evaluate $\text{Arg}\, z$ and the absolute value of z. This is illustrated by the following exercises.

17. (a) Apply Euler's identity, $e^{2i} = \cos 2 + i \sin 2$.
(b) Use Example 1(d): for $z = x + i\,y$,

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

So

$$\sin i = \sin 0 \cosh 1 + i \cos 0 \sinh 1 = i \sinh 1 = i\,\frac{e - e^{-1}}{2}.$$

(c) Use Example 1(e): for $z = x + i\,y$,

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

So

$$\cos i = \cos 0 \cosh 1 - i \sin 0 \sinh 1 = \cosh 1 \frac{e + e^{-1}}{2}.$$

(It is real!)
(d) Use Example 1(f): for $z = x + i\,y$,

$$\text{Log}\, z = \ln(|z|) + i\ \text{Arg}\, z.$$

For $z = i$, $|i| = 1$ and $\text{Arg}\, i = \frac{\pi}{2}$. So $\text{Log}\, i = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$, because $\ln 1 = 0$.

> Remember that there are many branches of the logarithm, $\log z$, and $\text{Log}\, z$ is one of them. All other values of $\log z$ differ from $\text{Log}\, z$ by an integer multiple of $2\pi i$. This is because the imaginary part of the logarithm is defined by using a branch of $\arg z$, and the branches of $\arg z$ differ by integer multiples of 2π. (See Applied Complex Analysis and PDE for more details on the logarithm.) In particular, the imaginary part of $\text{Log}\, z$, which is $\text{Arg}\, z$, is in the interval $(-\pi, \pi]$.
> You can use Mathematica to evaluate $\text{Log}\, z$ and e^z. This is illustrated by the following exercises.

21. We have $(-1)\cdot(-1) = 1$ but

$$0 = \text{Log}\, 1 \neq \text{Log}\,(-1) + \text{Log}\,(-1) = i\pi + i\pi = 2i\pi.$$

25. (a) By definition of the cosine, we have

$$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x.$$

(b) By definition of the since, we have

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = i\frac{e^{x} - e^{-x}}{2} = i\sinh x.$$

29. (a) Note that, for $z = x + i\,y \neq 0$,

$$\frac{x + i\,y}{x^2 + y^2} = \frac{z}{z \cdot \overline{z}} = \frac{1}{\overline{z}}.$$

We claim that this function is not analytic at any z. We have

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{y}{x^2 + y^2};$$

so

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad u_y = \frac{-2xy}{(x^2 + y^2)^2}, \quad v_x = \frac{-2xy}{(x^2 + y^2)^2}, \quad v_y = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Since the euqality $u_x = v_y$ and $u_y = -v_x$ imply that $(x, y) = (0, 0)$. Hence f is not analytic at any $z = x + i\,y$.
(b) Note that, for $z = x + i\,y \neq 0$,

$$\frac{x - i\,y}{x^2 + y^2} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{1}{z},$$

and ths function is anaytic for all $z \neq 0$. Using the Cauchy-Riemann equations, we have

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2};$$

so

$$u_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad u_y = \frac{-2xy}{(x^2 + y^2)^2}, \quad v_x = \frac{2xy}{(x^2 + y^2)^2}, \quad v_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

We have $u_x = v_y$ and that $u_y = -v_x$. Hence the Cauchy-Riemann equations hold. Also, all the partial derivatives are continuous functions of $(x, y) \neq (0, 0)$. Hence by Theorem 1, $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$ and

$$\begin{aligned} f'(z) &= u_x + iv_x = \frac{y^2 - x^2}{(x^2 + y^2)^2} + i\frac{2xy}{(x^2 + y^2)^2} \\ &= \frac{(y + ix)^2}{(x^2 + y^2)^2} = \frac{(y + ix)^2}{(x^2 + y^2)^2} \\ &= \frac{[i(x - iy)]^2}{[(x + i\,y)(x - i\,y)]^2} = \frac{i^2}{(x + i\,y)^2} = \frac{-1}{z^2}. \end{aligned}$$

33. We leave the verification that u is harmonic as a simple exercise. The function

$$u(x, y) = e^x \cos y$$

is the real part of the entire function

$$e^z = e^x \cos y + i\,e^x \sin y.$$

So a harmonic conjugate of $e^x \cos y$ is $e^x \sin y$. (By the same token, a harmonic conjugate of $e^x \sin y$ is $-e^x \cos y$.)

Let us now find the harmonic conjugate using the technique of Example 6. Write

$$u(x, y) = e^x \cos y \quad u_x(x, y) = e^x \cos y \quad u_y(x, y) = -e^x \sin y.$$

The first equation of the Cauchy-Riemann equations tells us that $u_x = v_y$. So

$$v_y(x,\, y) = e^x \cos y.$$

Integrating with respect to y (treating x as a constant), we find

$$v(x,\, y) = e^x \sin y + c(x),$$

where the constant of integration $c(x)$ is a function of x. The second equation of the Cauchy-Riemann equations tells us that $u_y = -v_x$. So

$$\begin{aligned} v_x(x,\, y) &= e^x \sin y + c'(x); \\ e^x \sin y + c'(x) &= e^x \sin y; \\ c'(x) &= 0; \\ c(x) &= C. \end{aligned}$$

Hence,

$$v(x,\, y) = e^x \sin y + C,$$

which matches the previous formula up to a additive constant.

37. (a) The level curves are given by

$$u(x,\, y) = \frac{y}{x^2 + y^2} = \frac{1}{2C},$$

where, for convenience, we have used $1/(2C)$ instead of the usual C for our arbitrary constant. The equation becomes

$$x^2 + y^2 - 2Cy = 0 \quad \text{or} \quad x^2 + (y - C)^2 = C^2.$$

Thus the level curves are circles centered at $(0,\, C)$ with radius C.
(b) By Exercise 35, a harmonic conjugate of $u(x,\, y)$ is

$$v(x,\, y) = \frac{x}{x^2 + y^2}.$$

Thus the orthogonal curves to the family of curves in (a) are given by the level curves of v, or

$$v(x,\, y) = \frac{x}{x^2 + y^2} = \frac{1}{2C} \quad \text{or} \quad (x - C)^2 + y = C^2.$$

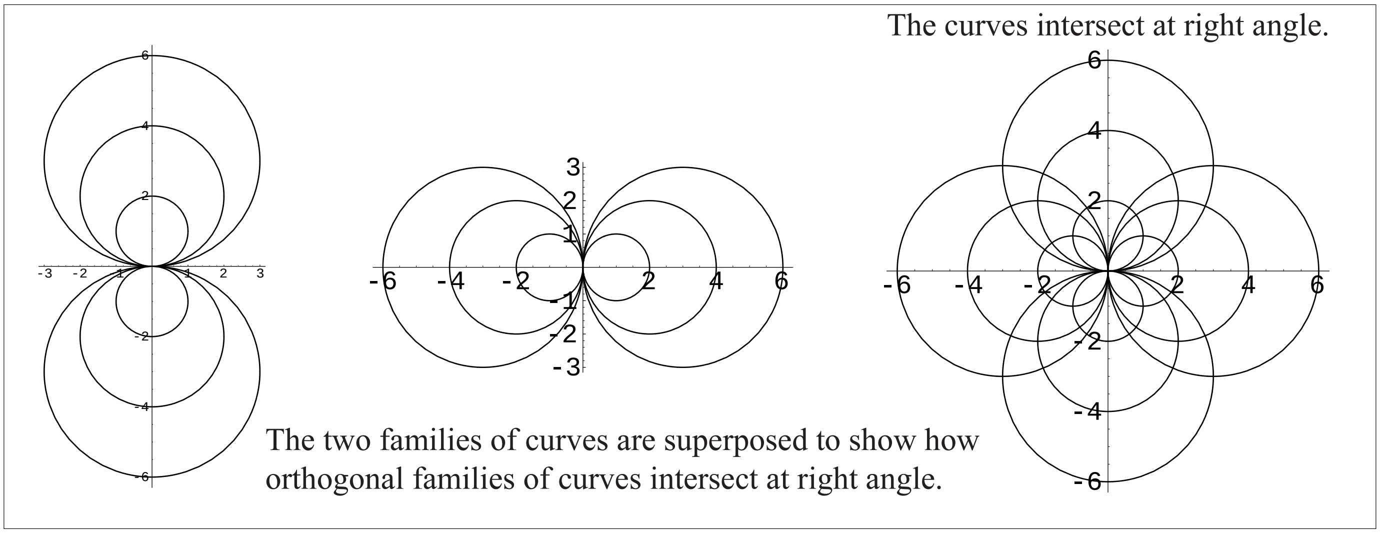
Thus the level curves are circles centered at $(C,\, 0)$ with radius C.

The curves in (a) and (b) are shown in the figure. Note how we defined the parametric equation of a circle centered at $(x_0,\, y_0)$ with radius $r > 0$: $x(t) = x_0 + r\cos t$, $y(t) = y_0 + r \sin t$, $0 \le t \le 2\pi$.

```
tt1 = Table[{Abs[r] Cos[t], r + Abs[r] Sin[t]}, {r, -3, 3}]
tt2 = Table[{r + Abs[r] Cos[t], Abs[r] Sin[t]}, {r, -3, 3}]
ParametricPlot[Evaluate[tt1], {t, 0, 2 Pi}, AspectRatio → Automatic]
ParametricPlot[Evaluate[tt2], {t, 0, 2 Pi}, AspectRatio → Automatic]
Show[{%, %%}]

{{3 Cos[t], -3 + 3 Sin[t]}, {2 Cos[t], -2 + 2 Sin[t]}, {Cos[t], -1 + Sin[t]},
 {0, 0}, {Cos[t], 1 + Sin[t]}, {2 Cos[t], 2 + 2 Sin[t]}, {3 Cos[t], 3 + 3 Sin[t]}}

{{-3 + 3 Cos[t], 3 Sin[t]}, {-2 + 2 Cos[t], 2 Sin[t]}, {-1 + Cos[t], Sin[t]},
 {0, 0}, {1 + Cos[t], Sin[t]}, {2 + 2 Cos[t], 2 Sin[t]}, {3 + 3 Cos[t], 3 Sin[t]}}
```



41. If $u(x, y)$ does not depend on y, then u is a function of x alone. We have $u_y = 0$ and so $u_{yy} = 0$. If u is also harmonic, then $u_{xx} + u_{yy} = 0$. But $u_{yy} = 0$, so $u_{xx} = 0$. Integrating with respect to x twice, we find $u(x, y) = a\,x + b$.

45. We reason as in Example 4 and try for a solution a function of the form

$$u(r, \theta) = a \text{ Arg}\, z + b,$$

where $z = x + i\,y$ and a and b are constant to be determined. Using the boundary conditions in Figure 18, we find

$$\begin{aligned}
u(r\frac{9\pi}{10}) = 60 \quad &\Rightarrow \quad a\frac{9\pi}{10} + b = 60;\\
u(r, \frac{3\pi}{5}) = 0 \quad &\Rightarrow \quad a\frac{3\pi}{5} + b = 0;\\
&\Rightarrow \quad a\left(\frac{9\pi}{10} - \frac{6\pi}{10}\right) = 60 \text{ or } a = \frac{200}{\pi};\\
&\Rightarrow \quad b = -120;\\
&\Rightarrow \quad u(r, \theta) = \frac{200}{\pi} \text{ Arg}\, z - 120.
\end{aligned}$$

In terms of (x, y), we can use (10) and conclude that, for $y > 0$,

$$u(x, y) = \frac{200}{\pi} \cot^{-1}\left(\frac{x}{y}\right) - 120.$$

49. As in Exercise 47, we think of the given problem as the sum of two simpler subproblems. Let u_1 be harmonic in the upper half-plane and equal to 100 on the x-axis for $0 < x < 1$ and 0 for all other values of x. Let u_2 be harmonic in the upper half-plane and equal to 20 on the x-axis for $-1 < x < 0$ and 0 for all other values of x. Let $u = u_1 + u_2$. Then u is harmonic in the upper half-plane and equal to 20 on the x-axis for $-1 < x < 0$; 100 for $0 < x < 1$; and 0 otherwise. (Just add the boundary values of u_1 and u_2.) Thus $u_1 + u_2$ is the solution to our problem.

Both u_1 and u_2 are given by Example 5. We have

$$\begin{aligned}
u_1(x, y) &= \frac{100}{\pi}\left[\cot^{-1}\left(\frac{x-1}{y}\right) - \cot^{-1}\left(\frac{x}{y}\right)\right];\\
u_2(x, y) &= \frac{20}{\pi}\left[\cot^{-1}\left(\frac{x}{y}\right) - \cot^{-1}\left(\frac{x+1}{y}\right)\right];\\
u(x, y) &= -\frac{80}{\pi}\cot^{-1}\left(\frac{x}{y}\right) + \frac{100}{\pi}\cot^{-1}\left(\frac{x-1}{y}\right) - \frac{20}{\pi}\cot^{-1}\left(\frac{x+1}{y}\right).
\end{aligned}$$

Solutions to Exercises 12.6

1. (a) $f(z) = 1/z$ is analytic for all $z \neq 0$, by Theorem 2, Section 12.5, since it is the quotient of two analytic functions. $U(u, v) = uv$ is harmonic since $U_{uu} = 0$, $U_{vv} = 0$, so $U_{uu} + U_{vv} = 0$.
(b) We have (this was done several times before)

$$f(z) = \frac{x - i\,y}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

So

$$\operatorname{Re}(f) = u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad \operatorname{Im}(f) = v(x, y) = -\frac{y}{x^2 + y^2}.$$

(c) We have

$$\begin{aligned} \phi(x, y) &= U \circ f(z) = U\big(u(x, y), v(x, y)\big) \\ &= U\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) \\ &= \frac{-xy}{(x^2 + y^2)^2}. \end{aligned}$$

You can verify directly that $\phi(x, y)$ is harmonic for all $(x, y) \neq (0, 0)$ or, better yet, you can apply Theorem 1.

5. (a) If z is in S, then $z = a + i, y$ where $b \leq y \leq c$. So

$$f(z) = e^z = e^{a+i\,y} = e^a e^{i\,y}.$$

The complex number $w = e^a e^{i\,y}$ has modulus e^a and argument y. As y varies from b to c, the point $w = e^a e^{i\,y}$ traces a circular arc of radius e^a, bounded by the two rays at angles b and c.
(b) According to (a), the image of $\{z = 1 + i\,y : \ 0 \leq y \leq \pi/2\}$ by the mapping e^z is the circular arc with radius e, bounded by the two rays at angles 0 and $\pi/2$. It is thus a quarter of a circle of radius e (see figure).

Similarly, the image of $\{z = 1 + i\,y : \ 0 \leq y \leq \pi\}$ by the mapping e^z is the circular arc with radius e, bounded by the two rays at angles 0 and π. It is thus a semi-circle of radius e, centered at the origin (see figure).

The most basic step is to define the complex variable z=x+iy, where x and y real. This is done as follows:

```
Clear[x, y, z, f]
<< Algebra`ReIm`
x /: Im[x] = 0
y /: Im[y] = 0
z = x + I y
```

You can now define any function of z and take its real and imaginary. For example:

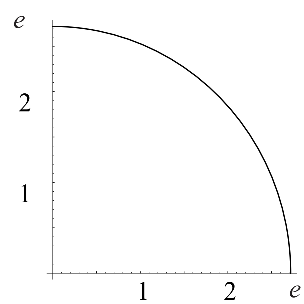
```
f[z_] = E^z
Re[f[z]]
Im[f[z]]
```

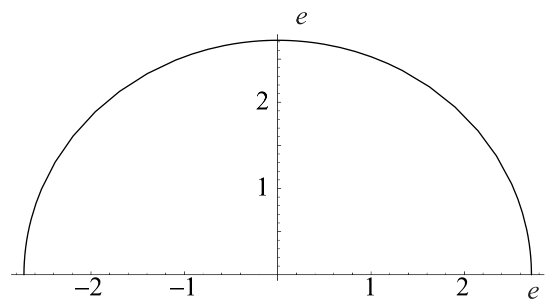
$e^{x+i\,y}$

$e^x \operatorname{Cos}[y]$

$e^x \operatorname{Sin}[y]$

```
f1[t_] = Re[f[z]] /. {x → 1, y → t}
f2[t_] = Im[f[z]] /. {x → 1, y → t}
ParametricPlot[Evaluate[{f1[t], f2[t]}], {t, 0, Pi / 2}, AspectRatio → Automatic]
ParametricPlot[Evaluate[{f1[t], f2[t]}], {t, 0, Pi}, AspectRatio → Automatic]
ⅇ Cos[t]
ⅇ Sin[t]
```





9. We map the region onto the upper half-plane using the mapping $f(z) = z^2$ (see Example 1). The transformed problem in the uv-plane is $\nabla^2 U = 0$ with boundary values on the u-axis given by $U(u, 0) = 100$ if $0 < u < 1$ and 0 otherwise. The solution in the uv-plane follows from Example 5, Section 12.5. We have

$$U(u, v) = \frac{100}{\pi}\left[\cot^{-1}\left(\frac{u-1}{v}\right) - \cot^{-1}\left(\frac{u}{v}\right)\right].$$

The solution in the xy-plane is $\phi(x, y) = U \circ f(z)$. To find the formula in terms of (x, y), we write $z = x + i\,y$, $f(z) = z^2 = x^2 - y^2 + 2ixy = (u, v)$. Thus $u = x^2 - y^2$ and $v = 2xy$ and so

$$\phi(x, y) = U \circ f(z) = U(x^2 - y^2,\, 2xy) = \frac{100}{\pi}\left[\cot^{-1}\left(\frac{x^2 - y^2 - 1}{2xy}\right) - \cot^{-1}\left(\frac{x^2 - y^2}{2xy}\right)\right].$$

13. We map the region onto the upper half-plane using the mapping $f(z) = e^z$ (see Example 2). The points on the x-axis, $z = x$, are mapped onto the positive u-axis, since $e^x > 0$ for all x, as follows: $f(x) \geq 1$ if $x \geq 0$ and $0 < f(x) < 1$ if $x < 0$. The points on the horizontal line $z = x + i\pi$ are mapped onto the negative u-axis, since $e^{x+i\,\pi} = -e^x < 0$, as follows: $f(x + i\,\pi) = -e^x \leq -1$ if $x \geq 0$ and $-1 < f(x + i\,\pi) = -e^x < 0$ if $x < 0$. With these observations, we see that the transformed problem in the uv-plane is $\nabla^2 U = 0$ with boundary values on the u-axis given by $U(u, 0) = 100$ if $-1 < u < 1$ and 0 otherwise. The solution in the uv-plane follows from Example 5, Section 12.5. We have

$$U(u, v) = \frac{100}{\pi}\left[\cot^{-1}\left(\frac{u-1}{v}\right) - \cot^{-1}\left(\frac{u+1}{v}\right)\right].$$

The solution in the xy-plane is $\phi(x, y) = U \circ f(z)$. To find the formula in terms of (x, y), we write $z = x + i\,y$,

$$(u, v) = f(z) = e^z = e^x \cos y + i\,e^x \sin y.$$

Thus $u = e^x \cos y$ and $v = e^x \sin y$ and so

$$\begin{aligned}\phi(x, y) &= U \circ f(z) = U(e^x \cos y,\, e^x \sin y) \\ &= \frac{100}{\pi}\left[\cot^{-1}\left(\frac{e^x \cos y - 1}{e^x \sin y}\right) - \cot^{-1}\left(\frac{e^x \cos y + 1}{e^x \sin y}\right)\right].\end{aligned}$$

17. We map the region onto the upper half-plane using the mapping $f(z) = z^2$ (see Example 1). The transformed problem in the uv-plane is $\nabla^2 U = 0$ with boundary values on the u-axis given by $U(u, 0) = u$ if $0 < u < 1$ and 0 otherwise. To solve the problem in the uv-plane, we apply the Poisson integral formula ((5), Section 7.5). We have

$$U(u, v) = \frac{v}{\pi}\int_{-\infty}^{\infty} \frac{U(s, 0)}{(u-s)^2 + v^2}\, ds = \frac{v}{\pi}\int_0^1 \frac{s}{(u-s)^2+v^2}\, ds.$$

Now use your calculus skills to compute this integral. We have

$$\begin{aligned}
\frac{v}{\pi}\int_0^1 \frac{s}{(u-s)^2+v^2}\, ds &= \frac{v}{\pi}\int_0^1 \frac{(s-u)}{(s-u)^2+v^2}\, ds + \frac{v}{\pi}\int_0^1 \frac{u}{(s-u)^2+v^2}\, ds \\
&= \frac{v}{\pi}\left[\frac{1}{2}\ln[(s-u)^2+v^2]\Big|_0^1 + \int_{-u}^{1-u} \frac{u}{t^2+v^2}\, dt\right] \quad (s-u=t) \\
&= \frac{v}{2\pi}\ln\frac{(1-u)^2+v^2}{u^2+v^2} + \frac{u}{\pi}\tan^{-1}\left(\frac{t}{v}\right)\Big|_{-u}^{1-u} \\
&= \frac{v}{2\pi}\ln\frac{(1-u)^2+v^2}{u^2+v^2} + \frac{u}{\pi}\left[\tan^{-1}\left(\frac{1-u}{v}\right) - \tan^{-1}\left(\frac{-u}{v}\right)\right] \\
&= \frac{v}{2\pi}\ln\frac{(1-u)^2+v^2}{u^2+v^2} + \frac{u}{\pi}\left[\tan^{-1}\left(\frac{1-u}{v}\right) + \tan^{-1}\left(\frac{u}{v}\right)\right].
\end{aligned}$$

The solution in the xy-plane is $\phi(x, y) = U \circ f(z)$. To find the formula in terms of (x, y), we write $z = x + i\,y$, $f(z) = z^2 = x^2 - y^2 + 2ixy = (u, v)$. Thus $u = x^2 - y^2$ and $v = 2xy$ and so

$$\begin{aligned}
\phi(x, y) &= U(x^2-y^2, 2xy) \\
&= \frac{xy}{\pi}\ln\frac{(1-(x^2-y^2))^2+(2xy)^2}{(x^2-y^2)^2+(2xy)^2} \\
&\quad + \frac{x^2-y^2}{\pi}\left[\tan^{-1}\left(\frac{1-x^2+y^2}{2xy}\right) + \tan^{-1}\left(\frac{x^2-y^2}{2xy}\right)\right] \\
&= \frac{xy}{\pi}\ln\frac{(1-(x^2-y^2))^2+(2xy)^2}{(x^2+y^2)^2} \\
&\quad + \frac{x^2-y^2}{\pi}\left[\tan^{-1}\left(\frac{1-x^2+y^2}{2xy}\right) + \tan^{-1}\left(\frac{x^2-y^2}{2xy}\right)\right].
\end{aligned}$$

(I verified this solution on Mathematica and it works! It is harmonic and has the right boundary values.)

21. The mapping $f(z) = \operatorname{Log} z$ maps the given annular region onto the $1 \times \pi$-rectangle in the uv-plane with vertices at $(0, 0)$, $(1, 0)$, $(1, \pi)$, and $(0, \pi)$. With the help of the discussion in Example 4, you can verify that the transformed Dirichlet problem on the rectangle has the following boundary conditions: $U(u, 0) = 0$ and $U(u, \pi) = 0$ for $0 < u < 1$ (boundary values on the horizontal sides), and $U(0, v) = 100$ and $U(1, v) = 100$ for $0 < v < \pi$ (boundary values on the vertical sides). To simplify the notation, we rename the variables x and y instead of u and v. Consider Figure 3, Section 3.8, and take $a = 1$, $b = \pi$, $f_1 = f_2 = 0$, and $f_3 = f_4 = 100$. The desired solution is the sum of two functions $u_3(x, y)$ and

$u_4(x,\, y)$, where

$$\begin{aligned}
u_4(x,\, y) &= \sum_{n=1}^{\infty} D_n \sinh nx \sin ny;\\
D_n &= \frac{2}{\pi \sinh n}\int_0^{\pi} 100 \sin ny\, dy\\
&= \frac{-200}{n\pi \sinh n}(\cos n\pi - 1) = \frac{200}{n\pi \sinh n}(1-(-1)^n);\\
u_3(x,\, y) &= \sum_{n=1}^{\infty} C_n \sinh[n(1-x)] \sin ny;\\
C_n &= \frac{2}{\pi \sinh n}\int_0^{\pi} 100 \sin ny\, dy\\
&= \frac{200}{n\pi \sinh n}(1-(-1)^n).
\end{aligned}$$

Thus (back to the variables u and v)

$$\begin{aligned}
U(u,\, v) &= u_3 + u_4\\
&= \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh n}(1-(-1)^n) \sinh nu \sin nv\\
&\quad + \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh n}(1-(-1)^n) \sinh[n(1-u)] \sin nv\\
&= \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{\sin nv}{n \sinh n}(1-(-1)^n)\big[\sinh nu + \sinh[n(1-u)]\big]\\
&= \frac{400}{\pi}\sum_{n=0}^{\infty} \frac{\sinh[(2n+1)u] + \sinh[(2n+1)(1-u)]}{(2n+1)\sinh(2n+1)} \sin[(2n+1)v].
\end{aligned}$$

To get the solution in the xy-plane, substitute $u = \frac{1}{2}\ln(x^2+y^2)$ and $v = \cot^{-1}\left(\frac{y}{x}\right)$. The solution takes on a neater form if we use polar coordinates and substitute $x^2 + y^2 = r^2$ and $\theta = \cot^{-1}\left(\frac{y}{x}\right)$. Then

$$\begin{aligned}
\phi(x,\, y) &= U\Big(\frac{1}{2}\ln(x^2+y^2),\, \cot^{-1}\left(\frac{y}{x}\right)\Big) = U(\ln r,\, \theta)\\
&= \frac{400}{\pi}\sum_{n=0}^{\infty} \frac{\sinh[(2n+1)\ln r] + \sinh[(2n+1)(1-\ln r)]}{(2n+1)\sinh(2n+1)} \sin[(2n+1)\theta]\\
&= \phi(r,\, \theta).
\end{aligned}$$

It is interesting to verify the boundary conditions for the solution. For example, when $r = 1$, we have

$$\begin{aligned}
\phi(1,\, \theta) &= \frac{400}{\pi}\sum_{n=0}^{\infty} \frac{\sinh[(2n+1)(1)]}{(2n+1)\sinh(2n+1)} \sin[(2n+1)\theta]\\
&= \frac{400}{\pi}\sum_{n=0}^{\infty} \frac{\sin[(2n+1)\theta]}{(2n+1)}.
\end{aligned}$$

This last Fourier sine series is equal to 100 if $0 < \theta < \pi$. (see, for example, Exercise 1, Section 2.3). Thus the solution equals to 100 on the inner semi-circular boundary. On the outer circular boundary, $r = e$, and we have

$$\begin{aligned}\phi(e,\,\theta) &= \frac{400}{\pi}\sum_{n=0}^{\infty}\frac{\sinh(2n+1)}{(2n+1)\sinh(2n+1)}\sin[(2n+1)\theta]\\ &= \frac{400}{\pi}\sum_{n=0}^{\infty}\frac{\sin[(2n+1)\theta]}{(2n+1)},\end{aligned}$$

which is the same series as we found previously; and thus it equals 100 for $0 < \theta < \pi$. Now if $\theta = 0$ or π (which corresponds to the points on the x-axis), then clearly $\phi = 0$. Hence ϕ satisfies the boundary conditions, as expected.

25 We have $f(z) = z + z_0 = x + i\,y + x_0 + i\,y_0 = x + x_0 + i\,(y + y_0)$. Thus f maps a point $(x,\,y)$ to the point $(x + x_0,\,y + y_0)$. Thus f is a translation by $(x_0,\,y_0)$.

37 We have

$$\phi + i\,\psi = U \circ f + i\,V \circ f = \overbrace{(U + i\,V)}^{g} \circ f.$$

Since V is a harmonic conjugate of U, $U + i\,V$ is analytic. Thus $g \circ f$ is analytic, being the composition of two analytic functions. Hence $\phi + i\,\psi$ is analytic and so ψ is a harmonic conjugate of ϕ.

Solutions to Exercises 12.6

1. The function $\phi(z) = z - 1$ is a conformal mapping of Ω, one-to-one, onto the unit disk. Apply Theorem 3; then for $z = x + i\,y$ and $z_0 = x_0 + i\,y_0$ in Ω, we have

$$\begin{aligned}
G(x,\,y,\,x_0,\,y_0) &= \ln\left|\frac{x + i\,y - 1 - x_0 - i\,y_0 + 1}{1 - \overline{(x_0 + i\,y_0 - 1)}\,(x + i\,y - 1)}\right| \\
&= \ln\left|\frac{(x - x_0) + i\,(y - y_0)}{1 - (x_0 - 1 - i\,y_0)\,(x - 1 + i\,y)}\right|. \\
&= \frac{1}{2}\ln\left[(x - x_0)^2 + (y - y_0)^2\right] - \frac{1}{2}\ln\left[1 - (x_0 - 1 - i\,y_0)\,(x - 1 + i\,y)\right]. \\
&= \frac{1}{2}\ln\left[(x - x_0)^2 + (y - y_0)^2\right] \\
&\quad -\frac{1}{2}\ln\left[(-x_0x + x_0 + x - y_0y)^2 + (y_0x + yx_0 - y_0 - y)^2)\right].
\end{aligned}$$

5. The function $\phi(z) = e^z$ maps Ω, one-to-one, onto the upper half-plane. Apply Theorem 4; then for $z = x + i\,y$ and $z_0 = x_0 + i\,y_0$ in Ω, we have

$$\begin{aligned}
G(x,\,y,\,x_0,\,y_0) &= \ln\left|\frac{e^z - e^{z_0}}{e^z - e^{\overline{z_0}}}\right| \quad (\text{Note that } \overline{e^{z_0}} = e^{\overline{z_0}}.) \\
&= \ln\left|\frac{e^x\cos y + ie^x\sin y - e^{x_0}\cos y_0 - ie^{x_0}\sin y_0}{e^x\cos y + ie^x\sin y - e^{x_0}\cos y_0 + ie^{x_0}\sin y_0}\right| \\
&= \frac{1}{2}\ln\left|\frac{\big(e^x\cos y - e^{x_0}\cos y_0\big)^2 + \big(e^x\sin y - e^{x_0}\sin y_0\big)^2}{\big(e^x\cos y - e^{x_0}\cos y_0\big)^2 + \big(e^x\sin y + e^{x_0}\sin y_0\big)^2}\right| \\
&= \frac{1}{2}\ln\frac{e^{2x} + e^{2x_0} - 2e^{x+x_0}\big(\cos y\cos y_0 + \sin y\sin y_0\big)}{e^{2x} + e^{2x_0} - 2e^{x+x_0}\big(\cos y\cos y_0 - \sin y\sin y_0\big)} \\
&= \frac{1}{2}\ln\frac{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y - y_0)}{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y + y_0)}.
\end{aligned}$$

9. We use the result of Exercise 5 and apply Theorem 2, Section 12.3. Accordingly,

$$u(x_0,\,y_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\,\frac{\partial G}{\partial y}\bigg|_{y=\pi} dx.$$

We have

$$\begin{aligned}
G(x, y, x_0, y_0) &= \frac{1}{2} \ln \frac{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y - y_0)}{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y + y_0)} \\
&= \frac{1}{2} \ln\left(e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y - y_0)\right) \\
&\quad -\frac{1}{2} \ln\left(e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y + y_0)\right) \\
\frac{\partial G}{\partial y}\Big|_{y=\pi} &= \frac{1}{2} \frac{2e^{x+x_0}\sin(y - y_0)}{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos(y - y_0)}\Big|_{y=\pi} \\
&\quad -\frac{1}{2} \frac{2e^{x+x_0}\cos(y + y_0)}{e^{2x} + e^{2x_0} - 2e^{x+x_0}\sin(y + y_0)}\Big|_{y=\pi} \\
&= \frac{e^{x+x_0}\sin y_0}{e^{2x} + e^{2x_0} - 2e^{x+x_0}\cos y_0} \\
&= \frac{\sin y_0}{e^{x-x_0} + e^{x_0-x} - 2\cos y_0}.
\end{aligned}$$

Thus the Poisson integral formula in this case is

$$u(x_0, y_0) = \frac{\sin y_0}{2\pi} \int_{-\infty}^{\infty} \frac{g(x)}{e^{x-x_0} + e^{x_0-x} - 2\cos y_0} dx.$$

Let us test this formula in a case where we know the solution. Take $g(x) = 1$ for all x. Then, we know that the solution is a linear function of y (see Exercise , Section 12.); in fact, it is easy to verify that the solution is

$$u(x_0, y_0) = \frac{y_0}{\pi}.$$

Take $g(x) = 1$ in the Poisson formula and ask: Do we have

$$\frac{y_0}{\pi} = \frac{\sin y_0}{2\pi} \int_{-\infty}^{\infty} \frac{1}{e^{x-x_0} + e^{x_0-x} - 2\cos y_0} dx?$$

Change variables in the integral: $x \leftrightarrow x - x_0$. Then the last equation becomes

$$\frac{y_0}{\pi} = \frac{\sin y_0}{2\pi} \int_{-\infty}^{\infty} \frac{1}{e^{x} + e^{-x} - 2\cos y_0} dx.$$

Evaluate the right side in the case $y_0 = \frac{\pi}{2}$. The answer should be 1/2. Then try $y_0 = \pi/4$. The answer should be 1/4. (I tried it on Mathematica. It works.) Out of this exercise, you can get the interesting integral formula

$$\frac{2\, y_0}{\sin y_0} = \int_{-\infty}^{\infty} \frac{1}{e^{x} + e^{-x} - 2\cos y_0} dx \quad (0 < y_0 < \pi).$$

Solutions to Exercises A.1

1. We solve the equation $y'+y=1$ in two different ways. The first method basically rederives formula (2) instead of just appealing to it.
Using an integrating factor. In the notation of Theorem 1, we have $p(x)=1$ and $q(x)=1$. An antiderivative of $p(x)$ is thus $\int 1\cdot dx = x$. The integrating factor is

$$\mu(x)=e^{\int p(x)\,dx}=e^x.$$

Multiplying both sides of the equation by the integrating factor, we obtain the equivalent equation

$$e^x[y'+y]=e^x;$$

$$\frac{d}{dx}[e^x y]=e^x,$$

where we have used the product rule for differentiation to set $\frac{d}{dx}[e^x y]=e^x[y'+y]$. Integrating both sides of the equation gets rid of the derivative on the left side, and on the right side we obtain $\int e^x\,dx = e^x + C$. Thus,

$$e^x y = e^x + C \quad \Rightarrow \quad y = 1 + Ce^{-x},$$

where the last equality follows by multiplying by e^{-x} the previous equality. This gives the solution $y=1+Ce^{-x}$ up to one arbitrary constant, as expected from the solution of a first order differential equation.
Using formula (2). We have, with $p(x)=1$, $\int p(x)\,dx = x$ (note how we took the constant of integration equal 0):

$$y=e^{-x}\left[C+\int 1\cdot e^x\,dx\right]=e^{-x}[C+e^x]=1+Ce^{-x}.$$

5. According to (2),

$$y=e^x\left[C+\int \sin x e^{-x} dx\right].$$

To evaluate the integral, use integration by parts twice

$$\begin{aligned}
\int \sin x e^{-x} dx &= -\sin x e^{-x} + \int e^{-x}\cos x\,dx \\
&= -\sin x e^{-x} + \cos x(-e^{-x}) - \int e^{-x}\sin x\,dx; \\
2\int \sin x e^{-x} dx &= -e^{-x}\big(\sin x+\cos x\big) \\
\int \sin x e^{-x} dx &= -\frac{1}{2}e^{-x}\big(\sin x+\cos x\big).
\end{aligned}$$

So

$$y=e^x\left[C-\frac{1}{2}e^{-x}\big(\sin x+\cos x\big)\right]=Ce^x-\frac{1}{2}\big(\sin x+\cos x\big).$$

9. We use an integrating factor

$$e^{\int p(x)\,dx}=e^{\int \tan x\,dx}=e^{-\ln(\cos x)}=\frac{1}{\cos x}=\sec x.$$

Then

$$\begin{aligned}
\sec x y' - \sec x \tan x\, y &= \sec x\cos x; \\
\frac{d}{dx}[y \sec x] &= 1 \\
y\sec x &= x + C = \\
y &= x\cos x + C\cos x.
\end{aligned}$$

13. An integrating factor is $e^{\frac{x^2}{2}}$, so

$$\begin{aligned} e^{\frac{x^2}{2}}y' + xe^{\frac{x^2}{2}}y = xe^{\frac{x^2}{2}} \quad &\Rightarrow \quad \frac{d}{dx}\left[e^{\frac{x^2}{2}}y\right] = e^{\frac{x^2}{2}}x \\ &\Rightarrow \quad ye^{\frac{x^2}{2}} = \int xe^{\frac{x^2}{2}}\,dx = e^{\frac{x^2}{2}} + C \\ &\Rightarrow \quad y = 1 + Ce^{-\frac{x^2}{2}}. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad 0 = 1 + C \\ &\Rightarrow \quad C = -1 \\ &\Rightarrow \quad y = 1 - e^{-\frac{x^2}{2}}. \end{aligned}$$

17. An integrating factor is $\sec x$ (see Exercise 9), so

$$\begin{aligned} \sec x y' + y\tan x \sec x = \tan x \sec x \quad &\Rightarrow \quad \frac{d}{dx}[y\sec x] = \sec x \tan x \\ &\Rightarrow \quad y\sec x = \int \tan x \sec x\,dx = \sec x + C \\ &\Rightarrow \quad y = 1 + C\cos x. \end{aligned}$$

We now use the initial condition:

$$\begin{aligned} y(0) = 1 \quad &\Rightarrow \quad 1 = 1 + C \\ &\Rightarrow \quad C = 0 \\ &\Rightarrow \quad y = 1. \end{aligned}$$

21. (a) Clear.
(b) e^x as a linear combination of the functions $\cosh x$, $\sinh x$: $e^x = \cosh x + \sinh x$.
(c) Let a, b, $c\,d$ be any real numbers such that $ad - bc \neq 0$. Let $y_1 = ae^x + be^{-x}$ and $y_2 = ce^x + de^{-x}$. Then y_1 and y_2 are solutions, since they are linear combinations of two solutions. We now check that y_1 and y_2 are linearly independent:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} ae^x + be^{-x} & ce^x + de^{-x} \\ ae^x - be^{-x} & ce^x - de^{-x} \end{vmatrix} \\ &= -ad + bc - (ad - bc) = -2(ad - bc) \neq 0. \end{aligned}$$

Hence y_1 and y_2 are linearly independent by Theorem 7.

25. The general solution is

$$y = c_1e^x + c_2e^{2x} + 2x + 3.$$

Let's use the initial conditions:

$$\begin{aligned} &y(0) = 0 && \Rightarrow \quad c_1 + c_2 + 3 = 0 && (*) \\ &y'(0) = 0 && \Rightarrow \quad c_1 + 2c_2 + 2 = 0 && (**) \\ &\text{Subtract (*) from (**)} && \Rightarrow \quad c_2 - 1 = 0;\ c_2 = 1 \\ &\text{Substitute into (*)} && \Rightarrow \quad c_1 + 4 = 0;\ c_1 = -4. \end{aligned}$$

Thus, $y = -4e^x + e^{2x} + 2x + 3$.

29. As in the previous exercise, here it is easier to start with the general solution

$$y = c_1 e^{x-2} + c_2 e^{2(x-2)} + 2x + 3.$$

From the initial conditions,

$$\begin{array}{lll}
y(2) = 0 & \Rightarrow & c_1 + c_2 + 7 = 0 \qquad (*) \\
y'(1) = 1 & \Rightarrow & c_1 + 2c_2 + 2 = 1 \qquad (**) \\
\text{Subtract (*) from (**)} & \Rightarrow & c_2 - 5 = 1;\ c_2 = 6 \\
\text{Substitute into (*)} & \Rightarrow & c_1 + 13 = 0;\ c_1 = -13.
\end{array}$$

Thus, $y = -13e^{x-2} + 6e^{2(x-2)} + 2x + 3$.

Using Mathematica **to solve ODE**

Let us start with the simplest command that you can use to solve an ode. It is the DSolve command. We illustrate by examples the different applications of this command. The simplest case is to solve y' = y

```
DSolve[y'[x] == y[x], y[x], x]
```

$\{\{y[x] \to e^{x} C[1]\}\}$

The answer is y= C e^x as you expect. Note how *Mathematica* denoted the constant by C[1]. The enxt example is a 2nd order ode

```
DSolve[y''[x] == y[x], y[x], x]
```

$\{\{y[x] \to e^{x} C[1] + e^{-x} C[2]\}\}$

Here we need two arbitrary constants C[1] and C[2]. Let's do an intial value problem.

Solving an Initial Value Problem

Here is how you would solve y'' = y, y(0)=0, y'(0)= 1

```
DSolve[{ y''[x] == y[x], y[0] == 0, y'[0] == 1}, y[x], x]
```

$\left\{\left\{y[x] \to \frac{1}{2} e^{-x} (-1 + e^{2x})\right\}\right\}$

As you see, the initial value problem has a unique solution (there are no arbitrary constants in the answer.

Plotting the Solution

First we need to learn to extract the solution from the output. Here is how it is done. First, solve the problem and call the output solution:

```
solution = DSolve[{ y''[x] == y[x], y[0] == 0, y'[0] == 1}, y[x], x]
```

$$\left\{\left\{y[x] \to \frac{1}{2} e^{-x} \left(-1 + e^{2x}\right)\right\}\right\}$$

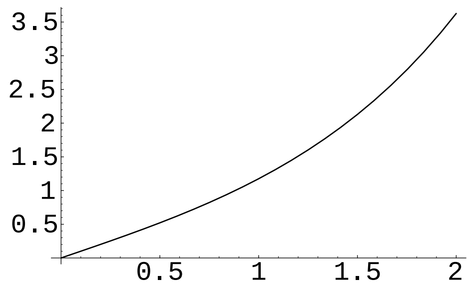
Extract the solution y(x) as follows:

```
y[x_] = y[x] /. solution[[1]]
```

$$\frac{1}{2} e^{-x} \left(-1 + e^{2x}\right)$$

Now you can plot the solution:

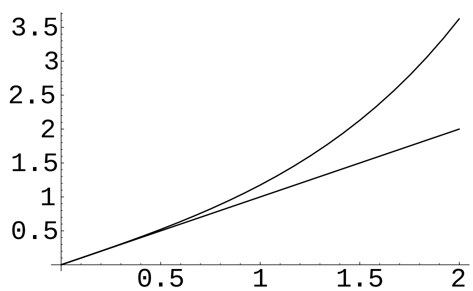
```
Plot[y[x], {x, 0, 2}]
```



The solution y(x) goes through the point (0, 0). This confirms the initial condition y(0) = 0.

Note the intial conditions on the graph: y(0)=0 and y'(0)=1. To confirm that y'(0)=1 (the slope of the graph at x=0 is 1), plot the tangent line (line with slope 1)

```
Plot[{y[x], x}, {x, 0, 2}]
```



The tangent to the solution y(x) at x = 0 is the line y = x, whose slope is 1. Thus y'(0) = 1, because the derivative is equal to the slope of the tangent line.

The Wronskian

The Wronskian is a determinat, so we can compute it using the Det command. Here is an illustration.

```
Clear[y]
sol1 = DSolve[y''[x] + y[x] == 0, y[x], x]
```

{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

Two solutions of the differential equation are obtained different values to the constants c1 and c2. For

```
Clear[ y1, y2 ]
y1[x_] = c1 Cos[x];
y2[x_] = c2 Sin[x];
```

Their Wronskian is

```
w[x_] = Det[{{y1[x], y2[x]}, {y1'[x], y2'[x]}}]
```

c1 c2 Cos[x]^2 + c1 c2 Sin[x]^2

Let's simplify using the trig identity cos^2 x + sin^2 x =1

```
Simplify[w[x]]
```

c1 c2

The Wronskian is nonzero if c1\= 0 and c2 \= 0. Let us try a different problem with a nonhomogeneous

```
Clear[y]
sol2 = DSolve[y''[x] + y[x] == 1, y[x], x]
```

{{y[x] → 1 + C[1] Cos[x] + C[2] Sin[x]}}

Two solutions of the differential equation are obtained different values to the constants c1 and c2. For

```
Clear[ y1, y2 ]
y1[x_] = 1 +  Cos[x];
y2[x_] = 1 + Sin[x];
```

These solutions are clearly linearly independent (one is not a multiple of the other). Their Wronskian is

```
Clear[w]
w[x_] = Det[{{y1[x], y2[x]}, {y1'[x], y2'[x]}}]
```

Cos[x] + Cos[x]^2 + Sin[x] + Sin[x]^2

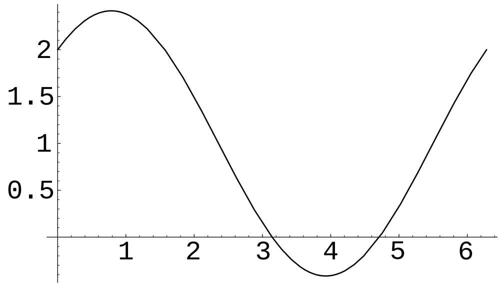
Let's simplify using the trig identity cos^2 x + sin^2 x =1

```
Simplify[w[x]]
```

1 + Cos[x] + Sin[x]

Let's plot w(x):

```
Plot[w[x], {x, 0, 2 Pi}]
```



The Wronskian does vanish at some values of x without being identically 0. Does this contradict Theorem 7 of Appendix A.1?

Solutions to Exercises A.2

1.

Equation: $y'' - 4y' + 3y = 0;$

Characteristic equation: $\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0;$

Characteristic roots: $\lambda_1 = 1; \quad \lambda_2 = 3$

General solution: $y = c_1 e^x + c_2 e^{3x}$

5

Equation: $y'' + 2y' + y = 0;$

Characteristic equation: $\lambda^2 + 2\lambda + 1 = 0$

$(\lambda + 1)^2 = 0;$

Characteristic roots: $\lambda_1 = -1$ (double root)

General solution: $y = c_1 e^{-x} + c_2 x\, e^{-x}$

9.

Equation: $y'' + y = 0;$

Characteristic equation: $\lambda^2 + 1 = 0$

Characteristic roots: $\lambda_1 = i \quad \lambda_2 = -i;$

Case III: $\alpha = 0, \quad \beta = 1;$

General solution: $y = c_1 \cos x + c_2 \sin x$

13.

Equation: $y'' + 4y' + 5y = 0;$

Characteristic equation: $\lambda^2 + 4\lambda + 5 = 0;$

Characteristic roots: $\lambda = \dfrac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i;$

$\lambda_1 = -2 + i, \quad \lambda_2 = -2 - i;$

Case III: $\alpha = -2, \quad \beta = 1;$

General solution: $y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$

17.

Equation: $y''' - 2y'' + y' = 0;$

Characteristic equation: $\lambda^3 - 2\lambda^2 + \lambda = 0$

$\lambda(\lambda - 1)^2 = 0;$

Characteristic roots: $\lambda_1 = 0; \quad \lambda_2 = 1$ (double root)

General solution: $y = c_1 + c_2\, e^x + c_3\, x\, e^x$

21.

Equation: $y''' - 3y'' + 3y' - y = 0;$

Characteristic equation: $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda - 1)^3 = 0;$

Characteristic roots: $\lambda_1 = 1$ (multiplicity 3);

General solution: $y = c_1\, e^x + c_2\, x\, e^x + c_3\, x^2 e^x$

25.

Equation:	$y'' - 4\,y' + 3\,y = e^{2x};$
Homogeneous equation:	$y'' - 4\,y' + 3\,y = 0;$
Characteristic equation:	$\lambda^2 - 4\lambda + 3 = 0$
	$(\lambda - 1)(\lambda - 3) = 0;$
Characteristic roots:	$\lambda_1 = 1,\ \lambda_2 = 3;$
Solution of homogeneous equation:	$y_h = c_1\, e^x + c_2\, e^{3x}.$

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned} y_p &= Ae^{2x};\\ y_p' &= 2Ae^{2x};\\ y_p'' &= 4Ae^{2x}. \end{aligned}$$

Plug into the equation $y'' - 4\,y' + 3\,y = e^{2x}$:

$$\begin{aligned} 4Ae^{2x} - 4(2Ae^{2x}) + 3Ae^{2x} &= e^{2x}\\ -Ae^{2x} &= e^{2x};\\ A &= -1. \end{aligned}$$

Hence $y_p = -e^{2x}$ and so the general solution

$$y_g = c_1\, e^x + c_2\, e^{3x} - e^{2x}.$$

29.

Equation:	$y'' - 4\,y' + 3\,y = x\, e^{-x};$
Homogeneous equation:	$y'' - 4\,y' + 3\,y = 0;$
Characteristic equation:	$\lambda^2 - 4\,\lambda + 3 = 0$
	$(\lambda - 1)(\lambda - 3) = 0;$
Characteristic roots:	$\lambda_1 = 1,\ \lambda_2 = 3;$
Solution of homogeneous equation:	$y_h = c_1\, e^x + c_2\, e^{3x}.$

To find a particular solution, we apply the method of undetermined coefficients. Accordingly, we try

$$\begin{aligned} y_p &= (Ax + B)e^{-x};\\ y_p' &= e^{-x}(Ax - B + A);\\ y_p'' &= e^{-x}(Ax - B - 2A). \end{aligned}$$

Plug into the equation $y'' - 4\,y' + 3\,y = e^{-x}$:

$$\begin{aligned} e^{-x}(Ax - B - 2A) - 4e^{-x}(Ax - B + A) + 3(Ax + B)e^{-x} &= e^{-x}\\ 8A &= 1;\\ -6A + 8B &= 0. \end{aligned}$$

Hence

$$A = 1/8, \quad B = 3/32; \quad y_p = (\frac{x}{8} + \frac{3}{32})e^{-x};$$

and so the general solution

$$y_g = c_1\, e^x + c_2\, e^{3x} + (\frac{x}{8} + \frac{3}{32})e^{-x}.$$

33.

Equation:	$y'' + y = \frac{1}{2} + \frac{1}{2}\cos 2x;$
Homogeneous equation:	$y'' + y = 0;$
Characteristic equation:	$\lambda^2 + 1 = 0$
Characteristic roots:	$\lambda_1 = -i,\ \lambda_2 = i;$
Solution of homogeneous equation:	$y_h = c_1\, \cos x + c_2\, \sin x.$

To find a particular solution, we apply the method of undetermined coefficients. We also use our experience and simplify the solution by trying

$$\begin{aligned} y_p &= \frac{1}{2} + A\cos 2x; \\ y_p' &= -2A\sin 2x; \\ y_p'' &= -4A\cos 2x. \end{aligned}$$

Plug into the equation $y'' + y = \frac{1}{2} + \frac{1}{2}\cos 2x$:

$$\begin{aligned} -3A\cos 2x + \frac{1}{2} &= \frac{1}{2} + \frac{1}{2}\cos 2x; \\ -3A &= \frac{1}{2}; \end{aligned}$$

$$A = -\frac{1}{6}.$$

Hence

$$y_p = -\frac{1}{6}\cos 2x + \frac{1}{2};$$

and so the general solution

$$y_g = c_1\, \cos 2x + c_2\, \sin 2x - \frac{1}{6}\cos 2x + \frac{1}{2}.$$

37.

Equation:	$y'' - y' - 2\,y = x^2 - 4;$
Homogeneous equation:	$y'' - y' - 2\,y = 0;$
Characteristic equation:	$\lambda^2 - \lambda - 2 = 0$
Characteristic roots:	$\lambda_1 = -1, \quad \lambda_2 = 2;$
Solution of homogeneous equation:	$y_h = c_1\, e^{-x} + c_2\, e^{2x}.$

For a particular solution, try

$$\begin{aligned} y_p &= A\,x^2 + B\,x + C; \\ y_p' &= 2A\,x + B; \\ y_p'' &= 2A. \end{aligned}$$

Plug into the equation $y'' - y' - 2\,y = x^2 - 4$:

$$\begin{aligned} 2A - 2A\,x - B - 2A\,x^2 - 2B\,x - 2C &= x^2 - 4; \\ -2A &= 1; \\ A = -\frac{1}{2}; & \\ -2A - 2B = 0 &\Rightarrow 1 - 2B = 0; \\ & B = \frac{1}{2}; \\ 2A - 2C - B = 4 &\Rightarrow -\frac{1}{2} - \frac{1}{2} - 2C = 4 \\ & C = \frac{5}{4}. \end{aligned}$$

Hence

$$y_p = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{5}{4};$$

and so the general solution

$$y_g = c_1\,e^{-x} + c_2\,e^{2x} - \frac{1}{2}x^2 + \frac{1}{2}x + \frac{5}{4}.$$

41. $2\,y' - y = e^{2x}$.

$$\begin{array}{lrcl} \text{Equation:} & 2\,y' - y &=& e^{2x}; \\ \text{Homogeneous equation:} & 2\,y' - y &=& 0; \\ \text{Characteristic equation:} & 2\,\lambda - 1 &=& 0 \\ \text{Characteristic root:} & \lambda_1 = \frac{1}{2}; && \\ \text{Solution of homogeneous equation:} & y_h = c_1\,e^{x/2}. && \end{array}$$

For a particular solution, try

$$\begin{aligned} y_p &= Ae^{2x}; \\ y_p' &= 2Ae^{2x}; \end{aligned}$$

Plug into the equation $2y' - y = e^{2x}$:

$$\begin{aligned} 4Ae^{2x} - Ae^{2x} &= e^{2x}; \\ 3A = 1 &\Rightarrow A = \frac{1}{3}. \end{aligned}$$

Hence

$$y_p = \frac{1}{3}e^{2x};$$

and so the general solution

$$y_g = c_1 e^{x/2} + \frac{1}{3}e^{2x}.$$

45. Write the equation in the form

$$\begin{aligned} y'' - 4\,y' + 3\,y &= e^{2x}\sinh x \\ &= e^{2x}\frac{1}{2}\big(e^x - e^{-x}\big) \\ &= \frac{1}{2}\big(e^{3x} - e^x\big). \end{aligned}$$

From Exercise 25, $y_h = c_1 e^x + c_2 e^{3x}$. For a particular solution try

$$y_p = Axe^{3x} + Bxe^x.$$

49. $y'' - 3\,y' + 2\,y = 3x^4 e^x + x\,e^{-2x}\cos 3x$. Characteristic equation

$$\lambda^2 - 3\,\lambda + 2 = 0 \;\Rightarrow\; \lambda_1 = 1,\ \lambda_2 = 2.$$

So $y_h = c_1 e^x + c_2 e^{2x}$. For a particular solution, try

$$y_p = x(Ax^4 + Bx^3 + Cx^2 + Dx + E)e^x + (Gx + H)\,e^{-2x}\cos 3x + (Kx + L)\,e^{-2x}\sin 3x.$$

53. $y'' - 2\,y' + y = 6x - e^x$. Characteristic equation

$$\lambda^2 - 2\,\lambda + 1 = 0 \;\Rightarrow\; \lambda_1 = 1 (\text{double root}).$$

So $y_h = c_1 e^x + c_2 x e^x$. For a particular solution, try

$$y_p = Ax + B + Cx^2 e^x.$$

57. $y'' + 4y = \cos \omega x$. We have

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

If $\omega \neq \pm 2$, a particular solution of

$$y'' + 4y = \cos \omega x$$

is $y_p = A\cos \omega x$. So $y_p'' = -A\omega^2 \cos \omega x$. Plugging into the equation, we find

$$\begin{aligned} A\cos\omega x(4-\omega^2) &= \cos\omega x; \\ A(4-\omega^2) &= 1; \\ A &= \frac{1}{4-\omega^2}. \end{aligned}$$

Note that $4 - \omega^2 \neq 0$, because $\omega \neq \pm 2$. So the general solution in this case is of the form

$$y_g = c_1 \cos 2x + c_2 \sin 2x + \frac{\cos \omega x}{4 - \omega^2}.$$

If $\omega = \pm 2$, then we modify the particular solution and use $y_p = x\big(A\cos \omega x + B\sin \omega x\big)$. Then

$$\begin{aligned} y_p' &= \big(A\cos\omega x + B\sin\omega x\big) + x\omega\big(-A\sin\omega x + B\cos\omega x\big), \\ y_p'' &= x\omega^2\big(-A\cos\omega x - B\sin\omega x\big) + 2\omega\big(-A\sin\omega x + B\cos\omega x\big). \end{aligned}$$

Plug into the left side of the equation

$$x\omega^2\big(-A\cos\omega x - B\sin\omega x\big) + 2\omega\big(-A\sin\omega x + B\cos\omega x\big) + 4x\big(A\cos\omega x + B\sin\omega x\big).$$

Using $\omega^2 = 4$, this becomes

$$2\omega\big(-A\sin\omega x + B\cos\omega x\big).$$

This should equal $\cos\omega x$. So $A = 0$ and $2\omega B = 1$ or $B = 1/(2\omega)$.

$$y_g = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2\omega} x \sin \omega x \quad (\omega = \pm 2).$$

Note that if $\omega = 2$ or $\omega = -2$, the solution is

$$y_g = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} x \sin 2x.$$

61. To solve $y'' - 4\,y = 0,\ y(0) = 0,\ y'(0) = 3$, start with the general solution

$$y(x) = c_1 \cosh 2x + c_2 \sinh 2x.$$

Then

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 \cosh 0 + c_2 \sinh 0 = 0 \\ &\Rightarrow \quad c_1 = 0;\ \text{so } y(x) = c_2 \sinh 2x. \\ y'(0) = 3 \quad &\Rightarrow \quad 2c_2 \cosh 0 = 3 \\ &\Rightarrow \quad c_2 = \frac{3}{2} \\ &\Rightarrow \quad y(x) = \frac{3}{2} \sinh 2x. \end{aligned}$$

65. To solve $y'' - 5\,y' + 6\,y = e^x, \quad y(0) = 0,\ y'(0) = 0$, use the general solution from Exercise 27 (modify it slightly):

$$y = c_1\, e^{2x} + c_2\, e^{3x} + \frac{1}{2}e^x.$$

Then

$$\begin{aligned} y(0) = 0 \quad &\Rightarrow \quad c_1 + c_2 = -\frac{1}{2}; \\ y'(0) = 0 \quad &\Rightarrow \quad 2c_1 + 3c_2 = -\frac{1}{2} \\ &\Rightarrow \quad c_2 = \frac{1}{2}; c_1 = -1 \\ &\Rightarrow \quad y(x) = -\,e^{2x} + \frac{1}{2}\, e^{3x} + \frac{1}{2}e^x. \end{aligned}$$

69. Because of the initial conditions, it is more convenient to take

$$y = c_1 \cos[2(x - \frac{\pi}{2})] + c_2 \sin[2(x - \frac{\pi}{2})]$$

as a general solution of $y'' + 4y = 0$. For a particular solution of $y'' + 4y = \cos 2x$, we try

$$y = Ax \sin 2x,\ y' = A \sin 2x + 2Ax \cos 2x,\ y'' = 4A \cos 2x - 4Ax \sin 2x.$$

Plug into the equation,

$$4A \cos 2x = \cos 2x \ \Rightarrow \ A = \frac{1}{4}.$$

So the general solution is

$$y = c_1 \cos[2(x - \frac{\pi}{2})] + c_2 \sin[2(x - \frac{\pi}{2})] + \frac{1}{4}x \sin 2x.$$

Using the initial conditions

$$\begin{aligned} y(\pi/2) = 1 \quad &\Rightarrow \quad c_1 = 1 \\ y'(\pi/2) = 0 \quad &\Rightarrow \quad 2c_2 + \frac{\pi}{4} \cos \pi = 0 \\ &\Rightarrow \quad c_2 = \frac{\pi}{8}; \end{aligned}$$

and so

$$y = \cos[2(x - \frac{\pi}{2})] + \frac{\pi}{8}\sin[2(x - \frac{\pi}{2})] + \frac{1}{4}x\sin 2x.$$

using the addition formulas for the cosine and sine, we can write

$$\cos[2(x - \frac{\pi}{2})] = -\cos 2x \quad \text{and} \quad \sin[2(x - \frac{\pi}{2})] = -\sin 2x,$$

and so

$$y = -\cos 2x - \frac{\pi}{8}\sin 2x + \frac{1}{4}x\sin 2x = -\cos 2x + \big(-\frac{\pi}{8} + \frac{1}{4}x\big)\sin 2x.$$

73. An antiderivative of $g(x) = e^{a\,x}\cos bx$ is a solution of the differential equation

$$y' = e^{a\,x}\cos bx.$$

We assume throughout this exercise that $a \neq 0$ and $b \neq 0$. For these special cases the integral is clear. To solve the differential equation we used the method of undetermined coefficients. The solution of homogeneous equation $y' = 0$ is $y = C$. To find a particular solution of $y' = e^{ax}\cos bx$ we try

$$\begin{aligned} y &= e^{ax}(A\cos bx + B\sin bx) \\ y' &= ae^{ax}(A\cos bx + B\sin bx) + e^{ax}(-bA\sin bx + bB\cos bx) \\ &= e^{ax}(Aa + bB)\cos bx + e^{ax}(aB - bA)\sin bx. \end{aligned}$$

Plugging into the equation, we find

$$\begin{aligned} e^{ax}(Aa + bB)\cos bx + e^{ax}(aB - bA)\sin bx &= e^{ax}\cos bx \\ Aa + bB = 1,\ aB - bA = 0 \quad \Rightarrow \quad A &= \frac{a}{a^2 + b^2},\ B = \frac{b}{a^2 + b^2}; \end{aligned}$$

so

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}\big(a\cos x + b\sin bx\big) + C.$$

Solutions to Exercises A.3

1. We apply the reduction of order formula and take all constants of integration equal to 0.

$$y'' + 2\,y' - 3\,y = 0, \qquad y_1 = e^x;$$

$$p(x) = 2, \int p(x)\,dx = 2x, \qquad e^{-\int p(x)\,dx} = e^{-2x};$$

$$\begin{aligned} y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= e^x \int \frac{e^{-2x}}{e^{2x}}\,dx \\ &= e^x \int e^{-4x}\,dx = e^x\left[-\frac{1}{4}e^{-4x}\right] = -\frac{1}{4}e^{-3x}. \end{aligned}$$

Thus the general solution is

$$y = c_1 e^x + c_2 e^{-3x}.$$

5. $y'' + 4\,y = 0, \quad y_1 = \cos 2x.$

$$y'' + 4\,y = 0, \qquad y_1 = \cos 2x;$$

$$p(x) = 0, \int p(x)\,dx = 0, \qquad e^{-\int p(x)\,dx} = 1;$$

$$\begin{aligned} y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= \cos 2x \int \frac{1}{\cos^2 2x}\,dx \\ &= \cos 2x \int \sec^2 2x\,dx \\ &= \cos 2x \left[\frac{1}{2}\tan 2x\right] = \frac{1}{2}\sin 2x. \end{aligned}$$

Thus the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

9. Put the equation in standard form:

$$(1 - x^2)y'' - 2xy' + 2y = 0, \qquad y_1 = x;$$

$$y'' - \frac{2x}{1 - x^2}y' + \frac{2}{1 - x^2}y = 0, \qquad p(x) = -\frac{2x}{1 - x^2};$$

$$\begin{aligned} \int p(x)\,dx &= \int -\frac{2x}{1 - x^2}\,dx = \ln(1 - x^2) \\ e^{-\int p(x)\,dx} &= e^{-\ln(1 - x^2)} = \frac{1}{1 - x^2}; \\ y_2 &= x \int \frac{1}{(1 - x^2)x^2}\,dx. \end{aligned}$$

To evaluate the last integral, we use the partial fractions decomposition

$$\begin{aligned}
\frac{1}{(1-x^2)x^2} &= \frac{1}{(1-x)(1+x)x^2} \\
&= \frac{A}{(1-x)} + \frac{B}{(1-x)} + \frac{C}{x} + \frac{D}{x^2}; \\
&= \frac{A(1+x)x^2 + B(1-x)x^2 + C(1-x^2)x + D(1-x^2)}{(1-x)(1+x)x^2} \\
1 &= A(1+x)x^2 + B(1-x)x^2 + C(1-x^2)x + D(1-x^2). \\
\text{Take } x=0 &\Rightarrow D = 1. \\
\text{Take } x=1 &\Rightarrow 1 = 2A,\ A = \frac{1}{2}. \\
\text{Take } x=-1 &\Rightarrow 1 = 2B,\ B = \frac{1}{2}.
\end{aligned}$$

Checking the coefficient of x^3, we find $C=0$. Thus

$$\begin{aligned}
\frac{1}{(1-x^2)x^2} &= \frac{1}{2(1-x)} + \frac{1}{2(1+x)} + + \frac{1}{x^2} \\
\int \frac{1}{(1-x^2)x^2}\,dx &= -\frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) - \frac{1}{x} \\
&= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x}.
\end{aligned}$$

So

$$y_2 = x\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x}\right] = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1.$$

Hence the general solution

$$y = c_1\,x + c_2\left[\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right].$$

13. Put the equation in standard form:

$$\begin{aligned}
&x^2y'' + xy' + y = 0, && y_1 = \cos(\ln x); \\
&y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0, && p(x) = \frac{1}{x};
\end{aligned}$$

$$\begin{aligned}
\int p(x)\,dx &= \int \frac{1}{x}\,dx = \ln x \\
e^{-\int p(x)\,dx} &= e^{-\ln x} = \frac{1}{x}; \\
y_2 &= \cos(\ln x)\int \frac{1}{x\,\cos^2(\ln x)}\,dx \\
&= \cos(\ln x)\int \frac{1}{\cos^2 u}\,du \quad \left(u = \ln x,\ du = \frac{1}{x}\,dx\right) \\
&= \cos u\,\tan u = \sin u = \sin(\ln u).
\end{aligned}$$

Hence the general solution

$$y = c_1\cos(\ln x) + c_2\sin(\ln u).$$

17. Put the equation in standard form:

$$x\,y'' + 2\,(1-x)y' + (x-2)\,y = 0, \qquad y_1 = e^x;$$

$$y'' + \frac{2(1-x)}{x}y' + \frac{x-2}{x}y = 0, \qquad p(x) = \frac{2}{x} - 2;$$

$$\begin{aligned}
\int p(x)\,dx &= 2\ln x - 2x \\
e^{-\int p(x)\,dx} &= e^{-2\ln x + 2x} = \frac{e^{2x}}{x^2}; \\
y_2 &= e^x \int \frac{e^{2x}}{x^2 e^{2x}}\,dx \\
&= e^x \int \frac{1}{x^2}\,dx = -\frac{e^x}{x}.
\end{aligned}$$

Hence the general solution

$$y = c_1 e^x + c_2 \frac{e^x}{x}.$$

21. $y'' - 4y' + 3\,y = e^{-x}$.

$$\begin{aligned}
\lambda^2 - 4\lambda + 3 = 0 \quad &\Rightarrow \quad (\lambda-1)(\lambda-3) = 0 \\
&\Rightarrow \quad \lambda = 1 \text{ or } \lambda = 3.
\end{aligned}$$

Linearly independent solutions of the homogeneous equation:

$$y_1 = e^x \quad \text{and} \quad y_2 = e^{3x}.$$

Wronskian:

$$W(x) = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 3e^{4x} - e^{4x} = 2e^{4x}.$$

We now apply the variation of parameters formula with

$$\begin{aligned}
g(x) &= e^{-x}; \\
y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\,dx + y_2 \int \frac{y_1 g(x)}{W(x)}\,dx \\
&= e^x \int \frac{-e^{3x}e^{-x}}{2e^{4x}}\,dx + e^{3x}\int \frac{e^x e^{-x}}{2e^{4x}}\,dx \\
&= -\frac{e^x}{2}\int e^{-2x}\,dx + \frac{e^{3x}}{2}\int e^{-4x}\,dx \\
&= \frac{e^x}{4}e^{-2x} - \frac{e^{3x}}{8}e^{-4x} = \frac{e^{-x}}{4} - \frac{e^{-x}}{8} \\
&= \frac{e^{-x}}{8}.
\end{aligned}$$

Thus the general solution is

$$y = c_1 e^x + c_2 e^{3x} + \frac{e^{-x}}{8}.$$

25. $y'' + y = \sec x$.

Two linearly independent solutions of the homogeneous equation are:

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x.$$

Wronskian:

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

We now apply the variation of parameters formula with

$$\begin{aligned} g(x) &= \sec x = \frac{1}{\cos x}; \\ y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\, dx + y_2 \int \frac{y_1 g(x)}{W(x)}\, dx \\ &= -\cos x \int \frac{\sin x}{\cos x}\, dx + \sin x \int dx \\ &= \cos x \cdot \ln(|\cos x|) + x \sin x. \end{aligned}$$

Thus the general solution is

$$y = c_1 \cos x + c_2 \sin x + \cos x \cdot \ln(|\cos x|) + x \sin x.$$

29. $x^2 y'' + 3\,xy' + y = \sqrt{x}$. The homogeneous equation is an Euler equation. The indicial equation is

$$r^2 + 2r + 1 = 0 \quad \Rightarrow (r+1)^2 = 0.$$

We have one double indicial root $r = -1$. Hence the solutions of the homogenous equation

$$y_1 = x^{-1} \quad \text{and} \quad y_2 = x^{-1} \ln x.$$

Wronskian:

$$W(x) = \begin{vmatrix} \frac{1}{x} & \frac{1}{x}\ln x \\ \frac{-1}{x^2} & \frac{1-\ln x}{x^2} \end{vmatrix} = \frac{1-\ln x}{x^3} + \frac{\ln x}{x^3} = \frac{1}{x^3}.$$

We now apply the variation of parameters formula with

$$\begin{aligned} g(x) &= \frac{\sqrt{x}}{x^2} = x^{-\frac{3}{2}}; \\ y_p &= y_1 \int \frac{-y_2 g(x)}{W(x)}\, dx + y_2 \int \frac{y_1 g(x)}{W(x)}\, dx \\ &= -\frac{1}{x} \int \frac{\ln x}{x} x^3 x^{-\frac{3}{2}}\, dx + \frac{\ln x}{x} \int \frac{1}{x} x^3 x^{-\frac{3}{2}}\, dx \\ &= -\frac{1}{x} \int \overbrace{\ln x}^{u}\, \overbrace{\sqrt{x}\, dx}^{dv} + \frac{\ln x}{x} \int x^{\frac{1}{2}}\, dx \\ &= -\frac{1}{x} \left[\frac{2}{3} x^{3/2} \ln x - \int \frac{2}{3} x^{3/2} \frac{1}{x} dx \right] + \frac{\ln x}{x} \frac{2}{3} x^{3/2} \\ &= \frac{1}{x} \frac{2}{3} \frac{2}{3} x^{3/2} = \frac{4}{9} x^{1/2}. \end{aligned}$$

Thus the general solution is

$$y = c_1 x^{-1} + c_2 x^{-1} \ln x + \frac{4}{9} x^{1/2}.$$

33. $x^2 y'' + 3xy' + y = 0$. See Exercise 29.

37. $x^2 y'' + 7xy' + 13\,y = 0$. Euler equation with $\alpha = 7$, $\beta = 13$, indicial equation $r^2 + 6\,r + 13 = 0$; indicial roots:

$$r = -3 \pm \sqrt{-4}; \qquad r_1 = -3 - 2i, \quad r_2 = -3 + 2i.$$

Hence the general solution

$$y = x^{-3}\left[c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right].$$

41. We have

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx.$$

Using the product rule for differentiation

$$\begin{aligned} y_2' &= y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + y_1 \frac{e^{-\int p(x)\,dx}}{y_1^2} \\ &= y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + \frac{e^{-\int p(x)\,dx}}{y_1}. \end{aligned}$$

So

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \\ y_1' & y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + \frac{e^{-\int p(x)\,dx}}{y_1} \end{vmatrix} \\ &= y_1 y_1' \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx + e^{-\int p(x)\,dx} - y_1' y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \\ &= e^{-\int p(x)\,dx} > 0. \end{aligned}$$

This also follows from Abel's formula, (4), Section A.1.

45. (a) From Abel's formula (Theorem 2, Section A.1), the Wronskian is

$$y_1 y_2' - y_1' y_2 = Ce^{-\int p(x)\,dx},$$

where y_1 and y_2 are any two solution of (2).

(b) Given y_1, set $C = 1$ in (a)

$$y_1 y_2' - y_1' y_2 = e^{-\int p(x)\,dx}.$$

This is a first-order differential equation in y_2 that we rewrite as

$$y_2' - \frac{y_1'}{y_1} y_2 = e^{-\int p(x)\,dx}.$$

The integrating factor is

$$e^{\int -\frac{y_1'}{y_1}\,dx} = e^{-\ln y_1} = \frac{1}{y_1}.$$

Multiply by the integrating factor:

$$\frac{y_2'}{y_1} - \frac{y_1'}{y_1^2} y_2 = \frac{1}{y_1} e^{-\int p(x)\,dx}$$

or

$$\frac{d}{dx}\left[\frac{y_2}{y_1}\right] = \frac{1}{y_1} e^{-\int p(x)\,dx}$$

Integrating both sides, we get

$$\begin{aligned} \frac{y_2}{y_1} &= \int \left[\frac{1}{y_1} e^{-\int p(x)\,dx}\right] dx; \\ y_2 &= y_1 \int \left[\frac{1}{y_1} e^{-\int p(x)\,dx}\right] dx, \end{aligned}$$

which implies (3).

49. $3\,y'' + 13\,y' + 10\,y = \sin x, \quad y_1 = e^{-x}.$
As in the previous exercise, let

$$y_1 = e^{-x},\; y = ve^{-x},\; y' = v'e^{-x} - ve^{-x},\; y'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}.$$

Then

$$\begin{aligned} 3\,y'' + 13\,y' + 10\,y = \sin x \quad &\Rightarrow \quad 3(v''e^{-x} - 2v'e^{-x} + ve^{-x}) \\ &\qquad +13(v'e^{-x} - ve^{-x}) + 10ve^{-x} = \sin x \\ &\Rightarrow \quad 3v'' + 7v' = e^{x}\sin x \\ &\Rightarrow \quad v'' + \frac{7}{3}v' = \frac{1}{3}e^{x}\sin x. \end{aligned}$$

We now solve the first order o.d.e. in v':

$$\begin{aligned} e^{7x/3}v'' + \frac{7}{3}e^{7x/3}v' &= e^{7x/3}\frac{1}{3}e^{x}\sin x \\ \frac{d}{dx}\left[e^{7x/3}v'\right] &= \frac{1}{3}e^{10x/3}\sin x \\ e^{7x/3}v' &= \frac{1}{3}\int \frac{1}{3}e^{10x/3}\sin x\,dx \\ &= \frac{1}{3}\frac{e^{10x/3}}{(\frac{10}{3})^2 + 1}(\frac{10}{3}\sin x - \cos) + C \\ v' &= \frac{e^{x}}{109}(10\sin x - \frac{9}{3}\cos) + C. \end{aligned}$$

(We used the table of integrals to evaluate the preceding integral. We will use it again below.) Integrating once more,

$$\begin{aligned} v &= \frac{10}{109}\int e^{x}\sin x\,dx - \frac{9}{327}\int e^{x}\cos x\,dx \\ &= \frac{10}{109}\frac{e^{x}}{2}(\sin x - \cos x) - \frac{9}{327}\frac{e^{x}}{2}(\cos x + \sin x) + C \\ y &= vy_1 = \frac{10}{218}(\sin x - \cos x) - \frac{9}{654}(\cos x + \sin x) + Ce^{-x} \\ &= -\frac{13}{218}\cos x + \frac{7}{218}\sin x + Ce^{-x}. \end{aligned}$$

Solutions to Exercises A.4

1. Using the ratio test, we have that the series

$$\sum_{m=0}^{\infty} \frac{x^m}{5m+1}$$

converges whenever the limit

$$\lim_{m\to\infty} \left| \frac{x^{m+1}}{5(m+1)+1} \Big/ \frac{x^m}{5m+1} \right| = \lim_{m\to\infty} \left(\frac{5m+1}{5m+6} \right) |x| = |x|$$

is less than 1. That is, $|x| < 1$. Thus, the interval of convergence is $|x| < 1$ or $(-1, 1)$. Since the series is centered at 0, the radius of convergence is 1.

5. Using the ratio test, we have that the series

$$\sum_{m=1}^{\infty} \frac{m^m\, x^m}{m!}$$

converges whenever the following limit is < 1:

$$\begin{aligned} \lim_{m\to\infty} \left| \frac{(m+1)^{m+1}\, x^{m+1}}{(m+1)!} \Big/ \frac{m^m\, x^m}{m!} \right| &= \lim_{m\to\infty} \frac{(m+1)^m(m+1)}{(m+1)!} \cdot \frac{m!}{m^m\, x^m} \cdot |x| \\ &= |x| \lim_{m\to\infty} \left(\frac{m+1}{m} \right)^m = e|x|. \end{aligned}$$

We have used the limit

$$\lim_{m\to\infty} \left(\frac{m+1}{m} \right)^m = \lim_{m\to\infty} \left(1 + \frac{1}{m} \right)^m = e$$

(see the remark at the end of the solution). From $|x|e < 1$ we get $|x| < 1/e$. Hence the interval of convergence is $(-1/e,\, 1/e)$. It is centered at 0 and has radius $1/e$.

One way to show

$$\lim_{m\to\infty} \left(\frac{m+1}{m} \right)^m = e$$

is to show that the natural logarithm of the limit is 1:

$$\ln \left(\frac{m+1}{m} \right)^m = m \ln \left(\frac{m+1}{m} \right) = m \left[\ln(m+1) - \ln m \right].$$

By the mean value theorem (applied to the function $f(x) = \ln x$ on the interval $[m,\, m+1]$), there is a real number c_m in $[m,\, m+1]$ such that

$$\ln(m+1) - \ln m = f'(c_m) = \frac{1}{c_m}$$

Note that

$$\frac{1}{m+1} \le \frac{1}{c_m} \le \frac{1}{m}.$$

So

$$\frac{m}{m+1} \le m \left[\ln(m+1) - \ln m \right] \le 1.$$

As $m \to \infty$, $\frac{m}{m+1} \to 1$, and so by the sandwich theorem,

$$m \left[\ln(m+1) - \ln m \right] \to 1.$$

Taking the exponential, we derive the desired limit.

9. Using the ratio test, we have that the series

$$\sum_{m=1}^{\infty} \frac{[10(x+1)]^{2m}}{(m!)^2}$$

converges whenever the following limit is < 1:

$$\begin{aligned}
\lim_{m\to\infty} \left| \frac{[10(x+1)]^{2(m+1)}}{((m+1)!)^2} \Big/ \frac{[10(x+1)]^{2m}}{(m!)^2} \right| &= 10^2|x+1|^2 \lim_{m\to\infty} \frac{(m!)^2}{((m+1)!)^2} \\
&= 10^2|x+1|^2 \lim_{m\to\infty} \frac{(m!)^2}{(m+1)^2(m!)^2} \\
&= 10^2|x+1|^2 \lim_{m\to\infty} \frac{1}{(m+1)^2} = 0.
\end{aligned}$$

Thus the series converges for all x, $R = \infty$.

13. We use the geometric series. For $|x| < 1$,

$$\begin{aligned}
\frac{3-x}{1+x} &= -\frac{1+x}{1+x} + \frac{4}{1+x} \\
&= -1 + \frac{4}{1-(-x)} \\
&= -1 + 4\sum_{n=0}^{\infty} (-1)^n x^n.
\end{aligned}$$

17. Use the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x < \infty.$$

Then

$$e^{u^2} = \sum_{n=0}^{\infty} \frac{u^{2n}}{n!} \qquad -\infty < u < \infty.$$

Hence, for all x,

$$\begin{aligned}
e^{3x^2+1} &= e \cdot e^{(\sqrt{3}x)^2} \\
&= e\sum_{n=0}^{\infty} \frac{(\sqrt{3}x)^{2n}}{n!} \\
&= e\sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}.
\end{aligned}$$

21. We have

$$\frac{1}{2+3x} = \frac{1}{2(1-(-\frac{3x}{2}))} = \frac{1}{2(1-u)},$$

where $u = -\frac{3x}{2}$. So

$$\frac{1}{2+3x} = \frac{1}{2}\sum_{n=0}^{\infty} u^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{3x}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{3^n}{2^{n+1}} x^n.$$

The series converges if $|u| < 1$; that is $|x| < \frac{2}{3}$.

25. Let a be any real number $\neq 0$, then

$$\begin{aligned}\frac{1}{x} &= \frac{1}{a-a+x}\\ &= \frac{1}{a}\cdot\frac{1}{1-\left(\frac{a-x}{a}\right)}\\ &= \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{a-x}{a}\right)^n\\ &= \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n\frac{(x-a)^n}{a^n}.\end{aligned}$$

The series converges if

$$\left|\frac{a-x}{a}\right| < 1 \quad \text{or} \quad |a-x| < |a|.$$

29. Recall that changing m to $m-1$ in the terms of the series requires shifting the index of summation up by 1. This is what we will do in the second series:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{x^m}{m} - 2\sum_{m=0}^{\infty} m x^{m+1} &= \sum_{m=1}^{\infty}\frac{x^m}{m} - 2\sum_{m=1}^{\infty}(m-1)x^m\\ &= \sum_{m=1}^{\infty} x^m\left[\frac{1}{m} - 2(m-1)\right]\\ &= \sum_{m=1}^{\infty}\frac{-2m^2+2m+1}{m}x^m.\end{aligned}$$

33. Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Then

$$\begin{aligned}y' + y &= \sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m\\ &= \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m + \sum_{m=0}^{\infty} a_m x^m\\ &= \sum_{m=0}^{\infty}\left[(m+1)a_{m+1} + a_m\right]x^m\end{aligned}$$

37. Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned}
x^2y'' + y &= x^2 \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=2}^{\infty} m(m-1)a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=2}^{\infty} m(m-1)a_m x^m + a_0 + a_1 x + \sum_{m=2}^{\infty} a_m x^m \\
&= a_0 + a_1 x + \sum_{m=2}^{\infty} \left[m(m-1)a_m + a_m \right] x^m \\
&= a_0 + a_1 x + \sum_{m=2}^{\infty} (m^2 - m + 1)a_m x^m.
\end{aligned}$$

Solutions to Exercises A.5

1. For the differential equation $y' + 2xy = 0$, $p(x) = 1$ is its own power series expansion about $a = 0$. So $a = 0$ is an ordinary point. To solve, let Let

$$y = \sum_{m=0}^{\infty} a_m x^m \qquad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}.$$

Then

$$\begin{aligned} y' + 2xy &= \sum_{m=1}^{\infty} m a_m x^{m-1} + 2x \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} 2 a_m x^{m+1} \\ &= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} 2 a_{m-1} x^m \\ &= a_1 + \sum_{m=1}^{\infty} \left[(m+1) a_{m+1} + 2 a_{m-1}\right] x^m. \end{aligned}$$

So $y' + 2xy = 0$ implies that

$$\begin{aligned} a_1 + \sum_{m=1}^{\infty} \left[(m+1) a_{m+1} + 2 a_{m-1}\right] x^m &= 0; \\ a_1 &= 0 \\ (m+1) a_{m+1} + 2 a_{m-1} &= 0 \\ a_{m+1} &= -\frac{2}{m+1} a_{m-1}. \end{aligned}$$

From the recurrence relation,

$$a_1 = a_3 = a_5 = \cdots = a_{2k+1} = \cdots = 0;$$

a_0 is arbitrary;

$$\begin{aligned} a_2 &= -\frac{2}{2} a_0 = -a_0, \\ a_4 &= -\frac{2}{4} a_2 = \frac{1}{2!} a_0, \\ a_6 &= -\frac{2}{6} a_4 = -\frac{1}{3!} a_0, \\ a_8 &= -\frac{2}{8} a_6 = \frac{1}{4!} a_0, \\ &\vdots \\ a_{2k} &= \frac{(-1)^k}{k!} a_0. \end{aligned}$$

So

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} = a_0 \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = a_0 e^{-x^2}.$$

5. For the differential equation $y''-y=0$, $p(x)=0$ is its own power series expansion about $a=0$. So $a=0$ is an ordinary point. To solve, let

$$y=\sum_{m=0}^{\infty} a_m x^m \qquad y'=\sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y''=\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned} y''-y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} - a_m\right] x^m. \end{aligned}$$

So $y''-y=0$ implies that

$$(m+2)(m+1)a_{m+2}-a_m=0 \quad \Rightarrow \quad a_{m+2}=\frac{a_m}{(m+2)(m+1)} \text{ for all } m\geq 0.$$

So a_0 and a_1 are arbitrary;

$$\begin{aligned} a_2 &= \frac{a_0}{2}, \\ a_4 &= \frac{a_2}{4\cdot 3} = \frac{a_0}{4!}, \\ a_6 &= \frac{a_4}{6\cdot 5} = \frac{a_0}{6!}, \\ &\vdots \\ a_{2n} &= \frac{a_0}{(2n)!}. \end{aligned}$$

Similarly,

$$a_{2n+1}=\frac{a_1}{(2n+1)!},$$

and so

$$y=a_0\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n}+a_1\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1}=a_0\cosh x + a_1 \sinh x.$$

9. For the differential equation $y''+2\,x\,y'+y=0$, $p(x)=2x$ is its own power series expansion about $a=0$. So $a=0$ is an ordinary point. To solve, let

$$y=\sum_{m=0}^{\infty} a_m x^m \qquad y'=\sum_{m=1}^{\infty} m a_m x^{m-1} \qquad y''=\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}.$$

Then

$$\begin{aligned} y''+2\,x\,y'+y &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=0}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} + (2m+1)a_m\right] x^m. \end{aligned}$$

So $y'' + 2\,x\,y' + \,y = 0$ implies that

$$(m+2)(m+1)a_{m+2} + (2m+1)a_m = 0$$

$$\Rightarrow \quad a_{m+2} = -\frac{(2m+1)}{(m+2)(m+1)}a_m \text{ for all } m \geq 0.$$

So a_0 and a_1 are arbitrary;

$$\begin{aligned}
a_2 &= -\frac{1}{2}a_0, \\
a_4 &= -\frac{5}{4\cdot 3}a_2 = \frac{5}{4!}a_0, \\
a_6 &= -\frac{9}{6\cdot 5}\frac{5}{4!}a_0 = \frac{-9\cdot 5}{6!}a_0, \\
&\vdots \\
a_3 &= -\frac{3}{3\cdot 2}a_1 \\
a_5 &= -\frac{7}{5\cdot 4}a_3 = \frac{7\cdot 3}{5!}a_1 \\
a_7 &= -\frac{11\cdot 7\cdot 3}{7!}a_1 \\
&\vdots
\end{aligned}$$

So

$$\begin{aligned}
y = a_0 &\left(1 - \frac{1}{2}x^2 + \frac{5}{4!}x^4 - \frac{9\cdot 5}{6!}x^6 + \cdots\right) \\
&+ a_1\left(x - \frac{3}{3!}x^3 + \frac{7\cdot 3}{5!}x^5 - \frac{11\cdot 7\cdot 3}{7!}x^7 + \cdots\right).
\end{aligned}$$

13. To solve $(1-x^2)y'' - 2xy' + 2y = 0,\ \ y(0) = 0,\ \ y'(0) = 3$, follow the steps in Example 5 and you will arrive at the recurrence relation

$$a_{m+2} = \frac{m(m+1)-2}{(m+2)(m+1)}a_m = \frac{(m+2)(m-1)}{(m+2)(m+1)}a_m = \frac{m-1}{m+1}a_m, \quad m \geq 0.$$

The initial conditions give you $a_0 = 0$ and $a_1 = 3$. So $a_2 = a_4 = \cdots = 0$ and, from the recurrence relation with $m = 1$,

$$a_3 = \frac{(1-1}{1+1}a_0 = 0.$$

So $a_5 = a_7 = \cdots = 0$ and hence $y = 3x$ is the solution.

17. Put the equation $(1-x^2)y'' - 2xy' + 2y = 0$ in the form

$$y'' - \frac{2x}{1-x^2} + \frac{2}{1-x^2}y = 0.$$

Apply the reduction of order formula with $y_1 = x$ and $p(x) = -\frac{2x}{1-x^2}$. Then

$$\begin{aligned} e^{-\int p(x)\,dx} &= e^{\int \frac{2x}{1-x^2}\,dx} \\ &= e^{-\ln(1-x^2)} = \frac{1}{1-x^2} \\ y_2 &= y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \\ &= x \int \frac{1}{x^2(1-x^2)}\,dx \end{aligned}$$

Use a partial fractions decomposition

$$\begin{aligned} \frac{1}{x^2(1-x^2)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x} \\ &= \frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)} \end{aligned}$$

So

$$\begin{aligned} y_2 &= x \int \left(\frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)} \right) dx \\ &= x \left[(-\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) \right] \\ &= -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

The following notebook illustrates how we can use Mathematica to solve a differential equations with poser series.

The solution is y and we will solve for the first 10 coefficients.

Let's define a partial sum of the Taylor series solution (degree 3) and set y[0]=1:

```
In[82]:= seriessol = Series[y[x], {x, 0, 3}] /. y[0] → 1
```

Out[82]= $1 + y'[0]\,x + \frac{1}{2} y''[0]\,x^2 + \frac{1}{6} y^{(3)}[0]\,x^3 + O[x]^4$

Next we set equations based on the given differential equation y'+y=0.

```
In[83]:= leftside = D[seriessol, x] + seriessol
         rightside = 0
         equat = LogicalExpand[leftside == rightside]
```

Out[83]= $(1 + y'[0]) + (y'[0] + y''[0])\,x + \left(\frac{y''[0]}{2} + \frac{1}{2} y^{(3)}[0]\right) x^2 + O[x]^3$

Out[84]= 0

Out[85]= $1 + y'[0] == 0 \;\&\&\; y'[0] + y''[0] == 0 \;\&\&\; \frac{y''[0]}{2} + \frac{1}{2} y^{(3)}[0] == 0$

This gives you a set of equations in the coefficients that *Mathematica* can solve

```
In[86]:= seriescoeff = Solve[equat]
```

Out[86]= $\{\{y'[0] \to -1,\ y''[0] \to 1,\ y^{(3)}[0] \to -1\}\}$

Next, we substitute these coefficients in the series solution. This can be done as follows

```
In[87]:= seriessol /. seriescoeff[[1]]
```

Out[87]= $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + O[x]^4$

To get a partial sum without the Big O, use

```
In[88]:= Normal[seriessol /. seriescoeff[[1]]]
```

Out[88]= $1 - x + \frac{x^2}{2} - \frac{x^3}{6}$

With the previous example in hand, we can solve Exercises 19-22 using Mathematica by repeating and modifying the commands. Here is an illustration with Exercise 19. We suppress some outcomes to save space.

19. $y'' - y' + 2y = e^x$, $y(0) = 0$, $y'(0) = 1$.

In[70]:=
```
Clear[y, seriessol, n, partsol]
n = 10
seriessol = Series[y[x], {x, 0, n}] /. {y[0] → 0, y'[0] → 1}
leftside = D[seriessol, {x, 2}] - D[seriessol, {x, 1}] + 2 seriessol;
rightside = Series[E^x, {x, 0, n}];
equat = LogicalExpand[leftside == rightside];
seriescoeff = Solve[equat];
partsol = Normal[seriessol /. seriescoeff[[1]]];
```

Out[71]= 10

Out[72]= $x + \frac{1}{2} y''[0]\, x^2 + \frac{1}{6} y^{(3)}[0]\, x^3 + \frac{1}{24} y^{(4)}[0]\, x^4 + \frac{1}{120} y^{(5)}[0]\, x^5 + \frac{1}{720} y^{(6)}[0]\, x^6 + \frac{y^{(7)}[0]\, x^7}{5040} + \frac{y^{(8)}[0]\, x^8}{40320} + \frac{y^{(9)}[0]\, x^9}{362880} + \frac{y^{(10)}[0]\, x^{10}}{3628800} + O[x]^{11}$

The equation can be solved using analytical methods (undetermined coefficients). The exact solution is

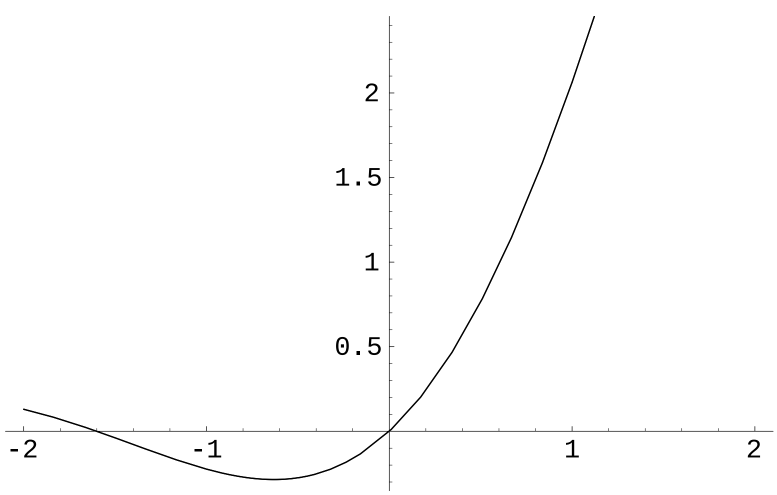
```
sol = DSolve[{ y''[x] - y'[x] + 2 y[x] == E^x, y[0] == 0, y'[0] == 1}, y[x], x];
```

In[46]:=
```
sss = sol[[1, 1, 2]]
```

$$\frac{1}{14} e^{x/2} \left(-7 \operatorname{Cos}\left[\frac{\sqrt{7}\, x}{2}\right] + 7 e^{x/2} \operatorname{Cos}\left[\frac{\sqrt{7}\, x}{2}\right]^2 + 3\sqrt{7} \operatorname{Sin}\left[\frac{\sqrt{7}\, x}{2}\right] + 7 e^{x/2} \operatorname{Sin}\left[\frac{\sqrt{7}\, x}{2}\right]^2\right)$$

Let's compare with the partial sum that we found earlier

In[67]:=
```
Plot[{sss, partsol}, {x, -2, 2}]
```



Out[67]= - Graphics -

We have a nice match on the interval [-2, 2]

Solutions to Exercises A.6

1. For the equation $y'' + (1-x^2)y' + xy = 0$, $p(x) = 1-x^2$ and $q(x) = x$ are both analytic at $a = 0$. So $a = 0$ is a ordinary point.

5. For the equation $x^2y'' + (1-e^x)y' + xy = 0$,

$$p(x) = \frac{1-e^x}{x^2}, \qquad x\,p(x) = \frac{1-e^x}{x};$$

$$q(x) = \frac{1}{x}, \qquad x^2q(x) = x.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. To see that $xp(x)$ is analytic at 0, derive its Taylor series as follows: for all x,

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ 1 - e^x &= -x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \\ &= x\left(-1 - \frac{x}{2!} - \frac{x^2}{3!} + \cdots\right) \\ \frac{1-e^x}{x} &= -1 - \frac{x}{2!} - \frac{x^2}{3!} + \cdots . \end{aligned}$$

Since $\frac{1-e^x}{x}$ has a Taylor series expansion about 0 (valid for all x), it is analytic at 0.

9. For the equation $4\,x^2\,y'' - 14\,x\,y' + (20-x)\,y = 0$,

$$p(x) = -\frac{7}{2x}, \qquad x\,p(x) = -\frac{7}{2}, \qquad p_0 = -\frac{7}{2};$$

$$q(x) = \frac{20-x}{4x^2}, \qquad x^2q(x) = 5 - \frac{x}{4}, \qquad q_0 = 5.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. Indicial equation

$$\begin{aligned} r(r-1) - \frac{7}{2}r + 5 = 0 \quad &\Rightarrow \quad 2r^2 - 9r + 10 = 0, \quad (r-2)(2r-5) = 0 \\ &= \quad r_1 = \frac{5}{2} \quad r_2 = 2. \end{aligned}$$

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, we are in Case I. The solutions are of the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^{m+2} \quad \text{and} \quad y_2 = \sum_{m=0}^{\infty} b_m x^{m+\frac{5}{2}},$$

with $a_0 \neq 0$ and $b_0 \neq 0$. Let us determine y_1. We use y instead of y to simplify the notation. We have

$$y = \sum_{m=0}^{\infty} a_m x^{m+2}; \quad y' = \sum_{m=0}^{\infty} (m+2)a_m x^{m+1}; \quad y'' = \sum_{m=0}^{\infty} (m+2)(m+1)a_m x^m.$$

Then

$$4\,x^2y'' - 14\,x\,y' + (20-x)y$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} 4(m+2)(m+1)a_m x^{m+2} - 14\sum_{m=1}^{\infty}(m+2)a_m x^{m+2} + (20-x)\sum_{m=0}^{\infty} a_m x^{m+2}\\
&= \sum_{m=0}^{\infty}[4(m+1)-14](m+2)a_m x^{m+2} + 20\sum_{m=0}^{\infty} a_m x^{m+2} - \sum_{m=0}^{\infty} a_m x^{m+3}\\
&= \sum_{m=0}^{\infty}[(4m-10)(m+2)+20]a_m x^{m+2} - \sum_{m=0}^{\infty} a_m x^{m+3}\\
&= \sum_{m=0}^{\infty}[4m^2-2m]a_m x^{m+2} - \sum_{m=1}^{\infty} a_{m-1} x^{m+2}\\
&= \sum_{m=1}^{\infty}[4m^2-2m]a_m - a_{m-1}]x^{m+2}
\end{aligned}$$

This gives the recurrence relation: For all $m \geq 0$,

$$a_m = \frac{a_{m-1}}{4m^2-2m} = \frac{a_{m-1}}{2m(2m-1)}.$$

Since a_0 is arbitrary, take $a_0 = 1$. Then

$$\begin{aligned}
a_1 &= \frac{1}{2};\\
a_2 &= \frac{1}{2(12)} = \frac{1}{4!};\\
a_3 &= \frac{1}{4!}\frac{1}{6\cdot 5} = \frac{1}{6!};\\
&\vdots\\
y_1 &= a_0x^2\left[1 + \frac{1}{2!}x + \frac{1}{4!}x^2 + \frac{1}{6!}x^3 + \cdots\right]
\end{aligned}$$

We now turn to the second solution:

$$y = \sum_{m=0}^{\infty} b_m x^{m+\frac{5}{2}};\quad y' = \sum_{m=0}^{\infty}(m+\frac{5}{2})b_m x^{m+\frac{3}{2}};\quad y'' = \sum_{m=0}^{\infty}(m+\frac{5}{2})(m+\frac{3}{2})b_m x^{m+\frac{1}{2}}.$$

So

$$4\,x^2\,y'' - 14\,x\,y' + (20-x)\,y$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty}\left[4(m+\frac{5}{2})(m+\frac{3}{2}) - 14(m+\frac{5}{2})\right] b_m x^{m+\frac{5}{2}} + (20-x)\sum_{m=0}^{\infty} b_m x^{m+\frac{5}{2}}\\
&= \sum_{m=0}^{\infty}\left[4(m+\frac{5}{2})(m+\frac{3}{2}) - 14(m+\frac{5}{2}) + 20\right] b_m x^{m+\frac{5}{2}} - \sum_{m=0}^{\infty} b_m x^{m+\frac{7}{2}}\\
&= \sum_{m=0}^{\infty}\left[4m^2+2m\right] b_m x^{m+\frac{5}{2}} - \sum_{m=1}^{\infty} b_{m-1} x^{m+\frac{5}{2}}\\
&= \sum_{m=1}^{\infty}\left[(4m^2+2m)b_m - b_{m-1}\right] x^{m+\frac{5}{2}}
\end{aligned}$$

This gives b_0 arbitrary and the recurrence relation: For all $m \geq 1$,

$$b_m = \frac{b_{m-1}}{2m(2m+1)}.$$

Since b_0 is arbitrary, take $b_0 = 1$. Then

$$\begin{aligned}
b_1 &= \frac{1}{3!};\\
b_2 &= \frac{1}{3!}\frac{1}{4\cdot 5} = \frac{1}{5!};\\
b_3 &= \frac{1}{5!}\frac{1}{6\cdot 7} = \frac{1}{7!};\\
&\vdots\\
y_2 &= b_0 x^{5/2}\left[1 + \frac{1}{3!}x + \frac{1}{5!}x^2 + \frac{1}{7!}x^3 + \cdots\right]
\end{aligned}$$

13. For the equation $x\,y'' + (1-x)\,y' + y = 0$,

$$p(x) = \frac{1-x}{x}, \qquad x\,p(x) = 1-x, \qquad p_0 = 1;$$

$$q(x) = \frac{1}{x}, \qquad x^2 q(x) = x, \qquad q_0 = 0.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2 q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. Indicial equation

$$r(r-1) + r = 0 \quad \Rightarrow \quad r = 0 \text{ (double root)}.$$

We are in Case II. The solutions are of the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^m \quad \text{and} \quad y_2 = y_1 \ln x + \sum_{m=0}^{\infty} b_m x^m,$$

with $a_0 \neq 0$. Let us determine y_1. We use y instead of y to simplify the notation. We have

$$y = \sum_{m=0}^{\infty} a_m x^m; \quad y' = \sum_{m=0}^{\infty} m a_m x^{m-1}; \quad y'' = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Plug into $x\,y'' + (1-x)\,y' + y = 0$:

$$\begin{aligned}
\sum_{m=1}^{\infty} m(m-1)a_m x^{m-1} + \sum_{m=1}^{\infty} m a_m x^{m-1} & \\
- \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m &= 0 \\
\sum_{m=0}^{\infty} (m+1)m a_{m+1} x^m + \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m & \\
- \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m &= 0 \\
\sum_{m=1}^{\infty} (m+1)m a_{m+1} x^m + a_1 + \sum_{m=1}^{\infty} (m+1) a_{m+1} x^m & \\
- \sum_{m=1}^{\infty} m a_m x^m + a_0 + \sum_{m=1}^{\infty} a_m x^m &= 0 \\
a_0 + a_1 + \sum_{m=1}^{\infty} \left[(m+1)m a_{m+1} + (m+1)a_{m+1} - m a_m + a_m\right] x^m &= 0 \\
a_0 + a_1 + \sum_{m=1}^{\infty} \left[(m+1)^2 a_{m+1} + (1-m)a_m\right] x^m &= 0
\end{aligned}$$

This gives $a_0 + a_1 = 0$ and the recurrence relation: For all $m \geq 1$,

$$a_{m+1} = -\frac{1-m}{(m+1)^2} a_m.$$

Take $a_0 = 1$. Then $a_1 = -1$ and $a_2 = a_3 = \cdots = 0$. So $y_1 = 1 - x$. We now turn to the second solution: (use $y_1 = 1 - x$, $y_1' = -1$, $y_1'' = 0$)

$$\begin{aligned}
y &= y_1 \ln x + \sum_{m=0}^{\infty} b_m x^m; \\
y' &= y_1' \ln x + \frac{y_1}{x} + \sum_{m=0}^{\infty} m b_m x^{m-1}; \\
y'' &= y_1'' \ln x + \frac{2}{x} y_1' - \frac{1}{x^2} y_1 + \sum_{m=0}^{\infty} m(m-1) b_m x^{m-2}.
\end{aligned}$$

Plug into $xy'' + (1-x)\,y' + y = 0$:

$$\begin{aligned}
& xy_1''\ln x + 2y_1' - \frac{1}{x}y_1 + \sum_{m=0}^{\infty} m(m-1)b_m x^{m-1} \\
+(1-x)y_1'\ln x + \frac{y_1}{x}(1-x) + (1-x)\sum_{m=0}^{\infty} m b_m x^{m-1} + y_1\ln x + \sum_{m=0}^{\infty} b_m x^m &= 0 \\
& -2 - \frac{1}{x}(1-x) + \sum_{m=1}^{\infty} m(m-1)b_m x^{m-1} \\
+\frac{(1-x)^2}{x} + \sum_{m=0}^{\infty} m b_m x^{m-1} - \sum_{m=0}^{\infty} m b_m x^m + \sum_{m=0}^{\infty} b_m x^m &= 0 \\
-3 + x + \sum_{m=0}^{\infty} [(m+1)m b_{m+1} + (m+1)b_{m+1} - m b_m + b_m]x^m &= 0 \\
-3 + x + \sum_{m=0}^{\infty} [(m+1)^2 b_{m+1} + (1-m)b_m]x^m &= 0
\end{aligned}$$

For the constant term, we get $b_1 + b_0 - 3 = 0$. Take $b_0 = 0$. Then $b_1 = 3$. For the term in x, we get

$$1 + 2b_2 + 2b_2 - b_1 + b_1 = 0 \quad \Rightarrow \quad b_2 = -\frac{1}{4}.$$

For all $m \geq 3$,

$$b_{m+1} = \frac{m-1}{(m+1)^2} b_m.$$

Then

$$\begin{aligned}
b_3 &= \frac{1}{9}(-\frac{1}{4}) = -\frac{1}{36}; \\
b_4 &= \frac{2}{16}(-\frac{1}{36}) = -\frac{1}{288}; \\
&\ \vdots \\
y_2 &= -3x - \frac{1}{4}x^2 - \frac{1}{36}x^3 + \cdots
\end{aligned}$$

17. For the equation $x^2y'' + 4\,x\,y' + (2-x^2)\,y = 0$,

$$p(x) = \frac{4}{x}, \qquad x\,p(x) = 4, \qquad p_0 = 4;$$

$$q(x) = \frac{2-x^2}{x^2}, \qquad x^2 q(x) = 2 - x^2, \qquad q_0 = 2.$$

$p(x)$ and $q(x)$ are not analytic at 0. So $a = 0$ is a singular point. Since $xp(x)$ and $x^2q(x)$ are analytic at $a = 0$, the point $a = 0$ is a regular singular point. Indicial equation

$$\begin{aligned}
r(r-1) + 4r + 2 = 0 \quad &\Rightarrow \quad r^3 + 3r + 2 = 0 \\
&\Rightarrow \quad r_1 = -2 \quad r_2 = -1.
\end{aligned}$$

We are in Case III. The solutions are of the form

$$y_1 = \sum_{m=0}^{\infty} a_m x^{m-1} \quad \text{and} \quad y_2 = k y_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-2},$$

with $a_0 \neq 0$, $b_0 \neq 0$. Let us determine y_1. We use y instead of y to simplify the notation. We have

$$y = \sum_{m=0}^{\infty} a_m x^{m-1}; \quad y' = \sum_{m=0}^{\infty} (m-1) a_m x^{m-2}; \quad y'' = \sum_{m=0}^{\infty} (m-1)(m-2) a_m x^{m-3}.$$

Plug into $x^2 y'' + 4\,x\,y' + (2 - x^2)\,y = 0$:

$$\begin{aligned}
\sum_{m=0}^{\infty} (m-1)(m-2) a_m x^{m-1} + \sum_{m=0}^{\infty} 4(m-1) a_m x^{m-1} & \\
\sum_{m=0}^{\infty} 2 a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^{m+1} &= 0 \\
(-1)(-2) a_0 x^{-1} + \sum_{m=2}^{\infty} (m-1)(m-2) a_m x^{m-1} & \\
+4(-1) a_0 x^{-1} + \sum_{m=2}^{\infty} 4(m-1) a_m x^{m-1} & \\
2 a_0 x^{-1} + 2 a_1 + \sum_{m=2}^{\infty} 2 a_m x^{m-1} - \sum_{m=2}^{\infty} a_{m-2} x^{m-1} &= 0 \\
2 a_1 + \sum_{m=2}^{\infty} [(m-1)(m-2) a_m + 4(m-1) a_m 2 a_m - a_{m-2}] x^{m-1} &= 0 \\
2 a_1 + \sum_{m=2}^{\infty} [(m^2 + m) a_m - a_{m-2}] x^{m-1} &= 0
\end{aligned}$$

This gives the recurrence relation: For all $m \geq 1$,

$$a_m = -\frac{1}{m^2 + m} a_{m-2}.$$

Take $a_0 = 1$ and $a_1 = 0$. Then $a_3 = a_5 = \cdots = 0$ and

$$\begin{aligned}
a_2 &= -\frac{1}{6} = -\frac{1}{3!} \\
a_4 &= -\frac{1}{4^2 + 4} a_2 = \frac{1}{20} \frac{1}{6} = \frac{1}{5!} \\
a_6 &= -\frac{1}{6^2 + 6} \frac{1}{5!} = -\frac{1}{7 \cdot 6} \frac{1}{5!} = \frac{1}{7!} \\
&\vdots \\
y_1 &= a_0 x^{-1} \left(1 - \frac{1}{3!} x^2 + \frac{1}{5!} x^4 - \frac{1}{7!} x^6 + \cdots \right)
\end{aligned}$$

We now turn to the second solution:

$$\begin{aligned} y &= ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-2}; \\ y' &= ky_1' \ln x + k\frac{y_1}{x} + \sum_{m=0}^{\infty} (m-2) b_m x^{m-3}; \\ y'' &= ky_1'' \ln x + \frac{2k}{x} y_1' - \frac{k}{x^2} y_1 + \sum_{m=0}^{\infty} (m-2)(m-3) b_m x^{m-4}. \end{aligned}$$

Plug into $xy'' + (1-x)\, y' + y = 0$:

$$\begin{aligned} 2kxy_1' + ky_1'' x^2 \ln x - ky_1 + \sum_{m=0}^{\infty} (m-2)(m-3) b_m x^{m-2} + & \\ 4ky_1' x \ln x + 4ky_1 + \sum_{m=0}^{\infty} 4(m-2) b_m x^{m-2} & \\ +(2-x^2) ky_1' \ln x + (2-x^2) \sum_{m=0}^{\infty} b_m x^{m-2} &= 0 \\ 2kxy_1' + 3ky_1 + \sum_{m=0}^{\infty} [(m+2)^2 - (m+2) b_{m+2} + b_m] x^m &= 0 \\ (m+2)^2 - (m+2) b_{m+2} + b_m &= 0 \end{aligned}$$

Take $k = 0$ and for all $m \geq 0$,

$$b_{m+2} = -\frac{b_m}{(m+2)(m+1)}.$$

Take $b_0 = 1$ and $b_1 = 0$. (Note that by setting $b_1 = 1$ and $b_0 = 0$, you will get y_1.) Then

$$\begin{aligned} b_2 &= -\frac{1}{2}; \\ b_4 &= \frac{1}{4!}; \\ &\vdots \\ y_2 &= b_0 x^{-2} \left(1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 - \cdots \right). \end{aligned}$$