A PRESENTATION THEOREM FOR CONTINUOUS LOGIC AND METRIC ABSTRACT ELEMENTARY CLASSES

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Contents

1. Introduction

In the spirit of Chang and Shelah's presentation results (from [Cha68] and [Sh88], respectively), we prove a presentation theorem for classes of continuous structures, both those axiomatized by first-order and beyond, in terms of a class of discrete structures. The thrust of this presentation theorem is the basic analytic fact that the behavior of continuous functions is determined by their values on a dense subset of their domain. Focusing on dense subsets is key because it allows us to drop the requirement that structures be complete, which is not a property expressible by discrete (classical) logic, even in the broader contexts of $L_{\lambda,\omega}$ or Abstract Elementary Classes.

The specific statements of the presentation theorems appear below (see Theorem 2.1 for continuous first-order logic and Theorem 6.1 for Metric Abstract Elementary Classes), but the general idea is the same in both cases: given a continuous language L, we define a discrete language L^+ that allows us to approximate the values of the functions and relations by a countable dense subset of values, namely $\mathbb{Q} \cap [0,1]$. Note that the specification that this dense set (and its completion) is standard already requires an $L^+_{\omega_1,\omega}$ sentence, even if we are working in continuous first-order logic. Then, given a continuous L-structure M and a nicely dense (see Definition 1.1 below) subset of it A, we can form a discrete L^+-structure $\mathcal{A}$ with universe A that encodes all of M.

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Given a class of continuous L-structures, we represent it by all dense approximations (in the way described above) to members of the class. In each presentation theorem, the way the continuous class is defined helps define the class of approximations. If the class is axiomatized by a continuous first-order theory, then the uniform continuity of the functions and relations (see Ben-Yaacov, Berenstein, Henson, and Usvyatsov [BBHU08]) allow an $L^+_{\omega_1,\omega}$ axiomatization of the dense approximations. If the continuous class is non-elementary and forms a Metric Abstract Elementary Class (see Hirvonen and Hyttinen [HH09] or Villaveces and Zambrano [ViZa]), then the class of dense approximations has no explicit axiomatization, but forms an Abstract Elementary Class.

We carry the relation between the class of continuous models and discrete approximations further by considering various model-theoretic properties and finding analogues for them in the class of approximations. We consider types, saturation, (and for Metric Abstract Elementary Classes), amalgamation, joint embedding, and d-tameness.

Throughout we assume that the reader is familiar with the basics of the continuous contexts–either first-order or Metric Abstract Elementary Classes–but try to provide references and reminders when discussing the concepts.

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Many of the arguments in the two cases are similar. In order to avoid repeating the same arguments twice in slightly different contexts, we provide the details only once. We have chosen to provide the details for continuous first-order logic because this context often allows more specific formulations of the correspondence.

Dense sets are not quite the right context because they need not be substructures of the larger structure. Instead, we introduce nicely dense sets to require them to be closed under functions.

Definition 1.1. *Given a continuous model M and a set $A \subset |M|$, we say that A is* nicely dense *iff A is dense in the metric structure $(|M|, d^M)$ and A is closed under the functions of M.*

In the what follows, we will often want to prove similar results for both "greater than" and "less than." In order to avoid writing everything twice, we often use $\Box$ to stand in for both $\geq$ and $\leq$. Thus, asserting a statement for "$r \Box s$" means that that statement is true both for "$r \geq s$" and for "$r \leq s$."

Our goal is to translate the real-valued formulas of L into classic, true/false formulas of L^+. We do this by encoding relations into L^+ that are intended to specify the value of ϕ by deciding if it is above or below each possible value.

To ensure that the size of the language doesn't grow, we take advantage of the separability or $\mathbb{R}$ and only compare each ϕ to the rationals in $[0,1]$. For notational ease, we set $\mathbb{Q}' := [0,1] \cap \mathbb{Q}$.

2. Models and Theories

The main thesis of the presentation of continuous first-order logic is that model-theoretic properties of continuous first order structures can be translated to model-theoretic (but typically quantifier free) properties of discrete structures that model a specific theory in an expanded language. We use $Fml^c L$ to denote the continuous formulas of the language L. The main theorem about this presentation is the following:

Theorem 2.1. *Let L be a continuous language. Then there is*

(a) *a discrete language L^+;*
(b) *an $L^+_{\omega_1,\omega}$ theory T_{dense};*
(c) *a map that takes continuous L-structures M and nicely dense subsets A to discrete L^+-structures M_A that model T_{dense};*
(d) *a map that takes discrete L^+ structures A that model T_{dense} to continuous L-structures $\overline{A}$*

with the properties that

(1) *$M_A \vDash T_{dense}$ has universe A and, for any $\mathbf{a} \in A$, $\phi(\mathbf{x}) \in Fml^c L$, $r \in \mathbb{Q}'$, and $\Box$ standing for $\geq$ and $\leq$, we have*
$$M_A \vDash R_{\phi\Box r}[\mathbf{a}] \iff \phi^M(\mathbf{a})\Box r$$
(2) *A is a dense subset of $\overline{A}$ and, for any $\mathbf{a} \in A$, $\phi(\mathbf{x}) \in Fml^c L$, $r \in \mathbb{Q}'$, and $\Box$ standing for $\geq$ and $\leq$, we have*
$$A \vDash R_{\phi\Box r}[\mathbf{a}] \iff \phi^{\overline{A}}(\mathbf{a})\Box r$$
(3) *these maps are (essentially) each other's inverse. That is, given any nicely dense $A \subset M$, we have $M \cong_A \overline{M_A}$ and, given any L^+-structure $A \vDash T_{dense}$, we have $(\overline{A})_A = A$.*

The "essentially" in the last clause comes from the fact that completions are not technically unique as the objects selected as limits can vary, but this fairly pedantic point is the only obstacle.

Restricting to dense subsets and their completions have already been considered in continuous first-order logic, where it goes by the name *prestructure* (see [BBHU08].§3). The key difference here is that, while prestructures are still continuous objects with uniformly continuous functions and relations, M_A is a discrete object with the relations on it being either true or false.

Proof: Our proof is long, but straightforward. First, we will define L^+ and T_{dense}. Then, we will introduce the map $(M, A) \mapsto M_A$ and prove it satisfies (1). After this, we will introduce the other map $A \mapsto \overline{A}$ and prove (2). Finally, we will prove that they satisfy (3).

Defining the new language and theory

We define the language L^+ to be

$$\langle F_i^+, R_{\phi(\mathbf{x})\geq r}, R_{\phi(\mathbf{x})\leq r}\rangle_{i<n_F, \phi(\mathbf{x})\in Fml^c(L), r\in\mathbb{Q}'}$$

with the arity of F_i^+ matching the arity of F_i and the arity of $R_{\phi(\mathbf{x})\square r}$ matching $\ell(\mathbf{x})$. Since we only use a full (dense) set of connectives (see [BBHU08].6.1), we have ensured that $|L^+| = |L| + \aleph_0$.

We define $T_{dense} \subset L^+_{\omega_1,\omega}$ to be the universal closure of all of the following formulas ranging over all continuous formulas $\phi(\mathbf{z})$ and $\psi(\mathbf{z}')$, all terms $\tau(z, \mathbf{z}'')$, and all $r, s \in \mathbb{Q}'$ and $t \in \mathbb{Q}' - \{0\}$. We have divided them into headings so that their meaning is (hopefully) more clear. When we refer to specific sentences of T_{dense} later, we reference the ordering in this list. As always, a $\square$ in a formula means that it should be included with both a '$\geq$' and a '$\leq$' replacing the $\square$.

(1) **The ordered structure of $\mathbb{R}$**
   (a) $\neg R_{\phi(\mathbf{z})\geq r}(\mathbf{x}) \implies R_{\phi(\mathbf{z})\leq r}(\mathbf{x})$
   (b) $\neg R_{\phi(\mathbf{z})\leq r}(\mathbf{x}) \implies R_{\phi(\mathbf{z})\geq r}(\mathbf{x})$
   (c) If $r > s$, then include $\neg R_{\phi(\mathbf{z})\geq r}(\mathbf{x}) \vee \neg R_{\phi(\mathbf{z})\leq s}(\mathbf{x})$
   (d) If $r \geq s$, then include
      - $R_{\phi(\mathbf{z})\leq s}(\mathbf{x}) \implies R_{\phi(\mathbf{z})\leq r}(\mathbf{x})$; and
      - $R_{\phi(\mathbf{z})\geq r}(\mathbf{x}) \implies R_{\phi(\mathbf{z})\geq s}(\mathbf{x})$
   (e) $R_{\phi(\mathbf{z})\geq r}(\mathbf{x}) \vee R_{\phi(\mathbf{z})\leq r}(\mathbf{x})$
   (f) $\wedge_{n<\omega} \vee_{r,s\in\mathbb{Q}', |r-s|<\frac{1}{n}} R_{\phi\leq r}(\mathbf{x}) \wedge R_{\phi\geq s}(\mathbf{x})$
   (g) $(\wedge_{n<\omega} R_{\phi(\mathbf{z})\geq r-\frac{1}{n}}(\mathbf{x})) \implies R_{\phi(\mathbf{z})\geq r}(\mathbf{x})$
   (h) $(\wedge_{n<\omega} R_{\phi(\mathbf{z})\leq r+\frac{1}{n}}(\mathbf{x})) \implies R_{\phi(\mathbf{z})\leq r}(\mathbf{x})$

(2) **Construction of formulas**
   (a) $R_{\phi(\mathbf{z})\geq 0}(\mathbf{x}) \wedge R_{\phi(\mathbf{z})\leq 1}(\mathbf{x})$
   (b) $\neg R_{0\geq t}(\mathbf{x}) \wedge \neg R_{1\leq 1-t}(\mathbf{x})$;
   (c) $R_{\frac{\phi(\mathbf{z})}{2}\geq r}(\mathbf{x}) \iff R_{\phi(\mathbf{z})\geq 2r}(\mathbf{x})$
   (d) $R_{\frac{\phi(\mathbf{z})}{2}\leq r}(\mathbf{x}) \iff R_{\phi(\mathbf{z})\leq 2r}(\mathbf{x})$;
   (e) $R_{\phi(\mathbf{z})\dot{-}\psi(\mathbf{z}')\geq r}(\mathbf{x}, \mathbf{x}') \iff \vee_{s\in\mathbb{Q}'}(R_{\psi(\mathbf{z}')\leq s}(\mathbf{x}') \wedge R_{\phi(\mathbf{z})\geq r+s}(\mathbf{x}))$
   (f) $R_{\phi(\mathbf{z})\dot{-}\psi(\mathbf{z}')\leq r}(\mathbf{x}, \mathbf{x}') \iff \vee_{s\in\mathbb{Q}'}(\neg R_{\psi(\mathbf{z}')\leq s}(\mathbf{x}') \wedge R_{\phi(\mathbf{z})\leq r+s}(\mathbf{x}))$;
   (g) $R_{\sup_y \tau(y,\mathbf{y})\leq r}(\mathbf{x}) \iff \forall x R_{\tau(y,\mathbf{y})\leq r}(x, \mathbf{x})$
   (h) $R_{\sup_y \tau(y,\mathbf{y})\geq r}(\mathbf{x}) \iff \wedge_{n<\omega} \exists x R_{\tau(y,\mathbf{y})\geq r-\frac{1}{n}}(x, \mathbf{x})$;
   (i) $R_{\inf_y \tau(y,\mathbf{y})\leq r}(\mathbf{x}) \iff \wedge_{n<\omega} \exists x R_{\tau(y,\mathbf{y})\leq r+\frac{1}{n}}(x, \mathbf{x})$
   (j) $R_{\inf_y \tau(y,\mathbf{y})\geq r}(\mathbf{x}) \iff \forall x R_{\phi(y,\mathbf{y})\geq r}(x, \mathbf{x})$;
   (k) $R_{\phi(y,\mathbf{y})\square r}(\tau(\mathbf{x}'), \mathbf{x}) \iff R_{\phi(\tau(\mathbf{y}'),\mathbf{x})\square r}(\mathbf{x}', \mathbf{x})$

(3) **Metric structure**
   (a) $R_{d(y,y')\leq 0}(x, x') \iff x = x'$;

(b) $R_{d(y,y')\Box r}(x, x') \iff R_{d(y,y')\Box r}(x', x)$;
(c) $\wedge_{r\in\mathbb{Q}'}(R_{d(y,y')\geq r}(x, x') \implies \forall x'' \vee_{s\in\mathbb{Q}'\cap[0,r]} R_{d(y,y')\geq s}(x, x'') \wedge R_{d(y,y')\geq r-s}(x'', x'))$

(4) **Uniform Continuity**
(a) For each $r, s \in \mathbb{Q}'$ and $i < L_F$ such that $s < \Delta_{F_i}(r)$, we include the sentence

$$\wedge_{i<n} R_{d(z,z')\leq s}(x_i, y_i) \implies R_{d(z,z')\leq r}(F_i(\mathbf{x}), F_i(\mathbf{y}))$$

(b) For each $r, s \in \mathbb{Q}'$ and $j < L_R$ such that $s < \Delta_{R_j}(r)$, we include the sentence

$$\wedge_{i<n} R_{d(z,z')\leq s}(x_i, y_i) \implies (R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}')\leq r}(\mathbf{x}, \mathbf{y}) \wedge R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}')\leq r}(\mathbf{y}, \mathbf{x}))$$

We have been careful about the specific enumeration of these axioms for a reason. If the original continuous language is countable, then T_{dense} is countable. In particular, we could take the conjunction of it and make it a single $L^+_{\omega_1,\omega}$ sentence. This means that it is expressible in a countable fragment of $L_{\omega_1,\omega}$. Countable fragments are the most well-studied infinitary languages and many of the results in, say, Keisler [Kei71] use these fragments. In general, T_{dense} is expressible in a $|L| + \aleph_0$ sized fragment of $L^+_{\omega_1,\omega}$.

From continuous to discrete...

This is the easier of the directions. We define the structure M_A so that all of the "intended" correspondences hold and everything works out well.

Suppose we have a continuous L-structure M and a nicely dense subset A. Now we define an L^+ structure M_A by

(1) the universe of M_A is A;
(2) $(F_i^+)^{M_A} = F_i^M \upharpoonright A$ for $i < n_F$; and
(3) for $r \in \mathbb{Q}'$ and $\phi(\mathbf{x}) \in Fml^c(L)$, set

$$R^{M_A}_{\phi\Box r} = \{\mathbf{a} \in A : \phi^M(\mathbf{a})\Box r\}$$

This is an L^+-structure since it is closed under functions. The real meat of this part is the following claim, which is (1) from the theorem.

Claim: $M_A \vDash T_{dense}$ and, for any $\mathbf{a} \in A$, $\phi(\mathbf{x}) \in Fml^c L$, $r \in \mathbb{Q}'$, and $\Box = \geq, \leq$, we have

$$M_A \vDash R_{\phi\Box r}[\mathbf{a}] \iff \phi^M(\mathbf{a})\Box r$$

Proof of Claim: This is all straightforward. From the definition, we know that, for any $\mathbf{a} \in A$ and formula $\phi(\mathbf{x}) \in Fml^c(L)$ and $\Box \in \{\geq, \leq\}$, we have

$$M_A \vDash R_{\phi\Box r}[\mathbf{a}] \iff \phi^M(\mathbf{a})\Box r$$

This gives an easy proof of the fact that $M_A \vDash T_{dense}$ because they are all just true facts if '$R_{\phi\Box r}(\mathbf{a})$' is replaced by '$\phi(\mathbf{a})\Box r$.' $\dagger_{Claim}$

...and back again

This is the harder direction. We want to 'read out' the L-structure that A is a dense subset of from the L^+ structure. First, we use the axioms of T_{dense} to show that we can read out the metric and relations of L from the relations of L^+ and that these are well-defined. Then we complete A and use the uniform continuity of the derived relations to expand them to the whole structure. In the first direction, T_{dense} could have been any collection of true sentences about continuous structures and the real line, but this direction makes it clear that the axioms chosen are sufficient.

Suppose that we have an L^+-structure A that models T_{dense}. The following claim is an important step in reading out the relations of the completion of A from A.

Claim 2.2. *For any $\phi(\mathbf{x}) \in Fml^c(L)$ and $\mathbf{a} \in A$, we have*

$$\sup\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\leq t}(\mathbf{a})\} = \inf\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\geq t}(\mathbf{a})\}$$

Proof: We show this equality by showing two inequalities.

- Let $r \in \{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\leq t}(\mathbf{a})\}$ and $s \in \{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\geq t}(\mathbf{a})\}$. Then
$$A \models R_{\phi(\mathbf{x})\geq r}(\mathbf{a}) \wedge R_{\phi(\mathbf{x})\leq s}(\mathbf{a})$$
Then, since M^+ satisfies 1c, we must have $r \leq s$. Thus $\sup\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\leq t}(\mathbf{a})\} \leq \inf\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\geq t}(\mathbf{a})\}$.
- By 1f, we have
$$A \models \wedge_{n<\omega} \vee_{r,s\in\mathbb{Q}';|r-s|<\frac{1}{n}} R_{\phi(\mathbf{x})\leq r}(\mathbf{a}) \wedge R_{\phi(\mathbf{x})\geq s}(\mathbf{a})$$
Let $\epsilon > 0$. Then there is $n_0 < \omega$ such that $\epsilon > \frac{1}{n_0}$. By the above, there are $r, s \in \mathbb{Q}'$ such that $|r - s| < \frac{1}{n_0}$ and
$$M^+ \models R_{\phi(\mathbf{x})\leq r}(\mathbf{a}) \wedge R_{\phi(\mathbf{x})\geq s}(\mathbf{a})$$
As above, 1c implies $r \geq s$, so we have $r - s < \frac{1}{n_0} < \epsilon$. Thus $r < s + \epsilon$ and $s \in \{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\leq t}(\mathbf{a})\}$ and $r \in \{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\geq t}(\mathbf{a})\}$. Then, $\inf\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\geq t}(\mathbf{a})\} \leq \sup\{t \in \mathbb{Q}' : A \models R_{\phi(\mathbf{x})\leq t}(\mathbf{a})\}$. $\dagger_{Claim}$

The first relation that we need is the metric. Given $a, b \in A$, we set

$$\begin{aligned} D(a,b) &:= \sup\{r \in \mathbb{Q}' : A \vDash R_{d(x,y)\geq r}[a,b]\} \\ &= \inf\{r \in \mathbb{Q}' : A \vDash R_{d(x,y)\leq r}[a,b]\} \end{aligned}$$

These definitions are equivalent by Claim 2.2. We show that this is indeed a metric on A.

Claim 2.3. $(|A|, D)$ *is a metric space.*

Proof: We go through the metric space axioms. Let $a, b \in |A|$.

(1)

$$\begin{aligned} D(a,b) = 0 \quad &\implies \quad \inf\{r \in \mathbb{Q}' : A \vDash R_{d(x,y)\leq r}(a,b)\} = 0 \\ &\implies \quad \forall n < \omega \exists r_n \in \mathbb{Q}' \text{ so } A \models R_{d(x,y)\leq r_n}(a,b) \text{ and } r_n \leq \frac{1}{n} \\ &\implies_{1d} \quad \forall n < \omega, A \models R_{d(x,y)\leq \frac{1}{n}}(a,b) \\ &\implies_{1h} \quad A \models R_{d(x,y)\leq 0}(a,b) \\ &\implies \quad a = b \end{aligned}$$

$$\begin{aligned} a = b \quad &\implies \quad A \models R_{d(x,y)\leq 0}(a,b) \\ &\implies \quad \inf\{r \in \mathbb{Q}' : A \models R_{d(x,y)\leq r}(a,b)\} = 0 \\ &\implies \quad D(a,b) = 0 \end{aligned}$$

(2)

$$D(a,b) = \sup\{r \in \mathbb{Q}' : A \models R_{d(x,y)\geq r}(a,b)\} =_{3b} \sup\{r \in \mathbb{Q}' : A \models R_{d(x,y)\geq r}(b,a)\} = D(b,a)$$

(3) Let $c \in |A|$. We want to show $D(a,c) \leq D(a,b) + D(b,c)$. It is enough to show

$$\forall r \in \mathbb{Q}'(D(a,c) \geq r \implies D(a,b) + D(b,c) \geq r)$$

Thus, let $r \in \mathbb{Q}'$ and suppose $D(a,c) \geq r$. Then $\sup\{s \in \mathbb{Q}' : A \models R_{d(x,y)\geq s}(a,c)\} \geq r$. By 3c, this means

$$\sup\{s \in \mathbb{Q}' : A \models \vee_{t\in\mathbb{Q}'\cap[0,s]} R_{d(x,y)\geq t}(a,b) \wedge R_{d(x,y)\geq s-t}(b,c)\} \geq r$$

Fix $n < \omega$. There is some $s_n \in \mathbb{Q}'$ such that $s_n \geq r - \frac{1}{n}$ and

$$A \models \vee_{t\in\mathbb{Q}'\cap[0,s_n]} R_{d(x,y)\geq t}(a,b) \wedge R_{d(x,y)\geq s_n-t}(b,c)$$

Thus, there is some $t_n \in \mathbb{Q}'$ such that $0 \leq t_n \leq s_n$ and

$$A \models R_{d(x,y)\geq t_n}(a,b) \wedge R_{d(x,y)\geq s_n-t_n}(b,c)$$

By the definition of D, this means that $D(a,b) \geq t_n$ and $D(b,c) \geq s_n - t_n$; thus, $D(a,b) + D(b,c) \geq s_n$. Since this is true for all $n < \omega$, we get that $D(a,b) + D(b,c) \geq r$ as desired.

Thus, D is a metric on $|A|$. $\dagger_{Claim}$

Now we define partial functions and relations on $(|A|, D)$ such that they are uniformly continuous. In particular,

(1) for $i < L_F$, set $f_i := F_i^A$ with modulus $\Delta_{f_i}(r) = \sup\{s \in \mathbb{Q}' : A \vDash \forall x_0, \ldots, x_{n(F_i)-1}; \forall y_0, \ldots, y_{n(F_i)-1} \left(\wedge_{i<n(F_i)} R_{d(z,z')\leq s}(x_i, y_i) \to R_{d(z,z')\leq r}(F_i(\mathbf{x}), F_i(\mathbf{y})\right)\}$.

(2) for $j < L_R$, set $r_j(\mathbf{a}) := \sup\{r \in \mathbb{Q}' : A \vDash R_{R_j(\mathbf{z}) \leq r}[\mathbf{a}]\}$ with modulus $\Delta_{r_j}(r) = \sup\{s \in \mathbb{Q}' : A \models \forall \mathbf{x} \forall \mathbf{y} (\wedge_{i<n(R_j)} R_{d(z,z') \leq s}(x_i, y_i) \to (R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}') \leq r}(\mathbf{x}, \mathbf{y}) \wedge R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}') \leq r}(\mathbf{y}, \mathbf{x})))\}$.

These functions are not defined on the desired structure (ie the completion of A), but they already fulfill our goal in terms of agreeing with the discrete relations in the following sense.

Claim: For all $\mathbf{a} \in A$ and all formulas $\phi(\mathbf{x})$ built up from these functions and D, we have that

$$\phi(\mathbf{a}) \Box r \iff A \vDash R_{\phi(\mathbf{z}) \Box r}[\mathbf{a}]$$

Proof: We proceed by induction on the construction of $\phi(\mathbf{x})$. We assume that $\Box$ is $\geq$ in our proofs, but the proofs for $\leq$ are the same.

- If ϕ is atomic, then it falls into one of the following cases.
  - Suppose $\phi(\mathbf{x}) \equiv R_j(\tau(\mathbf{x}))$ for some term τ. Then

$$\begin{aligned} R_j^M(\tau(\mathbf{a})) \geq r \quad & \iff \quad \inf\{s \in \mathbb{Q}' : A \models R_{R_j(\mathbf{x}) \geq s}[\tau(\mathbf{a})]\} \geq r \\ & \iff \quad \forall n < \omega, \exists s_n \in \mathbb{Q}' \text{ so } s_n \geq r - \frac{1}{n} \text{ and } A \models R_{R_j(\mathbf{x}) \geq s_n}[\tau(\mathbf{a})] \\ & \iff_{1d} \quad \forall n < \omega, A \models R_{R_j(\mathbf{x}) \geq r - \frac{1}{n}}[\tau(\mathbf{a})] \\ & \iff_{1g} \quad A \models R_{R_j(\mathbf{x}) \geq r}[\tau(\mathbf{a})] \\ & \iff_{2k} \quad A \models R_{R_j(\tau(\mathbf{y})) \geq r}[\mathbf{a}] \end{aligned}$$

  - Suppose that $\phi(\mathbf{x}, \mathbf{y}) \equiv d(\tau_1(\mathbf{x}), \tau_2(\mathbf{y}))$ for terms τ_1 and τ_2. The detail are essentially as above: $D^M(\tau_1(\mathbf{a}), \tau_2(\mathbf{b}))$ iff (by 1d, the definition of sup, and 1h and 1g) $M^+ \models R_{d(x,y) \geq r}[\tau_1(\mathbf{a}), \tau_2(\mathbf{b})]$ iff (by 2k) $M^+ \models R_{d(\tau_1(\mathbf{x}), \tau_2(\mathbf{y})) \geq r}(\mathbf{a}, \mathbf{b})$.
- For the inductive step, we deal with each connective (from our full set) in turn. The induction steps for $x \mapsto 0$, $x \mapsto 1$, and $x \mapsto \frac{x}{2}$ are clear.
  - Suppose $\phi \equiv \psi \dot{-} \tau$, where τ is a formula and not a term. Note if $r = 0$, then this is obvious. So assume $r \neq 0$. Recall that

$$\phi^M(\mathbf{a}) = \psi^M(\mathbf{a}) \dot{-} \tau^M(\mathbf{a}) = \begin{cases} \psi^M(\mathbf{a}) - \tau^M(\mathbf{a}) & \text{if } \psi^M(\mathbf{a}) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

    Thus, we can assume we are in the case that $\psi^M(\mathbf{a}) > \tau^M(\mathbf{a})$.
    * First, suppose $\psi^M(\mathbf{a}) - \tau^M(\mathbf{a}) \geq r$. Since $\psi^M(\mathbf{a}) > \tau^M(\mathbf{a})$, there is some $s \in \mathbb{Q}'$ such that $\psi^M(\mathbf{a}) > s > \tau^M(\mathbf{a})$. Then $\tau^M(\mathbf{a}) \leq s$ and $\psi^M(\mathbf{a}) \geq s + r$. By induction, we have that

$$M^+ \models R_{\tau(\mathbf{x}) \leq s}[\mathbf{a}] \wedge R_{\psi^M(\mathbf{x}) \geq s + r}[\mathbf{a}]$$

      Then, by 2e, we have that $M^+ \models R_{\psi \dot{-} \tau \geq r}[\mathbf{a}]$ as desired.

  * Now, suppose $M^+ \models R_{\psi \dot{-} \tau \geq r}[\mathbf{a}]$. Again, 2e implies there is is some $s \in \mathbb{Q}'$ such that

    $$M^+ \models R_{\tau \leq s}[\mathbf{a}] \wedge R_{\psi \geq r+s}[\mathbf{a}]$$

    By induction, we get $\tau^M(\mathbf{a}) \leq s$ and $\psi^M(\mathbf{a}) \geq r + s$. Then

    $$\phi^M(\mathbf{a}) = \psi^M(\mathbf{a}) - \tau^M(\mathbf{a}) \geq (r+s) - s = r$$

    as desired.

- Suppose $\phi(\mathbf{x}) \equiv \sup_x \psi(x, \mathbf{x})$. We will consider both sides of the inequality since they're not symmetrically axiomatized (see 2g and 2h), but we won't worry about inf.
  * Suppose that $\sup_x \phi^M(x, \mathbf{a}) \geq r$. Then for any $n < \omega$, there is some $a_n \in |M|$ such that $\phi^M(a_n, \mathbf{a}) > r - \frac{1}{2n}$. Since ϕ^M is uniformly continuous, there is some $\delta > 0$ such that, if $d(a_n, b) < \delta$, then $|\phi^M(a_n, \mathbf{a}) - \phi^M(b, \mathbf{a})| < \frac{1}{2n}$. Since M^+ is dense in M, there is some $a'_n \in M^+$ such that $d(a_n, a'_n) < \delta$. Thus, $\phi^M(a'_n, \mathbf{a}) > r - \frac{1}{n}$. By induction, we have that

    $$M^+ \models \wedge_{n<\omega} \exists x R_{\phi(y,\mathbf{y}) \geq r - \frac{1}{n}}(x, \mathbf{a})$$

    Then 2h says that $M^+ \models R_{\sup_y \phi(y,\mathbf{y}) \geq r}(\mathbf{a})$.
  * Suppose that $M^+ \models R_{\sup_y \phi(y,\mathbf{y}) \geq r}[\mathbf{a}]$. Then, by 2h, $M^+ \models \wedge_{n<\omega} \exists x R_{\phi(y,\mathbf{y}) \geq r - \frac{1}{n}}(x, \mathbf{a})$. So, for each $n < \omega$, there is some $a_n \in M^+$ such that $M^+ \models R_{\phi(y,\mathbf{y}) \geq r - \frac{1}{n}}[a_n, \mathbf{a}]$. By induction, we have that $\phi^M(a_n, \mathbf{a}) \geq r - \frac{1}{n}$. Since this is true for each $n < \omega$, we get $\sup_y \phi^M(y, \mathbf{a}) \geq r$.
  * The other direction is easier and we can combine the two parts

$$\begin{aligned}
\sup_x \phi^M(x, \mathbf{a}) \leq r \quad & \iff \quad \forall a \in M \phi^M(a, \mathbf{a}) \leq r \\
& \iff \quad \forall a \in M^+ \phi(a, \mathbf{a}) \leq r \\
& \iff_{Induction} \quad M^+ \models \forall x R_{\phi \leq r}(x, \mathbf{a}) \\
& \iff_{2g} \quad M^+ \models R_{\sup_x \phi(x,\mathbf{x})}[\mathbf{a}]
\end{aligned}$$

†

We have given these functions moduli, but do not know they are uniformly continuous. We show this now. It is also worth noting that these moduli might not be the same moduli in the original signature L. Instead, these are the optimal moduli, while the original language might have moduli that could be improved.

Claim: The functions f_i and r_j are continuous.

Proof: We do each of these cases separately.

- **Sub-Claim 1:** $F_i^{M^+}$ is uniformly continuous on $(|M^+|, D)$ with modulus Δ_{F_i}.
  Let $r \in \mathbb{Q}'$ and let $\mathbf{a}, \mathbf{b} \in |M^+|$ such that $\max_{i<n} D(a_i, b_i) < \Delta_{F_i}(r)$. Thus, for each $i < n$, $D(a_i, b_i) = \inf\{s \in \mathbb{Q}' : M^+ \models R_{d(x,y)\leq s}(a_i, b_i)\} < \Delta_{F_i}(r)$. Since this is strict, there is some $s_i \in \mathbb{Q}'$ such that $M^+ \models R_{d(x,y)\leq s_i}(a_i, b_i)$. Note that 1d implies that the set $\Delta_{F_i}(r)$ is supremuming over is downward closed. Thus, $s' = \max_{i<n} s_i$ is in it. Thus, we can conclude
  $$M^+ \models R_{d(x,y)\leq r}[F_i(\mathbf{a}), F_i(\mathbf{b})]$$
  This means that $D(F_i(\mathbf{a}), F_i(\mathbf{b})) \leq r$, as desired.
- **Sub-Claim 2:** $R_j^{M^+}$ is uniformly continuous on $([0,1], |\cdot|)$ with modulus Δ_{R_j}.
  Let $r \in \mathbb{Q}'$ and $\mathbf{a}, \mathbf{b} \in |M^+|$ such that $\wedge_{i<n} D(a_i, b_i) < \Delta_{R_j}(r)$. From the infimum definition of D, for each $i < n$, there is $s_i \in \mathbb{Q}'$ such that $s_i < \Delta_{R_j}(r)$ and $M^+ \models R_{d(z,z')\leq s_i}[a_i, b_i]$. Thus,
  $$M^+ \models R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}')\leq r}(\mathbf{a}, \mathbf{b}) \wedge R_{R_j(\mathbf{z}) \dot{-} R_j(\mathbf{z}')\leq r}(\mathbf{b}, \mathbf{a})$$
  For this next part, we need some of the future proofs, but essentially we have enough to show that this implies
  $$R_j^{M^+}(\mathbf{a}) \dot{-} R_j^{M^+}(\mathbf{b}) \leq r \text{ and } R_j^{M^+}(\mathbf{b}) \dot{-} R_j^{M^+}(\mathbf{a}) \leq r$$
  This implies $|R_j^{M^+}(\mathbf{a}) - R_j^{M^+}(\mathbf{b})| \leq r$, so $R_j^{M^+}$ is uniformly continuous. †

Now we have a prestructure, see [BBHU08].§3. Now we complete $|A|$ to $|\overline{A}|$ in the standard way; see Munkries [Mun00] for a reference for the topological facts. In particular, we define the continuous L structure $\overline{A}$ by

- $|\overline{A}|$ is the completion of $(|A|, D)$;
- the metric $d^{\overline{A}}$ is the extension of D to $|\overline{A}|$;
- for $i < n_F$, $F_i^{\overline{A}}$ is the unique extension of f_i to $|\overline{A}|$; and
- for $j < n_R$, $R_j^{\overline{A}}$ is the unique extension of r_j to $|\overline{A}|$.

Essential inverses

Proposition 2.4. *Given any continuous L-structure M and dense subset A, we have that $M \cong_A \overline{M}_A$ and, given any L^+ structure A that models T_{dense}, we have that $(\overline{A})_A = A$.*

Proof: First, let M be a continuous L-structure and $A \subset |M|$ be nicely dense. We define a map $f : M \to \overline{(M_A)}$ as follows: if $a \in A$, then $f(a) = a$. For $a \in M - A$, fix some (any) sequence $\langle a_n \in A : n < \omega\rangle$ such that $\lim_{n\to\infty} a_n = a$ (this limit computed in M). We know that $\langle a_n : n < \omega\rangle$ is Cauchy in M, so it's

Cauchy in $\overline{(M_A)}$. Then set $f(a) = \lim_{n\to\infty} a_n$, where that limit is computed in $\overline{(M_A)}$. This is well-defined and a bijection because A is dense in both sets. That this is an L-isomorphism follows from applying the correspondence twice: for all $\mathbf{a} \in A$ and $\phi(\mathbf{x}) \in Fml^c(L)$

$$\phi^M(\mathbf{a})\Box r \iff M_A \vDash R_{\phi(\mathbf{x})\Box r}[\mathbf{a}] \iff \phi^{\overline{(M_A)}}(\mathbf{a})\Box r$$

and the fact that the values of ϕ on A determines its values on M and $\overline{(M_A)}$.

Second, let A be a L^+ structure that models T_{dense}. Clearly, the universes are the same, ie, $|(\overline{A})_A| = |A|$. For any relation $R_{\phi\Box r}$ and $\mathbf{a} \in |A|$, we have

$$A \vDash R_{\phi\Box r}[\mathbf{a}] \iff \phi^{\overline{A}}(\mathbf{a})\Box r \iff (\overline{A})_A \vDash R_{\phi\Box r}[\mathbf{a}]$$

Given a function F_i^+ and $\mathbf{a}, a \in A$, we have that

$$\begin{aligned}(F_i^+)^A(\mathbf{a}) = a \quad &\iff \quad A \vDash R_{d(F_i^+(\mathbf{x}),x)\le 0}[\mathbf{a}, a] \\ &\iff \quad (\overline{A})_A \vDash R_{d(F_i^+(\mathbf{x}),x)\le 0}[\mathbf{a}, a] \iff (F_i^+)^{(\overline{A})_A}(\mathbf{a}) = a\end{aligned}$$

†

We can extend this correspondence to theories. Suppose that T is a continuous theory in L. Following [BBHU08].4.1, theories are sets of closed L-conditions; that is, a set of "$\phi = 0$," where ϕ is a formula with no free variables. The following is immediate from Theorem 2.1.

Corollary 2.5. *If "$\phi = 0$" is a closed L-condition, then*

$$\phi^M = 0 \iff M_A \vDash R_{\phi\le 0}$$

With our fixed theory T, set T^* to be $T_{dense} \cup \{R_{\phi\le 0} : \text{"}\phi = 0'' \in T\}$. Then our representation of continuous L-structures as discrete L^+-structures modeling T_{dense} can be extended to a representation of continuous models of T and discrete models of T^*.

3. Elementary Substructure

We now discuss translating the notion of elementary substructure between our two contexts. Depending on the generality needed, this is either easy or difficult.

For the easy case, we have the following.

Theorem 3.1. *Let M, N be continuous L structures. Then $M \prec_L N$ iff, for every nicely dense $A \subset M$ and $B \subset N$ such that $A \subset B$, we have that $M_A \subset_{L^+} N_B$.*

Note that the relation between M_A and N_B is just substructure. So even though they are models of infinitary theories, their relation just concerns atomic formulas. This is because we have built the quantifiers of L into the relations of L^+.

Proof: $\leftarrow$: Let $A = M$ and $B = N$. Then $M \subset N$, so $M_M \subset_{L^+} N_N$ by assumption. Thus they agree on all relations concerning elements of M. Now we want to show that $M \prec_L^c N$. Let $\phi(\mathbf{x}) \in Fml^c L$ and $\mathbf{a} \in M$. From the theorems proved last section, we have, for each $r \in \mathbb{Q}'$,

$$\begin{aligned} \phi^M(\mathbf{a})\Box r &\iff M_M \vDash R_{\phi(\mathbf{x})\Box r}[\mathbf{a}] && \text{Theorem 2.1} \\ &\iff N_N \vDash R_{\phi(\mathbf{x})\Box r}[\mathbf{a}] && M_M \subset_{L^+} N_N \\ &\iff \phi^N(\mathbf{a})\Box r && \text{Theorem 2.1} \end{aligned}$$

Thus $\phi^M(\mathbf{a}) = \phi^N(\mathbf{a})$ and $M \prec_L N$ as desired.
$\to$: Let $A \subset M$ and $B \subset N$ be nicely dense so $A \subset B$. We want to show that $M_A \subset_{L^+} N_B$.

- Let $F^+ \in L^+$ and $\mathbf{a} \in A$. Then, by definition of the structures,
$$(F^+)^{M_A}(\mathbf{a}) = F^M(\mathbf{a}) = F^N(\mathbf{a}) = (F^+)^{N_B}(\mathbf{a})$$
- Let $R_{\phi\Box r}(\mathbf{x}) \in L^+$ and $\mathbf{a} \in A$.
$$\begin{aligned} M_M \vDash R_{\phi(\mathbf{x})\Box r}[\mathbf{a}] &\iff \phi^M(\mathbf{a})\Box r && \text{Theorem 2.1} \\ &\iff \phi^N(\mathbf{a})\Box r && M \prec N \\ &\iff N_N \vDash R_{\phi(\mathbf{x})\Box r}[\mathbf{a}] && \text{Theorem 2.1} \end{aligned}$$

†

Similarly, we have the following.

Theorem 3.2. *Given $A, B \vDash T_{dense}$, if $A \subset_{L^+} B$, then $\overline{A} \prec_L \overline{B}$.*

However, these are not the best theorems possible. In particular, the requirement that $A \subset B$ limits the scope of this theorem. We would like to know when L^+ structures complete to L-elementary substructures even when the dense substructures are not subsets or each other; for instance, the completions of $\mathbb{Q} \cap [0, 1]$ and $\mathbb{Q} + \sqrt{2}$ are nicely related, but the previous theorem does not see that. We would like to develop a criterion for L^+ structures $A, B \vDash T_{dense}$ that is equivalent to $\overline{A} \prec_L \overline{B}$.

Our first attempt is the following.

Theorem 3.3. *Suppose $M \prec N$ are continuous L-structures and $A \subset M$ and $B \subset N$ are nicely dense. Then*

(1) *$M_A \subset_{L^+} N_C$, where C is the closure of $A \cup B$ under the functions of N.*
(2) *There is a nicely dense $B' \subset N$ such that $M_A \subset_{L^+} N_{B'}$ and $|B'| = |A| + dc(N) + |L|$.*

Proof:

(1) Note that $A \subset C$ and C is nicely dense in N. By Theorem 3.1, $M_A \subset_{L^+} N_C$.
(2) Let $B'' \subset N$ be dense of size $dc(N)$ and let B' be the closure of $B'' \cup A$ under the functions of N. By Theorem 3.1, $M_A \subset_{L^+} N_{B'}$.

While this is an improvement, it is still not the best desireable. In particular, it still makes reference to the continuous structures. We would prefer a correspondence that only involved L^+ structures. To that end, we give the following definition of inessential extensions.

Definition 3.4. *Given $A \subset_{L_A^+} B$, we say that B is an* inessential extension *of A iff for every $b \in |B|$ and $n < \omega$, there is some $a \in |A|$ such that $B \vDash R_{d(x,y)<\frac{1}{n}}[b,a]$.*

Proposition 3.5. *If B is an inessential extension of A, then $\overline{A} = \overline{B}$.*

This gives us the following theorem.

Theorem 3.6. *Let $A, B \vDash T_{dense}$. Then TFAE*

(1) *$\overline{A} \prec_L \overline{B}$.*
(2) *There is an L^+ structure C such that $A, B \prec_{L_A^+} C$ and C is an inessential extension of B.*
(3) *There is an L^+ structure $C' \vDash T_{dense}$ such that $A, B \subset_{L^+} C$ and C is an inessential extension of B.*
(4) *There is an extension of the functions and relations of L^+ to $A \cup B$ such that $A \cup B \vDash T_{dense}$ that are still uniformly continuous and such that for every $a \in |A \cup B|$ and $n < \omega$, there is some $b \in |B|$ such that $A \cup B \vDash R_{d(x,y)<\frac{1}{n}}[b,a]$.*

Proof:

(1) $\implies$ (2) Take $C = (\overline{B})_{A \cup B}$.

(2) $\implies$ (3) Immediate.

(3) $\implies$ (4) Take the extension inherited from C.

(4) $\implies$ (1) We have $\overline{A}, \overline{B} \prec \overline{A \cup B}$ from the first condition and $\overline{B} = \overline{A \cup B}$ form the second.

4. Types and Saturation

In this section, we will connect types in the continuous logic sense to types in the discrete sense. However, just as elements in the complete structure are represented by sequences of the discrete structure, we represent types by sequence types. Recall from [BBHU08].8.1 that a type over B is a consistent collection of conditions of the form "$\phi(\mathbf{x}, \mathbf{b}) = 0$" with $\mathbf{b} \in B$.

Definition 4.1.

- *We say that $\langle r_n : n < \omega \rangle$ is a* sequence ℓ-type over B' *iff $r_0(\mathbf{x})$ is an ℓ-type over B' and $r_{n+1}(\mathbf{x}, \mathbf{y})$ is a 2ℓ-type over B' such that there is some index set I, (possibly repeating) formulas $\langle \phi_i : i \in I \rangle$; and (possibly repeating) Cauchy sequences $\langle \langle \mathbf{b}^i_n \in B' \rangle_{n<\omega} : i \in I \rangle$ so $d(\mathbf{b}^i_n, \mathbf{b}^i_{n+1}) \leq \frac{1}{2^n}$ such that*
  - *$r_0(\mathbf{x}) = \{R_{\phi_i(\mathbf{z},\mathbf{z}') \leq w^{\phi_i}(2)}(\mathbf{x}, \mathbf{b}^i_0) : i \in I\}$; and*

  - $r_{n+1}(\mathbf{x}, \mathbf{y}) = \{R_{\phi_i(\mathbf{z},\mathbf{z}') \leq w^{\phi_i}(\frac{1}{2^n})}(\mathbf{x}, \mathbf{b}^i_{n+1}) : i \in I\} \cup \{R_{d(z,z') \leq \frac{1}{2^n}}(x_k, y_k) : k < \ell\}$

- *A* realization *of a sequence type $\langle r_n : n < \omega \rangle$ is $\langle \mathbf{a}_n : n < \omega \rangle$ such that*
  - $\mathbf{a}_0$ *realizes* r_0*; and*
  - $\mathbf{a}_{n+1}\mathbf{a}_n$ *realizes* r_{n+1}.

Note that the use of $\frac{1}{2^n}$ is not necessary; this could be replaced by any summable sequence for an equivalent definition (also replacing $\frac{1}{2^{n-1}}$ by the trailing sums). However, we fix $\frac{1}{2^n}$ for computational ease. The fundamental connection between continuous types and sequence types is the following.

Theorem 4.2. *Let $A \subset |M|$ be nicely dense.*

(1) *If $B \subset |M|$ and $r(\mathbf{x})$ is a partial ℓ-type over B, then for any $B' \subset A$ such that $\overline{B}' \supset B$, there is a sequence ℓ type $\langle r_n : n < \omega \rangle$ over B' such that*

$$M \text{ realizes } r \text{ iff } M_A \text{ realizes } \langle r_n : n < \omega \rangle$$

(2) *If $B' \subset A$ and $\langle r_n : n < \omega \rangle$ is a partial sequence ℓ-type over B', then there is a unique ℓ-type r over $\overline{B}'$ such that*

$$M \text{ realizes } r \text{ iff } M_A \text{ realizes } \langle r_n : n < \omega \rangle$$

We can denote the type in (2) by $\lim_{n\to\infty} r_n$. In each case, we have that $\langle \mathbf{a}_n \in M_A : n < \omega \rangle$ realizes $\langle r_n : n < \omega \rangle$ implies $\lim_{n\to\infty} \mathbf{a}_n$ realizes $\lim_{n\to\infty} r_n$.

Proof:

(1) Recall that $r(\mathbf{x})$ contains conditions of the form "$\phi(\mathbf{x}, \mathbf{b}) = 0$" for $\phi \in Fml^c(L)$ and $\mathbf{b} \in B$. For $n < \omega$ and $b \in B$, set $B'_n(b) = \{b' \in B' : d^M(b', b) < \frac{1}{2^n}\}$. To make the cardinality work out nicer, fix a choice function G, ie $G(B'_n(b)) \in B'_n(b)$. Then $B'_n(\mathbf{b})$ and $G(B'_n(\mathbf{b}))$ have the obvious meanings. Define

$$\begin{aligned} r_n^+(\mathbf{x}) &:= \{R_{\phi(\mathbf{z};\mathbf{y}) < w^\phi(\frac{1}{2^{n-1}})}(\mathbf{x}; G(B'_n(\mathbf{b}))) : \text{“}\phi(\mathbf{x}; \mathbf{b}) = 0\text{”} \in r\} \\ r_0(\mathbf{x}) &:= r_0^+(\mathbf{x}) \\ r_{n+1}(\mathbf{x}, \mathbf{y}) &:= r_{n+1}^+(\mathbf{x}) \cup \{R_{d(z,z') < \frac{1}{2^n}}(x_i, y_i) : i < \ell(\mathbf{x})\} \end{aligned}$$

Then $\langle r_n : n < \omega \rangle$ is a sequence type over B'; we can see this by taking r as the index set, $\phi_i = \phi$, and $\mathbf{b}^i = G(B'_n(\mathbf{b}))$ for $i =$ "$\phi(\mathbf{x}; \mathbf{b}) = 0$" $\in r$. To show it has the desired property, first suppose that $\langle \mathbf{a}_n : n < \omega \rangle$ from M_A realizes $\langle r_n : n < \omega \rangle$. We know that $\langle \mathbf{a}_n : n < \omega \rangle$ is a Cauchy sequence; in particular, for $m > n$,

$$d^M(\mathbf{a}_n, \mathbf{a}_m) \leq \sum_{i=n}^{m} \frac{1}{2^i} = \frac{2^{m+1-n} - 1}{2^{m+1}}$$

Since $\overline{M_A} \cong M$ is complete, there is $\mathbf{a} \in \overline{M_A}$ such that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$. We claim that $\mathbf{a} \vDash r$. Let "$\phi(\mathbf{x}; \mathbf{b}) = 0$" $\in r$. Then

$$d^{\overline{M}_A}(\mathbf{ab}; \mathbf{a}_n G(B'_n(\mathbf{b}))) = \max\{d^M(\mathbf{a}, \mathbf{a}_n), d^M(\mathbf{b}, G(B'_n(\mathbf{b}))\} \leq \max\{\sum_{i=n}^{\infty} \frac{1}{2^i}, \frac{1}{2^n}\} = \frac{1}{2^{n-1}}$$

Thus,

$$|\phi^M(\mathbf{a}; \mathbf{b}) - \phi^M(\mathbf{a}_n; G(B'_n(\mathbf{b})))| < w^\phi(\frac{1}{2^{n-1}})$$

Letting $n \to \infty$, we have that

$$\phi^{\overline{M}_A}(\mathbf{a}; \mathbf{b}) = \lim_{n\to\infty} \phi^M(\mathbf{a}_n; G(B'_n(\mathbf{b}))) \leq \lim_{n\to\infty} w^\phi(\frac{1}{2^{n-1}}) = 0$$

as desired.

Now suppose that $\mathbf{a} \in M$ realizes r. Since A is dense, we can find $\langle \mathbf{a}_n \in A : n < \omega \rangle$ such that $d^M(\mathbf{a}_n, \mathbf{a}_{n+1}) < \frac{1}{2^n}$ and $\mathbf{a}_n \to \mathbf{a}$. We want to show that $r_n^+(\mathbf{a}_n)$ holds. Let "$\phi(\mathbf{x}, \mathbf{b})$" $\in r$. We know that

$$d^M(\mathbf{a}_n G(B'_n(\mathbf{b})), \mathbf{ab}) = \frac{1}{2^{n-1}}$$

, so we get

$$\phi^{M_A}(\mathbf{a}_n, G(B'_n(\mathbf{b}))) = |\phi^M(\mathbf{a}, \mathbf{b}) - \phi^M(\mathbf{a}_n, G(B'_n(\mathbf{b})))| < w^\phi(\frac{1}{2^{n-1}})$$

as desired. †

(2) Let $\langle r_n : n < \omega \rangle$ be a partial sequence ℓ-type given by I, $\langle \phi_i : i \in I \rangle$, and $\langle \langle \mathbf{b}_n^i \rangle_{n<\omega} : i \in I \rangle$. Then set

$$r(\mathbf{x}) := \{\phi_i(\mathbf{x}, \lim_{n\to\infty} \mathbf{b}_n^i) = 0 : i \in I\}$$

First, suppose that $\langle \mathbf{a}_n \in M_A : n < \omega \rangle$ realizes $\langle r_n : n < \omega \rangle$. Then, since $\mathbf{a}_{n+1}\mathbf{a}_n \vDash r_{n+1}$, we have $d^{M_A}(\mathbf{a}_{n+1}, \mathbf{a}_n) < \frac{1}{2^n}$ and, thus, the sequence is Cauchy. Since M is complete, let $\mathbf{a} = \lim_{n\to\infty} \mathbf{a}_n \in M$. Then, by uniform continuity, we have

$$\begin{aligned}
\phi_i^M(\mathbf{a}, \mathbf{b}^i) &= \phi_i^M(\lim_{n\to\infty} \mathbf{a}_n, \lim_{n\to\infty} \mathbf{b}_n^i) \\
&= \lim_{n\to\infty} \phi_i^M(\mathbf{a}_n, \mathbf{b}_n^i) \\
&\leq \lim_{n\to\infty} w^{\phi_i}(\frac{1}{2^{n-1}}) \\
&= 0
\end{aligned}$$

So $\mathbf{a} \vDash r$.

Now suppose that $\mathbf{a} \in M$ realizes t. Then, by denseness, we can find a Cauchy sequence $\langle \mathbf{a}_n \in A : n < \omega \rangle$ such that $d(\mathbf{a}_{n+1}, \mathbf{a}_n) \leq \frac{1}{2^n}$. Then $d(\mathbf{ab}^i, \mathbf{a}_n\mathbf{b}_n^i) \leq \frac{1}{2^{n-1}}$. Then we can conclude

$$|\phi_i^M(\mathbf{a}, \mathbf{b}^i) - \phi_i^M(\mathbf{a}_n, \mathbf{b}_n^i)| \leq w^{\phi_i}(\frac{1}{2^{n-1}})$$

$$\phi^{M_A}(\mathbf{a}_n, \mathbf{a}_n^i) \leq w^{\phi_i}(\frac{1}{2^{n-1}})$$

So $\langle \mathbf{a}_n : n < \omega \rangle$ realizes $\langle r_n : n < \omega \rangle$. †

We now connect type-theoretic concepts in continuous logic (e.g. saturation and stability) with concepts in our discrete analogue.

Recall (see [BBHU08].7.5) that a continuous structure M is κ-saturated iff, for any $A \subset M$ of size $< \kappa$ and any continuous type $r(\mathbf{x})$ over A, if every finite subset of $r(\mathbf{x})$ is satisfiable in M, then so is $r(\mathbf{x})$.

Definition 4.3.
- *If $\langle r_n : n < \omega \rangle$ is a sequence type defined by an index set I and $I_0 \subset I$, then $\langle r_n : n < \omega \rangle^{I_0}$ is the sequence type defined by I_0.*
- *We say that $M_A \vDash T_{dense}$ is κ-saturated for sequence types iff, for all $B' \subset A$ and sequence type $\langle r_n : n < \omega \rangle$ over B' that is defined by I, if $\langle r_n : n < \omega \rangle^{I_0}$ is realized in M_A for all finite $I_0 \subset I$, then $\langle r_n : n < \omega \rangle$ is realized in M_A.*

Theorem 4.4.
1. *If M is κ-saturated and $\lambda^{\aleph_0} < \kappa$, then M_A is λ^+ saturated for sequence types.*
2. *If M_A is κ saturated, then M is κ saturated.*

Proof:

1. Let M be κ-saturated and $A \subset M$ be nicely dense. Let $B' \subset M_A$ of size λ and let $\langle r_n : n < \omega \rangle$ be a sequence type over B' that is finitely satisfiable in M_A. Set $r = \lim_{n\to\infty} r_n$ from Theorem 4.2; this is a type over $\overline{B'}$ where $|\overline{B'}| \leq \lambda^{\aleph_0} < \kappa$. We claim that r is finitely satisfiable in M. Any finite subset of r^- of
$$r = \{\phi_i(\mathbf{x}, \lim_{n\to\infty} \mathbf{b}_n^i) : i \in I\}$$
corresponds to a finite $I_0 \subset I$. Then, by Theorem 4.2, r_0 is realized in M iff $\langle r_n : n < \omega \rangle^{I_0}$ is realized in M_A. Then, since each $\langle r_n : n < \omega \rangle^{I_0}$ is realized in M_A by assumption, we have that r is finitely satisfiable in M. By the κ-saturation of M, r is realized in M. By Theorem 4.2, $\langle r_n : n < \omega \rangle$ is realized in M_A. So M_A is λ^+-saturated.
2. Let M_A be κ-saturated for sequence types. Let $B \subset M$ of size $< \kappa$ and r be a type over B that is finitely satisfied in B. Find $B' \subset A$ such that $\overline{B'} \supset B$; this can be done with $|B'| \leq |B| + \aleph_0 < \kappa$. Then form the sequence type $\langle r_n : n < \omega \rangle$ over B' that converges to r_n as in Theorem 4.2. As before, since r is finitely satisfiable in M, so is $\langle r_n : n < \omega \rangle$ in M_A. So $\langle r_n : n < \omega \rangle$ is realized in M_A by saturation. Thus, r is realized in M. †

We immediately get the following corollary.

Corollary 4.5. *If $\kappa = (\lambda^{\aleph_0})^+$ or, more generally, $\kappa = \sup_{\lambda<\kappa}(\lambda^{\aleph_0})^+$ and M is of size κ, then M is saturated iff M_A is saturated for some nicely dense $A \subset M$ of size κ.*

5. T_{dense} as an Abstract Elementary Class

In this section we view the discrete side of things as an Abstract Elementary Class; see Baldwin [Bal09] or Grossberg [Gro1X].

Theorem 5.1. *Let T be a complete, continuous first order L-theory. Then let L^+ and T_{dense} be from Theorem 2.1. Set $K = (Mod\ (T_{dense} \cup T^*, \subset_{L^+})$. Then*

1. *K is an AEC;*
2. *K has amalgamation, joint embedding, and no maximal models; and*
3. *Galois types in K correspond to sequence types (Definition 4.1).*

Note that if T were not complete, then amalgamation would not hold. However, the other properties will continue to hold, including the correspondence between Galois types and sequence types.

Proof: $T_{dense} \cup T^+$ is a $L_{\omega_1,\omega}$ theory, so all of the examples hold except perhaps the chain axioms. For those, consider a $\subset_{L^+}$-increasing chain $\langle M_{A_i} : i < \alpha\rangle$. Then, by Theorem 2.1 and Theorem 3.2, the sequence $\langle \overline{M_{A_i}} : i < \alpha\rangle$ is $\prec_L$-increasing chain that each model T. Then by the chain axiom for continuous logic, there is $M = \overline{\cup_{i<\alpha}(\overline{M_{A_i}})}$ that models T. Additionally, $A := \cup_{i<\alpha} A_i$ is nicely dense in M. Thus, $M_A = \cup_{i<\alpha} M_{A_i}$ is as desired. Additionally, if $M_{A_i} \subset_{L^+} M_B$ for some B, then $M \prec_L \overline{M_B}$, so $M_A \subset_{L^+} M_B$.

These properties all follow from the corresponding properties of continuous first-order logic. For instance, considering amalgamation, suppose $M_A \subset_{L^+} M_B, M_C$. Then we have $\overline{M_A} \prec_L \overline{M_B}, \overline{M_C}$. By amalgamation for continuous first-order logic, there is some $N \succ_L \overline{M_B}$ and elementary $f : \overline{M_C} \to_{\overline{M_A}} N$. Let $D \subset N$ be nicely dense that contains $B \cup C$. Then we have $M_B \subset_{L^+} N_D$ and $f \restriction M_C : M_C \to_{M_A} N_D$; this is an amalgamation of the original system.

Finally, we wish to show that Galois types are sequence types and vice versa. Note that there are monster models in each class. Further more, we may assume that, if $\mathfrak{C}$ is the monster model of T, that there is some nicely dense $U \subset \mathfrak{C}$ such that the monster model of K is M_U; in fact, we could take $U = |\mathfrak{C}|$. Let continuous $M \vDash T$ and $A \subset M$ be nicely dense. If we have tuples $\mathbf{a}$ and $\mathbf{b}$, then

$$\begin{aligned} gtp_K(\mathbf{a}/M_A) = gtp_K(\mathbf{b}/M_A) &\iff \exists f \in Aut_{M_A} M_U . f(\mathbf{a}) = \mathbf{b} \\ &\iff \exists f \in Aut_{\overline{M_A}} \mathfrak{C} . f(\mathbf{a}) = \mathbf{b} \\ &\iff tp(\mathbf{a}/M) = tp(\mathbf{b}/M) \\ &\iff \lim_{n\to\infty} r_n^{\mathbf{a}} = \lim_{n\to\infty} r_n^{\mathbf{b}} \end{aligned}$$

where $\langle r^x_n : n < \omega \rangle$ is the sequence type derived from $tp(x/M)$ as in Theorem 4.2 using A as the dense subset. †

6. Metric Abstract Elementary Classes

In this section, we extend the above representation to Metric Abstract Elementary Classes. Recall from Hirvonen and Hyttinen [HH09] or that a Metric Abstract Elementary Class (MAEC) is a class of continuous L-structures K and a strong substructure relation $\prec_K$ satisfying the following axioms:

(1) $\prec_K$ is a partial order on K;
(2) for every $M, N \in K$, if $M \prec_K N$, then $M \subseteq_L N$;
(3) $(K, \prec_K)$ respects $L(K)$ isomorphisms, if $f : N \to N'$ is an $L(K)$ isomorphism and $N \in K$, then $N' \in K$ and if we also have $M \in K$ with $M \prec_K N$, then $f(M) \in K$ and $f(M) \prec_K N'$;
(4) *(Coherence)* if $M_0, M_1, M_2 \in K$ with $M_0 \prec_K M_2$; $M_1 \prec_K M_2$; and $M_0 \subseteq M_1$, then $M_0 \prec M_1$;
(5) *(Tarski-Vaught chain axioms)* suppose $\langle M_i \in K : i < \alpha \rangle$ is a $\prec_K$-increasing continuous chain, then
    (a) $\overline{\cup_{i<\alpha} M_i} \in K$ and, for all $i < \alpha$, we have $M_i \prec_K \overline{\cup_{i<\alpha} M_i}$; and
    (b) if there is some $N \in K$ such that, for all $i < \alpha$, we have $M_i \prec_K N$, then we also have $\cup_{i<\alpha} M_i \prec_K N$; and
(6) *(Löwenheim-Skolem number)* There is an infinite cardinal $\lambda \geq |L(K)|$ such that for any $M \in K$ and $A \subset M$, there is some $N \prec_K M$ such that $A \subset |N|$ and $\|N\| \leq |A| + \lambda$. We denote the minimum such cardinal by $LS(K)$.

These axioms were first given in Hirvonen and Hyttinen [HH09].

A key difference is that the functions and relations L are no longer required to be *uniformly* continuous, but just continuous. This is due to the lack of compactness in the MAEC context. This initially seems problematic because functions must be uniformly continuous on a set to be guaranteed an extension to its closure. However, we get around this by simply defining K_{dense} to be all the structures that happen to complete to a member of K, then use the MAEC axioms to show that K_{dense} satisfies the AEC axioms.

For this reason, when we refer to continuous languages, structures, etc. in this section, we will not mean that they are uniformly continuous.

Theorem 6.1. *Let L be a continuous language. Then there is a discrete language L^+ such that, for every MAEC K with $LS(K) = L$, there is*

(1) *an AEC K_{dense} with $L(K_{dense}) = L^+$ and $LS(K_{dense}) = LS(K)$;*
(2) *a map from $M \in K$ and nicely dense subsets A of M to $M_A \in K_{dense}$; and*
(3) *a map from $\mathcal{A} \in K_{dense}$ to $\overline{\mathcal{A}} \in K$*

with the properties that

(1) *M_A has universe A and for each $\mathbf{a} \in A$, $r \in \mathbb{Q}'$, and $\Box \in \{\leq, \geq\}$, we have that*

$$M_A \vDash R_{R_j(\mathbf{z})\Box r}[\mathbf{a}] \iff R_j^M(\mathbf{a})\Box r$$

(2) *$\overline{\mathcal{A}}$ has universe that is the completion of A with respect to the derived metric and for each $\mathbf{a} \in A$, $r \in \mathbb{Q}'$, and $\Box \in \{\leq, \geq\}$, we have that*

$$R_j^{\overline{\mathcal{A}}}(\mathbf{a})\Box r \iff \mathcal{A} \vDash R_{R_j(\mathbf{z})\Box r}[\mathbf{a}]$$

(3) *The maps above are essentially inverses, in the since of Theorem 2.1.3*

(4)
- *Given $M_\ell \in K$ and A_ℓ nicely dense in M_ℓ for $\ell = 0, 1$, if $f : M_0 \to M_1$ is a K-embedding such that $f(A_0) \subset A_1$, then $f \upharpoonright A_0$ is a K_{dense}-embedding from $(M_0)_{A_0}$ to $(M_1)_{A_1}$.*
- *Given $\mathcal{A}, \mathcal{B} \in K_{dense}$ and a K_{dense}-embedding $f : \mathcal{A} \to \mathcal{B}$, this lifts canonically to a K-embedding $\overline{f} : \overline{\mathcal{A}} \to \overline{\mathcal{B}}$.*

Proof: The proof proceeds similar to the first-order version, Theorem 2.1. In particulalr, many of the definitions of continuous structures from discrete approximations (such as getting the metric and relations from their approximations) did not use compactness and only used uniform continuity to ensure that a completion existed, which will be guaranteed by the definition of K_{dense} in this case.

Given continuous $L = \langle F_i, R_j \rangle_{i<n_f, j<n_r}$, define

$$L^+ := \langle F_i^+, R_{R_j(\mathbf{z})\geq r}, R_{R_j(\mathbf{z})\leq r} \rangle_{i<n_f, j<n_r, r\in\mathbb{Q}'}$$

Given an MAEC K, we define the AEC K_{dense} as follows:

- $L(K_{dense}) = L^+$;
- Given an L^+ structure $\mathcal{A}$, we use the following procedure to determine membership in K_{dense}: define D on $A \times A$ by

$$\begin{aligned} D(a,b) &:= \sup\{r \in \mathbb{Q}' : \mathcal{A} \vDash R_{d(x,y)\geq r}[a,b]\} \\ &= \inf\{r \in \mathbb{Q}' : \mathcal{A} \vDash R_{d(x,y)\leq r}[a,b]\} \end{aligned}$$

  The proof that this is a well-defined and is a metric proceeds exactly as in the previous case. We can similarly define the relations R_j on A and complete the universe (A, D) to $\overline{A}$. We call the structure $\mathcal{A}$ *completable* iff
  1. for every $\mathbf{a} \in \overline{A}$ and every Cauchy sequence $\langle \mathbf{a}^n \in A : n < \omega \rangle$ converging to to $\mathbf{a}$, the value of $\lim_{n\to\infty} R_j(\mathbf{a}^n)$ is independent of the choice of the sequence; and similarly
  2. for every $\mathbf{a} \in \overline{A}$ and every Cauchy sequence $\langle \mathbf{a}^n \in A : n < \omega \rangle$ converging to to $\mathbf{a}$, the value of $\lim_{n\to\infty} F_i^+(\mathbf{a}^n)$ is independent of the choice of the sequence.

  If $\mathcal{A}$ is completable, then we define $\overline{\mathcal{A}}$ to be the L^+-structure where F_i and R_j are defined on $\overline{A}$ according to the independent value given above. Finally, we say that $\mathcal{A} \in K_{dense}$ iff
  1. $\mathcal{A}$ is completable; and

(2) $\overline{\mathcal{A}} \in K$.
- Given $\mathcal{A}, \mathcal{B} \in K_{dense}$, we say that $\mathcal{A} \prec_{dense} \mathcal{B}$ iff
  (1) $\mathcal{A} \subset_{L^+} \mathcal{B}$; and
  (2) $\overline{\mathcal{A}} \prec_K \overline{\mathcal{B}}$.

Now that we have the definition of K_{dense}, we must show that it is in fact an AEC. The verification of the axioms are routine; we give the arguments for coherence and the chain axioms as templates.

For coherence, suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in K_{dense}$ such that $\mathcal{A} \prec_{dense} \mathcal{C}$; $\mathcal{B} \prec_{dense} \mathcal{C}$; and $\mathcal{A} \subset_{L^+} \mathcal{B}$. Then, taking completions, we get that

$$\overline{\mathcal{A}} \prec_K \overline{\mathcal{C}}; \overline{\mathcal{B}} \prec_K \overline{\mathcal{C}}; \text{ and } \overline{\mathcal{A}} \subset_L \overline{\mathcal{B}}$$

By coherence in K, we then have that $\overline{\mathcal{A}} \prec_K \overline{\mathcal{B}}$. Then, by definition, $\mathcal{A} \prec_{dense} \mathcal{B}$, as desired.

For the chain axioms, suppose that $\langle \mathcal{A}_i \in K_{dense} : i < \alpha \rangle$ is a continuous, $\prec_{dense}$-increasing chain such that, for all $i < \alpha$, $\mathcal{A}_i \prec_{dense} \mathcal{B} \in K_{dense}$. Again, taking completions, we get that $\langle \overline{\mathcal{A}_i} \in K : i < \alpha \rangle$ is a continuous, $\prec_K$-increasing chain such that, for all $i < \alpha$, $\overline{\mathcal{A}_i} \prec_K \overline{\mathcal{B}} \in K$. Then, by the union axioms for K, we have that $\overline{\cup_{i<\alpha} \overline{\mathcal{A}_i}} \in K$ and $\overline{\cup_{i<\alpha} \overline{\mathcal{A}_i}} \prec_K \overline{\mathcal{B}}$. Note that the existence of $\overline{\cup_{i<\alpha} \overline{\mathcal{A}_i}}$ shows that $\cup_{i<\alpha} \mathcal{A}_i$ is completable and that an easy computation shows that $\overline{\cup_{i<\alpha} \overline{\mathcal{A}_i}} = \overline{\cup_{i<\alpha} \mathcal{A}_i}$. Thus, $\cup_{i<\alpha} \mathcal{A}_i \in K_{dense}$ and, for all $j < \alpha$,

$$\mathcal{A}_j \prec_{dense} \cup_{i<\alpha} \mathcal{A}_i \prec_{dense} \mathcal{B}$$

Once we have defined the maps and shown that K_{dense} is an AEC, the rest of the proof proceeds exactly as in the continuous first-order case, in some ways simpler since L^+ only has relations for each relation of L, rather than each formula of L.†

We now turn to an application. Both Hirvonen and Zambrano have proved versions of Shelah's Presentation Theorem for MAECs in their theses. The more general is Zambrano's Theorem 1.2.7 from [Zam]:

Theorem 6.2. *Let K be a MAEC. There is $L_1 \supset L$ and a continuous*[1] *L_1-theory T_1 and a set of T_1-types Γ such that $K = PC(T_1, \Gamma, L)$.*

An immediate corollary to our presentation theorem is a discrete presentation theorem.

Corollary 6.3 (Discrete Presentation Theorem for Metric AECs)**.** *Let K be a MAEC. Then there is a (discrete) language L_1 of size $LS(K)$, an L_1-theory T_1, and a set of T_1-types Γ such that $K = \{\overline{M_1 \upharpoonright L(K)} : M_1 \vDash T_1 \text{ and omits } \Gamma\}$, where the completion is taken with respect to a canonically definable metric.*

Proof: Apply Theorem 6.1 to represent K as a discrete AEC K_{dense} and then apply Shelah's Presentation Theorem from Shelah [Sh88].

[1]But not *uniformly* continuous

Additionally, Zambrano asks (as Question [Zam].1.2.9) if there exists is a Hanf number for model existence in MAECs. Using our presentation theorem, we can answer this questions in the affirmative. Furthermore, the Hanf number for MAECs is the same as for AECs.

Theorem 6.4. *If K is a MAEC with $LS(K) = \kappa$ models of size or density character cofinal in $\beth_{(2^\kappa)^+}$, then K has models with density character arbitrarily large.*

Proof: For every, $\lambda < \beth_{(2^\kappa)^+}$, let $M_\lambda \in K$ have size $\geq \lambda$. $|M_\lambda|$ is nicely dense in itself, so $(M_\lambda)_{|M_\lambda|} \in K_{dense}$ has size $\geq \lambda$. By the definition of Hanf number for discrete AECs, this means that K_{dense} has arbitrarily large models. Taking completions, this means K has arbitrarily large models. The proof for density character is the same. †

Given this representation, we can determine basic structural properties of K by looking at K_{dense} and vice versa. The above theorem already shows how to transfer arbitrary large models and the other properties transfer similarly.

Proposition 6.5. *Suppose K is an MAEC. For P being amalgamation, joint embedding, or no maximal models, K has P iff K_{dense} has P.*

The fourth clause of the conclusion of Theorem 6.1 is stated as it is to make the proof of this proposition easy to see.

We now look at the notion of type in K_{dense} that corresponds to Galois types in K. As before, we pass to a sequence of types representing a Cauchy sequence for a realization of the Galois type we wish to represent.

Definition 6.6.

- *Given $\mathcal{A} \in K_{dense}$, $\langle r_n : n < \omega\rangle$ is a* sequence Galois type *over $\mathcal{A}$ iff*
  1. *$r_0 \in gS^\ell(\mathcal{A})$; and*
  2. *$r_{n+1} \in gS^{\ell 2}(\mathcal{A})$ such that*
     - (a) *if $\mathbf{ab} \vDash r_{n+1}$, then $d(\mathbf{a}, \mathbf{b}) \leq \frac{1}{2^n}$; and*
     - (b) *$r_{n+1}^{\{\ell,\dots,\ell 2-1\}} = r_n^\ell$, i. e. the first ℓ coordinates of r_n are the same as the final ℓ coordinates of r_{n+1}.*
- *Given a sequence Galois type $\langle r_n : n < \omega\rangle$ over $\mathcal{A}$ and $\mathcal{A} \prec_{dense} \mathcal{B}$, we say that $\langle \mathbf{a}_n \in \mathcal{B} : n < \omega\rangle$* realizes *$\langle r_n : n < \omega\rangle$ iff*
  1. *$\mathbf{a}_0 \vDash r_0$; and*
  2. *$\mathbf{a}_{n+1}\mathbf{a}_n \vDash r_{n+1}$*
- *Given a sequence Galois type $\langle r_n : n < \omega\rangle$ over $\mathcal{A}$ and $\mathcal{A} \prec_{dense} \mathcal{B}$, we say that $\langle \mathbf{a}_n \in \mathcal{B} : n < \omega\rangle$* weakly realizes *$\langle r_n : n < \omega\rangle$ iff there is some $\mathcal{C}$ and $\langle \mathbf{b}_n \in \mathcal{C} : n < \omega\rangle$ such that*
  1. *$\mathcal{B} \prec_{dense} \mathcal{C}$;*
  2. *$\langle \mathbf{b}_n : n < \omega\rangle$ realizes $\langle r_n : n < \omega\rangle$; and*
  3. *$\lim_{n<\omega} \mathbf{a}_n = \lim_{n<\omega} \mathbf{b}_n$.*

Note that realizing a sequence Galois type is different than realizing each individual Galois type in the sequence separately. However, we can always realize sequence Galois types given amalgamation.

Lemma 6.7. *Suppose that K has amalgamation. Given any sequence Galois type $\langle r_n : n < \omega \rangle$ over $\mathcal{A} \in K_{dense}$, there is $\mathcal{A} \prec_{dense} \mathcal{B} \in K_{dense}$ that contains a realization of $\langle r_n : n < \omega \rangle$.*

Proof: By the definition of Galois type, we can write each r_n as $gtp_{K_{dense}}(\mathbf{a}_0^0/\mathcal{A}; \mathcal{B}_0)$ and $gtp_{K_{dense}}(\mathbf{a}_{n+1}^{n+1}\mathbf{a}_n^{n+1}/\mathcal{A}; \mathcal{B}_{n+1})$. Using amalgamation in K_{dense}, which follows from amalgamation in K, we can construct increasing $\langle \mathcal{C}_n : n < \omega \rangle$ and increasing $f_n : \mathcal{B}_n \to_{\mathcal{A}} \mathcal{C}_n$ such that f_0 is the identity and $f_n(a_n^n) = f_{n+1}(a_n^{n+1})$; this second part is due to the second clause in the definition of sequence types. Setting $\mathcal{C} = \cup_{n<\omega} \mathcal{C}_n$, we have that $\langle f_n(a_n^n) : n < \omega \rangle$ realizes $\langle r_n : n < \omega \rangle$. †

Unfortunately, we don't have the same tight connection between Galois types in K and sequence Galois types in K_{dense} as exists in Theorem 4.2. This is due to the fact that the Galois version of sequence types specifies a distance between consecutive members of the Cauchy sequence, rather than just specifying a bound on the distance. It is possible that perturbations (as in Hirvonen and Hyttinnen [HH12]) might be used to restore this connection. Instead, we have introduced the notion of weakly realizing a sequence Galois type because this is enough to prove a variant of Theorem 4.4 in this context.

Theorem 6.8. *Let $M \in K$ and $\mathcal{A} \in K_{dense}$ such that $\overline{\mathcal{A}} = M$.*

1. *If M is κ-Galois saturated and $\lambda^{\aleph_0} < \kappa$, then $\mathcal{A}$ is λ^+-weakly saturated for sequence Galois types (i.e. given any $\mathcal{A}_0 \prec_{dense} \mathcal{A}$ of size $< \lambda^+$, every sequence Galois type over $\mathcal{A}_0$ is weakly realized in $\mathcal{A}$).*
2. *If $\mathcal{A}$ is κ-weakly saturated for sequence Galois types, then M is κ-Galois saturated.*

Proof: First, suppose that M is κ saturated and $\lambda^{\aleph_0} < \kappa$. Let $\mathcal{A}_0 \prec_{dense} \mathcal{A}$ be of size $\leq \lambda$ and let $\langle r_n : n < \omega \rangle$ be a sequence Galois type over $\mathcal{A}_0$. By Lemma 6.7, there is some $\mathcal{B}_{dense} \succ \mathcal{A}_0$ and $\langle \mathbf{a}_n : n < \omega \rangle$ that realizes $\langle r_n : n < \omega \rangle$. After completing the members of K_{dense}, we have $\overline{\mathcal{A}_0} \prec M$ and $\overline{\mathcal{A}_0} \prec \overline{\mathcal{B}} \in K$. Since $\langle \mathbf{a}_n \in \mathcal{B} : n < \omega \rangle$ is a Cauchy sequence, there is some $\mathbf{a} \in \mathcal{B}$ such that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$. Since M is κ-saturated and $\|\overline{\mathcal{A}_0}\| \leq \lambda^{\aleph_0} < \kappa$, there is $\mathbf{b} \in M$ that realizes $gtp_K(\mathbf{a}/\overline{\mathcal{A}_0}; \overline{\mathcal{B}})$. Since $\mathcal{A}$ is dense in M, there is some Cauchy sequence $\langle \mathbf{b}_n \in \mathcal{A} : n < \omega \rangle$ that converges to $\mathbf{b}$. Then $\langle \mathbf{b}_n : n < \omega \rangle$ weakly realizes $\langle r_n : n < \omega \rangle$.

Second, suppose that $\mathcal{A}$ is κ-weakly saturated for sequence Galois types and let $M_0 \prec M$ of size $< \kappa$ and $r \in gS(M_0)$. Let $A_0 \subset A$ be nicely dense in M_0; then $\mathcal{A}_0 := (M_0)_{A_0} \prec_{dense} \mathcal{A}$. In K, we can write r as $gtp_K(\mathbf{a}/M_0; N)$. Then, in K_{dense}, $\overline{r} := \langle gtp_{K_{dense}}(\mathbf{a}/\mathcal{A}_0; N_{|N|}) : n < \omega \rangle$ is a sequence Galois type over $\mathcal{A}_0$. Since $\|\mathcal{A}_0\| < \kappa$, by $\mathcal{A}$'s weak saturation, there is some $\langle \mathbf{b}_n : n < \omega \rangle$ that weakly realizes

$\overline{r}$. This means that we can find an extension $\mathcal{B}_{dense} \succ \mathcal{A}$ and $f : N_{|N|} \to_{\mathcal{A}_0} \mathcal{B}$ such that

$$\lim_{n\to\infty} f(\mathbf{a}) = \lim_{n\to\infty} \mathbf{b}_n$$

Since $\mathbf{b}_n \in \mathcal{A}$, this means that $f(\mathbf{a}) \in M$. Since $N_{|N|}$ is complete, the K_{dense}-embedding f is in fact a K-embedding from N into $\overline{\mathcal{B}}$ and fixes $\overline{\mathcal{A}_0} = M_0$. Thus, we have that $f(\mathbf{a}) \in M$ realizes $gtp_K(\mathbf{a}/M_0; N)$. †

A crucial property in the study of MAECs is whether the natural notion of distance between Galois types defines a metric or not. This property is called the Pertubation Property by Hirvonen and Hyttinen [HH09] and the Continuity Type Property by Zambrano [Zam12]. For ease, we assume that K (equivalently, K_{dense}) has a monster model $\mathfrak{C}$.

Definition 6.9.
- *Given $M \in K$ and $p, q \in S(M)$, we define*

$$\mathbf{d}(p,q) = \inf\{d(a,b) : a \vDash p \text{ and } b \vDash q\}$$

- *K has the Pertubation Property (PP) iff, given any Cauchy sequence $\langle b_n \in \mathfrak{C} : n < \omega\rangle$ and $M \in K$, if, for all $n < m < \omega$,*

$$gtp(b_n/M) = gtp(b_m/M)$$

then, $gtp(\lim_{n\to\infty} b_n/M) = gtp(b_0/M)$.

Although these properties might not initially seem related, a little work shows that $\mathbf{d}$ is always a pseudometric and that it is a metric iff PP holds; this is due Hirvonen and Hyttinen. Then, similar to tameness from AECs, Zambrano [Zam12].2.9 defines a notion of tameness in MAECs that satisfy PP.

Definition 6.10. *K is* μ-d-tame *iff for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $M \in K$ of density character $\geq \mu$ and $p, q \in gS(M)$, if $\mathbf{d}(p,q) \geq \epsilon$, then there is some $N \prec M$ of density character μ such that $\mathbf{d}(p \upharpoonright N, q \upharpoonright N) \geq \delta_\epsilon$.*

Again these properties transfer to sequence types in K_{dense}. We can define a pseudometric on sequence Galois types in K_{dense} by

$$\begin{aligned}\mathbf{d}_{dense}(\langle r_n : n < \omega\rangle, \langle s_n : n < \omega\rangle) \quad := \quad & \inf\{\lim_{n\to\infty} d(a_n, b_n) : \langle a_n : n < \omega\rangle \text{ realizes} \\ & \langle r_n : n < \omega\rangle, \langle b_n : n < \omega\rangle \text{ realizes } \langle s_n : n < \omega\rangle\}\end{aligned}$$

Then $\mathbf{d}_{dense}$ is a metric iff $\mathbf{d}$ is, and μ-d-tameness transfers from K to K_{dense} in the obvious way.

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