Teaching for Mastery of Multiplication

As you enter Ms. Hand's fifth-grade classroom, you see students working at their desks on multidigit multiplication algorithms. A few students have taped multiplication tables on their desktops and look at these charts for each multiplication fact—even the zero-times and one-times facts. Across the hall in Mr. Smith's class, students are working on the same type of algorithms but several students are counting out small disks onto paper plates. Farther down the hall, the students in Mrs. Williams's class who are having trouble with the same algorithms are making little tally marks to count in groups.

A few students in each of these classrooms have not mastered the multiplication facts and are struggling with mathematics assignments dependent on understanding and fluency. Some of these students have identified learning disabilities that require more intensive instruction. Although they were introduced to multiplication concepts in second grade and are required to memorize their facts in third grade, these students still rely on external devices to compute. Why are students still lacking multiplication-fact skills in upper elementary and even secondary grades? How should multiplication facts be introduced to students? What strategies are effective for learning even the most difficult-to-recall facts?

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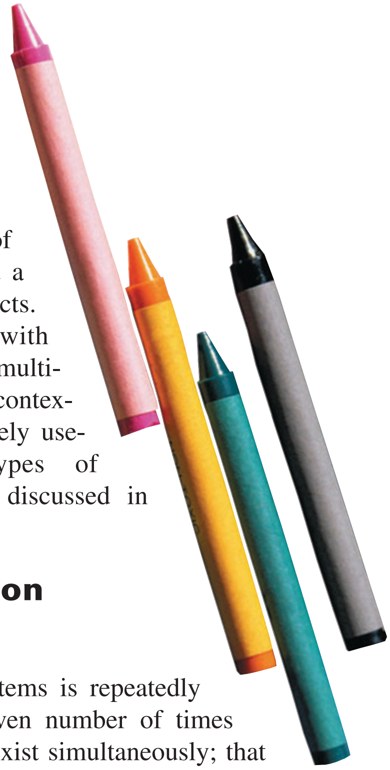
The Case for Requiring Multiplication-Fact Mastery

The National Council of Teachers of Mathematics (NCTM) makes a strong argument for computational fluency. In fact, *Principles and Standards for School Mathematics* states, "Developing fluency requires a balance and connection between conceptual understanding and computational proficiency" (NCTM 2000, p. 35). Rote memorization of basic facts is not fluency. Fluency with multiplication facts includes the deeper understanding of concepts and flexible, ready use of computation skills across a variety of applications.

Why should children learn the multiplication facts? Because children without either sound knowledge of their facts or a way of figuring them out are at a profound disadvantage in their subsequent mathematics achievement. Students without multiplication-fact fluency spend more time determining routine answers and less time on more meaningful applications. Students who know their facts build on these fundamental concepts, ultimately benefiting their later mathematical development.

For years, learning to compute has been viewed as a matter of following the teacher's directions and practicing until speedy execution is achieved.



The emphasis on paper-and-pencil computational skills is no surprise in light of educational realities. Teachers are held accountable by standardized tests whose primary emphasis is on students' computational abilities. The current high-stakes testing environment may at first compel teachers to deconstruct mathematics to its isolated skills. When skills such as multiplication facts are taught for conceptual understanding and connected to other mathematics concepts and real-world meaning, however, students actually perform better on standardized tests and in more complex mathematics applications (Campbell and Robles 1997). Finally, teaching multiplication facts effectively also promotes deeper conceptual understanding. For example, students taught isolated facts may not realize that multiplication has multiple meanings, and this lack of knowledge limits their ability to solve word problems.

A major conceptual challenge when teaching multiplication is helping children understand that multiplication has a variety of meanings and is not just a sequence of isolated facts. Providing experiences with the different meanings of multiplication, especially in contextual situations, is extremely useful. Consider the types of multiplication problems discussed in the following sections.

Repeated addition (grouping and partitioning)

A specified number of items is repeatedly arranged (grouped) a given number of times (see **fig. 1**). The groups exist simultaneously; that is, no item is in more than one group. One factor describes the number of items in each group; the other factor describes the number of groups. The product is the number of items contained in all the groups. (If the arrangements are in rows and columns, this is called an *array.*)

Figure 1

Repeated addition model

There are eight crayons in a box. How many crayons are there in three boxes?
8 + 8 + 8

Figure 2

Scalar model

Marcus has eight marbles. His brother has three times as many marbles. How many marbles does Marcus's brother have?

Scalar

Some quantity (a specified size or number of items) occurs a given number of times (the scalar multiple; see **fig. 2**). The product is a multiple of the original quantity. The scalar multiple expresses a relationship between the original quantity and the product, but the scalar multiple is not a visible quantity.

Rate

A value or distance is associated with a unit (see **fig. 3**). The product is the total value or distance associated with all the units. Because this problem involves a number of measured units, it can be represented with a number line.

Cartesian product

Two disjoint sets exist and the size of each set is known (see **fig. 4**). Each of the objects in one set is paired with each of the objects in the other set. The pairings do not occur simultaneously. The product indicates the number of possible pairings. **Figure 4** shows a pairing of main dishes (M) with side dishes (S).

Area

A rectangular region is defined in terms of units along its length and width (see **fig. 5**). The product is the number of square units in the region.

Why Do Some Children Fail to Learn the Facts?

Understanding the background and nature of a child's problems with learning multiplication facts can provide some clues for intervention strategies. Many students have not had access to sound instruction or a standards-based curriculum. These students may have been taught by teachers not familiar with NCTM's Standards or not using a teaching-for-understanding approach to multiplication. Student indicators of lack of access to standards-based instruction include a lack of understanding of basic concepts and their connections, rigidity in attempting algorithms or problem solving, and little or no exposure to related application topics such as measurement and geometry. Many special education teachers have had only limited training in mathematics instruction in preservice settings, a situation that is only beginning to change with the emphasis on access to the general curriculum and the Individuals with Disabilities Education Act of 1997. Students whose families have moved frequently or have changing parental roles for educational support are also at risk for lack of understanding and gaps in learning.

About 5 percent to 8 percent of children may have specific mathematics disabilities (Fuchs and Fuchs 2003). The nature of these disabilities varies considerably among children. Some of the more prevalent learning characteristics that may have an impact on multiplication fluency include the use of less mature procedures and strategies, a tendency to commit more procedural errors without self-monitoring, and more difficulty retrieving information from memory than their nondisabled peers (Geary 2003). Students with deficits in cognitive strategies may appear to their teachers to be giving up without trying, lazy, or even less able than their peers.

Some children with mathematics disabilities have memory or language deficits that have a direct impact on mathematics skills. Quick recall of previously learned mathematics facts may be related to a student's working memory, semantic retrieval, and ability to inhibit irrelevant information (Geary 2003). Studies using constant time-delay techniques, however, have demonstrated that students with mathematics disabilities can develop fluency with facts and maintain fluency over time (Koscin-

Figure 3

Rate model

Jeremy walks three miles an hour. How many miles does he walk in eight hours?

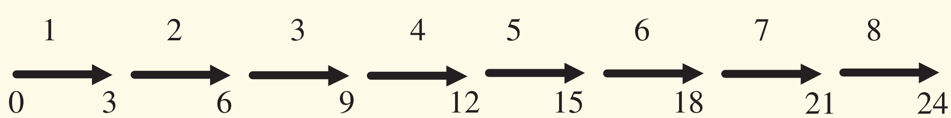
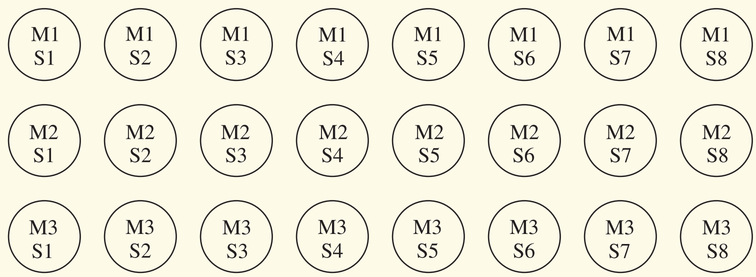


Figure 4

Cartesian product model

Kelly's Diner offers three main courses and eight side dishes. The Bargain Dinner is made up of one main dish and one side dish. How many different Bargain Dinners can Kelly make?



ski and Hoy 1993). Students with language disabilities may also have related problems with mathematics, especially with new vocabulary terms that have other, nonmathematical meanings, such as *difference, product, factor, set,* and *multiple* (Rubenstein and Thompson 2002). These students require more explicit instruction of vocabulary closely linked with mathematics materials and examples.

Other children may have developed ineffective learning habits. Students with specific learning disabilities tend to be passive, unengaged learners with few metacognitive strategies. They rarely set learning goals for themselves or monitor their own work. Unless taught explicitly how to engage with meaningful materials, link mathematics concepts, and study basic facts effectively, these students remain dependent on charts and calculators through middle and high school.

How Should Multiplication Facts Be Taught?

Multiplication instruction often begins with the 0 or 1 facts, probably because they are the easiest to learn, and ends with the 9 facts because they are considered the most difficult. Teachers support this structure because textbooks fashion their multiplication chapters in this sequence. With every lesson, teachers introduce students to additional number families between 0 and 9, and students are expected to memorize all the facts by these family groupings. An examination of third-grade textbooks depicted a similarity in their approach to multiplication-fact instruction that has changed little over the years (Addison-Wesley 1973; Houghton Mifflin 2002). Generally, they teach multiplication around the fifth or sixth chapter, and only after exploring addition and subtraction of multiple digits. The textbooks also teach multiplication in isolation, instead of relating it to addition or the reciprocal of division. The typical order for multiplication fact presentation is numbers 5 and fewer (in varying orders), followed by 6s, 7s, 8s, and 9s (predominantly in that order). The student practice pages are filled with isolated computational exercises. Only once all the facts have been "covered" do texts recommend that teachers incorporate multiplication applications and problem solving. Although this "textbook" method of teaching multiplication facts may promote computational proficiency, it lacks the conceptual understanding necessary for children to obtain computational fluency.

A growing body of evidence, including research on students with significant learning problems, indicates that the most effective sequence of instruction for multiplication facts is as follows:

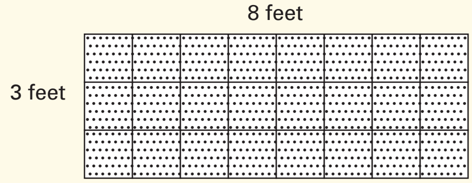
- Introducing the concepts through problem situations and linking new concepts to prior knowledge
- Providing concrete experiences and semiconcrete representations prior to purely symbolic notations
- Teaching rules explicitly
- Providing mixed practice (Fuson 2003; Mercer and Miller 1992b)

The teacher should introduce multiplication by presenting a realistic problem involving repeated

Figure 5

Area model

A sheet of wrapping paper is three feet wide and eight feet long. What is the area of the sheet of paper?



addition, such as adding the costs of markers in the school store. The first few lessons should be filled with a variety of hands-on materials for students to manipulate. A student could be asked to pass out four markers to each of the five members in her group (see **fig. 6**). How many markers are needed? Using the markers as manipulatives, the student is able to count the markers by 4s, or count them one at a time if necessary. Building on that problem, students learn to substitute blocks for the items in the problem; for example, "My father baked 3 cookies for each of my 6 friends. How many cookies did he bake?" Students use blocks to represent cookies instead of using actual cookies as manipulatives. They are still able to physically demonstrate the problem by giving three blocks to six friends and then determining that the total number of "cookies" is eighteen (see **fig. 7**).

The next segment of this instructional sequence promotes students' drawings. A student is asked to determine how many flowers are in a garden if there are six rows and five flowers in each row. The student draws a 6-by-5 representation of the garden, resulting in thirty total flowers (see **fig. 8**). Further drawings could include boxes or tallies within circles (see **fig. 9**). By working on multiplication using a variety of models and representations, students begin to develop thinking strategies to help determine the answers to basic facts. Using these strategies on simple problems can help them learn the more difficult facts. For example, to solve the problem 7×8, one third-grade student chose to "factor" 7×8 into 2×8 plus 5×8. She stated, "My 2s and 5s are easy for me." She was then able to determine that $2 \times 8 = 16$ and $5 \times 8 = 40$. She added $40 + 16$ to reach the answer 56. Another student in the class chose to "factor" the 8 into 4 and 4. He recalled that $7 \times 4 = 28$, so he added 28 to 28 to reach the correct answer of 56.

Another thinking strategy that students use is called *skip counting*. It involves counting by the second factor the number of times indicated by the first factor. For example, 3×4 could be found by skip counting 4, 8, 12. This method builds on the meaning of multiplication because it incorporates the concept of factors and promotes the understanding that counting by 4 three times is the same as counting three groups of 4. These informal strategies give children powerful resources for developing a deeper understanding of mathematics concepts and stronger skill with facts within applications.

Only after students have moved through concrete and representational experiences should time be devoted to fact drills important for speed and accuracy. Children with learning disabilities need dozens of repetitions of each fact so teachers must provide opportunities for varied and interesting practice, including games, flash cards, and dice as well as calculator- and computer-based practice. For example, a third-grade teacher uses "concen-

Figure 6

Realistic manipulatives

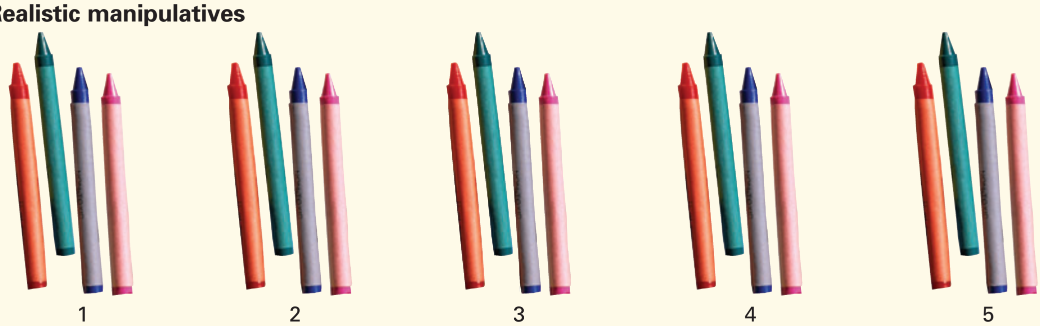


Table 1

Hypothetical known multiplication facts

X	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10
2	0	2	4	6	8	10	12	14	16	18	20
3	0	3	6	9	12	15	18	21	24	27	30
4	0	4	8	12	16	20	24	28	32	36	40
5	0	5	10	15	20	25	30	35	40	45	50
6	0	6	12	18	24	30	36	42	48	54	60
7	0	7	14	21	28	35	42	49	56	63	70
8	0	8	16	24	32	40	48	56	64	72	80
9	0	9	18	27	36	45	54	63	72	81	90
10	0	10	20	30	40	50	60	70	80	90	100

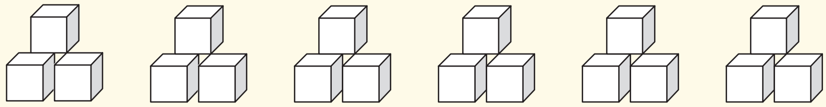
tration cards" to help her students relate array models to multiplication facts. A "match" would include one card presenting a fact, such as 3 × 4, and its pair presenting a 3 × 4 rectangular array with no numerals. The *Strategic Math Series* by Mercer and Miller (1992a) provides deceptively simple "pig dice" activities and multiplication minutes. Web sites such as illuminations.nctm.org/tools/index.aspx offer interactive on-line games such as the factor game. The Illuminations site also provides learning resources such as the product game that can be used off-line. That children memorize their multiplication facts is important, but this memorization should be based on an understanding of the operations and thinking strategies (Fuson 2003). Forcing memorization before children have moved through concept development results in acquiring knowledge that has little meaning or usefulness and often creates a dislike of mathematics.

How Can We Reach the Hard to Teach?

For the child who has good understanding of multiplication concepts and fluency with some facts but difficulty with others, we recommend pinpointing the problem facts through mixed probes. The teacher should share the results of the probes with the child by shading in a multiplication table to show the student which facts are fluent (no hesitations) and which require more effort. **Table 1** shows a multiplication grid with the easiest facts shaded. In our experience, students are surprised at how many facts they already know and how few remain.

Figure 7

Symbolic manipulatives



Make sure that understanding of the properties is firm. The most important "rules" for multiplication are the zero, one, and inverse rules, which mathematicians also call the zero, identity, and commutative properties of multiplication. We find that children understand the terminology "rule of the ones," or any number times one is that number, instead of memorizing the formal identity property. Some children will confuse their addition rules for zero and one. It is important for students to see the underlying difference between "Zero plus a number is that number" and "Zero times a number is zero—no groups or nothing in each group." Manipulatives, coupled with explicit student and teacher language, can bridge this understanding. When children really understand these rules, they begin to challenge one another with enormous problems such as "1,234,567 times one" or "5,500,000 times zero."

The commutative property—that x times y is the same as y times x—is also important for manipulating division statements and algebraic expressions. For multiplication facts, we typically teach that the first term is the number of groups and the second term is the number of items in each group. The product is the total number of items in all groups; so the first term differs in its label from the second. Reversing the terms changes the one that represents the groups, but not the total number of items. Comparing these groupings, and the other ways to show multiplication, using a variety of objects helps students see which terms have an effect on the product.

Promote varied practice to fluency. As discussed in the previous section, students need repeated and varied practice to achieve multiplication-fact fluency. Strong research supports developing and maintaining that fluency through the use of mnemonics (Greene 1999), computer-assisted practice (Irish 2002), constant time delay (Koscinski and Hoy 1993), and copy/cover/compare drills (Stading, Williams, and McLaughlin 1996). Students should be encouraged to set learning goals and chart their own progress.

Figure 8

Realistic representations

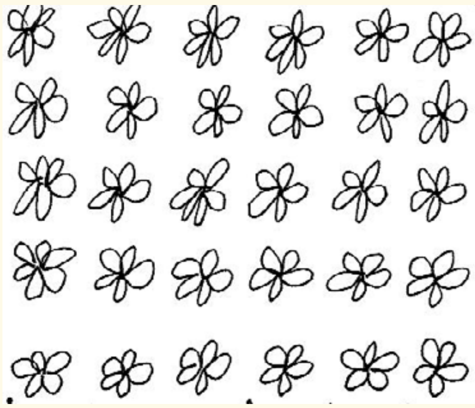
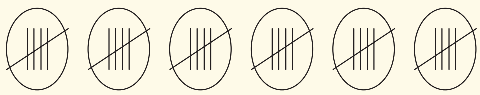


Figure 9

Symbolic representations



Teach students to think strategically. Children with deep concept understanding of multiplication will have an advantage when faced with a forgotten multiplication fact. The most common reconstructive strategies that proficient students use are repeated addition, addition sets, double a known fact, and subtraction from an anchor point (Erenberg 1995). Students with mathematics disabilities tend to use less systematic approaches and rely on guessing, but effective strategies can be taught to children who have immature approaches.

Repeated addition and doubling a known fact are strategies built on the concept of multiplication as repeated addition of factors. Subtraction from an anchor point also builds on known facts but reduces the number of required steps. If the student is asked 7 × 9 and knows that 7 × 10 = 70, then 7 × 9 would be 70 – 7, or 63. Students with mathematics disabilities have the most trouble with the four, six, seven, eight, and nine tables (Erenberg 1995). For the 6s through 9s, another conceptually based procedure is the finger multiplication that Barney (1970) describes. These procedures can also be viewed at www.cofc.edu/wallacegurganus. In our experience, students who learn reconstructive strategies enjoy teaching these strategies to other students, thereby gaining stronger concept understanding themselves.

Concluding Remarks

Teaching for mastery of multiplication facts no longer means rote memorization of basic facts. Making the connection between conceptual understanding and computational fluency is important for all students, not just students who are struggling. Learning through understanding helps students make connections across a variety of problem situations. Mathematics teachers, special educators, and remedial mathematics specialists can actively assist all students with multiplication-fact fluency by assessing their concept understanding and fluency, teaching for understanding across problem types, and encouraging students to use personal strategies for learning the facts and developing automaticity.

Students who master their multiplication facts have a more positive attitude about their mathematics abilities and further mathematics experiences. Teaching for understanding plus strategic learning equals a formula for success.

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