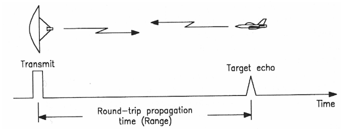
Radar Signal Processing

Ambiguity Function and Waveform Design
Golay Complementary Sequences (Golay Pairs)
Golay Pairs for Radar: Zero Doppler

Transmit a waveform $s(t)$ and analyze the radar return $r(t)$:

$$r(t) = hs(t-\tau_o)e^{-j\omega(t-\tau_o)}+n(t)$$



h: target scattering coefficient; $\tau_o = 2d_o/c$: round-trip time; $\omega = 2\pi f_o \frac{2v_o}{c}$: Doppler frequency; $n(t)$: noise

- Target detection: decide between target present ($h \neq 0$) and target absent ($h = 0$) from the radar measurement $r(t)$.
- Estimate target range d_0.
- Estimate target range rate (velocity) v_0.

Ambiguity Function

- Correlate the radar return $r(t)$ with the transmit waveform $s(t)$. The correlator output is given by

$$m(\tau - \tau_o, \omega) = \int_{-\infty}^{\infty} hs(t - \tau_o)\overline{s(t - \tau)}e^{-j\omega(t-\tau_o)}dt + \text{noise term}$$

- Without loss of generality, assume $\tau_o = 0$. Then, the receiver output is

$$m(\tau, \omega) = hA(\tau, \omega) + \text{noise term}$$

where

$$A(\tau, \omega) = \int_{-\infty}^{\infty} s(t)\overline{s(t - \tau)}e^{-j\omega t}dt$$

is called the *ambiguity function* of the waveform $s(t)$.

Ambiguity Function

- Ambiguity function $A(\tau, \omega)$ is a two-dimensional function of delay τ and Doppler frequency ω that measures the correlation between a waveform and its Doppler distorted version:

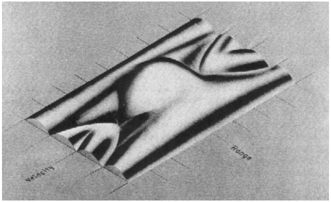
$$A(\tau, \omega) = \int_{-\infty}^{\infty} s(t)\overline{s(t-\tau)}e^{-j\omega t}dt$$

- The ambiguity function along the zero-Doppler axis ($\omega = 0$) is the autocorrelation function of the waveform:

$$A(\tau, 0) = \int_{-\infty}^{\infty} s(t)\overline{s(t-\tau)}dt = R_s(\tau)$$

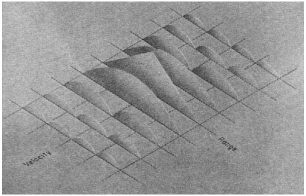
Ambiguity Function

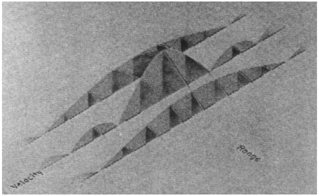
- Example: Ambiguity function of a square pulse



Picture: Skolnik, ch. 11

- Constant velocity (left) and constant range contours (right):





Pictures: Skolnik, ch. 11

Ambiguity Function: Properties

- Symmetry:

$$A(\tau, \omega) = \overline{A(-\tau, -\omega)}$$

- Maximum value:

$$|A(\tau, \omega)| \leq |A(0,0)| = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

- Volume property (Moyal's Identity):

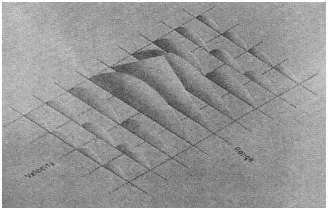
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A(\tau, \omega)|^2 d\tau d\omega = |A(0,0)|^2$$

Pushing $|A(\tau, \omega)|^2$ down in one place makes it pop out somewhere else.

Waveform Design

- Waveform Design Problem: Design a waveform with a good ambiguity function.
- A point target with delay τ_o and Doppler shift ω_o manifests as the ambiguity function $A(\tau, \omega_o)$ centered at τ_o.
- For multiple point targets we have a superposition of ambiguity functions.

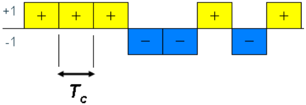
- A weak target located near a strong target can be masked by the sidelobes of the ambiguity function centered around the strong target.



Picture: Skolnik, ch. 11

Waveform Design

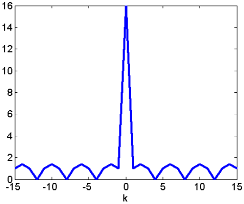
- Phase coded waveform:



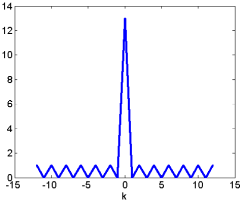
$$s(t) = \sum_{\ell=0}^{L-1} x(\ell) u(t - \ell \Delta T)$$

- The pulse shape $u(t)$ and the chip rate ΔT are dictated by the radar hardware.
- $x(\ell)$ is a length-L discrete sequence (or code) that we design.
- Control the waveform ambiguity function by controlling the autocorrelation function of $x(\ell)$.
- Waveform design: Design of discrete sequences with good autocorrelation properties.

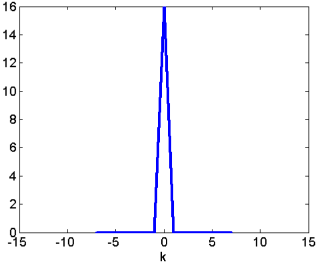
Phase Codes with Good Autocorrelations



Frank Code



Barker Code



Golay Complementary Codes

Waveform Design: Zero Doppler

- Suppose we wish to detect stationary targets in range.
- The ambiguity function along the zero-Doppler axis is the waveform autocorrelation function:

$$
\begin{aligned}
R_s(\tau) &= \int_{-\infty}^{\infty} s(t)\overline{s(t-\tau)}dt \\
&= \sum_{\ell=0}^{L-1}\sum_{m=0}^{L-1} x(\ell)\overline{x(m)} \int_{-\infty}^{\infty} u(t-\ell\Delta T)\overline{u(t-\tau-m\Delta T)}dt \\
&= \sum_{\ell=0}^{L-1}\sum_{m=0}^{L-1} x(\ell)\overline{x(m)} R_u(\tau+(m-\ell)\Delta T) \\
&= \sum_{k=-2(L-1)}^{2(L-1)}\sum_{\ell=0}^{L-1} x(\ell)\overline{x(\ell-k)} R_u(\tau-k\Delta T) \\
&= \sum_{k=-2(L-1)}^{2(L-1)} C_x(k) R_u(\tau-k\Delta T)
\end{aligned}
$$

Impulse-like Autocorrelation

- Ideal waveform for resolving targets in range (no range sidelobes):

$$R_s(\tau) = \sum_{k=-2(L-1)}^{2(L-1)} C_x(k) R_u(\tau - k\Delta T) \approx \alpha\delta(\tau)$$

- We do not have control over $R_u(\tau)$.
- Question: Can we find the discrete sequence $x(\ell)$ so that $C_x(k)$ is a delta function?
- Answer: This is not possible with a single sequence, but we can find a *pair* of sequences $x(\ell)$ and $y(\ell)$ so that

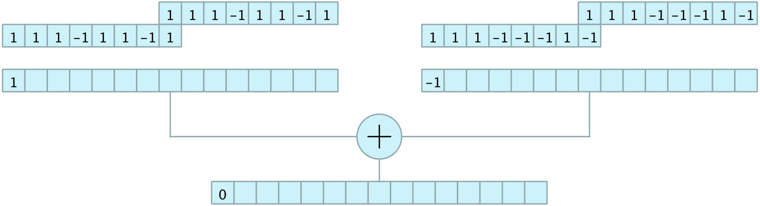
$$C_x(k) + C_y(k) = 2L\delta_{k,0}.$$

Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

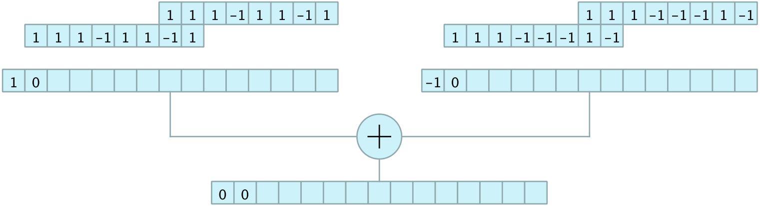
for all $-(L-1) \leq k \leq L-1$.



Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

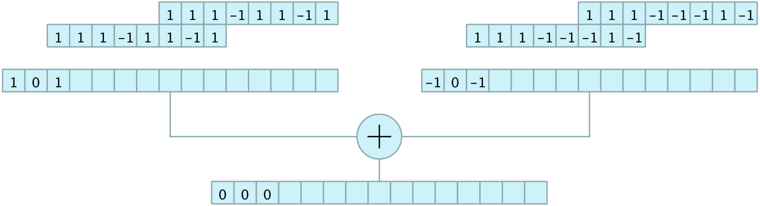


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

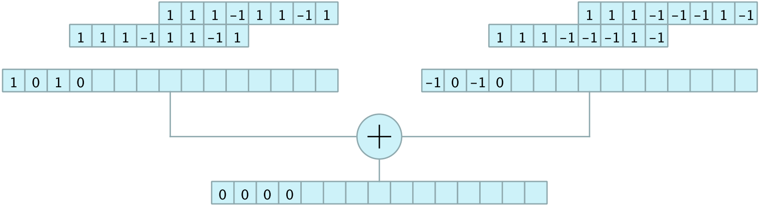


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

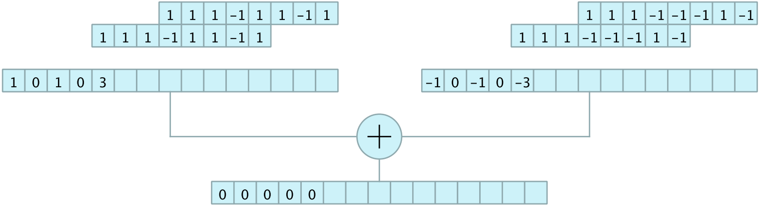


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

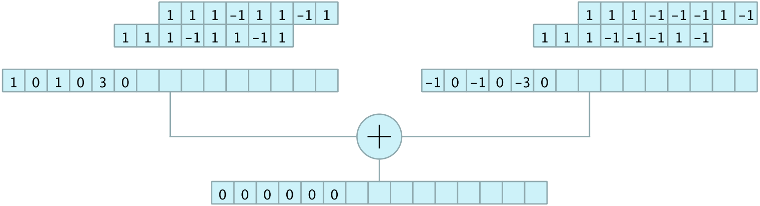
for all $-(L-1) \le k \le L-1$.



Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

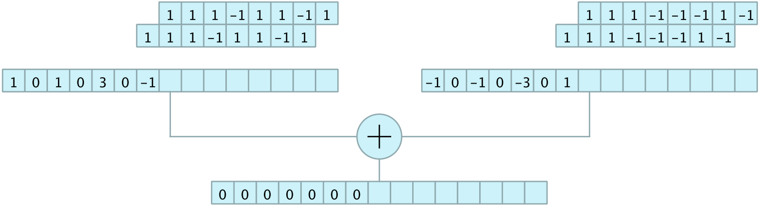


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

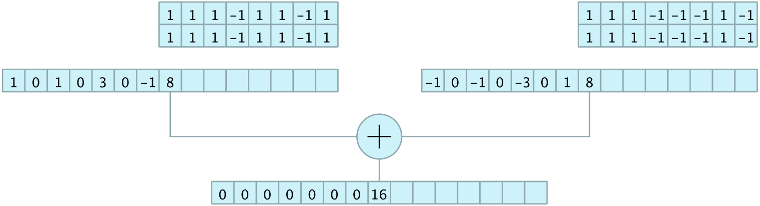


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

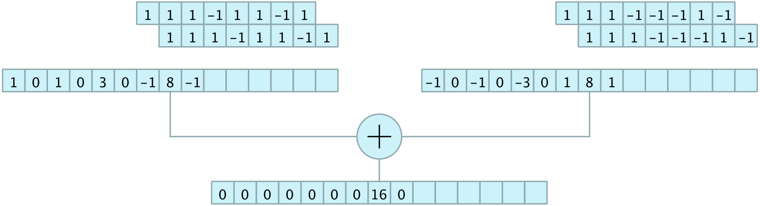


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

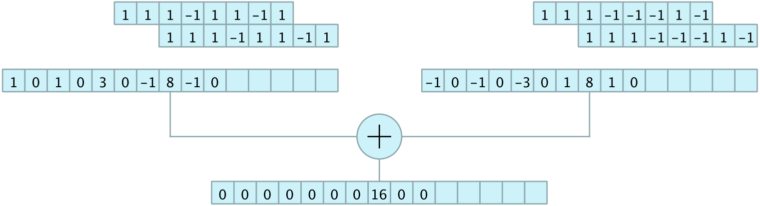


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

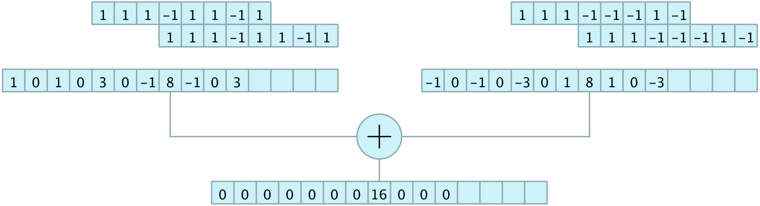


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

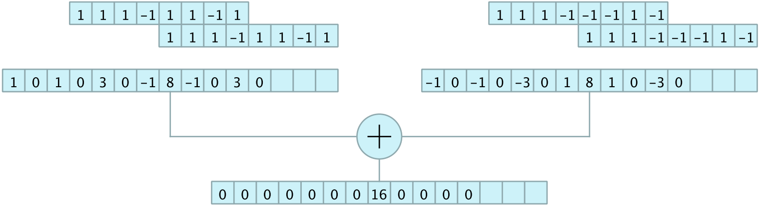


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

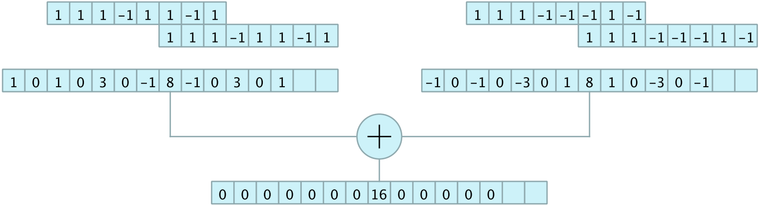


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

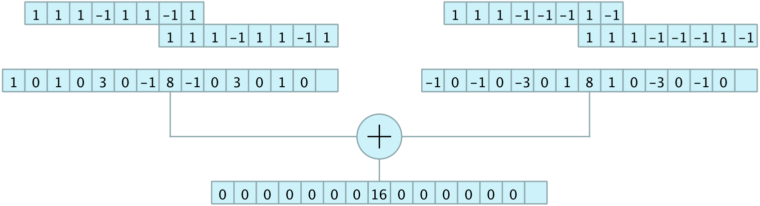


Golay Complementary Sequences (Golay Pairs)

Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.

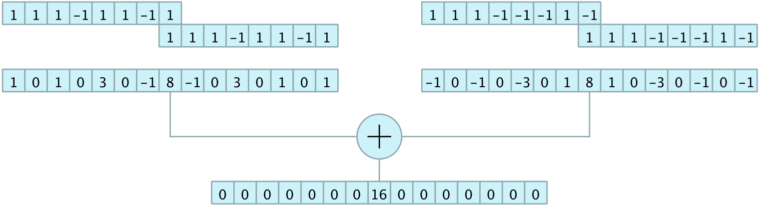


Golay Complementary Sequences (Golay Pairs)

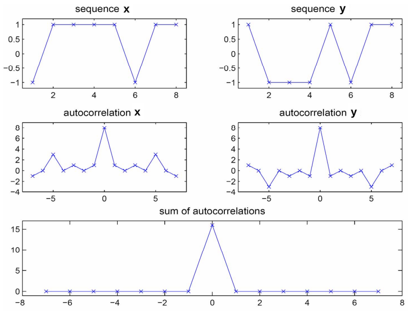
Definition: Two length L unimodular sequences $x(\ell)$ and $y(\ell)$ are Golay complementary if the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

for all $-(L-1) \leq k \leq L-1$.



Golay Pairs: Example



- Time reversal:

$$x: \quad -1 \quad 1 \quad 1 \quad 1 \quad 1 \quad -1 \quad 1 \quad 1$$

$$\widetilde{x}: \quad 1 \quad 1 \quad -1 \quad 1 \quad 1 \quad 1 \quad 1 \quad -1$$

- If (x, y) is a Golay pair then $(\pm x, \pm\widetilde{y})$, $(\pm\widetilde{x}, \pm y)$, and $(\pm\widetilde{x}, \pm\widetilde{y})$ are also Golay pairs.

Golay Pairs: Construction

- Standard construction: Start with $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and apply the construction

$$\begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} A & B \\ A & -B \\ B & A \\ B & -A \end{pmatrix}$$

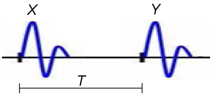
- Example:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

- Other constructions:
  - Weyl-Heisenberg Construction: Howard, Calderbank, and Moran, EURASIP J. ASP 2006
  - Davis and Jedwab: IEEE Trans. IT 1999

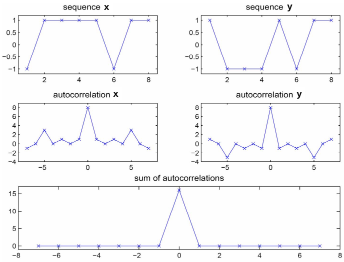
Golay Pairs for Radar: Zero Doppler

- The waveforms coded by Golay pairs x and y are transmitted over two Pulse Repetition Intervals (PRIs) T.

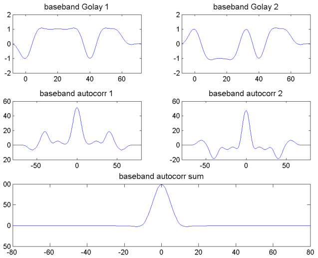


- Each return is correlated with it's corresponding sequence:

$$C_x(k) + C_y(k) = 2L\delta_{k,0}$$

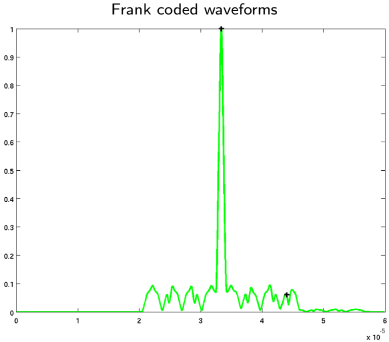


Discrete Sequence

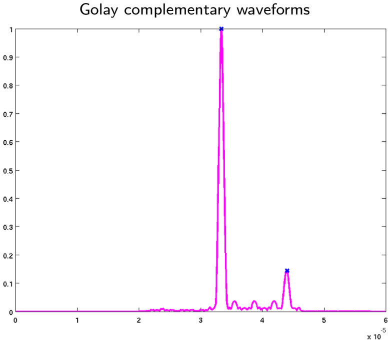


Coded Waveform

Golay Pairs for Radar: Advantage

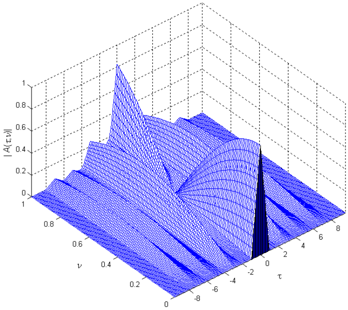


Weaker target is masked



Weaker target is resolved

Sensitivity to Doppler

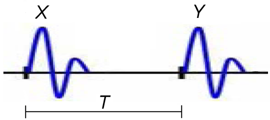


$$A_{s_x}(\tau,\nu) + e^{j2\pi\nu T} A_{s_y}(\tau,\nu)$$

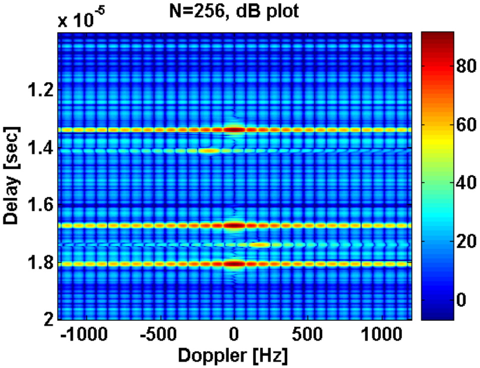
"Although the autocorrelation sidelobe level is zero, the ambiguity function exhibits relatively high sidelobes for nonzero Doppler." [Levanon, Radar Signals, 2004, p. 264]

Why? Roughly speaking

$$C_x(k) + C_y(k)e^{j\theta} \neq \alpha(\theta)\delta_{k,0}$$



Range Sidelobes Problem: A weak target located near a strong target can be masked by the range sidelobes of the ambiguity function centered around the strong target.



Range-Doppler image obtained with conventional pulse train

$x \quad y \quad \cdots \quad x \quad y$

References

1. M. I. Skolnik, "An introduction and overview of radar," in *Radar Handbook,* M. I. Skolnik, Ed. New York: McGraw-Hill, 2008.
2. M. R. Ducoff and B.W. Tietjen, "Pulse compression radar," in *Radar Handbook,* M. I. Skolnik, Ed. New York: McGraw-Hill, 2008.
3. S. D. Howard, A. R. Calderbank, and W. Moran, "The finite Heisenberg-Weyl groups in radar and communications," *EURASIP Journal on Applied Signal Processing,* Article ID 85685, 2006.
4. N. Levanon and E. Mozeson, *Radar Signals,* New York: Wiley, 2004.
5. M. Golay, "Complementary series," *IRE Trans. Inform. Theory,* vol. 7, no. 2, pp. 82-87, April 1961.