

EXISTENCE OF POLARIZED F-STRUCTURES ON COLLAPSED MANIFOLDS WITH BOUNDED CURVATURE AND DIAMETER

J. CHEEGER AND X. RONG

Abstract

We study the class of collapsed Riemannian n -manifolds with bounded sectional curvature and diameter. Our main result asserts that there is a constant, $\delta(n, d) > 0$, such that if a compact n -manifold has bounded curvature, $|K_{M^n}| \leq 1$, bounded diameter, $\text{diam}(M^n) \leq d$ and sufficiently small volume, $\text{Vol}(M^n) \leq \delta(n, d)$, then it admits a mixed polarized F-structure. As a consequence, $\inf_g \text{Vol}_g(M^n, g) = 0$, where the infimum is taken over all metrics with $|K_{(M^n, g)}| \leq 1$. This assertion can be viewed as a weakened version of Gromov's "critical volume" conjecture.

0. Introduction

We will begin by briefly recalling the notion of F-structure and some relevant related concepts; for further details, see [CG1], [CG2], [CR] and sections 1–3 below.

An *F-structure*, \mathcal{F} , on a manifold, M^n , is a kind of generalized torus action. Specifically, it is a sheaf of Lie algebras, together with a homomorphism of this sheaf onto a sheaf of abelian Lie algebras of vector fields, $e_{\mathcal{F}}$, for which a certain additional condition is satisfied. In the sequel, only the image sheaf $e_{\mathcal{F}}$ plays a role.

Let f denote a subsheaf of $e_{\mathcal{F}}$ and f_x its stalk at x . The additional condition on $e_{\mathcal{F}}$ is the following. For all $x \in M^n$, there exists an open neighborhood, $U(x)$, and a subsheaf, $f(x)$, of $e_{\mathcal{F}}|_{U(x)}$, such that $f(x)_x = (e_{\mathcal{F}})_x$ and such that for some finite normal covering space, $\pi: \tilde{U}(x) \rightarrow U(x)$, the lifted Lie algebra sheaf, $\tilde{f}(x)$, is a constant sheaf, which is isomorphic to the infinitesimal generators of the effective action of a torus, $T^{k(x)}$, on $\tilde{U}(x)$.

If all stalks, $(e_{\mathcal{F}})_x$, of the sheaf, $e_{\mathcal{F}}$, have the same dimension, $k(x) = k$, the structure is called *pure*. Otherwise, it is called *mixed*.

If for all x , one can choose $U(x)$ and $\tilde{U}(x)$, such that $\tilde{U}(x) = U(x)$, then the F-structure is called a *T-structure*. In this case, $e_{\mathcal{F}}$ is actually the Lie algebra sheaf of a *sheaf of tori*, $\mathcal{E}_{\mathcal{F}}$. If in addition, \mathcal{F} is a pure structure,

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then the sheaf, $\mathcal{E}_{\mathcal{F}}$, can be described alternatively as a flat torus bundle with holonomy in $SL(k, \mathbb{Z})$.

If M^n is simply connected, a pure F-structure is actually a T -structure for which the bundle, $\mathcal{E}_{\mathcal{F}}$, has trivial holonomy. Thus, in the simply connected case, modulo a choice of isomorphism of some fiber with the standard torus, a pure F -structure is just an ordinary torus action.

A *substructure* is defined by a subsheaf of $e_{\mathcal{F}}$, for which the action generated by each $\tilde{f}(x)$ is isomorphic to a torus action, i.e. the orbits are closed.

The action on each $\tilde{U}(x)$ of its covering group, preserves the orbits of the action generated by $\tilde{f}(x)$. Hence, the open set, $U(x)$, is partitioned into the projections of these orbits. Clearly, the projected orbit through a point, x , is independent of the choice of neighborhood, $U(x)$. It is denoted, \mathcal{O}_x , the *orbit* of x . It follows that M^n is the disjoint union of orbits, \mathcal{O}_x . Every such orbit is diffeomorphic to a compact flat Riemannian manifold, by a diffeomorphism which is unique up to affine equivalence of the flat manifold.

The *rank* of the structure is the dimension of the orbit, \mathcal{O}_x , of smallest dimension. An orbit, \mathcal{O}_x , is called *singular* if $\dim \mathcal{O}_x < k(x)$. The *singular set* S , is by definition, the union of the singular orbits. As with a group action the set S , has a canonical “coarse” stratification into strata, S_i . By definition, S_i consists of all orbits of dimension i . Note that S_i may contain exceptional orbits which are multiply covered.

If S is empty, the structure is said to be *polarized*.

A Riemannian metric, g , on M^n is called *invariant* for \mathcal{F} , if $e_{\mathcal{F}}$ is actually a sheaf of Killing fields of g . Every F-structure admits invariant metrics whose sectional curvatures satisfy the normalization, $|K| \leq 1$.

For additional background on the relation between F-structures and collapsed Riemannian manifolds with bounded curvature, see [CG1,2], [CR], [F1-4], [G1,2], [R1-3].

We now specialize to the situation which is the focus of this paper.

Let M^n be a compact Riemannian manifold, with bounded sectional curvature, say $|K_{M^n}| \leq 1$. By [CFG], [F1-4], there exists a constant $\epsilon(n, d) > 0$, such that if in addition, $\text{diam}(M^n) \leq d$ and $\text{Vol}(M^n) \leq \epsilon(n, d)$, then M^n admits a pure F-structure, \mathcal{F} , of positive rank, for which a metric, g' , close to the given one is invariant. After multiplying g' by a suitable constant (close to 1) we can assume that g' satisfies

$$|K_{(M^n, g')}| \leq 1, \quad \text{diam}(M^n, g') \leq d', \quad \text{Vol}(M^n, g') \leq \epsilon(n, d').$$

Moreover, we can assume that for the metric, g' , there are definite bounds on the higher covariant derivatives of the curvature tensor.

Our main result, Theorem 0.1, asserts that pure F-structures which arise in this way enjoy a significant property which is not shared by pure F-structures in general. Such an F-structure will be called a *sufficiently collapsible pure F-structure*.

Theorem 0.1. *There exists $\delta(n, d) > 0$, such that if M_n satisfies $|K_{M^n}| \leq 1$, $\text{diam}(M^n) \leq d$ and $\text{Vol}(M^n) \leq \delta(n, d)$, then the associated sufficiently collapsible F-structure \mathcal{F} admits a polarized substructure.*

For M^n simply connected, a pure F-structure is (up to choice of isomorphism) a torus action. If such a structure has positive rank, it follows that any 1-dimensional subgroup (with closed orbits) which does not intersect any nontrivial isotropy group defines a polarized substructure. Thus, in Theorem 0.1, implicitly our concern is with the nonsimply connected case.

Typically, the polarized substructure constructed in Theorem 0.1 will be mixed. In this connection, note that by Example 6.4 of [CR], there exist pure structures satisfying the assumptions of Theorem 0.1 (for fixed d and arbitrarily small δ) which admit no *pure* polarized substructure.

Gromov defined the *Minimal Volume* of a compact manifold by

$$\text{Min Vol}(M^n) = \inf_g \text{Vol}(M^n, g) ,$$

where the infimum is taken over all metrics, with bounded sectional curvature, $|K_{(M^n, g)}| \leq 1$; see [G2]. He conjectured the existence of a “gap” or “critical volume”, i.e. there exists $\delta(n) > 0$ such that $\text{Min Vol}(M^n) < \delta(n)$ implies $\text{Min Vol}(M^n) = 0$.

By the collapsing construction of [CG1], the existence of a polarized F-structure on M^n implies $\text{Min Vol}(M^n) = 0$. Thus, Theorem 0.1 implies the following weakened version of Gromov’s conjecture.

Theorem 0.2. *There exists $\delta(n, d) > 0$ such that if M^n admits a metric with*

$$|K_{M^n}| \leq 1 , \quad \text{diam}(M^n) \leq d , \quad \text{Vol}(M^n) \leq \delta(n, d) ,$$

then $\text{Min Vol}(M^n) = 0$.

It might seem natural to try to replace the conclusion, $\text{Min Vol}(M^n) = 0$, in Theorem 0.2, with the stronger assertion that M^n collapses with bounded curvature and diameter. However, Example 6.4 of [CR] indicates that this could well be false in general.

By the finiteness theorem of [C], for all $v > 0$, there are only finitely many diffeomorphism types of manifolds satisfying $|K_{M^n}| \leq 1$, $\text{diam}(M^n) \leq d$, for which in addition, $\text{Vol}(M^n) \geq v$; see also [Pe]. Hence, we obtain

COROLLARY 0.3. *For all $n, d > 0$, there are only a finite number of diffeomorphism classes of manifolds of nonvanishing minimal volume, which admit a metric with $|K_{M^n}| \leq 1$, $\text{diam}(M^n) \leq d$.*

Corollary 0.3 implies that there is a sense in which “most” manifolds with $|K_{M^n}| \leq 1$ have minimal volume zero. Indeed, according to [G1] for all $n \geq 3$, $d > 0$, there exist infinitely many manifolds admitting a metric with $|K_{M^n}| \leq 1$ and $\text{diam}(M^n) \leq d$. Moreover, it follows from the construction of [CR], Example 6.4, that given $n \geq 4$, there exists an increasing sequence, $d_i \rightarrow \infty$, such that for all i , there are infinitely many manifolds admitting a metric with $|K_{M^n}| \leq 1$, $\text{diam}(M^n) \leq d_{i+1}$, which admit no metric with $|K_{M^n}| \leq 1$, $\text{diam}(M^n) \leq d_i$.

If M^{2k} has some real characteristic number nonzero, then by Chern-Weil theory, there is a definite positive lower bound on $\text{Min Vol}(M^{2k})$; [C]. In [CG1], examples of pure positive rank F-structures on compact $4k$ -manifolds with nonvanishing Pontrjagin numbers are given (the first such example was due to T. Janusiewicz). These examples show that in order to obtain the existence of a polarized substructure, *some* additional geometric hypothesis on the pure F-structure is required.

It is possible however, that the bound on the diameter assumed in Theorem 0.1 is actually unnecessary and that a polarized substructure exists whenever $|K| \leq 1$, $\text{Vol}(M^n) \leq \delta(n)$, a sufficiently small positive constant. Presently, this is known to hold for $n = 2$ ([C]), $n = 3$ ([CG1, 2]) and $n = 4$ ([Bu1, 2], [R1, 2]); but compare Example 4.1 of [CG1]. If indeed, the bound on diameter is unnecessary, then by the collapsing construction of [CG1], the “critical volume” conjecture holds; in particular, it holds for $n \leq 4$.

We now briefly describe the contents of the remaining 5 sections of the paper.

As is explained in section 1, the proof of Theorem 0.1 will be carried out by working on the frame bundle, FM^n . In section 1, we also introduce a property of arbitrary pure F -structures and a property of pure F -structures which satisfy the geometric assumptions of Theorem 0.1. These two properties play a crucial role in the proof.

In section 2, we prove Theorem 0.1 modulo the above mentioned two properties.

In section 3, we establish the property of arbitrary pure F -structures; see Theorem 3.2. It concerns a certain canonical (mixed) substructure defined in a neighborhood of the singular set, S . This substructure, which is generated by the *kernels* of the local torus actions, turns out to be an F -structure of an extremely special type.

In section 4, we establish the property of pure F -structures which are

compatible with sufficiently collapsed metrics; see Theorem 4.1. Namely, over each stratum, S_i , there exists a pure polarized substructure, \mathcal{P}_i .

In section 5, we give a generalization of Theorem 0.1 to the case in which only a bound on the diameter of each component, $S_{i,j}$, of S is assumed (rather than on the diameter of M^n itself).

1. Outline of The Proof

In this section we give an indication of the proof of Theorem 0.1. Thus, unless we make explicit mention to the contrary, we will assume here that our structure, \mathcal{F} , is a sufficiently collapsible pure F-structure, equipped with an invariant metric.

Our discussion is simplified considerably by working on the frame bundle, FM^n , rather than on M^n itself; compare [F1–4]. Although this necessitates our making all constructions $O(n)$ -equivariant, in practice, for natural constructions, $O(n)$ -equivariance turns out to be automatic. For instance, a pure substructure defined over an $O(n)$ -invariant subset of FM^n is always $O(n)$ -equivariant; see [CR, Remark 0.1].

The advantage of working on FM^n lies in the fact that the canonical lift to FM^n of an F -structure is actually a T -structure, \mathcal{T} , of a particularly simple type – namely, one for which the local actions are free. (The lift is defined via the differentials of the local torus actions.) In particular, given a pure F -structure on M^n , we can regard FM^n as the total space of an $O(n)$ -invariant torus bundle, whose structural group lies in the group of affine automorphisms of the torus, T^k . Note that this group satisfies the exact sequence,

$$e \rightarrow T^k \rightarrow \text{Aff}(T^k) \rightarrow SL(k, \mathbb{Z}) \rightarrow e.$$

Before proceeding, we point out that the existence of pure F -structures of positive rank on sufficiently collapsed manifolds with bounded curvature and diameter was actually proved by working on the frame bundle; see [F1–4] and [CFG]; see also [CR] for further discussion.

In constructing a polarized substructure, it is clear that we can restrict attention to a neighborhood of the singular set, S ; outside such a neighborhood, our polarized structure will be chosen to coincide with \mathcal{F} itself.

Let D denote the inverse image of S in FM^n . Observe that D consists of those points for which the corresponding torus-fibre and $O(n)$ -fibre intersect in a subset of positive dimension. We denote by D_i , the inverse image of S_i in FM^n .

On each stratum, D_i , we define the *isotropy substructure*, \mathcal{I}_i , to be the unique maximal substructure, whose projection to M^n has rank zero. The

orbits of this structure are just the components of the intersections of torus-fibres and $O(n)$ -fibres.

An $O(n)$ -equivariant substructure, $\hat{\mathcal{T}}$, on FM^n , descends to a polarized substructure on M^n , if and only if on each D_i , it is *transversal* to \mathcal{I}_i , i.e. on each D_i , the intersection of an orbit of $\hat{\mathcal{T}}$ and an orbit of \mathcal{I}_i consists of a finite set of points. Equivalently, $\mathcal{E}_{\hat{\mathcal{T}}} \cap \mathcal{E}_{\mathcal{I}_i} = \mathcal{E}_0$, where \mathcal{E}_0 denotes the trivial subsheaf whose stalk at any point is the subgroup consisting of the identity element. A substructure of \mathcal{T} with this property will be called *nondegenerate*.

Let $1 \gg r_1 \gg r_2 \gg \cdots > 0$. Let $\eta > 0$.

Put $H_i(\eta) = T_{\eta r_i}(D_i) \setminus \bigcup_{l < i} T_{\frac{1}{2}r_l}(D_l)$, where $T_r(\cdot)$ denotes the r -tubular neighborhood. We can assume that the sequence, $\{r_i\}$, decreases so rapidly that if $\eta \leq 3$, then for every point, p , of $H_i(\eta)$, there is a *unique* point of S_i closest to p . Note that for $i \neq j$, the intersection, $H_i(\eta) \cap H_j(\eta)$, can be nonempty and might not be connected.

We also put $H'_i = H_i(1) \setminus \bigcup_{i > l} H_l(2)$ and note that $H'_i \subset H_i(1)$ and $H'_i \cap H'_j = \emptyset$, for all distinct i, j .

Our $O(n)$ -equivariant nondegenerate substructure of \mathcal{T} will be constructed on $\bigcup_i H_i(1)$. A priori, it is not clear why there should exist such a substructure over even a single $H_i(1)$. However, using our geometric hypothesis, we will show the following; see Theorem 4.1.

Property of sufficiently collapsible pure F-structures. On each $H_i(1)$, there exists a pure nondegenerate substructure, \mathcal{P}_i , of \mathcal{F} .

The existence of a pure nondegenerate substructure on each set, $H_i(1)$, is the only consequence of our geometric assumptions which is used in the proof. Indeed, we have the following refinement of Theorem 0.1.

Theorem 0.1'. Let \mathcal{F} be an arbitrary pure F -structure on M^n . If for all i , there is a pure nondegenerate substructure, \mathcal{P}_i , on $H_i(1)$, then there exists a canonical mixed polarized substructure, whose lift to the frame bundle, \mathcal{P} , satisfies $\mathcal{P}|_{H'_i} = \mathcal{P}_i$.

The sense in which the substructure, \mathcal{P} , is canonical will be made clear in the proof of Theorem 0.1'.

To construct an $O(n)$ -equivariant nondegenerate substructure on $\bigcup_i H_i(1)$, whose restriction to each H'_i coincides with \mathcal{P}_i , we will introduce a certain auxiliary substructure, \mathcal{I} , defined on $\bigcup_i H_i(2)$.

Since $D_i \cap H_i(2)$ is a deformation retract of $H_i(2)$, it follows that $\mathcal{I}_i|_{D_i \cap H_i(2)}$ extends naturally to a pure substructure, \mathcal{I}_i , on $H_i(2)$. The collection $\{(H_i(2), \mathcal{I}_i)\}$ determines a mixed structure, \mathcal{I} , on $\bigcup_i H_i(2)$, whose orbit at a point, x , is the orbit of \mathcal{I}_{i_0} , where i_0 is the maximal i , for which $x \in H_i(2)$.

Clearly, on H'_i , a pure substructure $\hat{\mathcal{T}} \subset \mathcal{T}$ is nondegenerate if and only if it is transversal to $\mathcal{I}|H'_i$. On the other hand, we claim that $\mathcal{I}[(\bigcup_i H_i(1) \setminus \bigcup_i H'_i)]$ has a *canonical mixed nondegenerate substructure*, \mathcal{C} . As will be explained in section 2, the nondegenerate substructure, on $\bigcup_i H_i(1)$ which we are seeking, is obtained by suitably combining a portion of \mathcal{C} with a collection of substructures derived from the nondegenerate substructures, $\{\mathcal{P}_i|H_i(1)\}$.

The existence of \mathcal{C} is a direct consequence of the following property of arbitrary pure F -structures; see Theorem 3.2.

Property of arbitrary pure F -structures. There exists a *canonical inner product* on the Lie algebra, $(e_{\mathcal{I}})_x$, of each stalk of the sheaf, $e_{\mathcal{I}}$, such that the pointwise inner product of two local sections of the sheaf, $e_{\mathcal{I}}$, is a constant function. Moreover, if a subspace of $(e_{\mathcal{I}})_x$ exponentiates to a closed subgroup, then so does its orthogonal complement.

We close this section by mentioning that the arguments used in establishing the above mentioned property of sufficiently collapsible pure F -structures are related to those of [CR], where collapsed manifolds with bounded diameter and bounded covering geometry are studied. Here instead, we exploit *local bounded covering geometry*; see [CFG, Theorem 1.7] and section 4.

2. Proof of Theorem 0.1 Modulo Two Properties of Pure F -structures

Let \mathcal{F} denote a pure F -structure on M^n with invariant metric and let \mathcal{T} denote the lifted T -structure on FM^n .

In the proofs of Theorems 0.1, 0.1', we will use the following procedure for constructing equivariant mixed substructures of \mathcal{T} .

Let $\{Z_\alpha\}$ be a covering of FM^n by $O(n)$ -invariant sets. Assume that over each Z_α , we are given a pure substructure, \mathcal{L}_α . Clearly, there is a unique smallest mixed substructure, \mathcal{L} , such that for all α , \mathcal{L}_α is a substructure of $\mathcal{L}|Z_\alpha$. Moreover, for any $\{\alpha\} = \{\alpha_1, \dots, \alpha_i\}$ the restriction of \mathcal{L} to $Z_{\alpha_1} \cap \dots \cap Z_{\alpha_i} \setminus \bigcup_{\alpha_j \notin \{\alpha\}} Z_{\alpha_j}$, is the smallest pure structure containing the restrictions of $\mathcal{L}_{\alpha_1}, \dots, \mathcal{L}_{\alpha_i}$, to this set.

Now assume that \mathcal{F} has nonempty singular set, S , with coarse stratification, S_1, \dots, S_k . Put $D_i = \pi^{-1}(S_i)$. Let $H_1(\eta), \dots, H_k(\eta)$ be defined as in section 1.

The proof of Theorem 0.1 consists of three steps: First, we construct a special invariant open cover for $\bigcup_i H_i(1)$. Then (as above) we assign to each open set of this cover, a pure substructure of \mathcal{T} . Finally, we verify that

on every nonempty multiple intersection, the assigned pure substructures generate a nondegenerate pure substructure (i.e. one which is transversal to the isotropy substructure on the intersection).

a. An invariant open cover. For $1 \leq i \leq k$, put

$$A_i = H_i(1) \setminus \overline{\bigcup_{i > \ell} H_\ell \left(\frac{3}{2} \right)}.$$

For any $1 \leq j < i \leq k$, define

$$B_{i,j} = \left(H_i(1) \setminus \overline{\bigcup_{i > \ell > j} H_\ell \left(\frac{3}{2} \right)} \right) \cap H_j(2).$$

Note that since $H_i(\eta)$ is invariant, so are A_i and $B_{i,j}$. Formally, A_i behaves like $B_{i,-1}$, although for this to be correct, we must define, $H_{-1}(2) = M^n$.

LEMMA 2.1.

$$(2.1.1) \quad H'_i \subset A_i \subset H_i(1).$$

$$(2.1.2) \quad H_i(1) = \left(\bigcup_{i > \ell} B_{i,\ell} \right) \cup A_i.$$

$$(2.1.3) \quad \text{If } B_{i,j} \cap B_{i',j'} \neq \emptyset \text{ and } i > i' \text{ then } j \geq i'.$$

$$(2.1.4) \quad \text{If } B_{i,j} \cap A_{i'} \neq \emptyset, \text{ then } i' = i \text{ or } j \geq i'.$$

$$(2.1.5) \quad A_i \cap A_{i'} = \emptyset, \text{ for } i \neq i'.$$

Proof. Since (2.1.1), (2.1.3), (2.1.4) and (2.1.5) can be seen directly from the definition, we will only check (2.1.2). Put $A_{i,j} = H_i(1) \setminus \overline{\bigcup_{i > \ell \geq j} H_\ell \left(\frac{3}{2} \right)}$. Then $A_i = A_{i,\ell}$, where ℓ is the smallest index such that D_ℓ is nonempty. It is easily checked that for $i-1 > j$, one has $B_{i,j} \cup A_{i,j} = A_{i,j+1}$ and $B_{i,i-1} \cup A_{i,i-1} = H_i(1)$. By an obvious inductive argument, the claim follows. \square

As a consequence of Lemma 2.1, every nonempty intersection of a subcollection of $\{B_{i,j}\} \cup \{A_i\}$ can be written in one of the following forms:

$$(2.1.6) \quad X = B_{i_1,j_1} \cap \cdots \cap B_{i_1,j_{k_1}} \cap B_{i_2,l_1} \cap \cdots \cap B_{i_2,l_{k_2}} \cap \cdots \cap B_{i_r,m_1} \cap \cdots \cap B_{i_r,m_{k_r}}, \text{ where } i_1 > j_1 > \cdots > j_{k_1} \geq i_2 > l_1 > \cdots > l_{k_2} \geq \cdots \geq i_r > m_1 > \cdots > m_{k_r}.$$

$$(2.1.7) \quad X \cap A_i, \text{ where } X \text{ is as in (2.1.6) and either } i = i_r \text{ or } m_{k_r} \geq i.$$

$$(2.1.8) \quad A_i, \text{ for some } i.$$

b. Assignment of pure structures. Assume that on each $H_i(1)$, there is a pure nondegenerate substructure, \mathcal{P}_i , of $T|H_i(1)$; compare Theorem 4.1.

On each nonempty intersection, $H_i(1) \cap H_j(1)$, where $i > j$, there is a canonical substructure, $\mathcal{I}_{i,j} \subset \mathcal{I}_j$, such that $\mathcal{I}_{i,j}$ is transversal to \mathcal{I}_i . By

definition, the Lie algebra of a stalk of $\mathcal{I}_{i,j}$ is the orthogonal complement of the Lie algebra of \mathcal{I}_i in the Lie algebra of \mathcal{I}_j , with respect to the inner product described in the property of arbitrary pure F -structures stated in section 1; see Theorem 3.2. Thus, if $H_i(1) \cap (\bigcap_{s=1}^{\ell} H_{j_s}(1)) \neq \emptyset$ (where $i > j_1 > j_2 > \cdots > j_{\ell}$) then on this set, $\mathcal{I}_{i,j_1} \subset \cdots \subset \mathcal{I}_{i,j_{\ell}}$.

We now assign to each element of the collection $\{B_{i,j}\} \cup \{A_i\}$, a pure nondegenerate substructure as follows.

(2.2.1) To each A_i , assign the nondegenerate substructure $\mathcal{P}_i|A_i$ (note that $A_i \subset H_1(1)$).

(2.2.2) To each $B_{i,j}$, assign a pure substructure, $\mathcal{P}_{i,j}$, where $\mathcal{P}_{i,j} = \mathcal{P}_i \cap \mathcal{I}_j$, provided this substructure is nontrivial, and $\mathcal{P}_{i,j} = \mathcal{I}_{i,j}|B_{i,j}$ otherwise.

Observe that a pure substructure on $B_{i,j}$ is nondegenerate if and only if it is transversal to $\mathcal{I}_j|B_{i,j}$. From the above definition, it is clear that $\mathcal{P}_{i,j}$ is nondegenerate.

As explained at the beginning of this section, the collection, $\{(A_i, \mathcal{P}_i|A_i)\} \cup \{(B_{i,j}, \mathcal{P}_{i,j})\}$, generates a substructure, \mathcal{P} , of $\mathcal{T}|\bigcup_i H_1(1)$. Clearly, $\mathcal{P}|H'_i(1) = \mathcal{P}_i$. In the next subsection we will show that the substructure, \mathcal{P} , is nondegenerate.

c. Nondegeneracy on multiple intersections. The remainder of the proof of Theorem 0.1 uses only elementary linear algebra.

LEMMA 2.3. *Assume $B_{i,j_1} \cap \cdots \cap B_{i,j_{\ell}}$ is nonempty, where $j_1 > \cdots > j_{\ell}$. Then on this subset the pure substructure generated by $\mathcal{P}_{i,j_1}, \dots, \mathcal{P}_{i,j_{\ell}}$ is nondegenerate. If in addition, $B_{i,j_1} \cap \cdots \cap B_{i,j_{\ell}} \cap A_{i'}$ is nonempty, where $i' = i$ or $j_k \geq i'$, then on this subset, the pure substructure generated by $\mathcal{P}_{i,j_1}, \dots, \mathcal{P}_{i,j_{\ell}}, \mathcal{P}_{i'}$ is nondegenerate.*

Proof. Since $\mathcal{I}_{j_1} \subset \cdots \subset \mathcal{I}_{j_{\ell}}$ either $\mathcal{P}_i \cap \mathcal{I}_{j_1} \neq \emptyset$ or for some j_t , we have $\mathcal{P}_i \cap \mathcal{I}_s = \emptyset$, for $s = j_1, \dots, j_t$, where j_t is the last such index. We will assume that the latter alternative holds, since the argument in the former case is entirely similar to the one that follows. For the same reason, we can assume $j_t < j_{\ell}$.

The substructures assigned to $B_{i,j_1}, \dots, B_{i,j_t}$ are $\mathcal{I}_{i,j_1}, \dots, \mathcal{I}_{i,j_t}$, respectively. The substructures assigned to $B_{i,j_{t+1}}, \dots, B_{i,j_{\ell}}$, are $\mathcal{P}_i \cap \mathcal{I}_{j_{t+1}}, \dots, \mathcal{P}_i \cap \mathcal{I}_{j_{\ell}}$, respectively. Thus, on $B_{i,j_1} \cap \cdots \cap B_{i,j_k}$ the pure substructure generated by $\mathcal{P}_{i,j_1}, \dots, \mathcal{P}_{i,j_{\ell}}$, is actually generated by \mathcal{I}_{i,j_t} and $\mathcal{P}_i \cap \mathcal{I}_{j_{\ell}}$. Moreover, \mathcal{I}_{i,j_t} is transversal to \mathcal{I}_i , $\mathcal{I}_{i,j_t} \subset \mathcal{I}_{j_t}$, and $\mathcal{P}_i \cap \mathcal{I}_{j_{\ell}}$ is transversal to \mathcal{I}_{j_t} . To verify the first assertion of Lemma 2.3, it suffices to check that the substructure generated by \mathcal{I}_{i,j_t} and $\mathcal{P}_i \cap \mathcal{I}_{j_{\ell}}$ is transversal to \mathcal{I}_i . In view of the above, this (pointwise) condition follows by elementary linear algebra.

We now verify the second assertion. If $i' = i$, our substructure is generated by \mathcal{I}_{i,j_t} and \mathcal{P}_i , where \mathcal{P}_i is transversal to $\mathcal{I}_{j_t} \supset \mathcal{I}_{i,j_t}$. On the other hand, if $i_k > i'$, our substructure is generated by $\mathcal{I}_{i,j_t}, \mathcal{P}_i \cap \mathcal{I}_{j_t}$ and $\mathcal{P}_{i'}$, where $\mathcal{P}_{i'}$ is transversal to \mathcal{I}_{j_t} . As above, in either case, the assertion follows. \square

Proof of Theorem 0.1'. Given the characterization of the substructure, \mathcal{L} , generated by a collection, $\{(Z_\alpha, \mathcal{L}_\alpha)\}$, which was stated at the beginning of this section, it suffices to check that over each nonempty intersection of sets taken from a subcollection of $\{B_{i,j}\} \cup \{A_i\}$, the substructure generated by the relevant subset of $\{\mathcal{P}_{i,j}\} \cup \{\mathcal{P}_i\}$ is nondegenerate. But in view of the description of the possible nonempty intersections given in (2.1.6)–(2.1.8), the nondegeneracy follows by repeated application of Lemma 2.3 (and the elementary linear algebra facts, employed in its proof). \square

Proof of Theorem 0.1. As explained in section 1, Theorem 0.1 follows directly from Theorem 0.1' and the property of sufficiently collapsible F -structures stated in that section (i.e. Theorem 4.1). \square

REMARK 2.4: Consider the lifted \mathcal{T} -structure associated to an arbitrary F -structure. As above, it follows that the collection, $\{(B_{i,j}, \mathcal{I}_{i,j})\}$ generates a canonical nondegenerate substructure, $\{\mathcal{C}\}$, over $\bigcup B_{i,j}$. Moreover, it is easy to check that $\bigcup B_{i,j} = \bigcup_i H_i(1) \setminus \bigcup_i H'_i$.

3. A Property of Arbitrary Pure F -structures

In this section we prove the property of arbitrary pure F -structures stated in section 1. Thus, throughout this section, we will consider an arbitrary pure F -structure, \mathcal{F} , on M^n , with nonempty singular set. We assume that the Riemannian metric on M^n is invariant, so that \mathcal{F} lifts to an $O(n)$ -invariant pure polarized T -structure, \mathcal{T} , on the frame bundle, $\pi : FM^n \rightarrow M^n$.

The inner products on stalks, $(e_{\mathcal{T}})_x$, arise from the isotropy representations of the local actions of the stalks of $\mathcal{E}_{\mathcal{T}}$ on finite covering spaces of neighborhoods in the base. For completeness, we will describe these local actions, in the process supplying further details of the description of F -structures given at the beginning of the introduction.

Let F denote the torus fibre of the T -structure, \mathcal{T} , and let $Aff_0(F)$ denote the identity component of the group of affine automorphisms, $Aff(F)$, of F . Recall that a choice of affine isomorphism, $F \simeq T^k$, induces an isomorphism, $Aff_0(F) \simeq T^k$, where k is the rank of \mathcal{F} .

Let $G(F) \subset O(n)$ denote the subgroup which preserves F under the natural action of $O(n)$ on FM^n . Thus, $G(F) = \{e\}$, the trivial subgroup, unless $F \subset D$. In particular there is a faithful representation, $\tau : G(F) \rightarrow$

$Aff(F)$. Let $G_0(F) \subset G(F)$ denote the identity component. Then $\tau : G_0(F) \rightarrow Aff_0(F)$.

Fix $\epsilon_1 > 0$ such that for every point $y \in T_{\epsilon_1}(F)$, the ϵ_1 -tubular neighborhood of F , there is a unique point, $x \in F$, closest to y . Fix $\epsilon_2, \delta > 0$, so small that every component of $T_\delta(G(F))$ intersects a unique component of $G(F)$, in addition, $g(T_{\epsilon_2}(F)) \subset T_{\epsilon_1}(F)$ and finally, if $g(T_{\epsilon_2}(F)) \cap T_{\epsilon_2}(F) \neq \emptyset$, then $g \in T_\delta(G(F))$.

The action of $Aff_0(F)$ extends canonically to a torus-fibre preserving action on $T_{\epsilon_1}(F)$; see [CR, Section 2]. Moreover, for $g \in T_\delta(G_0(F))$, the automorphism in $Aff(F)$ defined by, $A \rightarrow g^{-1}Ag$, is continuously deformable to the identity and hence is trivial. In particular the action of elements of $T_\delta(G_0(F))$ commutes with the action of $Aff_0(F)$ on $T_{\epsilon_2}(F)$.

Put $W = T_{\epsilon_2}(F)$. Then W is a disjoint union of equivalence classes, where $y_1 \sim y_2$ if and only if $y_2 = gy_1$, with $g \in T_\delta(G(F))$. Moreover, $\pi(W)$ can be identified with the corresponding quotient space with its natural topology. Similarly, the equivalence relation $y_1 \sim y_2$ if and only if $y_2 = gy_1$, with $g \in T_\delta(G_0(F))$, can be identified with a finite normal covering space, $\tilde{\pi} : \pi(\tilde{W}) \rightarrow \pi(W)$, with covering group, the group of components of $G(F)$.

Since the action of each element of $Aff_0(F)$ commutes with that of each element of $T_\delta(G_0(F))$, it follows that there is a canonical action of $Aff_0(F)$ on $\pi(\tilde{W})$.

Note the action of an element of $Aff_0(F)$ need not commute with that of an element of $T_\delta(G(F))$. Thus, $Aff_0(F)$ need not act naturally on $\pi(W)$ itself. Equivalently, an F -structure need not be a T -structure (nor in particular, is a flat manifold necessarily a torus).

Clearly, the isotropy group of any point of $\tilde{\pi}^{-1}(\pi(F)) \subset \pi(\tilde{W})$ is $\tau(G(F)) \subset Aff(F)$.

If $x \in F$, then by definition, the stalk of $\mathcal{E}_{\mathcal{T}}$ at x is $Aff_0(F)$. We have $x \in D_i$, for some i , if and only if $\dim G(F) > 0$. Let $x \in D$. By definition, $\tau(G_0(F))$ is the stalk of the subsheaf, \mathcal{E}_{I_1} , of $\mathcal{E}_{\mathcal{T}}$. Thus, there is a natural (faithful) isotropy representation of $(\mathcal{E}_{I_1})_x$ on the tangent space, $W_{\tilde{\pi}(x)}$, for any $x \in F$. The lifted isotropy representation, ρ , acts on the quotient of the tangent space, W_x , by the tangent space to the $O(n)$ -orbit, $O(n)_x$. Let $x_\ell \rightarrow x$, where $\{x_\ell\} \subset D_i$, $x \in D_j$ and $i > j$. Then the limit of the isotropy representations, $\lim_{\ell \rightarrow \infty} \rho(\mathcal{E}_{I_1})_{x_\ell}$, is the restriction of the representation, $\rho((\mathcal{E}_{I_j})_x)$, to the limit subgroup, $\lim_{\ell \rightarrow \infty} (\mathcal{E}_{I_1})_{x_\ell} \subset (\mathcal{E}_{I_j})_x$.

Let ρ_* denote the representation of Lie algebras induced by ρ . Since a torus is compact, the symmetric bilinear form,

$$\langle\langle A, B \rangle\rangle = -\frac{1}{2} \text{tr}(\rho_*(A)\rho_*(B)) ,$$

defines a *canonical inner product* on the Lie algebra, $(e_{\mathcal{I}})_x$, of the stalk, $(e_{\mathcal{I}})_x$, of $e_{\mathcal{I}}$ at $x \in D$. Recall that up to isomorphism, representations of a compact Lie group are isolated. Moreover, the bilinear form, $-\frac{1}{2}\text{tr}(\rho_*(A)\rho_*(B))$ is invariant under isomorphism. Thus, it follows that the inner product of two local sections of the sheaf, $e_{\mathcal{I}}$, is a constant function. Note that local sections of the *sheaf*, $e_{\mathcal{I}}$, can be described equivalently as local sections of the corresponding vector bundles which are parallel with respect to the canonical flat connection.

Observe that by the above discussion, if $x_{\ell} \rightarrow x$, where $x_{\ell} \in D_i$, $x \in D_j$ and $i > j$, then:

(3.1) The sequence of canonical inner products on Lie algebras, $(e_{\mathcal{I}_i})_{x_{\ell}}$, converges to an inner product on the limit Lie algebra, $\lim_{\ell \rightarrow \infty} (e_{\mathcal{I}_i})_{x_{\ell}} \subset (e_{\mathcal{I}_j})_x$. Moreover, the limiting inner product, coincides with the restriction to $\lim_{\ell \rightarrow \infty} (e_{\mathcal{I}_i})_{x_{\ell}}$, of the canonical inner product on $(e_{\mathcal{I}_j})_x$.

Recall that \mathcal{I} is the substructure of \mathcal{T} defined on $\bigcup_i H_i(2)$ by the collection, $\{(H_i(2), \mathcal{I}_i)\}$.

Now we can state the main result of this section.

Theorem 3.2. *For all i , there is a canonical pointwise inner product on stalks of $e_{\mathcal{I}_i}$ such that the inner product of two local sections is a constant function. Moreover, if $H_i(2) \cap H_j(2) \neq \emptyset$, where $i > j$, then:*

(3.2.1) *The canonical inner product on $e_{\mathcal{I}_i}|_{H_i(2) \cap H_j(2)}$ coincides with the restriction of the canonical inner product on $e_{\mathcal{I}_j}|_{H_i(2) \cap H_j(2)}$. In particular, the collection of inner products on the various $e_{\mathcal{I}_i}$, $i = 1, 2, \dots$, defines an inner product on $e_{\mathcal{I}}$.*

(3.2.2) *There is a pure substructure, $\mathcal{I}_{i,j}$, of $\mathcal{I}_j|_{H_i(2) \cap H_j(2)}$ such that each stalk $(e_{\mathcal{I}_{i,j}})_{x_i}$ is the orthogonal complement of $(e_{\mathcal{I}_i})_x$ in $(e_{\mathcal{I}_j})_x$.*

Proof. Clearly, the inner product on Lie algebras of stalks of \mathcal{I}_i , initially, defined over D_i , extends naturally over $H_i(2)$. As a consequence of the consistency condition implied by (3.1), it follows that if $x \in H_i(2) \cap H_j(2)$, where $i > j$, then the inner product on $(e_{\mathcal{I}_i})_x$, obtained by restricting the canonical inner product on $(e_{\mathcal{I}_j})_x$, coincides with the canonical inner product on $(e_{\mathcal{I}_i})_x$. This gives (3.2.1).

To verify (3.2.2), it suffices to consider an orthogonal representation of the standard k -torus, $T^k = S^1 \times \dots \times S^1$. Let e_i denote the vector in the Lie algebra of T^k such that the i -th circle factor is the 1-parameter subgroup generated by e_i , and $\exp 2\pi e_i$ is the identity element. Subtori of T^k are in 1-1 correspondence with subspaces of \mathbb{R}^k , which admit a basis, v_1, \dots, v_j , where $v_j = \sum_i a_{i,j} e_i$, and $a_{i,j}$ is rational, for all i, j . Thus, by elementary linear algebra, an inner product, $\langle \cdot, \cdot \rangle$ satisfies that $\langle e_i, e_{\ell} \rangle$ is rational, for all i, ℓ , if and only if it has the property that the orthogonal complement

of a subspace which exponentiates to a subtorus always exponentiates to a subtorus.

For any representation, ρ , of T^k , there is a decomposition,

$$\mathbb{R}^n = L_1 \oplus \cdots \oplus L_r \oplus K,$$

into ρ -invariant subspaces, where each L_j is 2-dimensional and $\rho(T^k)$ acts trivially on K . On L_j , we have $\rho(\exp te / [\exp 2\pi e_i]) = R_{m_{i,\ell}t}$, where $m_{i,\ell} \in \mathbb{Z}$ and R_s denotes rotation by s . From this, it follows immediately that the inner product, $\langle\langle A, B \rangle\rangle = -\frac{1}{2} \text{tr}(\rho_*(A)\rho_*(B))$, has the above mentioned rationality property. \square

4. A Property of Sufficiently Collapsible Pure F-structures

In this section, we will prove the property of sufficiently collapsible pure F-structures which was stated in section 1.

Theorem 4.1. *Let the assumptions be as in Theorem 0.1. If \mathcal{F} is a sufficiently collapsible pure F-structure, with lifted structure, \mathcal{T} , then for all i , $\mathcal{T}|D_i$ has a pure nondegenerate substructure.*

First we will recall from [CR], geometric conditions which guarantee the existence of a nondegenerate pure substructure on the frame bundle over a subset of M^n . In [CR], the assumptions were such that this subset could be taken to be M^n itself. Here, we will show that these conditions are actually satisfied when restricted to each set, D_i .

a. A criterion for the existence of transversal substructures. Let $p : E \rightarrow B$ be a fiber bundle with fiber a torus, T^k , and structural group $\text{Aff}(T^k)$. Assume that E is equipped with an invariant metric, for the local action described in section 3. In particular, the projection, p , is a Riemann submersion.

Recall that a *subfibration* of $p : E \rightarrow B$ is a fibration, $p_1 : E \rightarrow B_1$, such that each fiber of p_1 is a totally geodesic submanifold of a fiber of p . Let p_2 be another subfibration of p . We say that p_2 is *transversal* to p_1 if the fiber of the latter is transversal to that of the former at each point (cf. [CR]).

Theorem 4.2 [CR]. *There exists a constant, $\epsilon(n, d, \Lambda, \rho) > 0$, such that the following conditions imply the existence of a subfibration of p transversal to p_1 ,*

$$(4.2.1) \quad \text{diam}(E) \leq d,$$

$$(4.2.2) \quad \text{the second fundamental form of each } p\text{-fiber satisfies } \|II(F)\| \leq \Lambda,$$

$$(4.2.3) \quad \text{the injectivity radius of each } p_1\text{-fiber is greater than } \rho,$$

$$(4.2.4) \quad \text{the diameter of every } p\text{-fiber satisfies, } \text{diam}(F) < \epsilon(n, d, \Lambda, \rho).$$

Now let M^n be as in Theorem 4.1, with the lifted T-structure, \mathcal{T} , on FM^n and a degenerate set D . Let $\tilde{f} : FM^n \rightarrow B_{\tilde{f}}$ be the projection to the orbit space of the bundle, $F \rightarrow FM^n \rightarrow B_{\tilde{f}}$ defined by \mathcal{T} . Then $\tilde{f}_i : D_i \rightarrow B_i$, the restriction of \tilde{f} to D_i , is also an $O(n)$ -invariant torus bundle. Moreover, the substructure, \mathcal{I}_i , of D_i gives rise to an $O(n)$ -invariant subfibration, $p_i : D_i \rightarrow B_{p_i}$.

In view of Theorem 4.2, the following proposition implies Theorem 4.1.

PROPOSITION 4.3. *Let the assumptions be as in Theorem 4.1. Then, there exist constants, $h(n, d)$, $\Lambda(n)$ and $\rho(n)$, such that for all i , the following hold.*

(4.3.1) *The second fundamental form of each \tilde{f}_i -fiber satisfies $|II(\tilde{f}_i^{-1}(x))| \leq \Lambda(n)$,*

(4.3.2) *$\text{diam}(D_i) \leq h(n, d)$,*

(4.3.3) *the injectivity radius of each p_i -fiber is greater than $\rho_0(n)$.*

b. Proof of (4.3.1). By [CFG], there exists a constant, $\Lambda(n)$, such that the $O(n)$ -invariant fibration, $\tilde{f} : FM^n \rightarrow B_{\tilde{f}}$ satisfies (4.3.1). Hence, $\tilde{f}_i : D_i \rightarrow B_i$ satisfies (4.3.1).

c. Proof of (4.3.2). As in section 1, we have $S_i = \pi(D_i)$, where S_i is a singular stratum of $S = \pi(D)$. There is a universal constant, C , such that

$$C^{-1} \cdot \text{diam}(S_i) \leq \text{diam}(D_i) \leq C \cdot \text{diam}(S_i).$$

By the above discussion, (4.3.2) is equivalent to

LEMMA 4.4. *Let the assumptions be as in Proposition 4.3. There exists a constant, $h(n, d) > 0$, depending on n and d such that each singular stratum, S_i , has diameter $\leq h(n, d)$.*

Proof. We argue by contradiction. Assume that there is a sequence of n -manifolds, $\{M_j^n\}$, which satisfy the assumptions of Theorem 4.1 and such that the invariant pure structure on M_j^n has a singular stratum, $S_{i,j}(M_j^n)$, with $\text{diam}(S_{i,j}(M_j^n)) > j$.

As mentioned in the introduction, we can assume that the metric on FM_j^n has a uniform bound on the covariant derivative of the curvature tensor (see section 0 and [CFG]). Then, by Gromov's precompactness theorem, after passing to a subsequence, we can assume that $\{M_j^n\}$ converges to a metric space, B , and the sequence of the frame bundles, $\{FM_j^n\}$, converges to a Riemannian manifold \tilde{B} (of lower dimension) such that for j sufficiently large the following diagram commutes (compare [F2]).

$$\begin{array}{ccc}
 FM_j^n & \xrightarrow{\tilde{\eta}_j} & \tilde{B} \\
 \downarrow \pi & & \downarrow \tilde{\pi} \\
 M_j^n & \xrightarrow{\eta_j} & B
 \end{array}$$

Here $\tilde{\eta}_j : FM_j^n \rightarrow \tilde{B}$ is an $O(n)$ -invariant fibration with fiber affine isomorphic to a nilmanifold, and affine structural group; see [CFG] and compare section 1. In the language of [CFG], η_j defines a nilpotent Killing structure on M_j^n . The $O(n)$ -invariance implies that \tilde{B} admits an isometric $O(n)$ -action such that $B = \tilde{B}/O(n)$ and the fibration $\tilde{\eta}_j$ descends to a singular fibration projection, $\eta_j : M_j^n \rightarrow B$. It follows from Proposition A1.14 of [CFG] that the $O(n)$ -action on \tilde{B} is effective. The centers of the nilpotent fibers form an $O(n)$ -invariant torus bundle. This is the structure which was described in section 1 (see [CR]).

Note that the singular set of the nilpotent Killing structure coincides with that of the canonical F-structure; see [CR].

Let $\{Z_i\}$ denote the collection of all singular strata of the $O(n)$ -action on \tilde{B} . Then the above commutative diagram implies that $\{\tilde{\pi}(Z_i)\}$ is the collection of images under the projection, η_j , of all singular strata of the nilpotent Killing structure on M_j^n . Thus, $\{f_j^{-1}(\tilde{\pi}(Z_i))\}$ is the collection of all singular strata of the nilpotent Killing structure on M_j^n . By the above discussion, $\{f_j^{-1}(\tilde{\pi}(Z_i))\}$ is the collection of all singular strata of the canonical pure F-structure on M_j^n .

Since $\tilde{\pi}(Z_i)$ has a definite diameter, the diameter of $f_j^{-1}(\tilde{\pi}(Z_i))$ is bounded for all j . Since there are only finitely many singular strata for the $O(n)$ -action on \tilde{B} (see [B]), we conclude that the diameters of all $f_j^{-1}(\tilde{\pi}(Z_i))$ are uniformly bounded; a contradiction. \square

d. Proof of (4.3.3). Let M^n be as in Proposition 4.1 and let \mathcal{F} be a sufficiently collapsible pure F-structure on M^n .

LEMMA 4.5. *There exists $c(n, r) > 0$, such that for all $p \in M^n$, there exists $q \in B_r(p) \setminus S$, such that the second fundamental form of \mathcal{O}_q satisfies $\|II(\mathcal{O}_q)\| \leq c(n, r)$.*

Proof. It follows from Theorem 1.7 of [CFG] (*local bounded covering geometry*) that the norm of the second fundamental form of a nonsingular orbit of \mathcal{F} can be bounded above in terms of its distance from the singular set S . Thus, it suffices to show that each ball of radius r contains a nonsingular orbit lying at a definite distance (depending only on n and r) from S . This can be seen by an argument by contradiction analogous to the proof of

Lemma 4.4. In this connection, recall that the $O(n)$ -action on \tilde{B} is effective. Thus, the set of nonsingular orbits is dense. \square

For a subset U of M^n , we use $\pi_U : \tilde{U} \rightarrow U$ to denote the universal covering space of U equipped with the pullback metric.

LEMMA 4.6 [CFG]. *There exists a constant, $\rho(n) > 0$, such that for any $p \in M^n$, there is an invariant open subset, U , containing the ball, $B_{2\rho(n)}(p)$, and each point in $\pi_U^{-1}(B_{\rho(n)}(p))$ has injectivity radius $\geq \rho(n)$.*

Note that Lemma 4.6 is a version of *local bounded covering geometry* which suffices for our present purposes (for the full statement, see [CFG, Theorem 1.7]).

Proof of (4.3.3). Let $x \in D_i$. Put $\pi(x) = p$. For $\rho(n)$ as in Lemma 4.6, and $r = \rho(n)$, let q be as in Lemma 4.5. Clearly, there exists $y \in \pi^{-1}(q)$ and a minimal geodesic, γ , with $\gamma(0) = x$, $\gamma(1) = y$, such that $\pi(\gamma) \subset B_{\rho(n)}(p)$.

By light abuse of notation, let $\mathcal{O}_{\gamma(t)}^{\mathcal{I}_i}$ denote the orbit through $\gamma(t)$, of the parallel translate along γ , of the stalk, $(\mathcal{E}_{\mathcal{I}_i})_x$. Here the parallel translation is with respect to the canonical connection on $\mathcal{E}_{\mathcal{I}}$, viewed as a flat bundle. By Lemma 4.6, $\pi(\mathcal{O}_{\gamma(1)}^{\mathcal{I}_i}) = \pi(\mathcal{O}_y^{\mathcal{I}_i})$ has second fundamental form bounded in norm by $c(n, r)$. Moreover, for U as in Lemma 4.6, the family, $\pi(\mathcal{O}_{\gamma(t)}^{\mathcal{I}_i})$, provides a contraction in U , of $\pi(\mathcal{O}_y^{\mathcal{I}_i})$ to point x . Let $\hat{y} \in \pi_U^{-1}(y)$ and let $\hat{\mathcal{O}}_y^{\mathcal{I}_i}$ denote the component of $\pi_U^{-1}(\pi(\mathcal{O}_y^{\mathcal{I}_i}))$ through \hat{y} . Then $\pi_U|_{\hat{\mathcal{O}}_y^{\mathcal{I}_i}}$ is a homeomorphism. Thus, for the pull back metric, $\text{inj rad}(\hat{\mathcal{O}}_y^{\mathcal{I}_i}) = \text{inj rad}(\pi(\mathcal{O}_y^{\mathcal{I}_i}))$. Since also $\|II(\hat{\mathcal{O}}_y^{\mathcal{I}_i})\| = \|II(\pi(\mathcal{O}_y^{\mathcal{I}_i}))\| \leq c(n, \rho(n))$, it follows from Lemma 4.6 that $\text{inj rad}(\hat{\mathcal{O}}_y^{\mathcal{I}_i}) \geq \rho_0(n)$. The fact that $\pi : FM^n \rightarrow M^n$ is a Riemannian submersion, easily implies $\text{inj rad}(\mathcal{O}_y^{\mathcal{I}_i}) \geq \rho_0(n)$ as well.

By (4.3.1) and (4.3.2) metrics on orbits of \mathcal{I}_i are quasi-isometric, with the constant depending on n and d . Hence, it follows from the above that $\text{inj rad}(\mathcal{O}_x^{\mathcal{I}_i})$ has a lower bound depending only on n and d . \square

5. A Generalization of Theorem 0.1

In this section, we will give a generalization of Theorem 0.1; see Theorem 5.2.

DEFINITION 5.1: Let \mathcal{F} be a (possibly mixed) F-structure. A singular component, S_i , of \mathcal{F} is called *essential* if \mathcal{F} has no polarized substructure in any neighborhood of S_i .

By definition, an F-structure has a polarized substructure if and only all singular components are nonessential. Examples of positive rank F-structures with essential singularities were mentioned in the introduction.

Let M^n be a complete manifold with $|K_{M^n}| \leq 1$. Recall that for all sufficiently small $\epsilon > 0$, there is a natural decomposition, $M^n = \mathcal{B}(\epsilon) \cup \mathcal{C}(\epsilon)$, where $\mathcal{B}(\epsilon)$ consists of points at which the injectivity radii are not less than ϵ and $\mathcal{C}(\epsilon)$ is the complement. If $M^n = \mathcal{C}(\epsilon)$, then M^n is called ϵ -collapsed.

The main result in [CFG] asserts that there is a constant, $\epsilon(n) > 0$, such that (after a slight adjustment of its boundary) $\mathcal{C}(\epsilon(n))$ admits a (possibly mixed) positive rank F-structure, \mathcal{F} , which is *almost* compatible with the metric. We will also call \mathcal{F} the *associated F-structure*.

The following result can be viewed as a generalization of Theorem 0.1.

Theorem 5.2. *For all $d > 0$, there exists a constant, $0 < \epsilon(n, d) < \epsilon(n)$, such that the following holds. If M^n is an $\epsilon(n, d)$ -collapsed complete manifold with $|K_{M^n}| \leq 1$ such that the associated F-structure on M^n has essential singular components, then all such components have diameter $\geq d$.*

Note that the injectivity radius collapsed metric in Theorem 5.2 need not be *volume collapsed*, i.e. the volume need not be small and could be infinite.

COROLLARY 5.3. *Let M^n be a complete manifold with $|K| \leq 1$ and $\text{Vol}(M^n) < \infty$. Suppose that for the associated F-structure, \mathcal{F} , on $\mathcal{C}(\epsilon(n))$, all singular components have diameter $\leq d$. Then, there is a constant, $0 < \epsilon(n, d) < \epsilon(n)$, such that $\mathcal{F}[\mathcal{C}(\epsilon(n, d))]$ has a polarized substructure.*

Note that Corollary 5.3 means that \mathcal{F} has a polarized substructure near infinity.

REMARK 5.4: Theorem 5.2 provides a geometric constraint on essential singular components. Nonessential singular components can have arbitrarily small diameter; see Example 5.7.

REMARK 5.5: Recall that given a positive rank F-structure, \mathcal{F} , there exists a family of invariant metrics with $|K| \leq 1$ and injectivity radii uniformly converging to zero ([CG1]). An F-structure associated to each sufficiently collapsed metric is actually a substructure of \mathcal{F} . If, in addition, one assumes that \mathcal{F} has essential singularities, then such an F-structure will have an essential singular component ([CG1]). (Note that by definition, any substructure of an F-structure with essential singularities has essential singularities).

Assume that M^n is ϵ -collapsed with $0 < \epsilon < \epsilon(n)$. Consider an associated F-structure, \mathcal{F} , on M^n . Note that \mathcal{F} need not be a pure F-structure (see Example 0.1 of [CFG]). However, we have

LEMMA 5.6. *For all $d > 0$, there is a constant, $0 < \epsilon(n, d) < \epsilon(n)$, such that if $M^n = \mathcal{C}(\epsilon(n, d))$, then for all $x \in M^n$, the restriction of \mathcal{F} to a subset containing $B_d(x)$ has a pure positive rank substructure.*

Proof. The proof is based on an observation concerning the construction of sufficiently collapsible F-structures in [CFG].

Fix any $d > 0$. It follows from section 5 of [CFG], there is a constant, $0 < \epsilon(n, d) < \epsilon(n)$, depending only on n and d such that if $M^n = \mathcal{C}(\epsilon)$, $\epsilon \leq \epsilon(n, d)$, then for all $x \in M^n$, a subset containing $B_d(x)$ admits a pure positive rank F-structure, say $\mathcal{F}_{x,d}$, such that all orbits have diameter less than ϵ .

If, in addition, we choose $\epsilon(n, d) \ll \epsilon(n)$, then $\mathcal{F}_{x,d}$ is actually a pure substructure of the associated F-structure, \mathcal{F} , on M^n . This can be seen from the construction of \mathcal{F} in [CFG]. \square

Now the proof of Theorem 5.2 follows easily from Lemma 5.6 and Theorem 0.1.

We conclude this paper with an example mentioned in Remark 5.4.

EXAMPLE 5.7: Consider the standard T^2 -action on $S^2 \times S^1$. Using a standard method (see [CG1]), we will construct a (continuous) sequence of invariant metrics, g_ϵ , with $|K_{g_\epsilon}| \leq 1$ such that $(S^2 \times S^1, g_\epsilon)$ converges to a closed interval ($\epsilon \rightarrow 0$) in the Gromov-Hausdorff topology (see [GLP]). Clearly, the F-structure associated to any sufficiently collapsed metric coincides with the T^2 -action. Observe that the length of each of the two singular circle orbits (each one is a non-essential singular component) goes to zero as $\epsilon \rightarrow 0$.

Take a one parameter subgroup, R , of T^2 such that the closure of R is T^2 and take a T^2 -invariant metric, g , on $S^2 \times S^1$. At each point, write $g = g_R \oplus g_R^\perp$, where g_R is the restriction of g to the subspace tangent to the R -orbit and g_R^\perp is the orthogonal complement. Then $g_\epsilon = \epsilon^2 g_R \oplus g_R^\perp$, $0 < \epsilon \leq 1$.

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Jeff Cheeger
Courant Institute of Mathematical Sciences
New York University
New York, NY 10012
USA
E-mail: cheeger@cims.nyu.edu

Xiaochun Rong
Mathematics Department
University of Chicago
Chicago, IL, 60637
USA
E-mail: xr@math.uchicago.edu

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