Random Matrix Models of String Theory

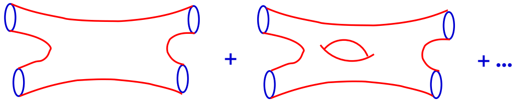
Sunil Mukhi,
Tata Institute of Fundamental Research, Mumbai

Mathematics of String Theory (MOST) Workshop
ANU Canberra, July 13-17 2006

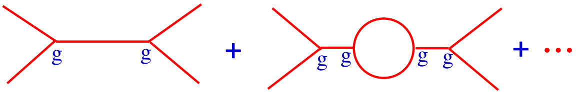
Outline

Introduction

- String theory was originally defined as a sum over worldsheets of ever-increasing genus.



- This is analogous to defining field theory by its expansion over Feynman diagrams.



- The number of handles in the surfaces, like the number of loops in the diagrams, count the order in perturbation theory.

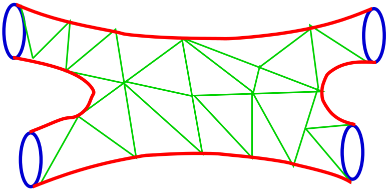
- In field theory, there is a non-perturbative formulation (e.g. Lagrangian path integral) that contains information about such things as solitons, tunnelling and confinement.
- There exists a non-perturbative formulation of string theory too – but so far, it is known only about rather specific spacetime backgrounds.
- This is the random matrix formulation describes strings propagating in very low dimensional spacetimes, such as two.
- Hence, strings propagating in two spacetime dimensions (one space, one time) will be the subject of these lectures.

- The road to the nonperturbative formulation is rather long. We will start with a theory that almost achieves this, but fails. This is called the $c = 1$ bosonic string.
- The theory is still rather interesting, in that we knows its partition function and scattering amplitudes to all orders in perturbation theory.
- Then we will turn our attention to the more recently understood noncritical type 0A and 0B strings. In perturbation theory these are very much like the bosonic string, but they are also non-perturbatively well-defined.

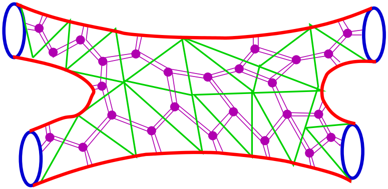
Outline

Random Matrices - Generalities

- There are two different ways to motivate the random matrix approach. Let us first start with the traditional motivation.
- The idea is to start with an action principle which generates, not Riemann surfaces but discrete (lattice-like) versions of them.
- This is quite easy to achieve. A discrete Riemann surface can be made by gluing together triangles:



- The next step would be to write a function that, on expanding, generates these triangles.
- This is achieved via a trick called lattice duality. Put a vertex at the centre of every triangle, and connect every pair of vertices by a line that cuts the common boundary of the triangles.



- In fact it's natural to thicken these new lines to double lines. One sees now that the Riemann surface is covered by polygons glued together at their common edges.

- The polygons can have different numbers of sides. But the dual diagram always has three lines meeting at a point, precisely because we did lattice duality on triangles.
- Now we are almost done. Double lines are generated by matrices because they have two indices.
- And three-point vertices are generated by cubic couplings among the matrices.
- This suggests a random matrix integral will do the job::

$$\mathcal{Z} = \int [dM]\, e^{-N\,\mathrm{tr}(\frac{1}{2}M^2 + gM^3)}$$

where M are $N \times N$ Hermitian random matrices.

- This is still a little vague. What do we mean “do the job”? And is this the unique action for the purpose? Please be patient...
- The random matrix integral we wrote should be thought of as a field theory path integral, except that instead of fields we have matrices. Instead of an integral over space and time, we have a trace.
- The integral can be evaluated using the very same technique we learn in field theory: solve the quadratic (Gaussian) part explicitly and treat the rest in perturbation theory.

- For this we need to develop some rules. First, let M be an $N \times N$ Hermitian matrix.
- The measure in the integral is then:

$$[dM] \equiv \prod_{i=1}^{N} dM_{ii} \prod_{i<j=1}^{N} dM_{ij} dM_{ij}^*$$

- Now we evaluate the Gaussian matrix integral in the presence of a source:

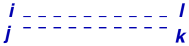
$$\int [dM]\, e^{-N\,\mathrm{tr}(\frac{1}{2}M^2+JM)} = \left(\frac{2\pi}{N}\right)^{\frac{N^2}{2}} e^{N\,\mathrm{tr}\frac{J^2}{2}}$$

- Next we use this to compute the propagator:

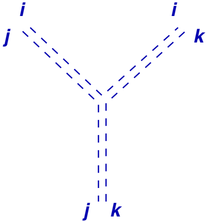
$$\langle M_{ij} M_{kl} \rangle \equiv \frac{\int [dM]\, M_{ij} M_{kl}\, e^{-N\,\mathrm{tr}\frac{1}{2}M^2}}{\int [dM]\, e^{-N\,\mathrm{tr}\frac{1}{2}M^2}} = \frac{1}{N}\delta_{il}\delta_{jk}$$

- By virtue of its structure, the propagator is naturally represented in terms of double lines:

$$\left\langle M_{ij} M_{kl} \right\rangle =$$



- Next, consider the cubic term. This can be used to generate a cubic vertex, as in field theory:



- Combining these elements we see that the perturbation expansion of our matrix model is a dual triangulated surface.

- The matrix integral generates all possible closed diagrams. Therefore it will produce all types of Riemann surfaces. The topology of the surface is defined by the particular diagram.
- Indeed we know that if:

$$\begin{aligned} \text{number of vertices} &= V \\ \text{number of edges} &= E \\ \text{number of faces} &= F \end{aligned}$$

one has the relation:

$$V - E + F = 2 - 2h$$

where h is the genus of the surface.

- The same relation is true on the dual graph, with

$$V \leftrightarrow F$$

- Now each vertex has a factor of gN, each propagator has $\frac{1}{N}$ and each face has a factor of N from the sum over matrix indices.
- Therefore a given graph in the expansion will be of order:

$$(gN)^V N^{-E} N^F = g^V N^{2-2h}$$

We learn that $\frac{1}{N^2}$ is the genus expansion parameter, and g is an additional coupling constant to be held fixed.

- Thus the partition function can be written:

$$\mathcal{Z}(g, N) = \sum_{h=0}^{\infty} \mathcal{Z}_h(g)\, N^{2-2h}$$

Eigenvalue Reduction and Vandermonde determinant

- A Hermitian matrix can always be diagonalised:

$$M = U \Lambda U^\dagger$$

  where U is a unitary matrix, and

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$$

  is a diagonal matrix of eigenvalues.
- The unitary matrix decouples from the action, which we can write as:

$$\mathrm{tr}(\tfrac{1}{2}M^2 + gM^3) = \sum_{i=1}^{N}(\tfrac{1}{2}\lambda_i^2 + g\lambda_i^3)$$

- Next we reduce the integration measure to eigenvalues:

$$[dM] = \prod_{i=1}^{N} d\lambda_i \; \Delta(\lambda)^2$$

where we see the appearance of the Vandermonde determinant:

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

- This arises as follows. We have:

$$\begin{aligned} dM &= dU\,\Lambda\,U^\dagger + U\,d\Lambda\,U^\dagger + U\,\Lambda\,dU^\dagger \implies \\ U^\dagger\,dM\,U &= d\Lambda + [U^\dagger\,dU, \Lambda] \end{aligned}$$

- Next we use two facts:
  (i) $d\alpha = U^\dagger \, dU$ is the infinitesimal element in the Lie algebra (tangent space to the unitary group).
  (ii) the measures $[dM]$ and $[dM'] = [U^\dagger \, dM \, U]$ are the same.
- Then we have:

$$dM'_{ij} = d\lambda_i \delta_{ij} + d\alpha_{ij}(\lambda_i - \lambda_j)$$

Geometrically, this means that the identity metric on the N^2-dimensional space with coordinates dM'_{ij} transforms to a nontrivial metric:

$$G_{AB} = \mathrm{diag}(1, 1, \cdots, 1, (\lambda_1 - \lambda_2)^2, (\lambda_1 - \lambda_3)^2, \cdots)$$

in the coordinates (λ_i, α_{ij}).

- To transform the measure, we compute

$$\sqrt{G} = \prod_{i \neq j} (\lambda_i - \lambda_j) = \Delta(\lambda)^2$$

and therefore

$$[dM] = [dU] \prod_{i=1}^{N} d\lambda_i \, \Delta(\lambda)^2$$

- The integral over dU is just a numerical factor since the integrand is independent of it. That completes the proof.

Outline

- Let us now return to our goal of extracting a string theory from the matrix integral.
- Recall that the expansion of the integral is:

$$\mathcal{Z}(g, N) = \sum_{h=0}^{\infty} \mathcal{Z}_h(g)\, N^{2-2h}$$

- We notice that the large-N limit picks out the genus-0 contribution. In string theory, this would be tree level.
- But this is still not string theory. The genus-0 partition function, $\mathcal{Z}_0(g)$, describes discrete surfaces with all possible numbers of vertices.

- We would like to take a continuum limit where $\mathcal{Z}_0(g)$ is dominated by graphs with very many vertices (the dual graph then has many triangles).
- Defining the area of a triangulation as the number of triangles (or in the dual graph, the number of vertices), we are looking for infinite-area surfaces.
- To achieve this we exploit the constant parameter g. As g is increased, the partition function undergoes a phase transition:

$$\mathcal{Z}_0(g) \sim (g - g_c)^{2-\gamma}$$

for some critical exponent γ.

- We have:

$$\mathcal{Z}_0(g) \sim (g - g_c)^{2-\gamma} \sim \sum_{n=1}^{\infty} n^{\gamma-3} \left(\frac{g}{g_c}\right)^n$$

and therefore

$$\langle n \rangle \sim \frac{1}{\mathcal{Z}_0(g)} \sum_{n=1}^{\infty} n \cdot n^{\gamma-3} \left(\frac{g}{g_c}\right)^n \sim \frac{\partial}{\partial g} \log \mathcal{Z}_0 \sim \frac{1}{g - g_c}$$

- Therefore, the average area diverges as $g \to g_c$.

- We see that to recover a continuum, tree-level theory we need to take the limit:

$$N \longrightarrow \infty, \qquad g \longrightarrow g_c$$

- Remarkably, by changing this limit slightly, we can get a continuum theory that includes all genus contributions.
- First of all we expect that the divergence as $g \longrightarrow g_c$ is a local phenomenon on the worldsheet. Therefore the value of g_c is the same in all genus.
- Next we claim that in genus h, the divergence goes as:

$$\mathcal{Z}_h(g) \sim (g - g_c)^{(2-\gamma)(1-h)}$$

- Thus the full partition function behaves near $g \to g_c$ as:

$$\mathcal{Z}(g,N) \sim \sum_h F_h \left[N(g-g_c)^{(2-\gamma)/2}\right]^{2-2h} = \sum_h F_h\, g_s^{2h-2}$$

where

$$g_s \equiv \left[N(g-g_c)^{(2-\gamma)/2}\right]^{-1}$$

- Thus, to obtain a continuum theory that includes all genus we simply take the limit:

$$N \to \infty,\ \ g \to g_c,\ \ g_s \equiv \left[N(g-g_c)^{(2-\gamma)/2}\right]^{-1}\ \ \text{fixed}$$

and it is g_s that will be the new genus expansion parameter, or string coupling.

- The above limit is called the double scaling limit.

- The next step is to carry out the genus expansion of this matrix model in the double-scaling limit and see if it has the properties expected of a string theory.
- In fact by varying the matrix potential, one finds a series of string theories. These can be identified by their susceptibility χ to be the $(q = 2, p)$ minimal CFT's coupled to worldsheet gravity (a Liouville field theory).
- Instead of pursuing this direction, I would like to introduce a somewhat different matrix model that leads to a more interesting string theory.

Outline

Matrix Quantum Mechanics

- Consider a Hermitian matrix $M(t)$ that depends on a parameter t. Let's write a matrix model:

$$\mathcal{Z} = \int [dM(t)]\, e^{-N \int dt\ \mathrm{tr}\left(\frac{1}{2} D_t M^2 + \frac{1}{2} M^2 - \frac{g}{3!} M^3\right)}$$

  where

$$D_t M \equiv \dot{M} + [A_t, M]$$

  This is a path integral for gauged matrix quantum mechanics.
- In terms of the genus expansion, this model has the same properties as the simpler model of constant matrices.
- However, it also has a parameter t that will endow the string theory with a time direction.

- Here, A_t is a $U(N)$ gauge field, due to which the matrix model has a local (in time) gauge symmetry:

$$M(t) \to U^\dagger(t)\, M(t)\, U(t)$$

- We can gauge fix $A_t = 0$, but must remember to impose its equation of motion ("Gauss Law"):

$$[M, \dot{M}] = 0$$

on physical states.

- The eigenvalue reduction comes about by diagonalising the matrix:

$$M(t) = U(t)\,\Lambda(t)\,U(t)^\dagger$$

- We appear to have a problem. The matrix model action does not reduce only to eigenvalues:

$$\begin{aligned}\mathrm{tr}(\dot{M}^2) &= \mathrm{tr}(\dot{\Lambda} + [U^\dagger \dot{U}, \Lambda])^2 = \mathrm{tr}(\dot{\Lambda}^2 + [U^\dagger \dot{U}, \Lambda]^2) \\ &= \sum_{i=1}^{N} \dot{\lambda}_i^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2\, \dot{\alpha}_{ij}\, \dot{\alpha}_{ji}\end{aligned}$$

where $\dot{\alpha}_{ij} = (U^\dagger \dot{U})_{ij}$.

- Moreover, the Vandermonde determinant will now appear in the measure at every time t.

- To avoid these two inconveniences, it is convenient to pass to the Hamiltonian, which acts on a Hilbert space of wave functions: $\Psi(M_{ij})$ or $\Psi(\lambda_i, \alpha_{ij})$.
- In terms of M, the Hamiltonian is just:

$$\begin{aligned} H &= -\frac{1}{2}\sum_i \frac{\partial^2}{\partial M_{ii}^2} - \sum_{i<j} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} - \frac{1}{2}\mathrm{tr} M^2 + \frac{g}{3!\sqrt{N}}\mathrm{tr} M^3 \\ &= H_{kin} + H_{int} \end{aligned}$$

  where we first scaled the matrix M by $\frac{1}{\sqrt{N}}$.
- However, because of the metric that we saw earlier, the kinetic term H_{kin} is nontrivial in the λ_i, α_{ij} coordinates.

- Indeed, the correct answer is:

$$H_{kin} = -\tfrac{1}{2}\frac{1}{\sqrt{G}}\frac{\partial}{\partial\lambda_i}\sqrt{G}\,\frac{\partial}{\partial\lambda_i} + \sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2}\frac{1}{\sqrt{G}}\,\Pi_{ij}\sqrt{G}\,\Pi_{ji}$$
$$= -\tfrac{1}{2}\frac{1}{\Delta(\lambda)^2}\frac{\partial}{\partial\lambda_i}\Delta(\lambda)^2\frac{\partial}{\partial\lambda_i} + \sum_{i<j}\frac{1}{(\lambda_i-\lambda_j)^2}\Pi_{ij}\Pi_{ji}$$

  where

$$\Pi_{ij} = [\Lambda,[\Lambda,\dot{\alpha}]]_{ij}$$

  is the canonical momentum conjugate to α_{ji}.

- However, the Gauss law constraint $[M,\dot{M}] = 0$ precisely implies that:

$$[\Lambda,[\Lambda,\dot{\alpha}]] = 0$$

  on physical states. Thus the second term in H vanishes.

- We are left with the kinetic Hamiltonian

$$H_{kin} = -\tfrac{1}{2}\sum_{i=1}^{N} \frac{1}{\Delta(\lambda)^2}\frac{\partial}{\partial \lambda_i}\Delta(\lambda)^2\frac{\partial}{\partial \lambda_i}$$

- Using the identity:

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2}\Delta(\lambda) = 0$$

we can re-write this Hamiltonian as:

$$H_{kin} = -\tfrac{1}{2}\sum_{i=1}^{N} \frac{1}{\Delta(\lambda)}\frac{\partial^2}{\partial \lambda_i^2}\Delta(\lambda)$$

- This acts on wave functions $\Psi(\lambda)$ that are symmetric under interchange of all the eigenvalues.

- The Schrödinger equation:

$$H\Psi(\lambda) = E\Psi(\lambda)$$

can now be re-written

$$\tilde{H}\tilde{\Psi}(\lambda) = E\tilde{\Psi}(\lambda)$$

where

$$\begin{aligned} \tilde{H} &= \Delta(\lambda)\, H \frac{1}{\Delta(\lambda)} = \sum_{i=1}^{N} \left(-\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2} - \frac{1}{2}\lambda_i^2 + \frac{g}{3!\sqrt{N}}\lambda_i^3 \right) \\ \tilde{\Psi}(\lambda) &= \Delta(\lambda)\Psi(\lambda) \end{aligned} \quad (1)$$

- Thus we are left with a system of mutually noninteracting particles with coordinates λ_i moving in a common potential. The extra Δ factor makes the wave functions fermionic.

Outline

Free Fermions and the $c = 1$ String

- We have reduced the Hamiltonian of Matrix Quantum Mechanics to a sum of one-particle Hamiltonians:

$$H = \sum_{i=1}^{N} h(\lambda_i)$$

where

$$h(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!\sqrt{\beta}}\lambda^3, \qquad \beta = \frac{N}{g^2}$$

- We now wish to study this free fermion system in a large-N, double-scaled limit.

- What do we want to know about the system?
- We would like to compute the partition function of the matrix model. In Hamiltonian formulation, this can be written:

$$\mathcal{Z} = {}_{out}\langle 0| e^{-HT} |0\rangle_{in}$$

- For large times T, it is the ground state energy that contributes:

$$\lim_{T\to\infty} \frac{\ln Z}{T} = -E_{gr}$$

- Therefore we will try to compute the ground state energy of the free fermions.
- First, it is convenient to redefine variables in a way that provides us some physical intuition.

- If we send $\lambda \to \sqrt{\beta}\,\lambda$ then the single-particle Schrödinger equation becomes:

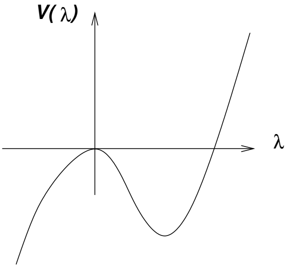
$$\left(-\frac{1}{2\beta^2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3\right)\Psi(\lambda) = \frac{1}{\beta}E\Psi(\lambda)$$

- The advantage of this is that we can interpret β^{-1} as $\hbar$, Planck's constant. The equation is then written:

$$\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3\right)\Psi(\lambda) = \hbar\,E\,\Psi(\lambda) = \varepsilon\,\Psi(\lambda)$$

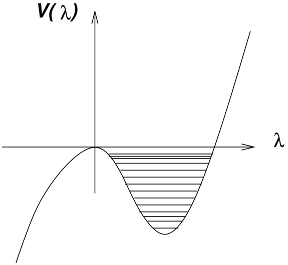
- The kinetic term has the usual form for quantum mechanics, and E on the RHS is the energy measured in units of Planck's constant.

- Now we can start to understand the double scaling limit. The potential looks like this:



- The Hamiltonian is actually unbounded below. However, eigenvalues localised on the right will tunnel through the barrier at a rate of order $e^{-\beta} = e^{-\frac{N}{g^2}}$.

- Therefore at this stage we have to bid farewell to our hopes of the theory being nonperturbatively well-defined.
- However, as long as we are only interested in perturbation theory in $\frac{1}{N^2}$, we can ignore tunneling.
- In this approximation, the Hamiltonian has discretely spaced levels on the right of the barrier, with typical spacing of order $\hbar = \beta^{-1}$.



- Very qualitatively, we see that the depth of the well is of order 1, and the level spacing is roughly of order

$$\frac{1}{\beta} = \frac{g^2}{N}$$

- We have to fill up the well with N fermions. Because of the Pauli principle, in the ground state they will fill the first N levels.
- Thus the topmost level ("Fermi level") will be at a height of order g^2 above the bottom of the well.
- And g is precisely the parameter in our control.
- For small g, the Fermi level can be below the barrier. But for large enough g, this level will rise above the barrier and eigenvalues will spill out to the other side.
- This is precisely the phase transition that makes continuum Riemann surfaces!

- To do better than this crude approximation, we use the WKB method to find the eigenvalues of this potential.
- This tells us that the n'th energy eigenvalue ε_n is given by:

$$\oint p_n(\lambda) d\lambda_n = \frac{2\pi}{\beta} n$$

where:

$$p_n(\lambda) = \sqrt{2(\varepsilon_n + \tfrac{1}{2}\lambda^2 - \tfrac{1}{3!}\lambda^3)}$$

and the integral is over a closed classical orbit.
- If the topmost orbit has turning points λ_+, λ_-, the Fermi level μ_F satisfies:

$$\int_{\lambda_-}^{\lambda_+} \sqrt{2(\mu_F + \tfrac{1}{2}\lambda^2 - \tfrac{1}{3!}\lambda^3)}\, d\lambda = \pi \frac{N}{\beta} = \pi g^2$$

- This confirms our qualitative guess that tuning g is responsible for tuning the Fermi level.
- Since we are going to take the limit of large N, it is convenient to analyse this problem in terms of the density of states of the system:

$$\rho(\varepsilon) = \frac{1}{\beta} \sum_{i=1}^{N} \delta(\varepsilon - \varepsilon_i)$$

- Then we have:

$$\begin{aligned} E_{gr} &= \beta\varepsilon_{gr} = \beta \sum_{i=1}^{N} \varepsilon_i = \beta^2 \int_{V_{min}}^{\mu_F} d\varepsilon \, \varepsilon \, \rho(\varepsilon) \\ g^2 &= \frac{N}{\beta} = \int_{V_{min}}^{\mu_F} d\varepsilon \, \rho(\varepsilon) \end{aligned}$$

- To compute the density of states, we equate the two expressions for g^2 to get:

$$g^2 = \int_{V_{min}}^{-\mu} d\varepsilon\, \rho(\varepsilon) = \frac{1}{\pi} \int_{\lambda_-}^{\lambda_+} \sqrt{2(-\mu + \tfrac{1}{2}\lambda^2 - \tfrac{1}{3!}\lambda^3)}\; d\lambda$$

  where we have defined the positive quantity $\mu = -\mu_F$.

- Differentiating in $-\mu$, we get:

$$\begin{aligned} -\frac{\partial g^2}{\partial \mu} \quad = \quad \rho(-\mu) \quad &= \quad \frac{1}{\pi} \int_{\lambda_-}^{\lambda_+} \frac{d\lambda}{\sqrt{2(-\mu + \frac{1}{2}\lambda^2 - \frac{1}{3!}\lambda^3)}} \\ &= \quad -\frac{1}{\pi} \log \mu + \mathcal{O}(\beta^{-2}) \end{aligned}$$

- We are looking for a singularity at a critical value g_c, so we define:

$$\Delta = \pi(g_c^2 - g^2)$$

and seek a relation between Δ and μ, given that both go to zero together.

- From the previous page we have:

$$\frac{\partial \Delta}{\partial \mu} = \pi\rho(-\mu) = -\log \mu$$

which can be integrated to give:

$$\Delta(\mu) = -\mu \log \mu$$

- The last step is to differentiate the equation

$$E_{gr} = \beta^2 \int_{V_{min}}^{-\mu} d\varepsilon\, \varepsilon\, \rho(\varepsilon)$$

to get:

$$\frac{\partial E_{gr}}{\partial \mu} = -\beta^2\, \mu\, \rho(-\mu)$$

which on integrating gives:

$$E_{gr} = \frac{1}{2\pi}(\beta\mu)^2 \log \mu$$

- With this we have performed the single-scaled limit of this matrix model and found the free energy (log of the partition function) in genus 0.
- Note that the key result was the logarithmic behaviour of the density of states as a function of μ as $\mu \to 0$.
- To leading order in the WKB approximation, this depended only on the quadratic part of the potential. In fact, this is true to all orders in the WKB approximation.

- To see this, let us go back to the original form of the one-particle Hamiltonian:

$$h(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2 + \frac{1}{3!\sqrt{\beta}}\lambda^3$$

- We see that as $\beta \to \infty$, the cubic term disappears completely. The states we are considering in this limit have energy $-\beta\mu$ which is kept finite.
- Thus from now on our single-particle Hamiltonian is:

$$h(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} - \frac{1}{2}\lambda^2$$

- Now we look at the double-scaled theory. We will see that the genus expansion parameter is $\beta\mu$.
- For this, the density of states will prove particularly useful. This time we need to know $\rho(\mu)$ to all orders in $\beta\mu$.
- We can write:

$$\rho(\mu) = \mathrm{tr}\,\delta(h + \beta\mu) = \frac{1}{\pi}\mathrm{Im}\,\mathrm{tr}\left[\frac{1}{h + \beta\mu - i\epsilon}\right] \quad (2)$$

$$= \frac{1}{\pi}\mathrm{Im}\int_0^\infty dT\, e^{-(\beta\mu - i\epsilon)T}\,\mathrm{tr} e^{-hT} \quad (3)$$

- Now we use the fact that our Hamiltonian is the continuation of a simple harmonic oscillator:

$$\tilde{h}(\lambda) = -\frac{1}{2}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{2}\omega^2\lambda^2$$

to the case $\omega = -i$. We easily see that:

$$\begin{aligned} \mathrm{tr} e^{-\tilde{h}T} &= e^{-\frac{\omega T}{2}} + e^{-\frac{3\omega T}{2}} + e^{-\frac{5\omega T}{2}} + \cdots \\ &= \frac{e^{-\frac{\omega T}{2}}}{1 - e^{-\omega T}} \\ &= \frac{1}{2\sinh \omega T/2} \end{aligned}$$

- Now we set $\omega \to -i$ and simultaneously use the $i\epsilon$ prescription to rotate $T \to iT$. Thus:

$$\rho(\mu) = \frac{1}{\pi} \mathrm{Re} \int_0^\infty dT\, e^{-i\beta\mu T} \frac{1}{2\sinh T/2}$$

- A small problem: this is logarithmically divergent at the lower limit of integration. This can be removed by differentiating and integrating back in $\beta\mu$.

- The result is best expressed in terms of the dilogarithm function:

$$\Psi(x) \equiv \frac{\partial}{\partial x} \log \Gamma(x)$$

and we find:

$$\begin{aligned} \rho(\mu) &= -\frac{1}{\pi} \Psi(\tfrac{1}{2} + i\beta\mu) \\ &= \frac{1}{\pi} \left(-\log \mu + \sum_{n=1}^{\infty} \frac{2^{2n-1} - 1}{n} \left| B_{2n} \right| (2\beta\mu)^{-2n} \right) \end{aligned}$$

- We clearly see that the genus expansion parameter in the double scaling limit is:

$$g_s = (\beta\mu)^{-1}$$

and it is held fixed as $\beta \to \infty, \mu \to 0$.

- Finally we recall that $E_{gr}(\mu) = \beta^2 \int d\mu\, \mu\, \rho(\mu)$ to write:

$$E_{gr}(g_s) = -\frac{1}{8\pi}\Big(-4g_s^{-2}\log g_s + \tfrac{1}{3}\log g_s + \sum_{h=2}^{\infty} \frac{2^{2h-1}-1}{2^{2n}h(h-1)(2h-1)} |B_{2h}|\, g_s^{2h-2}\Big)$$

- This is precisely the all-genus free energy of a string theory, the bosonic $c = 1$ string theory.

Outline

Continuum Approach to Noncritical Strings

- We have discovered a powerful result, but we don't yet have a clear picture of what string background we are studying!
- For this, let us briefly review the standard worldsheet approach to bosonic string theory.
- Start by fixing a Riemann surface, usually of genus 0, with local coordinates σ, τ.
- Introduce a metric $g_{ab}(\sigma, \tau)$ on the surface, and a map $X^\mu(\sigma, \tau)$ into a target space.

- The theory is defined via the worldsheet path integral:

$$Z_h = \int \frac{[\mathcal{D}_g g]\,[\mathcal{D}_g X]}{\mathrm{vol}\ (\mathrm{Diff})}\ e^{-S_M - S_c}$$

where

$$\begin{aligned} S_M &= \frac{1}{8\pi}\int d^2\xi\sqrt{g}g^{ab}\ \partial_a X^\mu \partial_b X^\mu \\ S_c &= \frac{\mu_0}{2\pi}\int d^2\xi\sqrt{g} \end{aligned}$$

- Next, we gauge-fix the worldsheet diffeomorphism invariance via conformal gauge:

$$g_{ab}(\sigma,\tau) = e^{\phi(\sigma,\tau)}\hat{g}_{ab}(\sigma,\tau)$$

- Gauge-fixing introduces Faddeev-Popov ghost fields on the worldsheet. A straightforward CFT computation shows that they carry a conformal anomaly (central charge) of -26.
- The final worldsheet theory has to be conformally invariant, which means the total central charge of all the fields should add up to zero.
- This can be achieved in two different ways, depending on the number of flat target-space coordinates, D.

- The central charge of a single flat spacetime direction is 1.
- Therefore when $D = 26$, the anomaly cancellation condition is met, the scale factor of the metric decouples, and we have the worldsheet action:

$$S_{total} = \frac{1}{4\pi} \int d^2 z \left(\partial_z X^\mu \partial_{\bar{z}} X_\mu + b_{zz} \partial_{\bar{z}} c_z + b_{\bar{z}\bar{z}} \partial_z c_{\bar{z}} \right)$$

Here we have taken the "reference metric" to be the identity.

- When $D < 26$, we find instead:

$$S_{total} = \frac{1}{4\pi}\int d^2z\Big(\partial_z X^\mu \partial_{\bar z} X_\mu + b_{zz}\partial_{\bar z} c_z + b_{\bar z\bar z}\partial_z c_{\bar z} + \partial_z\phi\partial_{\bar z}\phi + \mu\, e^{2b\phi}\Big)$$

  A new field, the "Liouville field", has appeared. It is the scale factor of the metric that failed to decouple.

- This field has a non-standard central charge:

$$c_\phi = 1 + 6Q^2$$

  where Q is a parameter that is self-consistently determined to cancel the total conformal anomaly:

$$Q = \sqrt{\frac{25-D}{6}}$$

- The non-standard central charge is because the parameter Q appears in the Lagrangian on a curved worldsheet:

$$\frac{1}{4\pi}\int d^2 z\sqrt{|g|}\,Q\,R(g)\,\phi$$

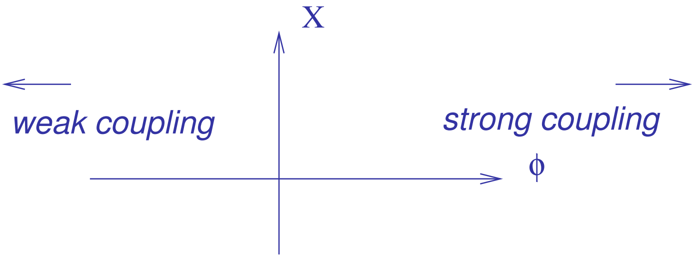
  and thereby modifies the energy-momentum tensor. Here, R is the Ricci scalar on the worldsheet.
- But Q has another physical effect. The worldsheet operator $\sqrt{|g|}\,R(g)\,\Phi$ describes the coupling of the dilaton field Φ in string theory.
- The dilaton governs the string coupling g_s. This follows because for constant Φ, the dilaton term is:

$$\frac{1}{4\pi}\Phi\int d^2 z\sqrt{|g|}\,R(g) = (2-2h)\Phi$$

- Therefore we have:

$$g_s = e^{\Phi}$$

With a linear dilaton, the spatial (Liouville) direction of spacetime effectively has a linearly varying string coupling:



- The Liouville field appears to be decoupled from everything else, but physical observables depend on it by the requirement of being in the BRST cohomology.
- More simply, a CFT vertex operator: $\int e^{2ik\cdot X}$ is generally not physical, since the integrand has conformal dimension:

$$\Delta_X = \bar{\Delta}_X = (k^2, k^2) \neq (1,1)$$

- Thus the typical observables are of the type: $\int e^{2ik\cdot X} e^{2k'\phi}$ where the rule for conformal dimensions of Liouville operators is:

$$\Delta_\phi = \bar{\Delta}_\phi = \big(k'(Q-k'), k'(Q-k')\big)$$

So we adjust k' to make the operators have dimension $(1,1)$.

- What about the term $\mu \int e^{2b\phi}$? This is called the cosmological operator, because the classical version of $e^{2b\phi}$ is $\sqrt{|g|}$.
- Its presence is necessary to "cut off" the strong-coupling region. And μ turns out to be the inverse string coupling that governs the genus expansion.
- To preserve conformal invariance, the operator must have dimension $(1,1)$ so:

$$b = \frac{1}{2}\left(Q \pm \sqrt{Q^2 - 4}\right)$$

- Now reality of b requires:

$$Q \geq 2 \implies D = 25 - 6Q^2 \leq 1$$

- Thus, noncritical strings only exist for dimension $D \leq 1$!
- For $D < 1$ we replace the X coordinates by an abstract CFT of central charge $c < 1$. A particular series of these, called the minimal models, have central charges:

$$c_{min.mod.} = 1 - \frac{6(p-q)^2}{pq}, \quad (p,q) \text{ co-prime integers}$$

- For these theories one can show that the fixed area partition function scales like:

$$\mathcal{Z}(A) \sim A^{\gamma - 3}$$

where

$$\gamma(h) = \frac{2h(p+q) - 2}{p+q-1}$$

- We finally have a point of contact with random matrices.
- γ is precisely the susceptibility that we encountered there, the critical exponent for the formation of continuous surfaces.
- From the above formula one sees that:

$$(\gamma(h) - 2) = (\gamma(0) - 2)(1 - h)$$

confirming the assumption we had made earlier.

- And indeed, the system of constant random matrices that we introduced earlier, precisely reproduces one infinite subset of the above $\gamma(h)$.
- For the most general class of matrix potentials:

$$\mathrm{tr}\left(M^2 + \sum_{i=1}^{p} a_i M^i\right)$$

  one finds by tuning coefficients that one can reproduce the susceptibilities of all the (q, p) minimal models for $q = 2$.
- The case $q > 2$ requires models with two coupled matrices.

- Among noncritical strings, that only leaves the case $D = 1$, where there is a genuine space (or rather, time) direction in the worldsheet formulation.
- Not surprisingly, this corresponds to matrix quantum mechanics.
- This time the continuum theory tells us that:

$$\gamma(h) = 2(h-1) + 2$$

- In matrix quantum mechanics, we saw that the genus-h contribution goes like $(\beta\mu)^{2-2h}$ which is like $(N(g_c - g))^{2-2h}$. This agrees perfectly with the above formula, since $2 - 2h = 2 - \gamma(h)$ for this model.

- In particular, we have:

$$\gamma(h=0)=0, \qquad \gamma(h=1)=2$$

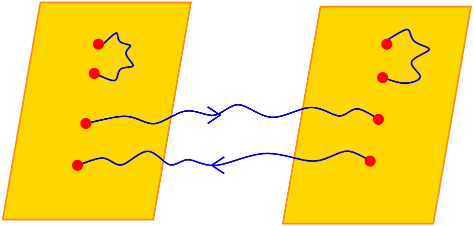
- This leads to an apparent puzzle. The behaviour $(g_c - g)^{2-\gamma}$ is singular only for genus $h > 2$, and nonsingular for $h = 0, 1$. Then how can the continuum limit arise at all in genus $h = 0, 1$?
- The theory cleverly cures this problem by developing logarithmic singularities precisely for $h = 0, 1$.

Outline

Random Matrices as D-branes

- We now understand that matrix quantum mechanics is a string theory in two dimensions with a linear dilaton in the spatial direction.
- An abiding mystery at this point is: where did the matrix description come from?
- We just proposed it and found that it reproduces the $c = 1$ string theory. But there is a deeper reason why this works: D-branes.

- Matrices generally arise in string theory from D-branes, which are dynamical objects on which open strings can end.
- In critical string theories there are stable, supersymmetric D-branes:



- In the case illustrated, the theory on the worldvolume of the D-branes will be a supersymmetric $SU(2)$ gauge theory.

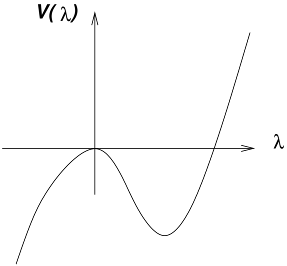
- More generally, when we have N D-branes then the fields living on them are always $N \times N$ matrices, transforming in the adjoint of the gauge group.
- In particular, the critical (10-dimensional) type IIB superstring, has stable D3-branes.
- As Maldacena found in 1997, the theory of open strings on a large number of these branes is equivalent to type IIB closed-string theory itself.

- Something a little different, but similar, occurs in the present case.
- As we will see, the $c = 1$ closed string theory has $D0$-branes among its excitations..
- We are going to claim that the theory on the worldline of these branes is precisely the matrix quantum mechanics of the $c = 1$ closed string theory!
- Therefore the full string theory can be reconstructed from a large number of its own D0-branes.

- The branes in question were found by the Zamolodchikovs many years ago as consistent boundary states of Liouville theory.
- It was shown that they are anchored at the strong-coupling end of the Liouville direction. Moreover, they are unstable.
- Since they are fixed in space and propagate in time, they are $D0$-branes.

- Let's now ask what the theory on their worldline looks like.
- D-branes that are unstable inevitably have a tachyonic scalar field on their worldvolume.
- This is just a field sitting at a maximum of its potential (like the Higgs field of the standard model).
- It turns out that this open-string tachyon, along with a 1d gauge field, are the only excitations on these ZZ D0-branes.
- If we put N ZZ branes together, the tachyon and the gauge field both become matrix-valued fields that we can call $M(t), A_t(t)$.

- What is the potential for the open-string tachyon?
- On very general grounds, it has been shown that the open-string tachyon potential in bosonic string theories is cubic:



- The curvature at the top of the potential is the $(mass)^2$ of the tachyon, which in string theory is known precisely to be -1 in units where the string tension parameter $\alpha' = 1$.

- Thus we have argued that the worldline action on N ZZ branes is:

$$S = N \int dt \ \mathrm{tr}(\frac{1}{2} D_t M^2 + \frac{1}{2} M^2 - \frac{g}{3!} M^3)$$

where

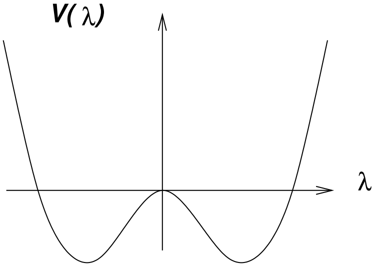
$$D_t M \equiv \dot{M} + [A_t, M]$$

- Polchinski showed long ago that the tension of a D-brane is g_s^{-1}.
- The Sen conjecture states that the the energy difference between the top and bottom of the tachyon potential is the tension of the D-brane.

- Therefore the tension of a D0-brane in the $c = 1$ background should be the energy required to lift an eigenvalue up from the Fermi level to the top of the barrier. This is just $\mu \sim g_s^{-1}$.
- Many other comparisons support the view that matrix quantum mechanics is precisely the theory on N D0-branes of $c = 1$ noncritical string theory.

- This connection between random matrices and D-branes provides several illuminations on how to understand string theory better.
- For example, one is led to ask if we can find a noncritical string theory whose D-branes have a tachyon potential which is bounded below. The answer is yes.
- Introducing fermionic coordinates is a significant modification of string theory. Some theories in this class have spacetime supersymmetry, then they are called "superstrings". Others do not, and are called "type 0 strings".
- In both cases the worldsheet theory is a supersymmetric Liouville theory.
- The D-branes of type 0 and of superstrings are quite similar.

- One of their known properties is that the tachyon potential is even. The simplest such potential is the quartic plus quadratic:



- Now we can recycle all our work on generating continuum surfaces, by filling both wells in this potential.

- The noncritical super-Liouville theory with this potential is called type 0B string theory.
- It has been known for over a decade that this theory has a doubled spectrum of states. But only about 3 years ago was that fact connected to the double-well tachyonic potential!
- In studying type 0B theory, the first step would be to fill up the wells on both sides to a common Fermi level, and then compute the perturbative free energy in the double-scaling limit.
- But we have already solved this problem! It is the same as for the bosonic theory. As we already argued, perturbation theory does not care about whether the well is single or double.

- Therefore the only thing left to do is to understand nonperturbative effects in this theory.
- Also we need to fill a gap in our treatment of the bosonic string. We did not identify the physical observables and study their scattering amplitudes.
- Therefore in the rest of this course, I will fix the discussion on the type 0B string, and discuss observables, scattering amplitudes and nonperturbative effects.

Outline

Observables of Noncritical Strings

- We start by re-focusing on the bosonic $c = 1$ string. We already found its partition function to all orders in the string coupling.
- But what are its physical observables? These are the particles which, in higher dimensional string theory, would be gravitons, gauge fields and all the rest.
- We already indicated that a class of observables in this theory are of the type:

$$T(k, k') = e^{2ikX} e^{2k'\phi}$$

where

$$k^2 + k'(Q - k') = 1$$

- Now we have $Q = 2$ and it's easy to see that this determines the observables to be:

$$T(k) = e^{2ikX} e^{2(1 \mp |k|)\phi}$$

- There is a slight subtlety here. The above is a "physical state" only in the Euclidean theory. Upon making X timelike, we find that the correct operator is:

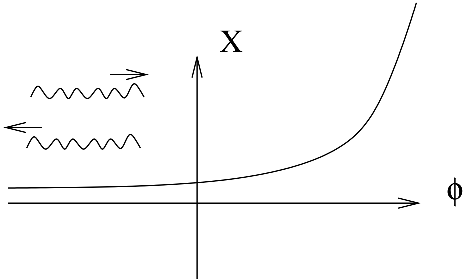
$$T(k) = e^{2ikX} e^{2(1 \mp i|k|)\phi}$$

- This operator creates states whose wave function is

$$\psi(k) = e^{2ikX} e^{\mp 2i|k|\phi}$$

which shows that we are dealing with a massless particle.
- For historical reasons it is called the closed string tachyon though it is not at all tachyonic!

- These wave functions correspond to plane waves moving to the right (into strong coupling) or left (into weak coupling) depending on the sign.



- Because of the Liouville wall, the correct physical wave functions are linear combinations of the two. But it is convenient to label them by the incoming (right-moving) wave function:

$$T(k) = e^{2ikX} e^{2(1-i|k|)\phi}$$

- Thus, in addition to the partition function, we want to calculate quantities like:

$$\langle T_{k_1} T_{k_2} \cdots T_{k_n} \rangle$$

as a function of the string coupling $g_s = \mu^{-1}$.

- From time translation invariance, these correlators are zero unless

$$\sum_{i=1}^{n} k_i = 0$$

- In what follows, we will consider the case where the time direction is compact with periodicity R. Then, the partition function and correlation functions all depend on the two parameters (μ, R).

- How are these observables defined in matrix quantum mechanics?
- We start by defining a macroscopic loop operator:

$$W(\ell, t) = \mathrm{tr}\, e^{-\ell M(t)}$$

which can be thought of as cutting a hole of length ℓ in the discretised Riemann surface.

- Next we take its Fourier transform in t:

$$W(\ell, k) = \int dt\, e^{ikt} W(\ell, t)$$

This is an operator with a definite energy k.

- As $\ell \to 0$, we can associate it with a pointlike puncture carrying momentum k. This must be just the tachyon T_k that we've encountered before.
- Therefore we propose that, as $\ell \to 0$,

$$\langle W(\ell_1, k_1) W(\ell_2, k_2) \cdots W(\ell_2, k_2) \rangle \to \prod_{i=1}^{n} \ell_i^{k_i} \langle \hat{T}_{k_1} \hat{T}_{k_2} \cdots \hat{T}_{k_n} \rangle$$

- We've kept a "hat" on T, to allow for the possibility that

$$\hat{T}_k = \mathcal{N}(k) T_k$$

where $\mathcal{N}(k)$ is a normalisation factor.

- The next step is to convert the matrix quantum mechanics into a double scaled field theory with Lagrangian:

$$\mathcal{L} = \int_{-\infty}^{\infty} d\lambda \, \Psi^\dagger(\lambda, t) \left(i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial \lambda^2} + \frac{1}{2}\lambda^2 + \mu \right) \Psi(\lambda, t)$$

- This formulation allows us to handle the many-body problem more efficiently.
- In this language, the density of states is simply:

$$\rho(\lambda, t) = \Psi^\dagger(\lambda, t)\Psi(\lambda, t)$$

  and we note that it is now an operator.
- Now we can easily convert correlators of loop operators into correlators of the density of states.

- The relation is easily seen to be:

$$
\begin{aligned}
W(\ell, k) &= \int dt\, d\lambda\, e^{ikt} e^{-\ell\lambda} \rho(\lambda, t) \\
&= \int d\lambda\, e^{-\ell\lambda} \tilde{\rho}(\lambda, k)
\end{aligned}
$$

- Hence we need to compute the correlator

$$
\langle \rho(\lambda_1, k_1)\rho(\lambda_2, k_2) \cdots \rho(\lambda_n, k_n) \rangle
$$

  from the double-scaled fermi field theory.

- In principle the calculation is straightforward since we have a quadratic Lagrangian and can use Wick's theorem.

- The results are rather complicated. However, they give a very interesting mathematical structure to the theory.
- To start with, the two-point function of the density of states comes out to be:

$$\begin{aligned}\langle\tilde{\rho}(\lambda_1,k_1)\tilde{\rho}(\lambda_2,k_2)\rangle &= \frac{1}{\sqrt{\lambda_1\lambda_2}}\delta(k_1+k_2,0)\times \\ &\int dk\Big[e^{(i\mu-k-k_1)|\log\lambda_1/\lambda_2|-\frac{i}{4}|\lambda_1^2-\lambda_2^2|} \\ &+R_{k+k_1}e^{(i\mu-k-k_1)\log\lambda_1\lambda_2-\frac{i}{4}(\lambda_1^2+\lambda_2^2)}\Big]\end{aligned}$$

- Here, R_k is the reflection coefficient that encodes all the scattering properties of this fermi field theory.

- For the bosonic string it can be shown that:

$$R_k = (-i\mu)^{-k} \frac{\Gamma(\frac{1}{2} - i\mu + k)}{\Gamma(\frac{1}{2} - i\mu)}$$

- Since we are in Euclidean compact time, on a circle of radius R, the momentum integral turns into a discrete sum. The fermion momenta along the time direction are:

$$k_m = \frac{m + \frac{1}{2}}{R}$$

- Calculating general density correlators and extracting tachyon amplitudes leads to a beautiful underlying mathematical structure for the theory, which we will now discuss in some detail.

Outline

The Kontsevich-Penner Matrix Model

- The basic result of the procedure described above is a "coherent state" representation for the generating functional $Z(t, \bar{t})$ of tachyon amplitudes.
- By "generating functional" we mean a function that satisfies:

$$\langle T_{k_1} T_{k_2} \cdots T_{k_n} T_{-j_1} T_{-j_2} \cdots T_{-j_m} \rangle = \frac{\partial}{\partial t_{k_1}} \frac{\partial}{\partial t_{k_2}} \cdots \frac{\partial}{\partial t_{k_n}} \frac{\partial}{\partial \bar{t}_{j_1}} \frac{\partial}{\partial \bar{t}_{j_2}} \cdots \frac{\partial}{\partial \bar{t}_{j_n}} \log Z(t, \bar{t})$$

and thereby defines all tachyon amplitudes to all genus.

- The construction involves a Fock space for bosonic creation and annihilation operators α_{-n} and α_n, satisfying the canonical commutation relations

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}$$

- The α_n can be summarised into a new scalar field

$$\partial\varphi(z) \equiv \sum_n \alpha_n z^{-n-1}$$

- This in turn can be fermionised by the familiar formula:

$$\partial\varphi(z) = \ :\bar{\psi}(z)\psi(z):$$

where the fermion mode expansion is:

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_{n+\frac{1}{2}} z^{-n-1} \qquad \bar{\psi}(z) = \sum_{n\in\mathbb{Z}} \bar{\psi}_{n+\frac{1}{2}} z^{-n-1}$$

- The fermionic oscillators obey canonical anticommutation relations:

$$\{\psi_r, \bar{\psi}_s\} = \delta_{r+s,0}, \quad r, s \in \mathbb{Z} + \frac{1}{2}$$

- Scattering amplitudes of the theory are then given by the master formula:

$$Z(t, \bar{t}) = \langle t|S|\bar{t}\rangle$$

where $\langle t|$ and $|\bar{t}\rangle$ are coherent states associated to the positive and negative tachyons:

$$\langle t| \equiv \langle 0|\mathrm{e}^{i\mu\sum_{n=1}^{\infty}\alpha_n t_n} \equiv \langle 0|U(t)$$

$$|\bar{t}\rangle \equiv \mathrm{e}^{i\mu\sum_{n=1}^{\infty}\alpha_{-n}\bar{t}_n}|0\rangle \equiv U(\bar{t})|0\rangle$$

- The operator S acts linearly on the fermionic fields:

$$S\psi_{-n-\frac{1}{2}}S^{-1} = R_{p_n}\psi_{-n-\frac{1}{2}} \qquad S\bar{\psi}_{-n-\frac{1}{2}}S^{-1} = R^*_{p_n}\bar{\psi}_{-n-\frac{1}{2}}$$

where R_{p_n} are the reflection coefficients, which satisfy a unitarity condition:

$$R_{p_n}R^*_{-p_n} = 1$$

- We have seen that these coefficients, in the bosonic string, are given by:

$$R_{p_n} = (-i\mu)^{-p_n}\frac{\Gamma(\frac{1}{2} - i\mu + p_n)}{\Gamma(\frac{1}{2} - i\mu)}$$

- We are going to find a generating functional from the coherent states formula by introducing the notion of semi-infinite forms:

$$|0\rangle = z^0 \wedge z^1 \wedge z^2 \ldots$$

- On this space the fermions act as:

$$\psi_{n+\frac{1}{2}} = z^n, \qquad \bar{\psi}_{-n-\frac{1}{2}} = \frac{\partial}{\partial z^n}$$

- It follows that:

$$S : z^n \to R_{-p_n} z^n$$

- Recall that

$$\left[\alpha_n, \psi_{m+\frac{1}{2}}\right] = \psi_{m+n+\frac{1}{2}}$$

Then the action of the coherent state operator $U(\bar{t})$ on the fermionic oscillators is:

$$U(\bar{t}) : \psi_{n+\frac{1}{2}} \rightarrow U(\bar{t})\psi_{n+\frac{1}{2}} U(\bar{t})^{-1}$$

- In the semi-infinite forms representation this reads:

$$\begin{aligned} U(\bar{t}) : z^n &\rightarrow \mathrm{e}^{i\mu \sum_{k>0} \bar{t}_k \alpha_{-k}} z^n \mathrm{e}^{-i\mu \sum_{k>0} \bar{t}_k \alpha_{-k}} \\ &= \mathrm{e}^{i\mu \sum_{k>0} \bar{t}_k z^{-k}} z^n = \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) z^{n-k} \end{aligned}$$

where $P_k(i\mu\bar{t})$ are the Schur polynomials.

- Therefore the combined action of S and $U(\bar{t})$ is

$$S \circ U(\bar{t}) : z^n \rightarrow w^{(n)}(z;\bar{t}) = S \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) z^{n-k} S^{-1}$$
$$= \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) R_{-p_{n-k}} z^{n-k}$$

- At this point we specialise to the selfdual value of the radius, $R = 1$. This is a special point in the theory where it acquires a topological character.

- Recalling the expression for R_{p_n} and rewriting the Γ-function in terms of its integral representation, one obtains:

$$\begin{aligned} w^{(n)}(z;\bar{t}) &= \frac{(-i\mu)^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-i\mu)} \int_0^\infty dm\, \mathrm{e}^{-m} m^{-i\mu-1} \sum_{k=0}^{\infty} P_k(i\mu\bar{t}) \left(\frac{-i\mu z}{m}\right)^{n-k} \\ &= c(\mu)\, z^{-i\mu} \int_0^\infty dm\ m^{-n} \mathrm{e}^{i\mu z m} m^{-i\mu-1} \mathrm{e}^{i\mu \sum_{k>0} \bar{t}_k m^k} \end{aligned}$$

where

$$c(\mu) \equiv \frac{(-i\mu)^{-i\mu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-i\mu)}$$

- From this we finally derive the expression for the state $S|\bar{t}\rangle$ in terms of semi-infinite forms:

$$\begin{aligned} S|\bar{t}\rangle &= S \circ U(\bar{t})\, z^0 \wedge z^1 \wedge z^2 \wedge \dots \\ &= w^{(0)}(z;\bar{t}) \wedge w^{(1)}(z;\bar{t}) \wedge w^{(2)}(z;\bar{t}) \wedge \dots \end{aligned}$$

- One also needs to make use of the parametrization for the coherent state $\langle t|$.
- For this, we summarise the parameters t_n into a new $N \times N$ matrix:

$$i\mu\, t_n = -\frac{1}{n}\mathrm{tr} A^{-n}$$

This is called the Kontsevich-Miwa transform.

- If $a_i,\ i = 1, \ldots, N$ are the eigenvalues of A then:

$$\langle t| = \langle 0| \prod_{i=1}^{N} \mathrm{e}^{-\sum_{n>0} \frac{\alpha_n}{n} a_i^n} = \langle N| \frac{\prod_{i=1}^{N} \psi(a_i)}{\Delta(a)}$$

where:

$$|N\rangle = z^N \wedge z^{N+1} \wedge z^{N+2} \ldots$$

- Putting together everything, we end up with:

$$\begin{aligned} Z(t,\bar{t}) &= \langle t|S|\bar{t}\rangle = \frac{\det\ w^{(j-1)}(a_i)}{\Delta(a)} \\ &= c(\mu)^N (\prod_j a_j)^{-i\mu} \\ &\times \int_0^\infty \prod_j \left(\frac{dm_j}{m_j}\ \mathrm{e}^{i\mu m_j a_j - i\mu \log m_j + i\mu \sum_{k>0} \bar{t}_k m_j^k} \right) \frac{\Delta(m^{-1})}{\Delta(a)} \end{aligned}$$

- Converting the Vandermonde depending on m_i^{-1} to the standard one, and using the famous Harish Chandra formula, one finally finds:

$$Z(t,\bar{t}) = (\det\ A)^{-i\mu} \int dM\ \mathrm{e}^{i\mu\, \mathrm{tr} MA - (i\mu+N)\mathrm{tr}\ \log M + i\mu \sum_{k>0} \bar{t}_k \mathrm{tr} M^k}$$

- Remarkably we have ended up with another matrix model!
- This model summarises all amplitudes, to all genus, of the bosonic $c = 1$ string – though only when the time direction is compactified with radius $R = 1$.
- In this model, we need N to be large only so that all the tachyon couplings are independent. Also, unlike MQM, it has an explicit parameter μ. And finally, it is a constant matrix model – there is no time parameter.
- A different, but related, matrix model can be constructed for $R \neq 1$, but we will not have time to discuss it here.

Outline

Type 0B String Theory

- If on the worldsheet of a string we put D fermions along with the D bosons, there is worldsheet supersymmetry.
- Coupling this to the super-extension of the worldsheet metric, namely worldsheet supergravity, and gauge-fixing, we end up (in the noncritical case) with a string whose worldsheet is:

$$\begin{aligned} S &= \frac{1}{4\pi}\int d^2z\Big(\partial_z X^\mu \partial_{\bar{z}} X_\mu + i\psi_{\bar{z}}\partial_z\psi_{\bar{z}} + i\psi_z\partial_{\bar{z}}\psi_z \\ &\qquad +\text{ghosts} + \text{super-ghosts} \\ &\qquad +\text{Liouville} + \text{super-Liouville}\Big) \end{aligned} \tag{4}$$

- This actually describes two different theories: type 0A and type 0B. This arises from a twofold ambiguity in projecting the spectrum.

- We will focus on type 0B because it closely resembles the bosonic $c = 1$ string.
- In its spectrum it has a massless "closed-string tachyon", just like the bosonic theory. But it also has another real massless field C, called the "Ramond-Ramond scalar".
- The equations of motion of this field allow the linear solution:

$$C(\phi, X) = \nu\phi + \tilde{\nu}X$$

which plays an important role.

- The tachyon potential in this background is a double well.
- Therefore filling the Fermi sea is done somewhat differently.
- First of all, since the potential is bounded below, it makes sense for the (scaled) Fermi level μ to be either positive or negative.
- Second, it makes sense to consider fermions that are moving towards the left or the right, and are on the left or the right. That makes four types of fermions (the bosonic case had two).

- It can be shown that the three parameters $\mu, \nu, \tilde{\nu}$ of the continuum theory are in correspondence with the four independent choices of the Fermi level (one relation holds between them due to conservation of fermion number).
- This is essentially the only new feature of this theory with respect to the bosonic string.
- In particular, the partition function and correlators will depend on $\mu, \nu, \tilde{\nu}$ (in addition to R if the time is compact, though here we are going back to $R = \infty$).
- I will conclude by exhibiting, for comparison, the partition functions of the bosonic $c = 1$ and type 0B strings, as a function of their respective parameters, in the noncompact case.

- For the bosonic case we have seen that:

$$\log \mathcal{Z}(\mu)_{c=1} = \frac{1}{\pi}\mathrm{Re}\int \frac{dT}{T} e^{-i\mu T} \frac{1}{2T \sinh T/2}$$

  and this expression is to be understood as a perturbation series in μ^{-2}.

- For the type 0B case, the answer is, instead,

$$\log \mathcal{Z}(\mu, \nu, \tilde{\nu})_{0B} = \frac{1}{\pi}\mathrm{Re}\int \frac{dT}{T}\left[e^{-\frac{yT}{2}} \frac{1}{2T \sinh T/2} - \frac{1}{t^2} + \frac{y}{2t} + \left(\frac{1}{24} - \frac{y^2}{8}\right)e^{-t}\right] + (y \leftrightarrow y')$$

where $y = 2i(\mu - |\nu|), y' = -2i(\mu + |\nu|)$

Outline

Concluding Remarks

- We have seen that dynamically triangulated random surfaces can reproduce string theories, albeit in low spacetime dimensions.
- Their origin lies in the existence of D-branes in the corresponding closed-string theory.
- Once we use this matrix, or open-string, description, we can extract very powerful results about string theory such as all-genus partition functions and correlation functions.
- The answers are fascinating both mathematically and physically.

Thank You