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RECENT ADVANCES

THEORY OF VARIATIONAL INEQUALITIES WITH APPLICATIONS TO PROBLEMS OF FLOW THROUGH POROUS MEDIA

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TABLE OF CONTENTS

PREFACE

Our interest in variational inequalities grew from studies of non-convex optimization theory and pseudomonotone operators as a basis for both the qualitative and numerical analysis of non-linear problems in continuum mechanics. The theory of variational inequalities is rich and exciting; within it, one can find a wealth of powerful ideas which not only reveal fundamental facts on the qualitative behavior of solutions to important classes of non-linear boundary-value problems, but which also provide a natural framework for a host of relatively new numerical methods. Equally important, the theory also enables one to construct a rather elaborate approximation theory which brings to light useful information on the behavior of numerical solutions—error estimates, convergence criteria, etc. Finally, at the heart of variational inequalities is their intrinsic inclusion of free boundaries; thus, they provide a natural and elegant framework for the study of the classical problem of flow through porous media. All of the applications of variational inequalities considered here are focused on problems of this type—the so-called seepage problems of slow irrotational flow of an incompressible fluid through a porous media characterized by Darcy's law.

Our aim in this monograph is to present a rather detailed survey of the theory of variational inequalities, their approximation and numerical analysis, and to demonstrate applications of these theories to the analysis of difficult free boundary problems encountered in the study of flow through porous media. Much of what we discuss here we owe to the principal developers of the subject: Stampacchia, Lions, Biaocchi, Mosco *et al.*, but several of the results we describe, particularly on the computational side, are new. Our account is by no means complete; among other things, we do not treat variational inequalities for evolution problems and we identify several open questions concerning quasi-variational inequalities. We hope that the introduction to these subjects presented here will provide a basis for those who wish to pursue these subjects in more detail.

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INTRODUCTION

Introductory comments

It is a well-known result in convex analysis that the minimization of a functional F defined on a closed convex set K leads to an inequality involving the derivative DF of F rather than the classical equality $DF(x) = 0$ which is valid when F is defined, for example, on a linear space. This fact has been exploited in the study of convex optimization problems for many years. What was not widely appreciated, however, until a decade ago, was that these ideas had far-reaching implications in many areas of non-linear mechanics; that, in particular, many free-boundary problems could be elegantly formulated using extensions of these ideas, and that, concomitantly, a variety of mathematical methods, both analytical and computational, could be used to study free-boundary problems which were formulated this way.

Modern work on the theory of variational inequalities began with the pioneering papers of Fichera[1], Stampacchia[2], Lions and Stampacchia[3] and Brezis[4], and was further developed by the French and Italian school of applied mathematicians during the last decade (e.g. Mosco[5], Glowinski, Lions and Trémolières[6], Fichera[7], Duvaut and Lions[8]). Excellent surveys of these ideas have been contributed by Mosco[5, 9], Stampachia[10] and Lions[11]; applications to a wide variety of free-boundary problems are discussed in the book of Duvaut and Lions[8]. Numerical methods based on variational inequalities are discussed in the two-volume text of Glowinski, Lions and Trémolières[6] and in the monograph of Glowinski[12]. The application of variational inequalities to free-boundary problems arising in the flow of fluids through porous media was studied by Baiocchi[13] and Baiocchi *et al.*[14], and a numerical analysis of such problems was investigated by Baiocchi *et al.*[15]. Theorems on the convergence of finite element approximation of certain classes of variational inequalities were developed by Falk[16], Brezzi, Hager and Raviart[17] *et al.* Additional references to literature on variational inequalities and their applications can be found in the works cited above. We will also cite other references relevant to our study later in this work.

Our objective here is five-fold:

1. To give a summary account of the general mathematical theory of variational inequalities set in the framework of non-linear operators defined on convex sets in real Banach spaces. We focus our attention on existence and uniqueness theorems for such abstract problems for, as

will be shown, these form the basis for the construction and analysis of numerical methods for such problems.

2. To study the approximation of variational inequalities by finite element methods, and to study various numerical schemes that can be used to solve discrete models of variational inequalities.

3. To describe the formulation of the seepage flow problem by variational inequalities using variants of the Baiocchi transformation, and to study the existence and regularity of solutions to such problems.

4. To develop finite element methods for the approximate solution of seepage flow problems. Here we are also concerned with the existence of solutions to the approximate problems, the convergence of finite element approximations, and the development of *a priori* error estimates.

5. To solve numerically several representative seepage flow problems and to discuss and compare various numerical schemes.

The theoretical foundations of variational inequalities are taken up in Chap. 1 following this introduction. There we give a rather complete account of the theory as it applies to monotone and pseudomonotone operators on reflexive Banach spaces. We also discuss the theory of quasi-variational inequalities, which we later show to be very important in the study of certain seepage problems.

Finite element approximations and various numerical methods are discussed in Chap. 2. We review the theory of Falk[16] for error estimation of certain classes of variational inequalities, and we describe algorithms for the solution of systems of inequalities; in particular, we examine fixed-point (contraction mapping) methods, S.O.R.-projection methods, Lagrange multiplier methods, and penalty methods. Some numerical experiments designed to test the validity of the theoretical estimates and to compare methods are also presented in this section.

For completeness, we give proofs to all of the major theorems discussed in Chaps. 1 and 2.

The formulation of seepage flow problems is discussed in Chap. 3. Here a rather general formulation is developed, using the notion of quasi-variational inequalities. We then consider a number of special cases, describe some numerical experiments, and compare results with those obtained by other methods.

Chapter 4 is devoted to the analysis of seepage in non-homogeneous dams in which the permeability k at a point (x, y) is given as either a function of only x or only y. Numerical examples are described and the results are compared with those obtained by other numerical techniques.

Selected special problems are treated in Chap. 5, including the effects of impermeable sheets as boundaries, and channel problems.

Conclusions reached during our study and comments on possible directions for future research are collected in Chap. 6.

Notation and conventions

Notations and terms common in literature on functional analysis, partial differential equations, and the mathematical theory of finite elements are used throughout this study. Introductory accounts of analysis sufficient to provide a background for this work can be found in the text of Oden[18]. For an introduction to the mathematical theory of finite elements, see, for example, the books of Oden and Reddy[19], Ciarlet[20], or Vol. 4 of the series by Becker, Carey and Oden[21]. The definitions of most symbols are given where they first appear in the text. The following conventions and definitions are used frequently in our study

$\mathcal{U}, \mathcal{V}$ = real Banach spaces with norms $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{V}}$, respectively.

$\mathcal{U}'$ = the (strong) topological dual of $\mathcal{U}$.

$\langle\cdot,\cdot\rangle$: $\mathcal{U}' \times \mathcal{U} \to \mathbf{R}$ = canonical duality pairing from $\mathcal{U}' \times \mathcal{U}$ into $\mathbf{R}$; thus, if f is a continuous linear functional on $\mathcal{U}$, we write

$$f(u) = \langle f, u\rangle$$

A: $K \subset \mathcal{U} \to \mathcal{U}'$ = an operator defined on a set K in $\mathcal{U}$ with its range in the dual of $\mathcal{U}$; A is monotone if

$$\langle A(u) - A(v), u - v\rangle \geq 0 \quad \forall u, v \in \mathcal{U}$$

A is hemicontinuous at u if $\varphi: [0, 1] \to \mathbf{R}$ is continuous for all $v, w \in \mathcal{U}$, where $\varphi(t) = \langle A(u + tv), w\rangle$; A is coercive on K if, for $u \in K$, there is a $v_0 \in K$ such that

$$\frac{\langle A(u), u - v_0\rangle}{\|u\|_{\mathcal{U}}} \to +\infty \quad \text{as } \|u\|_{\mathcal{U}} \to \infty$$

$\{u_m\} \in K$ = a sequence drawn from a set $K \subset \mathcal{U}$; a sequence converges strongly to $u \in \mathcal{U}$ whenever

$$\lim_{m\to\infty} \|u_m - u\|_{\mathcal{U}} = 0$$

and $\{u_m\}$ converges weakly to $u \in \mathcal{U}$ whenever

$$\lim_{m\to\infty} \langle f, u_m\rangle = \langle f, u\rangle \quad \forall f \in \mathcal{U}'$$

K = a subset of $\mathcal{U}$; K is convex if, $\forall u, v \in K$, the line segment

$$\theta u + (1 - \theta)v \quad \forall \theta \in [0, 1]$$

belongs to K. K is bounded if a constant $M < \infty$ exists such that $\|u\|_{\mathcal{U}} \leq M$ for all u in K. K is weakly sequentially closed if every weakly convergent sequence in K has its weak limit in K: K is closed if the limit of every strongly convergent sequence drawn from K is in K.

$F: K \subset \mathcal{U} \to \mathbf{R}$ = a real functional defined on a subset K of the Banach space $\mathcal{U}$.

$DF: K \to \mathcal{U}'$ = the Gâteaux derivative of F; i.e.. $DF(u)$ is the linear functional in $\mathcal{U}'$ satisfying, $\forall v \in \mathcal{U}$,

$$\lim_{t\to 0^+} \frac{\partial}{\partial t} F(u + tv) = \langle DF(u), v\rangle$$

$F: K \subset \mathcal{U} \to \mathbf{R}$ is convex if

$$F(\theta u + (1 - \theta)v) \leq \theta F(u) + (1 - \theta)F(v)$$

for all $u, v \in K$ and $\theta \in [0, 1]$; F is concave if $-F$ is convex.

$F: K \subset \mathcal{U} \to \mathbf{R}$ is weakly lower semicontinuous if, for every sequence $\{u_m\} \in K$ converging weakly to $u \in K$, we have

$$\liminf_{m\to\infty} F(u_m) \geq F(u)$$

If the reversed inequality holds ($\leq$), for $\limsup_{m\to\infty} F(u_m)$, F is weakly upper semicontinuous.

$W^{m,p}(\Omega)$ = the Sobolev space of order (m, p) for a bounded domain $\Omega \subset \mathbf{R}^n$, $m \geq 0$, $1 \leq p \leq \infty$, equipped with the norm

$$\|u\|_{m,p,\Omega} = \left\{\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^p \, dx\right\}^{1/p} \quad 1 \leq p < \infty$$

$$\|u\|_{m,\infty,\Omega} = \sup_{x\in\Omega} \sum_{|\alpha|\leq m} |D^\alpha u(x)|$$

where standard multi-index notations are used (see, e.g. Oden and Reddy[19]). When, from the context, the domain Ω is understood, we write $\|\cdot\|_{m,p}$ rather than $\|\cdot\|_{m,p,\Omega}$.

$W_0^{m,p}(\Omega)$ = the closure of the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω with respect to the Sobolev norm $\|\cdot\|_{m,p,\Omega}$.

$W^{-m,p'}(\Omega) = (W_0^{m,p}(\Omega))'$, the topological dual of $W_0^{m,p}(\Omega)$; here

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

$H^m(\Omega)$ = the Hilbert spaces $W^{m,2}(\Omega)$; $\|u\|_{m,2} \equiv \|u\|_m$.

$H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

$H^{-m}(\Omega) = (H_0^m(\Omega))'$.

We will also frequently make use of the fact that every bounded sequence $\{u_m\}$ in a reflexive Banach space has a weakly convergent subsequence. The spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are reflexive whenever $1 < p < \infty$; hence $H^m(\Omega)$ and $H_0^m(\Omega)$ are reflexive.

1. VARIATIONAL INEQUALITIES

1.1 *Introduction*

The modern theory of variational inequalities has its roots in the classical problem of minimizing a convex differentiable function on a convex set. Consider, for example, the elementary problem of determining the real number x_0 at which the quadratic function

$$F(x) = \frac{1}{2} bx^2 - ax + c, \; b > 0, \quad x \in \mathbf{R} \tag{1.1}$$

attains its minimum value. The minimizer is, of course, characterized by the condition

$$F'(x_0) = bx_0 - a = 0 \tag{1.2}$$

so that F attains its minimum at $x_0 = a/b$. The situation is, however, quite different if we add to the minimization problem a constraint on x, e.g.

$$x_0 \in K = \{x \in \mathbf{R}: 0 \le x \le 1\}. \tag{1.3}$$

Then (1.2) may not properly characterize the solution. A minimizer x_0 of F in K satisfies, by definition, the inequality

$$F(x_0) \le F(x) \quad x \in K$$

Since K is convex, $x_0 + \theta(x - x_0) = \theta x + (1-\theta)x_0 \in K$, $\theta \in [0, 1]$. Hence, for every $x \in K$

$$F(x_0 + \theta(x - x_0)) \ge F(x_0)$$

and

$$\lim_{\theta \to 0^+} \frac{1}{\theta} [F(x_0 + \theta(x - x_0)) - F(x_0)] = F'(x_0)(x - x_0) \ge 0$$

In other words, a minimizer x_0 of F is now characterized by the *inequality*

$$F'(x_0)(x - x_0) \ge 0 \quad \forall x \in K. \tag{1.4}$$

Examples are shown in Fig. 1.1. Notice that all of the minimizers indicated in Figs. 1.1(a–c) satisfy (1.4); only the case in 1.1(c) in which the minimizer x_0 falls on the interior of K is such that $F'(x_0) = 0$. But this also covered by (1.4) because if $x_0 \in \text{int } K$, an $\epsilon > 0$ can be found such that $x = x_0 \pm \epsilon y$ and $\pm \epsilon F'(x_0)y \ge 0$ for any $y \in K$, which is possible only if $F'(x_0) = 0$. Hence (1.4) includes the characterization (1.2) as a special case.

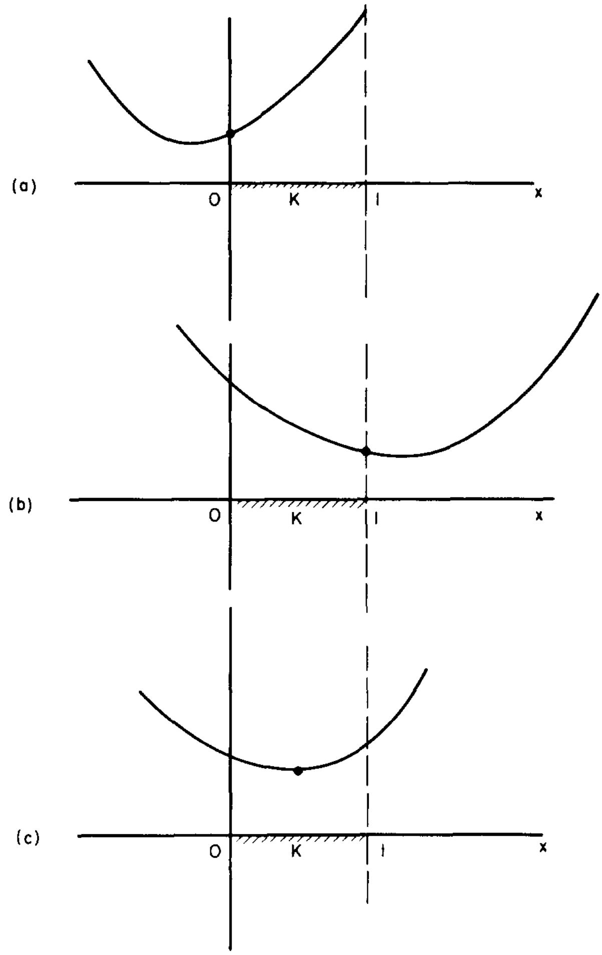


Fig. 1.1. Minimization of a convex function on a convex set.

Another important and interesting aspect of variational inequalities is, that in many cases, they can be shown to characterize so-called *free boundary-value problems*. This feature was exploited by Lions[22] in 1974 and has contributed to its popularity in studying a variety of physical problems. To illustrate this property, consider the following example:

Example 1-1.1. Consider the variational inequality: $u \in K$

$$\int_0^1 \{u'(v-u)' + (v-u)\}\,\mathrm{d}x \geq 0, \quad \forall v \in K$$

where $K = \{v \in H^1(0,1)\colon v(0) = 1/4,\ v(1) = 0 \text{ and } v \geq 0 \text{ in } (0,1)\}$, and $u' = \mathrm{d}u/\mathrm{d}x$.

Suppose that $u \in K$ is a solution of the variational inequality. Taking $v = u + \varphi$, $\varphi \in C_0^\infty(0,1)$ with $\varphi \geq 0$ in $(0,1)$ yields

$$\int_0^1 (u'\varphi' + \varphi)\,\mathrm{d}x \geq 0.$$

Since φ is an arbitrary positive function, the inequality implies that

$$-u'' + 1 \geq 0$$

in the sense of distributions. Further, suppose that the solution u is smooth enough, e.g. $u \in H^2(0,1)$. Then $-u'' + 1 \geq 0$ is satisfied, a.e. in (0,1).

By Sobolev's imbedding theorems, $H^1(0,1)\subset C[0,1]$. Then there exists an interval $(0,\delta)$, $\delta<1$, such that

$$u(x)\geq \epsilon \text{ in } x\in(0,\delta)$$

for an arbitrary given number $\epsilon>0$. For any function $\varphi\in C_0^\infty(0,\delta)$, there exists a constant $\hat{\epsilon}$ such that

$$(u\pm\hat{\epsilon}\varphi)(x)\geq 0 \text{ in } (0,\delta).$$

Extending φ to $(0,1)$ by zero outside of $(0,\delta)$, and substitution of $u\pm\hat{\epsilon}\varphi$ into the variational inequality leads to the conclusion that

$$\int_0^\delta (u'\varphi'+\varphi)\,\mathrm{d}x=0$$

i.e.

$$-u''+1=0 \text{ in } (0,\delta)$$

in the sense of distributions. Let φ_δ be a C^∞-function in $(0,\delta)$ such that $\varphi_\delta(0)=1/4$, $\varphi_\delta(\delta)=0$ and $u(x)\geq\varphi_\delta(x)\geq 0$. Taking $v=\varphi_\delta$ in $(0,\delta)$ and $v=0$ in $(\delta,1)$ implies

$$u'u|_\delta+\int_\delta^1 (u'u'+u)\,\mathrm{d}x\leq 0.$$

Taking $v=2u-\varphi_\delta$ in $(0,\delta)$ and $v=2u$ in $(\delta,1)$ shows that

$$u'u|_\delta+\int_\delta^1 (u'u'+u)\,\mathrm{d}x\geq 0.$$

Thus, we have

$$u'u|_\delta+\int_\delta^1 (u'u'+u)\,\mathrm{d}x=0$$

i.e.

$$(-u''+1)u=0 \quad \text{in} \quad (\delta,1).$$

Combining all results obtained above, the solution of the variational inequality satisfies the system

$$\left.\begin{array}{l} u\geq 0 \\ (-u''+1)u=0 \\ -u''+1\geq 0 \\ u(0)=1/4 \quad \text{and} \quad u(1)=0 \end{array}\right\} \text{ in } \quad (0,1)$$

provided that u is smooth enough; e.g. $u\in H^2(0,1)$.

We next make an important observation: the above system defines a natural partition of the domain $[0,1]$ into the subsets

$$\Omega^+=\{x\in[0,1]\colon u(x)>0\} \quad \text{and} \quad \Omega^0=\{x\in[0,1]\colon u(x)=0\}.$$

The point P of intersection, $P = \bar{\Omega}^+ \cap \bar{\Omega}^0$, defines a *free boundary* in the domain of the solution u and

$$\left.\begin{aligned} -u''+1=0 \quad &\text{in} \quad [0,P)=\Omega^+ \\ u=0 \quad &\text{in} \quad [P,1]=\Omega^0. \end{aligned}\right\}$$

In this case $P = 1/\sqrt{2}$. We can easily prove that there is only one free boundary P in $(0, 1)$. Suppose that p_1 and p_2 are free boundaries in $(0, 1)$ such that

$$\left.\begin{aligned} -u''+1=0 \quad &\text{in} \quad (p_1,p_2) \\ u(p_1)=u(p_2)=0. \end{aligned}\right\}$$

Then $u(x)=(1/2)(x-p_1)(x-p_2)$ in (p_1, p_2). This clearly satisfies the condition $u(x)<0$ in (p_1, p_2), i.e. $u \notin K$. Therefore, such p_1 and p_2 do not exist because of the constraint $u(x) \geq 0$ in $(0, 1)$. It is also worthwhile to note that if another boundary condition is imposed, the free boundary P may not occur. For example, if $K' = \{v \in H^1(0, 1)$: $v(0) = 1$, $v(1) = 0$, $v(x) \geq 0$ in $(0, 1)\}$, then the solution is

$$u(x)=\frac{1}{2}\left(x-\frac{3}{2}\right)^2-\frac{1}{8} \quad \text{in} \quad (0,1). \qquad \square$$

The importance of the elementary ideas just described is that they can be easily extended to very abstract situations involving operators defined on closed convex subsets of linear topological spaces. Indeed, if $A: K \to \mathscr{U}'$ is an operator defined on a non-empty closed convex set in a real linear topological space $\mathscr{U}$, the abstract problem of finding $u \in K$ such that, for given $f \in \mathscr{U}'$,

$$\langle A(u)-f, v-u\rangle \geq 0 \quad \forall v \in K \tag{1.5}$$

is called a *variational inequality* for the operator A (here $\langle \cdot, \cdot \rangle$ denotes duality pairing on $\mathscr{U}' \times \mathscr{U}$). The operator A need not be linear or even monotone and it need not be derivable from a potential functional $F: K \to \mathbf{R}$.

Although inequalities of the type (1.5) arise naturally in problems of minimization of convex differentiable functionals on convex sets, we will show that similar inequalities characterize minima of non-differentiable functionals as well. Thus, the theory of variational inequalities combines many of the elements of monotone operator theory and convex analysis in a way that generalizes both and has many significant applications in theoretical mechanics.

Our aim in this chapter is to give a brief account of the general theory of variational inequalities. Following this introduction, we describe a number of properties of variational inequalities on Hilbert and finite dimensional spaces. We then prove a general existence theorem for variational inequalities involving pseudomonotone operators defined on subsets of reflexive Banach spaces.

1.2 *Some preliminary results*

We will establish some very general results on abstract variational inequalities in the next section. However, in order to reinforce some of the geometrical concepts associated with certain types of variational inequalities and to record some preliminary results which are useful in studies of more general theorems, we will consider here several aspects of the theory which are clearer in a more restricted setting.

Let us first point out that the elementary example of minimizing a real-valued function of a real variable subject to a convex constraint which we sketched briefly in Section 1.1 is immediately extendable to a rather broad class of variational problems involving functionals on convex sets in normed linear spaces.

Theorem 1-2.1. Let K be a non-empty, closed, convex subset of a normed linear space $\mathscr{U}$ and $F: K \to \mathbf{R}$ a real Gâteaux-differentiable functional defined on K. Then any $u \in K$ which is a minimizer of F is characterized as a solution of the variational inequality

$$\langle DF(u), v-u\rangle \geq 0 \quad \forall v \in K. \tag{2.1}$$

If, in addition, F is convex, then any solution of (2.1) is also a minimizer of F, i.e.

$$F(u) \leq F(v) \quad \forall v \in K. \tag{2.2}$$

Proof. Since K is convex, $u + \theta(v-u) \in K$ for any $\theta \in [0, 1]$ and u, $v \in K$. If u is a minimizer of K, $F(u + \theta(v-u)) \geq F(u)$. Hence $\forall v \in K$

$$\lim_{\theta \to 0^+} \frac{1}{\theta} [F(u + \theta(v-u)) - F(u)] = \langle DF(u), v - u \rangle \geq 0.$$

If F is Gâteaux differentiable and convex

$$F(\theta v + (1-\theta)u) \leq \theta F(v) + (1-\theta)F(u) = F(u) + \theta(F(v) - F(u))$$

so that

$$F(v) - F(u) \geq \frac{1}{\theta}[F(u + \theta(v-u)) - F(u)].$$

Taking the limit as $\theta \to 0^+$ gives

$$F(v) - F(u) \geq \langle DF(u), v - u \rangle.$$

Thus, if u satisfies (2.1), $F(u) \leq F(v)$ for any $v \in K$. □

The question of existence of solutions to (2.1) is more difficult. Since (2.1) and (2.2) are "equivalent" for convex F, we can, in this special case, write down sufficient conditions for existence by simply calling on the existence theorems for minimizers of differential functionals.

First let us record the generalized Weierstrass minimization theorem:

Theorem 1-2.2. Let $\mathcal{U}$ be a reflexive Banach space and K a non-empty closed convex subset of $\mathcal{U}$. Let $F: K \to \mathbf{R}$ be a functional defined on K which is *weakly lower semicontinuous*, i.e.

If $\{u_m\} \in K$ converges weakly to $u \in K$, then

$$\liminf_{m \to \infty} F(u_m) \geq F(u).$$

Then F is bounded below on K and attains its minimum value on K whenever either of the following conditions hold:

(i) K is bounded, or

(ii) F is *coercive*, i.e.

$$F(v) \to +\infty \quad \text{as} \quad \|v\|_{\mathcal{U}} \to \infty.$$

Proof. First suppose that K is bounded but, contrary to the assertion, F is not bounded below on K. Then we can choose $\{u_m\} \in K$ so that $u_m \rightharpoonup u$ weakly, but $\liminf_{m\to\infty} F(u_m) \geq F(u)$, a contradiction. Hence, F is bounded below. Let $\mu_0 = \inf\{F(v): v \in K\}$ and let $\{u_k\}$ be such that $\mu_0 = \lim_{k\to\infty} F(u_k)$. Since K is bounded, $\{u_k\}$ contains a subsequence $\{u_{k'}\}$ which converges weakly to an element u in K. Hence $\mu_0 \leq F(u)$ and $\liminf_{k'\to\infty} F(u_{k'}) = \mu_0 \geq F(u)$, from which we conclude that $\mu_0 = F(u)$.

Next, suppose that K is unbounded but that F is coercive. Then for r a sufficiently large positive number, $F(v) > F(v_0)$, $v_0 \in K$, for $\|v\|_{\mathcal{U}} > r$. The ball $B_r = \{v \in K: \|v\|_{\mathcal{U}} \leq r\}$ is closed, bounded, and convex. Hence F attains its minimum on $K \cap B_r$. However, $\inf\{F(v): v \in K \cap B_r\} = \inf\{F(v): v \in K\}$, so that theorem is proved. □

We remark that the conclusions of this theorem also hold if K is only weakly sequentially closed (i.e. any weakly convergent sequence in K has as its limit an element of K), but every closed convex set in a reflexive Banach space is necessarily weakly sequentially closed.

Thus, for coercive F, we need only establish sufficient conditions for F to be weakly lower semicontinuous in order to guarantee the existence of minimizers on closed convex sets. There are several conditions we could impose which are sufficient to guarantee weak lower semicontinuity (see, e.g. Ekeland and Teman[23] or Vainberg[24]), but that which is used most frequently in convex analysis for differentiable F is that F be convex. To see this, we make use of the easily verified fact that the following are equivalent

(i) F is convex on K.

(ii) $F(v)-F(u)\geq\langle DF(u), v-u\rangle \quad \forall u, v\in K$.

(iii) DF is monotone, i.e.

$$\langle DF(u)-DF(v), u-v\rangle\geq 0 \quad \forall u, v\in K \tag{2.3}$$

where K is a non-empty, closed convex subset of a reflexive Banach space. Thus, if $\{u_m\}$ is a sequence drawn from K which converges weakly to u, $\liminf_{m\to\infty}(F(u_m)-F(u))\geq \liminf_{m\to\infty}\langle DF(u), u_m-u\rangle=0$. Hence $\liminf_{m\to\infty} F(u_m)\geq F(u)$.

Combining the above observations with Theorem 1-2.2, we have:

Theorem 1-2.3. Let F be a Gâteaux-differentiable, convex, coercive functional defined on a non-empty, closed, convex set K in a reflexive Banach space. Then there exists at least one minimizer u of F on K. Moreover, u is characterized as a solution of the variational inequality (2.1). □

Remark 1-2.1. If F is Gâteaux differentiable and *strictly* convex on K (i.e. $F(\theta u+(1-\theta)v)<\theta F(u)+(1-\theta)F(v)$, $\forall u, v\in K$, $u\neq v$, $\theta\in(0,1)$) then the minimizer of F is unique. Also DF is strictly monotone. □

Remark 1-2.1. If F is not convex but is Gâteaux differentiable, weaker conditions can be imposed in order to guarantee weak lower semicontinuity. For example, it is shown in Oden and Kikuchi[25] that F is weakly lower semicontinuous if

$$\langle DF(u)-DF(v), u-v\rangle\geq -H(\mu, \|u-v\|_{\mathcal{V}})$$

where H is a non-negative continuous function, $\|u\|_{\mathcal{U}}\leq\mu$, $\|v\|_{\mathcal{U}}\leq\mu$, $\mathcal{V}$ is a space on which $\mathcal{U}$ is compactly embedded, and $\lim_{\theta\to 0^+}(1/\theta)H(x,\theta y)=0$, $x, y\in\mathbf{R}^+$. □

Remark 1-2.3. Our results apply to functionals with values in $\mathbf{R}$. However, extensions of most of these results to functionals taking values in the extended real line $\mathbf{R}\cup\{+\infty\}$ or $\mathbf{R}\cup\{-\infty\}\cup\{+\infty\}$ are straightforward. Then we speak of *proper* functionals whenever $F(u)\not\equiv+\infty$ for all u and the *effective domain* of F, $\text{eff. dom } F=\{u\in K\colon F(u)<+\infty\}$. □

Since DF is, in general, a nonlinear operator, (2.1) represents a non-linear inequality in u. However, if F is convex and DF is hemicontinuous, a simple alternate formulation, equivalent to (2.1), that involves a linear inequality in u can be derived. Indeed, if F is convex and Gâteaux differentiable, DF is necessarily monotone, so that

$$\langle DF(u)-DF(v), u-v\rangle\geq 0.$$

But this implies that

$$\langle DF(v), v-u\rangle\geq\langle DF(u), v-u\rangle\geq 0 \quad \forall v\in K$$

whenever u satisfies (2.1). Conversely, suppose

$$\langle DF(w), w-u\rangle\geq 0 \quad \forall w\in K. \tag{2.4}$$

Set $w=u+\theta(v-u)$, $\theta\in(0,1)$, divide by θ and take the limit as $\theta\to 0$. This yields (2.1). Hence, we have proved:

Theorem 1-2.4. Let (2.3) hold and DF be hemicontinuous. Then problems (2.3) and (2.4) are equivalent. □

Let us interpret Theorems 1-2.1 and 1-2.4 geometrically in the case that $\mathcal{U}$ is a finite dimensional Euclidean space $\mathbf{R}^m$. In this case, $DF(u)$ is the gradient of F at the point $u=(u_1,\ldots,u_m)$ and

$$\langle DF(u), w\rangle = DF(u)\cdot w = \sum_{k=1}^{m} \frac{\partial F(u)}{\partial v_k} w_k$$

is the derivative of F at u in the direction of $w=(w_1,\ldots,w_m)$. The vector $DF(u)$ is oriented toward the direction of maximum increase of F at u and is normal to the surfaces $F=\text{constant}$.

Since F is convex, its level sets

$$L_\lambda = \{v\in K\colon F(v)\le\lambda\},\quad \lambda\in\mathbf{R}$$

are convex. Then, all vectors w such that

$$\langle DF(u), w-u\rangle\ge 0$$

define directions of increasing (or non-decreasing) F from the point u. This means that

$$F(w)\ge F(u).$$

Thus, if (2.1) holds, (2.2) must be valid.

Conversely, let $M(v)$ denote the set of all $w\in K$ from which $v\in\mathcal{U}$ is seen in a direction of increasing F, i.e.

$$M(v)=\{w\in K\colon \langle DF(w), v-w\rangle\ge 0\}.$$

If $u\in K$ is a minimizer of F on K

$$u\in \bigcap_{v\in K} M(v),$$

i.e.

$$\langle DF(u), v-u\rangle\ge 0,\quad \forall v\in K.$$

Thus, we have interpreted Theorem 1-2.1.

Similarly, let $N(v)$ denote the set of all $w\in K$ from which $v\in\mathcal{U}$ is seen in a direction of decreasing F, i.e.

$$N(v)=\{w\in K\colon \langle DF(w), v-w\rangle\le 0\}.$$

If $u\in K$ is a minimizer of F on K, $N(u)$ is identified with the set K, i.e.

$$\langle DF(w), u-w\rangle\le 0,\quad \forall w\in K.$$

Conversely, if (2.4) holds, $u\in K$ is seen in a direction of decreasing F from every point $v\in K$. This means that u is a minimizer of F on K.

1.3 *Projections in Hilbert spaces*

Another geometrical interpretation of some importance arises in the case in which $\mathcal{U}$ is a real Hilbert space $\mathcal{H}$ with an inner product $(\cdot,\cdot)$. Suppose that we wish to find the minimum distance between a given $f\in\mathcal{H}$ and a closed convex set $K\subset\mathcal{H}$, i.e. we wish to find $u\in K$ such that the (squared) distance function

$$F(v)=\|f-v\|^2,\quad \|v\|^2=(v,v)$$

is a minimum. Clearly, F is a lower semicontinuous functional defined on a closed convex set in a reflexive Banach space (any Hilbert space is reflexive). Hence, it attains its minimum on K. We define such a minimizer, denoted by $P_K f$, as the projection of f into K. Moreover, F is strictly convex on K so that the minimizer is unique and is characterized by

$$0 \le \langle DF(u), v-u\rangle; \quad \langle DF(u), v-u\rangle = 2(u-f, v-u)$$

i.e.

$$(u-f, v-u) \ge 0 \quad \forall v \in K. \tag{3.1}$$

Geometrically, (3.1) indicates that the angle between the vector $f-u$ and any vector $v-u$ in K is obtuse, as illustrated in Fig. 1.2.

The unique vector u satisfying (3.1) is thus the projection of f into K.

Clearly, if $f \in K$, then $P_K f = f$. Moreover, if K is a linear subspace of $\mathscr{H}$, then P_K is still surjective and

$$(f - P_K f, v) = 0 \quad \forall v \in K$$

i.e. the error $f - P_K f$ is, in this case, orthogonal to K. Note that if K is only a convex subset of $\mathscr{H}$, P_K need not be linear. However, it is continuous. Indeed, if $f_m \to f$ strongly in $\mathscr{H}$, then

$$\begin{aligned}\|P_K f_m - P_K f\|^2 &= (P_K f_m - P_K f, P_K f_m - P_K f)\\ &= (f - P_K f, P_K f_m - P_K f) - (P_K f_m - f_m, P_K f - P_K f_m)\\ &\quad + (f_m - f, P_K f_m - P_K f).\end{aligned}$$

In view of (3.1), the first two terms on the right-hand side of this last equality are seen to be non-positive. Thus, use of Schwartz's inequality reveals that

$$\|P_K f_m - P_K f\| \le \|f_m - f\|. \tag{3.2}$$

Hence P_K is continuous.

Example 1-3.1. Let $\mathscr{H} = \mathbf{R}^m$, and let the inner product $(\cdot,\cdot)$ be defined by

$$(u, v) = u_1 v_1 + \cdots + u_m v_m.$$

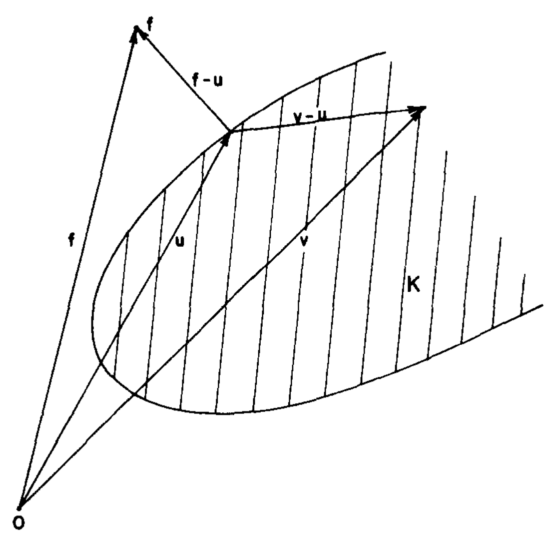


Fig. 1.2. Minimization of the distance from an arbitrary point f to a convex set K.

Suppose that

$$K = \{v \in \mathbf{R}^m : v_i \geq 0, i = 1, \ldots, m\}.$$

Then the projection P_K can be explicitly represented by

$$(P_K f)_i = \max(f_i, 0). \tag{3.3}$$

To show this, let us consider the (variational) inequality

$$u \in K: (u - f, v - u) \geq 0, \quad \forall v \in K. \tag{3.4}$$

Taking $v = (u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m)$, $v_i \geq 0$, in (3.4) yields

$$u_i \geq 0: (u_i - f_i)(v_i - u_i) \geq 0, \quad \forall v_i \geq 0. \tag{3.5}$$

It is easy to show that $u_i = \max(f_i, 0)$ satisfies the inequality (3.5). Indeed, if $f_i < 0$, $u_i = \max(f_i, 0) = 0$. Then

$$(u_i - f_i)(v_i - u_i) = (-f_i)v_i \geq 0, \quad v_i \geq 0.$$

If $f_i \geq 0$, $u_i = \max(f_i, 0) = f_i$. Then

$$(u_i - f_i)(v_i - u_i) = 0, \quad \forall v_i \geq 0.$$

Thus $u_i = \max(f_i, 0)$ is a solution of the inequality (3.5). Suppose that (3.5) has two solutions, say $\bar{u}_i$ and $\hat{u}_i$, i.e.

$$\bar{u}_i \geq 0: (\bar{u}_i - f_i)(v_i - \bar{u}_i) \geq 0, \quad \forall v_i \geq 0$$

$$\hat{u}_i \geq 0: (\hat{u}_i - f_i)(v_i - \hat{u}_i) \geq 0, \quad \forall v_i \geq 0.$$

Substituting $v_i = \hat{u}_i$ in the first inequality, $v_i = \bar{u}_i$ in the second inequality, and adding two inequalities, we have

$$(\bar{u}_i - \hat{u}_i)(\bar{u}_i - \hat{u}_i) \leq 0.$$

This implies $\bar{u}_i = \hat{u}_i$, i.e. the uniqueness of the solution of (3.5). Therefore

$$u_i = (P_K f)_i = \max(f_i, 0).$$

Applying the same arguments for

$$M = \{v \in \mathbf{R}^m : v_i \leq 0, \quad i = 1, \ldots, m\} \tag{3.6}$$

we have

$$(P_M f)_i = \min(f_i, 0). \tag{3.7}$$

Combining (3.3) and (3.7), for

$$N = \{v \in \mathbf{R}^m : a_i \leq v_i \leq b_i, \quad i = 1, \ldots, m\} \tag{3.8}$$

we have

$$(P_N f)_i = \min(\max(a_i, f_i), b_i). \quad \square \tag{3.9}$$

The properties of the projection P_K suggest an alternative to the formulation that is of some importance in the approximation and numerical analysis of variational inequalities as well as in proving the existence of solutions in certain special cases. Suppose that $u \in K$ is the minimizer of a convex Gâteaux-differentiable functional $F: K \to \mathbf{R}$ on a non-empty closed convex set K of a real Hilbert space $\mathscr{H}$. Then, by Theorem 1-2.1, u satisfies

$$\langle DF(u), v - u \rangle \geq 0 \quad \forall v \in K$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing on $\mathscr{H}' \times \mathscr{H}$. By Riesz representation theorem, every continuous linear functional on $\mathscr{H}$ can be identified with the element of $\mathscr{H}$, i.e.

$$\langle f, v \rangle = (\pi f, v), \quad \forall f \in \mathscr{H}' \quad \text{and} \quad v \in \mathscr{H}$$

where π is the Riesz map form $\mathscr{H}'$ into $\mathscr{H}$. Consequently

$$(\pi DF(u), v - u) \geq 0$$

and, for any $\rho > 0$

$$(u - u + \rho \pi DF(u), v - u) \geq 0, \quad \forall v \in K.$$

Therefore, $u \in K$ satisfies the equation

$$u = P_K(u - \rho \pi DF(u))$$

where P_K is the projection map of $\mathscr{H}$ onto K. This means that a minimizer $u \in K$ of F on K is a fixed point of the operator T defined by

$$T(\cdot) = P_K(I - \rho \pi DF)(\cdot) \tag{3.10}$$

As an application of these ideas, consider an operator $A: K \subset \mathscr{H} \to \mathscr{H}$, $\mathscr{H}$ being a real Hilbert space with an inner product $(\cdot, \cdot)$, which satisfies the conditions

$$\left.\begin{aligned} (A(u) - A(v), u - v) &\geq m\|u - v\|^2 \\ (A(u) - A(v), w) &\leq M\|u - v\|\|w\| \\ \forall u, v, w &\in K \end{aligned}\right\} \tag{3.11}$$

where m and M are positive constants and $\|\cdot\|^2 = (\cdot,\cdot)$. We define a new map $T: \mathscr{H} \to K$ by

$$T(w) = P_K(I - \rho A)(w), \quad \rho > 0. \tag{3.12}$$

Then, from (3.1)

$$(T(w) - (I - \rho A)(w), v - T(w)) \geq 0 \quad \forall v \in K.$$

Furthermore, if T has a fixed point u, then this inequality reduces to

$$(A(u), v - u) \geq 0 \quad \forall v \in K \tag{3.13}$$

i.e. a fixed point of T, if it exists, is a solution of the variational inequality (3.13).

We will show that if the number ρ in (3.12) is chosen properly that T is, in fact, a contraction mapping, i.e. that there exists a k, $0 < k < 1$, such that

$$\|T(u) - T(v)\| \leq k\|u - v\|. \tag{3.14}$$

Indeed, using (3.11) and (3.12)

$$\begin{aligned}\|T(u)-T(v)\|^2 &\le \|(I-\rho A)(u)-(I-\rho A)(v)\|^2\\ &= (u-\rho A(u)-v+\rho A(v), u-\rho A(u)-v+\rho A(v))\\ &= \|u-v\|^2-2\rho(A(u)-A(v), u-v)+\rho^2\|A(u)-A(v)\|^2\\ &\le (1-2\rho m+\rho^2M^2)\|u-v\|^2.\end{aligned}$$

Thus k in (3.14) satisfies $0<k<1$ whenever

$$k^2 = 1-2\rho m+\rho^2 M^2 \quad \text{and} \quad 0<\rho<\frac{2m}{M^2}. \tag{3.15}$$

Since we can always choose ρ so as to satisfy (3.15), T of (3.12) can always be constructed so as to satisfy (3.14); hence, there exists a unique solution to the variational inequality (3.13). Moreover, the solution to (3.13) can be obtained as the strong limit of the sequence generated by the classical iterative process

$$u^{n+1} = T(u^n) = P_K(u^n - \rho A(u^n)) \tag{3.16}$$

whenever ρ satisfies (3.15).

In summary, we have:

Theorem 1-3.1. Let $\mathcal{H}$ be a real Hilbert space, K a non-empty closed convex subset of $\mathcal{H}$, and $A: K \to \mathcal{H}$ an operator satisfying (3.11). Then there exists a unique solution $u \in K$ of the variational inequality (3.13). □

1.4 *The Hartman–Stampacchia theorem*

In the previous section, we have proved constructively that the map T defined by (3.12)

$$T(w) = P_K(I-\rho A)(w), \quad \rho>0 \tag{4.1}$$

has a unique fixed point for suitable $\rho>0$ when the condition (3.15) holds. Here we will show that the map T has a fixed point for every $\rho>0$ under the continuity condition of A, if the space $\mathcal{U}$ is finite dimensional.

Let $\mathcal{U}$ be a finite dimensional space, and let K be a non-empty compact convex subset of $\mathcal{U}$. We first recall the modified Brouwer fixed-point theorem:

Proposition 1-4.1. Let K be a non-empty compact convex subset of a finite dimensional space $\mathcal{U}$. Suppose that a map $T: K \to K$ is continuous. Then there exists at least one fixed point $u \in K$ such that

$$u = T(u). \quad \square$$

Because of the projection P_K, the map T defined by (4.1) satisfies the condition that $T: K \to K$. Furthermore, continuity of A and P_K implies the continuity of T. Therefore, by the Brouwer fixed-point theorem, there exists at least one fixed point $u \in K$ such that

$$u = T(u) = P_K(u-\rho A(u)), \quad \rho>0. \tag{4.2}$$

By the characterization of the projection P_K, (4.2) is equivalent to the inequality

$$(u-(u-\rho A(u)), v-u) \ge 0, \quad \forall v \in K$$

i.e.

$$(A(u), v-u) \ge 0, \quad \forall v \in K.$$

Therefore, we can conclude that:

Theorem 1-4.1. Let K be a non-empty compact convex subset of a finite dimensional space $\mathcal{U}$. Suppose that A is a continuous map of $\mathcal{U}$ into itself. Then there exists at least one solution $u \in K$ to the variational inequality

$$u \in K: (A(u), v-u) \geq 0, \quad \forall v \in K \tag{4.3}$$

where $(\cdot, \cdot)$ is the inner product of $\mathcal{U}$. □

The above theorem is due to Hartman and Stampacchia [26].

Remark 1-4.1. We note that the Brouwer fixed-point theorem can be obtained from the Hartman–Stampacchia theorem. Let K be a non-empty compact convex subset of a finite dimensional space $\mathcal{U}$, and let T be a continuous map of K into itself. Then $A = I - T$ is continuous on $\mathcal{U}$. By the Hartman–Stampacchia theorem, there exists at least one $u \in K$ such that

$$(A(u), v-u) \geq 0, \quad \forall v \in K.$$

Since $T(u) \in K$, we have

$$(u - T(u), T(u) - u) \geq 0.$$

This means that $u = T(u)$.

Thus, the Brouwer theorem precipitates as a corollary to the Hartman–Stampacchia theorem. □

1.5 *Variational inequalities of the second kind*

The variational inequalities described up to this point involve the search for elements u in a closed convex set $K \subset \mathcal{U}$ such that $\langle A(u), v-u\rangle \geq 0$ for all $v \in K$, A being an operator from K into $\mathcal{U}'$. We will call such problems ***variational inequalities of the first kind.*** Here we take $F: \mathcal{U} \to \bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$. (Recall Remark 1-1.3.).

It is possible to reformulate such inequalities so that they are defined on the totality of the space $\mathcal{U}$ rather than K by introducing an *indicator functional* ψ_K defined by

$$\psi_K: \mathcal{U} \to \bar{\mathbf{R}}, \quad \psi_K(v) = \begin{cases} +\infty & \text{if} \quad v \notin K \\ 0 & \text{if} \quad v \in K \end{cases} \tag{5.1}$$

where K is a non-empty closed convex subset of $\mathcal{U}$. If $A: \mathcal{U} \to \mathcal{U}'$, it is clear that if u satisfies

$$u \in K: \langle A(u), v-u\rangle \geq 0, \quad \forall v \in K \tag{5.2}$$

then u also is a solution of

$$\langle A(u), v-u\rangle + \psi_K(v) - \psi_K(u) \geq 0, \quad \forall v \in \mathcal{U}. \tag{5.3}$$

Conversely, any solution $u \in K$ of (5.3) is also a solution of (5.2).

The variational problem (5.3) is an example of a ***variational inequality of the second kind.*** Note that the indicator function ψ_K is convex but not differentiable. We will now show that we can enlarge on the class of variational problems of this type by considering minimization problems involving general convex non-differentiable functionals. The typical setting for the class of variational inequalities of the second kind which we consider here is as follows:

$$\left.\begin{array}{l} \mathcal{U} \text{ is a reflexive Banach space.} \\ F: \mathcal{U} \to \bar{\mathbf{R}} \text{ is a proper Gâteaux-differentiable convex functional.} \\ \phi: \mathcal{U} \to (-\infty, \infty] \ (\phi \not\equiv +\infty) \text{ a convex functional not necessarily differentiable.} \end{array}\right\} \tag{5.4}$$

We wish to find minima of the functional

$$G = F + \phi \tag{5.5}$$

i.e. we wish to find $u \in \mathcal{U}$ such that

$$G(u) \leq G(v) \quad \forall v \in \mathcal{U}. \tag{5.6}$$

The characterization of such minima as solutions of a variational inequality is laid down in the following theorem:

Theorem 1-5.1. Let (5.4) hold. Then any minimizer $u \in \mathcal{U}$ of the functional G of (5.5) satisfies

$$\langle DF(u), v-u\rangle + \phi(v) - \phi(u) \geq 0 \quad \forall v \in \mathcal{U}. \tag{5.7}$$

Conversely, if $u \in \mathcal{U}$ satisfies (5.7), then it minimizes G. □

Proof. Suppose u is a minimizer of G and $\theta \in (0, 1)$. Since F and ϕ are convex

$$\begin{aligned} F(u) + \phi(u) &\leq F(u + \theta(v-u)) + \phi(u + \theta(v-u)) \\ &\leq F(u + \theta(v-u)) + \theta\phi(v) + (1-\theta)\phi(u) \quad \forall v \in \mathcal{U}. \end{aligned}$$

Hence

$$\frac{1}{\theta}[F(u + \theta(v-u)) - F(u)] \geq \phi(u) - \phi(v)$$

so that (5.7) is obtained in the limit as $\theta \to 0^+$.

Next, suppose that u satisfies (5.7). Since F is convex

$$\langle DF(u), v-u\rangle \leq F(v) - F(u) \quad \forall v \in \mathcal{U}$$

Thus, for every $v \in \mathcal{U}$, $F(v) - F(u) + \phi(v) - \phi(u) \geq 0$, or $G(u) \leq G(v)$, $\forall v \in \mathcal{U}$. □

Inequality (5.7) is an example of a variational inequality of the second kind. Such inequalities need not involve gradients of differentiable functions. We will study variational inequalities of this type in more detail later.

1.6 *A general theorem on variational inequalities*

We will now consider a general existence theory for solutions of abstract variational inequalities on Banach spaces.† Let

$$\left.\begin{array}{l} \mathcal{U} \text{ be a separable‡ reflexive real Banach space.} \\ K \subset \mathcal{U} \text{ a non-empty closed convex subset of } \mathcal{U}. \\ A\colon K \to \mathcal{U}' \text{ an operator defined from } K \text{ into the (strong)} \\ \text{topological dual } \mathcal{U}' \text{ of } \mathcal{U}. \end{array}\right\} \tag{6.1}$$

We will establish conditions under which solutions exist to the variational inequality: find $u \in K$ such that

$$\langle A(u), v-u\rangle \geq 0, \quad v \in K. \tag{6.2}$$

Theorem 1-6.1. Let conditions (6.1) hold and let the operator $A\colon K \to \mathcal{U}'$ be

(i) Bounded.

(ii) Pseudomonotone: if $\{u_m\}$ is a sequence from K converging weakly to $u \in K$ and if $\limsup_{m\to\infty} \langle A(u_m), u_m - u\rangle \leq 0$, then

$$\liminf_{m\to\infty} \langle A(u_m), u_m - v\rangle \geq \langle A(u), u-v\rangle \quad \forall v \in K.$$

†These results can be further generalized to inequalities on linear topological spaces. See Brezis[4].

‡The assumption of separability is not essential and is introduced only for simplicity in certain arguments to follow.

Moreover, suppose that

(iii) K is bounded.

Then there exists at least one solution of the variational inequality (6.2).

Proof. The proof follows standard compactness arguments common in pseudomonotone operator theory, except that now we must resolve the finite dimensional problem using the Hartman–Stampacchia Theorem 1-4.2.

Suppose that $[w_1, w_2, \ldots]$ is a countable everywhere dense set in $\mathcal{U}$ and $[w_1, w_2, \ldots, w_m]$ is a basis for a finite dimensional subspace $\mathcal{U}_m$ of $\mathcal{U}$. The family of such subspaces obtained as m takes on all positive integers is such that $\bigcup_{m\geq 1} \mathcal{U}_m$ is everywhere dense in $\mathcal{U}$. Without loss of generality, suppose $0 \in K$ and consider the family of sets

$$\left.\begin{aligned} K_m &= \mathcal{U}_m \cap K \quad m \geq 1 \\ \overline{\bigcup_{m\geq 1} K_m} &= K. \end{aligned}\right\} \tag{6.3}$$

Each set K_m is a non-empty bounded closed convex subset of $\mathcal{U}$. Moreover, $A: K_m \to \mathcal{U}'_m$ is continuous since A is pseudomonotone and bounded. Thus, by Theorem 1-4.1, there exists a solution $u_m \in K_m$ of the finite dimensional variational inequality

$$\langle A(u_m), v_m - u_m \rangle \geq 0 \quad \forall v_m \in K_m. \tag{6.4}$$

Now we recall that any closed, bounded, convex set K in a reflexive Banach space is weakly sequentially compact. Hence, if $\{u_m\}$ is a sequence of solutions of the finite dimensional problems (6.4) obtained as $m \to \infty$, there exists a subsequence $\{u_{m_k}\}$ which converges weakly to an element $u \in K$. In view of (6.4), $\liminf_{m\to\infty} \langle A(u_m), v - u_m \rangle \geq 0$, $\forall v \in K$, so that, putting $u = v$ reveals that

$$\limsup_{m_k\to\infty} \langle A(u_{m_k}), u_{m_k} - u \rangle \leq 0. \tag{6.5}$$

Hence, by (6.5) and the pseudomonotonicity of A, we have for any $v \in K$

$$0 \geq \liminf_{m_k\to\infty} \langle A(u_{m_k}), u_{m_k} - v \rangle \geq \langle A(u), u - v \rangle$$

which implies that $\langle A(u), v - u \rangle \geq 0$, $\forall v \in K$. □

The more interesting cases involve sets K which are unbounded. The theory of pseudomonotone operator equations suggests that what is needed to complete an existence theorem for (6.2) for unbounded K is that A be coercive. This, in fact, is quite true, but the structure of a variational inequality, as opposed to an equality, provides for some alternative weaker forms of coerciveness. We first establish a useful lemma:

Lemma 1-6.1. Let (6.1) hold. Then a necessary and sufficient condition for a solution to exist to the variational inequality

$$u \in K: \langle A(u), v - u \rangle \geq 0 \quad \forall v \in K \tag{6.6}$$

is that there exist a real number $r > 0$ such that at least one solution of the inequality

$$u_r \in K_r: \langle A(u_r), v - u_r \rangle \geq 0 \quad \forall v \in K_r \tag{6.7}$$

satisfies

$$\|u_r\|_{\mathcal{U}} < r \tag{6.8}$$

where $K_r = K \cap \overline{B_r(0)}$ and $\overline{B_r(0)}$ is the closed ball of radius r centered at the origin ($\overline{B_r(0)} = \{v \in \mathcal{U}: \|v\|_{\mathcal{U}} \leq r\}$).

Proof. If there exists a solution u of (6.6), we need only choose r so that $\|u\|_{\mathcal{U}} < r$ for u to satisfy (6.7). Conversely, if u_r satisfies (6.7) and (6.8) for some $r > 0$, then there is a $v \in K_r$ such that $v - u_r = \epsilon(w - u_r)$ for $w \in K$ and ϵ sufficiently small. Then $\langle A(u_r), v - u_r\rangle = \epsilon\langle A(u_r), w - u_r\rangle \geq 0$, $\forall w \in K$, i.e. u_r satisfies (6.6). □

Now a careful examination of the above lemma and a comparison with Theorem 1-6.1 reveals that solutions u_r always exist to (6.7), whenever conditions (i) and (ii) of Theorem 1-6.1 are satisfied, because K_r is convex, closed, and bounded. Thus, we need to furnish an additional condition on A that will guarantee that (6.8) holds. One such condition is

$$\left.\begin{array}{l}\text{There exists a } v_0 \in K \text{ and an } r > 0 \text{ with} \\ \|v_0\|_{\mathcal{U}} < r \text{ such that for all } v \in K \text{ with} \\ \|v\|_{\mathcal{U}} = r \text{ we have} \\ \qquad\qquad \langle A(v), v - v_0\rangle > 0.\end{array}\right\} \tag{6.9}$$

For suppose (6.9) holds and u_r is a solution of (6.7). Then if $\|u_r\|_{\mathcal{U}} = r$ and (6.9) holds, we have $\langle A(u_r), u_r - v_0\rangle > 0$, which contradicts (6.7). Hence, $\|u_r\|_{\mathcal{U}} < r$ when (6.9) holds.

Note that condition (6.9) is satisfied whenever the following coerciveness condition of A holds

$$\left.\begin{array}{l}\text{There exists a } v_0 \in K \text{ such that} \\ \dfrac{\langle A(v), v - v_0\rangle}{\|v\|} \to +\infty \quad \text{as} \quad \|v\| \to +\infty \\ \text{for } v \in K.\end{array}\right\} \tag{6.10}$$

It is important to realize the difference between surjectivity theorems for pseudomonotone operator equations and existence theorems for pseudomonotone variational inequalities. If f is *arbitrary* data given in $\mathcal{U}'$ and we wish to solve the problem of finding $u \in K$ such that

$$\langle A(u) - f, v - u\rangle \geq 0 \quad \forall v \in K \tag{6.11}$$

then condition (6.9) is *not* sufficient to conclude the existence of solutions to (6.11), assuming conditions (i) and (ii) of Theorem 1-6.1 hold. If only (i), (ii) and (6.9) hold, we will generally need to impose additional conditions on f in order to guarantee solutions to (6.11). This also means that if (6.9) holds and A *is not coercive in the sense of* (6.10), the existence of a solution to (6.11) may still be established provided we add a suitable condition on the choice of the data $f \in \mathcal{U}'$. Note, however, that the stronger coercivity conditions (6.10), together with (i) and (ii) of Theorem 1-6.1 are sufficient for the solvability of (6.11) for unbounded K.

We now summarize these results:

Theorem 1-6.2. Let conditions (6.1) hold with K unbounded. Let $A: K \to \mathcal{U}'$ satisfy the following conditions:

(i) A is bounded.

(ii) A is pseudomonotone.

(iii) A satisfies the weak coercivity condition (6.9) or the coercivity conditions (6.10).

Then there exists at least one solution $u \in K$ of the variational inequality (6.2).

Moreover, if conditions (6.1) and conditions (i) and (ii) above hold and if A is coercive in the sense of (6.10), then a solution exists to (6.11) for any $f \in \mathcal{U}'$. □

Many useful corollaries of Theorem 1-6.2 can be obtained by replacing condition (ii) by conditions which imply the pseudomonotonicity of A. For example:

Corollary 1-6.2. Let (6.1) hold, K being unbounded, and let $A: K \to \mathcal{U}'$ be bounded and coercive in the sense of (6.10). Then there exists at least one solution $u \in K$ of (6.11) if any one of the following conditions hold:

(i) $A: K \to \mathcal{U}'$ is monotone and hemicontinuous.

(ii) $A: K \to \mathcal{U}'$ satisfies.

$$\langle A(u) - A(v), u - v\rangle \geq -H(\mu, \|u - v\|_{V}) \quad \forall u, v \in B_\mu(0) \cap K \tag{6.12}$$

where $\mathcal{V}$ is a Banach space in which $\mathcal{U}$ is compactly embedded and $H: [0,\infty)\times[0,\infty)\to\mathbf{R}$ is a non-negative valued continuous function satisfying

$$\lim_{\theta\to 0^+}\frac{1}{\theta}H(x,\theta y)=0,\quad x,y\in[0,\infty) \tag{6.13}$$

(iii) $A: K\to\mathcal{U}'$ satisfies

$$\langle A(u)-A(v),u-v\rangle\geq-\langle B(u)-B(v),u-v\rangle\quad\forall u,v\in K \tag{6.14}$$

where $B: K\to\mathcal{U}'$ is a completely continuous operator.

(iv) $A: K\to\mathcal{U}'$ is expressible in the form $A(u)=A(u,u)$, where $(u,v)\to A(u,v)$ is a map from $K\times K$ into $\mathcal{U}'$ satisfying

(iv.1) $\forall u\in K$, $v\to A(u,v)$ is bounded and hemicontinuous

(iv.2) $\forall u,v\in B_\mu(0)\cap K$,

$$\langle A(u,u)-A(u,v),u-v\rangle\geq-H(\mu,\|u-v\|_{\mathcal{V}}) \tag{6.15}$$

where H is a function of the type described in (ii) above.

(iv.3) If $\{u_m\}$ is a sequence converging weakly to $u\in K$, then

$$\left.\begin{aligned}&\liminf_{m\to\infty}\langle A(v,u_m)-A(v,u),u_m-u\rangle\geq 0\quad\forall v\in K\\ \text{and}\qquad&\\&\liminf_{n\to\infty}\langle A(v,u_m)-A(v,u),w\rangle=0\quad\forall v,w\in K.\end{aligned}\right\} \tag{6.16}$$

□

Several other conditions could, of course, also be listed.

1.7 *Pseudomonotone variational inequalities of the second kind*

We recall from Theorem 1-5.1 that so-called variational inequalities of the second kind arise in minimization problems involving non-differentiable functionals. We will now describe some results for general inequalities of this type for pseudomonotone operators. The major theorem is as follows:

Theorem 1-7.1. Let $\mathcal{U}$ be a reflexive Banach space and $A:\mathcal{U}\to\mathcal{U}'$ a bounded, pseudomonotone operator. Let $\phi:\mathcal{U}\to(-\infty,\infty]$ ($\phi\not\equiv+\infty$) be a convex lower semicontinuous functional on $\mathcal{U}$. In addition, let the following condition hold

$$\left.\begin{aligned}&\text{There exists a } v_0\in\mathcal{U}\text{ and a real number}\\&r>0\text{ with }\|v_0\|_{\mathcal{U}}<r\text{ such that}\\&\qquad\langle A(v),v-v_0\rangle+\phi(v)-\phi(v_0)>0\\&\text{for all } v\in\mathcal{U}\text{ such that }\|v\|_{\mathcal{U}}=r.\end{aligned}\right\} \tag{7.1}$$

Then there exists at least one solution $u\in\mathcal{D}_\phi=\{v\in\mathcal{U}:\phi(v)<+\infty\}$ to the variational inequality

$$\langle A(u),v-u\rangle+\phi(v)-\phi(u)\geq 0\quad\forall v\in\mathcal{U}. \tag{7.2}$$

Proof. Let $B:(\mathcal{U}\times\mathbf{R})\to(\mathcal{U}\times\mathbf{R})'$ be an operator defined on the product space $\mathcal{U}\times\mathbf{R}$ by

$$B(v,\alpha)=(A(v),1);\quad v\in\mathcal{U},\quad\alpha\in\mathbf{R}.$$

The operator B is easily seen to be pseudomonotone on $\mathcal{U}\times\mathbf{R}$ because A is pseudomonotone:

Indeed, if $(u_m, \alpha_m) \to (u, \alpha)$ and $\limsup_{m\to\infty} \langle B(u_m, \alpha_m), (u_m, \alpha_m) - (u, \alpha)\rangle_{\mathcal{U}\times\mathbf{R}} \le 0$, then

$$\begin{aligned}
&\liminf_{n\to\infty} \langle B(u_m, \alpha_m), (u_m, \alpha_m) - (v, \beta)\rangle_{\mathcal{U}\times\mathbf{R}} \\
&\quad = \liminf_{m\to\infty} [\langle A(u_m), u_m - v\rangle + 1\cdot(\alpha_m - \beta)] \\
&\quad \ge \langle A(u), u - v\rangle + 1(\alpha - \beta) \\
&\quad = \langle B(u, \alpha), (u, \alpha) - (v, \beta)\rangle_{\mathcal{U}\times\mathbf{R}} \quad \forall (v, \beta) \in \mathcal{U}\times\mathbf{R}.
\end{aligned}$$

Since ϕ is proper and lower semicontinuous, the set

$$K = \{(v, \beta) \in \mathcal{U}\times\mathbf{R}: \beta \ge \phi(v)\}$$

is a non-empty closed convex subset of $\mathcal{U}\times\mathbf{R}$. Moreover, condition (7.1) for A implies that B satisfies condition (6.8) on $\mathcal{U}\times\mathbf{R}$; in fact

$$\begin{aligned}
\langle A(v), v - v_0\rangle + \beta - \beta_0 &= \langle (A(v), 1), (v - v_0, \beta - \beta_0)\rangle_{\mathcal{U}\times\mathbf{R}} \\
&= \langle B(v, \beta), (v, \beta) - (v_0, \beta_0)\rangle_{\mathcal{U}\times\mathbf{R}} > 0
\end{aligned}$$

for $\|(v_0, \beta_0)\|_{\mathcal{U}\times\mathbf{R}} < r$ and $\|(v, \beta)\|_{\mathcal{U}} = r$. Thus, from Theorem 1-6.2, there exists at least one solution $(u, \alpha) \in K$ of the variational inequality

$$\langle B(u, \alpha), (v, \beta) - (u, \alpha)\rangle_{\mathcal{U}\times\mathbf{R}} \ge 0 \quad \forall (v, \beta) \in K \tag{7.3}$$

or

$$\langle A(u), v - u\rangle + \beta - \alpha \ge 0.$$

If $v \notin \mathcal{D}_\phi$, (7.2) is obviously satisfied. Take $v \in \mathcal{D}_\phi$ and $\beta = \phi(v)$. Next, note that from the definition of K, $\phi(u) \le \alpha$. However, upon setting $v = u$ in (7.3) with $\beta = \phi(v)$ we obtain $\phi(u) \ge \alpha$. Hence $\alpha = \phi(u)$. Thus (7.2) is obtained from (7.3). □

As in the case of Theorem 1-6.2, condition (7.1) can be replaced by the stronger coercivity conditions

$$\left.\begin{array}{l}
\text{There exists a } v_0 \in \mathcal{U} \text{ such that} \\[1ex]
\dfrac{\langle A(v), v - v_0\rangle + \phi(v) - \phi(v_0)}{\|v\|_{\mathcal{U}}} \to +\infty \\[1ex]
\text{for } \|v\|_{\mathcal{U}} \to \infty.
\end{array}\right\} \tag{7.4}$$

We also remark that if the functional ϕ in Theorem 1-7.1 is Gateaux differentiable on $\mathcal{D}_\phi$, then (7.2) reduces to the variational equation

$$u \in \mathcal{D}_\phi: \langle A(u), w\rangle + \langle D\phi(u), w\rangle = 0 \quad \forall w \in \mathcal{U}. \tag{7.5}$$

This is obtained from (7.2) by replacing v by $u + \theta w$, $\theta > 0$, dividing by θ, and taking the limit as $\theta \to 0$. This results in an inequality (≥ 0) instead of (7.5). The converse (≤ 0) is concluded using the convexity of ϕ (i.e. $\phi(v) - \phi(u) - \langle D\phi(u), v - u\rangle \ge 0$).

We emphasize that (7.2) holds for non-differential ϕ. However, a useful technique for solving inequalities of the type (7.2) consists of approximating ϕ by a sequence $\{\phi_n\}$ of differentiable functionals such that $\phi_n(v) \to \phi(v)$, $\forall v \in \mathcal{U}$ and $\liminf_{n\to\infty} \phi_n(v_n) \ge \phi(v)$ for any sequence $\{v_n\} \in \mathcal{U}$ which converges weakly to v. By Theorem 1-7.1, a bounded sequence $\{u_m\} \in \mathcal{D}_{\phi_m}$ of solutions exists, for any m, to the system

$$\langle A(u_m), v - u_m \rangle + \phi_m(v) - \phi_m(u_m) \geq 0 \quad \forall v \in \mathcal{U}.$$

Thus, there exists a subsequence, also denoted $\{u_m\}$, which converges weakly to a point $u \in \mathcal{U}$. Then $\liminf_{m\to\infty} \langle A(u_m), v - u_m \rangle \geq \liminf_{m\to\infty} (\phi_m(u_m) - \phi_m(v)) \geq 0$; i.e. $\limsup_{m\to\infty} \langle A(u_m), u_m - u \rangle \leq 0$. Hence, from the assumed pseudomonotonicity of A,

$$\begin{aligned}\langle A(u), u - v \rangle - \phi(v) + \phi(u) &\leq \liminf_{m\to\infty} \langle A(u_n), u - v \rangle \\ &\quad + \liminf_{m\to\infty} \phi_m(u_m) - \lim_{m\to\infty} \phi_m(v) \leq 0,\end{aligned}$$

i.e. u satisfies (7.2).

1.8 *Quasi-variational inequalities*

In several important classes of physical problems, we encounter cases in which the admissible set K depends upon the solution of the problem which is, of course, not known in advance. Variational inequalities associated with problems of this type are called *quasi-variational inequalities.*

In our present study of quasi-variational inequalities, we are only interested in cases in which existence theorems are derived from so-called comparison theorems and maximum principles of variational inequalities. To this end, the concept of ordering relations associated with positive cones in linear topological spaces is necessary.

We will first establish some preliminary concepts. A subset C of a linear space $\mathcal{U}$ is called a *pointed cone* with vertex 0 if $tC \subset C$ for every $t > 0$, $t \in \mathbf{R}$ and $0 \in C$. A *partial ordering relation*, denoted by $\leq$, can be defined on a pointed cone C by setting

$$p \leq q \quad \text{if and only if} \quad q - p \in C. \tag{8.1}$$

It is clear that

$$\left.\begin{aligned} &p \leq p \quad \forall p \in \mathcal{U}, \\ &p \leq q \quad \text{and} \quad q \leq r \quad \text{implies} \quad p \leq r \quad \forall p, q, r \in \mathcal{U} \end{aligned}\right\} \tag{8.2}$$

These relations imply that the partial ordering is compatible with the structure of a linear space in the sense that

$$\left.\begin{aligned} &0 \leq p \quad \text{implies} \quad 0 \leq tp, \quad \forall t > 0, \quad t \in \mathbf{R} \\ &q \leq p \quad \text{implies} \quad q + r \leq p + r \quad \forall r. \end{aligned}\right\} \tag{8.3}$$

Conversely, for any partial ordering $\leq$, the set

$$C = \{p \in \mathcal{U} : p \geq 0\} \tag{8.4}$$

is the *positive cone* with respect to the ordering relation. Its *negative cone* is given by

$$-C = \{p \in \mathcal{U} : p \leq 0\}. \tag{8.5}$$

If $C \cap (-C) = \{0\}$, the relation $\leq$ is an *ordering relation.*

Let $\mathcal{U}^*$ be the algebraic dual space of $\mathcal{U}$. The *polar cone* C^* of a cone C is then defined by

$$C^* = \{p^* \in \mathcal{U}^* : \langle p^*, p \rangle \geq 0 \quad \forall p \in C\} \tag{8.6}$$

Since C^* is a pointed cone with vertex 0 in $\mathcal{U}^*$, it defines a partial ordering relation $\leq$; i.e.

$$p^* \leq q^* \quad \text{if and only if} \quad q^* - p^* \in C^*. \tag{8.7}$$

If $\mathcal{U}^*$ is a (topological) dual space of a linear topological space $\mathcal{U}$, and if C is a pointed closed convex cone with vertex 0, then we have the properties

$$\left.\begin{array}{l} p \in C \quad \text{if and only if} \quad 0 \leq p \quad \text{and} \\ 0 \leq p \text{ if and only if} \quad 0 \leq \langle p^*, p\rangle \quad \forall p^* \in C^*. \end{array}\right\} \tag{8.8}$$

Since $C^{**} = (C^*)^* = C$

$$\left.\begin{array}{l} p \in C \quad \text{if and only if} \quad p \in C^{**} \quad \text{and} \\ p \in C^{**} \quad \text{if and only if} \quad 0 \leq \langle p, p^*\rangle \quad \forall p^* \in C^*. \end{array}\right\} \tag{8.9}$$

If the set $\{p, g\}$ of elements of a partially ordered set L have a least upper bound and a greatest lower bound, they are called the *join*, denoted by $p \vee q$, and the *meet*, denoted $p \wedge q$, of $\{p, q\}$, respectively. If, for every two elements in L, both of the join and the meet belong to L, the partially ordered set L is said to be a *lattice ordered set.* If we define

$$p^+ = p \vee 0 \quad \text{and} \quad p^- = (-p) \vee 0, \tag{8.10}$$

any element $p \in L$ can be decomposed according to

$$p = p^+ - p^-. \tag{8.11}$$

We note that, for every $p, q \in L$,

$$\left.\begin{array}{l} \sup(p, q) = p \vee q = p + (q - p)^+ = q + (p - q)^+ \\ \inf(p, q) = p \wedge q = p - (p - q)^+ = q - (q - p)^+. \end{array}\right\} \tag{8.12}$$

A typical example is the Sobolev space $W^{m,s}(\Omega)$, $m \geq 0$, $s > 1$, defined on a bounded open domain in $\mathbf{R}^n$ whose boundary is smooth enough. Under the "natural" ordering

$$u \leq v \quad \text{if and only if} \quad u(x) \leq v(x), \quad \text{a.e. } x \in \Omega$$

the space $W^{m,s}(\Omega)$ is a lattice ordered linear space, and the positive cone $W_+^{m,s}(\Omega)$, defined by

$$W_+^{m,s}(\Omega) = \{v \in W^{m,s}(\Omega) : 0 \leq v\}$$

is closed; see Littman, Stampacchia and Weinberger[27].

Let $\mathcal{U}$ be a lattice ordered real reflexive Banach space. An operator A from $\mathcal{U}$ into its dual $\mathcal{U}'$ is said to be *T-monotone* if

$$\langle A(u) - A(v), (u - v)^+\rangle \geq 0 \tag{8.13}$$

for every $u, v \in \mathcal{U}$ such that $(u - v)^+ \in \mathcal{U}$. If the equality is satisfied by the only $(u - v)^+ = 0$, A is *strictly T-monotone.*

Lemma 1.8.1 (Comparison Theorem 1). Let K be a non-empty closed convex set in a lattice ordered real reflexive Banach space $\mathcal{U}$ and let A be a strictly T-monotone operator on $\mathcal{U}$. Suppose that $u_1 \in K$ and $u_2 \in K$ are solutions to problems

$$u_1 \in K: \langle A(u_1), v - u_1\rangle \geq \langle f_1, v - u_1\rangle \quad \forall v \in K$$

$$u_2 \in K: \langle A(u_2), v - u_2\rangle \geq \langle f_2, v - u_2\rangle \quad \forall v \in K$$

for given data f_1 and f_2 in $\mathscr{U}'$. In addition, suppose that $u_1-(u_1-u_2)^+\in K$ and $u_2+(u_1-u_2)^+\in K$. Then

$$u_1\le u_2$$

whenever $f_1\le f_2$.

Proof. Substituting $u_1-(u_1-u_2)^+$ and $u_2+(u_1-u_2)^+$ for v,

$$\langle A(u_1),-(u_1-u_2)^+\rangle \ge \langle f_1,-(u_1-u_2)^+\rangle$$
$$\langle A(u_2),(u_1-u_2)^+\rangle \ge \langle f_2,(u_1-u_2)^+\rangle.$$

Adding these two inequalities gives

$$\langle A(u_1)-A(u_2),(u_1-u_2)^+\rangle \le \langle f_1-f_2,(u_1-u_2)^+\rangle \le 0.$$

Since A is strictly T-monotone in $\mathscr{V}$, this implies that

$$u_1\le u_2. \quad \square$$

Lemma 1.8.2 (Comparison Theorem 2). Let K_1 and K_2 be non-empty, closed convex subsets of a lattice ordered real reflexive Banach space $\mathscr{U}$ and let A be a strictly T-monotone on $\mathscr{U}$. Suppose that u_1 and u_2 are solutions such that

$$u_1\in K_1\colon \langle A(u_1),v-u_1\rangle \ge \langle f,v-u_1\rangle \quad \forall v\in K_1$$
$$u_2\in K_2\colon \langle A(u_2),v-u_2\rangle \ge \langle f,v-u_2\rangle \quad \forall v\in K_2$$

for some $f\in \mathscr{V}$. If $u_1-(u_1-u_2)^+\in K_1$ and $u_2+(u_1-u_2)^+\in K_2$, then

$$u_1\le u_2$$

Proof. Substituting $u_1-(u_1-u_2)^+$ and $u_2+(u_1-u_2)^+$ for v, gives

$$\langle A(u_1),-(u_1-u_2)^+\rangle \ge \langle f,-(u_1-u_2)^+\rangle$$
$$\langle A(u_2),(u_1-u_2)^+\rangle \ge \langle f,(u_1-u_2)^+\rangle.$$

Adding these two inequalities gives

$$\langle A(u_1)-A(u_2),(u_1-u_2)^+\rangle \le 0.$$

Since A is strictly T-monotone, this result implies

$$(u_1-u_2)^+=0, \quad \text{i.e.} \quad u_1\le u_2. \quad \square$$

Lemma 1.8.3 (Maximum Principle). Let K be a non-empty closed convex subset of a lattice ordered real reflexive Banach space $\mathscr{U}$. Let A be a strictly T-monotone operator form $\mathscr{U}$ into its dual $\mathscr{U}'$. Let $u\in K$ be a solution of the variational inequality

$$u\in K\colon \langle A(u),v-u\rangle \ge 0 \quad \forall v\in K. \tag{8.14}$$

Suppose that there exists functions $\bar{k}$ and k such that

and

$$\left.\begin{aligned} -\langle A(\bar{k}),(v-\bar{k})^+\rangle &\le 0 \quad \forall v\in K\\ \langle A(\underline{k}),(\underline{k}-v)^+\rangle &\le 0 \quad \forall v\in K\\ \inf(u,\bar{k})\in K \quad \text{and} \quad &\sup(u,\underline{k})\in K. \end{aligned}\right\} \tag{8.15}$$

Then

$$\underline{k} \le u \le \bar{k}. \tag{8.16}$$

Proof. Taking $v = \inf(u, \bar{k}) = u - (u - \bar{k})^+$,

$$\langle A(u), -(u-\bar{k})^+\rangle \ge 0, \quad \text{i.e.} \quad \langle A(u) - A(\bar{k}), (u-\bar{k})^+\rangle \le \langle A(\bar{k}), (u-\bar{k})^+\rangle \le 0.$$

Since A is strictly T-monotone

$$(u-\bar{k})^+ = 0, \quad \text{i.e.} \quad u \le \bar{k}.$$

Similarly, by taking $v = \sup(u, \underline{k})$,

$$\underline{k} \le u$$

can be obtained. □

Example 1-8.1. Let A be strictly T-monotone operator. Let $\mathcal{U}$ be a linear space, the elements of which are measurable functions defined on some domain $\Omega \subset \mathbf{R}^n$. Let $K \subset \mathcal{U}$ be defined by

$$K = \{v \in \mathcal{U}: v \le \psi, \quad \text{a.e. in} \quad \Omega\}$$

where ψ is a given function such that $\psi \ge 0$, a.e. in Ω. Let f be given data such that $f \ge 0$, a.e. in Ω. Let $u \in K$ be a solution of the variational inequality

$$\langle A(u), v-u\rangle \ge \langle f, v-u\rangle \quad \forall v \in K.$$

Suppose that

$$A(k) = \alpha k^p, \quad \alpha \ge 0 \quad \text{and} \quad p \ge 0$$

for every constant function k. Then

$$u \ge 0.$$

Indeed

$$\langle A(0) - f, (0-u)^+\rangle = -\langle f, (-u)^+\rangle \le 0. \quad □$$

Theorem 1-8.1. Let $\mathcal{U}$ be a lattice ordered real reflexive Banach space and A be an operator from $\mathcal{U}$ into its dual $\mathcal{U}'$ such that

$$\left.\begin{aligned} &\langle A(u) + B(u) - A(v) - B(v), u - v\rangle \ge m(\|u-v\|) \\ &\langle A(u) - A(v), w\rangle \le N(\|u-v\|)\|w\|. \end{aligned}\right\} \tag{8.16}$$

Here B is a completely continuous operator from $\mathcal{U}$ into $\mathcal{U}'$ (i.e. $B(u^n)$ converges strongly to $B(u)$ for every weakly convergent sequence u^n whose limit is u), the function $m: [0, \infty) \to \mathbf{R}$ is strictly increasing, continuous, and is such that $m(0) = 0$, and the function $N: [0, \infty) \to \mathbf{R}$ is continuous. Also let

$$B(u_1) \le B(u_2) \quad \text{in} \quad \mathcal{U}' \quad \text{if} \quad u_1 \le u_2.$$

Let M be a function defined on $\mathcal{U}$ such that

(i) $M(u_1) \le M(u_2)$ in $\mathcal{U}$ if $u_1 \le u_2$.

(ii) $M(v) \geq 0$ in $\mathscr{U}$ if $v \geq 0$.

(iii) For any monotonically decreasing sequence u^n in $\mathscr{U}$ satisfying $M(u^{n-1}) \geq u^n$, its weak limit u satisfies

$$M(u) \geq u.$$

Further, suppose that

$$\langle A(0) + B(0), (0 - v)^+ \rangle \leq 0.$$

Then, there exists at least one solution of the quasi-variational inequality.

$$u \leq M(u): \langle A(u), v - u \rangle \geq 0, \quad \forall v \leq M(u) \tag{8.17}$$

Proof. Let u^0 be a solution of the problem

$$\langle A(u^0), v \rangle = 0 \quad \forall v \in \mathscr{U}$$

such that $u^0 > 0$. We denote that the existence of such a solution is assured by (8.16).

Let us define u^n, $n \geq 1$, as a solution of the variational inequality

$$u^n \in K(u^{n-1}): \langle A(u^n) + B(u^n), v - u^n \rangle \geq \langle B(u^{n-1}), v - u^n \rangle$$

for every

$$v \in K(u^{n-1})$$

where

$$K(u^{n-1}) = \{v \in \mathscr{U}: v \leq M(u^{n-1})\}.$$

We shall show that

$$u^0 \geq \cdots \geq u^{n-1} \geq u^n \geq \cdots \geq 0$$
$$\|u^n\| \leq C.$$

We first prove that $u^0 \geq u^1$ and $u^1 \geq 0$. In fact

$$\langle A(u^0) + B(u^0), v - u^0 \rangle \geq \langle B(u^0), v - u^0 \rangle \quad \forall v \in \mathscr{U}$$
$$\langle A(u^1) + B(u^1), v - u^1 \rangle \geq \langle B(u^0), v - u^1 \rangle \quad \forall v \in K(u^0).$$

Since $A + B$ is strictly T-monotone, and since $K(u^0) \subset \mathscr{U}$

$$u^0 \geq u^1$$

as shown in Lemma 1.8.2. Since $u^0 \geq 0$, $M(u^0) \geq 0$. By the maximum principle. Lemma 1.8.3,

$$u^1 \geq 0.$$

For u^{n-1} and u^n

$$u^{n-1} + (u^n - u^{n-1})^+ \leq M(u^{n-1}) \leq M(u^{n-2})$$
$$u^n - (u^n - u^{n-1})^+ \leq M(u^{n-1}).$$

That is, $u^{n-1}+(u^n-u^{n-1})^+\in K(u^{n-2})$ and $u^n-(u^n-u^{n-1})^+\in K(u^{n-1})$.

Moreover, since $u^{n-2}\geq u^{n-1}$

$$B(u^{n-2})\geq B(u^{n-1}).$$

By Lemmas 1.8.1 and 1.8.2,

$$u^n\leq u^{n-1}.$$

Since $u^{n-1}\geq 0$, i.e. $B(u^{n-1})\geq 0$,

$$u^n\geq 0$$

by the maximum principle (Lemma 1.8.3). Thus

$$u^0\geq u^1\geq\cdots\geq u^{n-1}\geq u^n\geq\cdots\geq 0.$$

On the other hand, putting $v=0\leq M(u^{n-1})$ gives

$$\langle A(u^n),u^n\rangle+\langle B(u^n),u^n\rangle\leq\langle B(u^{n-1}),u^n\rangle\leq\langle B(u^0),u^n\rangle.$$

This implies

$$\|u^n\|\leq C.$$

Since any monotonically decreasing bounded sequence converges weakly to a unique limit, the sequence u^n converges weakly to u in $\mathcal{U}$.

By the hypothesis (iii)

$$u\leq M(u).$$

For every $v\leq M(u)\leq\cdots\leq M(u^n)\leq M(u^{n-1})\leq\cdots$,

$$\langle A(u^n)+B(u^n),v-u^n\rangle\geq\langle B(u^{n-1}),v-u^n\rangle.$$

By the hypothesis on B, $A+B$ is pseudomonotone. Since B is completely continuous

$$\langle A(u),v-u\rangle\geq 0\quad\forall v\leq M(u).$$

Therefore, u is a solution of the quasi-variational inequality (8.17). □

Note that the above sequence $\{u^n\}$ converges to a unique limit u, but this does not imply the uniqueness of the solution to the quasi-variational inequality, since the initial element u^0 can be chosen arbitrarily.

Existence of solutions of the quasi-variational inequality of the second kind, e.g.

$$u\in\mathcal{U}:\langle A(u),v-u\rangle+j(u;v)-j(u;u)\geq 0\quad\text{for every}\quad v\in\mathcal{U}\tag{8.18}$$

follows from similar arguments with Theorem 1-8.1. To this end, we recall the comparison theorem for variational inequalities of the second kind of Duvaut and Lions[8].

Lemma 1-8.4 (Comparison Theorem 3). Let $\mathcal{U}$ be a lattice ordered real reflexive Banach space, and let A be a strictly T-monotone operator on $\mathcal{U}$ into its dual $\mathcal{U}'$. Suppose that u_1 and u_2 in $\mathcal{U}$ are solutions to problems

$$u_1\in\mathcal{U}:\langle A(u_1),v-u_1\rangle+j_1(v)-j_1(u_1)\geq 0,\quad\forall v\in\mathcal{U}$$

$$u_2\in\mathcal{U}:\langle A(u_2),v-u_2\rangle+j_2(v)-j_2(u_2)\geq 0,\quad\forall v\in\mathcal{U}$$

for given proper convex semicontinuous functionals j_1 and j_2 defined on $\mathcal{U}$. Then

$$u_1 \leq u_2$$

whenever

$$j_1(\inf(v_2, v_1)) + j_2(\sup(v_2, v_1)) \leq j_1(v_1) + j_2(v_2) \tag{8.19}$$

for every $v_1, v_2 \in \mathcal{U}$.

Proof. We introduce

$$w_2 = \sup(u_2, u_1) = u_2 + (u_1 - u_2)^+$$
$$w_1 = \inf(u_2, u_1) = u_1 - (u_1 - u_2)^+.$$

Substituting of w_1 and w_2 into v of the first and second inequalities, respectively, we have

$$\langle A(u_1), -(u_1 - u_2)^+\rangle + j_1(\inf(u_2, u_1)) - j_1(u_1) \geq 0$$
$$\langle A(u_2), (u_1 - u_2)^+\rangle + j_2(\sup(u_2, u_1)) - j_2(u_2) \geq 0.$$

Adding above two inequalities, and applying the assumption (8.19), we obtain

$$-\langle A(u_1) - A(u_2), (u_1 - u_2)^+\rangle \geq 0.$$

Since A is assumed to be strictly T-monotone,

$$(u_1 - u_2)^+ = 0, \quad \text{i.e.} \quad u_1 \leq u_2. \qquad \square$$

Theorem 1-8.2. Let $\mathcal{U}$ be a lattice ordered, real, reflexive Banach space, and let A be a hemicontinuous, strictly T-monotone, and coercive operator of $\mathcal{U}$ into its dual $\mathcal{U}'$. Suppose that if $a \leq b$ in $\mathcal{U}$

$$j(a; \inf(v, w)) + j(b; \sup(v, w)) \leq j(a; w) + j(b; v) \tag{8.20}$$

where $(v, w) \rightarrow j(v; w)$ is proper convex lower semicontinuous from $\mathcal{U}$ into $\bar{\mathbf{R}}$, and that there exists a non-negative solution $u_0 \in \mathcal{U}$ of the non-linear equation

$$\langle A(u_0), v\rangle = 0, \quad \forall v \in \mathcal{U}.$$

Then there exists at least one solution to the quasi-variational inequality of the second kind

$$u \in \mathcal{U}: \langle A(u), v - u\rangle + j(u; v) - j(u; u) \geq 0 \tag{8.21}$$

for every $v \in \mathcal{U}$.

Proof. We define the iterative solutions $\{u^n\}$ by the variational inequalities of the second kind

$$u^n \in \mathcal{U}: \langle A(u^n), v - u^n\rangle + j(u^{n-1}; v) - j(u^{n-1}; u^n) \geq 0 \tag{8.22}$$

for every $v \in \mathcal{U}$.

We will show that if $u^{n-1} \leq u^{n-2}$, then $u^n \leq u^{n-1}$. By the definition of $\{u^n\}$

$$\langle A(u^{n-1}), v - u^{n-1}\rangle + j(u^{n-2}; v) - j(u^{n-2}; u^{n-1}) \geq 0$$
$$\langle A(u^n), v - u^n\rangle + j(u^{n-1}; v) - j(u^{n-1}; u^n) \geq 0.$$

Taking $v = \sup(u^{n-1}, u^n) = u^{n-1} + (u^n - u^{n-1})^+$ and $v = \inf(u^{n-1}, u^n) = u^n - (u^n - u^{n-1})^+$ in the above first and second inequalities, respectively, we obtain that

$$\langle A(u^{n-1}), (u^n - u^{n-1})^+\rangle + j(u^{n-2}; \sup(u^{n-1}, u^n)) - j(u^{n-2}; u^{n-1}) \geq 0$$
$$\langle A(u^n), -(u^n - u^{n-1})^+\rangle + j(u^{n-1}; \inf(u^{n-1}, u^n)) - j(u^{n-1}; u^n) \geq 0.$$

Adding the above two inequalities, and applying the assumption (8.20), we have

$$-\langle A(u^n) - A(u^{n-1}), (u^n - u^{n-1})^+\rangle \geq 0.$$

Since A is strictly T-monotone

$$(u^n - u^{n-1})^+ = 0, \quad \text{i.e.} \quad u^n \leq u^{n-1}.$$

Thus, we can conclude that

$$\cdot \leq u^n \leq u^{n-1} \leq \cdot \cdot \leq u^1 \leq u^0.$$

Furthermore, since A is coercive, the sequence $\{u_n\}$ is uniformly bounded in $\mathcal{U}$. Therefore, the sequence $\{u^n\}$ converges weakly to u in $\mathcal{U}$. Since $(v, w) \to j(v; w)$ is convex and lower semicontinuous on $\mathcal{U}$, we can pass to the limit $n \to +\infty$ in (8.22), i.e.

$$\langle A(u), v - u\rangle + j(u; v) - j(u; u) \geq 0$$

for every $v \in \mathcal{U}$. □

Remark 1-8.1. The ordering $u \leq v$ in Sobolev spaces deserves some additional comments. Suppose $u \in W^{m,p}(\Omega)$. Then u is an equivalence class of functions with generalized derivatives in $L^p(\Omega)$. The notation $u \leq 0$ (for example) means that we can find a representative $\hat{u}$ of u in this class with the following property: there is a sequence $\varphi_k \in C^\infty(\Omega)$ such that φ_k converges strongly to $\hat{u}$ in $W^{m,p}(\Omega)$ and $\varphi_k \leq 0$, $\forall k \geq 1$. Similar arguments and orderings apply to traces of $W^{m,p}$-functions on the boundary. For instance, if γ_j: $W^{m,p}(\Omega) \to W^{m-j-1/p,p}(\partial\Omega)$, $0 \leq j \leq m-1$, are the trace operators, the notation "$\partial^j v/\partial n^j \leq 0$, a.e. on $\partial\Omega$" is used to signify that $\gamma_j(v) \leq 0$ where $(\leq)$ is interpreted in the sense just described.

Throughout the remainder of this study (particularly in Chaps. 3–5) orderings on Sobolev spaces and on boundary traces will be interpreted in the sense described here. Thus, for example, "$v \leq 0$, a.e. on $\Gamma \subset \partial\Omega$" will be understood to apply to partial orderings of traces of, e.g. $H^1(\Omega)$ on $\partial\Omega$ restricted to $H^{1/2}(\Gamma)$. For additional details on this subject, see Oden and Kikuchi[25] and Littman, Stampacchia and Weinberger[27]. □

1.9 *Comments*

The theories discussed in this chapter summarize many of the fundamental results on variational inequalities developed over roughly the last decade. Our development follows principally the works of Mosco[5, 9], Lions[11, 28], Brezis[4, 29] and Stampacchia[10]. More details of the theory of variational inequalities can be also found in Oden[30] and Kikuchi[31] together with various examples from solid mechanics.

The introductory explanation of the concept of variational inequalities in Section 1.1 follows from Stampacchia[10]. Example 1-1.1 is used in Kikuchi[32] in order to show a relationship of variational inequalities to free boundary problems.

Theorem 1-2.1 is found in, for example, Mosco[5]. Theorem 1-2.2, i.e. the existence theorem of minimizers of functionals, follows from Vainberg[24]. Theorem 1-2.3 is an obvious result of Theorems 1-2.1 and 1-2.3, and equivalent properties of monotonicity of the gradient operator DF of a convex functional F, (2.3). Geometrical interpretations of a minimizer of a convex functional on $\mathbf{R}^m$ can be found in Mosco[5].

The inequality representation of the projection of a Hilbert space into a closed convex subset follows from Lions and Stampacchia[3], Stampacchia[10] and also Brezis[4].

Construction of the contraction T for a strongly monotone Lipschitzian operator A was first introduced by Lions and Stampacchia[3].

The Hartman–Stampacchia theorem, Theorem 1-4.1, was proved by Hartman and Stampacchia[26] in 1965. This is the first existence theorem on solutions to variational inequalities involving non-linear operators, and it had a significant impact on the development of the non-linear theory of monotone operators on subsets of reflexive Banach spaces.

Variational inequalities of the second kind were introduced by Browder[33] in applying the notion of an indicator functional for closed convex subsets of Banach spaces. Further results along these lines were contributed by Brezis[4].

Pseudo monotone theories of variational inequalities of the first and second kind, Sections 1.6 and 1.7, are found in Brezis[4], Lions[11] and Stampacchia[10].

The theory of quasi-variational inequalities discussed in Section 1.8 was studied by Mosco[9] and Lions[28]. Comparison theorems and a maximum principle follow from the works of Brezis[29] and Mosco[9]. Theorems 1-8.1 and 1-8.2 can be found in Lions[28] together with several examples. □

2. APPROXIMATION AND NUMERICAL ANALYSIS OF VARIATIONAL INEQUALITIES

2.1 *Convergence of approximations*

In this chapter, we discuss theories of approximation of variational inequalities with special emphasis on those theories applicable to finite element methods. We will be primarily concerned with the general variational inequality: find $u \in K$ such that

$$\langle A(u), v - u\rangle \geq 0 \quad \forall v \in K. \tag{1.1}$$

An approximation of (1.1) generally involves seeking a function u_h in a set K_h which is a subset of a finite dimensional subspace $\mathcal{U}_h$ of $\mathcal{U}$, h being an appropriate index. The approximation of (1.1) will then involve seeking $u_h \in K_h$ such that

$$\langle A(u_h), v_h - u_h\rangle \geq 0 \quad \forall v_h \in K_h. \tag{1.2}$$

In general, $\mathcal{U}_h$ is a member of a family of closed subspaces $\{\mathcal{U}_h\}_{0<h\leq 1}$ of $\mathcal{U}$, each containing a set K_h so that $\{K_h\}_{0<h<1}$ is a family of subsets of $\mathcal{U}$ approximating in some sense the constraint set K. We are interested in determining sequences of solutions $\{u_h\}$ to (1.2), $u_h \in K_h \subset \mathcal{U}_h$, and in investigating the behavior of the approximations as $h \to 0$. In particular, we wish to determine conditions under which $\{u_h\}$ converges in some sense to a solution to (1.1) and in estimating the error $u - u_h$.

The first question that arises is what is meant by a consistent approximation of the set K_h? For our purposes, the following general condition is sufficient:

> Let K be a subset of a normed linear space $\mathcal{U}$. A sequence of subsets $\{K_h\}$ in $\mathcal{U}$ is said to *converge to a set* K if
> (i) for every $v \in K$, there exists a sequence $v_h \in K_h$ which converges strongly to v, and
> (ii) for every weakly convergent sequence $\{u_h\}$, $u_h \in K_h$, its weak limit u belongs to K. (1.3)

Our first result is a general existence and approximation theorem for problems of the type (1.1) in which the operator A is of the Gårding type (i.e. it satisfies a generalized Gårding inequality in the sense of Oden[34]), and is coercive. Operators of this type can be shown to be pseudomonotone (see Oden[34] for a proof of this fact). The following approximation theorem and *a priori* estimates are discussed in Kikuchi[31].

Theorem 2-1.1. Let K be a non-empty closed convex subset of a reflexive Banach space $\mathcal{U}$. Let $\{K_h\}$ be a sequence of closed convex sets in $\mathcal{U}$ convergent to the set K in the sense of (1.3).

Let $\tilde{K}$ be a subset of $\mathcal{U}$ such that, for every $0<h\leq 1$, K and K_h belongs to $\tilde{K}$. Let $A\colon \tilde{K}\to\mathcal{U}'$ be an operator on $\tilde{K}$ satisfying the conditions,

(i) A is coercive.

(ii)

$$\langle A(u)-A(v), u-v\rangle \geq \alpha\|u-v\|_{\mathcal{U}}^p - \beta(M)\|u-v\|_{\mathcal{V}}^q, \quad \alpha>0 \tag{1.4}$$

for every $u, v\in\tilde{K}$ with $\|u\|_{\mathcal{U}}, \|v\|_{\mathcal{U}}\leq M$, $p>1$, $q>1$, where $\mathcal{V}$ is a normed linear space in which $\mathcal{U}$ is compactly imbedded and $\langle\cdot,\cdot\rangle$ is the duality pairing on $\mathcal{U}'\times\mathcal{U}$.

(iii) A is bounded in the sense that $\|A(v)\|_{\mathcal{U}'}<+\infty$ for $\|v\|_{\mathcal{U}}<+\infty$.

Then there exists at least one solution $u\in K$ of (1.1). Moreover, let $\{u_h\}$ be a sequence of solutions to (1.2) obtained as $h\to 0$. Then there exists a subsequence $\{u_{h'}\}$ of $\{u_h\}$ which converges strongly to a solution $u\in K$ of (1.1).

Proof. The existence of solutions $u\in K$ of (1.1) and $u_h\in K_h$ of (1.2) follow from results established in the previous chapter (see Theorem 1.6.1). Owing to the coerciveness of A on $\tilde{K}$, any solutions of (1.2) are bounded in $\mathcal{U}$. Since $\mathcal{U}$ is reflexive, there exists a subsequence $\{u_h'\}$ of the sequence of solutions $\{u_h\}$, $u_h\in K_h$, which converges weakly to $\bar{u}\in\mathcal{U}$.

Using the first condition (i) of convergent sets K_h in (1.3)

$$\langle A(u_h), v-u_h\rangle = \langle A(u_h), v_h-u_h\rangle + \langle A(u_h), v-v_h\rangle \geq -\|A(u_h)\|_{\mathcal{U}'}\|v-v_h\|_{\mathcal{U}}$$

for every $v\in K$ with $v_h\to v$ strongly in $\mathcal{U}$. Then

$$\lim_{h\to 0}\langle A(u_h), u_h-v\rangle \geq 0 \quad \forall v\in K. \tag{1.5}$$

By condition (ii) in (1.3), the weak limit of u_h' belongs to K, i.e. $\bar{u}\in K$. Since A is pseudomonotone on $\tilde{K}$, it can be easily shown that $\bar{u}$ is also a solution of (1.1). Indeed

$$0\geq \liminf\,\langle A(u_h'), u_h'-v\rangle \geq \langle A(\bar{u}), \bar{u}-v\rangle.$$

That is

$$\langle A(\bar{u}), v-\bar{u}\rangle \leq 0 \quad \forall v\in K.$$

We shall show that a subsequence $\{u_h''\}$ of $\{u_h'\}$ converges strongly to $\bar{u}\in K$. By the Gårding inequality, we have

$$\langle A(u_h), u_h-\bar{u}\rangle \geq \langle A(\bar{u}), u_h-\bar{u}\rangle + \alpha\|u_h-\bar{u}\|_{\mathcal{U}}^p - \hat{\beta}\|u_h-\bar{u}\|_{\mathcal{V}}^q$$

where $\hat{\beta}=\max_h\{\beta(\|u_h\|_{\mathcal{U}}), \beta(\|\bar{u}\|_{\mathcal{U}})\}$. Then, by (1.5) and since $\bar{u}\in K$

$$\begin{aligned}0 &\geq \lim_{h\to 0}\langle A(u_h''), u_h-\bar{u}\rangle\\ &\geq \lim_{h\to 0}(\alpha\|u_h''-\bar{u}\|_{\mathcal{U}}^p - \hat{\beta}\|u_h''-\bar{u}\|_{\mathcal{V}}^q)\\ &\geq \alpha\lim_{h\to 0}\|u_h''-\bar{u}\|_{\mathcal{U}}^p.\end{aligned} \tag{1.6}$$

Here we have used the fact that $\mathcal{U}$ is compactly embedded in $\mathcal{V}$ and, therefore, any subsequence $\{u_h''\}$ of $\{u_h'\}$ converging weakly in $\mathcal{U}$ must converge strongly in $\mathcal{V}$. This completes the proof. □

We observe that if $\beta(M)\leq 0$, then A is strongly monotone on $\tilde{K}$. In this case, the solutions of (1.1) and (1.2) are unique and $\{u_h\}$ converges strongly to the solution of (1.1).

We remark that the conclusions of Theorem 2-1.1 can easily be extended to cases of variational inequalities in the second kind which involve the sum of the gradient DF of a weakly lower semicontinuous functional F and a coercive, convex, non-differentiable functional $\phi: K \to \bar{\mathbf{R}}$. Indeed, let $A = DF$, $F: K \to \mathbf{R}$, and let ϕ be continuous and coercive on $\tilde{K}$. Then (1.2) becomes

$$u_h \in K_h: \langle A(u_h), v_h - u_h\rangle + \phi(v_h) - \phi(u_h) \geq 0 \quad \forall v_h \in K_h.$$

By the convergence condition (i) of (1.3), for every $v \in K$,

$$\langle A(u_h), v - u_h\rangle + \phi(v) - \phi(u_h) \geq -\|A(u_h)\|_{\mathcal{U}'}\|v - v_h\|_{\mathcal{U}} + \phi(v) - \phi(v_h).$$

Then

$$\lim_{h\to 0} \{\langle A(u_h), v - u_h\rangle + \phi(v) - \phi(u_h)\} \geq 0 \quad \forall v \in K.$$

By the Gårding-type inequality

$$\langle A(u_h), u_h - u\rangle \geq \langle A(u), u_h - u\rangle + \alpha\|u_h - u\|_{\mathcal{U}}^p - \hat{\beta}\|u_h - u\|_{\mathcal{V}}^q.$$

Then

$$\begin{aligned} 0 &\geq \lim_{h\to 0} \{\langle A(u_h), u_h - u\rangle + \phi(u_h) - \phi(u)\} \\ &\geq \lim_{h\to 0} \{\langle A(u), u_h - u\rangle + \phi(u_h) - \phi(u) + \alpha\|u_h - u\|_{\mathcal{U}}^p - \hat{\beta}\|u_h - u\|_{\mathcal{V}}^q\}. \end{aligned}$$

Since ϕ is convex and continuous on $\tilde{K}$, ϕ is weakly lower semicontinuous on $\tilde{K}$. Then

$$0 \geq \alpha \lim_{h\to 0} \|u_h'' - u\|_{\mathcal{U}}^p,$$

which means that the results stated in Theorem 2-1.1 can be extended to variational inequalities of the second kind.

Another interesting result is the following estimate which is useful for obtaining an *a priori* error estimate of approximations. Let K and K_h be closed convex subsets of a reflexive Banach space. Let u and u_h be solutions of the respective variational inequalities

$$u \in K: \langle A(u), v - u\rangle \geq 0 \quad \forall v \in K \tag{1.7}$$

$$u_h \in K_h: \langle A(u_h), v_h - u_h\rangle \geq 0 \quad \forall v_h \in K_h. \tag{1.8}$$

Then, for every $v \in K$ and $v_h \in K_h$,

$$\begin{aligned} \langle A(u), u - u_h\rangle &\leq \langle A(u), v - u_h\rangle \\ &= \langle A(u), u - v_h\rangle + \langle A(u), v - u_h + v_h - u\rangle \\ -\langle A(u_h), u - u_h\rangle &\leq \langle A(u_h), v_h - u\rangle \end{aligned}$$

Adding these two inequalities gives

$$\begin{aligned} \langle A(u) - A(u_h), u - u_h\rangle \leq{}& \langle A(u) - A(u_h), u - v_h\rangle \\ &+ \langle A(u), v_h - u\rangle + \langle A(u), v - u_h\rangle. \end{aligned} \tag{1.9}$$

If $K_h \subset K$ for every $h > 0$, we can take $v = u_h$. Thus

$$\langle A(u) - A(u_h), u - u_h \rangle \le \langle A(u) - A(u_h), u - v_h \rangle + \langle A(u), v_h - u \rangle. \tag{1.10}$$

Theorem 2-1.2. Let K and K_h be closed convex subsets of a reflexive Banach space $\mathcal{U}$. Let $u \in K$ and $u_h \in K_h$ be the solutions of variational inequalities (1.7) and (1.8). Then, for every $v \in K$ and $v_h \in K_h$,

$$\langle A(u) - A(u_h), u - u_h \rangle \le \langle A(u) - A(u_h), u - v_h \rangle + \langle A(u), v_h - u + v - u_h \rangle.$$

If $K_h \subset K$

$$\langle A(u) - A(u_h), u - u_h \rangle \le \langle A(u) - A(u_h), u - v_h \rangle + \langle A(u), v_h - u \rangle. \quad \square$$

Estimates (1.9) or (1.10) and the Gårding type inequality are useful in obtaining *a priori* estimates for u and u_h.

For variational inequalities of the second kind, (1.7) and (1.8) become

$$u \in K: \langle A(u), v - u \rangle + \phi(v) - \phi(u) \ge 0 \quad \forall v \in K \tag{1.11}$$

$$u_h \in K_h: \langle A(u_h), v_h - u_h \rangle + \phi(v_h) - \phi(u_h) \ge 0 \quad \forall v \in K_h. \tag{1.12}$$

Then, for $v \in K$ and $v_h \in K_h$,

$$\begin{aligned} \langle A(u), u - u_h \rangle &\le \langle A(u), u - v_h \rangle + \langle A(u), v - u_h + v_h - u \rangle + \phi(v) - \phi(u) \\ -\langle A(u_h), u - u_h \rangle &\le \langle A(u_h), v_h - u \rangle + \phi(v_h) - \phi(u_h). \end{aligned}$$

Adding these two inequalities gives

$$\begin{aligned} \langle A(u) - A(u_h), u - u_h \rangle \le{}& \langle A(u) - A(u_h), u - v_h \rangle + \langle A(u), v_h - u \rangle \\ &+ \langle A(u), v - u_h \rangle + (\phi(v_h) - \phi(u)) + (\phi(v) - \phi(u_h)). \end{aligned} \tag{1.13}$$

If $K_h \subset K$ for every $h > 0$, then

$$\begin{aligned} \langle A(u) - A(u_h), u - u_h \rangle \le{}& \langle A(u) - A(u_h), u - v_h \rangle + \langle A(u), v_h - u \rangle \\ &+ \phi(v_h) - \phi(u). \end{aligned} \tag{1.14}$$

Theorem 2-1.3. Let K and K_h be closed convex subsets of a reflexive Banach space $\mathcal{U}$. Let $u \in K$ and $u_h \in K_h$ be the solutions of variational inequalities (1.11) and (1.12), respectively. Then, for every $v \in K$ and $v_h \in K_h$,

$$\begin{aligned} \langle A(u) - A(u_h), u - u_h \rangle \le{}& \langle A(u) - A(u_h), u - v_h \rangle \\ &+ \langle A(u), v_h - u + v - u_h \rangle + \phi(v_h) - \phi(u) + \phi(v) - \phi(u_h). \end{aligned}$$

If $K_h \subset K$ is assumed for every $h > 0$,

$$\begin{aligned} \langle A(u) - A(u_h), u - u_h \rangle \le{}& \langle A(u) - A(u_h), u - v_h \rangle \\ &+ \langle A(u), v_h - u \rangle + \phi(v_h) - \phi(u). \quad \square \end{aligned}$$

2.2 *Error estimates for finite element approximations of variational inequalities*

We will now consider cases in which the subspace $\mathcal{U}_h$ and the subsets K_h have a structure typical of that found in finite element approximations. The parameter h can be regarded as the mesh parameter, which is typically the largest diameter of a finite element in a given mesh. The families $\{\mathcal{U}_h\}_{0<h\le 1}$ and $\{K_h\}_{0<h\le 1}$ are generated by appropriate refinements of the finite element

mesh. In all applications of our results, the space $\mathcal{U}$ is generally a Sobolev space $W^{m,p}(\Omega)$ or $H^m(\Omega)$, Ω being a bounded open domain in $\mathbf{R}^n$, and $m \geq 0$, $1 < p < \infty$.

For simplicity, we will restrict our attention to variational inequalities involving linear operators defined on a non-empty closed convex set K of a real Hilbert space H.

Let $a(\cdot, \cdot)$ be a continuous coercive bilinear form defined on H such that

$$\left.\begin{aligned} a(u, v) &\leq M\|u\|_H\|v\|_H \\ a(u, u) &\geq m\|u\|_H^2. \end{aligned}\right\} \quad \forall u, v \in H \tag{2.1}$$

Our first major result is an important theorem due to Falk[16]:

Theorem 2-2.1. Let (2.1) hold and let $u \in K$ and $u_h \in K_h$ be respective solutions of variational inequalities

$$\begin{aligned} u \in K&: a(u, v-u) \geq f(v-u) \quad \forall v \in K \\ u_h \in K_h&: a(u_h, v_h - u_h) \geq f(v_h - u_h) \quad \forall v_h \in K_h \end{aligned} \tag{2.2}$$

where K and K_h are non-empty closed convex sets in a Hilbert space H and a finite dimensional subspace H_h of H, respectively. Let $A: H \to H'$ be an operator defined by

$$\langle A(u), v\rangle = a(u, v) - f(v)$$

where $\langle\cdot,\cdot\rangle$ denotes duality pairing on $H' \times H$ and $f \in H'$. Then the following inequality holds for every $v \in K$ and for every $v_h \in K_h$

$$\|u - u_h\|_H^2 \leq \frac{M^2}{m^2}\|u - v_h\|_H^2 + \frac{2}{m}\langle A(u), v_h - u + v - u_h\rangle. \tag{2.3}$$

Proof. From the estimate (1.9)

$$\langle A(u) - A(u_h), u - u_h\rangle \leq \langle A(u) - A(u_h), u - v_h\rangle + \langle A(u), v_h - u + v - u_h\rangle.$$

That is

$$a(u - u_h, u - u_h) \leq a(u - u_h, u - v_h) + \langle A(u), v_h - u + v - u_h\rangle.$$

Then, from (2.1)

$$m\|u - u_h\|_H^2 \leq M\|u - u_h\|_H\|u - v_h\|_H + \langle A(u), v_h - u + v - u_h\rangle. \tag{2.4}$$

Using Young's inequality

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2 \quad \forall \epsilon > 0; \quad a, b \in \mathbf{R}$$

we have

$$M\|u - u_h\|_H\|u - v_h\|_H \leq \frac{m}{2}\|u - u_h\|_H^2 + \frac{M^2}{2m}\|u - v_h\|_H^2.$$

Substituting this into (2.4) yields (2.3). □

The next corollary follows immediately from Theorem 2-2.1.

Corollary 2-2.1.1. Let the conditions of Theorem 2-2.1 hold and in addition let H be continuously embedded in a Banach space $\mathcal{G}$. Suppose that $K_h \subset K$ (so that one can take $v = u_h$ in (2.3)) and $A(u) \in \mathcal{G}'$, the dual of $\mathcal{G}$. Then

$$\|u - u_h\|_H^2 \leq \frac{M^2}{m^2}\|u - v_h\|_H^2 + \frac{2}{m}\|A(u)\|_{\mathcal{G}'}\|u - v_h\|_{\mathcal{G}}. \tag{2.5}$$

Moreover, if $K = H$ and $K_h = H_h$ so that $A(u) = 0$ in H', then

$$\|u - u_h\|_H \leq \frac{M}{m}\|u - v_h\|_H. \quad \square \tag{2.6}$$

We recognize the estimate (2.6) as that of the usual finite element estimates for linear elliptic problems; see, for example, Oden–Reddy [19].

In general, solutions of elliptic variational inequalities are not expected to be smoother than to belong to $W^{2,p}(\Omega)$. However, as shown by Baiocchi [13] there are cases in which the solution of the variational inequality belongs to $W^{2.5-\epsilon,2}(\Omega)$, ϵ being a sufficiently small positive number. In such cases, we may obtain higher-order rates of convergence than h, h being the mesh-size parameter, by using finite elements of higher order than linear elements. The following example problem follows Brezzi and Sacchi [35] and Kikuchi [36].

Example 2-2.1. Let Ω be a bounded open convex domain in $\mathbf{R}^2$ and let its boundary Γ be smooth enough; for example, Γ can be piecewise C^2. According to Baiocchi [13], the solution u of the variational inequality

$$u \in K: \int_\Omega \nabla u \cdot \nabla(v - u)\,\mathrm{d}x + \int_\Omega (v - u)\,\mathrm{d}x \geq 0 \quad \forall v \in K \tag{2.7}$$

$$K = \{v \in H^1(\Omega): v = g, \quad \text{a.e. on} \quad \Gamma, \quad v \geq 0, \quad \text{a.e. in} \quad \Omega\} \tag{2.8}$$

can be characterized by

$$\begin{gathered} (-\nabla \cdot \nabla u + 1)u = 0, \quad -\nabla \cdot \nabla u + 1 \geq 0, \quad u \geq 0, \quad \text{in} \quad \Omega \\ u|_\Gamma = g \quad \text{and} \quad u \in H^{2.5-\epsilon}(\Omega) \end{gathered} \tag{2.9}$$

if the data g are smooth enough; for example, if g belongs to $C^2(\Gamma)$. Then

$$-\nabla \cdot \nabla u + 1 \in H^{0.5-\epsilon}(\Omega) \subset L^\infty(\Omega). \tag{2.10}$$

Let Ω be exactly triangulated by a finite element mesh and let Σ and Σ_Γ denote the sets of all nodal points in Ω and on the boundary Γ, respectively. We consider the following cases:

(a) *Linear case.* We first consider the case in which linear polynomial approximations are used over each finite element. Suppose that the admissible set K defined by (2.8) is approximated by

$$K_h = \{v_h \in S_h: v(\Sigma_\Gamma) = g(\Sigma_\Gamma), \quad v(\Sigma) \geq 0\}. \tag{2.11}$$

Then

$$K_h \subset K.$$

For this case, we take H to be the linear manifold

$$H = \{v \in H^1(\Omega): v = g, \quad \text{a.e. on} \quad \Gamma\}$$

and set

$$\mathscr{G} = \mathscr{G}' = L^2(\Omega).$$

Then

$$\|v\|_H = |v|_1 \geq C\|v\|_1 \quad \forall v \in H$$

so that Corollary 2-2.1.1 yields

$$\|u-u_h\|_H^2 \leq \frac{M^2}{m^2}\|u-v_h\|_H^2 + \frac{2}{m}\|-\nabla\cdot\nabla u+1\|_0\|u-v_h\|_0 \quad \forall v_h\in K_h. \tag{2.12}$$

Here $\|\cdot\|$ and $|\cdot|_1$ are the Sobolev norm and semi-norm on $H^1(\Omega)$, respectively, and $\|\cdot\|_0$ is the $L^2(\Omega)$ norm. It is well known that for regular refinements of piecewise linear finite elements, the following interpolation estimates hold (see Ciarlet[20] or Oden and Reddy[19] and Falk[16])

$$\begin{aligned}\inf_{v_h\in K_h} |v-v_h|_1 &\leq C_1 h|v|_2\\ \inf_{v_h\in K_h} \|v-v_h\|_0 &\leq C_2h^2|v|_2.\end{aligned} \tag{2.13}$$

Introducing these estimates into (2.12), we obtain for the final error estimate

$$\|u-u_h\|_H \leq Ch \tag{2.14}$$

where C is independent of h.

(b) *Quadratic case.* Next, suppose that quadratic polynomial approximations are used over each finite element and let K_h be defined by (2.11). Then it is clear that $K_h \not\subset K$. This implies that the term

$$\langle A(u), v_h-u+v-u_h\rangle, \quad v\in K, \quad v_h\in K_h \tag{2.15}$$

in (2.3) has to be estimated in order to obtain the rate of convergence of the method. Toward this end, let us consider the integral

$$I=\int_\Omega(-\nabla\cdot\nabla u+1)(u_h-v)\,\mathrm{d}x \tag{2.16}$$

from which the term (2.15) is derived. If

$$v=\sup(u_h,0)\in K \tag{2.17}$$

then u_h-v vanishes in Ω except in finite elements where the value of u_h is zero on at least one but not all of the nodal points. By examining the structure of the matrix induced by the bilinear form $a(u_h, v_h)$, we observe that the number of such finite elements in the model is at most Ch^{-1}, where C is some constant which depends only upon the boundary data g. Then

$$I\leq Ch^{-1}\int_{\Omega_e}(-\nabla\cdot\nabla u+1)(u_h-v)\,\mathrm{d}x$$

where Ω_e is a representative finite element of the type described above. By the regularity of the solution described in (2.10),

$$I\leq Ch^{-1}\|-\nabla\cdot\nabla u+1\|_{0,\infty,\Omega_e}\|u_h-v\|_{0,1,\Omega_e}$$

where $\|\cdot\|_{0,p,\Omega_e}$ is the L^p-norm on Ω_e. Since $u_h-v=0$ on the boundary of Ω_e, the following estimate is known to hold for $v\in H^1(\hat\Omega)$, $\hat\Omega\subset\mathbf{R}^n$

$$\|v\|_{0,p,\hat\Omega}\leq C(\text{mes}\,\hat\Omega)^{(1+(1/p)-(1/q))}\|v\|_{1,q,\hat\Omega} \tag{2.18}$$

$1\leq p\leq\infty$, $1\leq q\leq\infty$. Then

$$\|u_h-v\|_{0,1,\Omega_e}\leq Ch^4\|u_h-v\|_{1,\infty,\Omega_e}$$

Thus

$$I \le C\|-\nabla\cdot\nabla u+1\|_{0,\infty,\Omega_e}\|u_h-v\|_{1,\infty,\Omega_e}h^{-1+4}$$

Since $\|-\nabla\cdot\nabla u+1\|_{0,\infty,\Omega_e}\le\|-\nabla\cdot\nabla u+1\|_{0,\infty,\Omega}$, and since $\|u_h-v\|_{1,\infty,\Omega_e}$ is bounded, we obtain the estimate

$$I \le Ch^3$$

where C is a constant independent of h. Thus

$$\int_\Omega(-\nabla\cdot\nabla u+1)(u_h-v)\,\mathrm{d}x\le Ch^3. \tag{2.19}$$

Next, we will use essentially the same procedure used to obtain (2.19) to estimate the term

$$\int_\Omega(-\nabla\cdot\nabla u+1)(u-v_h)\,\mathrm{d}x$$

We observe that the integral

$$J=\int_\Omega(-\nabla\cdot\nabla u+1)(u-v_h)\,\mathrm{d}x$$

where v_h is the interpolant of u, vanishes in Ω except on finite elements in which the value of u is zero on at least one but not all of the nodal points. Then

$$\begin{aligned}J&\le Ch^{-1}\int_{\Omega_e}(-\nabla\cdot\nabla u+1)(u-v_h)\,\mathrm{d}x\\&\le Ch^{-1}\|-\nabla\cdot\nabla u+1\|_{0,\infty,\Omega_e}\|u-v_h\|_{0,1,\Omega_e}.\end{aligned}$$

Applying (2.18)

$$\begin{aligned}J&\le Ch^{-1}\|-\nabla\cdot\nabla u+1\|_{0,\infty,\Omega_e}h^{2(1+1-1/2)}\|u-v_h\|_{1,2,\Omega_e}\\&=Ch^2\|-\nabla\cdot\nabla u+1\|_{0,\infty,\Omega_e}\|u-v_h\|_{1,2,\Omega}.\end{aligned}$$

By the interpolation property (Oden and Reddy[20])

$$\inf_{v_h\in S_h}\|u-v_h\|_1\le Ch^{3/2-\epsilon}\|u\|_{2.5-\epsilon} \tag{2.20}$$

we obtain

$$J\le Ch^{7/2-\epsilon}\|u\|_{2.5-\epsilon}. \tag{2.21}$$

Finally, collecting the estimates (2.19) and (2.21) and introducing them into (2.3), we have

$$\begin{aligned}\|u-u_h\|^2&\le C\left(\frac{M^2}{m^2}h^{3-\epsilon}\|u\|_{2.5-\epsilon}+\frac{2}{m}(Ch^{7/2-\epsilon}\|u\|_{2.5-\epsilon}+Ch^3)\right)\\&\le Ch^{3-\epsilon}\end{aligned}$$

Thus, the final error estimate is

$$\|u-u_h\|_1\le 0(h^{3/2-\epsilon}).\quad\square \tag{2.22}$$

Example 2-2.2. We describe briefly some numerical results obtained by solving a 1-dimensional version of problems (2.7) and (2.8) studied in Example 2-2.1. We are particularly interested in verifying the rates of convergence of the finite element approximations derived in Example 1-1.1. In this simple case, $\Omega = (0, 1)$ and we take $g(0) = 0.25$ and $g(1) = 0$. The problem was solved for several uniform meshes using both linear and quadratic finite elements. The results are shown in Fig. 2.1. The computed rates of convergence are seen to be $0(h^{1.5-\epsilon})$ for quadratic finite elements in the H^1-norm, in perfect agreement with the theoretical estimates (2.14) and (2.22). □

2.3 *Solutions methods*

The approximation of variational inequalities by finite element methods leads to finite systems of inequalities in the nodal values of the approximate solution. For example, consider again the problem

$$u_h \in K_h: \langle A(u_h), v_h - u_h \rangle \geq 0 \quad \forall v_h \in K_h \tag{3.1}$$

where K_h is a subset of the finite-dimensional space $\mathcal{U}_h$ spanned by the collection of basis functions $\{\varphi_i\}_{i=1}^N$ generated using finite elements for a fixed partition of a bounded domain Ω. A is, for example, a strongly monotone operator from $K \subset \mathcal{U}$ into $\mathcal{U}'$. If $\{\mathbf{x}_i\}_{i=1}^N$ are nodal points in Ω_h, the approximation of Ω, then the functions $\{\varphi_j\}$ are designed so as to have the property

$$\varphi_j(\mathbf{x}_i) = \delta_{ij}, \quad 1 \leq i, j \leq N.$$

Then u_h and v_h are of the form

$$\left.\begin{aligned} u_h(\mathbf{x}) &= \sum_{i=1}^N u_i\varphi_i(\mathbf{x}), \quad v_h(\mathbf{x}) = \sum_{i=1}^N v_i\varphi_i(\mathbf{x}), \quad \mathbf{x} \in \Omega_h \\ \text{where} \quad u_i &= u_h(\mathbf{x}_i) \quad \text{and} \quad v_i = v_h(\mathbf{x}_i) \end{aligned}\right\} \tag{3.2}$$

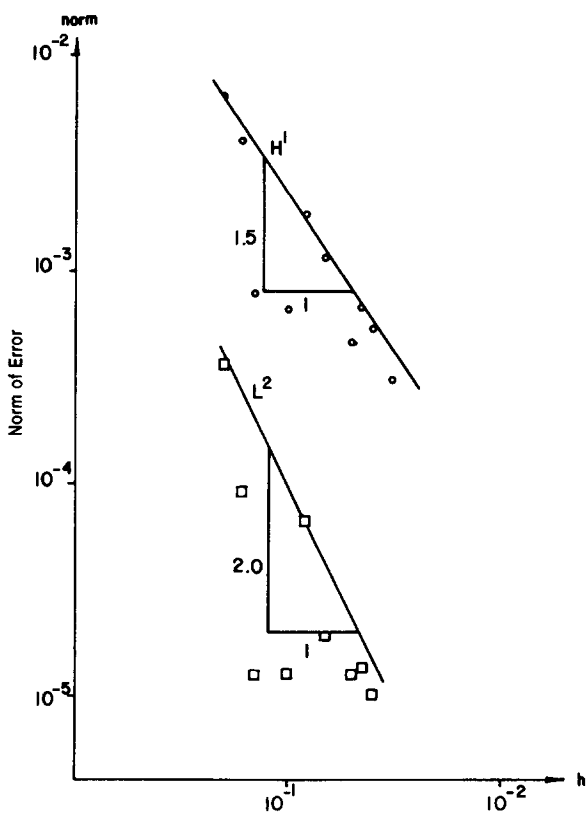


Fig. 2.1. Rates of convergence of a quadratic finite element approximation of Example 1-1.1.

Substitution of (3.2) into (3.1) yields the system of N inequalities in the N unknowns $u_1, u_2, \ldots, u_N$:

$$u_h \in K_h: \left\langle A\left(\sum_{i=1}^{N} u_i\varphi_i\right),\quad v_h - \sum_{k=1}^{N} u_k\varphi_k\right\rangle \geq 0,\quad v_h \in K_h,\quad 1 \leq j \leq N. \tag{3.3}$$

To proceed further, we must solve numerically the system (3.3) in such a way that the constraint $u_h \in K_h$ is fulfilled.

The purpose of this section is to outline and discuss several numerical techniques for solving systems of inequalities of the type (3.3). Since variational inequalities are closely related to constrained minimization problems, several standard algorithms in use in the theory of constrained optimization problems are directly applicable to our study.

Here we discuss four major methods of this type: fixed point methods (i.e. successive approximation), pointwise relaxation methods, penalty methods and Lagrange multiplier methods. We follow the works of Cryer[37], Cea–Glowinski[38], Levitin–Polyak[39] and Glowinski–Lions–Tremolieres[6].

We will adopt the following conventions. Let $\mathcal{U}$ be an N-dimensional Euclidean inner-product space with inner product $(\cdot, \cdot)$ given by

$$(u, v) = \sum_{i=1}^{N} u_i v_i \tag{3.4}$$

and the natural norm on $\mathcal{U}$ induced by $(\cdot, \cdot)$ is

$$\|u\| = (u, u)^{1/2} \tag{3.5}$$

Let K be a non-empty closed convex subset of $\mathcal{U}$, and let a map A be continuous and strongly monotone from $\mathcal{U}$ into $\mathcal{U}'$. (Here $\mathcal{U}'$ is identified with $\mathcal{U}$ itself.) That is, we assume constants m and M exist such that

$$\left.\begin{array}{c}(A(u) - A(v), u - v) \geq m\|u - v\|^2, \quad m > 0 \\ (A(u) - A(v), w) \leq M\|u - v\|\|w\|\end{array}\right\} \tag{3.6}$$

for every $u, v, w \in \mathcal{U}$. We will investigate solution methods for (3.3) for cases in which A satisfies (3.6).

(i) *Fixed-point methods.* Our first method involves a simple reiteration of the contraction mapping ideas developed in Section 1.3. Recall that if P_K is a projection of $\mathcal{U}$ onto a set K, the mapping

$$T(u) = P_K(u - \rho A(u))$$

is a contraction mapping for operators satisfying (3.6) whenever ρ satisfies $0 < \rho < 2m/M^2$. For this choice of ρ, T has a unique fixed point in K which can be calculated using the classical method of successive approximations: $u^t = T(u^{t-1})$, $t = 1, \ldots$.

We summarize the essential ideas in the following theorem.

Theorem 2-3.1. Let $\mathcal{U}$ be an N-dimensional inner product space and let A: $\mathcal{U} \to \mathcal{U}'$ satisfy (3.6). Then the variational inequality

$$u \in K: (A(u), v - u) \geq 0 \quad \forall v \in K \tag{3.7}$$

admits a unique solution which can be calculated as the limit of the sequence $\{u^t\}$ where

$$u^{t+1} = P_K(u^t - \rho A(u^t)),\quad \rho > 0,\quad u^0 \in K. \tag{3.8}$$

Here P_K is the projection map of $\mathcal{U}$ onto K, and

$$0 < \rho < 2m/M^2. \quad \square \tag{3.9}$$

Use of this iteration scheme to obtain solutions of variational inequalities has been discussed by Brezis and Sibony[39].

(ii) *Pointwise relaxation methods.* Suppose that the non-empty closed convex set K is representable in the form

$$K = \prod_{i=1}^{N} K_i, \quad K_i = [a_i, b_i] \tag{3.10}$$

where a_i and b_i are some real numbers. We only consider here the case in which the map A is potential, that is, there exists a potential F: $\mathcal{U} \to \mathbf{R}$ such that its Gâteaux derivative DF coincides with A. Let

$$\begin{gathered} {}^{i}u^{t+1} = (u_1^{t+1}, \ldots, u_i^{t+1}, u_{i+1}^{t}, \ldots, u_N^{t}) \\ {}^{i}v^{t+1} = (u_1^{t+1}, \ldots, u_{i-1}^{t+1}, v_i, u_{i+1}^{t}, \ldots, u_N^{t}) \\ t = 0, 1, 2, \ldots \end{gathered}$$

where we use the convention

$${}^{0}u^{t+1} = u^t = (u_1^t, \ldots, u_N^t).$$

We note that the variational inequality

$$u \in K: (DF(u), v-u) \geq 0 \quad \forall u \in K \tag{3.11}$$

is now equivalent to the minimization problem

$$u \in K: F(u) \leq F(v) \quad \forall v \in K. \tag{3.12}$$

The pointwise relaxation scheme is based on the algorithm

$$u_i^{t+1} \in K_i: F({}^{i}u^{t+1}) \leq F({}^{i}v^{t+1}) \quad \forall v_i \in K_i \tag{3.13}$$

for $i = 1, \ldots, N$ and $t = 0, 1, \ldots$. Here we assume that $u^0 \in K$.

Theorem 2-3.2. Let F be a continuously differentiable functional on K such that†

$$\left.\begin{aligned} &(DF(u) - DF(v), u-v) \geq r_M(\|u-v\|) \\ &\|u\| < M, \quad \|v\| < M, \quad \forall u, v \in K \end{aligned}\right\} \tag{3.14}$$

where $t \to r_M(t)$: $[0, 2M] \to \mathbf{R}^+$ is continuous strictly increasing function such that $r_M(0) = 0$. Suppose either that K is bounded or that F is coercive on K, i.e.,

$$F(v) \to +\infty \quad \text{as} \quad \|v\| \to +\infty \quad \forall v \in K.$$

Then, the pointwise relaxation procedure (3.13), with K given by (3.10), converges to the solution $u \in K$ of the problem (3.11).

Proof. Setting $v_i = u_i^t$ in (3.13) gives

$$F(u^{t+1}) \leq F({}^{N-1}u^{t+1}) \leq \cdots \leq F(u^t) \leq \cdots \leq F(u^0). \tag{3.15}$$

The coerciveness of F or the boundedness of K implies the boundedness of u^t; i.e.

$$\|u^t\| \leq M \quad \text{for any} \quad t.$$

†Condition (3.14) is guaranteed by the strict convexity of the continuously differentiable functional F *on* $\mathbf{R}^N$. See Cea-Glowinski[38].

By integrating the expression

$$(DF(u+s(v-u))-DF(u), v-u)\geq \frac{1}{s} r_M(s\|v-u\|)$$

from 0 to 1 in s, with

$$\hat{r}_M(s)=\int_0^s r_M(x)\frac{\mathrm{d}x}{x}$$

we obtain

$$F(v)-F(u)\geq (DF(u), v-u)+\hat{r}_M(\|v-u\|).$$

Thus

$$F({}^{i-1}u^{t+1})-F({}^{i}u^{t+1})\geq (u_i^t-u_i^{t+1})D_iF({}^{i}u^{t+1})+\hat{r}_M(|u_i^t-u_i^{t+1}|)$$

where $D_iF(v)$ is the i-directional derivative of F at v. By the definition (3.13) of u_i^{t+1},

$$(v_i-u_i^{t+1})D_iF({}^{i}u^{t+1})\geq 0 \quad \forall v_i\in K_i. \tag{3.16}$$

Thus, we have

$$F({}^{i-1}u^{t+1})-F({}^{i}u^{t+1})\geq \hat{r}_M(|u_i^t-u_i^{t+1}|).$$

Summing from $i=1$ to $i=N$ gives

$$F(u^t)-F(u^{t+1})\geq \sum_{i=1}^{N}\hat{r}_M(|u_i^t-u_i^{t+1}|).$$

By (3.15), $F(u^t)-F(u^{t+1})\to 0$ as $t\to\infty$, which implies $|u_i^t-u_i^{t+1}|\to 0$ as $t\to\infty$ for every i, i.e.

$$u^t-u^{t+1}\to 0 \quad \text{as} \quad t\to+\infty.$$

We now show that u^t converges to the solution $u\in K$ of the problem (3.7). According to (3.14)

$$(DF(u^{t+1})-DF(u), u^{t+1}-u)\geq r_M(\|u^{t+1}-u\|).$$

Let u be the solution of (3.7), i.e. u satisfies $(DF(u), v-u)\geq 0$, $\forall v\in K$. Then

$$(DF(u^{t+1}), u^{t+1}-u)\geq r_M(\|u^{t+1}-u\|). \tag{3.17}$$

Using (3.16), we have

$$\sum_{i=1}^{N}(v_i-u_i^{t+1})D_iF({}^{i}u^{t+1})\geq 0 \quad v_i\in K_i, \quad i=1,\ldots,N. \tag{3.18}$$

Under the condition (3.10), $u_i\in K_i$ since $u\in K$. Adding (3.17) and (3.18) yields

$$\sum_{i=1}^{N}(u_i^{t+1}-u_i)(D_iF(u^{t+1})-D_iF({}^{i}u^{t+1}))\geq r_M(\|u^{t+1}-u\|).$$

Since F is continuously differentiable and coercive, and since $\|u^{t+1} - {}^i u^{t+1}\| \leq \|u^{t+1} - u^t\|$, we can conclude that

$$\|u^{t+1} - u\| \to 0 \quad \text{as} \quad t \to \infty$$

This completes the proof of the theorem. □

It is important to note that property (3.10) is crucial to the proof of the above theorem. In many simple cases (e.g. if $K = \{(v_1, v_2) \in \mathbf{R}^2,\ v_1 \geq 0,\ v_2 \geq 0,\ v_1 + v_2 \geq 1\}$) this convergence theorem is not directly applicable.

A final question must be resolved if we are to use this algorithm in actual computations; namely, how can we compute the intermediate minimizers u_i^{t+1} in (3.13), i.e. the solution $u_i^{t+1} \in K_i$ of the inequality (3.16)? To answer this we recall Example 1-3.1. Let

$$D_i F({}^i u^{t+1}) = \sum_{j=1}^{i} \widehat{DF}_{ij} u_j^{t+1} + \sum_{j=i+1}^{N} \widehat{DF}_{ij} u_j^t.$$

Dividing (3.16) by DF_{ii}, we have

$$(v_i - u_i^{t+1})\Big(u_i^{t+1} - \sum_{j=1}^{i-1} \widehat{DF}_{ij} u_j^{t+1} - \sum_{j=i+1}^{N} \widehat{DF}_{ij} u_j^t\Big) \geq 0, \quad \forall v_i \in [a_i, b_i]$$

where

$$\widehat{DF}_{ij} = DF_{ij}/DF_{ii}.$$

Then, applying (1-3.9), we have

$$u_i^{t+1} = \min\Big\{\max\Big(a_i, \sum_{j=1}^{i-1} \widehat{DF}_{ij} u_j^{t+1} + \sum_{j=i+1}^{N} \widehat{DF}_{ij} u_j^t\Big), b_i\Big\}$$

for $1 \leq i \leq N$.

Remark 2-3.1. As we mentioned, if the constraint set K is given by, e.g.

$$K = \{(v_1, v_2) \in \mathbf{R}^2 \colon v_1 + v_2 \geq 1\}$$

we cannot apply the above convergence theorem. However, in this case, we change the variable u_2 by the affine transformation

$$u_2 = v_1 + v_2 - 1.$$

Then the constraint becomes simply

$$u_2 \geq 0.$$

For example, if the minimization of the functional

$$F(v_1, v_2) = \frac{1}{2} v_1^2 + \frac{1}{2} v_2^2$$

is considered on K, using the new variables (v_1, u_2), we need to minimize

$$\begin{aligned} F(v_1, u_2) &= \frac{1}{2} v_1^2 + \frac{1}{2}(u_2 - v_1 + 1)^2 \\ &= v_1^2 - v_1 u_2 + \frac{1}{2} u_2^2 - v_1 + u_2 + \frac{1}{2} \end{aligned}$$

on the set $\hat{K}$ given by

$$\hat{K} = \{(v_1, u_2): u_2 \geq 0\}.$$

Thus, we can apply the algorithm described in Theorem 2-3.2. □

(iii) *Penalty methods.* Both of the methods discussed up to this point involve the construction of a projection map P_K. In the pointwise relaxation method, P_K is constructed only when the non-empty closed convex subset K has the form $K = \prod_{i=1}^{N} K_i$. One alternative approach which avoids the construction of a projection P_K is provided by the so-called penalty methods.

To describe the general ideas underlying penalty methods, let us consider the case in which K is of the form

$$K = \{v \in \mathcal{U}: M_j(v) \geq 0; \quad j = 1, \ldots, m, \quad m < N\}. \tag{3.19}$$

Here $M_j(\cdot)$ is a continuously differentiable concave function on $\mathcal{U}$ for each j, $1 \leq j \leq m$. Further, suppose that the operator A (appearing in (3.1) and satisfying (3.6)) is derivable from a convex potential F: $\mathcal{U} \to \mathbf{R}$; i.e. $A = DF$. Then the variational inequality is equivalent to the minimization problem: find $u \in K$ such that

$$F(u) \leq F(v) \quad \forall v \in K. \tag{3.20}$$

In this case, A is continuous, monotone and symmetric on $\mathcal{U}$.

Penalty methods for this class of problems involve the construction of special auxiliary functionals which depend on an arbitrary real parameter ϵ and are constructed as the sum of F and a "penalty term" which depends on ϵ and the constraint K. In the present case, we may introduce the penalized functional

$$E(v, \epsilon) = F(v) + \frac{1}{\epsilon} M(v)^- \tag{3.21}$$

where

$$M_j(v)^- = \sup\{-M_j(v), 0\}, \; M(v)^- = \sum_{j=1}^{m} M_j(v)^-. \tag{3.22}$$

Then

$$M(u)^- = 0 \quad \text{if and only if} \quad u \in K.$$

Next, instead of (3.20), we consider, for fixed ϵ, the auxiliary minimization problem

$$u_\epsilon \in \mathcal{U}: E(u_\epsilon, \epsilon) \leq E(v, \epsilon) \quad \forall v \in \mathcal{U}. \tag{3.22}$$

Since $M(v)$ is concave and continuous on $\mathcal{U}$, the penalized functional $E(\cdot, \epsilon)$ is also convex and continuous. Since $M(v) \geq 0$, $\forall v \in \mathcal{U}$ and since $F(\cdot)$ is coercive, there exists a solution $u_\epsilon \in \mathcal{U}$ of the penalized minimization problem (3.22) for each $\epsilon > 0$. If DF (or A) is strictly monotone, F is strictly convex. Then the solution is unique for every $\epsilon > 0$. Coerciveness of $F(\cdot)$ on $\mathcal{U}$ implies the uniform boundedness of u_ϵ in ϵ.

The importance of the solutions u_ϵ of the penalized problem is made clear in the following theorem:

Theorem 2-3.3. Let F: $\mathcal{U} \to \mathbf{R}$ be coercive, strictly convex, and differentiable on $\mathcal{U}$ and let $K \subset \mathcal{U}$ be given by (3.19). Then

(i) There exists a unique solution $u \in K$ of the minimization problem

$$F(u) \leq F(v), \quad v \in K. \tag{3.23}$$

(ii) The solution u of (3.23) satisfies the variational inequality

$$(DF(u), v-u) \geq 0, \quad v \in K. \tag{3.24}$$

(iii) For every $\epsilon > 0$, there exists a unique solution $u_\epsilon \in \mathcal{U}$ of the penalized minimization problem

$$E(u_\epsilon, \epsilon) \leq E(v, \epsilon) \quad \forall v \in \mathcal{U} \tag{3.25}$$

(iv) The sequence $\{u_\epsilon\}$ of solutions to (3.25) obtained as $\epsilon \to 0$ converges strongly to the solution u of problems (3.23) or, equivalently, (3.24).

Proof. Since

$$F(u_\epsilon) \leq E(u_\epsilon, \epsilon) \leq \inf_{v \in K} E(v, \epsilon) = \inf_{v \in K} F(v) = F(u)$$

coerciveness of $F(\cdot)$ on $\mathcal{U}$ implies the existence of a convergent subsequence $u_{\epsilon'}$ of u_ϵ whose limit is $w \in \mathcal{U}$. Since F is convex and differentiable, it is lower semicontinuous and, therefore

$$F(w) \leq F(u_{\epsilon'}) \leq F(u).$$

Moreover

$$E(u_{\epsilon'}, \epsilon') \leq F(u); \quad \text{i.e.} \quad M(u_{\epsilon'})^- \leq \epsilon'(F(u) - F(u_{\epsilon'})).$$

Taking the limit as $\epsilon \to 0$ yields

$$M(u)^- \leq 0.$$

Since $M(w)^- \geq 0$, $M(w)^- = 0$; i.e. $w \in K$. Since F is strictly convex, its minimizer in K is unique; i.e. $u = w$. This conclusion is reached for every convergent subsequence $u_{\epsilon''}$ of u_ϵ. Thus, the original sequence u_ϵ converges to $w = u \in K$ as $\epsilon \to 0$. To show strong convergence, note that under the conditions of the theorem, $(3.14)_1$ holds. Since u is the solution of (3.24)

$$\begin{aligned} 0 &\geq (DF(u), u - P_K(u_\epsilon)) \\ &= (DF(u) - DF(u_\epsilon), u - u_\epsilon) + (DF(u_\epsilon), u - u_\epsilon) \\ &\quad + (DF(u), u_\epsilon - P_K(u_\epsilon)) \end{aligned}$$

Since u_ϵ is the solution of (3.25), and since the penalty functional M^- is convex, we have

$$(DF(u_\epsilon), u - u_\epsilon) \geq M^-(u_\epsilon) - M^-(u) \geq 0.$$

Then

$$\begin{aligned} 0 &\geq \lim_{\epsilon \to 0} \{ r_M(\|u - u_\epsilon\|) + (DF(u), u_\epsilon - P_K(u_\epsilon)) \} \\ &= \lim_{\epsilon \to 0} r_M(\|u - u_\epsilon\|). \end{aligned}$$

This indicates that the sequence $\{u_\epsilon\}$ converges strongly to u as $\epsilon \to 0$. □

In general, penalty methods can be constructed for quite general minimization problems in which

$F: \mathcal{U} \to \mathbf{R}$ is weakly lower semicontinuous and coercive.
$P: \mathcal{U} \to \mathbf{R}$ is the penalty functional, and P satisfies,

(i) $P: \mathcal{U} \to \mathbf{R}$ is differentiable (in the sense of Gâteaux) and weakly lower semicontinuous,
(ii) $P(v) \geq 0$ and $P = 0$ if and only if $v \in K$; $v \notin K$ implies $P(v) > 0$.

Then the penalty functional

$$E(v, \epsilon) = F(v) + \frac{1}{\epsilon} P(v) \tag{3.26}$$

has a minimizer u_ϵ (not necessarily unique) for every $\epsilon > 0$, and a subsequence of minimizers $\{u_\epsilon\}$ converges weakly to a solution u of the minimization problem

$$\inf_{v \in K} F(v) = F(u).$$

We further generalize the penalty method to the variational inequality (3.7). Let A be a continuous map from a non-empty closed convex subset K of a finite dimensional Euclidean space $\mathcal{U}$ into $\mathcal{U}'$. Let P_K be the projection map from $\mathcal{U}$ onto K. Then the map β: $\mathcal{U}\to\mathcal{U}'$ given by

$$\beta(v) = v - P_K(v) \tag{3.27}$$

is monotone and continuous on $\mathcal{U}$ and

$$\beta(v) = 0 \quad \text{if and only if} \quad v \in K.$$

Indeed, using (1–3.2)

$$\begin{aligned}(\beta(u) - \beta(v), u - v) &= (u - v, u - v) - (P_K(u) - P_K(v), u - v)\\ &\geq \|u - v\|^2 - \|P_K(u) - P_K(v)\|\|u - v\|\\ &\geq 0\end{aligned}$$

and

$$(\beta(u) - \beta(v), w) = (u - v, w) - (P_K(u) - P_K(v), w) \leq 2\|u - v\|\|w\|$$

Then, for every $\epsilon > 0$, there exists a unique solution $u_\epsilon \in \mathcal{U}$ ($\mathcal{U}$, we recall, is now finite dimensional) to the problem

$$u_\epsilon \in \mathcal{U}: (A(u_\epsilon) + \frac{1}{\epsilon}\beta(u_\epsilon), v) = 0 \quad \forall v \in \mathcal{U} \tag{3.28}$$

provided A is assumed to be strictly monotone, and coercive in the sense that

$$(A(v) - v_0, v)/\|v\| \to +\infty \quad \text{as} \quad \|v\| \to \infty, \quad v_0 \in K$$

Furthermore, u_ϵ is uniformly bounded in $\epsilon > 0$. Then there exists a convergent subsequence $u_{\epsilon'}$ which converges to $w \in \mathcal{U}$. By the definition of u_ϵ

$$(\beta(u_\epsilon), v) \leq \epsilon\|A(u_\epsilon)\|\|v\| \quad \forall v \in \mathcal{U}$$

i.e.

$$\beta(u_\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0$$

Since β is continuous, $\beta(w) = 0$, i.e. $w \in K$. Moreover, for every $v \in K$

$$(A(u_\epsilon), v - u_\epsilon) + \frac{1}{\epsilon}(\beta(u_\epsilon) - \beta(v), v - u_\epsilon) = 0$$

i.e.

$$(A(u_\epsilon), v - u_\epsilon) = \frac{1}{\epsilon}(\beta(u_\epsilon) - \beta(v), u_\epsilon - v) \geq 0$$

Taking $\epsilon \to 0$, we have

$$(A(w), v - w) \geq 0 \quad \forall v \in K$$

because of continuity of A. Since the solution of the variational inequality

$$u \in K: (A(u), v - u) \geq 0 \quad \forall v \in K$$

is unique, $w = u$, and for every convergent subsequence $u_{\epsilon''}$ the conclusions are the same. Thus the sequence u_ϵ converges to the solution $u \in K$ of the variational inequality.

Summarizing, we have:

Theorem 2-3.4. Under the conditions stated above, the sequence $u_\epsilon \in \mathcal{U}$ such that

$$u_\epsilon \in \mathcal{U}: (A(u_\epsilon)+\frac{1}{\epsilon}\beta(u_\epsilon), v)=0 \quad \forall v \in \mathcal{U}$$

converges to the unique solution $u \in K$ of the variational inequality

$$u \in K: (A(u), v-u) \geq 0 \quad \forall v \in K$$

as $\epsilon \to 0$. □

More general cases in reflexive Banach spaces are discussed by Lions[11].

If K is defined by an *equality constraint* such as

$$M(v)=0, \quad M \text{ is continuous}$$

then the penalized functional $E(v, \epsilon)$ can be defined, for example, by

$$E(v, \epsilon)=F(v)+\frac{1}{2\epsilon}(M(v), M(v)) \tag{3.29}$$

(iv) *Lagrange multiplier methods.* A method closely related to penalty methods is the classical Lagrange multiplier method. Again the idea is to release constraint conditions defining a closed convex set K by amending the "cost" functional F. If equality constraints are involved, Lagrange multipliers methods can be applied in the usual way with no restrictions. However, if the constraint conditions are the inequality type, some restrictions must be also imposed on the Lagrange multipliers. It is notable that the admissible set K for such restricted Lagrange multiplier problems can always be represented in the form

$$N=\prod_{i=1}^{m} N_i, \quad N_i=[c_i, d_i]$$

even though the set K which characterizes the original constraint condition cannot be represented by the form

$$K=\prod_{i=1}^{n} K_i, \quad K_i=[a_i, b_i].$$

This means that the pointwise relaxation scheme described earlier is applicable to formulations based on Lagrange multiplier methods and, as is well known, the optimization problem can be formulated in such a way that the unknowns are free from any constraint conditions. While penalty methods often lead to non-linear equations and sometimes even non-differentiable functionals for linear operator equations, Lagrange multiplier methods need not have such difficulties. Computationally, however, the Lagrange multiplier methods often lead to iterative schemes whose rate of convergence is slower than other methods for problems which can be resolved by all the methods discussed earlier. We shall study this feature in some detail in the next section.

Let us consider the problem

$$\left.\begin{aligned} &u \in K: (A(u), v-u) \geq 0 \quad \forall v \in K \\ &K=\{v \in \mathcal{U}: M_j(v) \leq 0, \quad j=1, \ldots, m\} \end{aligned}\right\} \tag{3.27}$$

where the operator A maps $\mathcal{U}$ into itself and is such that (3.6) holds and M: $\mathcal{U} \to \mathcal{V}$; $\mathcal{V}=\mathbf{R}^m$, is an operator satisfying

$$\|M(u)-M(v)\|_m \leq c\|u-v\|_n \tag{3.28}$$

where $\|\cdot\|_m$ denotes the norm in $\mathbf{R}^m$ $(m \leq N)$. Introducing the notation

$$(p, M(u))_m = \sum_{j=1}^{m} p_j M_j(u)$$

we introduce Lagrange multipliers p and replace (3.27) by the equivalent system

$$\left.\begin{aligned}(A(u), v-u)-(p, M(v)-M(u))_m \geq 0 \quad \forall v \in \mathcal{U} \\ (q-p, M(u))_m \geq 0 \quad \forall q \in N\end{aligned}\right\} \tag{3.29}$$

where N is the set

$$N = \{q \in \mathbf{R}^m: q_i \leq 0, \quad i = 1, 2, \ldots, m\}. \tag{3.30}$$

We now establish the following iterative procedure for the numerical solution of (3.29).

(i) Pick a starting value $p^0 = 0$.

(ii) Determine the tth iterate $u^t \in \mathcal{U}$ as the solution of the unconstrained problem

$$(A(u^t), v) - (p^t, M(v))_m = 0, \quad \forall v \in \mathcal{U}.$$

(iii) Using u^t, p^{t+1} is defined by

$$p^{t+1} = P_N(p^t - \rho M(u^t))$$

where $P_N: \mathcal{V} \to N$ a projection of $\mathcal{V}$ onto N. Then (u^t, p^t) converges to the solution $(u, p) \in \mathcal{U} \times N$ of problem (3.29). In fact, by (ii)

$$(A(u^t), v-u^t) - (p^t, M(v) - M(u^t))_m \geq 0 \quad \forall v \in \mathcal{U},$$

i.e.

$$(A(u^t), u - u^t) - (p^t, M(u) - M(u^t))_m \geq 0.$$

Also

$$(A(u), u^t - u) - (p, M(u^t) - M(u))_m \geq 0.$$

Adding these two inequalities gives

$$(A(u^t) - A(u), u^t - u) - (p^t - p, M(u^t) - M(u))_m \leq 0.$$

Putting $e^t = u^t - u$ and $r^t = p^t - p$, we have

$$m\|e^t\|^2 \leq (r^t, M(u^t) - M(u))_m \tag{3.31}$$

and, according to (ii) and the definition of p

$$\begin{aligned} p^{t+1} &= P_N(p^t - \rho M(u^t)) \\ p &= P_N(p - \rho M(u)). \end{aligned}$$

Then

$$r^{t+1} = P_N(p^t - \rho M(u^t)) - P_N(p - \rho M(u)).$$

Since P_N is non-expansive

$$\|r^{t+1}\|_m^2 \leq \|r^t - \rho(M(u^t) - M(u))\|_m^2$$
$$\leq \|r^t\|_m^2 - 2\rho(r^t, M(u^t) - M(u))_m + \rho^2\|M(u^t) - M(u)\|_m^2.$$

By (3.28) and (3.31)

$$\|r^{t+1}\|_m^2 \leq \|r^t\|_m^2 - 2\rho m\|e^t\|^2 + \rho^2 c^2\|e^t\|^2.$$

If $\rho(\rho c^2 - 2m) < 0$ (i.e. $0 < \rho < 2m/c^2$), $\|r^t\|_m$ is decreasing as $t \to +\infty$. Thus u^t converges to u as $t \to \infty$, indeed,

$$\|r^{t+1}\|_m^2 + \beta\|e^t\|^2 \leq \|r^t\|_m^2$$

where $\beta = \rho(\rho c^2 - 2m)$. For additional details, see Glowinski *et al.* [6].

Summarizing, we have

Theorem 2-3.5. Suppose that A and M satisfy (3.6) and (3.28). Then the sequence (u^t, p^t) defined by the above algorithm converges to the solution (u, p) of (3.29) as $t \to \infty$. □

The obvious computational procedure suggested by (i) and (ii) is generally known as Uzawa's method (see Arrow *et al.* [40]).

We finally show that the solution $(u, p) \in \mathcal{U} \times N$ to the Lagrangian problem (3.29) also satisfies the variational inequality (3.27).

Since $(3.29)_2$ is equivalent to the system

$$(p, M(u)) = 0, \quad p \leq 0, \quad \text{and} \quad M(u) \leq 0$$

we have, from $(3.29)_1$

$$(A(u), v - u) \geq (p, M(v) - M(u)) = (p, M(v)) \geq 0, \quad v \in K$$

2.4 *Numerical experiments*

In this section, we will consider a 1-dimensional version of the problem of seepage flow through a homogeneous rectangular dam. We will solve this problem numerically using the four solution methods discussed in the previous section. See Chap. 3.

The example problem involves finding a solution $u \in K$ of the variational inequality considered in Example 1-1.1

$$\int_0^1 u'(v - u)' \, dx + \int_0^1 (v - u) \, dx \geq 0 \quad \forall v \in K \tag{4.1}$$

where

$$K = \{v \in H^1(0, 1): v(0) = 1/4, \quad v(1) = 0, \quad \text{and} \quad v(x) \geq 0 \quad \text{in} \quad (0, 1)\} \tag{4.2}$$

and $u' = du/dx$.

Let the domain $\Omega = (0, 1)$ be discretized by a uniform mesh containing N-1 finite elements. Within a finite element, every function $v \in H^1(\Omega)$ is approximated by functions of the form

$$v = v^1\varphi_1(\xi) + v^2\varphi_2(\xi) \tag{4.3}$$

where v^i is the value of v at ith local nodal point, $\varphi_i(\xi)$ is the local interpolation function at ith local nodal point, and ξ is a local coordinate in the finite element. For a unit linear element

$$\varphi_1(\xi) = 1 - \xi, \quad \varphi_2(\xi) = \xi. \tag{4.4}$$

Upon assembling the elements, a global model is obtained for which every function $v \in H^1(\Omega)$ is approximated by

$$v_h(x)=\sum_1^n v_i\psi^i(x) \tag{4.5}$$

where v_i is the value of v_h at ith (global) nodal point, and $\psi^i(x)$ is the global basis function corresponding to the ith (global) nodal point. Introducing the approximation (4.5) into (4.1) and (4.2), the variational inequality (4.1) on the admissible set (4.2) reduces to an optimization problem in $\mathbf{R}^N$

$$\{u_i\}\in R_h: (v_i-u_i)(K^{ij}u_j-F^i)\geq 0 \quad \forall\{v_i\}\in R_h \tag{4.6}$$

where repeated indices are summed and

$$R_h=\{\{v_i\}\in\mathbf{R}^N: v_1=1/4, \quad v_N=0, \quad v_i\geq 0, \quad i=2,\dots,N-1\} \tag{4.7}$$

$$K^{ij}=\int_0^1(\psi^i)'(\psi^j)'\,\mathrm{d}x, \;\; F^i=-\int_0^1\psi^i\,\mathrm{d}x \tag{4.8}$$

and N is the total number of nodal points in the finite element model.

We now solve (4.6) and (4.7) using the four methods discussed in the previous section.

(i) *Fixed point methods.* The iterative scheme defined by (3.8) becomes

$$u_i^{t+1}=\max(0., u_i^t-\rho_i(K^{ij}u_j^t-F^i)), \quad i=1,\dots,N \tag{4.9}$$

assuming we have constructed some initial approximation $u_i^1\geq 0$, $i=1,\dots,N$. That is, the operator $A(\cdot)$ is defined by

$$A(\cdot)=[K]\{\cdot\}-\{F\} \tag{4.10}$$

and the projection P_K is defined pointwise by

$$P_K(\{\cdot\})=\{\max(0.,\cdot)\}. \tag{4.11}$$

In (4.10), $[K]$ is the $N\times N$-matrix defined by $(4.8)_1$. We note that the pointwise expression (4.11) of the projection is implied by the special structure of the admissible set R_h defined by (4.7); that is, R_h can be represented by the product of R_h^i

$$R_h=\prod_{i=1}^N R_h^i \tag{4.12}$$

$$R_h^i=\{v^i\in\mathbf{R}: v^i\geq 0, \quad v^i=1/4 \;\text{ if }\; i=1, \quad v^i=0 \;\text{ if }\; i=N\}. \tag{4.13}$$

The iteration factor ρ_i appearing in (4.9) has to be chosen so that condition (3.9) is satisfied.

In general, it is preferable to use a modification of the iteration scheme (4.9) given by

$$u_i^{t+1}=\max\left(0., u_i^t-\rho_i\left(\sum_{j=1}^{i-1}K^{ij}u_j^{t+1}+\sum_{j=i}^{N}K^{ij}u_j^t-F^i\right)\right). \tag{4.14}$$

That is, the terminal values u_j^{t+1}, $1\leq j<i$, are used in calculating the value u_i^{t+1}. Moreover, if the iteration factor ρ_i is chosen so that

$$\rho_i=\alpha/K_{ii} \tag{4.15}$$

then (4.14) reduces to the same form as the pointwise projectional S.O.R. method to be discussed later. For this case, (4.14) becomes

$$u_i^{t+1}=\max\left(0., (1-\alpha)u_i^t+\alpha\left(-\sum_{j=1}^{i-1}K^{ij}u_j^{t+1}-\sum_{j=i+1}^{N}K^{ij}u_j^t+F^i\right)\Big/K_{ii}\right). \tag{4.16}$$

This suggests that as a choice for the value of ρ_i, we take

$$\rho_i = \alpha / K^{ii}, \quad 0 < \alpha < 2 \tag{4.17}$$

since the S.O.R. method for positive definite linear systems converges for $0 < \alpha < 2$.

(ii) *Pointwise relaxation methods.* Since the matrix $[K]$ defined by (4.8) is symmetric, there exists a functional $F(v)$ such that

$$F(v) = \frac{1}{2} v_i K^{ij} v_j - F^i v_i \quad \text{and} \quad \frac{\partial F(v)}{\partial v_i} = K^{ij} v_j - F^i \tag{4.18}$$

(here repeated indices are summed; $1 \le i, j \le N$). Then, the discrete variational inequality (4.6) is equivalent to the constrained minimization problem

$$\{u_i\} \in R_h : F(u) \le F(v), \quad \forall \{v_i\} \in R_h \tag{4.19}$$

where R_h is defined by (4.7). We note that the admissible set R_h can be represented by (4.12) and (4.13) which is consistent with (3.10).

Then, the general relaxation scheme (3.13) becomes

$$u_i^{\ell+1} = \max(0., u_i^{\ell+(1/2)}) \tag{4.20}$$

where

$$u_i^{\ell+(1/2)} = \left(-\sum_{j=1}^{i-1} K^{ij} u_j^{\ell+1} - \sum_{j=i+1}^{N} K^{ij} u_j^{\ell} + F^i\right) \Big/ K^{ii}. \tag{4.21}$$

In general

$$u_i^{\ell+(1/2)} = (1-\alpha) u_i^{\ell} + \alpha\left(-\sum_{j=1}^{i-1} K^{ij} u_j^{\ell+1} - \sum_{j=i+1}^{N} K^{ij} u_j^{\ell} + F^i\right) \Big/ K^{ii} \tag{4.22}$$

is taken instead of (4.21). The scheme (4.21) is called the Gauss–Seidel algorithm, and the scheme (4.22) is called the S.O.R. algorithm for $\alpha > 1$. The pointwise relaxation (4.20) and (4.22) is called the pointwise projectional S.O.R. method here, after the S.O.R. method for systems of linear equalities. As mentioned above, the scheme (4.20) with (4.21) coincides with the special choice of ρ_i, (4.15), in (4.14). That is, the fixed-point method is equivalent to the pointwise relaxation method for the specific example (4.6) if the parameter ρ_i is chosen as in (4.15).

(iii) *Penalty methods.* Since the admissible set R_h can be represented as the product of componentwise sets in $\mathbf{R}$ as shown in (4.12) and (4.13), the penalty functional P can be constructed by the rule

$$P(v) = \frac{1}{2}(v_i^-)(v_i^-), \quad v_i^- = \sup(0, -v_i). \tag{4.23}$$

Indeed, $v \to P(v)$ is convex, continuous and differentiable. Its gradient is given by

$$DP(v) = \{-v_i^-\}. \tag{4.24}$$

Moreover, $P(v) \ge 0$ and $P(v) = 0$ if and only if $v_i^- = 0$ for every $i = 1, \ldots, N$, i.e. $v_i \ge 0$ for every $i = 1, \ldots, N$.

Since $P(\cdot)$ is convex, the gradient $DP(\cdot)$ is monotone. It is clear that $DP(\cdot)$ is continuous and $DP(v) = 0$ if and only if $v_i^- = 0$, i.e. $v_i \ge 0$ for every $i = 1, \ldots, N$. Thus the gradient $DP(\cdot)$ can be used as a penalty operator, i.e.

$$\beta(v) = DP(v) = \{-v_i^-\}. \tag{4.25}$$

It is worth noting that

$$\left.\begin{aligned} P_{R_h}(v) &= \{v_i^+\}, \quad v_i^+ = \sup(0, v_i) \\ v - P_{R_h}(v) &= \{v_i^+ - v_i^- - v_i^+\} = \{-v_i^-\}, \end{aligned}\right\} \tag{4.26}$$

that is

$$DP(v) = (I - P_{R_h})(v). \tag{4.27}$$

Thus, the variational inequality (4.6) can be approximated by the penalized equations

$$\{u_i^\epsilon\} \in \mathbf{R}^N: K^{ij}u_j^\epsilon - \frac{1}{\epsilon}(u_i^\epsilon)^- = F^i, \quad i = 1, \ldots, N \tag{4.28}$$

with the boundary condition

$$u_1^\epsilon = 1/4 \quad \text{and} \quad u_N^\epsilon = 0. \tag{4.29}$$

The non-linear non-differentiable system (4.28) can be solved by the modified S.O.R. method. That is, at tth increment, $u_i^{\epsilon,t}$ can be obtained by the algorithm

$$\left.\begin{aligned} &R_i^t = -\sum_{j=1}^{i-1} K^{ij}u_j^{\epsilon,t} - \sum_{j=i+1}^{N} K^{ij}u_j^{\epsilon,t-1} + F^i \\ &D_i = K^{ii} + \frac{1}{\epsilon} \quad \text{if} \quad u_i^{\epsilon,t-1} < 0 \qquad\qquad (i\text{: no sum}) \\ &D_i = K^{ii} \quad \text{if} \quad u_i^{\epsilon,t-1} \geq 0 \\ &u_i^{\epsilon,t} = (1 - \omega_i)u_i^{\epsilon,t-1} + \omega_i R_i^t / D_i. \end{aligned}\right\} \tag{4.30}$$

Here the iteration factor ω_i is defined, for $0<\omega<2$, by

$$\omega_i = \omega \quad \text{if} \quad u_i^{\epsilon,t-1} \geq 0, \quad \omega_i = 1.0 \quad \text{if} \quad u_i^{\epsilon,t-1} < 0. \tag{4.31}$$

(iv) *Lagrange multiplier methods*. The Lagrange multiplier p^i is introduced to release the constraint

$$v_i \geq 0, \quad i = 1, \ldots, N$$

in the admissible set R_h defined by (4.7). The vector $\{p\}$ is expected to satisfy

$$p^i v_i = 0, \quad p^i \geq 0, \quad i = 1, \ldots, N. \tag{4.32}$$

The problem corresponding to (3.29) is

$$\left.\begin{aligned} K^{ij}u_j - p^i &= F^i \\ (q^i - p^i)u_i \geq 0 \quad \forall\{q^i\} &\in \mathbf{R}^N \quad \text{with} \quad q^i \geq 0. \end{aligned}\right\} \tag{4.33}$$

The solution $(\{u\}, \{p\})$ to the problem (4.33) is obtained by the iterative scheme

$$\left.\begin{aligned} &\text{(i)} \quad u_j^t = (K^{ij})^{-1}(F^i + p_t^i) \\ &\text{(ii)} \quad p_{t+1}^i = \max(0., p_t^i - \rho u_i^t). \end{aligned}\right\} \tag{4.34}$$

Here, the initial vector $\{p_1\}$ is given so that $p_1^i \geq 0$, $i = 1, \ldots, N$. In (4.34), (i) can be solved for unknown vector $\{p_t\}$ at each iteration step. However, (4.34) is sometimes solved by the full iterative scheme

$$\left.\begin{aligned} u_i^t &= (1-\alpha)u_i^{t-1} + \alpha\left(-\sum_{j=1}^{i-1} K^{ij}u_j^t - \sum_{j=i+1}^{N} K^{ij}u_j^{t-1} + F^i + p_t^i\right)\Big/K^{ii} \\ p_{t+1}^i &= \max(0., p_t^i - \rho u_i^t). \end{aligned}\right\} \tag{4.35}$$

While the factor α is in $(0, 2)$ as in the usual S.O.R. method, the iteration factor ρ for the Lagrange multiplier $\{p\}$ has to be sufficiently small. One suggestion for the choice of ρ is

$$\rho = 0.05 \times (\min_{1\le i\le N} K^{ii}) \tag{4.36}$$

since the dimension of ρu_i^t becomes the same as that of $\{p^i\}$.

Example 2-4.1. We solve the variational inequality (4.6) for $N = 20$ using the four methods described above.

In Fig. 2.2, the exact solution and approximate finite element results on the variational inequality (4.6) are shown.

In Table 2.1, the convergence of the penalty method with respect to ϵ is also shown. A rather large ϵ gives a reasonable approximation ($\epsilon = 10^{-2}$) to the solution of the variational inequality (4.6).

In Table 2.2, numerical results obtained by using the schemes (4.14), (4.22), (4.30) and (4.35) are listed and compared with the exact solution of (4.6). According to these numerical results, Lagrange multiplier methods give the poorest results in accuracy of the solution as well as in the speed of the convergence. Fixed-point methods and pointwise relaxation methods give results the same to four significant figures.

Table 2.1. Convergence of penalty method. Over relaxation factor, $\omega = 1.55$; Tolerance for convergence, $e = \sum_{i=1}^{n} |u_i^{t+1} - u_i^t|/|u_i^{t+1}| \le 10^{-5}$

Node	Computed results				
	$\epsilon = 1.0E\text{-}1$	$\epsilon = 1.0E\text{-}2$	$\epsilon = 1.0E\text{-}3$	$\epsilon = 1.0E\text{-}4$	$\epsilon = 1.0E\text{-}5$
1	0.250000	0.250000	0.250000	0.250000	0.250000
2	0.215704	0.215861	0.215877	0.215878	0.215878
3	0.183912	0.184226	0.184257	0.184260	0.184260
4	0.154624	0.155095	0.155141	0.155145	0.155146
5	0.127838	0.128467	0.128529	0.128534	0.128534
6	0.103555	0.104342	0.104419	0.104425	0.104426
7	0.081774	0.082720	0.082811	0.082819	0.082820
8	0.062495	0.063600	0.063706	0.063715	0.063715
9	0.045718	0.046981	0.047102	0.047111	0.047112
10	0.031441	0.032864	0.032998	0.033009	0.033010
11	0.019665	0.021247	0.021397	0.021408	0.021409
12	0.010389	0.012130	0.012293	0.012306	0.012307
13	0.003613	0.005513	0.005689	0.005704	0.005705
14	−0.000664	0.001395	0.001586	0.001601	0.001603
15	−0.002774	−0.000223	−0.000019	−0.000002	−0.000000
16	−0.003770	−0.000460	−0.000049	−0.000005	−0.000000
17	−0.004152	−0.000494	−0.000050	−0.000005	−0.000000
18	−0.004111	−0.000498	−0.000050	−0.000005	−0.000000
19	−0.003624	−0.000489	−0.000050	−0.000005	−0.000000
20	−0.002450	−0.000427	−0.000049	−0.000005	−0.000000
21	−0.000000	−0.000000	−0.000000	−0.000000	−0.000000

ϵ	No. of Iterations
10^{-1}	30
10^{-2}	31
10^{-3}	31
10^{-4}	31
10^{-5}	31

Table 2.2. Comparison of numerical methods. Iteration factors: Pointwise relaxation, $\omega = 1.6$; Lagrange multiplier, $\rho = 0.04$, $\omega = 1.0$; Successive approximation, $\rho = 0.04$; Penalty, $\epsilon = 10^{-5}$, $\omega = 1.55$. Tolerance for convergence, $1.0E\text{-}5$

	Computed results				
Node	S.O.R.	Lagrange	Fixed point	Penalty	Exact
1	0.250000	0.250000	0.250000	0.250000	0.250000
2	0.215895	0.215927	0.215895	0.215878	0.215895
3	0.184291	0.184352	0.184291	0.184260	0.184289
4	0.155187	0.155277	0.155187	0.155146	0.155184
5	0.128581	0.128699	0.128581	0.128534	0.128579
6	0.104475	0.104620	0.104475	0.104426	0.104473
7	0.082867	0.083039	0.082867	0.082820	0.082868
8	0.063758	0.063956	0.063758	0.063715	0.063763
9	0.047148	0.047371	0.047148	0.047112	0.047157
10	0.033040	0.033283	0.033040	0.033010	0.033052
11	0.021431	0.021693	0.021431	0.021409	0.021447
12	0.012324	0.012601	0.012324	0.012307	0.012341
13	0.005716	0.006008	0.005716	0.005705	0.005736
14	0.001608	0.001912	0.001608	0.001603	0.001631
15	0.000000	0.000140	0.000000	−0.000000	0.000025
16	0.000000	−0.000538	0.000000	−0.000000	0.000000
17	0.000000	−0.000807	0.000000	−0.000000	0.000000
18	0.000000	−0.001001	0.000000	−0.000000	0.000000
19	0.000000	−0.001178	0.000000	−0.000000	0.000000
20	0.000000	−0.001083	0.000000	−0.000000	0.000000
21	−0.000000	0.000000	0.000000	0.000000	0.000000

	No. of Iterations
Pointwise relaxation,	17
Lagrange multiplier,	271
Successive approximation,	17
Penalty,	31

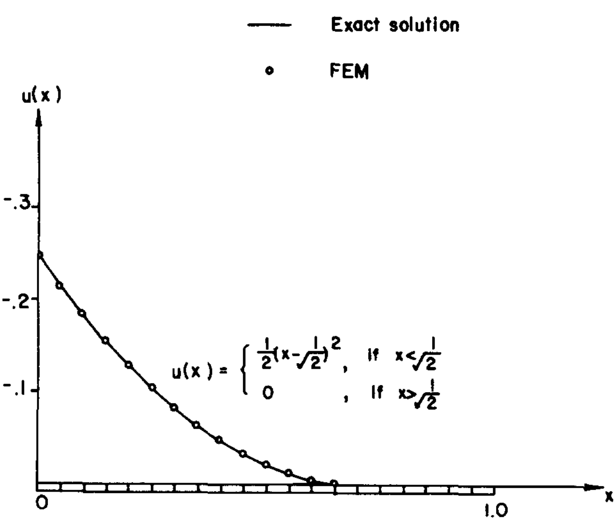


Fig. 2.2. Numerical results obtained for a one-dimensional variational inequality.

3. APPLICATIONS TO SEEPAGE PROBLEMS FOR HOMOGENEOUS DAMS

3.1 *Problem setting and Baiocchi's transformation*

In this section, we consider applications of the theory of variational inequalities and their approximation, discussed in Chaps. 1 and 2, to the problem of seepage of fluids through a porous media. For simplicity, we confine ourselves to 2-dimensional problems and we choose the classical model for such phenomena in which the flow is governed by Darcy's law (see, e.g. Bear[41]).

For this class of problems a rather elegant formulation is possible which fits the analysis conveniently into the framework of the theory of variational inequalities discussed thus far. We begin with a study of flow through non-rectangular homogeneous dams; we later extend these results to rectangular dams with variable permeability.

Derivations of quasi-variational inequalities associated with the seepage flow analysis mainly follow the work of Baiocchi *et al.*[14], Baiocchi[42], Baiocchi, Brezzi and Comincioli[43] and Lions[44].

Consider the case of a homogeneous isotropic dam on a horizontal impervious foundation, through which water is filtered so as to produce a steady irrotational incompressible 2-dimensional flow field. For simplicity, we take the specific weight γ of the water as unity. Then the problem is to find the pressure field $p = p(x, y)$ in the domain $\Omega \subset \mathbf{R}^2$ representing the flow region such that the following conditions hold

$$\left.\begin{aligned} p > 0 \quad \text{in} \quad \Omega, \quad p = 0 \quad \text{in} \quad D/\bar{\Omega} \\ -\Delta p = 0 \quad \text{in} \quad \Omega, \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} p = H - y \quad \text{on} \quad AF, \quad p = h - y \quad \text{on} \quad BC, \quad p = 0 \quad \text{on} \quad CD \\ (p + y)_n = 0 \quad \text{i.e.} \quad (p + y)_y = 0 \quad \text{on} \quad AB \\ p = 0, \quad (p + y)_n = 0 \quad \text{on} \quad S = FD \end{aligned}\right\} \tag{1.2}$$

The geometry of our problem, including the definition of terms in (1.1) and (1.2), is defined in Fig. 3.1. The symbol Δ denotes the Laplacian operator. The subscripts $(\cdot)_n$ and $(\cdot)_y$ denote the normal derivative

$$(\cdot)_n = n_x \partial \cdot / \partial x + n_y \partial \cdot / \partial y$$

and the derivative with respect to y, where $n = \{n_x, n_y\}$ is the outward unit vector normal to the boundary of the domain Ω. S is the free surface which is unknown *a priori*, and the dam D is made up of three parts, $D = D_1 \cup D_2 \cup D_3$, as shown in Fig. 3.1.

The pressure p can be characterized by the relation

$$\varphi = p/\gamma + y \tag{1.3}$$

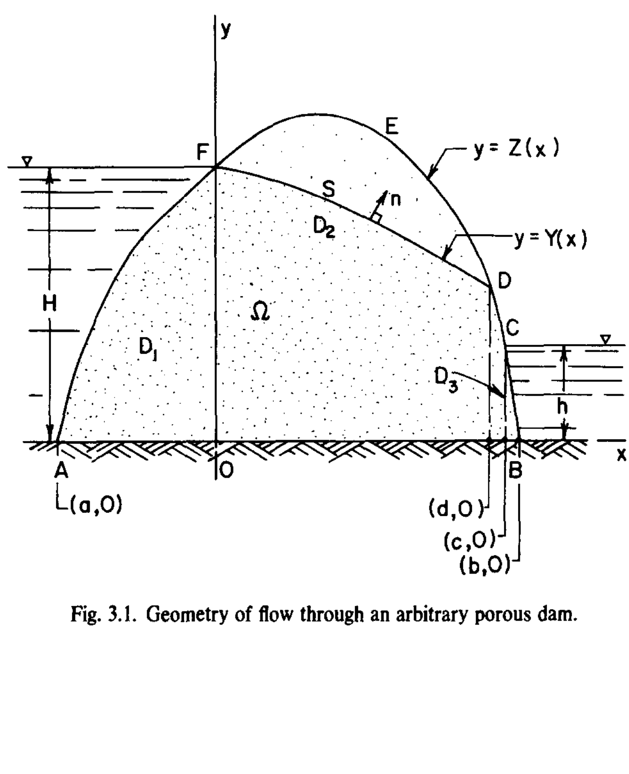


Fig. 3.1. Geometry of flow through an arbitrary porous dam.

where φ is a velocity potential with the property that

$$\vec{v} = -k \operatorname{grad} \varphi \tag{1.4}$$

$\vec{v}$ being the velocity of the water, k the permeability which, for the moment, is assumed to be constant, and γ is the specific weight of the water which is assumed to be unity. Equation (1.4) is referred to as Darcy's law for seepage flow.

Theorem 3-1.1. Let χ_Ω be the characteristic function for the domain Ω defined by

$$\chi_\Omega(x, y) = \begin{cases} 1 & \text{if} \quad (x, y) \in \Omega \\ 0 & \text{if} \quad (x, y) \notin \Omega. \end{cases} \tag{1.5}$$

If (1.1) and (1.2) hold, the pressure p satisfies the equation

$$-\Delta p = (\chi_\Omega)_y \tag{1.6}$$

in the sense of distributions on the domain D.

Proof. Let $\varphi \in C_0^\infty(D)$. Then, by (1.1),

$$\int_D \nabla p \cdot \nabla \varphi \, dx \, dy = \int_\Omega \nabla p \cdot \nabla \varphi \, dx \, dy = \int_S p_n \varphi \, ds$$

By $(1.2)_3$

$$\int_S p_n \varphi \, ds = -\int_S y_n \varphi \, ds = -\int_S n_y \varphi \, ds = -\int_\Omega \varphi_y \, dx \, dy$$

Then

$$\int_D \nabla p \cdot \nabla \varphi \, dx \, dy = -\int_\Omega \varphi_y \, dx \, dy = -\int_D \chi_\Omega \varphi_y \, dx \, dy,$$

i.e.

$$-\Delta p = (\chi_\Omega)_y$$

in the sense of distributions on D. □

Suppose that (1.6) is satisfied in the usual sense. Integrating (1.6) from 0 to y yields

$$-\Delta\left(\int_0^y p(x, t) \, dt\right) = \chi_\Omega(x, y) \tag{1.7}$$

Indeed

$$\int_0^y -\Delta p(x, t) \, dt = \int_0^y (\chi_\Omega)_t(x, t) \, dt$$

$$-\Delta\left(\int_0^y p(x, t) \, dt\right) + (p_y)(x, 0) = \chi_\Omega(x, y) - \chi_\Omega(x, 0).$$

By $(1.2)_2$

$$-\Delta\left(\int_0^y p(x, t) \, dt\right) - 1 = \chi_\Omega(x, y) - 1,$$

that is, (1.7) is implied. Here we have used the fact that the line AB is always saturated by water. Thus

Theorem 3-1.2. Let u be defined by

$$u(x, y) = \int_0^y p(x, t)\,dt. \tag{1.8}$$

If p is a solution of (1.1) and (1.2), then u satisfies

$$-\Delta u = \chi_\Omega. \quad \square \tag{1.9}$$

Equation (1.9) is defined on the whole domain D, while $(1.1)_2$ is satisfied only on the unknown domain Ω. The relation (1.8) is called the *Baiocchi transformation.* This novel change of variables makes it possible to reformulate the problem in terms of functions defined on the entire domain D; see Baiocchi[13].

We now introduce the following notations: the free boundary FDC in Fig. 3.1 is represented by

$$y = Y(x) \quad \text{on} \quad (0, c) \tag{1.10}$$

and the surface of the dam $AFEDCB$ is given by

$$y = Z(x) \quad \text{on} \quad (a, b). \tag{1.11}$$

By $(1.1)_1$

$$u(x, y) = \int_0^{Y(x)} p(x, t)\,dt \quad \text{for every} \quad y \in (Y(x), Z(x)) \tag{1.12}$$

that is, $u(x, y)$ is constant with respect to y in the non-flow domain $D/\bar{\Omega}$. Moreover, $(1.1)_1$ also implies that $u(x, y)$ is a strictly increasing function with respect to y in the flow domain Ω

$$u(x, y_1) < u(x, y_2) \quad \text{if} \quad y_1 < y_2 \quad \text{in} \quad \Omega \tag{1.13}$$

Combining (1.12) and (1.13)

$$\begin{aligned} 0 \le u(x, y) &< u(x, Z(x)) \quad \text{for} \quad (x, y) \in \Omega \\ u(x, y) &= u(x, Z(x)) \quad \text{for} \quad (x, y) \in D/\bar{\Omega}. \end{aligned} \tag{1.14}$$

The fact that $u(x, Y(x)) = u(x, y) = u(x, Z(x))$ in $D/\bar{\Omega}$ has been used in (1.14). We note that (1.11) is given, while the free surface (1.10) is unknown *a priori*. From the definition of (1.8) of u

$$p(x, y) = u_y(x, y).$$

This leads to the boundary conditions

$$u_y = H - y \quad \text{on} \quad AF, \quad u_y = h - y \quad \text{on} \quad BC, \quad u_y = 0 \quad \text{on} \quad CD. \tag{1.15}$$

Moreover, by (1.8)

$$u = 0 \quad \text{on} \quad AB. \tag{1.16}$$

The conditions $(1.2)_2$ and $(1.2)_3$ have already been incorporated into (1.9), etc.

Summing up, we have proved the following:

Theorem 3-1.3. Let u be defined by (1.8). If p is a solution of (1.1) and (1.2), then u satisfies

$$\left.\begin{array}{l} -\Delta u = \chi_\Omega \quad \text{in} \quad D \\ 0 \le u(x,y) < u(x, Z(x)) \quad \text{for} \quad (x,y) \in \Omega \\ u(x,y) = u(x,Z(x)) \quad \text{for} \quad (x,y) \in D/\bar{\Omega} \\ u_y = H - y \quad \text{on} \quad AF, \quad u_y = h - y \quad \text{on} \quad BC, \quad u_y = 0 \quad \text{on} \quad CD \\ u = 0 \quad \text{on} \quad AB. \quad \square \end{array}\right\} \tag{1.17}$$

3.2 *A variational formulation*

In the previous section, the seepage flow problem (1.1) and (1.2) was transformed into the equivalent problem (1.17) through Baiocchi's transformation (1.8). Here a variational formulation associated with (1.17) will be discussed. We will show that this formulation leads to a quasi-variational inequality.

Lemma 3-2.1. For every $Z(x) \in C^2(a,b)$, $u, v \in C^2(\bar{D})$ with $v = 0$ on AB, the relation

$$\int_D (-Z'u_y v_x + Z'u_x v_y - Z''u_y v)\,dx\,dy = -\int_{\hat{\Gamma}} u_n v\,ds + \int_a^b \sqrt{1+(Z')^2}\,u_y v\,dx \tag{2.1}$$

is satisfied, where $Z'(x) = dZ(x)/dx$ and $\hat{\Gamma} = \Gamma/AB$ and Γ is the boundary of D.

Proof. By integration by parts

$$\begin{aligned} &\int_D (-Z'u_y v_x + Z'u_x v_y - Z''u_y v)\,dx\,dy \\ &= \int_{\hat{\Gamma}} (-Z'u_y n_x + Z'u_x n_y)v\,ds - \int_D (-Z''u_y - Z'u_{yx} + Z'u_{xy} + Z''u_y)v\,dx\,dy \\ &= \int_{\hat{\Gamma}} (-u_y n_x + u_x n_y)Z'v\,dx \end{aligned}$$

Let $f(x,y) = y - Z(x)$. Then the outward normal unit vector $\mathbf{n} = \{n_x, n_y\}$ can be represented by

$$\mathbf{n} = \operatorname{grad} f / |\operatorname{grad} f|$$

Thus

$$\begin{aligned} \{n_x, n_y\} &= \{-Z', 1\}/\sqrt{1+Z'^2} \\ -n_x Z' &= (Z')^2 n_y, \quad n_y Z' = -n_x \\ (-u_y n_x + u_x n_y)Z' &= u_y n_y (Z')^2 - u_x n_y = -(u_y n_y + u_x n_y) + (1+(Z')^2)u_y n_y \\ &= -u_n + \sqrt{1+(Z')^2}\,u_y. \end{aligned}$$

Substituting this result into the above boundary integral gives (2.1). $\square$

Suppose that u satisfies (1.17). Let v be an arbitrary function in $H^1(D)$ such that $v = 0$ on AB. Then

$$\begin{aligned} &\int_D \{\nabla u \cdot \nabla(v-u) - Z'u_y(v-u)_x + Z'u_x(v-u)_y - Z''u_y(v-u)\}\,dx\,dy \\ &= \int_D (-\Delta u)(v-u)\,dx\,dy + \int_a^b \sqrt{1+(Z')^2}\,u_y(v-u)\,dx \end{aligned}$$

and

$$(-\Delta u)(v-u) = (v-u) - (1-\chi_\Omega)(v-u) = (v-u) - H(u-u_Z)(v-u).$$

Here

$$u_Z(x, y) = u(x, Z(x)), \tag{2.2}$$

and

$$H(v)(x, y) = \begin{cases} 1 & \text{if} \quad v(x, y) > 0 \\ 0 & \text{if} \quad v(x, y) < 0. \end{cases} \tag{2.3}$$

Setting

$$a(u, v) = \int_D (\nabla u \cdot \nabla v - Z' u_y v_x + Z' u_x v_y - Z'' u_y v) \, \mathrm{d}x \, \mathrm{d}y \tag{2.4}$$

$$L(v) = \int_D v \, \mathrm{d}x \, \mathrm{d}y + \int_a^0 (H - y)\sqrt{1 + (Z')^2} v \, \mathrm{d}x + \int_c^b (h - y)\sqrt{1 + (Z')^2} v \, \mathrm{d}x \tag{2.5}$$

yields

$$a(u, v - u) + \int_D H(u - u_Z)(v - u) \, \mathrm{d}x \, \mathrm{d}y = L(v - u).$$

Indeed,

$$a(u, v - u) = \int_D \{(v - u) - H(u - u_Z)(v - u)\} \, \mathrm{d}x \, \mathrm{d}y$$
$$+ \int_a^0 (H - y)\sqrt{1 + (Z')^2}(v - u) \, \mathrm{d}x + \int_c^b (h - y)\sqrt{1 + (Z')^2}(v - u) \, \mathrm{d}x$$

since

$$u_y = 0 \quad \text{on} \quad CD \quad \text{and} \quad u_y = 0 \quad \text{on} \quad FED.$$

Putting

$$j(u; v) = \int_\Omega (v - u_Z)^+ \, \mathrm{d}x, \quad \varphi^+ = \sup\{0., \varphi\} \tag{2.6}$$

we have

$$j(u; v) - j(u; u) \geq \int_\Omega H(u - u_Z)(v - u) \, \mathrm{d}x \, \mathrm{d}y.$$

Indeed, for every $a, b \in \mathbf{R}$

$$a^+ - b^+ \geq H(b)(a - b) \tag{2.7}$$

where

$$\begin{gathered} H(b) = 0 \quad \text{if} \quad b < 0, \quad H(b) = 1 \quad \text{if} \quad b > 0 \\ 0 \leq H(b) \leq 1 \quad \text{if} \quad b = 0. \end{gathered} \tag{2.8}$$

Therefore, the problem (1.17) can be transformed to the variational form

$$u \in K: a(u, v - u) + j(u; v) - j(u; u) \geq L(v - u) \quad \text{for every} \quad v \in V_0 \tag{2.9}$$

where (recall Remark 1-8.1)

$$K = \{v \in V_0: v \le v_Z, \quad v_Z(x, y) = v(x, Z(x))\} \tag{2.10}$$

$$V_0 = \{v \in H^1(D): v = 0 \quad \text{a.e. on} \quad AB\}. \tag{2.11}$$

The reason every element of V_0 belongs to the Sobolev space $H^1(D)$ is that

$$a(u, v) \le (1 + \|Z'\|_{0,\infty} + \|Z''\|_{0,\infty})\|u\|_1\|v\|_1 < +\infty, \tag{2.12}$$

if $Z \in C^2(a, b)$. The virtual work $a(u, v)$ is finite in $H'(D)$ if Z is smooth enough. The estimate (2.12) further means that the bilinear form $a(\cdot, \cdot)$ is continuous on $H^1(D) \times H^1(D)$. For every $v \in V_0$,

$$\begin{aligned} a(v, v) &= \int_D \left\{\nabla v \cdot \nabla v - Z'' \frac{1}{2}(v^2)_y\right\} dx\, dy \\ &= \int_D \nabla v \cdot \nabla v \, dx\, dy - \frac{1}{2}\int_a^b Z'' v^2 \, dx \\ &\ge c\|v\|_{1,D}^2 - \frac{1}{2}\int_a^b Z'' v^2 \, dx \end{aligned}$$

(by Poincaré's inequality). If the dam D is assumed to be convex, i.e. $Z''(x) \le 0$ on $[a, b]$, the bilinear form $a(\cdot, \cdot)$ is coercive. In this analysis, we will assume the convexity of D for simplicity so that

$$a(v, v) \ge c\|v\|_{1,D}^2. \tag{2.13}$$

Theorem 3-2.1. Suppose that the domain D is convex and that there exists a solution $u \in V_0$ of the variational inequality

$$u \in V_0: a(u, v - u) + j(u; v) - j(u; u) \ge L(v - u) \quad \forall v \in V_0. \tag{2.14}$$

Then the solution $u \in V_0$ satisfies

(i) $$u \ge 0, \quad \text{a.e. on} \quad D, \tag{2.15}$$

(ii) $$-\Delta u \in L^\infty(D), \quad \text{and} \quad 0 \le -\Delta u \le 1, \tag{2.16}$$

(iii) $$u - u_Z \le 0, \quad \text{a.e. on} \quad D, \quad \text{and} \tag{2.17}$$

(iv) $$\begin{aligned} &u_y = 0 \quad \text{on} \quad (x, Z(x)), \quad x \in (0, c), \\ &u_y = H - y \quad \text{on} \quad (x, Z(x)), \quad x \in (a, 0), \\ &u_y = h - y \quad \text{on} \quad (x, Z(x)), \quad x \in (c, b). \end{aligned} \tag{2.18}$$

In other words, the solution $u \in V_0$ of the variational inequality (2.14) is also a solution of (1.17).

Proof. (i) $u \ge 0$: Taking $v = u^+ \in V_o$ in (2.9), yields

$$-a(u^-, u^-) + \int_D \{(u^+ - u_Z)^+ - (u - u_Z)^+\} dx\, dy \ge \int_D u^- \, dx\, dy,$$

since $u = u^+ - u^-$, $a(u^+, u^-) = 0$, $H - y \ge 0$, and $h - y \ge 0$. By the following inequality

$$(u^+ - u_Z)^+ = (u^- + u - u_Z)^+ \le (u^-)^+ + (u - u_Z)^+ = u^- + (u - u_Z)^+,$$

we have

$$-a(u^-, u^-) \geq 0, \quad \text{i.e.} \quad u^- = 0 \quad \text{so that} \quad u \geq 0$$

(ii) $0 \leq -\Delta u \leq 1$: Taking $v = u + \varphi \in V_0$, $\varphi \in C_0^\infty(D)$, in (2.9), we obtain

$$a(u, \varphi) + \int_D \{(\varphi + u - u_Z)^+ - (u - u_Z)^+\} \, dx \, dy \geq \int_D \varphi \, dx \, dy.$$

Since $(p+q)^+ \leq p^+ + q^+$

$$a(u, \varphi) + \int_D \varphi^+ \, dx \, dy \geq \int_D \varphi \, dx \, dy.$$

Taking $\varphi \leq 0$ in D, we obtain

$$a(u, \varphi) \geq \int_D \varphi \, dx \, dy,$$

that is

$$-\Delta u - 1 \leq 0$$

in the sense of distributions on D. Taking $\varphi \geq 0$ in D, we have

$$a(u, \varphi) \geq 0,$$

that is

$$-\Delta u \geq 0$$

in the sense of distributions on D. Thus, we conclude that $0 \leq -\Delta u \leq 1$. This further implies that $-\Delta u \in L^\infty(D)$.

(iii) $u \leq u_Z$: Since we have already proved that $u \geq 0$ in D, $u_Z \geq 0$ must hold. Then

$$(u - u_Z)^+ = 0 \quad \text{on} \quad \Gamma$$

where Γ is the boundary of the domain D. Substitution of $v = u - (u - u_Z)^+ \in V_0$ into (2.9), and integrating by parts gives

$$-\int_D (-\Delta u)(u - u_Z)^+ \, dx \, dy + \int_D \{(u - u_Z - (u - u_Z)^+)^+ - (u - u_Z)^+\} \, dx \, dy \geq -\int_D (u - u_Z)^+ \, dx \, dy,$$

i.e.

$$\int_D (-\Delta u)(u - u_Z)^+ \, dx \, dy \leq 0.$$

Since $-\Delta u \geq 0$, this inequality implies $(u - u_Z)^+ \leq 0$, i.e. $u - u_Z \leq 0$.

(iv) Boundary conditions: Taking $v = u \pm w$, $w \in V_0$, in (2.19) and integrating by parts, we can obtain the following estimate

$$-\int_D (-\Delta u - 1) w \, dx \, dy + \int_D (-w)^+ \, dx \, dy \geq \int_a^b \sqrt{1 + Z'^2} u_y w \, dx - \int_a^0 (H - y)\sqrt{1 + Z'^2} w \, dx$$
$$-\int_c^b (h - y)\sqrt{1 + Z'} w \, dx \geq -\int_D (-\Delta u - 1) w \, dx \, dy - \int_D (w)^+ \, dx \, dy.$$

Since

$$\left|\iint_d (-\Delta u - 1) w \, dx \, dy\right| \leq |w|_{1,D} \equiv \int_D |w| \, dx \, dy,$$

$$\left|\iint_D (-w)^+ \, dx \, dy\right| \leq |w|_{1,D},$$

$$\left|\iint_D (w)^+ \, dx \, dy\right| \leq |w|_{1,D},$$

we have

$$\left|\int_a^b \sqrt{1+Z'^2} u_y w \, dx - \int_a^0 (H-y)\sqrt{1+Z'^2} w \, dx - \int_c^b (h-y)\sqrt{1+Z'^2} w \, dx\right| \leq 2|w|_{1,D}.$$

Since it is possible to take arbitrary values on $(x, Z(x))$, $x \in (a, b)$ while $|w|_{1,D} \leq \epsilon$ for an arbitrary given small number $\epsilon > 0$, we can conclude that

$$u_y = H - y \quad \text{on} \quad (a, 0), \quad u_y = h - y \quad \text{on} \quad (c, b), \quad \text{and} \quad u_y = 0 \quad \text{on} \quad (0, c). \qquad \square$$

We now consider the problem of determining sufficient conditions for the existence of solutions of the variational inequality (2.14).

Let us investigate properties of the functional j defined by (2.6). To this end, we introduce a function φ defined by

$$\varphi(u; v) = (v - u_Z)^+.$$

Suppose that $a \leq b$. Then we define χ by

$$\begin{aligned}\chi &= \varphi(a, v) + \varphi(b, w) - \varphi(a, \inf(v, w)) - \varphi(b, \sup(v, w)) \\ &= (v - a_Z)^+ + (w - b_Z)^+ - (\inf(v, w) - a_Z)^+ - (\sup(v, w) - b_Z)^+.\end{aligned}$$

If $v \geq w$,

$$\chi = (v - a_Z)^+ + (w - b_Z)^+ - (w - a_Z)^+ - (v - b_Z)^+.$$

Then

$$\chi = \begin{cases} 0 & (v \geq w \geq b \geq a) \\ b_Z - w & (v \geq b \geq w \geq a) \\ b_Z - a_Z & (v \geq b \geq a \geq w) \\ v - w & (b \geq v \geq w \geq a) \\ v - a_Z & (b \geq v \geq a \geq w) \\ 0 & (b \geq a \geq v \geq w), \end{cases}$$

i.e. $\chi \geq 0$. If $v \leq w$

$$\chi = (v - a_Z)^+ + (w - b_Z)^+ - (v - a_Z)^+ - (w - b_Z)^+ = 0.$$

Therefore, we have

Lemma 3-2.2. For $p \leq q$,

$$j(p; v) + j(q; w) - j(p; \inf(v, w)) - j(q; \sup(v, w)) \geq 0 \tag{2.19}$$

for every $v, w \in V_0$. $\square$

As shown above, the bilinear form $a(\cdot,\cdot)$ is continuous and coercive (recall (2.12) and (2.13)) on a closed subspace V_0 of $H^1(D)$, if convexity of the domain is assumed. Moreover, the non-differentiable convex functional $j(u;\cdot)$, which depends upon the solution itself, satisfies the "monotonicity" condition (2.19). Thus, we can establish the following existence theorem by a direct application of the general existence theorem, Theorem 1-8.2, discussed in Section 1.8.

Theorem 3-2.2. Suppose that the domain D is convex so that the bilinear form $a(\cdot,\cdot)$ is coercive on V_0. Then there exists a solution $u \in V_0$ of the variational inequality (2.14). □

Continuing, we further note that the solution $u \in V_0$ of the variational inequality (2.14), referred to as a variational inequality of the second kind in Chap. 1, also satisfies a variational inequality of the first kind. That is, $u \in V_0$ is a solution of

$$u \in K(u)\colon a(u, v-u) \geq L(v-u), \quad \forall v \in K(u) \tag{2.20}$$

where

$$K(u) = \{v \in V_0\colon v - u_Z \leq 0\}. \tag{2.21}$$

Indeed, as shown in (2.17), $u - u_Z \leq 0$, i.e.

$$j(u;u) = 0.$$

For every $v \in K(u)$, $j(u; v) = 0$. Thus $u \in V_0$ satisfies (2.20).

Solutions to (2.14) will not, in general, be unique. This fact implies considerable difficulties in obtaining approximate solutions, as indicated in the following example.

Example 3-2.1. We will consider an example problem of seepage flow through a non-reactangular quadrilateral isotropic homogeneous dam as shown in Fig. 3.2. The foundation is horizontal and is assumed to be impermeable. Physical dimensions and a discretization of the domain D are also shown in Fig. 3.2.

In this case, the variational inequality (2.9) and the admissible sets (2.10) and (2.11) become

$$u \in K\colon a(u, v-u) + j(u;v) - j(u;u) \geq L(v-u) \quad \forall v \in V_0 \tag{2.22}$$

$$K = \{v \in V_0\colon v(x,y) \leq v(x, Z(x)), \quad \text{a.e.} \quad x \in (0,c)\} \tag{2.23}$$

$$V_0 = \{v \in H^1(D)\colon v(x,0) = 0, \quad \text{a.e.} \quad x \in (0,b),$$

$$v(0,y) = \frac{1}{2}(2Hy - y^2), \quad \text{a.e.} \quad y \in (0,H)\} \tag{2.24}$$

where

$$a(u,v) = \int_D (\nabla u \cdot \nabla v - Z' u_y v_x + Z' u_x v_y)\,dx\,dy, \tag{2.25}$$

$$L(v) = \int_D v\,dx\,dy + \int_c^b (h-y)\sqrt{1+(Z')^2}\,v\,dx \tag{2.26}$$

$$j(u;v) = \int_D (v(x,y) - u(x, Z(x)))^+\,dx\,dy. \tag{2.27}$$

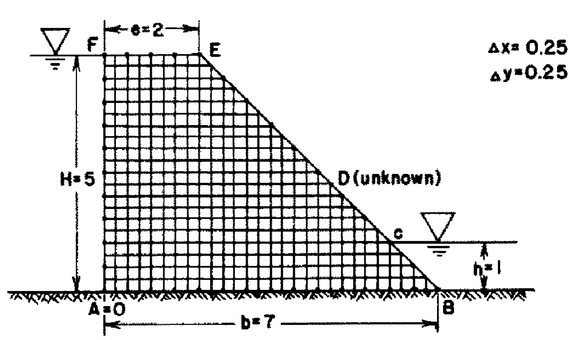


Fig. 3.2. Discretization of domain in Example 3-2.1.

An interesting fact is that if we seek sequential solutions of the variational inequalities

$$u_{n+1} \in V_0\colon a(u_{n+1}, v - u_{n+1}) + j(u_n; v) - j(u_n; u_{n+1}) \geq L(v - u_{n+1}), \quad \forall v \in V_0 \tag{2.28}$$

only "trivial" solutions would be obtained in the following sense: if the initial function u_1 is the solution of the "Neumann" problem

$$u_1 \in V_0\colon a(u_1, v) = L(v), \quad \forall v \in V_0 \tag{2.29}$$

then for all $n = 2, 3, \ldots,$ the solution u_n of the variational inequality (2.28) always coincides with u_1. This fact suggests that the variational inequality (2.14) may not be capable of predicting physically meaningful solutions to the seepage flow problem (1.1) and (1.2). In turn, this suggests that some additional conditions are needed in the model in order to preserve physical consistency which has apparently been lost in the process of deriving the quasi-variational inequality (2.14). We postpone a fuller exploration of such conditions until the end of this section. □

Recently, Brezis, Kinderlehrer and Stampacchia[45] introduced a new formulation to the seepage flow problem (1.1) and (1.2) using an (extended) pressure $p(x, y)$ defined on the whole domain of dam D described in (1.1). Their weak formulation is based on Theorem 3-1.1 instead of Theorem 3-1.2. Indeed, if an additional condition

$$(p + y)_n \leq 0 \quad \text{on} \quad CD \tag{2.30}$$

is assumed, the solution $p(x, y)$ of (1.1), (1.2) and (2.30) satisfies

$$\int_D \nabla p \cdot \nabla \varphi \, dx\, dy + \int_\Omega \varphi_y \, dx\, dy = \int_{CD} (p + y)_n \varphi \, ds \leq 0 \tag{2.31}$$

for every $\varphi \in C^1(\bar{D})$ with $\varphi = 0$ on $AF \cup BC$ and $\varphi \geq 0$ on CD. Using the Heaviside function, (2.31) can be written by

$$\int_D (\nabla p \cdot \nabla \varphi + H(p)\varphi)\, dx\, dy \leq 0 \tag{2.32}$$

where $H(p) = 1$ if $p > 0$, $H(p) = 0$ if $p < 0$, and $0 \leq H(p) \leq 1$ if $p = 0$. Then the mathematical problem is defined as follows:

> Find $p \in H^1(D)$ and $g \in L^\infty(D)$ such that $p \geq 0$, a.e. on D, $g(x, y) = 1$ if $p(x, y) > 1$, $0 \leq g(x, y) \leq 1$ if $p(x, y) = 0$, $p(x, y) = 0$ on $FEDC$, $p(x, y) = H - y$ on AF, $p(x, y) = h - y$ on BC, and $\int_D (\nabla p \cdot \nabla \varphi + g\varphi_y)\, dx\, dy \leq 0$, for every $\varphi \in H^1(D)$ such that $\varphi \geq 0$ on $FEDC$, $\varphi = 0$ on $AF \cup BC$. (2.33)

To show existence of solutions to the problem (2.33), Brezis *et al.*[45] introduce a penalized problem:

> Find $p_\epsilon \in H^1(D)$ with $p = 0$ on $FEDC$, $p_\epsilon = H - y$ on AF, and $p_\epsilon = h - y$ on BC such that $\int_D (\nabla p_\epsilon \cdot \nabla \varphi + H_\epsilon(p_\epsilon)\varphi_y)\, dx\, dy = 0$ for every $\varphi \in H^1(D)$ with $\varphi = 0$ on $FEDC \cup AF \cup BC$ where (2.34)

$$H_\epsilon(p) = \begin{cases} 0 & \text{if} \quad p \leq 0 \\ \dfrac{1}{\epsilon} p & \text{if} \quad 0 \leq p \leq \epsilon \\ 1 & \text{if} \quad \epsilon \leq p, \end{cases} \tag{2.35}$$

Then, applying the Schauder fixed-point theorem, it can be easily proved that the problem (2.34) has a unique solution $p_\epsilon \in H^1(D) \cap W^{1,s}_{\text{loc}}(D)$ for every $\epsilon > 0$ and $s < \infty$, and that p_ϵ is uniformly bounded in $H^1(D)$. Since $H^1(D)$ is reflexive, there exists a sequence $\{\epsilon_n\}$, $\epsilon_n \to 0$ as $n \to \infty$, such that

$$\begin{cases} p_{\epsilon_n} \rightharpoonup p & \text{weakly in } H^1(D) \\ p_{\epsilon_n} \to p & \text{strongly in } L^2(D) \\ H_{\epsilon_n}(p_{\epsilon_n}) \rightharpoonup g & \text{weakly in } L^\infty(D). \end{cases}$$

Moreover, since $p_\epsilon \geq 0$ in D, we have

$$(p_\epsilon)_n \leq 0 \quad \text{on} \quad CD.$$

Thus, for $\varphi \in H^1(D)$ with $\varphi = 0$ on $AF \cup BC \cup FE$ and $\varphi \geq 0$ on EC, we have

$$\int_D (\nabla p_\epsilon \cdot \nabla \varphi + H_\epsilon(p_\epsilon)\varphi_y)\, dx\, dy = \int_{EC} (p_\epsilon)_n \varphi\, ds \leq 0.$$

Passing the limit $\epsilon_n \to 0$, (2.33) is obtained. In summary, we have

Theorem 3-2.3. There exists a solution $p \in H^1(D) \cap W^{1,s}_{\text{loc}}(D)$, $s < +\infty$, to the problem (2.33). Furthermore, the solution p is the limit of the sequence $\{p_\epsilon\}$ which is a unique solution of the penalized problem (2.34) for each positive $\epsilon > 0$. □

Example 3-2.1. (*Continued*). Using the discrete model described in Fig. 3.2, we now solve the penalized problem (2.34) for $\epsilon = 0.1$ using the S.O.R. method discussed in Chap. 2. We plot saturated nodal points, which are identified whenever nodal values of the pressure exceed 10^{-3}, in Fig. 3.3.

The same problem is solved by a conventional *adaptive mesh method*, details of which will be discussed in Appendix 1,† using the discrete model shown in Fig. 3.4. Numerical results are given in Fig. 3.5, and the convergence characteristics of the adaptive mesh method are described in Fig. 3.6.

The position of the free surface obtained by the penalized formulation, (2.34), is slightly higher than the one by the adaptive mesh method.

We note that the penalty parameter $\epsilon > 0$ in (2.35) cannot be taken independently of the parameter of discretization of the model. One suggestion for the choice of the penalty parameter ϵ is that the order of ϵ should not be smaller than the one of the discretization parameter h (or $\Delta x, \Delta y$). □

We will call the method in which the penalized formulation (2.34) of the problem (2.35) is used to obtain the free surface the *extended pressure method*.

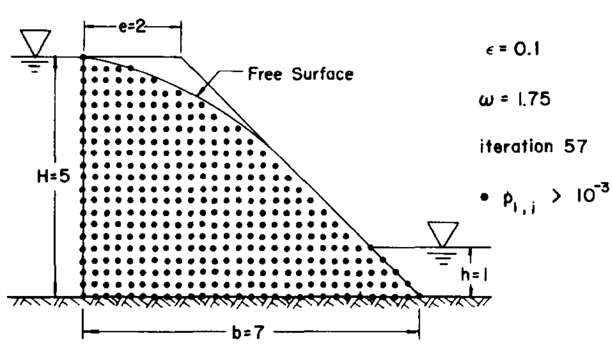


Fig. 3.3. Free surface calculated using the extended free pressure formulation.

†Even though we will frequently cite results obtained using the adaptive mesh method for comparison with other techniques, we delegate a more detailed discussion of this method to an appendix since it is not based on variational inequality formulations and since an analysis of its convergence properties is not known (and, in fact, may not exist).

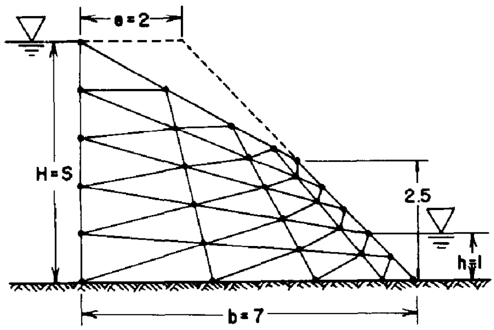


Fig. 3.4. Initial finite element mesh for adaptive mesh method for Example 3-2.1.

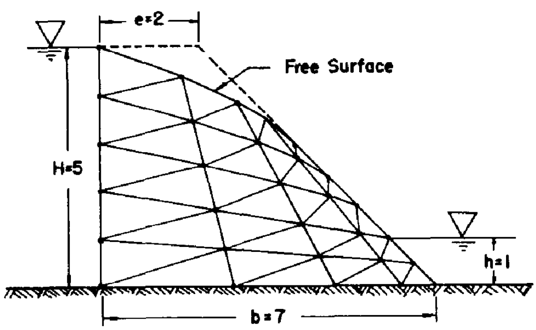


Fig. 3.5

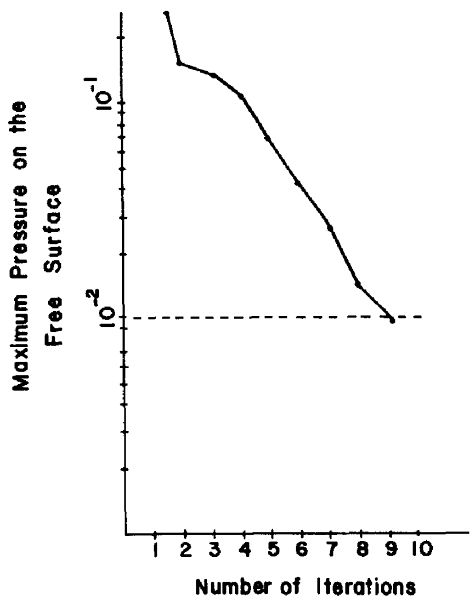


Fig. 3.6

Fig. 3.5. Free surface calculated by adaptive mesh method.

Fig. 3.6. Convergence characteristics of adaptive mesh method for Example 3-2.1.

3.3 *Special cases*

Suppose that BDE, in Fig. 3.1, is vertical, i.e. $b = c = d$. Then the bilinear form $a(\cdot,\cdot)$ and the linear form $L(\cdot)$, defined by (2.4) and (2.5), respectively, have to be modified since Z' and Z'' now do not exist on BDE. Toward developing a formulation appropriate for this case, we note that from the definition (1.8) of $u(x, y)$, on the vertical line BDE, we have

$$\left.\begin{aligned} u(b,y) &= \int_0^y p(b,t)\,\mathrm{d}t = \int_0^y (h-t)\,\mathrm{d}t = \frac{1}{2}(2hy - y^2) \quad \text{for} \quad 0 \le y \le h \\ u(b,y) &= \frac{1}{2}h^2 \quad \text{for} \quad h \le y \end{aligned}\right\} \tag{3.1}$$

where h is the level of the fluid downstream.

Since there is no outflow from the foundation AB and the free surface FDC, the conservation law asserts that

$$q(x) = \int_0^{Y(x)} v_x(x,t)\,\mathrm{d}t = \text{constant} \tag{3.2}$$

for every $x \in (0, d)$, where $v_x(x, y)$ is the x-component of the velocity vector of flow at the point (x, y), and q is the net flow from upstream to downstream through the dam. By the definition (1.8) of $u(x, y)$,

$$v_x(x, y) = -(p + y)_x = -p_x = -(u_y)_x = -(u_x)_y(x, y)$$

in the flow domain Ω. Since u is an extended function defined on the whole domain D, (3.2) can be written

$$-q(x) = \int_0^{Z(x)} (u_x)_t(x, t)\,dt = \text{constant}$$

for every $x \in (0, d)$, i.e.

$$-q = u_x(x, Z(x)) - u_x(0, 0).$$

Since $u = 0$ on AB, we obtain

$$-q = u_x(x, Z(x)) \quad \text{in} \quad (0, d).$$

Integrating with respect to x gives

$$u(x, Z(x)) = -qx + c \quad \text{in} \quad (0, d) \tag{3.3}$$

where c is a proper constant. In (3.3), q and c are unknown.

We record this result as a lemma:

Lemma 3-3.1. In the domain D_2, i.e. for $x \in (0, d)$, the condition

$$u(x, Z(x)) = -qx + c \tag{3.4}$$

is satisfied. Here q is the net seepage flow and c is a finite constant. □

Continuing, we observe that if BDE is vertical, then according to $(3.1)_2$

$$u(b, Z(b)) = -qb + c = \frac{1}{2}h^2, \quad \text{i.e.} \quad c = \frac{1}{2}h^2 + qb.$$

Here we have set $d = b$. Thus, in $(0, d)$,

$$u(x, Z(x)) = -q(x - b) + \frac{1}{2}h^2. \tag{3.5}$$

This means that the boundary condition on FED (see Fig. 3.1) is represented explicitly by (3.5) to within an unknown flow q. In this case we introduce the following definitions

$$a(u, v) = \int_D (\nabla u \cdot \nabla v - Z' u_y v_x + Z' u_x v_y - Z'' u_y v)\,dx\,dy \tag{2.4a}$$

$$L(v) = \int_D v\,dx\,dy + \int_a^0 (H - y)\sqrt{1 + (Z')^2}\,v\,dx \tag{2.5a}$$

$$j(p; v) = \int_\Omega \left(v - \left(\frac{1}{2}h^2 - p(x - b)\right)\right)^+ dx \tag{2.6a}$$

$$V_q = \Big\{ v \in H^1(D) \colon v = 0, \quad \text{a.e. on} \quad AB, \quad v = \frac{1}{2}(2hy - y^2),$$

$$\text{a.e. on} \quad BC, \quad v = \frac{1}{2}h^2, \quad \text{a.e. on} \quad CD,$$

and

$$v(x, Z(x)) = -q(x-b) + \frac{1}{2}h^2, \quad \text{a.e. on} \quad FED\Big\} \tag{2.11a}$$

$$K(q) = \{v \in V_q : v(x, y) \leq v(x, Z(x)), \quad \text{a.e. on} \quad D\}. \tag{2.14a}$$

Thus, problems (2.9) and (2.20) now assume the forms

$$u \in K(p) \colon a(u, v-u) + j(p; v) - j(p; u) \geq L(v-u), \quad \text{for every} \quad v \in V_p \tag{2.9a}$$

and

$$u \in K(p) \colon a(u, v-u) \geq L(v-u) \quad \text{for every} \quad v \in K(p) \tag{2.20a}$$

respectively.

A question that arises for the variational inequalities (2.9a) and (2.20a) is that of finding an unknown discharge p which is compatible with the seepage flow problem. Since the pressure at the point F in Fig. 3.1 is expected to be zero, we may employ the condition

$$f(p) = \frac{\partial}{\partial y} u(p)|_F = 0 \tag{3.6}$$

as the compatibility condition for determining the discharge p of the seepage flow. That is, the variational problem associated with the seepage flow problem for the vertical wall BDE is the pair of {(2.9a), (3.6)} or {(2.20a), 3.6)}.

In the discrete system, such as encountered in finite difference or element approximations, the compatibility condition (3.6) can be approximately written

$$\hat{f}(p) = u_{N_0,N_J} - u_{N_0,N_J-1} - \frac{1}{2}(\Delta y)^2 = 0 \tag{3.7}$$

using the notation described in Fig. 3.7. Condition (3.7) means that the pressure at $\hat{F}$ in Fig. 3.7 is assumed to be $(1/2)\Delta y$, i.e. that velocity of the flow is assumed to be zero at $\hat{F}$.

One way of solving problem {(2.9a), (3.6)} or {(2.20a), (3.6)} is to employ *the discharge descent method* described below. First, for properly chosen approximations q_1 and q_2 of discharge of the seepage flow, we solve the variational inequality (2.9a) or (2.20a) and obtain

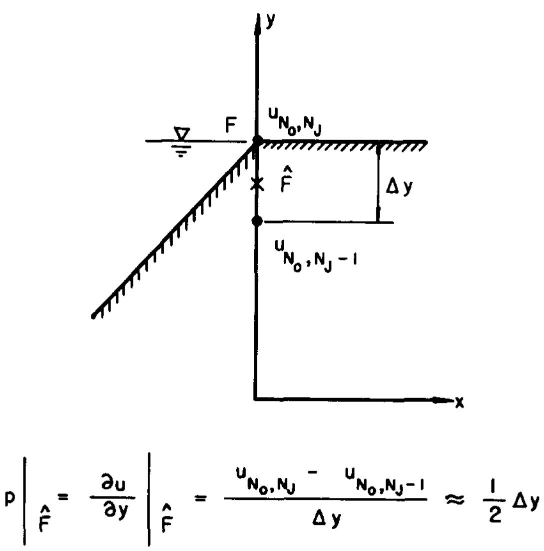


Fig. 3.7. Discrete compatibility condition.

solutions $u(q_1)$ and $u(q_2)$, respectively. Then $\hat{f}(q_1)$ and $\hat{f}(q_2)$ are calculated by the discrete compatibility condition (3.7). Using these values of q_1, q_2, $\hat{f}(q_1)$ and $\hat{f}(q_2)$, the third approximation q_3 of the discharge is obtained by

$$\begin{cases} \hat{f}(q_3)-\hat{f}(q_2)=\dfrac{\hat{f}(q_1)-\hat{f}(q_2)}{q_1-q_2}(q_3-q_2) \\ \hat{f}(q_3)=0 \end{cases}$$

i.e.

$$q_3=q_2-\frac{q_1-q_2}{\hat{f}(q_1)-\hat{f}(q_2)}\hat{f}(q_2). \tag{3.8}$$

Using q_3, we solve (2.9a) or (2.20a) and get $u(q_3)$ and $\hat{f}(q_3)$. If $\hat{f}(q_3)$ is far from zero, we calculate the fourth approximation q_4 through an equation similar to (3.8). We repeat these procedures until convergence is obtained.

If AF and BDF are vertical, i.e. if the cross section of the dam is rectangular, the quasi-variational inequalities (2.9a) and (2.20a) reduce to variational inequalities. In fact, as in (3.1), on the line AF

$$\left.\begin{aligned} u(0,y)&=\frac{1}{2}(2Hy-y^2) \quad \text{for} \quad 0\le y\le H \\ &=\frac{1}{2}H^2 \quad\quad\quad\quad \text{for} \quad y\ge H. \end{aligned}\right\} \tag{3.9}$$

Combining (3.5) and (3.9)

$$\begin{aligned} &qb+\frac{1}{2}h^2=\frac{1}{2}H^2, \quad \text{i.e.} \quad q=\frac{1}{2b}(H^2-h^2), \quad \text{i.e.} \\ &u(x,Z(x))=-\frac{1}{2b}(H^2-h^2)(x-b)+\frac{1}{2}h^2. \end{aligned} \tag{3.10}$$

Then

$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x\,\mathrm{d}y \tag{2.4a}$$

$$L(v)=\int_D v\,\mathrm{d}x\,\mathrm{d}y \tag{2.5a}$$

$$j(v)=\int_\Omega \left\{v-\left(\frac{1}{2}h^2-\frac{1}{2b}(H^2-h^2)(x-b)\right)\right\}^+ \mathrm{d}x\,\mathrm{d}y \tag{2.6b}$$

$$\begin{aligned} V_0=\Big\{&v\in H^1(D);\quad v=0,\quad \text{a.e. on}\quad AB,\quad v=\frac{1}{2}(2hy-y^2),\quad \text{a.e. on}\quad BC, \\ &v=\frac{1}{2}h^2,\quad \text{a.e. on}\quad CD,\quad v=\frac{1}{2}(2Hy-y^2),\quad \text{a.e. on}\quad AF, \\ &v=-\frac{1}{2b}(H^2-h^2)(x-b)+\frac{1}{2}h^2,\quad \text{a.e. on}\quad FED\Big\} \end{aligned} \tag{2.11b}$$

$$K=\{v\in V_0\colon v(x,y)\le v(x,Z(x)),\quad \text{a.e. in}\quad D\}. \tag{2.14b}$$

Problems (2.9a) and (2.15a) now become

$$u \in K: a(u, v-u) + j(v) - j(u) \geq L(v-u) \quad \forall v \in V_0 \tag{2.9b}$$

$$u \in K: a(u, v-u) \geq L(v-u) \quad \forall v \in K \tag{2.20b}$$

respectively.

Existence of solutions to the problems {(2.9a), (3.6)} and (2.9b) follows from the general existence Theorem 3-2.2 under the assumption that the dam D is convex. Thus, we need only show uniqueness of the solutions of {(2.9a), (3.6)} and (2.9b).

Suppose that p is the true discharge of the seepage flow in (2.9a), and suppose that u_1 and u_2 are solutions of (2.9a). Then, since $u_1, u_2 \in K(p)$

$$a(u_1, u_2 - u_1) + j(p; u_2) - j(p; u_1) \geq L(u_2 - u_1)$$

$$a(u_2, u_1 - u_2) + j(p; u_1) - j(p; u_2) \geq L(u_1 - u_2).$$

Adding the above two inequalities, we obtain

$$a(u_1 - u_2, u_1 - u_2) \leq 0.$$

Thus, by (2.13), $u_1 = u_2$.

By the same arguments, we also can establish uniqueness of the solution of (2.9b). Thus, we have

Theorem 3-3.1. Suppose that the domain of dam D is convex, and that $Z \in C^2(a, b)$. Then there exists a unique solution to the variational inequality (2.9a) or (2.20a) for a fixed discharge p. Furthermore, if the dam D is rectangular, there exists a unique solution to the variational inequalities (2.9b) or (2.20b). □

Example 3-3.1. This is a continuation of Example 3-2.1. Here we derive an additional condition which apparently overcomes the inconsistencies in the variational inequality (2.9) discussed earlier. Let us first suppose that the functions $u(x, y)$, $p(x, y)$ and $\mathbf{n} = (n_x, n_y)$ are sufficiently smooth. The discharge of seepage flow through the line connecting $(x, 0)$ and (x, y) is

$$q(x, y) = \int_0^y (-u_{tx}(x, t))\, dt = -\int_0^y u_{xt}(x, t)\, dt$$
$$= -u_x(x, y).$$

If discharge through the wall connected with $(x, Z(x))$ and $(e, Z(e))$ on CE is assumed to be zero, then

$$-u_x(x, Z(x)) = q \quad \text{on} \quad x \in (0, c),$$

where q is the true discharge of seepage flow. By integration with respect to x, we obtain

$$u(x, Z(x)) = -qx + \frac{1}{2}H^2 \quad \text{on} \quad x \in (0, c).$$

We note that the discharge q is certainly unknown *a priori*. However, an additional equation can be obtained by imposing the "compatibility" condition

$$\begin{aligned} f(q) &= p(c, Z(c)) \\ &= \lim_{\epsilon \to 0} \{u(c, Z(c)) - u(c-\epsilon, Z(c-\epsilon))\}/(Z(c) - Z(c-\epsilon)) \\ &= 0, \end{aligned}$$

where $u(x, y)$ is the solution for a "given" discharge q.

Here we employ the discharge descent method discussed earlier in this section. We start the process with

$$q_1 = (H^2 - h^2)/2b = 1.7143$$

$$q_2 = q_1 - \epsilon = 1.5143. \text{(say)}$$

The numerical results obtained are shown in Fig. 3.8. □

Example 3-3.2. We consider the problem of flow through a quadrilateral dam as shown in Fig. 3.9. Let the level of upstream and downstream be $H = 5$ and $h = 1$, respectively. The dam is isotropic and homogeneous and the horizontal foundation is impermeable. Physical dimensions and a discrete model for the variational inequality (2.9a) are also shown in Fig. 3.9. The solution method used is the projectional S.O.R. method discussed in Chap. 2. As an initial discharge q_1, we take

$$q_1 = \frac{1}{2b}(H^2 - h^2) = 6.$$

Using the compatibility condition (3.6) or (3.7), we correct the discharge through the descent method (3.8). Convergence and numerical results are shown in Fig. 3.10.

The same problem is also solved by the adaptive mesh method using the discrete model in Fig. 3.11. Numerical results and convergence of the adaptive mesh method are given in Figs. 3.12 and 3.13, respectively.

Both methods provide almost the same configuration for the free surface. It is noted that if a uniform element mesh is used in the discretization of the variational inequality (2.9a), the projectional S.O.R. method can be applied in the same manner as finite difference methods. This leads to rather short calculation times compared with the adaptive mesh method even though the number of unknowns is quite large. □

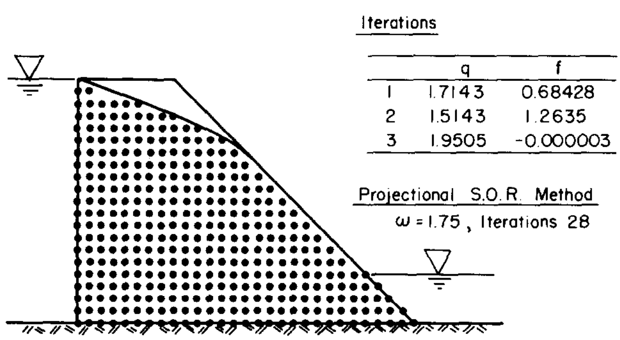


Fig. 3.8. Numerical results for Example 3-3.1 obtained by discharge descent method.

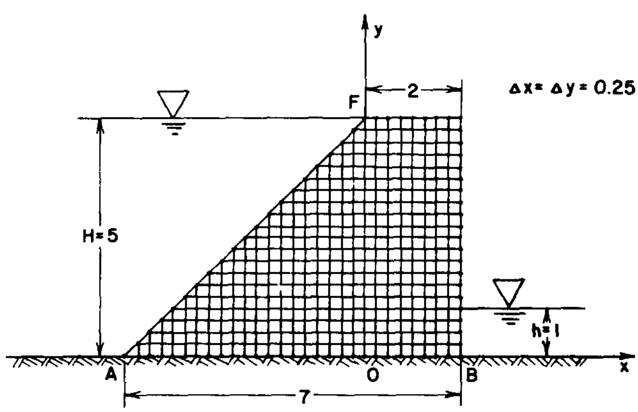


Fig. 3.9. Discrete model for Example 3-3.2.

ITERATIONS

	q	f
1	6.000	2.0099
2	5.900	1.9350
3	3.317	0.0287
4	3.278	0.0056
5	3.269	-0.0003

PROJECTIONAL S.O.R.

$\omega = 1.75$
Iterations 40~43
tolerance 10^{-3}

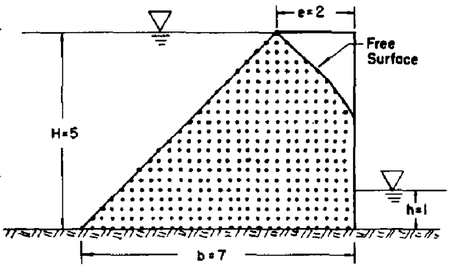


Fig. 3.10. Numerical results for Example 3-3.2 obtained by solving finite system of variational inequalities.

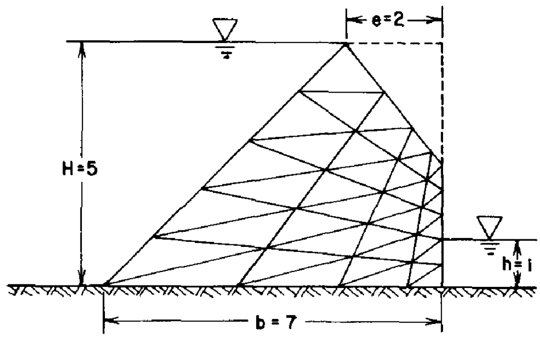


Fig. 3.11. Initial mesh for calculation of Example 3-3.2 by adaptive mesh method.

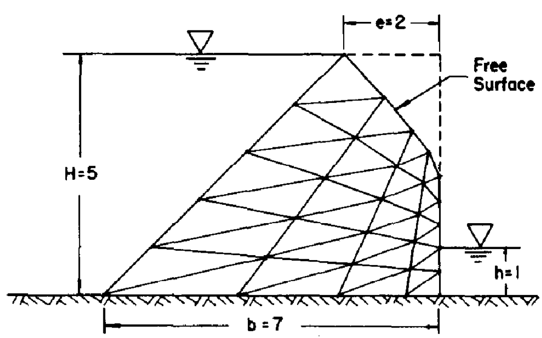


Fig. 3.12. Free surface in Example 3-3.2 calculated using adaptive mesh method.

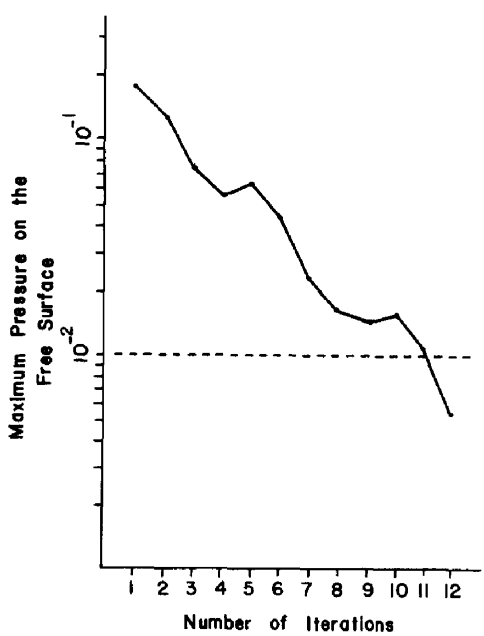


Fig. 3.13. Convergence characteristics of the adaptive mesh method for Example 3-3.2.

Example 3-3.3. In this example, we consider finite element models of three different formulations of the problem of determining the free streamline of the flow through an isotropic homogeneous rectangular dam: a variational inequality obtained using Baiocchi's transformation, the extended pressure method, and the adaptive mesh method. Physical dimensions and a discrete model for the method of variational inequalities and the extended pressure method are given in Fig. 3.14.

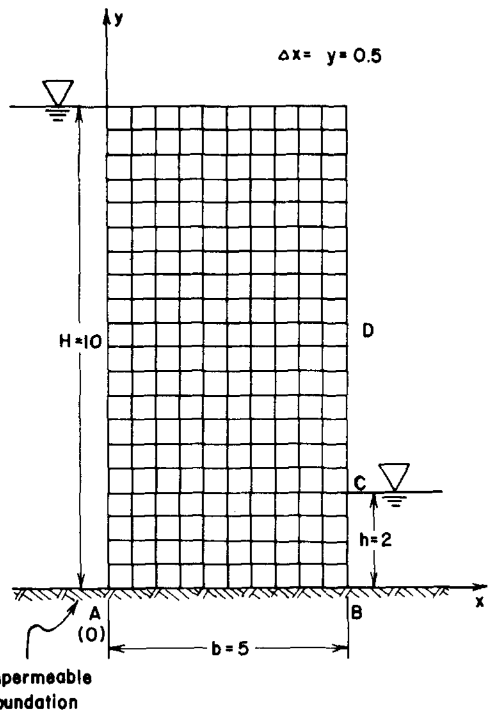


Fig. 3.14. Mesh used in discretization of a rectangular dam—Example 3-3.3.

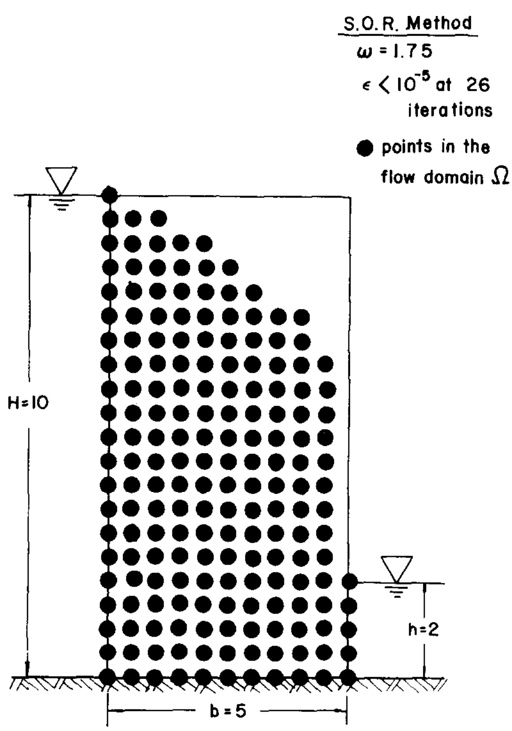


Fig. 3.15. Results of numerical solution of variational inequalities for Example 3-3.3.

Since the dam is rectangular, we choose to solve the variational inequality (2.9b) or (2.20b), using the projectional S.O.R. method discussed in Chap. 2. However, in this case the discharge of seepage is known *a priori*. This leads to no iterative calculations of the type in Example 3-3.2. Numerical results are shown in Fig. 3.15.

For the extended pressure method, we choose the penalty parameter $\epsilon = 0.1$ for a mesh size $h = \Delta x = \Delta y = 0.5$. As mentioned earlier, the penalty parameter ϵ depends strongly upon the mesh size h. To solve the non-linear system obtained by the discretization of the penalized formulation (2.34), we again apply the S.O.R. algorithm. Numerical results are shown in Fig. 3.16. The position of the free streamline obtained by the extended pressure method coincides with that obtained by variational inequalities.

Using the discrete model given in Fig. 3.17, we have also solved the same problem by the

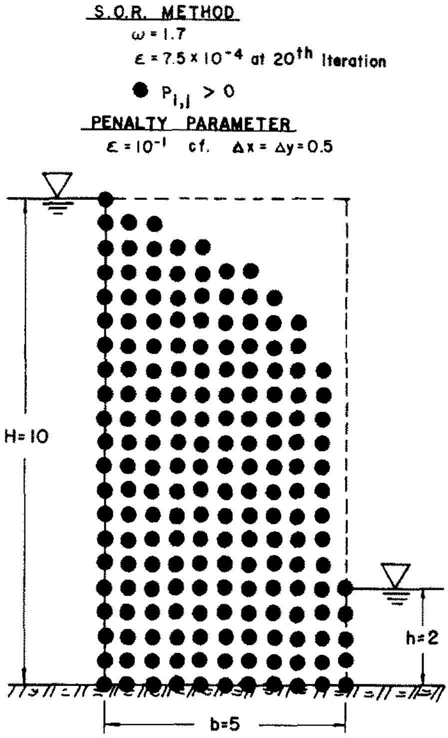


Fig. 3.16. Flow domain in Example 3-3.3 computed by extended pressure method.

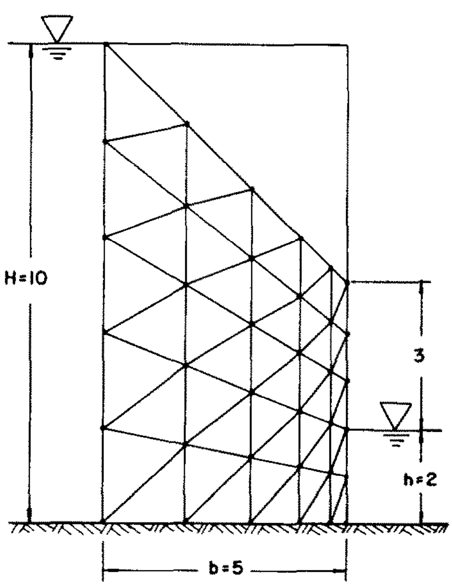


Fig. 3.17. Initial mesh for adaptive mesh calculation of Example 3-3.3.

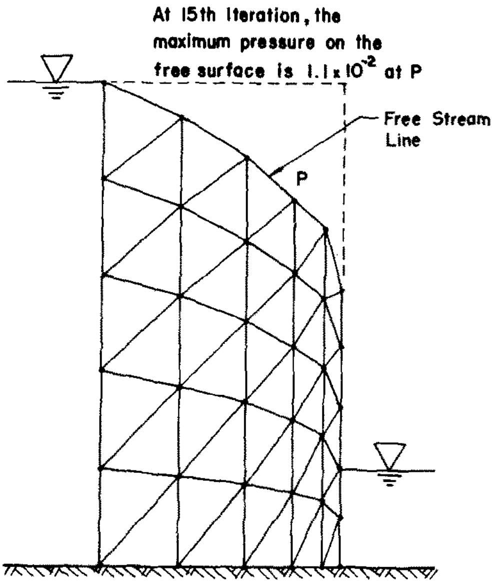


Fig. 3.18. Flow domain in Example 3-3.3 computed by adaptive mesh method.

adaptive mesh method. After 15 iterations the adaptive method converged to the solution indicated in Fig. 3.18. The results obtained by this method also agree well with those obtained by the other two methods. □

4. NON-HOMOGENEOUS DAMS

4.1 *Seepage flow problems in non-homogeneous dams*

Suppose that the dam is non-homogeneous, isotropic, and that the permeability of the dam is denoted by $k(x, y)$. If the seepage flow is governed by Darcy's law, the problem can be represented as the following boundary-value problem:

Find $\{p(x, y), Y(x)\}$ such that

$$\left.\begin{aligned} p(x,y)>0 \quad \text{in} \quad \Omega, \quad p(x,y)=0 \quad \text{in} \quad D/\bar{\Omega} \\ -\nabla\cdot(k\nabla(p+y))=0 \quad \text{in} \quad \Omega \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} p=H-y \quad \text{on} \quad AF, \quad p=h-y, \quad \text{on} \quad BC \\ k(p+y)_n=0 \quad \text{on} \quad AB \end{aligned}\right\} \tag{1.2}$$

$$p=0 \quad \text{and} \quad k(p+y)_n=0 \quad \text{on} \quad FD \tag{1.3}$$

$$p=0 \quad \text{and} \quad k(p+y)_n\le 0 \quad \text{on} \quad CD \tag{1.4}$$

$$\Omega=\{(x,y)\in D\colon\ y\le Y(x)\} \tag{1.5}$$

Here, the same notations as in Chap. 3 are used, see Fig. 3.1. The function $Y(\cdot)$ indicates the position of the free surface FD.

Suppose that $p(x, y)$ satisfies the above boundary-value problem (1.1) ~ (1.5). Then, for every $\varphi\in C^1(\bar{D})$ with $\varphi=0$ on $AF\cup BC$, and $\varphi\ge 0$ on CD, we have

$$\begin{aligned} &\int_D k\nabla p\cdot\nabla\varphi\,dx\,dy+\int_\Omega k\nabla y\cdot\nabla\varphi\,dx\,dy \\ &=\int_\Omega k\nabla(p+y)\cdot\nabla\varphi\,dx\,dy+\int_{D\setminus\Omega}k\nabla p\cdot\nabla\varphi\,dx\,dy \\ &=\int_\Omega\{-\nabla\cdot k\nabla(p+y)\}\varphi\,dx\,dy+\int_\Gamma k(p+y)_n\varphi\,ds \\ &=\int_{CD}k(p+y)_n\varphi\,ds\le 0. \end{aligned}$$

Here Γ denotes the boundary of the (unknown) flow domain Ω. Thus, the results by Brezis, Kinderleher and Stampacchia[45] given in Section 3.2 can be extended for the case of non-homogeneous dams. That is, using the extended pressure $p(x, y)$, the following variational problem is derived from the boundary-value problem (1.1) ~ (1.5).

Find $\{p(x, y), g(x, y)\} \in H^1(D) \times L^\infty(D)$ such that

$$\left.\begin{array}{c} g(x, y) = 1 \quad \text{if} \quad p(x, y) > 0 \\ 0 \le g(x, y) \le 1 \quad \text{if} \quad p(x, y) = 0, \end{array}\right\} \tag{1.6}$$

$$\left.\begin{array}{c} p = H - y \quad \text{on} \quad AF, \quad p = h - y \quad \text{on} \quad BC, \\ p = 0 \quad \text{on} \quad FED, \end{array}\right\} \tag{1.7}$$

$$\int_D (k\nabla p \cdot \nabla \varphi + gk\varphi_y)\, dx\, dy \le 0 \tag{1.8}$$

for every $\varphi \in H^1(D)$ with $\varphi \ge 0$ on CD and $\varphi = 0$ on $AF \cup BC$.

The penalized problem corresponding to (1.8) is then defined by:

Find $p_\epsilon \in H^1(D)$ with $p_\epsilon = H - y$ on AF, $p_\epsilon = h - y$ on BC, $p_\epsilon = 0$ on FED, and

$$\int_D (k\nabla p_\epsilon \cdot \nabla \varphi + H_\epsilon(p_\epsilon)k\varphi_y)\, dx\, dy = 0 \tag{1.9}$$

for every $\varphi \in H^1(D)$ with $\varphi = 0$ on $AF \cup BC \cup CD$ where

$$H_\epsilon(p) = \begin{cases} 0 & \text{if} \quad p \le 0 \\ p/\epsilon & \text{if} \quad 0 \le p \le \epsilon \\ 1 & \text{if} \quad p \ge \epsilon. \end{cases} \tag{1.10}$$

Applying the same arguments as Theorem 3-2.3, we can establish the following results:

Theorem 4.1.1. Suppose that $k(x, y)$ is a bounded positive function defined on D. Then there exists at least one solution $\{p(x, y), g(x, y)\}$ to the problem (1.6) ~ (1.8). Moreover, the solution $p(x, y)$ is the weak limit of the solution $\{p_\epsilon(x, y)\}$ of the penalized problem (1.9) as $\epsilon \to 0$, where p_ϵ is the unique solution to (1.9) of each fixed $\epsilon > 0$. □

We now recast the seepage flow problem (1.1) ~ (1.5) using Baiocchi's transformation. However, to obtain the formulation by variational inequalities through Baiocchi's transformation, we cannot consider general cases of non-homogeneity $k(x, y)$ as shown below. We will study only two cases; $k = k(x)$ and $k = k(y)$.

4.2 *The case* k = k(x)

We first consider the case in which $k = k(x)$. Again we introduce Baiocchi's transformation

$$u(x, y) = \int_0^y p(x, t)\, dt. \tag{2.1}$$

Let $v \in C_0^\infty(D)$ and $w(x, y) = \int_0^y v(x, t)\, dt$, i.e. $v = w_y$ and $w \in C_0^\infty(D)$. Suppose that (1.1) holds. Then

$$\begin{aligned} \int_\Omega k\nabla u \cdot \nabla v\, dx\, dy &= \int_\Omega k\nabla u \cdot (\nabla w)_y\, dx\, dy \\ &= -\int_\Omega (k\nabla u)_y \cdot \nabla w\, dx\, dy + \int_\Gamma (k\nabla u \cdot \nabla w)n_y\, ds \\ &= -\int_\Omega k\nabla u_y \cdot \nabla w\, dx\, dy + \int_\Gamma (k\nabla u \cdot \nabla w)n_y\, ds \\ &= \int_\Omega kw_y\, dx\, dy + \int_\Gamma (k\nabla u \cdot \nabla w)n_y\, ds \end{aligned}$$

and

$$\int_{D|\Omega} k\nabla u \cdot \nabla v \, dx \, dy = -\int_{D|\Omega} k\nabla u_y \cdot \nabla w \, dx \, dy - \int_\Gamma (k\nabla u \cdot \nabla w) n_y \, ds = -\int_\Gamma (k\nabla u \cdot \nabla w) n_y \, ds.$$

These imply

$$\int_D k\nabla u \cdot \nabla v \, dx \, dy = \int_\Omega kv \, dx \, dy + \int_S [\![(k\nabla u \cdot \nabla w)]\!] n_y \, ds$$

where $[\![\varphi]\!] = \varphi^+ - \varphi^-$ is the jump in φ on S. Since $k\nabla u$ is continuous on D,

$$\int_D k\nabla u \cdot \nabla v \, dx \, dy = \int_D k\chi_\Omega v \, dx \, dy. \tag{2.2}$$

Here χ_Ω is the characteristic function of the domain Ω. Continuity of $k\nabla u$ is verified by

$$\begin{aligned} k\nabla u &= \left(k \int_0^y p_x \, dt\right)\mathbf{i} + (kp)\mathbf{j} \\ &= \left(\int_0^y kp_x \, dt\right)\mathbf{i} + (kp)\mathbf{j}. \end{aligned}$$

If we define

$$z(x, y) = \int_0^y k(x) p(x, t) \, dt. \tag{2.3}$$

then, instead of (2.2), we obtain

$$\int_D k\nabla\left(\frac{z}{k}\right) \cdot \nabla v \, dx \, dy = \int_D \chi_\Omega v \, dx \, dy. \tag{2.4}$$

Summarizing, we have:

***Theorem* 4-2.1. Let the permeability k depend upon only the x-coordinate. Let the pressure p satisfy (1.1) ~ (1.4). Then u, defined by (2.1), satisfies**

$$-\nabla \cdot (k\nabla u) = k\chi_\Omega \tag{2.5}$$

in the sense of distributions on D. □

Boundary conditions, described in Fig. 3.1, are written

$$\begin{aligned} &u_y = H - y \quad \text{on} \quad AF, \quad u_y = h - y \quad \text{on} \quad BC, \\ &u_y = 0 \quad \text{on} \quad CD. \end{aligned} \tag{2.6}$$

Impermeability of the bottom implies

$$k(x)(p_x n_x + p_y n_y + n_y) = 0.$$

Since $k(x) \neq 0$

$$u_{xy} n_x + (u_{yy} + 1) n_y = 0.$$

Since the bottom of the dam is assumed to be flat,

$$u_{yy} + 1 = 0.$$

From (2.5),

$$-(ku_x)_x = 0 \quad \text{i.e.} \quad ku_x = \text{const.} \tag{2.7}$$

By the definition (2.1)

$$u = 0 \quad \text{on} \quad AB. \tag{2.8}$$

This means that (2.7) is automatically satisfied.

Let the set $K(u)$ be defined by

$$K(u) = \{v \in H^1(D): v = 0 \quad \text{a.e. on} \quad AB, \quad 0 \le v(x, y) \le u(x, Z(x)), \quad \text{a.e. in} \quad D\}. \tag{2.9}$$

Suppose that $k(x)$ is differentiable. Then

$$\begin{aligned}\int_D (-Z'ku_yv_x + Z'ku_xv_y - (Z'k)'u_yv)\,\mathrm{d}x\,\mathrm{d}y \\ = \int_a^b k(-u_yn_x + u_xn_y)Z'v\,\mathrm{d}x \\ = -\int_\Gamma k(u_kn_x + u_yn_y)v\,\mathrm{d}s + \int_a^b ku_y\sqrt{1+Z'^2}v\,\mathrm{d}x\end{aligned}$$

for every $u, v \in C^\infty(\bar{D})$ with $v = 0$ on AB. If, in addition, we require that u be a solution of (2.5), (2.6), (2.8), then

$$u(x, y) = u(x, Z(x)) \quad \forall (x, y) \in D/\Omega \tag{2.10}$$

and if we denote

$$\left.\begin{aligned} a(u, v) &= \int_D \{k((u_x - Z'u_y)v_x + (u_y - Z'u_x)v_y) - (kZ')'u_yv\}\,\mathrm{d}x\,\mathrm{d}y \\ L(v) &= \int_D kv\,\mathrm{d}x\,\mathrm{d}y + \int_a^0 k(H-y)\sqrt{1+Z'^2}v\,\mathrm{d}x + \int_c^b k(h-y)\sqrt{1+Z'^2}v\,\mathrm{d}x \end{aligned}\right\} \tag{2.11}$$

then, for every $v \in K(u)$,

$$a(u, v-u) = L(v-y) + \int_D (-\nabla \cdot (k\nabla u) - k)(v-u)\,\mathrm{d}x\,\mathrm{d}y \ge L(v-u).$$

Therefore, the *quasi-variational inequality*

$$u \in K(u): a(u, v-u) \ge L(v-u) \quad \forall v \in K(u) \tag{2.12}$$

is obtained for the case in which $k = k(x)$.

4.3 *Special cases for* k = k(x)

By arguments similar to those used in Lemma 3-3.1, we can show that the function q given by

$$q = \int_0^{Y(x)} kp_x(x, t)\,\mathrm{d}t = \int_0^{Z(x)} kp_x(x, t)\,\mathrm{d}t \tag{3.1}$$

is constant, a.e. on $(0, d)$. Since $p = u_y$

$$q = \int_0^{Z(x)} (k(x)u_x)_t\,\mathrm{d}t = k(x)u_x(x, Z(x)) - k(x)u_x(x, 0).$$

Since $u(x, 0) = 0$ for all $x \in [0, d]$

$$u_x(x, Z(x)) = \frac{q}{k(x)}.$$

By integration with respect to x, we get

$$u(x, Z(x)) = C_1 + \int_0^x \frac{q}{k(s)} \, ds. \tag{3.2}$$

(i) *The case that* EDCB *is vertical*: In this case

$$\begin{aligned} u(d, y) &= \int_0^y u_t \, dt = \int_0^y (h - t) \, dt \\ u(d, Z(d)) &= \int_0^h (h - t) \, dt = \frac{1}{2} h^2. \end{aligned} \tag{3.3}$$

Then

$$u(x, Z(x)) = \frac{1}{2} h^2 - \int_x^d \frac{q}{k(s)} \, ds \quad x \in [0, d] \tag{3.4}$$

The quasi-variational inequality (2.12) under the "moving" admissible set $K(u)$ defined by (2.9) becomes

$$u \in K(q): a(u, v - u) \geq \tilde{L}(v - u) \quad \forall v \in K(q) \tag{3.5}$$

where $a(u, v)$ is same as $(2.11)_1$, $\tilde{L}(v)$ is

$$\tilde{L}(v) = \int_D kv \, dx \, dy + \int_a^0 k(H - y)\sqrt{1 + (Z')^2} \, dx, \tag{3.6}$$

$$\begin{aligned} K(q) = \{ v \in H^1(D)&: v = 0, \quad \text{a.e.} \quad AB, \quad v = \frac{1}{2} y(2h - y), \\ &\text{a.e. on } BC, \quad v(x, Z(x)) = g(q) \quad \text{a.e.} \quad FDC, \\ &v(x, y) \leq g(q), \quad \text{a.e.} \quad (x, y) \in D_2 \}. \end{aligned} \tag{3.7}$$

Here

$$g(q) = \frac{1}{2} h^2 - \int_x^d q/k(s) \, ds = u(x, Z(x)). \tag{3.8}$$

We note that there is only one parameter q physically representing the discharge of the flow through the dam in the admissible set $K(q)$ of (3.7). That is, for physically meaningful discharges q the solution u of (3.5) has to satisfy the "compatibility" condition

$$f(q) = p = u_y = 0 \quad \text{at} \quad F. \tag{3.9}$$

(ii) *The case of a dam with two vertical layers.* In this case, the discharge q can be obtained explicitly. By the definition of u,

$$u(0, Z(0)) = \frac{1}{2} H^2. \tag{3.10}$$

From (3.4),

$$u(0, Z(0)) = \frac{1}{2} h^2 - \int_0^d q/k(s) \, ds. \tag{3.11}$$

Let t_1 be the thickness of the first layer. Let k_1 and k_2 be permeabilities for the first and second thickness of the rectangular two-layered dam, where k_1 and k_2 are constants. Then

$$\left.\begin{aligned} q &= -k_1k_2(H^2-h^2)/2(t_1k_2+(d-t_1)k_1) \\ g &= \frac{1}{2}H^2+qx/k(x). \end{aligned}\right\} \quad (3.12)$$

The quasi-variational inequality (3.5) is then reduced to the variational inequality

$$u \in K\colon \hat{a}(u, v-u) \geq \hat{L}(v-u) \quad \forall v \in K \quad (3.13)$$

$$\hat{a}(u, v) = \int_D k\nabla u \cdot \nabla v \, dx\, dy, \quad \hat{L}(v) = \int_D kv \, dx\, dy \quad (3.14)$$

$$K = \{v \in H^1(D)\colon v = 0, \text{ a.e. on } AB, \; v = \frac{1}{2}y(H-y), \text{ a.e. on } AF,$$

$$v = \frac{1}{2}y(h-y), \text{ a.e. on } BC, \; v = g, \text{ a.e. on } FDC,$$

$$0 \leq v \leq g, \text{ a.e. in } D\}. \quad (3.15)$$

(iii) *The case of a dam with three vertical layers.* In this case

$$\left.\begin{aligned} q &= -k_1k_2k_3(H^2-h^2)/2(t_1k_2k_3+(t_2-t_1)k_3k_1+(d-t_2)k_1k_2) \\ g &= \frac{1}{2}H^2+qx/k(x) \end{aligned}\right\} \quad (3.16)$$

where k_1, k_2 and k_3 are constant permeabilities of three layers, respectively, t_1 and t_2 are thickness of the first and second layers, respectively. The variational inequality (3.13) now represents the problem after an adjustment using (3.16).

Example 4-3.1. We solve numerically the variational inequality (3.13) on (3.15) for the case of a dam with two vertical layers using the methods discussed in Chap. 2. For simplicity, the dam is assumed to be isotropic and rectangular. Physical dimensions and a discretized model of the example problem (3.10) are shown in Fig. 4.1. The discretized problem (3.10) is solved by the projectional S.O.R. method described in Chap. 2. The iteration parameter ω is taken as 1.75, and the convergence of the S.O.R. method is obtained within 35 iterations. The numerical results are shown in Fig. 4.2.

The same example problem was also solved by the extended pressure method using a similar discrete model for the method of variational inequalities. The parameter of the penalization is $\epsilon = 10^{-1}$ for a mesh size of $h = \Delta x = \Delta y = 0.25$. Numerical results are given in Fig. 4.3. The position of the free streamline obtained by the extended pressure method almost coincides with that of the variational inequality (3.13).

The discrete model described in Fig. 4.4 was obtained by the adaptive mesh method. Numerical results are shown in Fig. 4.5. It is noteworthy that the iterative scheme used in the adaptive mesh method may not converge if the number of mesh divisors of material II in the vertical direction is the same as that of material *I*, and if the free streamline turns out to be coincident with the interface of the two materials. □

4.4 *The case* k = k(y)

For the case that $k = k(y)$, the Baiocchi's transformation becomes

$$u(x, y) = \int_0^y k(t)p(x, t)\, dt. \quad (4.1)$$

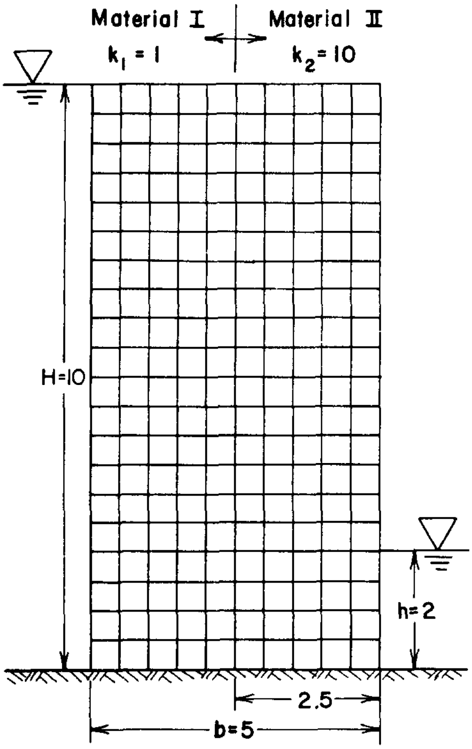


Fig. 4.1

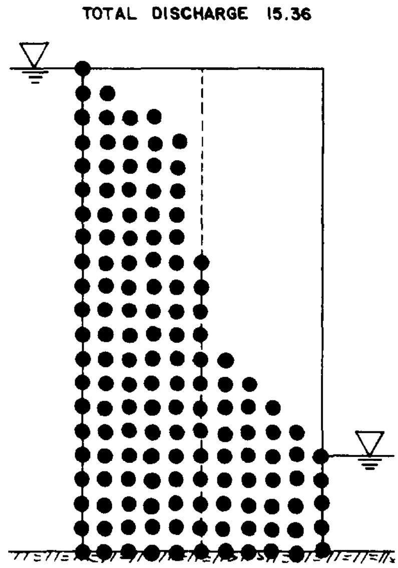


Fig. 4.2

Fig. 4.1. Discrete model for Example 4-3.1.

Fig. 4.2. Numerical solution of the discrete variational inequality of Example 4-3.1.

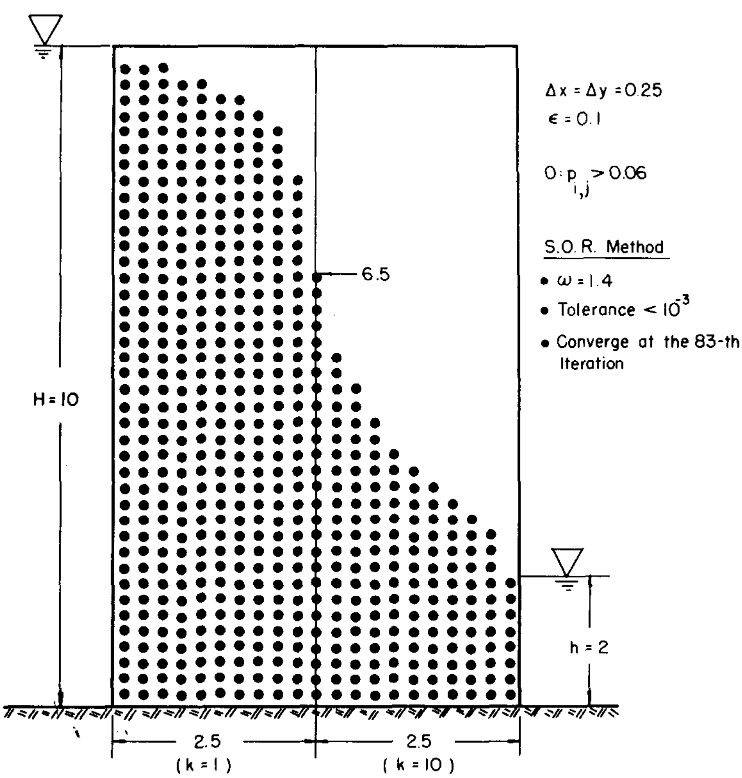


Fig. 4.3. Numerical results obtained by the extended pressure formulation (Example 4-3.1).

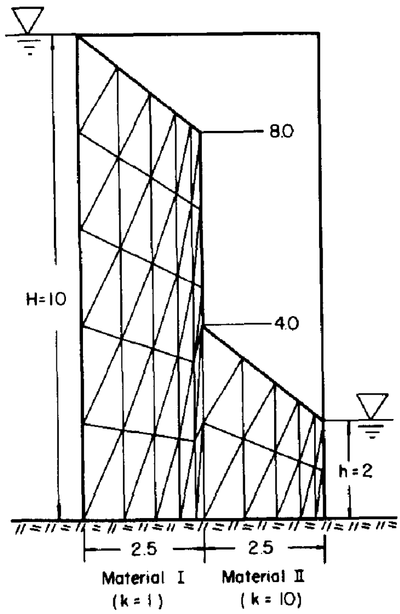


Fig. 4.4

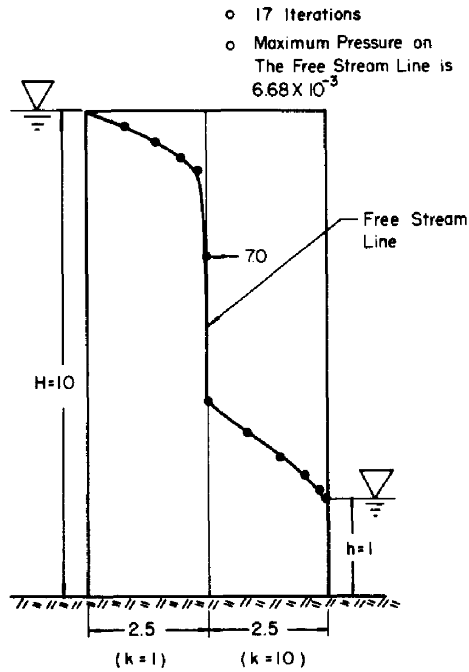


Fig. 4.5

Fig. 4.4. Initial mesh for adaptive mesh calculation of Example 4-3.1.

Fig. 4.5. Numerical results obtained by the adaptive mesh method for Example 4-3.1.

Then, as in the proof of Theorem 4-2.1, for $v \in C_0^\infty(D)$,

$$\int_\Omega \frac{1}{k}\nabla u \cdot \nabla(kv)\,dx\,dy = \int_\Omega \left\{u_x w_{xy} + \left(\frac{1}{k}u_y\right)(kw_y)_y\right\} dx\,dy$$
$$= -\int_\Omega \left(u_{yx}w_x + \left(\frac{1}{k}u_y\right)_y kw_y\right) dx\,dy + \int_{FD}\left(u_x w_x n_y + \frac{1}{k}u_y k w_y n_y\right) ds$$
$$= -\int_\Omega k(p_x w_x + p_y w_y)\,dx\,dy + \int_{FD}(u_x w_x + u_y w_y)n_y\,ds$$
$$= \int_\Omega k w_y\,dx\,dy + \int_{FD}(\nabla u \cdot \nabla w)n_y\,ds$$

$$\int_{D/\Omega}\frac{1}{k}\nabla u \cdot \nabla(kv)\,dx\,dy = -\int_{FD}(\nabla u \cdot \nabla w)n_y\,ds.$$

Using the continuity of ∇u on FD

$$\int_D \frac{1}{k}\nabla u \cdot \nabla(kv)\,dx\,dy = \int_\Omega k\chi_\Omega v\,dx\,dy, \quad \text{i.e.} \quad -k\nabla\cdot\left(\frac{1}{k}\nabla u\right) = k\chi_\Omega \tag{4.2}$$

in the sense of distributions on D.

Theorem 4-4.1. After the Baiocchi transformation (4.1), the solution u of (4.1) satisfies

$$-k\nabla\cdot\left(\frac{1}{k}\nabla u\right) = k\chi_\Omega \quad \text{in} \quad D \tag{4.3}$$

in the sense of distributions. □

Boundary conditions, described in Fig. 3.1, are written

$$\begin{aligned} &u_y = k(H-y) \quad \text{on} \quad AF, \quad u_y = k(h-y) \quad \text{on} \quad BC \\ &u_y = 0 \quad \text{on} \quad CD. \end{aligned} \tag{4.4}$$

Since the bottom foundation is horizontal

$$u=0 \quad \text{on} \quad AB. \tag{4.5}$$

Impermeability of the bottom is given by

$$p_x n_x + (p_y + 1)n_y = 0, \quad \text{i.e.} \quad p_y + 1 = 0$$

which implies that

$$\left(\frac{1}{k}u_y\right)_y + 1 = 0$$

and then

$$u_{xx} = 0 \quad \text{on} \quad AB. \tag{4.6}$$

This requirement is automatically satisfied by the boundary condition (4.5). That is, the transformation (4.1) is compatible with the assumed impermeability of the flat bottom.

Let the set $K(u)$ be defined by

$$\begin{aligned} K(u) = \{v \in H^1(D)\colon v = 0, \quad &\text{a.e. on} \quad AB, \quad v(x, y) = u(x, y), \quad \text{a.e. on} \quad FDC, \\ 0 \le v(x, y) \le u(x, Z(x)), \quad &\text{a.e. in} \quad D\}. \end{aligned} \tag{4.7}$$

We note that

$$\begin{aligned} u(x, y) &= u(x, Z(x)), \quad \text{a.e.} \quad (x, y) \in D/\Omega \\ -k\nabla \cdot \left(\frac{1}{k}\nabla u\right) &= 0, \quad \text{a.e.} \quad (x, y) \in D/\Omega \end{aligned} \tag{4.8}$$

Suppose that u is a solution of (4.3), (4.4), (4.5) and (4.8) and $k(y)$ is differentiable on D. Then, for every $v \in K(u)$,

$$\begin{aligned} &\int_D \left\{\frac{1}{k}(u_x - Z'u_y)(kv - ku)_x + \frac{1}{k}(u_y + Z'u_x)(kv - ku)_y \right. \\ &\qquad \left. - Z''u_y(v - u) - \frac{k'}{k}Z'u_x(v - u)\right\} \mathrm{d}x\,\mathrm{d}y \\ &= \int_D \left(k\nabla \cdot \left(\frac{1}{k}\nabla u\right) - k\right)(v - u)\,\mathrm{d}x\,\mathrm{d}y + \int_D k(v - u)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_a^0 u_y\sqrt{1 + Z'^2}(v - u)\,\mathrm{d}x + \int_c^b u_y\sqrt{1 + Z'^2}(v - u)\,\mathrm{d}x. \end{aligned}$$

By putting

$$a(u, v) = \int_D \left\{\frac{1}{k}(u_x - Z'u_y)(kv)_x + \frac{1}{k}(u_y + Z'u_x)(kv)_y - Z''u_y v - \frac{k'}{k}Z'u_x v\right\} \mathrm{d}x\,\mathrm{d}y \tag{4.9}$$

$$L(v) = \int_D kv\,\mathrm{d}x\,\mathrm{d}y + \int_a^0 k(H - y)\sqrt{(1 + Z'^2)}v\,\mathrm{d}x + \int_c^b k(h - y)\sqrt{1 + Z'^2 v\,\mathrm{d}x} \tag{4.10}$$

we obtain, for arbitrary v in $K(u)$,

$$\begin{aligned} a(u, v - u) &= L(v - u) + \int_D \left(k\nabla \cdot \left(\frac{1}{k}\nabla u\right) - k\right)(v - u)\,\mathrm{d}x\,\mathrm{d}y \\ &\ge L(v - u). \end{aligned}$$

Thus, we arrive at the quasi-variational inequality

$$u \in K(u)\colon a(u, v-u) \geq L(v-u), \quad \forall v \in K(u) \tag{4.11}$$

4.5 *Special cases for* k = k(y)

By arguments similar to those used in the proof of Lemma 3-3.1, we conclude that the function

$$q = -\int_0^{Z(x)} kp_x(x,t)\,\mathrm{d}t \tag{5.1}$$

is constant, a.e. on $(0, d)$. Since $kp = u_y$

$$-q = \int_0^{Z(x)} (u_x)_t\,\mathrm{d}t = u_x(x, Z(x)) - u_x(x, 0).$$

Since $u(x,0) = 0$ on $(0, d)$

$$-q = u_x(x, Z(x)).$$

Integrating with respect to x yields

$$u(x, Z(x)) = C_1 - qx. \tag{5.2}$$

(i) *The case that* EDCB *is vertical*: By the definition of u,

$$u(d, y) = \int_0^y k(t)(h-t)\,\mathrm{d}t \tag{5.3}$$

Then

$$C_1 = \int_0^h k(t)(h-t)\,\mathrm{d}t + qd.$$

This implies

$$g(q) = u(x, Z(x)) = \int_0^h k(t)(h-t)\,\mathrm{d}t + q(d-x). \tag{5.4}$$

The quasi-variational inequality (4.11) under the admissible set (4.7) is reduced to

$$u \in K(q)\colon a(u, v-u) \geq \tilde{L}(v-u) \quad \forall v \in K(q) \tag{5.5}$$

$$K(q) = \Big\{ u \in H^1(D)\colon v = 0, \quad \text{a.e. on} \quad AB,$$

$$v = \int_0^y k(t)(h-t)\,\mathrm{d}t, \quad \text{a.e. on} \quad BC, \quad v = g(q), \quad \text{a.e. on} \quad FDC,$$

$$0 \leq v \leq g(q), \quad \text{a.e. in} \quad D_2 \Big\} \tag{5.6}$$

$$\tilde{L}(v) = \int_D kv\,\mathrm{d}x\,\mathrm{d}y + \int_a^0 k(H-y)\sqrt{1+Z'^2}\,v\,\mathrm{d}x. \tag{5.7}$$

We note that for a physically meaningful q the compatibility condition

$$f(q) = p = u_y = 0 \quad \text{at} \quad F \tag{5.8}$$

must be satisfied.

(ii) *The case of a dam with two horizontal layers*: Suppose that the thickness of the lower layer is given by $b > h$ and that k_1 and k_2 are the permeabilities of the upper and lower layers, respectively. Then (5.3) becomes

$$u(d, y) = g_d = \begin{cases} \frac{1}{2} k_2(2hy - y^2) & \text{for} \quad y \le h \\ \frac{1}{2} k_2 h^2 & \text{for} \quad y \ge h \end{cases} \tag{5.9}$$

and also we have

$$u(0, y) = g_0 = \begin{cases} \frac{1}{2} k_2(2Hy - y^2) & \text{for} \quad y \le b \\ \frac{1}{2}(k_2 - k_1)(2Hb - b^2) + \frac{1}{2} k_1(2Hy - y^2) & \text{for} \quad y \ge b \end{cases} \tag{5.10}$$

and

$$\begin{aligned} g(x) = &\frac{1}{2}(k_2 - k_1)(2Hb - b^2) + \frac{1}{2} k_1 H^2 \\ &- \frac{x}{d}\left(\frac{1}{2}(k_2 - k_1)(2Hb - b^2) + \frac{1}{2} k_1 H^2 - \frac{1}{2} k_2 h^2\right). \end{aligned} \tag{5.11}$$

Let

$$\hat{a}(u, v) = \int_D \left\{\frac{1}{k} u_x(kv)_x + \frac{1}{k} u_y(kv)_y\right\} \mathrm{d}x\, \mathrm{d}y \tag{5.12}$$

$$\hat{L}(v) = \int_D kv \, \mathrm{d}x\, \mathrm{d}y. \tag{5.13}$$

Then the quasi-variational inequality (5.5) is reduced to the variational inequality

$$u \in K: \hat{a}(u, v - u) \ge \hat{L}(v - u), \quad \forall v \in K \tag{5.14}$$

$$\begin{aligned} K = \{&v \in H^1(D): v = 0, \quad \text{a.e. on} \quad AB, \quad v = g_d, \\ &\text{a.e. on} \quad BC \text{ and } CE, \quad v = g_0, \quad \text{a.e. on} \quad AF, \\ &0 \le v(x, y) \le g(x), \quad \text{a.e. in} \quad D\} \end{aligned} \tag{5.15}$$

where g_d, g_0 and $g(x)$ are defined by (5.9) ~ (5.11).

4.6 *Comments*

The formulations using extended pressures follow from Brezis, Kinderlehrer and Stampacchia[45] with slight modifications. The inequality (1.8) simply becomes an equation if the test function $\varphi \in H^1(D)$ satisfies $\varphi = 0$ on the unknown seepage line CD. This also holds for the penalized problem (1.9) in this case. We emphasize again that the penalty parameter ϵ cannot be selected arbitrarily for the discrete model of (1.8); it strongly depends upon the parameter h of the discretization of the domain. Moreover, the flow region Ω is defined by, e.g.

$$\Omega = \{(x, y) \in D, \quad p_\epsilon(x, y) \ge \alpha_\epsilon\}$$

where α is a proper constant. Discretizations of (1.8) by finite element methods have also been studied by Le Tallec[46].

Our derivations of variational inequalities in Sections 4.2 ~ 4.5 mainly follow the work of Baiocchi *et al.*[14] and Kikuchi[36]. A different type of non-homogeneous dam is analyzed by Benci[47], but his treatment is essentially the same as that given here.

We have not considered issues such as existence, uniqueness and regularity of solutions in this chapter. However, several results on these questions follow easily from arguments given in Chap. 3.

5. SEEPAGE FLOW PROBLEMS IN WHICH DISCHARGE IS UNKNOWN

5.1 *Dam with an impermeable sheet*

We will now consider the case of a rectangular, homogeneous, isotropic dam through which the flow is restricted by the presence of an impermeable sheet on part of the upstream side of the dam, as shown in Fig. 5.1. While such a problem can be formulated in terms of the pressure, as in the previous two chapters, the problem is described here in terms of the extended velocity potential φ. This problem is formally stated as follows:

Problem 5-1.1 Find (φ, Y) such that

$$\begin{aligned} \Delta\varphi = 0 \quad &\text{in} \quad \Omega = (0, a)\times(0, Y) \\ \varphi = y \quad &\text{in} \quad D/\Omega, \quad D = (0, a)\times(0, y_1) \end{aligned} \tag{1.1}$$

$$\varphi = y_1 \quad \text{on} \quad AH, \quad \varphi = y_2 \quad \text{on} \quad BC, \quad \varphi = y \quad \text{on} \quad CD \tag{1.2}$$

$$\frac{\partial\varphi}{\partial x} = 0 \quad \text{on} \quad GH \quad \text{and} \quad \frac{\partial\varphi}{\partial y} = 0 \quad \text{on} \quad AB \tag{1.3}$$

$$\varphi = y \quad \text{and} \quad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad GD \tag{1.4}$$

where

$$\frac{\partial\varphi}{\partial n} = n_x\frac{\partial\varphi}{\partial x} + n_y\frac{\partial\varphi}{\partial y},$$

$\mathbf{n} = \{n_x, n_y\}$ is the unit vector normal to the boundary $\partial\Omega$ of the domain Ω. □

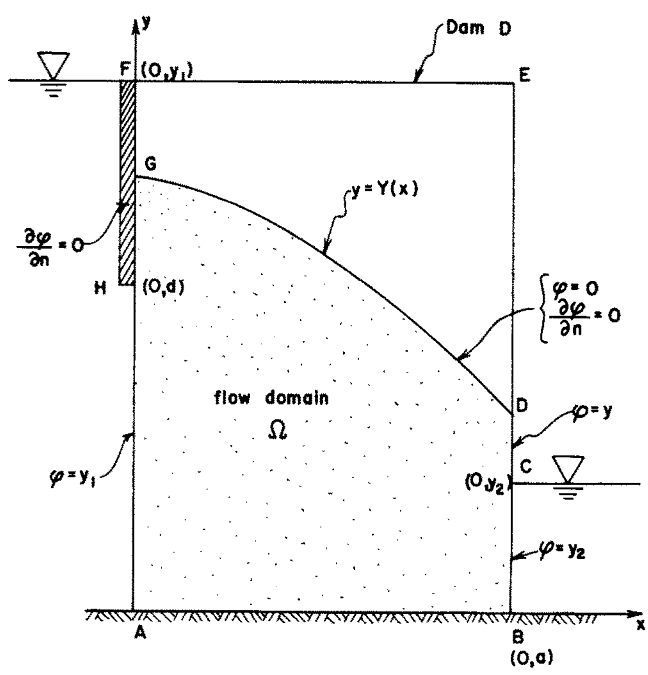


Fig. 5.1. Geometry of flow through a rectangular dam.

We note that the velocity potential φ appearing in problem (1.1) is the *extended* potential, in the sense that

$$\varphi(x, y) = \begin{cases} \hat{\varphi}(x, y) & \text{if } (x, y) \in \Omega \\ y & \text{if } (x, y) \in D/\Omega \end{cases} \tag{1.5}$$

where $\hat{\varphi}(x, y)$ is the usual velocity potential defined on the flow region Ω (recall 3-1.3).

Definition 5-1.1 Let us define the new function w by

$$w(x, y) = \int_y^{y_1} (\varphi(x, t) - t)\, dt. \tag{1.6}$$

□

If $w \in C^1(\bar{D})$, then

$$\varphi(x, y) = y - \frac{\partial w}{\partial y}(x, y). \tag{1.7}$$

Moreover, since the pressure $p(x, y)$ defined by

$$p(x, y) = \varphi(x, y) - y \tag{1.8}$$

is always non-negative on the whole dam D, and is strictly positive on the flow region Ω, we must have

$$w(x, y) \geq 0 \quad \text{in} \quad D \quad \text{and} \quad w(x, y) > 0 \quad \text{in} \quad \Omega \tag{1.9}$$

if φ is the solution of Problem 5-1.1.

We next establish governing equations for $w(x, y)$ when φ is the solution of Problem 5-1.1. Let u be the function defined in (2.1). Then

$$u(x, y_1) = u(x, y) + w(x, y)$$

Thus, since u satisfied (2.5) for $k = \text{constant}$,

$$-\Delta w + \chi_\Omega = \frac{d^2 u(x, y_1)}{dx^2} = 0.$$

This implies that

$$-\Delta w + \chi_\Omega = 0 \tag{1.10}$$

in the sense of distributions on D, where

$$\chi_\Omega(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \Omega \\ 0 & \text{if } (x, y) \in D/\Omega. \end{cases} \tag{1.11}$$

In the process of obtaining (1.10), the boundary conditions (1.4) on the unknown free boundary $y = Y(x)$ have been used.

We shall consider the impervious condition $\partial\varphi/\partial y = 0$ on AB. By (1.7)

$$\frac{\partial \varphi}{\partial y} = 1 - \frac{\partial^2 w}{\partial y^2} = 0 \text{ on } AB.$$

In view of (1.10), we shall choose w so that

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{on} \quad AB.$$

Since the foundation AB is horizontal,

$$\frac{\partial w}{\partial x} = -q \quad \text{on} \quad AB \tag{1.12}$$

where q is a constant. Integration with respect to x yields

$$w(x, 0) = -qx + C_1 \tag{1.13}$$

where C_1 is again a constant. From (1.12) and (1.6),

$$\frac{\partial}{\partial x}\int_0^{y_1} (\varphi(x, t) - t)\,\mathrm{d}t = \int_0^{y_1} \frac{\partial \varphi}{\partial x}(x, t)\,\mathrm{d}t = -q. \tag{1.14}$$

This means that q is the discharge of flow from upstream to downstream. It is obvious that q is unknown in this model problem.

The boundary conditions (1.2) now become

$$\begin{aligned} w(0, y) &= \int_y^{y_1} (\varphi(0, t) - t)\,\mathrm{d}t \\ &= \int_{\mathrm{d}}^{y_1} (\varphi(0, t) - t)\,\mathrm{d}t + \int_y^{\mathrm{d}} (y_1 - t)\,\mathrm{d}t \\ &= C_2 + y_1(d - y) - \frac{1}{2}(d^2 - y^2) \quad \text{for} \quad y \in (0, d) \end{aligned} \tag{1.15}$$

$$w(a, y) = \int_y^{y_1} (\varphi(a, t) - t)\,\mathrm{d}t = 0 \quad \text{for} \quad y \in (y_2, y_1) \tag{1.16}$$

$$w(a, y) = \int_y^{y_2} (y_2 - t)\,\mathrm{d}t = \frac{1}{2}y_2^2 - y_2 y + \frac{1}{2}y^2 \quad \text{for} \quad y \in (0, y_1). \tag{1.17}$$

By (1.13) and (1.17),

$$w(a, 0) = -qa + C_1 = \frac{1}{2}y_2^2.$$

That is

$$C_1 = \frac{1}{2}y_2^2 + qa. \tag{1.18}$$

By (1.13) and (1.15),

$$w(0, 0) = C_1 = C_2 + y_1 d - \frac{1}{2}d^2.$$

That is,

$$C_2 = \frac{1}{2}y_2^2 + qa - y_1 d + \frac{1}{2}d^2. \tag{1.19}$$

Then the condition (1.15) becomes

$$\begin{aligned} w(0, y) &= \frac{1}{2}y_2^2 + qa - y_1 y + \frac{1}{2}y^2 \\ &= \frac{1}{2}y^2 - y_1 y + qa + \frac{1}{2}y_2^2. \end{aligned} \tag{1.20}$$

On the impervious sheet GH

$$\frac{\partial w}{\partial x}(0, y) = -\frac{\partial w}{\partial n}(0, y) = \int_y^{y_1} \frac{\partial \varphi}{\partial x}(0, t)\,dt = 0 \tag{1.21}$$

because of $(1.3)_1$.

Summarizing these results, we have:

Theorem 5-1.1. Suppose that φ is the solution of Problem 5-1.1. Then the new variable w defined by $w(x, y) = \int_y^{y_1} (\varphi(x, t) - t)\,dt$ satisfies

$$-\Delta w + \chi_\Omega = 0 \quad \text{in} \quad D \tag{1.22}$$

$$w > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad w = 0 \quad \text{in} \quad D/\Omega \tag{1.23}$$

$$w|_{\Gamma/\Gamma_N} = g_q \quad \text{and} \quad \left.\frac{\partial w}{\partial n}\right|_{\Gamma_N} = 0 \tag{1.24}$$

where $\Gamma_N = FH$, Γ is the total boundary of the dam $D = (0, a) \times (0, y_1)$ and

$$g_q = \begin{cases} \frac{1}{2}y^2 - y_1 y + qa + \frac{1}{2}y_2^2 & \text{on} \quad HA \\ -q(x-a) + \frac{1}{2}y_2^2 & \text{on} \quad AB \\ \frac{1}{2}(y_2 - y)^2 & \text{on} \quad BC \\ 0 & \text{on} \quad CE \quad \text{and} \quad EF. \quad \square \end{cases} \tag{1.25}$$

It is important to note that since $w(x, y) \geq 0$ in the whole domain D, the condition

$$0 \leq g_q$$

must be imposed. That is

$$\begin{aligned} 0 &\leq \frac{1}{2}y^2 - y_1 y + qa + \frac{1}{2}y_2^2 \\ &= \frac{1}{2}(y - y_1)^2 - \frac{1}{2}y_1^2 + qa + \frac{1}{2}y_2^2 \qquad 0 \leq y \leq d \end{aligned}$$

and

$$0 \leq -q(x - a) - \frac{1}{2}y_2^2 \qquad 0 \leq x \leq a.$$

These are satisfied if

$$q \geq \frac{1}{a}\max\left\{y_1 d - \frac{1}{2}(d^2 + y_2^2), -\frac{1}{2}y_2^2\right\}. \tag{1.26}$$

Moreover, the discharge q for the case with the impervious sheet is less than the one for no impervious sheet, i.e.

$$q \leq \frac{1}{2a}(y_1^2 - y_2^2). \tag{1.27}$$

Thus, the fixed parameter q, which represents the total discharge of seepage, must satisfy the conditions (1.26) and (1.27).

We next seek a variational formulation in terms of the new variable w, using the results in Theorem 5-1.1. Let the set V_q be defined by

$$V_q = \{v \in H^1(D): v = g_q, \quad \text{a.e. on} \quad \Gamma/\Gamma_N\}. \tag{1.28}$$

Suppose that w satisfies (1.22) ~ (1.24) for fixed q, which is now restricted by (1.26) and (1.27). Then, for every $v \in V_q$,

$$\int_D \nabla w \cdot \nabla(v - w)\, dx\, dy = -\int_D \chi_\Omega(v - w)\, dx.$$

Since

$$\begin{aligned} \chi_\Omega(v - w) &= v^+ - w^+ + \chi_\Omega(v - w) - v^+ + w^+ \\ &= v^+ - w^+ - (v^+ - \chi_\Omega v) + w^+ - \chi_\Omega w \\ &= v^+ - w^+ - (v^+ - \chi_\Omega v) \leq v^+ - w^+ \end{aligned}$$

we obtain

$$\int_D \nabla w \cdot \nabla(v - w)\, dx\, dy + \int_D (v^+ - w^+)\, dx\, dy \geq 0 \quad \forall v \in V_q \tag{1.29}$$

where, as usual

$$\varphi^+ = \sup(\varphi, 0).$$

Thus, we have

Theorem 5-1.2. Suppose that w satisfies (1.22) ~ (1.25), and that q satisfies the restrictions (1.26) and (1.27). Then, w satisfies the variational inequality (1.29). □

An alternative principle can also be formulated:

Theorem 5-1.3. If w is a solution of the variational inequality (1.29), then

$$w \geq 0, \quad \text{a.e. in} \quad D \tag{1.30}$$

and w satisfies the inequality

$$\int_D \nabla w \cdot \nabla(v - w)\, dx\, dy + \int_D (v - w)\, dx\, dy \geq 0 \quad \forall v \in K_q \tag{1.31}$$

where

$$K_q = \{v \in V_q: v \geq 0, \quad \text{a.e. in} \quad D\}. \tag{1.32}$$

Proof. Since the conditions (1.26) and (1.27) are satisfied

$$g_q \geq 0$$

is assured. Then $v = w^+$, $w^+ = \sup[0, w\}$, satisfies the boundary condition on Γ/Γ_N, i.e. $w^+ \in V_q$. Substituting this into (1.29) implies

$$\int_D \nabla w \cdot \nabla(w^+ - w)\, dx\, dy \geq 0,$$

i.e.

$$\int_D \nabla w^- \cdot \nabla w^- dx\, dy \leq 0.$$

Since $w^- = 0$ on Γ/Γ_N, we can conclude that $w^- = 0$, a.e. in D, i.e. $w \geq 0$, a.e. in D. This means that $w \in K_q$. The inequality (1.31) then follows from $w \in K_q$. □

The above theorem shows that the variational formulation (1.29) is equivalent to (1.31). Using this fact, we will show the existence of a unique solution of (1.29) for a fixed value q which satisfies the conditions (1.26) and (1.27).

Theorem 5-1.4. Suppose that the value q satisfies the conditions (1.26) and (1.27). Then the set K_q is non-empty. Further, this implies the existence of a unique solution of the variational inequality (1.31) and, therefore, of the variational inequality (1.29).

Proof. As shown above, the condition (1.26) is set so that $g_q \geq 0$ on the boundary Γ/Γ_N. Moreover, $g_q \in C^\infty(\Gamma/\Gamma_N)$. Then the extension of g_q by zero to the interior of the domain D certainly belongs to K_q. Convexity and closedness of K_q are clear since the trace from $H^1(D)$ onto $H^{1/2}(I)$ is continuous.

The bilinear form

$$a(u, v) = \int_D \nabla u \cdot \nabla v\, dx\, dy$$

is continuous on $H^1(D) \times H^1(D)$; indeed

$$a(u, v) \leq \|u\|_1 \|v\|_1$$

where $\|\cdot\|_1$ is the Sobolev norm of $H^1(D)$. Using Friedrich's inequality, it can be shown that

$$a(u - v, u - v) \geq C\|u - v\|_1^2$$

for every $u, v \in V_1$. This shows the strong ellipticity of the bilinear form.

The linear form

$$L(v) = \int_D v\, dx\, dy$$

is continuous on $H^1(D)$. Thus a unique solution of the variational inequality (1.31) follows the general existence theorems on variational inequalities discussed in Chap. 1 (See Theorem 1-3.1). □

Up to this point, we have developed a formulation of problem (1.1) in terms of the variable w. We next discuss some of the properties of the solution of the variational inequality (1.29), or, equivalently, of the variational inequality (1.31).

Theorem 5-1.5. Suppose that w is a solution of the variational inequality (1.29). Then

$$-\Delta w \in L^\infty(D) \tag{1.33}$$

$$0 \leq \Delta w \leq 1, \quad \text{a.e. in} \quad D \tag{1.34}$$

$$\frac{\partial w}{\partial x} = 0 \quad H^{-1/2}(\Gamma_N) \tag{1.35}$$

In particular, $w \in C(\tilde{D})$, and on setting

$$\Omega_q = \{(x, y) \in D: w(x, y) > 0\} \tag{1.36}$$

it follows that

$$-\Delta w + 1 = 0 \tag{1.37}$$

in the sense of distributions in Ω_q.

Proof. Setting $v = w + \varphi$, $\varphi \in C_0^\infty(D)$ and $\varphi \geq 0$, yields

$$\int_D \nabla w \cdot \nabla \varphi \, dx \, dy + \int_D \varphi \, dx \, dy \geq 0.$$

This means that

$$-\Delta w + 1 \geq 0$$

in the sense of distributions on D.

Putting $v = w - \varphi$, $\varphi \in C_0^\infty(D)$ with $\varphi \geq 0$, implies

$$-\int_D \nabla w \cdot \nabla \varphi \, dx \, dy + \int_D \{(w - \varphi)^+ - w^+\} \, dx \, dy \geq 0$$

since $(w + (-\varphi))^+ \leq w^+ + (-\varphi)^+ = w^+$

$$-\int_D \nabla w \cdot \nabla \varphi \, dx \, dy \geq 0.$$

That is

$$\Delta w \geq 0$$

in the sense of distributions on D. Thus

$$0 \leq \Delta w \leq 1.$$

This also implies that $\Delta w \in L^\infty(D)$.

The natural boundary condition (1.35) then follows in the sense of $H^{-1/2}(\Gamma)$ from the fact that $w \in H^1(D)$ and $\Delta w \in L^\infty(D) \subset L^2(D)$: Let

$$V_0 = \{v \in H^1(D): v = 0, \quad \text{a.e. on} \quad \Gamma/\Gamma_N\}.$$

Putting $v = w \pm u$, $u \in V_0$, in (1.29), and integrating by parts, we obtain

$$-\int_D (-\Delta w) u \, dx \, dy - \int_D u^+ \, dx \, dy \leq \left\langle \frac{\partial w}{\partial x}, u \right\rangle_{\Gamma_N}$$
$$\leq -\int_D (-\Delta w) u \, dx \, dy + \int_D u^+ \, dx \, dy$$

i.e.

$$\left\langle \frac{\partial w}{\partial x}, u \right\rangle_{\Gamma_N} \leq 2 \int_D |u| \, dx \, dy$$

where $\langle\cdot,\cdot\rangle_{\Gamma_N}$ denotes duality pairing on $H^{-1/2}(\Gamma_N)\times H^{1/2}(\Gamma_N)$. Since we can find functions u such that u has non-zero values on Γ_N but vanishes in D, we can conclude (1.35).

For the moment, we set $f=\Delta w\in L^\infty(D)$. Then the solution w of (1.29) is a solution of the mixed boundary value problem

$$w\in H^1(D):\Delta w=f,\quad w=g_q \quad \text{on} \quad \Gamma/\Gamma_N,\quad \frac{\partial w}{\partial x}=0 \quad \text{on} \quad \Gamma_N.$$

By standard regularity results for second-order partial differential equations (see, e.g. Lions and Magenes[48]), it can be shown that

$$w\in H^{(3/2)-\epsilon}(D),\quad \epsilon>0.$$

By the Sobolev imbedding theorem, we know $w\in C(\bar{D})$. Thus the definition

$$\Omega_q=\{(x,y)\in D: w(x,y)>0\}$$

is meaningful.

The differential eqn (1.37) now follows by taking $v=w\pm\epsilon\varphi$, $\varphi\in C_0^\infty(\Omega_q)$ and $\varphi=0$ in D/Ω_q, for sufficiently small $\epsilon>0$. □

In the above discussions, we have assumed that the discharge q is given. However, the quantity q is, in fact, unknown in this problem. Thus, the question remains as to how we can determine the true discharge $\bar{q}$ using the variational solution w for a given value q. To resolve this difficulty, suppose that $\bar{w}\in C^1(\bar{D})$, where $\bar{w}$ is the solution corresponding to the true discharge $\bar{q}$. Since the impervious sheet covers the portion FH, the condition

$$\frac{\partial\bar{w}}{\partial x}=0 \quad \text{at} \quad (0,y) \quad \text{for} \quad d<y<y_1$$

holds, as shown in (1.21). If $\bar{w}$ is the solution, this condition must be satisfied at the point H, i.e. $(0,d)$ where the Dirichlet boundary condition is also imposed. That is, the condition

$$\left.\frac{\partial\bar{w}}{\partial x}\right|_{y=d}=\lim_{C\to 0}\frac{1}{C}\{\bar{w}(C,d)-\bar{w}(0,d)\}=0 \tag{1.38}$$

must be satisfied. We easily see that this "compatibility" condition may not be satisfied for an arbitrary assumed discharge q. Thus, we must demand that $\bar{q}$ be such that $\bar{w}$ satisfies the compatibility condition (1.38) in order to obtain the variational solution of problem (1.2). In Baiocchi *et al.*[14], the following facts are proved:

$$\left.\begin{aligned}&\text{if}\quad f(q)=\left.\frac{\partial w(q)}{\partial x}\right|_{\substack{y=d\\x=0}},\quad\text{and if}\quad f(q')\quad\text{and}\quad f(q'')\quad\text{are finite,}\\&\text{then}\quad f(q_1)\le 0\le f(q_0)\quad\text{and}\\&\qquad f(q')<f(q'')\quad\text{for}\quad q_0\le q''<q'\end{aligned}\right\} \tag{1.39}$$

where $w(q)$ is the solution of (1.29) for a fixed q, and

$$\left.\begin{aligned}q_1&=\frac{1}{2a}(y_1^2-y_2^2)\\q_0&=\max\left\{0,\frac{1}{a}\left(y_1d-\frac{1}{2}(d^2+y_2^2)\right)\right\}.\end{aligned}\right\} \tag{1.40}$$

That is, if $\partial w/\partial x$ at $H = (0, d)$ is finite, the function $f(q) = \partial w(q)/\partial x$ is a strictly decreasing function of q. From this we have the existence of a proper q which satisfies the compatibility condition (1.38). This, at the same time, guarantees the existence of solutions to problem (5-1.1). Moreover, the inequalities (1.39) suggest again the discharge descent method for obtaining approximations of the discharge q which satisfy condition (1.38). Suppose that, for given numbers $q_{(1)}$ and $q_{(2)}$, $f(q_{(1)})$ and $f(q_{(2)})$ are known. In general, we may take $q_{(1)} = q_1$ and $q_{(2)} = q_1 - \epsilon$, $\epsilon > 0$. Then, the third iterate $q_{(3)}$ is defined so that

$$\begin{cases} f(q_{(3)}) - f(q_{(2)}) = \dfrac{f(q_{(1)} - q_{(2)})}{q_{(1)} - q_{(2)}}(q_{(3)} - q_{(2)}) \\ f(q_{(3)}) = 0 \end{cases}$$

i.e.

$$q_{(3)} = q_{(2)} - \frac{f(q_{(2)})}{f(q_{(1)} - q_{(2)})}(q_{(1)} - q_{(2)}). \tag{1.41}$$

If $f(q_{(3)})$ is still far from zero, we use (1.41) to construct the fourth approximation $q_{(4)}$ (by replacing $q_{(2)}$ with $q_{(3)}$, $f(q_{(2)})$ with $f(q_{(3)})$, etc.). This iterative process can yield good approximations to the variational problem (1.29). □

Example 5-1.1. Here we wish to calculate the free surface of seepage flow through a homogeneous, isotropic, rectangular dam with an impermeable sheet on a upper part of the upstream wall, as shown in Fig. 5.2.

Following the arguments of Chap. 2, a discrete model for solving the variational inequality (1.29) is shown in Fig. 5.2. The method of solution of (1.29) is again the projectional S.O.R. method with the iteration factor $\omega = 1.75$. We note that uniformity of the mesh makes it possible to develop finite difference schemes equivalent to the projectional S.O.R. method. Numerical results are indicated in Fig. 5.3.

Rapid convergence of the iterative scheme (1.41) for the discharge q is observed. The projectional S.O.R. method converges within 30 iterates for each iteration on q.

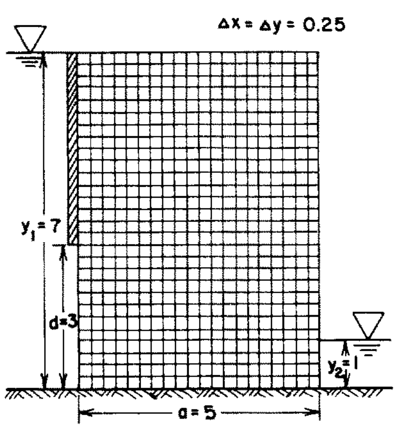
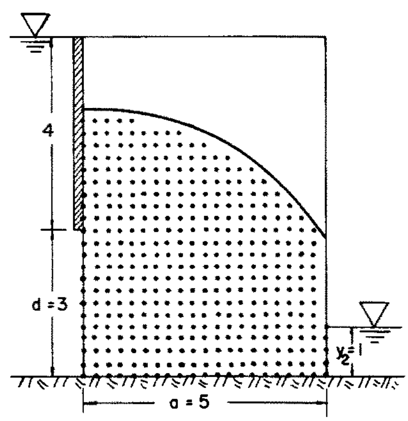


Fig. 5.2



ITERATIONS

	q	f
1	4.8	-0.72652
2	4.6	-0.54319
3	4.0074	-0.02072
4	3.9839	-0.00097

Fig. 5.3

Fig. 5.2. Geometry and discrete model of dam in Example 5-1.1.

Fig. 5.3. Numerical results obtained for Example 5-1.1 using projectional S.O.R. method for solving system of variational inequalities.

The same example problem is solved by the adaptive mesh method using the discrete model given in Fig. 5.4. Numerical results are shown in Fig. 5.5 and its convergence is described in Fig. 5.6. Eleven iterations are necessary to obtain convergence which is indicated by the absolute value of the pressure on the free streamline. Since the system of linear equations is solved twice at each step of iteration, 22 systems of equations are solved. However, the total number of degrees of freedom for this discrete model is very small, whereas many nodal points are necessary if we chose to solve the system of variational inequalities.

The total discharge of seepage flow calculated is 4.14 as opposed to 3.98 calculated using variational inequality (1.29). The position of the free boundary obtained by the adaptive mesh method is slightly higher than that obtained by solving the variational inequality (1.29). □

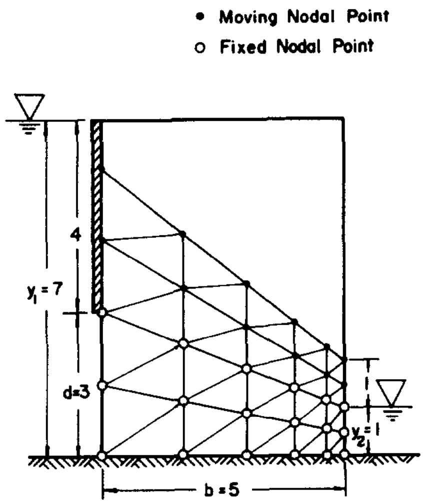


Fig. 5.4

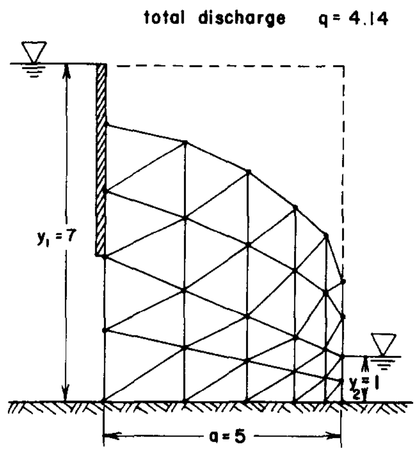


Fig. 5.5

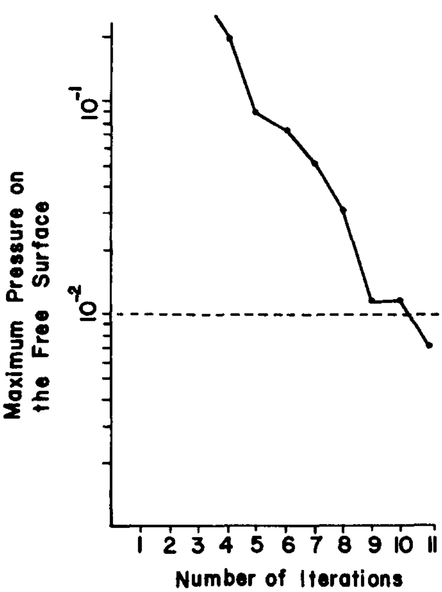


Fig. 5.6

Fig. 5.4. Initial mesh geometry for adaptive mesh method for Example 5-1.1.

Fig. 5.5. Computed free surface profile for Example 5-1.1 computed using adaptive mesh method.

Fig. 5.6. Convergence characteristics of adaptive mesh calculations of Example 5-1.1.

5.2 *Free surface from a symmetric channel*

We now consider the problem of 2-dimensional seepage flow from a symmetric channel into a permeable isotropic homogeneous foundation in which a horizontal drain is embedded as shown in Fig. 5.7. Let $\hat{\varphi}$ be the velocity potential and let $\hat{\psi}$ be the stream function defined on the unknown flow region $\Omega = ABEF$ which is assumed to be contained in the domain $D = ACDEF$. The domain D is chosen artificially in order to define the "fixed" region for the problem. This leads us to:

Problem 5-2.1. Find the triplet $(\hat{\varphi}, \hat{\psi}, \Omega)$ such that

$$\hat{\varphi}_x - \hat{\psi}_y = 0 \quad \text{and} \quad \hat{\varphi}_y + \hat{\psi}_x = 0 \quad \text{in} \quad \Omega \tag{2.1}$$

$$\hat{\varphi} = y_1 \quad \text{on} \quad EF, \quad \hat{\varphi} = 0 \quad \text{on} \quad AB \tag{2.2}$$

$$\hat{\psi} = \frac{q}{2} \quad \text{and} \quad \hat{\varphi}_x = 0 \quad \text{on} \quad AF \tag{2.3}$$

$$\hat{\psi} = 0, \quad \hat{\varphi} = y, \quad \hat{\varphi}_n = 0 \quad \text{on} \quad EB. \quad \square \tag{2.4}$$

Here $\varphi_x = \partial\varphi/\partial x$, $\varphi_y = \partial\varphi/\partial y$, $\varphi_n = n_x\varphi_x + n_y\varphi_y$, $\mathbf{n} = \{n_x, n_y\}$ is the unit vector outward normal to the boundary $\partial\Omega$ of the flow region Ω.

We next extend the above problem defined only on the unknown flow region Ω into the fixed domain D. Let

$$\varphi = \begin{cases} \hat{\varphi} & \text{in} \quad \Omega \\ y & \text{in} \quad D/\Omega \end{cases} \qquad \psi = \begin{cases} \hat{\psi} & \text{in} \quad \Omega \\ 0 & \text{in} \quad D/\Omega. \end{cases} \tag{2.5}$$

The new functions φ and ψ are called the *extended velocity potential* and *extended stream function*, respectively.

Theorem 5-2.1. Suppose that the pair $(\hat{\varphi}, \hat{\psi})$ satisfies (2.1) and (2.4). Then the equations

$$(y - \varphi)_x + \psi_y = 0 \quad \text{and} \quad (y - \varphi)_y - \psi_x = \chi_\Omega$$

are satisfied in the sense of distributions defined on the fixed domain D. Here χ_Ω is the characteristic function of Ω, i.e.

$$\chi_\Omega(x, y) = 1 \quad \text{if} \quad (x, y) \in \Omega, \quad \chi_\Omega(x, y) = 0 \quad \text{if} \quad (x, y) \notin \Omega. \tag{2.6}$$

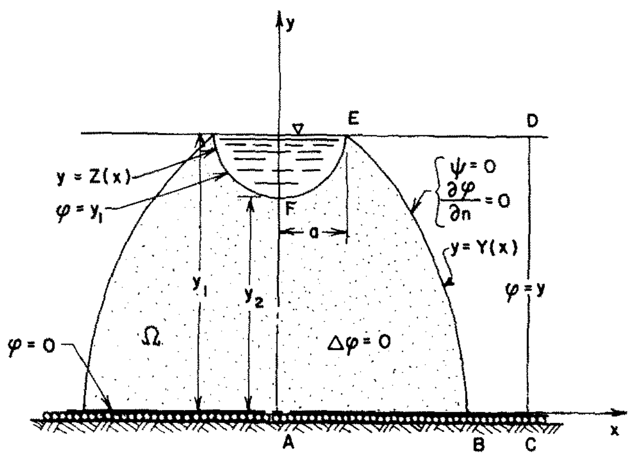


Fig. 5.7. Geometry of flow from a symmetric channel in a porous media.

Proof. Let $v \in C_0^\infty(D)$. Then, by $(2.1)_1$ and (2.4),

$$-\int_D \{(y-\varphi)v_x + \psi v_y\}\,dx\,dy = -\int_\Omega \{(y-\hat{\varphi})v_x + \hat{\psi} v_y\}\,dx\,dy$$
$$= \int_\Omega \{(y-\hat{\varphi})_x + \hat{\psi}_y\}v\,dx\,dy - \int_{EB} \{(y-\hat{\varphi})n_x + \hat{\psi} n_y\}v\,dx$$
$$= 0.$$

This means that

$$(y-\varphi)_x + \psi_y = 0$$

in the sense of distributions on D. Similarly

$$\int_D \{-(y-\varphi)v_y + \psi v_x\}\,dx\,dy = \int_\Omega \{-(y-\varphi)v_y + \psi v_x\}\,dx\,dy$$
$$= \int_\Omega \{(y-\varphi)_y - \psi_x\}v\,dx\,dy + \int_{EB} \{(y-\varphi)n_y + \psi n_x\}v\,ds$$
$$= \int_\Omega v\,dx\,dy = \int_D \chi_\Omega v\,dx\,dy.$$

This means that

$$(y-\varphi)_y - \psi_x = \chi_\Omega$$

in the sense of distributions on D. □

Thus, problem (5-2.1) can be rewritten in terms of the extended velocity potential and stream function in the fixed domain D as follows:

Problem 5-2.2. Find the triplet (φ, ψ, Ω) such that

$$(y-\varphi)_x + \psi_y = 0 \quad \text{and} \quad (y-\varphi)_y - \psi_x = \chi_\Omega \quad \text{in} \quad D \tag{2.7}$$

$$\varphi = y \quad \text{and} \quad \psi = 0 \quad \text{in} \quad D/\Omega \tag{2.8}$$

$$\varphi = y_1 \quad \text{on} \quad EF, \quad \varphi = 0 \quad \text{on} \quad AB \tag{2.9}$$

$$\psi = \frac{q}{2} \quad \text{and} \quad \varphi_x = 0 \quad \text{on} \quad AF. \quad \square \tag{2.10}$$

We note that the conditions on the free boundary EB are now imbedded in eqn (2.7). Let the scalar-valued function $w(x, y)$ be defined by

$$w(x, y) = \int_{EP} [-\psi\,dx + (y-\varphi)\,dy], \tag{2.11}$$

where the integral is considered as the line integral from the point E to the point $P = (x, y)$ of $\bar{D}$. Since the value $w(x, y)$ does not depend upon the path of integration, and since

$$y - \varphi < 0 \quad \text{in} \quad \Omega, \qquad y - \varphi = 0 \quad \text{in} \quad D/\Omega$$

it is easily verified that

$$\left.\begin{aligned} w(x, y) > 0 \quad &\text{if} \quad (x, y) \in \Omega \\ w(x, y) = 0 \quad &\text{if} \quad (x, y) \in D/\Omega. \end{aligned}\right\} \tag{2.12}$$

Moreover, we have

$$w_y = y - \varphi \quad \text{and} \quad w_x = -\psi. \tag{2.13}$$

Substituting (2.13) into (2.7) shows that the first equation in (2.13) is automatically satisfied by the new scalar-valued function w, and that the second equation of (2.13) now reduces to

$$\Delta w = \chi_\Omega \tag{2.14}$$

where, as usual, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The condition (2.8) is imbedded into the condition $(2.12)_2$ in terms of the variable $w(x, y)$. Boundary conditions (2.9) and (2.10) become

$$w_y = y - y_1 \quad \text{on} \quad EF, \quad w_y = 0, \quad \text{i.e.} \quad w_n = 0 \quad \text{on} \quad AC \tag{2.15}$$

$$w_x = -\frac{q}{2}, \quad \text{i.e.} \quad w_n = \frac{q}{2} \quad \text{on} \quad AF. \tag{2.16}$$

Thus, we can conclude the following:

Theorem 5-2.2. Suppose that the triplet (φ, ψ, Ω) is the solution of problem (5-2.2). Then the scalar-valued function w defined by (2.11) satisfies

$$\left.\begin{aligned} &-\Delta w + \chi_\Omega = 0 \quad \text{in} \quad D \\ &w > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad w = 0 \quad \text{in} \quad D/\Omega \\ &w_y = y - y_1 \quad \text{on} \quad EF, \quad w_n = 0 \quad \text{on} \quad AC, \quad w_n = \frac{q}{2}. \end{aligned}\right\} \quad \square \tag{2.17}$$

We next consider a variational formulation to the problem (2.17). In this case, a major difference between this problem and those in Chap. 3 is that a proper "Green's" formula like (3-2.1) cannot be obtained. The identity (3-2.1) has been used to express the *natural* boundary condition $w_y = y - y_1$ or $w_y = y - y_2$ in Chap. 3. The special identity (3-2.1) is, thus, introduced to equip the associated variational (or weak) formulation with the proper ingredients to cover the classical formulation.

In the present case, the bilinear form includes a term which consists of the first derivative of the function w on the part of the boundary Γ of the domain D. However, the first derivatives of this function in $H^1(D)$ (which is the proper space for this variational setting of the problem) cannot be defined on the boundary.

Let V_0 be defined by

$$V_0 = \{v \in H^1(D) : v = 0, \quad \text{a.e. on} \quad ED \quad \text{and} \quad DC\}. \tag{2.18}$$

Suppose that w satisfies (2.17). Then, for $\forall v \in V_0$

$$\int_D \nabla w \cdot \nabla(v - w)\, dx\, dy = \int_D (-\nabla w)(v - w)\, dx\, dy + \int_\Gamma w_n(v - w)\, ds$$

$$-\chi_\Omega(v - w) \geq -(v^+ - w^+) \quad a.e. \text{ in } D$$

$$\int_\Gamma w_n(v - w)\, ds = -\frac{q}{2}\int_{AF}(v - w)\, dy + \int_{FE}(w_x n_x + (y - y_1)n_y)(v - w)\, dy.$$

Then we have (by purely formal manipulations)

$$\int_D \nabla w \cdot \nabla(v - w)\, dx\, dy - \int_{FE} w_x n_x(v - w)\, ds + \int_D (v^+ - w^+)\, dx\, dy$$
$$\geq -\frac{q}{2}\int_{AF}(v - w)\, dy + \int_{FE}(y - y_1)n_y(v - w)\, ds. \tag{2.19}$$

If $n_x = 0$, i.e. if FE is horizontal, then the variational inequality (2.19) becomes

$$\int_D \nabla w \cdot \nabla(v-w)\,dx\,dy + \int_D (v^+ - w^+)\,dx\,dy \geq -\frac{q}{2}\int_{AF}(v-w)\,dy + (y-y_1)\int_{FE}(v-w)\,dx. \tag{2.20}$$

This problem represents the case shown in Fig. 5.8.

As mentioned above, the variational form (2.19) is improper in the space $H^1(D)$, since the second term of (2.19) includes the first derivative of w on the boundary FE. Thus, the inequation (2.19) is just a formal one. To make the development more precise, we introduce a selection map S defined by the following way: Let w_u be the solution of the variational inequality

$$\left.\begin{aligned} w_u \in V_0\text{:}\ \int_D \nabla w_u \cdot \nabla(v-w_u)\,dx\,dy + \int_D (v^+ - w_u^+)\,dx\,dy \geq -\frac{q}{2}\int_{AF}(v-w_u)\,dy \\ + \int_{FE}(u_x n_x + (y-y_1)n_y)(v-w_u)\,dy \end{aligned}\right\} \tag{2.21}$$

$$\text{for every}\quad v \in V_0$$

for a given data u, which is an element in $H^2(D)$. The map S: $H^2(D) \to H^2(D)$ is then defined by

$$S(u) = w_u. \tag{2.22}$$

Thus, the solution of (2.19) is the fixed point of the selection map S defined by (2.22).

It is not difficult to show that only a single solution $w_u \in H^2(D)$ can exist for each given $u \in H^2(D)$. However, the existence of a fixed point of the selection map is *still open.*

From the governing eqns (2.17), we can obtain a relationship which may indicate how proper data $u \in H^2(D)$ in the auxiliary problem (2.21) are selected. Let C be an arbitrary constant function defined on the domain D. Then

$$\int_D (-\Delta w + \chi_\Omega) C\,dx\,dy = \int_\Omega \nabla w \cdot \nabla C\,dx\,dy + C\,\text{mes}\,\Omega + \int_{\partial\Omega} w_n C\,ds$$

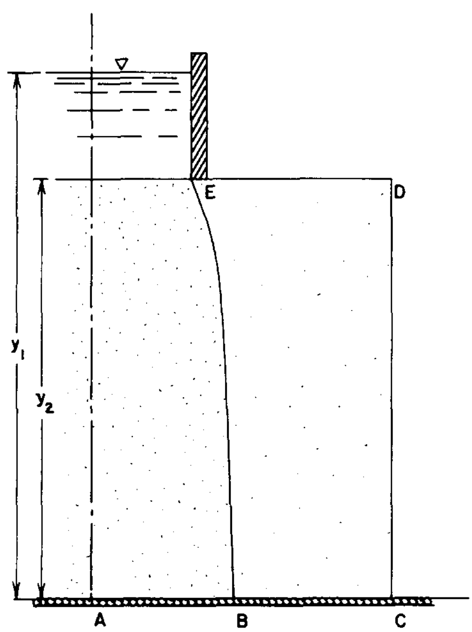


Fig. 5.8. Special geometry of flow from a rectangular channel.

where $\partial\Omega$ is the boundary of the flow region Ω. Applying the boundary conditions in (2.17)

$$\text{mes}\,\Omega - \frac{q}{2}\,\text{mes}\,(AF) + \int_{FE} w_x n_x \, ds + \int_{FE} (y - y_1) n_y \, ds = 0,$$

i.e.

$$\int_{FE} w_x n_x \, dx = \frac{q}{2}\,\text{mes}\,(AF) + \int_{FE} (y_1 - y) n_y \, ds - \text{mes}\,\Omega. \tag{2.23}$$

Therefore, the following procedure may be adopted to obtain the solution of (2.19): for an arbitrary element $u \in V_0$, e.g. $u = 0$, the variational inequality (2.21) is solved. Let this solution be denoted by w_u. Using w_u, we next obtain mes Ω which depends upon the data u. Then substitute this into (2.23), and obtain an approximation of $\int_{FE} w_x n_x \, ds$. Finally, this approximation is substituted into the fourth term of the ineqn (2.21). Then by solving (2.21), a second approximation of w_u is obtained. This process is repeated until convergence is (hopefully) obtained.

Remark 5-2.1. Despite the difficulties of the variational formulation (2.19) mentioned above, Bruch and Sloss[50, 51] have obtained numerical solutions which are in good agreement with analytical solutions available for special cases. In their work, only a formal variational framework is presented. Numerical solutions are obtained using a finite-difference discretization of the system (2.17). The question of conditions for the existence of solutions to problem (2.1), specifically the variational inequality (2.19), for the case in which the first derivative of the unknown on the boundary is included, appears to be open. □

Example 5-2.1. Although some open theoretical questions on the variational inequality (2.19) remain for the case $n_x \neq 0$ on the bottom of the channel, reasonable numerical solutions can be obtained without much difficulty. Following Bruch and Sloss[50], the unknown discharge q is obtained by the compatibility condition

$$\hat{f}(q) = \lim_{c \to 0} \frac{1}{c} (w(a, y_1) - w(a, y_1 - c)) = 0. \tag{2.24}$$

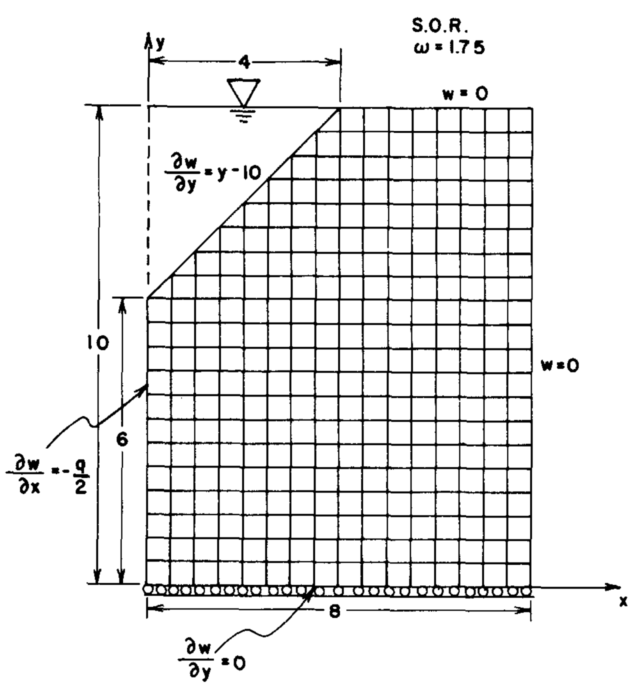
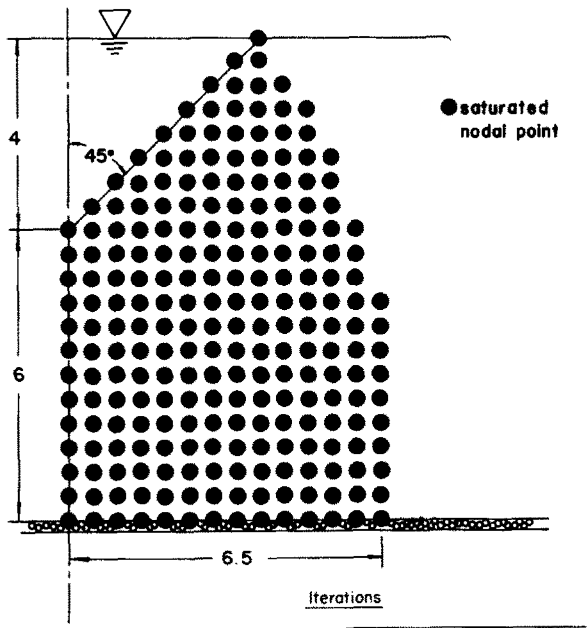


Fig. 5.9. Geometry and mesh for calculations in Example 5-2.1.

The condition (2.24) means that the pressure $p(x, y) = \partial w(x, y)/\partial y$ is zero at the point E in Fig. 5.8. The same iterative procedure as (1.41), which is called the discharge descent method, is adopted to obtain the proper discharge q.

Geometry and a discrete model of a model problem with a triangular channel are shown in Fig. 5.9. Numerical results are indicated in Fig. 5.10. We note that a rather large starting value of



Iterations

	q	f
1	166.67	-243.96
2	161.67	-235.74
3	182.11	-1.456
4	173.19	-0.487
5	168.71	-0.053
6	168.16	-0.000

Fig. 5.10. Flow domain in Example 5-2.1 calculated by solving system of variational inequalities by discharge descent method plus S.O.R.

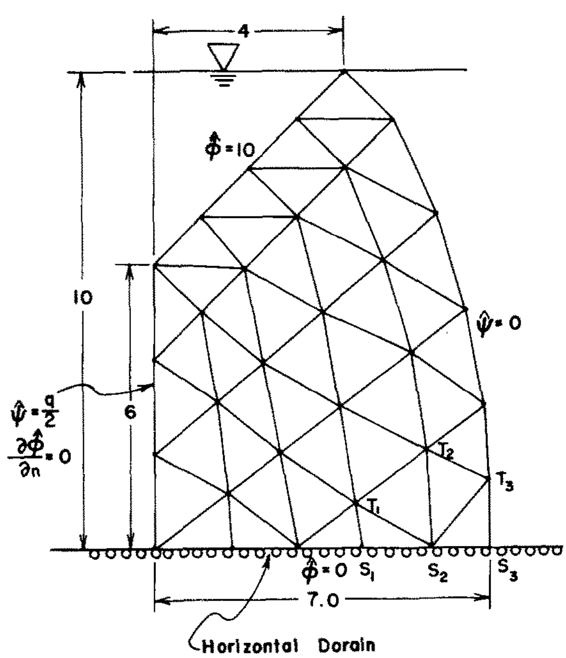


Fig. 5.11. Initial mesh for solving Example 5-2.1 by adaptive mesh method.

the assumed discharge is needed in this problem to bring the descent process within the radius of attraction of the solution to the discrete problem.

The adaptive mesh method is also applied to solve the same problem. In this case, the seepage point B is unknown *a priori*. Thus, a discrete model for initiating the adaptive mesh method is not easily obtained. Moreover, the method is very sensitive to the position of the seepage point B. The numerical results obtained by solving the variational inequality (2.19) were used to define the mesh described in Fig. 5.11. Numerical results and its convergence are shown in Figs. 5.12 and 5.13. At each iteration, points S_1, S_2 and S_3 (see Fig. 5.11) are defined by neighboring points T_1, T_2 and T_3 so that the x-coordinate of the point S_i is same as that of T_i, $i = 1$, 2 and 3. As seen in Fig. 5.13, convergence of this method is, at best, very slow. Some improvement in the maximum pressure is obtained up to values of order 10^{-1}, but the method appears to diverge if sharper tolerances are imposed. Nevertheless, the calculated free streamline is in reasonable agreement with that shown in Fig. 5.10. However, in this relatively simple example it is possible to start the process with a mesh which comes close to fitting the actual flow domain.

Numerical results such as these indicate that the adaptive mesh technique should be used with great care—if at all—for problems of this type. □

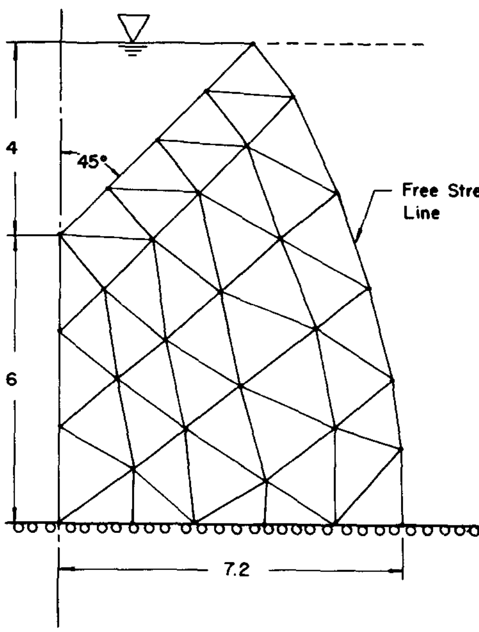


Fig. 5.12. Free streamline in Example 5-2.1 calculated after six iterations using adaptive mesh method.

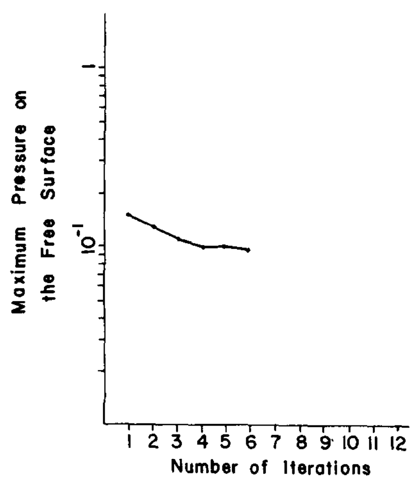


Fig. 5.13. Convergence characteristics of adaptive mesh method for Example 5-2.1.

5.3 A seepage flow problem with a horizontal drain

Let a rectangular dam $D=(0, a)\times(0, y_1)$ be homogeneous and isotropic, and let the flow through the dam D be diverted by a horizontal drain on part of the foundation, as shown in Fig. 5.14. Let $\hat{\varphi}$ and $\hat{\psi}$ denote the velocity potential and the stream functions, respectively, for the unknown flow region Ω. Then the problem can be expressed by

Problem 5-3.1. Find a triplet $(\hat{\varphi}, \hat{\psi}, \Omega)$ such that

$$\hat{\varphi}_y+\hat{\psi}_x=0, \quad \hat{\varphi}_x-\hat{\psi}_y=0 \quad \text{in} \quad \Omega \tag{3.1}$$

$$\hat{\varphi}=y_1 \quad \text{on} \quad AF, \quad \hat{\varphi}=0 \quad \text{on} \quad BC, \quad \hat{\psi}=q \quad \text{on} \quad AB \tag{3.2}$$

$$\hat{\psi}=0, \quad \hat{\varphi}=y, \quad \hat{\varphi}_n=0 \quad \text{on} \quad FC \tag{3.3}$$

where $\hat{\varphi}_n=n_x\hat{\varphi}_x+n_y\hat{\varphi}_y$, $\mathbf{n}=\{n_x, n_y\}$ is the outward unit vector normal to the boundary $\partial\Omega$ of Ω, $\hat{\varphi}_x=\partial\hat{\varphi}/\partial x$, etc. □

As in Section 5.2, $\hat{\varphi}$ and $\hat{\psi}$ onto the whole dam D in the same manner as in (2.5); that is

$$\varphi=\begin{cases}\hat{\varphi} & \text{in} \quad \Omega\\ y & \text{in} \quad D/\Omega\end{cases} \qquad \psi=\begin{cases}\hat{\psi} & \text{in} \quad \Omega\\ 0 & \text{in} \quad D/\Omega.\end{cases} \tag{3.4}$$

Then, paralleling Theorem 5-2.1., we have

Theorem 5-3.1. Let $\hat{\varphi}$ and $\hat{\psi}$ satisfy (3.1) and (3.3). Then the extended velocity potential φ and stream function ψ satisfy

$$(y-\varphi)_y-\psi_x=\chi_\Omega \quad \text{and} \quad (y-\varphi)_x+\psi_y=0 \quad \text{in} \quad D. \quad □ \tag{3.5}$$

Following the plan of previous sections, we next introduce a new scalar-valued function w defined by

$$w(x, y)=\int_{FP}[-\psi\,\mathrm{d}x+(y-\varphi)\,\mathrm{d}y] \tag{3.6}$$

where P is a point in $\bar{D}[P=(x, y)]$.

Since

$$w_x=-\psi \quad \text{and} \quad w_y=y-\varphi \quad \text{in} \quad D \tag{3.7}$$

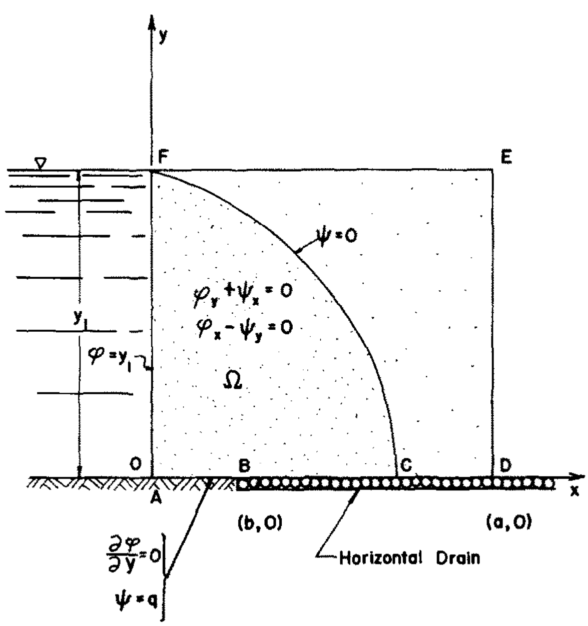


Fig. 5.14. Geometry of a rectangular dam with a horizontal drain.

the governing eqn (3.5) can be reduced to

$$\Delta w = \chi_\Omega. \tag{3.8}$$

Moreover, by the definition of φ and ψ,

$$w = 0 \quad \text{in} \quad D/\bar{\Omega} \tag{3.9}$$

and

$$w > 0 \quad \text{in} \quad \Omega \tag{3.10}$$

because $y - \varphi < 0$ in Ω. The boundary condition (3.2) is changed as follows

$$w_y = y - y_1 \quad \text{on} \quad AF. \tag{3.11}$$

Since AF coincides with the y-axis, (3.11) can be integrated to give

$$w(0, y) = \frac{1}{2} y^2 - y_1 y + C$$

where C is a constant. Since $w = 0$ at F,

$$w(0, y_1) = -\frac{1}{2} y_1^2 + C = 0, \quad \text{i.e.} \quad C = \frac{1}{2} y_1^2.$$

Then

$$w = \frac{1}{2}(y_1 - y)^2 \quad \text{on} \quad AF. \tag{3.12}$$

From $\hat{\psi} = q$ on AB, we have

$$w_x = -q \tag{3.13}$$

and since AB coincides with the x-axis, (3.13) can be integrated to give

$$w(x, 0) = -qx + C, \quad C = \text{constant}.$$

Finally, since $w(0, 0) = (1/2)y_1^2$ in accordance with (3.12),

$$w = -qx + \frac{1}{2} y_1^2 \quad \text{on} \quad AB \tag{3.14}$$

and since $\hat{\varphi} = 0$ on BC

$$w_y = 0 \quad \text{on} \quad BC. \tag{3.15}$$

Summing up, we have:

Theorem 5-3.2. Suppose that the triplet $(\hat{\varphi}, \hat{\psi}, \Omega)$ is a solution of Problem 5-3.1. Then the variable w defined by (3.6) must satisfy

$$\left.\begin{aligned} -\Delta w + \chi_\Omega = 0, \quad w \geq 0 \quad \text{in} \quad D \\ w > 0 \quad \text{in} \quad \Omega \quad \text{and} \quad w = 0 \quad \text{in} \quad D/\Omega \\ w = g_q \quad \text{on} \quad \Gamma/\Gamma_N \\ w_y = 0 \quad \text{on} \quad \Gamma_N \end{aligned}\right\} \tag{3.16}$$

where $\Gamma_N = BD$, Γ is the boundary of the dam D, and g_q is given by

$$g_q = \begin{cases} \frac{1}{2}(y - y_1)^2 & \text{on} \quad AF \\ -qx + \frac{1}{2}y_1^2 & \text{on} \quad AB \\ 0 & \text{on} \quad EF \cup DE. \end{cases} \tag{3.17}$$

We now construct a variational formulation of (3.16). Let us define an admissible set V_q by

$$V_q = \{v \in H^1(D): w = g_q \quad \text{on} \quad \Gamma/\Gamma_N\}. \tag{3.18}$$

Let w satisfy (3.16). Then, for each $v \in V_q$,

$$\begin{aligned} \int_D \nabla w \cdot \nabla(v - w)\, dx\, dy &= \int_D (-\Delta w)(v - w)\, dx\, dy + \int_{\Gamma_N} w_n(v - w)\, ds \\ &= -\int_D \chi_\Omega(v - w)\, dx\, dy - \int_{\Gamma_N} w_y(v - w)\, dx \\ &\geq -\int_D (v^+ - w^+)\, dx\, dy \end{aligned}$$

since, *a.e.* in D,

$$-\chi_\Omega(v - w) \geq -v^+ + w^+.$$

Therefore, we obtain the variational inequality

$$\int_D \nabla w \cdot \nabla(v - w)\, dx\, dy + \int_D (v^+ - w^+)\, dx\, dy \geq 0, \quad \forall v \in V_q. \tag{3.19}$$

It is clear that the discharge q cannot be arbitrary since $w \geq 0$ has to be imposed on $\bar{D}$. From (3.17), the condition

$$-qx + \frac{1}{2}y_1^2 \geq 0 \quad \text{for} \quad \forall x \in (0, b)$$

should be satisfied, i.e.

$$q \leq \frac{1}{2b}y_1^2. \tag{3.20}$$

It is also not so difficult to show that there exists a unique solution w to the variational problem (3.19) for a given fixed number $q > 0$ which satisfies the restriction (3.20), since (1) the bilinear form $(w, v) \rightarrow \int_D \nabla w \cdot \nabla v\, dx\, dy$ is continuous, (2) the form $v \rightarrow \int_D v^+ dx\, dy$ is a convex and continuous functional on $H^1(D)$, and (3) the form $(w, v) \rightarrow \int_D \nabla w \cdot \nabla v\ dx\, dy + \int_D v^+\ dx\, dy$ is coercive and strictly monotone on $V_q \times V_q$. We record this fact in the following theorem:

Theorem 5-3.2. There exists a unique solution to (3.19) for any choice of q, including the physically possible q's satisfying (3.20). □

The remaining question is how to decide the proper constant q so that the corresponding solution w of (3.19) is the solution to the problem (3.16), i.e. to Problem 5-3.1. As discussed in Section 5.1, we may take as a compatibility condition at the point B the condition

$$\lim_{C \to 0} \frac{1}{C}\{w(b + C, 0) - w(b, 0)\} = -q. \tag{3.21}$$

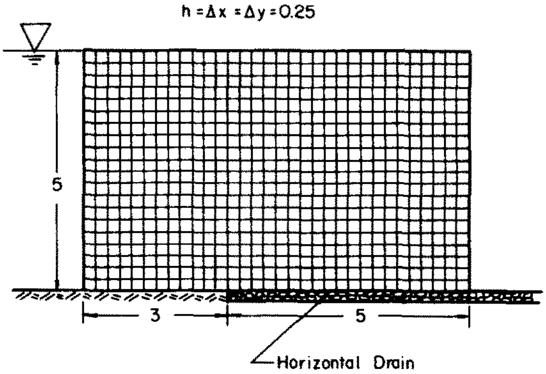


Fig. 5.15. Geometry and mesh for Example 5-3.1.

ITERATIONS

	q	f
1	3.125	-0.53421
2	2.925	- 1.0479
3	3.333	-0.00773
4	3.33	-0.00013

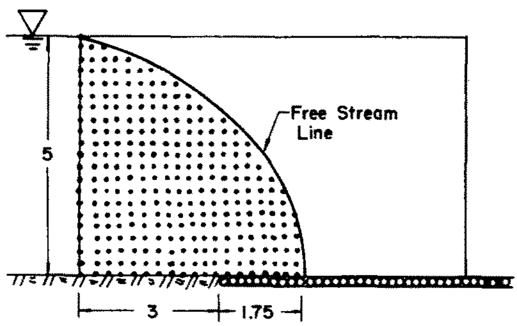


Fig. 5.16. Numerical results for Example 5-3.1 obtained by discharge descent iteration together with the projectional S.O.R. method.

This means that the derivative of w with respect to the x-direction is continuous at the point B, or equivalently, the stream function $\hat{\psi}$ is continuous at B.

Characterizations of the solution of variational inequality (3.18) can be established by following the same procedures given in Section 5.1.

Example 5-3.1. As a final example, we solve a problem of seepage flow through a dam with a horizontal drain which is attached on a part of the foundation. Suppose that the dam is homogeneous isotropic and rectangular, and that the foundation is horizontal and impervious, as shown in Fig. 5.15. Physical dimensions of the model and a discrete model for the variational inequality (3.19) and for the extended pressure method are also given in Fig. 5.15.

The compatibility condition (3.21) is used in order to obtain the proper discharge q. That is

$$f(q) = \lim_{C\to 0} \frac{1}{C}\{w(x+C,0) - w(b,0)\} + q = 0$$

has to be satisfied by the discharge q. Numerical results are shown in Fig. 5.16. Convergence for the discharge q is obtained within 4 iterations.

Again the projectional S.O.R. method is used to solve the variational inequality (3.19) for each given q.

The same problem is also solved by the extended pressure method using the same discrete model. The penalty parameter is assumed to be $\epsilon = 10^{-1}$ for a mesh parameter $h = \Delta x = \Delta y = 0.25$. Numerical results are shown in Fig. 5.17. The flow region Ω is here identified with the portion of D on which the pressure exceeds $\epsilon/2$.

As mentioned in Example 5-2.1, it is difficult to apply the adaptive mesh method when a horizontal drain is situated along the foundation. Using the numerical results obtained by solving the variational inequality (3.19), the initial flow domain shown in Fig. 5.18 is constructed. Then the adaptive mesh method is applied together with the strategy used in Example 5-2.1 for positioning points S_1 and S_2, see Fig. 5.19; i.e. the x-coordinate of S_1 and S_2 is assumed to be same as that of T_1 and T_2. Our numerical solution is shown in Fig. 5.19 and convergence characteristics are indicated in Fig. 5.20. While the rate of convergence indicated is a remarkable improvement over that experienced in Example 5-2.1, it was necessary to start the process with a mesh very closely resembling the final flow domain. Once again, this indicates the delicacy of the adaptive mesh process for problems of this type. □

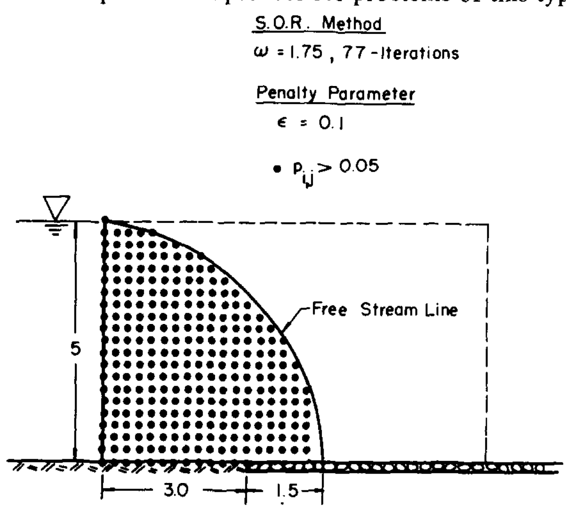


Fig. 5.17. Flow domain and free surface profile calculated using the extended pressure formulation with penalty and projectional S.O.R. iterations.

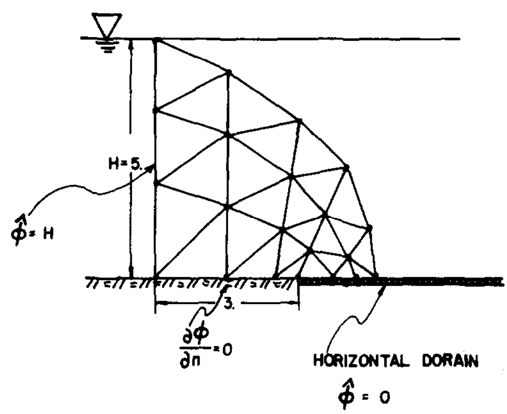


Fig. 5.18. Initial mesh for the adaptive mesh method for Example 5-3.1.

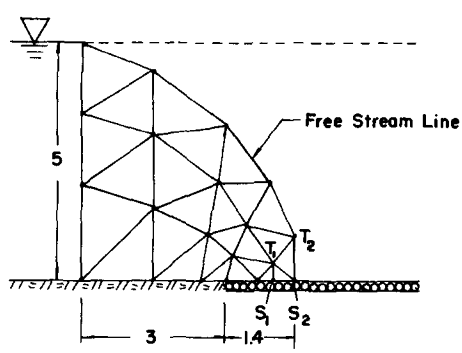


Fig. 5.19. Free streamline for Example 5-3.1 calculated using adaptive mesh method.

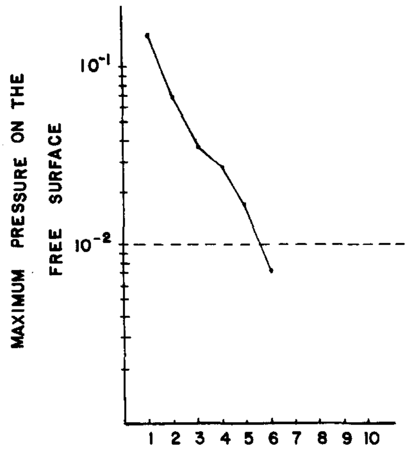


Fig. 5.20. Convergence characteristics of adaptive mesh method for Example 5-3.1.

5.4 *Comments*

The problem discussed in Section 5.1 was first discussed by Baiocchi[42]. Our mathematical treatment of the problem follows mainly the work of Baiocchi[42] and Baiocchi *et al.*[15],

The problem in Section 5.2 was first studied numerically by Bruch and Sloss[50]. As mentioned earlier, several important mathematical questions (e.g. the existence and uniqueness of solutions) for such problems appear to remain open.

The problem in Section 5.3 is a simplified version of a problem which has been discussed in Sloss and Bruch[51].

A survey of seepage flow problems by variational inequalities is also given in Bruch[52].

6. CONCLUDING REMARKS

While our principal concern in this study has been the application of variational inequalities and compatible numerical techniques to problems of flow through porous media, the foundation we have laid is quite broad. The general theories surveyed in Chaps. 1 and 2 can be used to formulate variational principles and computational methods for a wide range of free boundary problems in mechanics. These include problems in elastoplasticity, wherein the elastic–plastic interface is unknown, optimal control problems in the dynamics of distributed systems, Stefan problems in heat conduction such as those in which the interface between ice and water in a melting or freezing medium is unknown, contact problems in elasticity in which the contact surface is unknown, and many others. It is true that each of these areas of application requires special consideration of peculiarities of the physical problem at hand and the inequalities that model it. But many of the concepts and methods we have covered are fundamental to all of these applications.

Nevertheless, there are several topics that we have not dealt with here that pertain to variational inequalities for seepage problems. For example, we have not discussed evolution problems characterized by variational inequalities in which the solution also depends upon time. Time-dependent seepage flow problems fall into the category of Stefan problems mentioned above. Much work has been done on the numerical analysis of problems of this type. However, less is available on finite element methods for variational inequalities of evolution. The similarity between the classical Stefan problem of freezing and thawing of ice and seepage flow problems should be mentioned in this regard. The free streamline of seepage flow on which the pressure p is zero is analogous to the interface of the solid and melted phase of ice on which the temperature field θ is zero. There exists a discontinuity of the gradient of p and θ on this free boundary. If we replace the y-coordinate in our formulations of Chap. 3 with time t, our velocity potential φ with the temperature field and the (extended) pressure p by the heat potential u (which amounts to replacing Baiocchi's transformation by Duvaut's transformation), then the two problems are formally the same. Use of finite elements and variational inequalities for studying such time-dependent free boundary problems has been investigated by Ichikawa and Kikuchi[53] and Kikuchi and Ichikawa[54].

There are many open questions deserving further study in the analysis of seepage problems by the methods discussed here. In particular, the theory of quasi-variational inequalities does not appear to be developed to an extent that it provides a complete framework for studying seepage flow in arbitrary non-homogeneous dams. This, of course, means that a complete understanding of approximate methods based on such formulations must await further development of the mathematical theory itself. The fact that our introduction of the discharge conditions into the formulations discussed in Chap. 3 lead to acceptable numerical schemes suggests that similar conditions might be necessary in the quasi-variational inequality formulations in order that these problems be well-posed.

In particular, for homogeneous dams, the quasi-variational inequality formulation may not provide physically meaningful results without special consideration of certain conservation properties of the flow. However, the extended pressure method together with penalty arguments produces an approximation which is applicable to non-homogeneous cases and which leads to efficient schemes.

More work is also needed on penalty methods for free boundary problems of the type considered here. Our numerical results indicate a dependence of the penalty parameter ϵ on the mesh size h, but precisely how ϵ depends upon h is unknown. The effects of "reduced integration" in such penalty methods is still not well understood and the absence of *a priori* error estimates for complicated constrained problems of the type considered here stands in the way of a complete understanding of the qualitative behavior of finite-element approximations of these problems.

We also note that all of the numerical techniques we have described herein are merely examples of methods selected from a long list of optimization techniques that could be applied to variational inequalities. It is likely that many more efficient techniques are available. We leave the exploration of these to the interested reader. One interesting observation has resulted from our sample comparisons, however: the adaptive mesh methods popular in engineering literature should be used with caution. Our results indicate that they are often divergent (even though the computed results may look reasonable). Moreover, when they work, one must select a "starting mesh" very close to that approximating the actual flow domain. Care must also be taken in the case of inhomogeneous dams which have interfaces of materials with large differences in permeability. There it often appears to be necessary to use a "boundary layer" of elements to model the interface in order to avoid oscillations in the approximation of the free surface (see the Appendix). A complete numerical analysis of such difficulties is not yet available.

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Notes added in proof—A number of additional important works on variational inequalities and seepage flow problems have recently come to the author's attention. We mention, in particular, the work of Comincioli, Torelli, Friedman, Caffrey and Bruch. A supplemental list of references to this work is given below.

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APPENDIX

THE ADAPTIVE MESH METHOD

A popular method for solving free boundary problems in the study of flow through porous media consists of developing a trial finite-element model approximating only the flow domain Ω and then changing the geometry of the mesh through an iterative process designed to converge to the correct flow domain. Such adaptive mesh methods were developed by Taylor[1], Finn[2], Neuman *et al.*[3].

The underlying philosophy in the adaptive mesh method that we will employ is that the $(n+1)$-th approximation Ω^{n+1} of the flow domain Ω is defined by the nth iterate φ^n of an approximation of the velocity potential. This can be accomplished by a fixed-point algorithm of the type

$$Y^{n+1}(x) = Y^n(x) + \alpha p^n(x, Y^n(x)) \tag{A1}$$

or

$$X^{n+1}(y) = X^n(y) - \beta d^n(X^n(y), y). \tag{A2}$$

Here $y = Y(x)$ or $x = X(x)$ represent the position of the free boundary (free streamline), $p(x, y)$ is the pressure field defined by $p = \varphi - y$, $d(x, y)$ is the discharge of the point (x, y) which is obtained by multiplication of the stiffness matrix and the velocity potential in the finite element approximation, and (α, β) are proper iterative factors. We now show the procedure to obtain the free streamline $y = Y(x)$ by the scheme (A1).

(i) We begin by assuming a trial mesh Ω^1 involving an approximation of the flow region, $y = Y^1$, and an initial discharge D^1 from the seepage point D.

(ii) Suppose that $y = Y^n$, i.e. Ω^n and D^n are known.

Step 1. (1) Solve the boundary-value problem

$$-\Delta\varphi = 0 \quad \text{in} \quad \Omega^n$$
$$\varphi = H \quad \text{on} \quad AF, \quad \varphi = h \quad \text{on} \quad BC, \quad \underline{\varphi = y \quad \text{on} \quad CG}$$
$$\underline{\varphi_n = D^n \quad \text{at} \quad D} \quad \varphi_n = 0 \text{ on } FD \cup AB$$

(2) Obtain the pressure on FDC, and calculate the $(n+1)$-th position of the free streamline FD by the eqn (A1).

Step 2. (1) Solve the boundary-value problem

$$-\Delta\varphi = 0 \quad \text{in} \quad \Omega^{n+1}$$
$$\varphi = H \quad \text{on} \quad AF, \quad \varphi = h \quad \text{on} \quad BC, \quad \underline{\varphi = y \quad \text{on} \quad CD}$$
$$\varphi_n = 0 \quad \text{on} \quad FD \cup AB$$

(2) Obtain the discharge D^{n+1} at the point D.

Here domains and boundary conditions are described in Fig. A1.

When the dam is non-homogeneous, some special modifications are necessary in order to determine the free streamline along the interface of different material zones. For example, oscillating results such as those shown in Fig. A2 may be obtained if the ratio k_1/k_2 of permeabilities is very small. Such difficulties can be overcome by taking part of the interface of

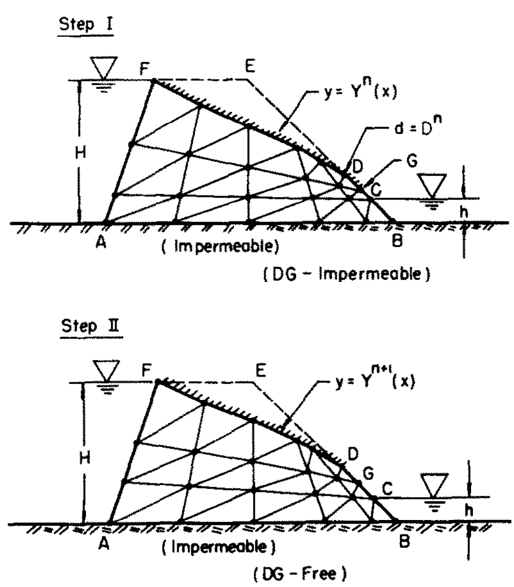

Fig. A1. Domains used in adaptive mesh method.

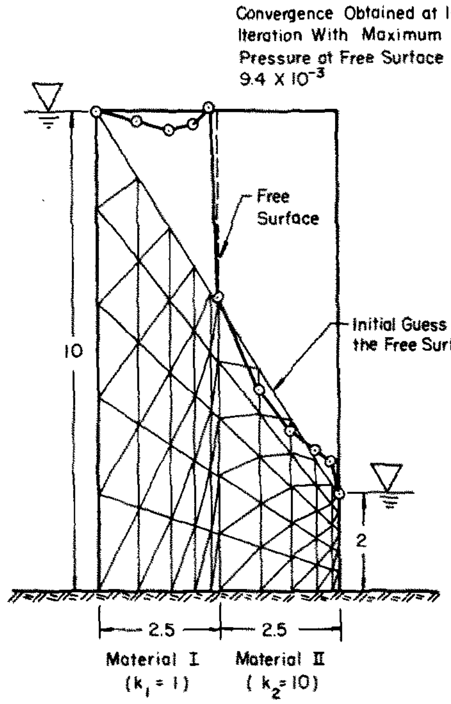

Fig. A2. Oscillation of free surface.

two different materials to be a part of the free streamline. Non-oscillatory results of the type shown in Fig. A3 are obtained when this assumption is introduced. The idea is similar to "upwinding" methods employed in the calculation of convection–diffusion problems.

Algorithm (A1) is not, in general, adequate for geometries of the type shown in Fig. A4, since the free streamline is very steep around the horizontal drain. To overcome this difficulty, we apply (A1) to a portion of the assumed free streamline and use the algorithm

$$X^{n+1}(y) = X^n(y) + \gamma p^n(X^n(y), y) \tag{A3}$$

on the remaining part of the free streamline. Here γ is a proper positive constant. A numerical example is shown in Fig. A5. □

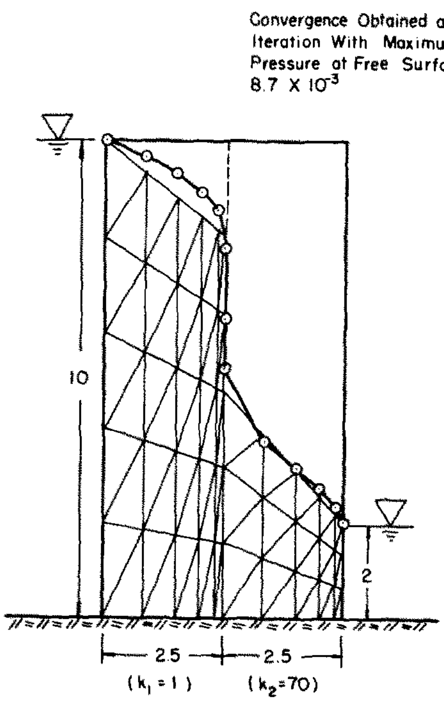

Fig. A3. Smooth free surface obtained by modelling interface with boundary-layer of elements.

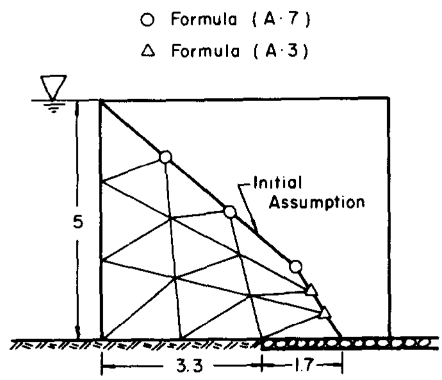

Fig. A.4

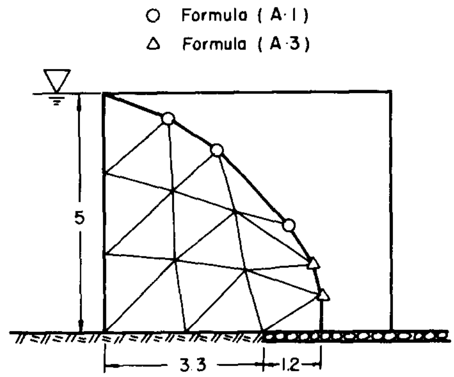

Fig. A.5

Fig. A4. Discharge model.

Fig. A5. Final profile of free surface calculated by adaptive mesh method.

References for Appendix

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