DIFFUSION AND MIXING IN FLUID FLOW: A REVIEW

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Abstract. This note is a review of a series of results on the interaction between diffusion and fluid flow that have been presented by the author at the International Congress in Mathematical Physics in Rio, 2006. The main object of study is the enhancement of diffusive mixing by a fast incompressible flow. Due to its physical relevance, the subject has been studied in detail from different angles. Here, we describe some of the recent work which combines PDE, functional analysis and dynamical systems theory by trying to establish links between diffusion enhancement and mixing properties inherent to the dynamical system generated by the flow. The proofs are based on a general criterion for the decay of the semigroup generated by an operator of the form $\Gamma + iAL$ with a negative unbounded self-adjoint operator Γ, a self-adjoint operator L, and parameter $A \gg 1$. In particular, they employ the RAGE theorem describing evolution of a quantum state belonging to the continuous spectral subspace of the hamiltonian (related to a classical theorem of Wiener on Fourier transforms of measures).

1. Introduction

Let M be a smooth compact d-dimensional Riemannian manifold. The main subject of this paper is the effect of a strong incompressible flow on diffusion on M. Namely, we consider solutions of the passive scalar equation

$$\phi_t^A(x,t) + Au \cdot \nabla \phi^A(x,t) - \Delta \phi^A(x,t) = 0, \quad \phi^A(x,0) = \phi_0(x). \tag{1.1}$$

Here Δ is the Laplace-Beltrami operator on M, u is a divergence free vector field, ∇ is the covariant derivative, and $A \in \mathbb{R}$ is a parameter regulating the strength of the flow. We are interested in the behavior of solutions of (1.1) for $A \gg 1$ at a fixed time $\tau > 0$.

It is well known that as time tends to infinity, the solution $\phi^A(x,t)$ will tend to its average,

$$\overline{\phi} \equiv \frac{1}{|M|} \int\limits_M \phi^A(x,t)\, d\mu = \frac{1}{|M|} \int\limits_M \phi_0(x)\, d\mu,$$

with $|M|$ being the volume of M. We would like to understand how the speed of convergence to the average depends on the properties of the flow and determine which flows are efficient in enhancing the relaxation process.

The question of the influence of advection on diffusion is very natural and physically relevant, and the subject has a long history. The passive scalar model is one of the most studied PDEs in both mathematical and physical literature. One important direction of research focused on homogenization, where in a long time–large propagation distance limit

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the solution of a passive advection-diffusion equation converges to a solution of an effective diffusion equation. Then one is interested in the dependence of the diffusion coefficient on the strength of the fluid flow. We refer to [27] for more details and references. The main difference with the present work is that here we are interested in the flow effect in a finite time without the long time limit.

On the other hand, the Freidlin-Wentzell theory [15, 16, 17, 18] studies (1.1) in $\mathbb{R}^2$ and, for a class of Hamiltonian flows, proves the convergence of solutions as $A \to \infty$ to solutions of an effective diffusion equation on the Reeb graph of the hamiltonian. The graph, essentially, is obtained by identifying all points on any streamline. The conditions on the flows for which the procedure can be carried out are given in terms of certain non-degeneracy and growth assumptions on the stream function. The Freidlin-Wentzell method does not apply, in particular, to ergodic flows or in odd dimensions.

Perhaps the closest to our setting is the work of Kifer and more recently a result of Berestycki, Hamel and Nadirashvili. Kifer's work (see [20, 21, 22, 23] where further references can be found) employs probabilistic methods and is focused, in particular, on the estimates of the principal eigenvalue (and, in some special situations, other eigenvalues) of the operator $-\epsilon\Delta + u \cdot \nabla$ when ϵ is small, mainly in the case of the Dirichlet boundary conditions. In particular, the asymptotic behavior of the principal eigenvalue λ_0^ϵ and the corresponding positive eigenfunction ϕ_0^ϵ for small ϵ has been described in the case where the operator $u \cdot \nabla$ has a discrete spectrum and sufficiently smooth eigenfunctions. It is well known that the principal eigenvalue determines the asymptotic rate of decay of the solutions of the initial value problem, namely

$$\lim_{t\to\infty} t^{-1} \log \|\phi^\epsilon(x,t)\|_{L^2} = -\lambda_0^\epsilon \tag{1.2}$$

(see e.g. [21]). In a related recent work [2], Berestycki, Hamel and Nadirashvili utilize PDE methods to prove a sharp result on the behavior of the principal eigenvalue λ_A of the operator $-\Delta + Au \cdot \nabla$ defined on a bounded domain $\Omega \subset \mathbb{R}^d$ with the Dirichlet boundary conditions. The main conclusion is that λ_A stays bounded as $A \to \infty$ if and only if u has a first integral w in $H_0^1(\Omega)$ (that is, $u \cdot \nabla w = 0$). An elegant variational principle determining the limit of λ_A as $A \to \infty$ is also proved. In addition, [2] provides a direct link between the behavior of the principal eigenvalue and the dynamics which is more robust than (1.2): it is shown that $\|\phi^A(\cdot,1)\|_{L^2(\Omega)}$ can be made arbitrarily small for any initial datum by increasing A if and only if $\lambda_A \to \infty$ as $A \to \infty$ (and, therefore, if and only if the flow u does not have a first integral in $H_0^1(\Omega)$). We should mention that there are many earlier works providing variational characterization of the principal eigenvalues, and refer to [2, 23] for more references.

Many of the studies mentioned above also apply in the case of a compact manifold without boundary or Neumann boundary conditions, which is the primary focus of this paper. However, in this case the principal eigenvalue is simply zero and corresponds to the constant eigenfunction. Instead one is interested in the speed of convergence of the solution to its average, the relaxation speed. A recent work of Franke [14] provides estimates on the heat kernels corresponding to the incompressible drift and diffusion on manifolds, but these estimates lead to upper bounds on $\|\phi^A(1) - \overline{\phi}\|$ which essentially do not improve as $A \to \infty$. One way to study the convergence speed is to estimate the spectral gap – the difference between the principal eigenvalue and the real part of the next eigenvalue. To the best of our knowledge, there is very little known about such estimates in the context of (1.1); see [21] p. 251 for a discussion. Neither probabilistic methods nor PDE methods of [2] seem to apply in this situation, in particular because the eigenfunction corresponding to the eigenvalue(s) with the second smallest real part is no longer positive and the eigenvalue itself does not need to be real. Moreover, even if the spectral gap estimate were available, generally it only yields a limited asymptotic in time dynamical information of type (1.2), and how fast the long time limit is achieved may depend on A. Part of our motivation for studying the advection-enhanced diffusion comes from the applications to quenching in reaction-diffusion equations (see e.g. [3, 11, 25, 30, 32]). For these applications, one needs estimates on the A-dependent L^∞ norm decay at a fixed positive time, the type of information the bound like (1.2) does not provide. We are aware of only one case where enhanced relaxation estimates of this kind are available. It is the recent work of Fannjiang, Nonnemacher and Wolowski [9, 10], where such estimates are provided in the discrete setting (see also [21] for some related earlier references). In these papers a unitary evolution step (a certain measure preserving map on the torus) alternates with a dissipation step, which, for example, acts simply by multiplying the Fourier coefficients by damping factors. The absence of sufficiently regular eigenfunctions appears as a key for the lack of enhanced relaxation in this particular class of dynamical systems. In [9, 10], the authors also provide finer estimates of the dissipation time for particular classes of toral automorphisms (that is, they estimate how many steps are needed to reduce the L^2 norm of the solution by a factor of two if the diffusion strength is ϵ).

Our main goal in this paper is to provide a review of recent work that addresses a question of sharp characterization of incompressible flows that are relaxation enhancing, in a quite general setup. The following natural definition has been introduced in [4] as a measure of the flow efficiency in improving the solution relaxation.

Definition 1.1. *Let M be a smooth compact Riemannian manifold. The incompressible flow u on M is called relaxation enhancing if for every $\tau > 0$ and $\delta > 0$, there exist $A(\tau, \delta)$ such that for any $A > A(\tau, \delta)$ and any $\phi_0 \in L^2(M)$ with $\|\phi_0\|_{L^2(M)} = 1$ we have*

$$\|\phi^A(\cdot, \tau) - \overline{\phi}\|_{L^2(M)} < \delta, \tag{1.3}$$

where $\phi^A(x,t)$ is the solution of (1.1) and $\overline{\phi}$ the average of ϕ_0.

Remarks. 1. In [4] it was shown that the choice of the L^2 norm in the definition is not essential and can be replaced by any L^p-norm with $1 \le p \le \infty$.

2. It follows from the proofs of our main results that the relaxation enhancing class is not changed even when we allow the flow strength that ensures (1.3) to depend on ϕ_0, that is, if we require (1.3) to hold for all $\phi_0 \in L^2(M)$ with $\|\phi_0\|_{L^2(M)} = 1$ and all $A > A(\tau, \delta, \phi_0)$.

The main approach is to bypass the issue of the spectral gap, and work directly with dynamical estimates. The first result we describe has been proved in [4].

Theorem 1.2. *Let M be a smooth compact Riemannian manifold. A Lipschitz continuous incompressible flow $u \in \mathrm{Lip}(M)$ is relaxation enhancing if and only if the operator $u \cdot \nabla$ has no eigenfunctions in $H^1(M)$, other than the constant function.*

Any incompressible flow $u \in \mathrm{Lip}(M)$ generates a unitary evolution group U^t on $L^2(M)$, defined by $U^t f(x) = f(\Phi_{-t}(x))$. Here $\Phi_t(x)$ is a measure preserving transformation associated with the flow, defined by $\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x))$, $\Phi_0(x) = x$. Recall that a flow u is called weakly mixing if the corresponding operator U has only continuous spectrum. The weakly mixing flows are ergodic, but not necessarily mixing (see e.g. [5]). There exist fairly explicit examples of weakly mixing flows (see e.g. [1, 12, 13, 26, 31, 29]). A direct consequence of Theorem 1.2 is the following Corollary.

Corollary 1.3. *Any weakly mixing incompressible flow $u \in \mathrm{Lip}(M)$ is relaxation enhancing.*

Theorem 1.2 in its turn follows from quite general abstract criterion, which we are now going to describe. Let Γ be a self-adjoint, positive, unbounded operator with a discrete spectrum on a separable Hilbert space H. Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of Γ, and e_j the corresponding orthonormal eigenvectors forming a basis in H. The (homogenous) Sobolev space $H^m(\Gamma)$ associated with Γ is formed by all vectors $\psi = \sum_j c_j e_j$ such that

$$\|\psi\|^2_{H^m(\Gamma)} \equiv \sum_j \lambda_j^m |c_j|^2 < \infty.$$

Note that $H^2(\Gamma)$ is the domain $D(\Gamma)$ of Γ. Let L be a self-adjoint operator such that, for any $\psi \in H^1(\Gamma)$ and $t > 0$ we have

$$\|L\psi\|_H \leq C\|\psi\|_{H^1(\Gamma)} \text{ and } \|e^{iLt}\psi\|_{H^1(\Gamma)} \leq B(t)\|\psi\|_{H^1(\Gamma)} \tag{1.4}$$

with both the constant C and the function $B(t) < \infty$ independent of ψ and $B(t) \in L^2_{\mathrm{loc}}(0,\infty)$. Here e^{iLt} is the unitary evolution group generated by the self-adjoint operator L.

Consider a solution $\phi^A(t)$ of the Bochner differential equation

$$\frac{d}{dt}\phi^A(t) = iAL\phi^A(t) - \Gamma\phi^A(t), \quad \phi^A(0) = \phi_0. \tag{1.5}$$

Similarly to the Definition 1.1 above, we say

Definition 1.4. *We call evolution corresponding to* (1.5) *relaxation enhancing if for any $\tau, \delta > 0$ there exists $A(\tau,\delta)$ such that for any $A > A(\tau,\delta)$ and any $\phi_0 \in H$ with $\|\phi_0\|_H = 1$, the solution $\phi^A(t)$ of the equation* (1.5) *satisfies $\|\phi^A(\tau)\|_H < \delta$.*

Theorem 1.5. *Let Γ be a self-adjoint, positive, unbounded operator with a discrete spectrum and let a self-adjoint operator L satisfy conditions* (1.4)*. Then the following two statements are equivalent:*

- *The evolution is relaxation enhancing*
- *The operator L has no eigenvectors lying in $H^1(\Gamma)$.*

Remark. Here L corresponds to $iu \cdot \nabla$ (or, to be precise, a self-adjoint operator generating the unitary evolution group U^t which is equal to $iu\cdot\nabla$ on $H^1(M)$), and Γ to $-\Delta$ in Theorem 1.2, with $H \subset L^2(M)$ the subspace of mean zero functions.

Theorem 1.5 provides a sharp answer to the general question of when a combination of fast unitary evolution and dissipation produces a significantly stronger dissipative effect than dissipation alone. It can be useful in any model describing a physical situation which involves fast unitary dynamics with dissipation (or, equivalently, unitary dynamics with weak dissipation). The proof uses ideas from quantum dynamics, in particularly the RAGE theorem (see e.g., [6]) describing evolution of a quantum state belonging to the continuous spectral subspace of a self-adjoint operator.

A natural concern is if the existence of rough eigenvectors of L is consistent with the condition (1.4) which says that the dynamics corresponding to L preserves $H^1(\Gamma)$. In [4], this question was answered in the affirmative by providing examples where rough eigenfunctions exist yet (1.4) holds. One of the examples involved a discrete version of the celebrated Wigner-von Neumann construction of an imbedded eigenvalue of a Schrödinger operator [28]. Moreover, another example constructed in [4] involved a smooth flow on the two dimensional torus $\mathbb{T}^2$ with discrete spectrum and rough (not $H^1(\mathbb{T}^2)$) eigenfunctions – the idea of this example essentially goes back to Kolmogorov [26]. Thus, the issue of rough eigenfunctions is not moot and result of Theorem 1.5 is precise.

The third result we are going to describe is a natural extension of Theorem 1.5 and Theorem 1.2 to the case of time periodic flows [24]. Clearly, most flows in practice are time dependent, and it is important to understand this more general case. The time periodic situation is a natural first step. Without loss of generality, we will assume that the period in time is equal to one, and will state only the general result. Let $L(t)$ be a self-adjoint (for any t) operator, periodic with respect to t with period 1 and such that the following two conditions hold.

Condition 1. For any $\psi \in H^1(\Gamma)$ *we have*

$$\|L(t)\psi\| \leq C_0\|\psi\|_1 \tag{1.6}$$

with constant C_0 *independent of* t.

Denote by $U(t,s)$ unitary group associated with equation

$$\frac{d}{dt}\psi(t) = iL(t)\psi(t). \tag{1.7}$$

Thus, $U(t,s)\psi(s) = \psi(t)$. Due to periodicity of $L(t)$ we have $U(t+1,s+1) = U(t,s)$. For period operator $U(1,0)$ we use the notation V.

Condition 2. For any $\psi \in H^1(\Gamma)$ *we have*

$$\|U(s+t,s)\psi\|_1 \leq B(t,s)\|\psi\|_1 \tag{1.8}$$

with constant $B(t,s) < \infty$ *(periodic in* s*) such that for any* $X > 0$

$$\sup_{t\in[0,X]}\sup_{s\in[0,1]} B(t,s) \leq C_*(X) < \infty. \tag{1.9}$$

Consider the equation

$$\frac{d}{dt}\phi^A(t) = iL(t)\phi^A(t) - \Gamma\phi^A(t), \quad \phi^A(0) = \phi_0, \tag{1.10}$$

where Γ is as before. Then the main result is the following [24]:

Theorem 1.6. *Under conditions 1 and 2 evolution is relaxation enhancing (in the sense of Definition 1.4) if and only if the unitary operator V has no eigenfunctions in $H^1(\Gamma)$.*

Thus the relaxation enhancement for time periodic flows is equivalent to the investigation of the eigenfunctions of the time one map V. It is interesting to note that in the case of a fluid flow $u(x,t)$, the problem reduces to studying a different flow of special form in one extra space dimension. Namely, let us denote the unit circle by $\mathbb{T}$. In the Hilbert space $\mathcal{H} := L_2(\mathbb{T}, H)$ - functions on $\mathbb{T}$ which are L_2 with values in H consider unitary evolution $e^{iK\sigma}$ defined by

$$e^{iK\sigma} f(t) := U(t, t-\sigma) f(t-\sigma). \tag{1.11}$$

We denote by K the self-adjoint generator of the unitary group $e^{iK\sigma}$. Formally,

$$K := i\frac{d}{dt} + L(t). \tag{1.12}$$

Then one can prove [24]

Theorem 1.7. *Operator V has an eigenfunction in $H^1(\Gamma)$ if and only if operator K has an eigenfunction $f(x,t)$ in $\mathcal{H}^1 := L_2(\mathbb{T}, H^1(\Gamma)) \cap H^1(\mathbb{T}, H)$.*

Finally, the last result we are going to mention deals with relaxation enhancement in non-compact regions (specifically, $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{T}$). Given an incompressible Lipshitz flow v in a domain D, let us denote $P_t(v)$ the solution operator for the equation

$$\psi_t + v \cdot \nabla \psi = \Delta \psi, \;\; \psi(0) = \psi_0 \tag{1.13}$$

on D. That is, $P_t(v)\psi_0 = \psi(\cdot, t)$ when ψ solves (1.13). The following theorem has been proved in [33]:

Theorem 1.8. *Let u be a periodic, incompressible, Lipshitz flow on $D = \mathbb{R}^2$ or $D = \mathbb{R} \times \mathbb{T}$ with a cell of periodicity C, and let ϕ^A solve (1.1) in D. The the following are equivalent.*
(i) For some $1 \le p \le q \le \infty$ and each $\tau > 0$, $\phi_0 \in L^p(D)$,

$$\|\phi^A(\cdot, \tau)\|_{L^p(D)} \to 0 \;\; as \; A \to \infty. \tag{1.14}$$

(ii) For any $1 \le p \le q \le \infty$ such that $p < \infty$ and $q > 1$, and each $\tau > 0$, $\phi_0 \in L^p(D)$,

$$\|\phi^A(\cdot, \tau)\|_{L^p(D)} \to 0 \;\; as \; A \to \infty.$$

(iii) For any $1 \le p \le q \le \infty$ and each $\tau > 0$,

$$\|P_\tau(Au)\|_{L^p(D) \to L^q(D)} \to 0 \;\; as \; A \to \infty.$$

(iv) No bounded open subset of D is invariant under u and any eigenfunction of u on C that belongs to $H^1(C)$ is a first integral of u.

The first three statements in the above theorem provide essentially different equivalent definitions of relaxation enhancement (which may be more reasonable to call dissipation enhancement in this setting, since the limiting value is going to be equal to zero). The fourth is a sharp characterization of flows that provide such dissipation enhancement on D. An interesting aspect of this theorem is that in the non-compact setting the class of relaxation enhancing flows includes some flows with first integrals on the cell, such as shear flows in the infinite direction which have a plateau. The paper [33] contains some other interesting examples and generalizations.

In the following section we sketch some of the main ideas behind the proof of the Theorems 1.2, 1.5, 1.6 and 1.8.

2. The Heart of the Matter

While we are not going to present detailed proofs, we would like to outline, or perhaps even just illustrate, the main idea behind the general criterion Theorem 1.5 and its connection with estimates on wavepacket spreading in quantum mechanics. For this purpose, it is convenient to switch to an equivalent formulation with small parameter $\epsilon = A^{-1}$. Namely, we will look at the equation

$$\phi^\epsilon_t = iL\phi^\epsilon - \epsilon\Gamma\phi^\epsilon, \;\; \phi^\epsilon(0) = \phi_0. \tag{2.1}$$

The question then becomes under what conditions on L for any $\tau, \delta > 0$ we can find sufficiently small $\epsilon(\tau, \delta)$ such that for any $\epsilon < \epsilon(\tau, \delta)$ we have $\|\phi^\epsilon(\tau/\epsilon)\| < \delta$?

One direction of the Theorem 1.5 is rather straightforward. If there exists $\phi \in H^1$ such that $L\phi = \lambda\phi$, then it is not difficult to show that L cannot be relaxation enhancing. It suffices to take the initial data ϕ_0 equal to ϕ and carry out a few elementary estimates on the equation [4].

The converse direction is trickier. The evolution due to L is unitary, while unperturbed dissipative part due to $\epsilon\Gamma$ delivers decay by just a fixed factor on the time scale of the order ϵ^{-1} - if there is no mixing by L. Thus enhanced dissipation can only happen if the unitary evolution transports the initial data to progressively higher harmonics of Γ. Since at any time t we have

$$\partial_t\|\phi^\epsilon\|^2 = \epsilon\|\phi^\epsilon(s)\|_1^2, \tag{2.2}$$

we just need to obtain sufficiently strong lower bound (perhaps on average) for the H^1 norm of the solution. Let us recall the following statement, well known in mathematical quantum mechanics as the RAGE theorem (in honor of Ruelle, Ahmrein, Georgescu and Enss [6]).

Theorem 2.1. *Assume L is a self-adjoint operator, and denote P_c the projector on the continuous spectral subspace. Let C be a compact operator. Then for any $\phi \in H$, we have*

$$\frac{1}{T}\int_0^T \|C\exp(iLt)P_c\phi\|^2\,dt \to 0 \tag{2.3}$$

as $T \to \infty$.

This is a precise formulation of a well known informal principle saying that the quantum evolution corresponding to continuous spectrum is unbounded. Indeed, think of a discrete case, where $H = l^2(Z^d)$, and take K equal to a projection on a ball of radius R. Then the statement of the Theorem says that the wavepacket, on average, will stay out of this ball for large times. The proof is based on a well known and simple Wiener theorem saying that if μ

is a probability measure with point masses at a_i and $\hat{\mu}$ is its Fourier transform, then

$$\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\,dt \to \sum_i \mu(a_i)^2$$

as $T \to \infty$. Clearly, the unboundedness of the unitary dynamics should be very relevant in our case, where the natural basis is given by the eigenfunctions of Γ and the goal is to show that the evolution migrates to high modes and dissipates.

Let us now consider the case where our operator L has purely continuous spectrum. In this case, the proof is especially transparent. The key are the following two lemmas. The first lemma ensures that the solution of (2.1) stays close for a while to the unitary evolution $\phi^0(t) = \exp(iLt)\phi_0$.

Lemma 2.2. *Assume the conditions* (1.4) *hold. Let* $\phi^0(t), \phi^\epsilon(t)$ *be solutions of*

$$(\phi^0)'(t) = iL\phi^0(t), \;\; (\phi^\epsilon)'(t) = (iL - \epsilon\Gamma)\phi^\epsilon(t),$$

satisfying $\phi^0(0) = \phi^\epsilon(0) = \phi_0 \in H^1$. *Then we have*

$$\frac{d}{dt}\|\phi^\epsilon(t) - \phi^0(t)\|^2 \le \frac{1}{2}\epsilon\|\phi^0(t)\|_1^2 \le \frac{1}{2}\epsilon B^2(t)\|\phi_0\|_1^2. \tag{2.4}$$

As a consequence,

$$\|\phi^\epsilon(t) - \phi^0(t)\|^2 \le \frac{1}{2}\epsilon\|\phi_0\|_1^2 \int_0^\tau B^2(t)\,dt$$

for any time $t \le \tau$.

This lemma can be proved by elementary arguments [4].

The second lemma is an upgraded version of the RAGE theorem. Recall that we denote by $0 < \lambda_1 \le \lambda_2 \le \dots$ the eigenvalues of the operator Γ and by $e_1, e_2, \dots$ the corresponding orthonormal eigenvectors. Let us also denote by P_N the orthogonal projection on the subspace spanned by the first N eigenvectors $e_1, \dots, e_N$ and by $S = \{\phi \in H : \|\phi\| = 1\}$ the unit sphere in H. The following lemma shows that if the initial data lies in the continuous spectrum of L then the L-evolution will spend most of time in the higher modes of Γ.

Lemma 2.3. *Let* $K \subset S$ *be a compact set. For any* $N, \sigma > 0$, *there exists* $T_c(N, \sigma, K)$ *such that for all* $T \ge T_c(N, \sigma, K)$ *and any* $\phi \in K$, *we have*

$$\frac{1}{T}\int_0^T \|P_N e^{iLt} P_c \phi\|^2\,dt \le \sigma\|\phi\|^2. \tag{2.5}$$

An important aspect of this lemma is the uniformity of the estimate in $\phi \in K$.

Given these two lemmas, here is a sketch of the proof of Theorem 1.5.

Fix $\delta, \tau > 0$. Take $\sigma = 1/10$, and choose N so that

$$\exp(-\lambda_N \tau/10) < \delta. \tag{2.6}$$

We will also assume that λ_N is chosen to be large than one. Define a compact set

$$K = \left\{ \phi \in H \,\middle|\, \|\phi\|_1^2 \le \lambda_N \|\phi\|^2 \right\}.$$

Define $\tau_1 = T_c(N, 1/10, K)$. Finally, take any $\epsilon < \epsilon(\tau, \delta)$ where the latter is defined by a condition

$$\epsilon(\tau, \delta) \int_0^{\tau_1} B^2(t)\, dt \le \frac{1}{20\lambda_N}. \tag{2.7}$$

Assume that we have $\|\phi^\epsilon(t)\|_1^2 > \lambda_N \|\phi^\epsilon(t)\|^2$ for any t in some interval $[a, b] \subset [0, \tau/\epsilon]$. Then from (2.2), it follows that

$$\|\phi^\epsilon(b)\|^2 \le \exp(-\epsilon\lambda_N(b-a))\|\phi^\epsilon(a)\|^2. \tag{2.8}$$

In particular, if we could take $[a, b] = [0, \tau/\epsilon]$, then by (2.6) the norm of the solution will be less than δ at $t = \tau/\epsilon$.

Now let us examine what happens if at any time τ_0 we have $\|\phi^\epsilon(\tau_0)\|_1^2 \le \lambda_N \|\phi^\epsilon(\tau_0)\|^2$. For the sake of transparency, henceforth we will denote $\phi^\epsilon(\tau_0) = \phi_0$. On the interval $[\tau_0, \tau_0 + \tau_1]$, consider the function $\phi^0(t)$ satisfying $\frac{d}{dt}\phi^0(t) = iL\phi^0(t)$, $\phi^0(\tau_0) = \phi_0$. Note that by the choice of ϵ, τ_0, (2.7), and Lemma 2.4, we have

$$\|\phi^\epsilon(t) - \phi^0(t)\|^2 \le \frac{1}{10}\|\phi_0\|^2 \tag{2.9}$$

for all $t \in [\tau_0, \tau_0 + \tau_1]$. Our choice of τ_1 implies that

$$\frac{1}{\tau_1} \int_{\tau_0}^{\tau_0+\tau_1} \|P_N \phi^0(t)\|^2\, dt \le \frac{1}{10}\|\phi_0\|^2. \tag{2.10}$$

Taking into account that the evolution $\phi^0(t)$ is unitary, it follows that

$$\frac{1}{\tau_1} \int_{\tau_0}^{\tau_0+\tau_1} \|(I - P_N)\phi^0(t)\|^2\, dt \ge \frac{9}{10}\|\phi_0\|^2. \tag{2.11}$$

Using (2.9), we conclude that

$$\frac{1}{\tau_1} \int_{\tau_0}^{\tau_0+\tau_1} \|(I - P_N)\phi^\epsilon(t)\|^2\, dt \ge \frac{1}{2}\|\phi_0\|^2. \tag{2.12}$$

This estimate implies that

$$\int_{\tau_0}^{\tau_0+\tau_1} \|\phi^\epsilon(t)\|_1^2\, dt \ge \frac{\lambda_N \tau_1}{2}\|\phi_0\|^2. \tag{2.13}$$

Combining (2.13) with (2.2) yields

$$\|\phi^\epsilon(\tau_0 + \tau_1)\|^2 \le \left(1 - \frac{\lambda_N \epsilon \tau_1}{2}\right)\|\phi^\epsilon(\tau_0)\|^2 \le e^{-\lambda_N \epsilon \tau_1/2}\|\phi^\epsilon(\tau_0)\|^2. \tag{2.14}$$

The whole interval $[0, \tau/\epsilon]$ can now be split into a union of intervals such that either (2.8) or (2.14) applies. Thus we obtain (assuming $\|\phi_0\| = 1$)

$$\|\phi^\epsilon(\tau/\epsilon)\|^2 \leq \exp(-\lambda_N \tau/2) < \delta^2,$$

finishing the proof in this case.

Including the point spectrum case is technically tricky, and we refer to [4] for the complete treatment. However, the argument that we just provided illustrates some key ideas well. The overall plan of the argument is flexible enough to also apply in the time periodic case [24].

3. Open Questions

In this section, we briefly discuss some open questions. There are many natural directions in which one can pursue further developments. For example, discrete time version, more precise quantitative estimates of the enhancement and links to relevant properties of the dynamical systems and nonlinear dissipation are all of interest. However here we will focus on describing in detail just two questions - which are most likely hardest but are also in our opinion very interesting.

1. The spectral analog. The first question is to obtain estimates on the spectral gap in this truly non-selfajoint situation. The issue is really twofold. Consider the operator

$$H_A = iAL - \Gamma.$$

Let is denote λ_1^A the eigenvalue with the minimal real part (one of those eigenvalues if it is not unique). Obtaining any analytical estimate on the $\Re\lambda_1^A$ of the operator H_A would require completely new ideas. As we mentioned in the introduction, the current estimates available in the Dirichlet boundary condition setting for the flow operator depend critically on the fact that the principal eigenvalue is real and the corresponding eigenfunction is positive. Clearly, such properties have no analog in general for (3). The second question is if the link between dynamical behavior and spectral gap remains true in this case. A natural conjecture is that $\limsup_{A\to\infty} \Re\lambda_1^A < \infty$ if and only if L has $H^1(\Gamma)$ eigenfunctions and if and only if evolution is not relaxation enhancing. In fact, the "only if" direction can be proved similarly to [2]; it is the "if" direction that looks difficult. Finding what the above limit (if it exists and finite) is going to be equal to is another interesting problem.

Generally, it is well known that in non-selfadjoint spectral analysis, "anything can happen". The question is if the natural problem that we are looking at has sufficient structure to still possess some nice properties.

2. Examples of time-dependent flows. Theorem 1.6 provides a simple characterization of the time-periodic relaxation enhancing flows. Intuitively, it is clear that time dependence of the flow is likely to improve its mixing properties in many situations. A very reasonable question is therefore the following:
Find an example of a 2D incompressible flow $u(x, y, t)$, periodic both in space and time with period 1, such that for each fixed t_0, $u(x, y, t_0)$ is not relaxation enhancing (for instance, the mean of u is zero) but the time time dependent flow $u(x, y, t)$ is relaxation enhancing.

We were unable to find such examples in the existing literature. One approach suggested by Theorem 1.6, would be to start from a relaxation enhancing (for example, mixing) time one map, and try to build a flow leading to it. However, most explicit mixing maps on the torus that appear in the literature, such as simple Anosov diffeomorphisms, are not homotopic to the identity map - and so cannot be realized by a smooth flow on the torus. We believe that the above question is interesting purely from the dynamical systems point of view, independently of its applications to the advection-diffusion.

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