

QUENCHING OF COMBUSTION BY SHEAR FLOWS

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ABSTRACT. We consider a model describing premixed combustion in the presence of fluid flow: reaction diffusion equation with passive advection and ignition type nonlinearity. What kinds of velocity profiles are capable of quenching (suppressing) any given flame, provided the velocity's amplitude is adequately large? Even for shear flows, the solution turns out to be surprisingly subtle. In this paper, we provide a sharp characterization of quenching for shear flows: the flow can quench any initial data if and only if the velocity profile does not have an interval larger than a certain critical size where it is identically constant. The efficiency of quenching depends strongly on the geometry and scaling of the flow. We discuss the cases of slowly and quickly varying flows, proving rigorously scaling laws that have been observed earlier in numerical experiments. The results require new estimates on the behavior of the solutions to advection-enhanced diffusion equation (also known as passive scalar in physical literature), a classical model describing a wealth of phenomena in nature. The technique involves probabilistic and PDE estimates, in particular applications of Malliavin calculus and central limit theorem for martingales.

1. INTRODUCTION

A mathematical model that describes a chemical reaction in a fluid is a system of two equations for concentration n and temperature T of the form

$$\begin{aligned} T_t + u \cdot \nabla T &= \kappa \Delta T + Mg(T)n \\ n_t + u \cdot \nabla n &= \frac{\kappa}{\text{Le}} \Delta n - Mg(T)n. \end{aligned} \tag{1.1}$$

Here κ is the thermal diffusivity, Le the Lewis number (ratio of thermal and material diffusivities), and M the reaction strength. The equations (1.1) are coupled to the reactive Navier-Stokes equations for the advection velocity $u(x, y, t)$. Two assumptions are usually made to simplify the problem: the first is a constant density approximation [3] that allows to decouple the Navier-Stokes equations from the system (1.1) and to consider $u(x, y, t)$ as a prescribed quantity that does not depend on T and n . The second assumption is that $\text{Le} = 1$ (equal thermal and material diffusivities). These two assumptions reduce the above system to a single scalar equation for the temperature T . We assume in addition that the advecting flow is unidirectional. Then the system (1.1) becomes

$$\begin{aligned} T_t + Au(y)T_x &= \kappa \Delta T + Mf(T) \\ T(0, x, y) &= T_0(x, y) \end{aligned} \tag{1.2}$$

with $f(T) = g(T)(1 - T)$. We are interested in strong advection, and accordingly have written the velocity as a product of the amplitude A and the profile $u(y)$. In this paper we consider a nonlinearity $f \not\equiv 0$ of the ignition type

- (i) $f(0) = f(1) = 0$ and $f(T)$ is Lipschitz continuous on $[0, 1]$,
- (ii) $\exists \theta_0 > 0$ such that $f(T) = 0$ for $T \in [0, \theta_0]$, $f(T) \geq 0$ for $T \in (\theta_0, 1)$,
- (iii) $f(T) \leq T$.

The last condition in (1.3) is just a normalization. We consider the reaction-diffusion equation (1.2) in the strip $D = \{x \in \mathbb{R}, y \in [0, h]\}$. We take $u(y)$ to be periodic with period h and with mean equal to zero:

$$\int_0^h u(y) dy = 0. \quad (1.4)$$

A constant non-zero mean can be easily taken into account by translation. For the temperature, we impose periodic boundary conditions

$$T(t, x, y) = T(t, x, y + h) \quad (1.5)$$

in y and decay in x . We will always assume that initial data $T_0(x, y)$ is such that $0 \leq T_0(x, y) \leq 1$. Then we have $0 \leq T \leq 1$ for all $t > 0$ and $(x, y) \in D$. For simplicity, we will usually assume that the initial data coincide with characteristic function of some set. More generally, we may assume that for some L and $\eta > 0$ we have

$$\begin{aligned} T_0(x, y) &> \theta_0 + \eta \text{ for } |x| \leq L/2, \\ T_0(x, y) &= 0 \text{ for } |x| \geq L. \end{aligned} \quad (1.6)$$

Equation (1.2) may be considered as a simple model of flame propagation in a fluid [2], advected by a shear (unidirectional) flow. The physical literature on the subject is vast, and we refer to the recent review [18] for an extensive bibliography. The main physical effect of advection on front-like solutions is the speed-up of the flame propagation due to the large scale distortion of the front. The role of the advection term in (1.2) for the front-like initial data was also a subject of intensive mathematical scrutiny recently, see [1, 18] for the references.

The present paper considers advection effects for a different physically interesting situation, where initial data are compactly supported. In this case, two generic scenarios are possible. If the support of the initial data is large enough, then two fronts form and propagate in opposite directions. Fluid advection speeds up the propagation, accelerating the burning. However, if the support of the initial data is small, then the advection exposes the initial hot region to diffusion which cools it below the ignition temperature θ_0 , ultimately extinguishing the flame. The main purpose of this paper is to study the possibility of quenching of flames by strong fluid advection in the model (1.2). The phenomena associated with flame quenching are of great interest for physical, astrophysical and engineering applications. For example, modelling of quenching and propagation of reaction fronts in fluid flow are relevant to studies of internal combustion engines, nuclear burning in stars and forest fires. Mathematically, the problem reduces to studying the advection-enhanced dissipation rate for the passive scalar. The passive scalar equation is one of the most studied PDE models, and has been the subject of extensive

research by both physicists and mathematicians. However, the question that we address here - controlling the rate of decay of the L^∞ norm in terms of the amplitude and geometric properties of the flow - while extremely natural, remained largely open until some very recent work.

The problem of extinction and flame propagation in the mathematical model (1.2) was first studied by Kanel [8] in one dimension and with no advection. He showed that, in the absence of fluid motion, there exist two length scales $L_0 < L_1$ such that the flame becomes extinct for $L < L_0$, and propagates for $L > L_1$. More precisely, he has shown that there exist L_0 and L_1 such that

$$\begin{aligned} T(t, x, y) &\rightarrow 0 \text{ as } t \rightarrow \infty \text{ uniformly in } D \text{ if } L < L_0 \\ T(t, x, y) &\rightarrow 1 \text{ as } t \rightarrow \infty \text{ for all } (x, y) \in D \text{ if } L > L_1. \end{aligned} \tag{1.7}$$

We note in passing that it has been only very recently established by one of the authors that $L_0 = L_1$ in this situation [20]. In the absence of advection, the flame extinction is achieved by diffusion alone, given that the support of initial data is small compared to the scale of the laminar front width $l = \sqrt{\kappa/M}$. However, in many applications quenching is the result of a strong wind, intense fluid motion, and operates on larger scales. Kanel's result was extended to non-zero advection by shear flows by Roquejoffre [13] who has shown that (1.7) holds also for $u \neq 0$ with L_0 and L_1 depending, in particular, on A and $u(y)$. However the interesting question about more explicit quantitative dependence of L_0, L_1 on A and $u(y)$ remained completely open until recent work [4]. The following definition was given in [4].

Definition 1.1. We say that the profile $u(y)$ is *quenching* if for any L and any initial data $T_0(x, y)$ supported inside the interval $[-L, L] \times [0, h]$, there exists $A_0 = A_0(M, \kappa, f, u, L)$ such that the solution of (1.2) becomes extinct:

$$T(t, x, y) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ uniformly in } D \tag{1.8}$$

for all $|A| \geq A_0$. We call the profile $u(y)$ *strongly quenching* if the critical amplitude of advection A_0 satisfies $A_0 \leq CL$ for some constant $C = C(M, \kappa, f, u)$ (which has the dimension of inverse time).

The quenching property has been linked in [4] to hypoellipticity of a certain degenerate diffusion equation. In particular, one of the main results showed that $u(y)$ is strongly quenching if there is no point y where all derivatives of u vanish. On the other hand, if $u(y)$ has a plateau larger than a certain critical size, then u is not quenching. However hypoellipticity does not provide a precise solution of the problem at hand: a shear flow $u(y)$ with a small plateau leads to an auxiliary equation which is not hypoelliptic, yet it is quenching. The first main result of this paper, Theorem 3.1, provides a sharp characterization of quenching shear flows. It states that a shear flow is quenching if and only if it has a plateau exceeding certain critical size. Nearby plateaux of smaller size will not lead to the same effect. This critical scale can be described in terms of existence of solutions to a nonlinear Dirichlet problem. The main new technical ingredient involves estimates on certain stochastic integrals, in particular application of Malliavin calculus to derive absolute continuity of the relevant random variables.

The second goal is to study the dependence of quenching on scaling of the flow. Numerical experiments [16] suggest that there is a certain scale of the flow for which quenching is most

efficient. Namely, if $u(y) = \sin \alpha y$, then the size L_A of initial data that can be quenched by flow $Au(y)$ satisfies $L_A \sim C_\alpha A$ with C_α achieving maximum for some α_0 . Moreover, the constant C_α satisfies $C_\alpha \sim \alpha^{-1}$ for large α and $C_\alpha \sim \alpha^2$ for small α . Our Theorems 4.1 and 4.2, which apply to general shear flows, prove that in the small and large α asymptotic regimes one indeed has quenching for the initial data satisfying the above scaling. Central limit-type theorem for martingales is instrumental in obtaining the large α result.

We mention that in a separate work [19] one of us investigates the phenomenon of quenching in the presence of combustion-type reaction functions that do not have an ignition cutoff θ_0 , but are allowed to be positive for all $T \in (0, 1)$. An important example of such function is the Arrhenius type reaction $f(T) = e^{-A/T}(1 - T)$, which is used in modeling of many chemical reactions. One of the main results of that paper, Theorem 1.3, is related to our Theorem 3.1. It states that the quenching property of shear flows is in this case linked not only to the size of their plateaux but also to the decay rate of f at $T = 0$. Namely, if (in our setting) $f(T) \geq cT^p$ with $p < 3$, then no flow is quenching, whereas if $f(T) \leq cT^p$ with $p > 3$, then flows with small enough plateaux are quenching and those with large plateaux are not. This result, however, does not provide a sharp characterization of quenching flows, and should be understood as an extension to non-ignition reactions of the results of [4] rather than of our Theorem 3.1.

As a final remark we note that proving results for the system (1.1) is typically much harder than for a single equation (1.2), due to the lack of appropriate comparison principles. This is not so in our case. While in the paper we discuss quenching for a single equation (1.2), all our quenching results (including Theorems 3.1(i), 4.1, and 4.2) extend immediately to the case of the system (1.1). This is a consequence of a remnant of the maximum principle: the concentration $n(t, x)$ remains bounded above by one for all times. Then $T_t + u \cdot \nabla T \leq \kappa \Delta T + Mg(T)$ in (1.1), and all the bounds we prove for quenching in the single equation model apply.

The paper is organized as follows. In Section 2 we establish some auxiliary technical estimates on stochastic integrals. In Section 3 we prove results on quenching by shear flows and provide a characterization of the critical plateau size in terms of the corresponding Dirichlet problem. In Section 4 we deal with the scaling question.

2. STOCHASTIC INTEGRALS

Results from this section will be used to obtain upper bounds on the solutions of (1.2) without the non-linear term, which can be expressed in terms of the Brownian motion. See the beginning of Section 3 for details and how this translates into estimates on the temperature T .

We call a *plateau* of a function $u \in C(\mathbb{R})$ any maximal (w.r.t. inclusion) interval on which u is constant. We start by proving

Lemma 2.1. *Let $u \in C^1(\mathbb{R})$ be bounded along with its first derivative and let W_s^y denote the normalized one-dimensional Brownian motion starting at y . Then for any $a \in \mathbb{R}$ we have*

$$\mathbb{P}\left(\int_0^t u(W_s^y) ds = a\right) = \mathbb{P}\left(u(W_s^y) = \frac{a}{t} \text{ for } s \in [0, t]\right). \quad (2.1)$$

Remarks. 1. In other words, the first probability is zero unless y is an interior point of a plateau of u with $u(y) = \frac{a}{t}$, in which case it equals the probability of W_s^y staying inside this plateau for all $s \in [0, t]$.

2. This lemma for $u \in C^\infty$ and y not in a plateau of u follows from a probabilistic version of Hörmander's theorem (see, e.g., [11, Theorem 2.3.2]). Here we extend it to all $u \in C^1$ and all y .

3. We believe that the same result holds for $u \in C(\mathbb{R})$ but we were unable to locate an appropriate reference in the literature.

Proof. By Theorem 2.1.3 in [11] with $F(W^y) \equiv \int_0^t u(W_s^y) ds$, the law of the random variable F is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} whenever

$$\left(\|DF\|_2^2 = \int_0^t \left(\int_s^t u'(W_r^y) dr \right)^2 ds > 0 \right) \quad (2.2)$$

almost surely. We note that with the notation of [11, p.24-26], if $u \in C^1$, then $F \in \mathbb{D}^{1,1}$ is the limit of $F_n(W^y) \equiv \frac{1}{n} \sum_{k=1}^n u(W_{tk/n}^y)$, and $DF(s) = \int_s^t u'(W_r^y) dr$ is the limit of

$$DF_n(s) = \frac{1}{n} \sum_{k=1}^n u'(W_{tk/n}^y) \chi_{[0, tk/n)}(s) = \sum_{k=1}^n \left[\frac{1}{n} \sum_{j=k}^n u'(W_{tj/n}^y) \right] \chi_{[\frac{t(k-1)}{n}, \frac{tk}{n})}(s).$$

Eq. (2.2) is obviously true if u' is not identically zero on an interval around y , that is, when y is not inside a plateau. In particular, for such y and all a ,

$$\mathbb{P} \left(\int_0^t u(W_s^y) ds = a \right) = 0. \quad (2.3)$$

Now assume y to be inside a plateau I . For any open interval J with rational end points not intersecting any plateau of u , and any rational $\tau \in (0, t)$, let $B_{J,\tau}$ be the set of Brownian paths W^y such that $W_\tau^y \in J$. Notice that every W^y that exits I before time t , belongs to some such $B_{J,\tau}$.

We have for any a

$$\mathbb{P} \left(\int_0^t u(W_s^y) ds = a \mid W^y \in B_{J,\tau} \right) = 0. \quad (2.4)$$

This follows from (2.3) applied to the \int_τ^t portion of the integral. Indeed — since $U_s \equiv W_{s+\tau}^y$ (for $s \geq 0$) is just Brownian motion starting at W_τ^y , given any history $\{W_s^y\}_{s \leq \tau}$, the probability of $\int_\tau^t u(W_s^y) ds (= \int_0^{t-\tau} u(U_s) ds)$ being $a - \int_0^\tau u(W_s^y) ds$ is zero because W_τ^y is not in a plateau of u if $W^y \in B_{J,\tau}$. By Fubini's theorem, (2.4) holds. Since there are only countably many sets $B_{J,\tau}$, the result follows. \square

The main result of this section is

Lemma 2.2. *Let $u \in C^1(\mathbb{R})$ be periodic. Then for any compact interval $S \subset (0, \infty)$ we have*

$$\mathbb{P} \left(\int_0^t u(W_s^y) ds \in [a, a + \varepsilon] \setminus \{tu(y)\} \right) \rightarrow 0 \quad (2.5)$$

as $\varepsilon \rightarrow 0$, uniformly in $(t, y, a) \in S \times \mathbb{R} \times \mathbb{R}$.

Remarks. 1. Note that non-uniform convergence is an obvious consequence of Lemma 2.1.

2. The importance of this lemma lies in the fact that for large A it gives us a uniform (in $(t, y, x) \in S \times \mathbb{R} \times \mathbb{R}$) estimate on the solution of (3.3),(3.4) below, using (3.10). Through (3.8),(3.9) this translates into an upper bound on the temperature T .

To prove the lemma, consider the function

$$p(t, y, a, \varepsilon) \equiv \mathbb{P}\left(\int_0^t u(W_s^y) ds \in [a, a + \varepsilon] \setminus \{tu(y)\}\right),$$

that is, the probability of $\int_0^t u(W_s^y) ds \in [a, a + \varepsilon]$ and $\{u(W_s^y)\}_{s \leq t}$ not constant.

Lemma 2.3. *Under the conditions of Lemma 2.2, p is jointly continuous in $\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}_0^+$.*

Proof. For $\delta_1 \in \mathbb{R}_0^+$ and $\delta_2, \delta_3, \delta_4 \in \mathbb{R}$ let

$$\delta \equiv \|u\|_\infty |\delta_1| + t \|u'\|_\infty |\delta_2| + |\delta_3| + |\delta_4|.$$

Then

$$\left| \int_0^{t+\delta_1} u(W_s^y + \delta_2) ds - \int_0^t u(W_s^y) ds \right| \leq \|u\|_\infty |\delta_1| + t \|u'\|_\infty |\delta_2|$$

and we have

$$\begin{aligned} & |p(t+\delta_1, y + \delta_2, a + \delta_3, \varepsilon + \delta_4) - p(t, y, a, \varepsilon)| \\ & \leq \mathbb{P}\left(\int_0^t u(W_s^y) ds \in [a - \delta, a + \delta] \cup [a + \varepsilon - \delta, a + \varepsilon + \delta] \setminus \{tu(y)\}\right) \\ & \quad + \mathbb{P}(\text{exactly one of } \{u(W_s^y)\}_{s \leq t} \text{ and } \{u(W_s^y + \delta_2)\}_{s \leq t + \delta_1} \text{ is constant}). \end{aligned}$$

As $\delta \rightarrow 0$, the first probability goes to zero because by Lemma 2.1,

$$\mathbb{P}\left(\int_0^t u(W_s^y) ds \in \{a, a + \varepsilon\} \setminus \{tu(y)\}\right) = 0.$$

The second probability goes to zero because

$$\mathbb{P}(\{u(W_s^y)\}_{s \leq t} \text{ is constant})$$

is continuous in (t, y) . □

Proof of Lemma 2.2. By Lemma 2.1, $p(t, y, a, 0) = 0$. Hence by Lemma 2.3, $p(t, y, a, \varepsilon) \downarrow 0$ as $\varepsilon \rightarrow 0$, for any (t, y, a) . By joint continuity of p we then have $p(t, y, a, \varepsilon) \downarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in $(t, y, a) \in K$, for any compact $K \subset \mathbb{R}^+ \times \mathbb{R}^2$. But p is periodic in y and $p(t, y, a, \varepsilon) = 0$ for $|a| > t \|u\|_\infty + \varepsilon$. Thus $p(t, y, a, \varepsilon) \downarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in $(t, y, a) \in S \times \mathbb{R} \times \mathbb{R}$, for any compact $S \subset \mathbb{R}^+$. □

3. THE QUENCHING FLOWS

Let $u(y) \in C^1(\mathbb{R})$ be a periodic function and let $f(T)$ be an ignition-type non-linearity satisfying (i)-(iii) of (1.3). Let $T(t, x, y)$, $\Phi(t, x, y)$, and $\Psi(t, x, y)$ be the solutions of

$$T_t = \kappa \Delta T - Au(y)T_x + Mf(T) \quad (3.1)$$

$$\Phi_t = \kappa \Delta \Phi - Au(y)\Phi_x \quad (3.2)$$

$$\Psi_t = \kappa \Psi_{yy} - Au(y)\Psi_x \quad (3.3)$$

with $(t, x, y) \in \mathbb{R}_0^+ \times \mathbb{R}^2$ and initial conditions

$$T(0, x, y) = \Phi(0, x, y) = \Psi(0, x, y) = \chi_{[-L, L]}(x). \quad (3.4)$$

Notice that to prove quenching, one only needs to show

$$\|T(\tau, \cdot, \cdot)\|_\infty \leq \theta_0 \quad (3.5)$$

for some $\tau > 0$. Indeed, the maximum principle then implies $T(t, x, y) \leq \theta_0$ for all $t \geq \tau$. Hence we have

$$T_t = \kappa \Delta T - Au(y)T_x \quad (3.6)$$

for $t \geq \tau$. To show (1.8) we first notice that by integrating (3.1) in $(x, y) \in \mathbb{R} \times [0, h]$ (where h is the period of u) we have for $\|\cdot\|_p \equiv \|\cdot\|_{L^p(\mathbb{R} \times [0, h])}$

$$\frac{\partial}{\partial t} \|T(t, \cdot, \cdot)\|_1 = M \int f(T) dx dy \leq M \|T(t, \cdot, \cdot)\|_1$$

by (1.3), and so $\|T(\tau, \cdot, \cdot)\|_1 < \infty$. One can then, for instance, use the estimates on the parabolic kernel of the operator $\Delta - u \cdot \nabla$ for periodic divergence-free flow u on \mathbb{R}^n from [10] to show that the kernel for $\Delta - u \cdot \nabla$ on $\mathbb{R} \times [0, h]$ with periodic boundary conditions is bounded above by $Ct^{-1/2}$ for some C and all $t > 1$ (see [19]). Therefore

$$\|T(\tau + t, \cdot, \cdot)\|_\infty \leq Ct^{-1/2} \|T(\tau, \cdot, \cdot)\|_1 \quad (3.7)$$

for $t > 1$ and (1.8) follows. Alternatively, there is a more elementary proof of (3.7) based on proving a Nash-type inequality for the evolution of (3.6), namely that

$$\|T(\tau + t, \cdot, \cdot)\|_2 \leq \tilde{C}t^{-1/2} \|T(\tau, \cdot, \cdot)\|_1$$

with \tilde{C} independent of the flow. Such an estimate also leads to (3.7) by a duality argument. See [7, 9] for more details.

The functions Φ , Ψ can be used to estimate the non-linear evolution:

$$T(t, x, y) \leq \Phi(t, x, y)e^{Mt} \quad (3.8)$$

$$\sup_x \Phi(t, x, y) \leq \sup_x \Psi(t, x, y). \quad (3.9)$$

The first bound is achieved by replacing $f(T)$ with T in (3.1), while the second bound follows from the equality

$$\Phi(t, x, y) = \int_{-\infty}^{\infty} G(t, x - x') \Psi(t, x', y) dx'$$

where

$$G(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-x^2/4\kappa t}$$

is the fundamental solution of the one-dimensional heat equation (using that $\|G(t, \cdot)\|_{L_x^1} = 1$). The equality is verified by plugging it into (3.2).

Since Φ and Ψ satisfy the above linear equations, we can apply the results from the previous section to obtain the following estimates. Let (W^x, W^y) be the normalized 2-dimensional Brownian motion starting at (x, y) and let (X_t^x, Y_t^y) be the random process starting at (x, y) and given by

$$\begin{aligned} dX_t^x &= \sqrt{2\kappa} dW_t^x - Au(Y_t^y)dt, \\ dY_t^y &= \sqrt{2\kappa} dW_t^y. \end{aligned}$$

Thus, $Y_t^y = y + \sqrt{2\kappa}(W_t^y - y) = W_{2\kappa t}^y$ and

$$X_t^x = x + \sqrt{2\kappa}(W_t^x - x) - \int_0^t Au(Y_s^y)ds = W_{2\kappa t}^x - \frac{A}{2\kappa} \int_0^{2\kappa t} u(W_s^y)ds.$$

Then we have by (3.2), (3.4), and Lemma 7.8 in [12],

$$\Phi(t, x, y) = \mathbb{E}(\Phi(0, X_t^x, Y_t^y)) = \mathbb{P}\left(W_{2\kappa t}^x - \frac{A}{2\kappa} \int_0^{2\kappa t} u(W_s^y)ds \in [-L, L]\right).$$

Similarly,

$$\Psi(t, x, y) = \mathbb{P}\left(x - \frac{A}{2\kappa} \int_0^{2\kappa t} u(W_s^y)ds \in [-L, L]\right). \quad (3.10)$$

The following result provides a sharp characterization of the quenching flows (see Definition 1.1).

Theorem 3.1. *With the above notation, there exists $0 < \ell < \infty$, depending only on M , κ , and f , such that the following hold.*

- (i) *If the longest plateau of u is shorter than ℓ , then u is quenching.*
- (ii) *If the longest plateau of u is longer than ℓ , then u is not quenching.*

Moreover, this ℓ is the infimum of all l such that the equation

$$\phi_t = \kappa\Delta\phi + Mf(\phi) \quad (3.11)$$

on $(x, y) \in \mathbb{R} \times [0, l]$ with Dirichlet boundary conditions at $y = 0, l$, has a solution ϕ with $\phi(0, \cdot, \cdot)$ compactly supported (and taking values in $[0, 1]$) such that ϕ does not go uniformly to zero as $t \rightarrow \infty$.

The key step in the proof is the following proposition.

Proposition 3.2. *For any $l, L \geq 0$ let $\tau(l, L)$ be the minimal time such that any solution ϕ of (3.11) on $(x, y) \in \mathbb{R} \times [0, l]$ with Dirichlet boundary conditions at $y = 0, l$ and $\phi(0, \cdot, \cdot)$ supported in $[-L, L] \times [0, l]$ (and taking values in $[0, 1]$), satisfies $\phi(t, x, y) \leq \theta_0/2$ for $t \geq \tau(l, L)$. If such a time does not exist, we set $\tau(l, L) = \infty$. Then with the above notation and l the length of the longest plateau of u we have the following.*

- (i) If $\tau(l, L) < \infty$ for every $L < \infty$, then u is quenching.
 (ii) If $\tau(l, L_0) = \infty$ for some $L_0 < \infty$, then u is not quenching (and for all $L \geq L_0$ and any A the temperature $T(t, x, y)$ does not go uniformly to zero as $t \rightarrow \infty$).

Remark. Note that this result also applies in the case $l = \ell$. Whether quenching happens in this case depends not only on whether solutions of (3.11) with Dirichlet boundary conditions at $y = 0, \ell$, initially compactly supported, go uniformly to zero, but on this decay being uniform in all $\phi(0, \cdot, \cdot)$ supported in $[-L, L] \times [0, \ell]$ (for each L).

Proof. (ii) Without loss of generality we can assume that the longest plateau of u is $I = [0, l]$. Also without loss of generality, let $u(0) = 0$. Indeed — if $u(0) \neq 0$ and \tilde{T} is the solution of (3.1) with $u(y)$ replaced by $\tilde{u}(y) = u(y) - u(0)$, then $T(t, x, y) = \tilde{T}(t, x - Au(0)t, y)$ and the result for \tilde{T} translates directly to T .

Assume that for $L = L_0$ the temperature T (with initial condition (3.4) and some A) goes uniformly to zero and let $\tau_0 < \infty$ be such that $\|T(\tau_0, \cdot, \cdot)\|_\infty \leq \theta_0/2$. Since by comparison theorems any ϕ from the statement of the proposition must satisfy $\phi(x, y, t) \leq T(x, y, t)$ for all t , we obtain $\tau(l, L_0) \leq \tau_0$, a contradiction. Therefore T does not go uniformly to zero for $L \geq L_0$ and any A , and so u is not quenching.

(i) Fix $L < \infty$, choose any $0 < \delta < \min\{L, 1\}$, and let $\tau_2 \equiv 1 + \tau(l, L + \delta)$. We will show that for large enough $|A|$ and all x, y one has

$$T(\tau_2, x, y) \leq \theta_0. \quad (3.12)$$

This is (3.5) and so (1.8) will follow.

Our strategy to show (3.12) will be to estimate T by Φ via Lemma 2.2 for y outside of the plateaux of u (with large enough $|A|$), and by a suitable ϕ from the statement of the proposition for y inside a plateau.

Let d be the Lipschitz constant for f , define $c \equiv M \max\{d, 1\}$, and pick $\tau_1 \in (0, 1)$ such that

$$\mathbb{P}\left(|W_{2\kappa\tau_1}^0| \geq \frac{\delta}{2}\right) \leq \frac{\theta_0}{4e^{c\tau_2}}. \quad (3.13)$$

This choice will become clear later.

Let A_0 be large enough so that for all $|A| \geq A_0$, $t \in [\tau_1, \tau_2]$, $x \in \mathbb{R}$, and y not in the interior of a plateau of u we have

$$\Psi(t, x, y) = \mathbb{P}\left(x - \frac{A}{2\kappa} \int_0^{2\kappa t} u(W_s^y) ds \in [-L, L]\right) \leq \frac{\theta_0}{2e^{2c\tau_2}}. \quad (3.14)$$

This is possible by Lemma 2.2 and the first remark after Lemma 2.1. Then by (3.8) and (3.9), for $t \in [\tau_1, \tau_2]$ and y not in the interior of a plateau,

$$\sup_x T(t, x, y) \leq e^{Mt} \sup_x \Psi(t, x, y) \leq e^{Mt} \frac{\theta_0}{2e^{2c\tau_2}} \leq \frac{\theta_0}{2e^{c\tau_2}}. \quad (3.15)$$

In particular, (3.12) holds when y is outside of all plateaux of u .

We are left with the case of y inside a plateau. Hence consider a plateau I of maximal length l . All the following arguments will also apply to any other plateau of length $\tilde{l} \leq l$ because

$\tau(\tilde{l}, L) \leq \tau(l, L)$ by comparison theorems. Therefore proving (3.12) for $y \in I$ will yield the same statement for all other plateaux of u and the proof will be finished.

Again assume without loss of generality that $I = [0, l]$ and $u(0) = 0$. Increase A_0 (if necessary) so that for any $|A| \geq A_0$,

$$\mathbb{P}\left(x - \frac{A}{2\kappa} \int_0^{2\kappa\tau_1} u(W_s^y) ds \in \left[-L - \frac{\delta}{2}, L + \frac{\delta}{2}\right]\right) \leq \frac{\theta_0}{4e^{c\tau_2}} \quad (3.16)$$

whenever $y \in I$ and $|x| \geq L + \delta$. Such A_0 exists because by Lemma 2.2,

$$\mathbb{P}\left(\int_0^{2\kappa\tau_1} u(W_s^y) ds \in \frac{2\kappa}{A} \left[-L - \frac{\delta}{2} + x, L + \frac{\delta}{2} + x\right]\right) \rightarrow 0$$

as $|A| \rightarrow \infty$, uniformly in $y \in I$ and $x \notin [-L - \frac{\delta}{2}, L + \frac{\delta}{2}]$ (because $u(0) = 0$). Using (3.8), (3.13), and (3.16), it follows that for $y \in I$ and $|x| \geq L + \delta$,

$$\begin{aligned} T(\tau_1, x, y) &\leq e^{M\tau_1} \Phi(\tau_1, x, y) \\ &= e^{M\tau_1} \mathbb{P}\left(W_{2\kappa\tau_1}^x - \frac{A}{2\kappa} \int_0^{2\kappa\tau_1} u(W_s^y) ds \in [-L, L]\right) \\ &\leq e^{M\tau_1} \mathbb{P}\left(x - \frac{A}{2\kappa} \int_0^{2\kappa\tau_1} u(W_s^y) ds \in \left[-L - \frac{\delta}{2}, L + \frac{\delta}{2}\right]\right) + e^{M\tau_1} \frac{\theta_0}{4e^{c\tau_2}} \\ &\leq \frac{\theta_0}{2e^{c(\tau_2 - \tau_1)}} \end{aligned} \quad (3.17)$$

Next consider a function $\phi(t, x, y)$ defined on $[\tau_1, \infty) \times \mathbb{R} \times I$, taking values in $[0, 1]$, and satisfying (3.11) with Dirichlet boundary conditions at $y = 0, l$ and initial data

$$T(\tau_1, x, y) - \frac{\theta_0}{2e^{c(\tau_2 - \tau_1)}} \leq \phi(\tau_1, x, y) \leq \chi_{[-L - \delta, L + \delta]}(x) \chi_I(y). \quad (3.18)$$

Such a ϕ exists because of (3.15) and (3.17), and by the definition of τ_2 we have

$$\phi(\tau_2, x, y) \leq \frac{\theta_0}{2}. \quad (3.19)$$

Now let $\omega \equiv T - \phi$ for $(t, x, y) \in [\tau_1, \infty) \times \mathbb{R} \times I$. Then

$$\omega_t = \Delta\omega + M[f(T) - f(\phi)] \leq \Delta\omega + c|\omega| \quad (3.20)$$

By (3.18),

$$\omega(\tau_1, x, y) \leq \frac{\theta_0}{2e^{c(\tau_2 - \tau_1)}} \quad (3.21)$$

and by (3.15),

$$\sup_x \{\omega(t, x, 0), \omega(t, x, l)\} \leq \frac{\theta_0}{2e^{c(\tau_2 - \tau_1)}} \quad (3.22)$$

for $t \in [\tau_1, \tau_2]$. Now for $\tilde{\omega} \equiv e^{-ct}\omega$ we have

$$\tilde{\omega}_t \leq \Delta\tilde{\omega} + c(|\tilde{\omega}| - \tilde{\omega}).$$

Thus by (3.21), (3.22), and the maximum principle,

$$\tilde{\omega}(t, x, y) \leq \frac{\theta_0}{2e^{c\tau_2}},$$

for $t \in [\tau_1, \tau_2]$. Thus $\omega(\tau_2, x, y) \leq \theta_0/2$ whenever $y \in I$. So by (3.19), $T(\tau_2, x, y) \leq \theta_0$ for $y \in I$. As mentioned before, this also holds for any other plateau of u . Together with (3.15) this gives (3.12), and the result follows. \square

Proof of Theorem 3.1. Let ℓ be defined as in the statement of Theorem 3.1. The fact that $\ell < \infty$ is proved in [4] by constructing a subsolution of (3.11) on $\mathbb{R} \times [0, l]$ for large enough l . Proposition 3.3 below shows that $\ell > 0$. Notice that by comparison theorems (see e.g. [15, Chapter 10]), a solution ϕ described in the statement of Theorem 3.1 exists when $l > \ell$ and does not exist when $l < \ell$.

Let l be the length of the longest plateau of u and let $\tau(l, L)$ be as in Proposition 3.2.

(ii) If $l > \ell$, then there exists a solution ϕ described above. This means that $\|\phi(t, \cdot, \cdot)\|_\infty > \theta_0$ for all t . Since $\phi(0, x, y)$ is supported in $[-L_0, L_0] \times [0, l]$ for some $L_0 < \infty$, we obtain $\tau(l, L_0) = \infty$. Proposition 3.2(ii) then gives (ii).

(i) If $l < \ell$, then take $\delta \equiv (\ell - l)/3$. For any $L < \infty$ let ϕ be a solution of (3.11) on $\mathbb{R} \times [-\delta, l + \delta]$ with Dirichlet boundary conditions at $y = -\delta, l + \delta$ and

$$\chi_{[-L, L]}(x)\chi_{[0, l]}(y) \leq \phi(0, x, y) \leq \chi_{[-L-1, L+1]}(x)\chi_{[-\delta, l+\delta]}(y).$$

Since $l + 2\delta < \ell$, there is $\tau_L < \infty$ such that $\|\phi(\tau_L, \cdot, \cdot)\|_\infty \leq \theta_0/2$. By comparison theorems this implies $\tau(l, L) \leq \tau_L < \infty$. The result follows from Proposition 3.2(i). \square

In [4] an upper bound on ℓ was provided by constructing a non-zero compactly supported $\phi(x, y)$ such that

$$\kappa\Delta\phi + Mf(\phi) \geq 0$$

in the sense of distributions. By comparison theorems, ℓ is at most the diameter of the support of ϕ . Here we give a lower bound on ℓ , in terms of the existence of a stationary 1D solution of (3.11).

Proposition 3.3. *Let $\tilde{\ell}$ be the length of the shortest interval I such that there exists a non-zero $\psi : I \rightarrow [0, 1]$, vanishing at the edges of I , such that inside I*

$$\kappa\psi'' + Mf(\psi) = 0. \tag{3.23}$$

Then $\ell \geq \tilde{\ell}$.

Proof. Assume ϕ is a solution of (3.11) on $(x, y) \in \mathbb{R} \times [0, l]$ with Dirichlet boundary conditions at $y = 0, l$ and $\phi(0, \cdot, \cdot)$ compactly supported (and taking values in $[0, 1]$), such that ϕ does not go uniformly to zero as $t \rightarrow \infty$. Let $\tilde{\phi}$ be the solution of (3.11) with the same boundary conditions, but with $\tilde{\phi}(0, x, y) \equiv \sup_x \phi(0, x, y)$. By comparison theorems, $\tilde{\phi} \geq \phi$, and so $\tilde{\phi}$ also does not go uniformly to zero as $t \rightarrow \infty$.

Moreover, obviously $\tilde{\phi}(t, x_1, y) = \tilde{\phi}(t, x_2, y)$ for any t, y, x_1, x_2 , and so $\tilde{\psi}(t, y) \equiv \tilde{\phi}(t, x, y)$ is well-defined and solves

$$\tilde{\psi}_t = \kappa\tilde{\psi}_{yy} + Mf(\tilde{\psi}). \tag{3.24}$$

Since $\tilde{\psi}$ does not go uniformly to 0, Proposition 3.6 provides us ψ solving (3.23), defined on $[0, l]$. A simple shooting argument can be used to prove that the set of all l for which solution of (3.23) does not exist is open. Thus the set of all l for which such solution exists has a minimum $\tilde{\ell}$, and the result follows. \square

Corollary 3.4. *With the above notation (and $f(T) \leq T$) we have $\ell > \pi\sqrt{\kappa/M}$.*

Remark. From results in [4] it follows that $\ell \leq c\sqrt{\kappa/M}$ for some constant c depending on f . It follows that the critical plateau length ℓ is of the order of the *laminar front width* $\sqrt{\kappa/M}$.

Proof. By Proposition 3.3, there exists a solution ψ of (3.23) on $[0, \tilde{\ell}]$ vanishing at $0, \tilde{\ell}$. Since $f(\psi) \leq \psi$, we then have

$$\kappa\psi'' + M\psi \geq 0.$$

That is, the lowest eigenvalue of $-\Delta$ on $[0, \tilde{\ell}]$ is at most $M\kappa$. Hence $\tilde{\ell} \geq \pi\sqrt{\kappa/M}$. But if $\tilde{\ell} = \pi\sqrt{\kappa/M}$, then necessarily $\psi(y) = c \sin(\sqrt{M/\kappa}y)$. This contradicts (3.23) because $f(\psi) = 0$ for small ψ . Hence $\tilde{\ell} > \pi\sqrt{\kappa/M}$ and Proposition 3.3 gives the result. \square

The above provides also a criterion for strong quenching (recall Definition 1.1).

Theorem 3.5. *If the longest plateau of u is shorter than $\tilde{\ell}$ from Proposition 3.3, then u is strongly quenching.*

Proof. Going through the proof of Proposition 3.2, one observes that A_0 depends on τ_1, τ_2 , and L . For $L \geq 1$ one can make τ_1 only depend on τ_2 and from the conditions on A_0 one then sees that as long as τ_2 is bounded, A_0 only depends on L . Moreover, this dependence is linear by Lemma 2.2 (and Lemma 2.1) since the lengths of the intervals in (3.14) and (3.16) are $O(L)$. Therefore as long as $\tau(l, L)$ is bounded in L , we have $A_0 \leq O(L)$ and so u is strongly quenching.

Assume that the longest plateau of u has length l and let $\delta \equiv (\tilde{\ell} - l)/3$. Let $\tilde{\psi}$ solve (3.24) on $[-\delta, l + \delta]$ with Dirichlet boundary conditions at $y = -\delta, l + \delta$ and $\psi(0, y) \geq \chi_{[0, l]}(y)$. By $l + 2\delta < \tilde{\ell}$, the definition of $\tilde{\ell}$, and Proposition 3.6, $\tilde{\psi}$ goes uniformly to 0 and so there is $\tilde{\tau} < \infty$ such that $\|\tilde{\psi}(\tilde{\tau}, \cdot)\|_\infty \leq \theta_0/2$. As in the proof of Proposition 3.3, comparison theorems show that if $\phi(t, x, y)$ is any solution of (3.11) on $\mathbb{R} \times [0, l]$ with Dirichlet boundary conditions at $y = 0, l$ and taking values in $[0, 1]$, then $\phi(t, x, y) \leq \tilde{\psi}(t, y)$. Hence $\tau(l, L) \leq \tilde{\tau}$ for all L , and the result follows. \square

The following proposition relates dynamical properties of reaction-diffusion equation with Dirichlet boundary conditions to existence of stationary solutions. Since we were not able to find this simple and natural result in the literature, we provide the proof in a slightly more general setting than needed for our application. Let Ω be a bounded domain in \mathbb{R}^n with a smooth boundary. We also assume for the sake of simplicity that the reaction function is smooth. In one dimension this requirement can be removed and f only continuous is sufficient. This can be done by approximation from above with smooth f , comparison principles, and a simple ODE shooting argument.

Proposition 3.6. *Assume that there is a solution ϕ of*

$$\phi_t = \kappa \Delta \phi + Mf(\phi) \quad (3.25)$$

on $(x, t) \in \Omega \times \mathbb{R}^+$, with Dirichlet boundary conditions at $\partial\Omega$ and $\phi(\cdot, 0)$ compactly supported (and taking values in $[0, 1]$), such that ϕ does not go uniformly to zero as $t \rightarrow \infty$. Then there exists a positive solution $\psi : \Omega \rightarrow [0, 1]$ of

$$\kappa \Delta \psi + Mf(\psi) = 0 \quad (3.26)$$

satisfying Dirichlet boundary conditions on $\partial\Omega$.

Proof. For the sake of simplicity we let $\kappa = M = 1$. Since by the maximum principle $\phi(x, t) \leq 1$ for any t , standard regularity estimates imply that all Sobolev norms of $\phi(x, t)$ are uniformly bounded in time: $\|\phi(x, t)\|_{H^s(\Omega)} \leq C_s$. Define $\phi_-(x, t) = \limsup_{t \rightarrow \infty} \phi(x, t)$ at every $x \in \Omega$. We claim that $\phi_-(x, t)$ is Lipschitz continuous and is moreover a weak subsolution, that is

$$\int_{\Omega} D\phi_-(x) Dv(x) dx \leq \int_{\Omega} f(\phi_-(x))v(x) dx$$

for any $v \in C_0^\infty(\Omega)$. To avoid certain degenerate cases, we define here Lipschitz continuity as $|\phi_-(x, t) - \phi_-(y, t)| \leq C|x - y|$ for any x, y which belong to some ball $B \subset \Omega$, with the constant C independent of x, y and B . Indeed, let C_1 be a uniform upper bound on $|\nabla \phi(x, t)|$. Assume there exist $x, y \in B \subset \Omega$ with $|\phi_-(x) - \phi_-(y)| > 2C_1|x - y|$. From the definition of ϕ_- it follows that there exist $t_n \rightarrow \infty$ such that either $\phi(y, t_n) - \phi_-(x) > 2C_1|x - y|$ or $\phi(x, t_n) - \phi_-(y) > 2C_1|x - y|$. But this implies that for any $\epsilon > 0$, for all sufficiently large n we have $|\phi(y, t_n) - \phi(x, t_n)| > 2C_1|x - y| - \epsilon$, which contradicts the bound on the gradient of ϕ .

Notice also that compactness of Ω and uniform boundedness of $|\nabla \phi|$ show that ϕ_- is not identically zero and vanishes on $\partial\Omega$.

Define $\Delta_\delta \phi_-(x) = \delta^{-2} \sum_{j=1}^n (\phi_-(x + \delta e_j) + \phi_-(x - \delta e_j) - 2\phi_-(x))$, where e_j are unit vectors in coordinate directions. Next, we claim that for any x such that $\text{dist}(x, \partial\Omega) > \delta$, we have $\Delta_\delta \phi_-(x) \geq -f(\phi_-(x)) - \gamma(\delta)$, where $\gamma(\delta)$ converges to zero when δ goes to zero. Indeed, by definition of $\phi_-(x)$, we have that for any $\epsilon > 0$, there exists a sequence $t_n \rightarrow \infty$ such that $|\phi_-(x) - \phi(x, t_n)| < \epsilon$ and $\phi_-(y) \geq \phi(y, t_n) - \epsilon$ for any y . Moreover, we can choose t_n so that $|\phi_t(x, t_n)| < \epsilon$. Now

$$\begin{aligned} \Delta_\delta \phi_-(x) &= \delta^{-2} \sum_j (\phi_-(x + \delta e_j) + \phi_-(x - \delta e_j) - 2\phi_-(x)) \\ &\geq -C\epsilon\delta^{-2} + \delta^{-2} \sum_j (\phi(x + \delta e_j, t_n) + \phi(x - \delta e_j, t_n) - 2\phi(x, t_n)). \end{aligned}$$

Using the mean value theorem and uniform upper bounds on derivatives of ϕ , it is not hard to show that

$$\delta^{-2} \sum_j (\phi(x + \delta e_j, t_n) + \phi(x - \delta e_j, t_n) - 2\phi(x, t_n)) \rightarrow \phi_{x_j x_j}(x, t_n)$$

uniformly in x and t_n as $\delta \rightarrow 0$, with an error bounded by $C\delta$. Therefore,

$$\Delta_\delta \phi_-(x) \geq -C\epsilon\delta^{-2} - C\delta + \Delta\phi(x, t_n)$$

$$\begin{aligned} &\geq -f(\phi(x, t_n)) - C(\epsilon\delta^{-2} + \delta) - \epsilon \\ &\geq -f(\phi_-(x)) - C(\epsilon\delta^{-2} + \delta + \epsilon). \end{aligned}$$

Since ϵ is arbitrary, this leads to $\Delta_\delta\phi_-(x) \geq -f(\phi_-(x)) - C\delta$.

Given $v \in C_0^\infty(\Omega)$, $v \geq 0$, such that $\text{dist}(\text{supp}(v), \partial\Omega) \geq \delta$, we have

$$-\int_{\Omega} \Delta_\delta\phi_-(x)v(x) dx \leq \int_{\Omega} f(\phi_-(x))v(x) dx + C\delta\|v\|_{L^1(\Omega)}.$$

Carrying out discrete integration by parts on the left hand side and passing to the limit $\delta \rightarrow 0$, we get

$$\int_{\Omega} D\phi_-(x)Dv(x) dx \leq \int_{\Omega} f(\phi_-(x))v(x) dx.$$

Passage to the limit is justified since we know that $\phi_-(x)$ is Lipschitz and therefore belongs to the Sobolev space $W^{1,\infty}$. Thus we see that $\phi_-(x)$ is a weak subsolution of (3.26).

Now consider initial data $\tilde{\phi}(x, 0)$ such that $\phi_-(x) \leq \tilde{\phi}(x, 0) \leq 1$. By the maximum principle, for all t we have $\tilde{\phi}(x, t) \geq \phi_-(x)$. Consider $\phi_+(x) = \liminf_{t \rightarrow \infty} \tilde{\phi}(x, t) \geq \phi_-(x)$. By repeating the same arguments as above, we find that $\phi_+(x)$ is a weak supersolution. Then by well-known results (see e.g. [6], Theorem 9.3.1), there exists a weak solution $\psi(x)$ of (3.26), satisfying $\phi_-(x) \leq \psi(x) \leq \phi_+(x)$. By boundary regularity results, $\psi(x)$ is regular on all of Ω . \square

Results in this section extend without change to the case of shear flows in higher dimensions. The proofs are identical to those above, this time using higher dimensional Brownian motion. Assume that $T(t, x, y)$ is a solution of (3.1), (3.4) on $\mathbb{R}_0^+ \times \mathbb{R} \times \mathbb{R}^n$ with the C^1 shear flow u satisfying $u(y) = u(y + h_j e_j)$ for $j = 1, 2, \dots, n$ and some $h_j > 0$ ($\{e_1, \dots, e_n\}$ being the standard basis in \mathbb{R}^n).

The definition of quenching flows is identical to that for $n = 1$. A plateau of u is any maximal domain Ω on which u is constant. We also say that a domain $\Omega \subseteq \mathbb{R}^n$ is *quenching* if for every $L < \infty$ there is $\tau(\Omega, L) < \infty$ such that any solution ϕ of (3.11) on $(x, y) \in \mathbb{R} \times \Omega$ with Dirichlet boundary conditions on $\partial\Omega$ and $\phi(0, \cdot, \cdot)$ supported in $[-L, L] \times \Omega$ (and taking values in $[0, 1]$), satisfies $\phi(t, x, y) \leq \theta_0/2$ for $t \geq \tau(\Omega, L)$. Of course, the quenching property again depends on M, κ, f . We then have

Theorem 3.7. *With the above notation the following hold.*

- (i) *If every plateau of u is quenching, then u is quenching.*
- (ii) *If u has a plateau that is not quenching, then u is not quenching.*
- (iii) *If $\partial\Omega$ and f are smooth, and there is no non-zero $\psi : \Omega \rightarrow [0, 1]$ satisfying*

$$\kappa\Delta\psi + Mf(\psi) = 0$$

and vanishing on $\partial\Omega$, then the domain Ω is quenching. Moreover, if each plateau of u is contained in some such domain, then u is strongly quenching.

Remark. Note that if $n \geq 2$, then even non-constant u can have unbounded plateaux.

Finally we note that we only considered initial conditions (3.4) for the sake of simplicity of presentation. It is obvious that our results apply also in the case of smooth initial conditions

satisfying, for instance,

$$\chi_{[-L,L]}(x) \leq T(0, x, y) \leq \chi_{[-L-c_L, L+c_L]}(x).$$

If we wish to consider initial temperatures that are not maximal (but still above the ignition temperature θ_0) on an increasing family of regions, for example,

$$\eta \chi_{[-L,L]}(x) \leq T(0, x, y) \leq \eta \chi_{[-L-c_L, L+c_L]}(x).$$

for some $\eta \in (\theta_0, 1)$, then there is only one change — ℓ in Theorem 3.1 is defined in terms of Dirichlet solutions ϕ initially compactly supported and initially bounded above by η . The above method actually applies in the case of any family of compactly supported initial conditions $T_L(0, x, y)$ as long as these are such that for any L_1 and $\delta_1 > 0$ there are L_2 and $\delta_2 > 0$ so that $T_{L_2}(0, x_2, y_2) \geq T_{L_1}(0, x_1, y_1) - \delta_1$ whenever $|(x_2, y_2) - (x_1, y_1)| < \delta_2$ (in particular, $T_L(0, \cdot, \cdot)$ continuous will do). This last condition is necessary for our proof of part (i) of Theorem 3.1 because now we have

$$\Phi(t, x, y) = \mathbb{E} \left(T_L \left(0, W_{2\kappa t}^x - \frac{A}{2\kappa} \int_0^{2\kappa t} u(W_s^y) ds, W_{2\kappa t}^y \right) \right).$$

Here ℓ is defined in terms of ϕ initially bounded above by the T_L 's.

4. SCALING

In this section we study the dependence of the “quenching amplitude”, that is, the infimum of all A such that initial temperature distribution

$$T(0, x, y) = \chi_{[-L,L]}(x) \tag{4.1}$$

leads to quenching, on the scaling of the profile of the shear flow u . Hence we consider

$$T_t = \kappa \Delta T - Au(\alpha y)T_x + Mf(T) \tag{4.2}$$

with u periodic and $\alpha > 0$. The results of this section are motivated by and agree well with numerical simulations performed in [16]. The first is

Theorem 4.1. *Let $u \in C(\mathbb{R})$ be a periodic function with period h . Then there is $C > 0$ such that for large enough α and $|A| \geq C\alpha L$, the solution of (4.2) with initial condition (4.1) satisfies $T(t, x, y) \rightarrow 0$ as $t \rightarrow \infty$, uniformly in \mathbb{R}^2 .*

Remark. The necessity of this bound can be explained by the fact that fast oscillations in the advection homogenize propagation of the flame (w.r.t. y) and so larger advection amplitudes are needed to expose the hot region to diffusion.

Proof. Notice that we have

$$\begin{aligned} \sup_x \Psi \left(\frac{1}{2\kappa}, x, y \right) &= \sup_x \mathbb{P} \left(x - \frac{A}{2\kappa} \int_0^1 u(\alpha W_s^y) ds \in [-L, L] \right) \\ &= \sup_a \mathbb{P} \left(\int_0^1 u(\alpha W_s^y) ds \in \left[a, a + \frac{4\kappa L}{|A|} \right] \right) \end{aligned}$$

$$\begin{aligned}
&= \sup_a \mathbb{P} \left(\frac{1}{\alpha^2} \int_0^{\alpha^2} u(W_s^{\alpha y}) ds \in \left[a, a + \frac{4\kappa L}{|A|} \right] \right) \\
&= \sup_a \mathbb{P} \left(\frac{1}{\alpha} \int_0^{\alpha^2} u(W_s^{\alpha y}) ds \in \left[a, a + \frac{4\kappa\alpha L}{|A|} \right] \right).
\end{aligned}$$

Let us estimate the last integral.

First, we can assume $\int_0^h u(y) dy = 0$, since, as before, changing u by a constant does not change the result. Second, let $v(y)$ be such that $v'(y) = u(y)$ and $\int_0^h v(y) dy = 0$, and define $z(y) \equiv \int_0^y v(s) ds$. Hence, all three functions are periodic with period h .

Now by the Itô formula (see, e.g., [11, Proposition 1.1.4]),

$$z(W_t^y) - z(y) = \int_0^t v(W_s^y) dW_s^y + \frac{1}{2} \int_0^t u(W_s^y) ds$$

almost surely. Thus,

$$\frac{1}{\alpha} \int_0^{\alpha^2} u(W_s^y) ds = \frac{2}{\alpha} (z(W_{\alpha^2}^y) - z(y)) - 2\mathcal{M}(y, \alpha, W^y)$$

with

$$\mathcal{M}(y, \alpha, W^y) \equiv \frac{1}{\alpha} \int_0^{\alpha^2} v(W_s^y) dW_s^y.$$

Therefore with $c \equiv \|z\|_\infty$ we have

$$\sup_x \Psi \left(\frac{1}{2\kappa}, x, y \right) \leq \sup_a \mathbb{P} \left(\mathcal{M}(\alpha y, \alpha, W^{\alpha y}) \in \left[a, a + \frac{2\kappa\alpha L}{|A|} + \frac{4c}{\alpha} \right] \right).$$

From (3.8) and (3.9) we can see that to obtain (3.5) for $\tau = (2\kappa)^{-1}$ (and hence (1.8)), we only need to prove

$$\sup_{y,a} \mathbb{P} \left(\mathcal{M}(y, \alpha, W^y) \in \left[a, a + \frac{2\kappa}{C} + \frac{4c}{\alpha} \right] \right) \leq \theta_0 e^{-M/2\kappa}$$

for some C and all large enough α . That is,

$$\sup_{y,a} \mathbb{P}(\mathcal{M}(y, \alpha, W^y) \in [a, a + \varepsilon]) \leq \theta_0 e^{-M/2\kappa} \quad (4.3)$$

for small ε and large α . However, for each y , the family $\alpha\mathcal{M}(y, \alpha, W^y)$ is a martingale with respect to α . It is not difficult to check that the central limit theorem for martingales (see, e.g. [5], Theorem 7.7.3, or [14]) applies to $\mathcal{M}(y, \alpha, W^y)$ giving convergence in distribution to the normal random variable with variance

$$\sigma^2 = \frac{1}{h} \int_0^h \mathbb{E} \left[\int_0^1 v(W_s^z)^2 ds \right] dz = \frac{1}{h} \int_0^h |v(z)|^2 dz > 0,$$

where \mathbb{E} denotes expectation with respect to the Brownian motion starting at z . Moreover the convergence can be shown to be uniform in y since all the estimates entering the proof are uniform in y . This implies the estimate (4.3). \square

Next, we consider scaling in the opposite direction, that is $\alpha \rightarrow 0$.

Theorem 4.2. *If $u \in C^{n+1}(\mathbb{R})$ is periodic and $|u'(y)| + |u''(y)| + \dots + |u^{(n)}(y)| > 0$ for some n and all y , then there is $C > 0$ such that for small enough $\alpha > 0$ and $|A| \geq C\alpha^{-n}L$, the solution of (4.2) with initial condition (4.1) satisfies $T(t, x, y) \rightarrow 0$ as $t \rightarrow \infty$, uniformly in \mathbb{R}^2 .*

Let us give a short explanation of this result. Consider first the situation as in [16], where $u(y) = \sin y$ was analyzed numerically. When there is no flow, the critical quenching size, according to results of Kanel', is of the order ℓ . Therefore one expects that to quench initial data of size L , the flow should be able to thin it down to width ℓ , given by Theorem 3.1, in time $\tau \sim M^{-1}$ (before the reaction picks up). The differential of velocities near the tip at points which are distance $\sim \ell$ apart is $A\alpha^2\ell^2$, so we get the condition for quenching $A\alpha^2\ell^2\tau \sim L$, which is consistent with our theorem (since the assumptions are satisfied with $n = 2$ when $u(y) = \sin y$). In a more general setting, assume u is smooth enough and $u'(0) = u''(0) = \dots = u^{(n-1)}(0) = 0$ (and u' does not vanish to a higher degree elsewhere). If then A grows slower than $O(\alpha^{-n})$ as $\alpha \rightarrow 0$, the functions $Au(\alpha y)$ become very flat on intervals around 0 with increasing lengths. The heuristic reasoning above then suggests that one should not expect quenching for small α 's.

To prove the theorem, we will need an auxiliary lemma. For $b \in \mathcal{S}^{n-1}$, the unit sphere in \mathbb{R}^n , we define

$$P_b(y) \equiv b_n y^n + b_{n-1} y^{n-1} + \dots + b_1 y.$$

Lemma 4.3. *Given any $t > 0$ and $K < \infty$ we have*

$$\sup_{b \in \mathcal{S}^{n-1}, a} \mathbb{P} \left(\int_0^t P_b(W_s^0) ds \in [a, a + \varepsilon] \mid |W_s^0| \leq K \text{ for } s \in [0, t] \right) \rightarrow 0 \quad (4.4)$$

as $\varepsilon \rightarrow 0$.

Proof. We define

$$q(b, a, \varepsilon) \equiv \mathbb{P} \left(\int_0^t P_b(W_s^0) ds \in [a, a + \varepsilon] \mid |W_s^0| \leq K \text{ for } s \in [0, t] \right),$$

and we let $N \equiv K^n + K^{n-1} + \dots + K$ so that $|P_{b+\delta}(y) - P_b(y)| \leq N|\delta|$ whenever $|y| \leq K$. Hence we need to show that, just as p in Section 2, $q \rightarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in $(b, a) \in \mathcal{S}^{n-1} \times \mathbb{R}$. Notice that we do not need to exclude the value $tP_b(0) = 0$ in the above probability because the P_b 's have no plateaux.

The proof is identical to that of Lemma 2.2. First, the absence of plateaux in the P_b 's gives $q(b, a, 0) = 0$. Then with $\delta \equiv tN(|\delta_1| + |\delta_2| + |\delta_3|)$ we have

$$\begin{aligned} & |q(b + \delta_1, a + \delta_2, \varepsilon + \delta_3) - q(b, a, \varepsilon)| \\ & \leq \mathbb{P} \left(\int_0^t P_b(W_s^0) ds \in [a - \delta, a + \delta] \cup [a + \varepsilon - \delta, a + \varepsilon + \delta] \mid |W_s^0| \leq K \text{ for } s \in [0, t] \right) \end{aligned}$$

which goes to zero as $\delta \rightarrow 0$ because $q(b, a, 0) = q(b, a + \varepsilon, 0) = 0$. Thus, q is jointly continuous in (b, a, ε) . This means that $q(b, a, \varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in any compact subset of $\mathcal{S}^{n-1} \times \mathbb{R}$. Finally, $q(b, a, \varepsilon) = 0$ for $|a| > tN + \varepsilon$, finishing the proof. \square

Proof of Theorem 4.2. Since $u \in C^{n+1}(\mathbb{R})$ and is periodic, $|u'(y)| + |u''(y)| + \dots + |u^{(n)}(y)| > \rho$ for some $\rho > 0$ and all y . Let K be such that

$$\mathbb{P}\left(|W_s^0| \leq K \text{ for } s \in [0, 1]\right) \geq 1 - \frac{\theta_0}{2} e^{-M/2\kappa}.$$

Let $C > 0$, $|A| \geq C\alpha^{-n}L$, and $c \equiv \|u^{(n+1)}\|_\infty/(n+1)!$. Then if $b_k \equiv u^{(k)}(\alpha y)/k!$ for $k = 1, \dots, n$, Taylor's theorem gives us

$$u(\alpha(y + \delta)) = u(\alpha y) + P_b(\alpha\delta) + \tilde{c}\alpha^{n+1}|\delta|^{n+1}$$

for some $|\tilde{c}| \leq c$. Notice that b need not be a unit vector here.

With all the following probabilities conditioned by $|W_s^0| \leq K$ for $s \in [0, 1]$, we have

$$\begin{aligned} & \sup_{x,y} \Psi\left(\frac{1}{2\kappa}, x, y\right) \\ & \leq \sup_{x,y} \mathbb{P}\left(x - \frac{A}{2\kappa} \int_0^1 u(\alpha(y + W_s^0)) ds \in [-L, L]\right) + \frac{\theta_0}{2} e^{-M/2\kappa} \\ & \leq \sup_{a,y} \mathbb{P}\left(\int_0^1 \alpha^{-n} u(\alpha(y + W_s^0)) ds \in \left[a, a + \frac{4\kappa}{C}\right]\right) + \frac{\theta_0}{2} e^{-M/2\kappa} \\ & \leq \sup_{a,y} \mathbb{P}\left(\int_0^1 \alpha^{-n} P_b(\alpha W_s^0) ds \in \left[a, a + \frac{4\kappa}{C} + 2c\alpha K^{n+1}\right]\right) + \frac{\theta_0}{2} e^{-M/2\kappa} \\ & = \sup_{a,y} \mathbb{P}\left(\int_0^1 P_d(W_s^0) ds \in \left[a, a + \frac{4\kappa}{C} + N\alpha\right]\right) + \frac{\theta_0}{2} e^{-M/2\kappa} \end{aligned} \quad (4.5)$$

with $d_k \equiv b_k \alpha^{k-n}$ and $N \equiv 2cK^{n+1}$. If we take $\alpha < 1$, then $|d| \geq |b| \geq \rho/(n+1)!$ and so there are $e \in \mathcal{S}^{n-1}$ and $r \geq \rho/(n+1)!$ such that $d = re$. The last expression in (4.5) is then at most

$$\sup_{e \in \mathcal{S}^{n-1}, a} \mathbb{P}\left(\int_0^1 P_e(W_s^0) ds \in \left[a, a + \frac{(n+1)!}{\rho} \left(\frac{4\kappa}{C} + N\alpha\right)\right]\right) + \frac{\theta_0}{2} e^{-M/2\kappa}.$$

Lemma 4.3 ensures that for some $C < \infty$ and all small α the supremum is smaller than $\theta_0 e^{-M/2\kappa}/2$, and then (3.8) and (3.9) give (3.5) for $\tau = (2\kappa)^{-1}$. The result follows. \square

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