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Commutativity of Γ-Generalized Boolean Semirings with Derivations

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Abstract

In this paper the notion of derivations on Γ-generalized Boolean semiring are established, namely Γ-(f, g) derivation and Γ-(f, g) generalized derivation. We also investigate the commutativity of prime Γ-generalized Boolean semiring admitting Γ-(f, g) derivation and Γ-(f, g) generalized derivation satisfying some conditions.

Keywords: Γ-generalized Boolean semiring, semiring, commutativity, derivation

1. Introduction

There has been a great deal of work concerning commutativity of prime rings and prime near rings with derivations or generalized derivations satisfying certain differential identity (Ali, 2012; Asci, 2007; Bell, 2012; Rehman, 2011; Quadri, 2003). The notion of semiring was first introduced by H.S. Vandiver (Vandiver, 1934) in 1934 and a generalization of semiring, Γ-semiring was first studied by M.K. Rao (Rao, 1995).

In 1987, H.E. Bell and G. Mason (Bell & Mason, 1987) introduced derivations on Γ-near rings and studied some basic properties. The concept of Γ-derivations in Γ-near ring was introduced by Jun, Kim and Cho (Jun, 2003). Then Asci(Asci, 2007) investigated some commutativity conditions for Γ-near rings with derivations. Kazaz and Alkan (Kazaz & Alkan, 2008) introduced the notion of two-side Γ-α derivation of Γ-near rings and investigated some commutativity of prime and semiprime Γ-near rings. In 2011, the notion of derivations in prime Γ-semiring was introduced by M.A. Javed et al (Javed et al, 2013). In 2013, K.K. Dey and A.C. Paul (Dey & Paul, 2013) studied on generalized derivations of prime gamma ring. Later in 2014, M.R. Khan and M.M. Hasnain (Khan & Hasnain, 2014) introduced the notion of generalized Γ-derivation in Γ-near rings and investigated some basic properties.

In this paper, we introduce the notion of Γ-(f, g) derivations and Γ-(f, g) generalized derivations on Γ-generalized Boolean semirings, and investigate some related properties. We also investigate some commutativity results for Γ-generalized Boolean semiring involving Γ-(f, g) derivation and Γ-(f, g) generalized derivation.

2. Preliminaries

We first recall some definitions and prove lemmas use in proving our main results.

A Γ-generalized Boolean semiring (or simply Γ-GB-semiring) is a triple $(R, +, \Gamma)$, where
(1) $(R, +)$ is an abelian group.

(2) Γ is a nonempty finite set of binary operations satisfying the following properties

(i) $a\alpha b \in R$ for all $a, b \in R$ and $\alpha \in \Gamma$,

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$ for all $a, b, c \in R$ and $\alpha \in \Gamma$,

(iii) $a\alpha(b\beta c) = (a\alpha b)\beta c = (b\alpha a)\beta c$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$,

(iv) $a\alpha(b\beta c) = a\beta(b\alpha c)$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$.

The following are some basic properties on Γ-GB-semiring then the proof is straightforward and hence omitted. For any $a, b, c \in R$ and $\alpha \in \Gamma$, we have

(i) $-(-a) = a$,

(ii) $a\alpha 0 = 0$,

(iii) $a\alpha(-b) = -(a\alpha b)$,

(iv) $a\alpha(b - c) = (a\alpha b) - (a\alpha c)$,

(v) $-(a + b) = -a - b$,

(vi) $-(a-b) = -a+b$,

(vii) $-(a\alpha(b+c)) = -(a\alpha b) - (a\alpha c)$,

(viii) $-(a\alpha(b-c)) = -(a\alpha b) + (a\alpha c)$.

A nonempty subset I of R is said to be a Γ-ideal of R if

(1) $(I, +)$ is a subgroup of $(R, +)$,

(2) $r\alpha a \in I$ for all $r \in R$, $a \in I$, and $\alpha \in \Gamma$ (i.e. $R\Gamma I \subseteq I$),

(3) $(r+a)\alpha s - r\alpha s \in I$ for all $r, s \in R$, $a \in I$, and $\alpha \in \Gamma$.

An automorphism f on R is a $\Gamma\Gamma$-isomorphism from R onto R if

(1) $f(a+b) = f(a) + f(b)$,

(2) $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R$ and $\alpha \in \Gamma$.

R is a prime Γ-GB-semiring if $x\Gamma R\Gamma y = \{0\}$ for all $x, y \in R$, then $x = 0$ or $y = 0$.

For any $x, y \in R$ and $\alpha \in \Gamma$, the symbol $[x, y]_\alpha$ will represent the commutator $x\alpha y - y\alpha x$ and the symbol $(x \circ y)_\alpha$ stands for skew-commutator $x\alpha y + y\alpha x$.

Next, the following are some basic properties of commutator and skew-commutator. The proofs of these properties are straightforward and hence omitted.

(i) $[x\alpha y, z]_\beta = x\alpha[y, z]_\beta = y\alpha[x, z]_\beta + z\alpha[y, x]_\beta$,

(ii) $[x, y\alpha z]_\beta = y\alpha[x, z]_\beta = z\alpha[x, y]_\beta + x\alpha[y, z]_\beta$,

(iii) $(x \circ y\alpha z)_\beta = y\alpha(x \circ z)_\beta = z\alpha(x \circ y)_\beta + x\alpha[y, z]_\beta$,

(iv) $(x\alpha y \circ z)_\beta = x\alpha(y \circ z)_\beta = y\alpha(x \circ z)_\beta + z\alpha[x, y]_\beta$.

The center of R, written $Z(R)$, is defined to be the set

$$Z(R) = \{a \in R | a\alpha b = b\alpha a \text{ for all } b \in R \text{ and } \alpha \in \Gamma\}$$

Next, we start with following lemmas which will be used extensively.

Lemma 2.1. Let R be a Γ-GB-semiring. If $x \in Z(R)$ then $y\alpha x \in Z(R)$ and $x\alpha y \in Z(R)$ for all $y \in R$ and $\alpha \in \Gamma$.

Proof. Let $x \in Z(R), y, z \in R$, and $\alpha \in \Gamma$. Then

$(y\alpha x)\beta z = x\alpha(y\beta z) = (y\beta z)\alpha x = z\beta(y\alpha x)$ for all $\beta \in \Gamma$. So $y\alpha x \in Z(R)$. Since $x \in Z(R)$, $x\alpha y = y\alpha x \in Z(R)$. This completes the proof.

Lemma 2.2. Let R be a prime Γ-GB-semiring such that $0\alpha a = a$ for all $a \in R$ and $\alpha \in \Gamma$ and let $I \neq \{0\}$ be a Γ-ideal of R. Then for any $x, y \in R$

(i) If $x\Gamma I = \{0\}$, then $x = 0$.

(ii) If $I\Gamma x = \{0\}$, then $x = 0$.

(iii) If $x\Gamma I\Gamma y = \{0\}$, then $x = 0$ or $y = 0$.

Proof. (i) Let $x \in R$ be such that $x\Gamma I = \{0\}$. Since $I \neq \{0\}$, there exists nonzero z in I. We have $x\Gamma R\Gamma z \subseteq x\Gamma I = \{0\}$ and so $x\Gamma R\Gamma z = \{0\}$. Since R is prime and $z \neq 0$, it follows that $x = 0$.

(ii) Let $x \in R$ be such that $I\Gamma x = \{0\}$. Since $I \neq \{0\}$, there exists nonzero z in I and since $z\beta r = (0+z)\beta r - 0\beta r \in I$ for all $r \in R$ and $\beta \in \Gamma$, $z\Gamma R \subseteq I$. We have $z\Gamma R\Gamma x \subseteq I\Gamma x = \{0\}$ and so $z\Gamma R\Gamma x = \{0\}$. Since R is prime and $z \neq 0$, it follows that $x = 0$

(iii) Let $x, y \in R$ be such that $x\Gamma I\Gamma y = \{0\}$. Then $x\Gamma R\Gamma I\Gamma y \subseteq x\Gamma I\Gamma y = \{0\}$ and so $x\Gamma R\Gamma I\Gamma y = \{0\}$. Since R is prime, it follows that $x = 0$ or $I\Gamma y = \{0\}$. By (ii) we get $y = 0$.

Lemma 2.3. Let R be a prime Γ-GB-semiring and Δ be a nonzero function from R into R. Then $\Delta(x) \in Z(R)$ for all $x \in R$ if and only if R is commutative.

Proof. If R is commutative, then it is obvious that $\Delta(x) \in Z(R)$ for all $x \in R$. Suppose that $\Delta(x) \in Z(R)$ for all $x \in R$. By Lemma 2.1, we have $\Delta(x)\alpha y \in Z(R)$ for all $y \in R$ and $\alpha \in \Gamma$. It follows that $[\Delta(x)\alpha y, t]_\beta = 0$ for all $t, x, y \in R$ and $\alpha, \beta \in \Gamma$.

To show that R is commutative, let $x, y \in R$ and $\alpha \in R$. Since Δ is a nonzero function on R, there exists $z \in R$ such that

$\Delta(z) \neq 0$. For any $t \in R$ and $\beta, \gamma \in \Gamma$ we have

$\Delta(z)\beta t\gamma[x, y]_\alpha = [\Delta(z)\beta(t\gamma x), y]_\alpha = 0$. So, $\Delta(z)\Gamma R\Gamma[x, y]_\alpha = \{0\}$.

Since R is prime and $\Delta(z) \neq 0$, $[x, y]_\alpha = 0$. It follows that $x\alpha y = y\alpha x$. Thus R is commutative. This completes the proof.

Lemma 2.4. Let R be a prime Γ-GB-semiring and Δ be a nonzero function from R into R.

If $[\Delta(x), y]_\alpha \in Z(R)$ or $(\Delta(x) \circ y)_\alpha \in Z(R)$ for all $x, y \in R$ and $\alpha \in \Gamma$ then R is commutative.

Proof. First, assume that $[\Delta(x), y]_\alpha \in Z(R)$ for all $x, y \in R$ and $\alpha \in \Gamma$. Then we have $[[\Delta(x), y]_\alpha, t]_\beta = 0$ for all $t \in R$ and $\beta \in \Gamma$. Replacing y by $\Delta(z)\gamma y$, we obtain $[[\Delta(x), \Delta(z)\gamma y]_\alpha, t]_\beta = 0$ for all $t, x, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$[\Delta(x), y]_\alpha\gamma[\Delta(z), t]_\beta = [\Delta(x), y]_\alpha\gamma[\Delta(z), t]_\beta + t\gamma[[\Delta(x), y]_\alpha, \Delta(z)]_\beta = [\Delta(z)\gamma[\Delta(x), y]_\alpha, t]_\beta = 0.$ for all $t, x, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$.

Now to show that R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. We obtain

$[\Delta(x), y]_\alpha\beta t\gamma[\Delta(x), y]_\alpha = [\Delta(x), y]_\alpha\beta[\Delta(x), t\gamma y]_\alpha = 0$, for all $t \in R$ and $\beta, \gamma \in \Gamma$.

So $[\Delta(x), y]_\alpha\Gamma R\Gamma[\Delta(x), y]_\alpha = \{0\}$. Since R is prime, $[\Delta(x), y]_\alpha = 0$.

It follows that $\Delta(x) \in Z(R)$. By Lemma 2.3, we get required result.

Next, assume that $(\Delta(x) \circ y)_\alpha \in Z(R)$ for all $x, y \in R$ and $\alpha \in \Gamma$. Then we have $[(\Delta(x) \circ y)_\alpha, t]_\beta = 0$ for all $t \in R$ and $\beta \in \Gamma$. Replacing y by $\Delta(z)\gamma y$, we obtain

$[(\Delta(x) \circ \Delta(z)\gamma y)_\alpha, t]_\beta = 0$ for all $t, x, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$[\Delta(z), t]_\beta\gamma(\Delta(x) \circ y)_\alpha = (\Delta(x) \circ y)_\alpha\gamma[\Delta(z), t]_\beta + t\gamma[(\Delta(x) \circ y)_\alpha, \Delta(z)]_\beta = [\Delta(z)\gamma(\Delta(x) \circ y)_\alpha, t]_\beta = [(\Delta(x) \circ \Delta(z)\gamma y)_\alpha, t]_\beta = 0$ for all $t, x, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$. Replacing y by $y\delta w$, we get $[\Delta(z), t]_\beta\gamma(\Delta(x) \circ y\delta w)_\alpha = 0$ for all $w \in R$ and $\delta \in \Gamma$. Then

$\Delta(x)\gamma[\Delta(z), t]_\beta\delta[y, w]_\alpha = [\Delta(z), t]_\beta\gamma\Delta(x)\delta[y, w]_\alpha = [\Delta(z), t]_\beta\gamma(w\delta(\Delta(x) \circ y)_\alpha + \Delta(x)\delta[y, w]_\alpha) = [\Delta(z), t]_\beta\gamma(\Delta(x) \circ y\delta w)_\alpha = 0$ for all $w, t, x, y, z \in R$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Now to show R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. We obtain

$\Delta(z)\beta t\gamma[\Delta(x), y]_\alpha\delta[s\pi y, x]_\alpha = \Delta(z)\beta[\Delta(x), y]_\alpha\gamma[t\delta(s\pi y), x]_\alpha = 0$ for all $s, t, z \in R$ and $\beta, \delta, \pi \in \Gamma$. So $\Delta(z)\Gamma R\Gamma[\Delta(x), y]_\alpha\delta[s\pi y, x]_\alpha = \{0\}$. Since $\Delta \neq 0$, there exists $z \in R$ such that $\Delta(z) \neq 0$ and R is prime, $[\Delta(x), y]_\alpha\delta[s\pi y, x]_\alpha = 0$. And we have $[\Delta(x), y]_\alpha\delta s\pi[y, x]_\alpha = 0$. So $[\Delta(x), y]_\alpha\Gamma R\Gamma[y, x]_\alpha = \{0\}$. Since R is prime, $[\Delta(x), y]_\alpha = 0$ or $[y, x]_\alpha = 0$.

If $[\Delta(x), y]_\alpha = 0$. It follows that $\Delta(x) \in Z(R)$ by Lemma 2.3, R is commutative.

If $[y, x]_\alpha = 0$, R is commutative. This completes the proof.

Lemma 2.5. Let R be a prime Γ-GB-semiring and ζ be an automorphism on R. If there exists a nonzero z in R such that $z\beta\zeta[x, y]_\alpha = 0$ or $z\beta\zeta(x \circ y)_\alpha = 0$ for all $x, y \in R$ and $\alpha, \beta \in \Gamma$. Then R is commutative.

Proof. Case 1 Assume that there exists a nonzero z in R such that $z\beta\zeta[x, y]_\alpha = 0$ for all $x, y \in R$ and $\alpha, \beta \in \Gamma$. Then for any $t, x, y \in R$ and $\alpha, \beta, \gamma \in \Gamma$, we have

$z\beta\zeta(t)\gamma\zeta[x, y]_\alpha = z\beta\zeta(t\gamma[x, y]_\alpha) = z\beta\zeta[t\gamma x, y]_\alpha = 0$ for all $t \in R$ and $\beta, \gamma \in \Gamma$.

Since ζ is surjective, $z\Gamma R\Gamma\zeta[x, y]_\alpha = 0$. Since R is prime and $z \neq 0$, $\zeta[x, y]_\alpha = 0$

Since $\zeta(0) = 0$ and ζ is injective, $[x, y]_\alpha = 0$, it follows that R is commutative.

Case 2 Assume that there exists a nonzero z in R such that $z\beta\zeta[x, y]_\alpha = 0$ for all $x, y \in R$ and $\alpha, \beta \in \Gamma$. Similarly to case 1, we have

$z\beta\zeta(t)\gamma\zeta[x, y]_\alpha = z\beta\zeta(t)\gamma\zeta[x, y]_\alpha + 0 = z\beta\zeta(t)\gamma\zeta[x, y]_\alpha + z\beta\zeta(y)\zeta(x \circ t)_\alpha = z\beta\zeta(y\gamma(x \circ t)_\alpha + t\gamma[x, y]_\alpha) = z\beta\zeta(x\gamma y \circ t)_\alpha = 0$, for all $t \in R$ and $\beta, \gamma \in \Gamma$. We use the same argument in the proof of case 1, we conclude that R is commutative. This completes the proof.

3. Derivations on Γ-Generalized Boolean Semirings

In this section we establish derivations on Γ- generalized Boolean semiring and investigate some results satisfying certain identities involving these derivations.

Definition 3.1. Let R be a Γ-GB-semiring and let f and g be automorphisms on R. An additive mapping $d : R \to R$ is called a Γ-(f, g) derivation

$$d(x\alpha y) = f(x)\alpha d(y) + d(x)\alpha g(y) \text{ for all } x, y \in R \text{ and } \alpha \in \Gamma.$$

An additive mapping $D : R \to R$ is called a left (resp. right) Γ-(f, g) generalized derivation if there exists nonzero Γ-(f, g) derivation d on R satisfiying

$$D(x\alpha y) = f(x)\alpha d(y) + D(x)\alpha g(y) \text{ (resp. } D(x\alpha y) = f(x)\alpha D(y) + d(x)\alpha g(y))$$

for all $x, y \in R$ and $\alpha \in \Gamma$.

Lemma 3.2. Let R be a Γ-GB-semiring and D be a left Γ-(f, g) generalized derivation on R. Then

$[f(x)\alpha d(y) + D(x)\alpha g(y)]\beta g(z) = f(x)\alpha d(y)\beta g(z) + D(x)\alpha g(y)\beta g(z)$.

Proof. Let $x, y, z \in R$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned} D((x\alpha y)\beta z) &= f(x\alpha y)\beta d(z) + D(x\alpha z)\beta g(z) \\ &= f(x)\alpha f(y)\beta d(z) + (f(x)\alpha d(y) + D(x)\alpha g(y))\beta g(z) \text{ and} \\ D(x\alpha(y\beta z)) &= f(x)\alpha d(y\beta z) + D(x)\alpha g(y\beta z) \\ &= f(x)\alpha(f(y)\beta d(z) + d(y)\beta g(z)) + D(x)\alpha g(y)\beta g(z) \\ &= f(x)\alpha f(y)\beta d(z) + f(x)\alpha d(y)\beta g(z) + D(x)\alpha g(y)\beta g(z). \end{aligned}$$

Since $D((x\alpha y)\beta z) = D(x\alpha(y\beta z))$,

$(f(x)\alpha d(y) + D(x)\alpha g(y))\beta g(z) = f(x)\alpha d(y)\beta g(z) + D(x)\alpha g(y)\beta g(z)$. This completes the proof.

Corollary 3.3. Let R be a Γ-GB-semiring. Let d be a Γ-(f, g) derivation on R and f, g be automorphisms on R. Then

$[f(x)\alpha d(y) + d(x)\alpha g(y)]\beta g(z) = f(x)\alpha d(y)\beta g(z) + d(x)\alpha g(y)\beta g(z)$.

Lemma 3.4. Let R be a prime Γ-GB-semiring. Let D be a nonzero Γ-(f, g) generalized derivation on R and f, g be automorphisms on R. If $f(x)\alpha d(y) + D(x)\alpha g(y) \in Z(R)$ for all $x, y \in R$ and $\alpha \in \Gamma$ then R is commutative.

4. Commutativity of Γ-generalized Boolean Semirings

In this section, we show that Γ-generalized Boolean semiring with derivations satisfying certain conditions are commutative.

Theorem 4.1. Let R be a prime Γ-GB-semiring and let f, g be automorphisms on R. If d is a nonzero Γ-(f, g) derivation on R satisfying any one of the following

(i) $[d(x), g(y)]_\alpha = [f(x), g(y)]_\alpha$,

(ii) $d[x, y]_\alpha = [f(x), g(y)]_\alpha$,

(iii) $(d(x) \circ g(y))_\alpha = (f(x) \circ g(y))_\alpha$,

(iv) $d(x \circ y)_\alpha = (f(x) \circ g(y))_\alpha$,

(v) $d(x \circ y)_\alpha = [f(x), g(y)]_\alpha$,

(vi) $d[x, y]_\alpha = (f(x) \circ g(y))_\alpha$,

for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is commutative.

Proof. (i) Assume that $[d(x), g(y)]_\alpha = [f(x), g(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$. Replacing x by $z\beta x$, we obtain $[d(z\beta x), g(y)]_\alpha = [f(z\beta x), g(y)]_\alpha$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} d(z\beta x)\alpha g(y) - g(y)\alpha d(z\beta x) &= [f(z\beta x), g(y)]_\alpha \\ f(z)\beta d(x)\alpha g(y) + d(z)\beta g(x)\alpha g(y) - g(y)\alpha f(z)\beta d(x) - g(y)\alpha d(z)\beta g(x) &= f(z)\beta[f(x), g(y)]_\alpha \\ f(z)\beta[d(x), g(y)]_\alpha + d(z)\beta[g(x), g(y)]_\alpha &= f(z)\beta[f(x), g(y)]_\alpha \\ d(z)\beta g[x, y]_\alpha &= 0. \end{aligned}$$

Since $d \neq 0$, there exists $z \in R$ such that $d(z) \neq 0$. By Lemma 2.5, it follows that R is commutative.

The proof of (ii) - (vi) are obtained similarly to that of (i).

Theorem 4.2. Let R be a prime Γ-GB-semiring and f, g be automorphisms on R. If d is a nonzero Γ-(f, g) derivation on R such that

(i) $[d(x), y]_\alpha \in Z(R)$, or

(ii) $(d(x) \circ y)_\alpha \in Z(R)$,

for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is commutative.

Proof. This follows directly from Lemma 2.4.

Theorem 4.3. Let R be a prime Γ-GB-semiring such that $0\alpha a = 0$ for all $a \in R$ and $\alpha \in \Gamma$. Let d be a nonzero Γ-(f, f) derivation on R where f is a nonzero automorphism on R. If

(i) $d[x, y]_\alpha = [d(x), f(y)]_\alpha$, or

(ii) $d(x \circ y)_\alpha = (d(x) \circ f(y))_\alpha$,

for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is commutative.

Proof. (i) Assume that $d[x, y]_\alpha = [d(x), f(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$. Replacing x by $z\beta x$, we obtain $d[z\beta x, y]_\alpha = [d(z\beta x), f(y)]_\alpha$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned} d((z\beta x)\alpha y - y\alpha(z\beta x)) &= d(z\beta x)\alpha f(y) - f(y)\alpha d(z\beta x) \\ f(z\beta x)\alpha d(y) + d(z\beta x)\alpha f(y) - f(y)\alpha d(z\beta x) - d(y)\alpha f(z\beta x) &= d(z\beta x)\alpha f(y) - f(y)\alpha d(z\beta x) \\ f(z)\beta[f(x), d(y)]_\alpha &= 0. \end{aligned}$$

Hence $f(z)\beta[f(x), d(y)]_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show that R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since $f \neq 0$, there exists $z \in R$ such that $f(z) \neq 0$. We have

$f(z)\beta f(t)\gamma[f(x), d(y)]_\alpha = f(z)\beta[f(t\gamma x), d(y)]_\alpha = 0$ for all $t \in R$ and $\beta, \gamma \in \Gamma$.

Since f is surjective, $f(z)\Gamma R\Gamma[f(x), d(y)]_\alpha = \{0\}$.

Since R is prime and $f(z) \neq 0$, $[f(x), d(y)]_\alpha = 0$.

And since f is surjective on R, $d(y) \in Z(R)$. By Lemma 2.3, it follows that R is commutative.

(ii) Using similar techniques as above, we obtain $f(z)\beta(f(x) \circ d(y))_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since $f \neq 0$, there exists $z \in R$ such that $f(z) \neq 0$. We have

$f(z)\beta f(t)\gamma(f(x) \circ d(y))_\alpha = f(z)\beta(f(t\gamma x) \circ d(y))_\alpha = 0$ for all $t \in R$ and $\beta, \gamma \in \Gamma$.

Since f is surjective, $f(z)\Gamma R\Gamma(f(x) \circ d(y))_\alpha = \{0\}$.

Since R is prime and $f(z) \neq 0$, $(f(x) \circ d(y))_\alpha = 0 \in Z(R)$.

By Theorem 4.2(ii), it follows that R is commutative. This completes the proof.

Theorem 4.4. Let R be a nonzero prime Γ-GB-semiring such that $0\alpha a = 0$ for all $a \in R$ and $\alpha \in \Gamma$ and f, g be automorphism on R. Let D be a left Γ-(f, g) generalized derivation on R satisfying

(i) $[D(x), g(y)]_\alpha = [f(x), g(y)]_\alpha$, or

(ii) $(D(x) \circ g(y))_\alpha = (f(x) \circ g(y))_\alpha$,

for all $x, y \in R$ and $\alpha \in \Gamma$. If there exists $0 \neq z \in R$ such that $D(z) = 0$, then R is commutative.

Proof. (i) Assume that $[D(x), g(y)]_\alpha = [f(x), g(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Replacing x by $z\beta x$ we obtain $[D(z\beta x), g(y)]_\alpha = [f(z\beta x), g(y)]_\alpha$. For each $x, y, z \in R$ and $\alpha, \beta \in \Gamma$ we have

$$\begin{aligned} D(z\beta x)\alpha g(y) - g(y)\alpha D(z\beta x) &= [f(z)\beta f(x), g(y)]_\alpha \\ (f(z)\beta d(x) + D(x)\beta g(x))\alpha g(y) - g(y)\alpha(f(z)\beta d(x) + D(x)\beta g(x)) &= f(z)\beta[f(x), g(y)]_\alpha \\ f(z)\beta d(x)\alpha g(y) + D(x)\beta g(x)\alpha g(y) - g(y)\alpha f(z)\beta d(x) - g(y)\alpha D(x)\beta g(x) &= f(z)\beta[f(x), g(y)]_\alpha \\ f(z)\beta[d(x), g(y)]_\alpha + D(z)\beta[g(x), g(y)]_\alpha &= f(z)\beta[f(x), g(y)]_\alpha \\ f(z)\beta([d(x), g(y)]_\alpha - [f(x), g(y)]_\alpha) + D(z)\beta[g(x), g(y)]_\alpha &= 0. \end{aligned}$$

Hence $f(z)\beta([d(x), g(y)]_\alpha - [f(x), g(y)]_\alpha) + D(z)\beta[g(x), g(y)]_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since there exists $0 \neq z \in R$ such that $D(z) = 0$, we have

$$\begin{aligned} f(z)\beta g(t)\gamma([d(x), g(y)]_\alpha - [f(x), g(y)]_\alpha) &= f(z)\beta(g(t)\gamma[d(x), g(y)]_\alpha - g(t)\gamma[f(x), g(y)]_\alpha) \\ &= f(z)\beta([d(x), g(t\gamma y)]_\alpha - [f(x), g(t\gamma y)]_\alpha) \\ &= f(x)\beta([d(x), g(t\gamma y)]_\alpha - [f(x), g(t\gamma y)]_\alpha) \\ &\quad +D(z)\beta[g(x), g(t\gamma y)]_\alpha \\ &= 0 \end{aligned}$$

Thus $f(z)\beta g(t)\gamma([d(x), g(y)]_\alpha - [f(x), g(y)]_\alpha) = 0$ for all $t, x, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$.

Since g is surjective, $f(z)\Gamma R\Gamma([d(x), g(y)]_\alpha - [f(x), g(y)]_\alpha) = \{0\}$.

Since f is injective, $f(z) \neq 0$. And R is prime, we have $[d(x), g(y)]_\alpha - [f(x), g(y)]_\alpha = 0$.

Thus $[d(x), g(y)]_\alpha = [f(x), g(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$.

By Theorem 4.1(i), it follows that R is commutative.

(ii) Using similar techniques as above, we obtain

$f(z)\beta((d(x) \circ g(y))_\alpha - (f(x) \circ g(y))_\alpha) + D(z)\beta(g(x) \circ g(y))_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since there exists $0 \neq z \in R$ such that $D(z) = 0$, we have

$$\begin{aligned} f(z)\beta g(t)\gamma((d(x) \circ g(y))_\alpha - (f(x) \circ g(y))_\alpha) &= f(z)\beta(g(t)\gamma(d(x) \circ g(y))_\alpha - g(t)\gamma(f(x) \circ g(y))_\alpha) \\ &= f(z)\beta((d(x) \circ g(t\gamma y))_\alpha - (f(x) \circ g(t\gamma y))_\alpha) \\ &= f(x)\beta((d(x) \circ g(t\gamma y))_\alpha - (f(x) \circ g(t\gamma y))_\alpha) \\ &\quad + D(z)\beta(g(x) \circ g(t\gamma y))_\alpha \\ &= 0 \end{aligned}$$

Thus $f(z)\beta g(t)\gamma((d(x) \circ g(y))_\alpha - (f(x) \circ g(y))_\alpha) = 0$ for all $t, x, y, z \in R$ and $\alpha, \beta, \gamma \in \Gamma$.

The same argument in the proof of (i) and by Theorem 4.1(iii) we conclude that R is commutative. This completes the proof.

Theorem 4.5. Let R be a prime Γ-GB-semiring and f, g be automorphisms on R. Let D be a right Γ-(f, g) generalized derivation on R satisfying any one of the following

(i) $D[x, y]_\alpha = [f(x), g(y)]_\alpha$,

(ii) $D(x \circ y)_\alpha = (f(x) \circ g(y))_\alpha$,

(iii) $D(x \circ y)_\alpha = [f(x), g(y)]_\alpha$,

(iv) $D[x, y]_\alpha = (f(x) \circ g(y))_\alpha$,

for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is commutative.

Proof. (i) Assume that $D[x, y]_\alpha = [f(x), g(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$. Replacing x by $z\beta x$, we obtain $D[z\beta x, y]_\alpha = [f(z\beta x), g(y)]_\alpha$. For each $x, y, z \in R$ and $\alpha, \beta \in \Gamma$ we have

$$\begin{aligned} D(z\beta[x, y]_\alpha) &= [f(z)\beta f(x), g(y)]_\alpha \\ f(z)\beta D[x, y]_\alpha + d(z)\beta g[x, y]_\alpha &= f(z)\beta[f(x), g(y)]_\alpha \\ f(z)\beta(D[x, y]_\alpha - [f(x), g(y)]_\alpha) + d(z)\beta g[x, y]_\alpha &= 0 \\ d(z)\beta g[x, y]_\alpha &= 0. \end{aligned}$$

To show that R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since $d \neq 0$, there exists $z \in R$ such that $d(z) \neq 0$, we have $d(z)\beta g[x, y]_\alpha = 0$. By Lemma 2.5, it follows that R is commutative.

The proof of (ii) - (iv) are obtained similarly to that of (i).

Theorem 4.6 Let R be a prime Γ-GB-semiring and f, g be automorphisms on R. Let D be a nonzero left (resp. right) Γ-(f, g) generalized derivation on R such that

(i) $[D(x), y]_\alpha \in Z(R)$, or

(ii) $(D(x) \circ y)_\alpha \in Z(R)$,

for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is commutative.

Proof. This follows directly from Lemma 2.4.

Theorem 4.7. Let R be a prime Γ-GB-semiring such that $0\alpha a = 0$ for all $a \in R$ and $\alpha \in \Gamma$. Let f be a nonzero automorphism on R. If D is a left Γ-(f, f) generalized derivation on R such that

(i) $D[x, y]_\alpha = [D(x), f(y)]_\alpha$, or

(ii) $D(x \circ y)_\alpha = (D(x) \circ f(y))_\alpha$,

for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is commutative.

Proof. (i) Assume that $D[x, y]_\alpha = [D(x), f(y)]_\alpha$ for all $x, y \in R$ and $\alpha \in \Gamma$

Replacing x by $z\beta x$, we obtain $D[z\beta x, y]_\alpha = [D(z\beta x), f(y)]_\alpha$.

For each $x, y, z \in R$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned} D(z\beta[x,y]_\alpha) &= D(z\beta x)\alpha f(y) - f(y)\alpha D(z\beta x) \\ f(z)\beta d[x,y]_\alpha + D(z)\beta f[x,y]_\alpha &= d(z\beta x)\alpha f(y) - f(y)\alpha D(z\beta x) \end{aligned}$$

$f(z)\beta f(x)\alpha d(y)+f(z)\beta d(x)\alpha f(y)-f(z)\beta f(y)\alpha d(x)-f(z)\beta d(y)\alpha f(x)+D(z)\beta f(x)\alpha f(y)-D(z)\beta f(y)\alpha f(x) = f(z)\beta d(x)\alpha f(y)+ D(z)\beta f(x)\alpha f(y) - f(y)\alpha f(z)\beta d(x) - f(y)\alpha D(z)\beta f(x)$

so, $f(z)\beta[f(x), d(y)]_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show that R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since $f \neq 0$, there exists $z \in R$ such that $f(z) \neq 0$, we have

$f(z)\beta f(t)\gamma[f(x), d(y)]_\alpha = f(z)\beta[f(t\gamma x), d(y)]_\alpha = 0$ for all $t \in R$ and $\beta, \gamma \in \Gamma$

Since f is surjective, $f(z)\Gamma R\Gamma[f(x), d(y)]_\alpha = 0$.

Since R is prime and $f(z) \neq 0$, $[f(x), d(y)]_\alpha = 0$.

Since f is surjective, $d(y) \in Z(R)$. By Lemma 2.3, it follows that R is commutative.

(ii) Using similar techniques as above, we have, $f(z)\beta(f(x) \circ d(y))_\alpha = 0$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

To show R is commutative, let $x, y \in R$ and $\alpha \in \Gamma$. Since $f \neq 0$, there exists $z \in R$ such that $f(z) \neq 0$, we have

$f(z)\beta f(t)\gamma(f(x) \circ d(y))_\alpha = f(z)\beta(f(t\gamma x) \circ d(y))_\alpha = 0$ for all $t \in R$ and $\beta, \gamma \in \Gamma$

Since f is surjective, $f(z)\Gamma R\Gamma(f(x) \circ d(y))_\alpha = 0$.

Since R is prime and $f(z) \neq 0$, $(f(x) \circ d(y))_\alpha = 0 \in Z(R)$.

By Theorem 4.2(ii), it follows that R is commutative. This completes the proof.

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