
CHAPTER 5

POLARIZATION[†]

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GLOSSARY

c	velocity of light
d	thickness
E	electric field
\mathbf{k}	wave vector ($k = 2\pi/\lambda$)
k	extinction coefficient
m	number of reflections
N	retardation per wavelength
n	real refractive index
\tilde{n}	complex refractive index
$\hat{\mathbf{n}}$	unit normal vector
p	degree of polarization
p	parallel polarization
R	intensity reflection coefficient
r	amplitude reflection coefficient
\mathbf{r}	position vector
s	senkrecht or perpendicular polarization
t	amplitude transmission coefficient
t	time
z	cartesian coordinate

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$\alpha, \beta, a, b, c, d$	intermediate parameters
α	absorption coefficient
γ	$2\pi nd \cos \theta / \lambda$
δ	phase angle
ε	dielectric constant
η	effective refractive index
θ_B	Brewster angle
θ	angle
κ	absorption index
λ	wavelength
ρ	extinction ratio
σ	conductivity
ω	radian or angular frequency
∇	laplacian operator
0	first medium
1	second medium

The material on polarization is abridged from the much more complete treatment by Bennett and Bennett.¹ Information on polarizers is found in Volume 2, Chapter 3, “Polarizers.”

1. Basic Concepts and Conventions Optical polarization was discovered by E. L. Malus in 1808. A major triumph of nineteenth- and early twentieth-century theoretical physics was the development of electromagnetic theory and the demonstration that optical polarization is completely described by it. This theory is phenomenological in that instead of trying to explain why materials have certain fundamental characteristics, it concentrates on the resulting properties which any material with those characteristics will display. In the optical case, the polarization and all other optical properties of a material are determined by two or more phenomenological parameters called *optical constants*. Electromagnetic theory has little or nothing to say about *why* a material should have these particular optical constants or *how* they are related to its atomic character. This problem has been extensively investigated in twentieth-century solid-state physics and is still only partially understood. It is clear, however, that the optical constants are a function not only of the atomic nature of the material, i.e., its position in the periodic table, but are also quite sensitive to how it is prepared. Perhaps *optical parameters* would be a better term than optical constants. Nevertheless, the concept of optical constants is an extremely useful one and makes it possible to predict quantitatively the optical behavior of a material and, under certain conditions, to relate this behavior to nonoptical parameters.

Since the optical constants are so fundamental, differences in their definition are particularly unfortunate. The most damaging of these differences arises from an ambiguity in the initial derivation. Maxwell's equations, which form the basis of electromagnetic theory, result in the wave equation, which in mks units is

$$\nabla^2 \mathbf{E} = \frac{\varepsilon}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi\sigma}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (1)$$

where ∇^2 = laplacian operator

\mathbf{E} = electric field vector of traveling wave

t = time

c = velocity of light

σ = conductivity of material at frequency of wave motion

ε = dielectric constant of material at frequency of wave motion

A solution to this equation is

$$\mathbf{E} = \mathbf{E}_0 \exp [i(\omega t + \delta)] \exp (-i\mathbf{k} \cdot \mathbf{r}) \exp \left(-\frac{\alpha z}{2} \right) \quad (2)$$

where \mathbf{E}_0 = amplitude of wave

ω = angular frequency of wave

δ = phase vector

\mathbf{k} = wave vector

\mathbf{r} = position vector

z = direction wave is traveling

α = absorption coefficient

The wave vector \mathbf{k} is assumed to be real and equal to $(2\pi/\lambda_m)\hat{\mathbf{n}}$, where λ_m is the wavelength in the medium in which the wave is traveling and \mathbf{n} is a unit vector in the k direction.[†] Equation (2) can also be written in terms of \tilde{n} , the complex index of refraction, defined as

$$\tilde{n} = n - ik \quad (3)$$

where n is the index of refraction and k the extinction coefficient. In this form, Eq. (2) is

$$E = E_0 \exp \left[i\omega \left(t - \frac{\tilde{n}z}{c} \right) \right] \quad (4)$$

when $\delta = 0$. By setting the imaginary part of the exponent equal to zero one obtains

$$z = \frac{c}{n} t \quad (5)$$

To show that Eq. (4) represents a wave traveling in the positive z direction with phase velocity c/n , we note that the phase ϕ of the wave in Eq. (4) is $\omega t - (\omega \tilde{n} z)/c = \phi$. For a wave propagating with a constant phase, $d\phi = 0$, so that $\omega dt - (\omega \tilde{n}/c) dz = d\phi = 0$, and hence the phase velocity $v_p = dz/dt = c/n$.² The amplitude of the wave at z is, from Eq. (4),

$$|E| = E_0 e^{-2\pi k z / \lambda} \quad (6)$$

where λ is the wavelength in vacuum. The wave is thus exponentially damped, and the amplitude penetration depth, or distance below an interface at which the *amplitude* of the wave falls to $1/e$ times its initial value, is $z = \lambda/2\pi k$. The absorption coefficient α , or

[†] Frequently the wave vector is taken to be complex, that is, $\tilde{\mathbf{k}} = (2\pi/\lambda_m - i\alpha/2)\mathbf{n}$, and Eq. (2) is written $\mathbf{E} = \mathbf{E}_0 \exp [i(\omega t + \delta)] \exp (-i\tilde{\mathbf{k}} \cdot \mathbf{r})$.

the reciprocal of the distance in which the *intensity* of the wave falls to $1/e$ times its initial value, is

$$\alpha = \frac{4\pi k}{\lambda} \quad (7)$$

This development follows that commonly given by those working at optical or radio frequencies. The confusion in the definition of the optical constants arises because an equally valid solution to Eq. (1) is

$$E' = E_0 \exp \left[-i\omega \left(t - \frac{\tilde{n}'z}{c} \right) \right] \quad (8)$$

which also represents an exponentially damped wave traveling in the $+z$ direction *provided that the complex index of refraction is defined to be*

$$\tilde{n}' = n + ik \quad (9)$$

where the primes indicate the alternative solution. When the wave equation arises in quantum mechanics, the solution chosen is generally the negative exponential, i.e. Eq. (8) rather than Eq. (4). Solid-state physicists working in optics thus often define the complex index of refraction as the form given in Eq. (9) rather than that in Eq. (3). Equally valid, self-consistent theories can be built up using either definition, and as long as only intensities are considered, the resulting expressions are identical. However, when phase differences are calculated, the two conventions usually lead to contradictory results. Even worse, an author who is not extremely careful may not consistently follow either convention, and the result may be pure nonsense. Some well-known books might be cited in which the authors are not even consistent from chapter to chapter.

There are several other cases in optics in which alternative conventions are possible and both are found in the literature. Among these, the most distressing are the use of a left-handed rather than a right-handed coordinate system, which makes the p and s components of polarized light have the same phase change at normal incidence (see Par. 2), and defining the optical constants so that they depend on the angle of incidence, which makes the angle of refraction given by Snell's law real for an absorbing medium. There are many advantages to be gained by using a single set of conventions in electromagnetic theory. In any event, an author should *clearly* state the conventions being used and then *stay with them*.

Finally, the complex index of refraction is sometimes written

$$\tilde{n} = n(1 - i\kappa) \quad (10)$$

In this formulation the symbol κ is almost universally used instead of k , which is reserved for the imaginary part of the refractive index. Although k is more directly related to the absorption coefficient α than κ [see Eq. (7)] and usually makes the resulting expressions slightly simpler, in areas such as attenuated total reflection the use of κ results in a simplification. To avoid confusion between k and κ , if Eq. (10) is used, κ could be called the *absorption index* to distinguish it from the extinction coefficient k , and the absorption coefficient α .

2. Fresnel Equations The Fresnel equations are expressions for the reflection and transmission coefficients of light at nonnormal incidence. In deriving these equations, the coordinate system assumed determines the signs in the equations and therefore the phase changes on reflection of the p and s components. In accordance with the Muller

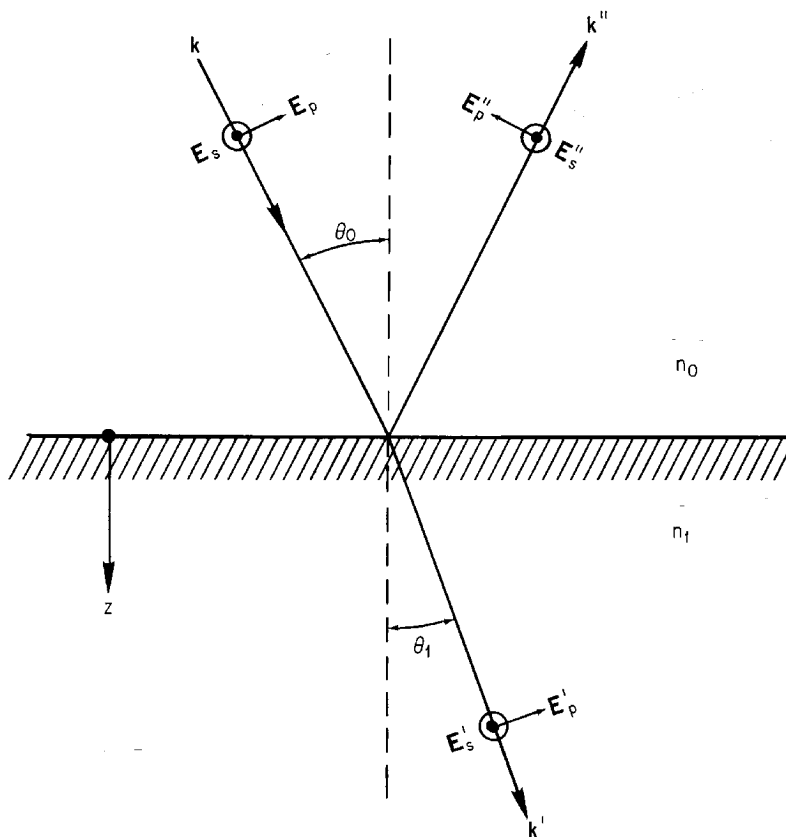


FIGURE 1 Coordinate system for measuring the E vectors of a plane wave reflected and refracted at a boundary between a medium of refractive index n_0 and a medium of refractive index n_1 (may be absorbing). The positive direction for the coordinates of the E_s , E'_s , and E''_s components is out of the paper, and that for the coordinates of the E_p components is in the plane of the paper, as indicated by the arrows. The wave vector \mathbf{k} , the direction the wave is traveling z , and angles of incidence and refraction θ_0 and θ_1 are also shown. [Modified from Muller, Ref. 3.]

convention,³ we shall assume that the coordinate system is as shown in Fig. 1. In this system, the angle of incidence is θ_0 , and the angle of refraction is θ_1 . The s component of polarization is the plane of vibration of the E wave which is perpendicular to the plane of the paper, and the p component is the plane of vibration which is in the plane of the paper.[†] (The plane of incidence is in the plane of the paper.) The positive directions for the vibrations are indicated in Fig. 1 by the dots for E_s , E'_s , and E''_s and by the arrows for the corresponding p components. Note that the positive direction for E''_p is as shown in the

[†] Unfortunately, when Malus discovered that light reflected at a certain angle from glass is, as he said, “polarized,” he defined the plane of polarization” of the reflected light as the plane of incidence. Since the reflected light in this case has its E vector perpendicular to the plane of incidence, the “plane of polarization” is perpendicular to the plane in which the E vector vibrates. This nomenclature causes considerable confusion and has been partially resolved in modern terminology by discussing the *plane of vibration* of the E vector and avoiding, insofar as possible, the term *plane of polarization*. In this chapter, when specifying the direction in which light is polarized, we shall give the direction of vibration, *not* the direction of polarization.

figure because of the *mirror-image effect*. By convention, one always looks *against the direction of propagation of the wave* so that the positive direction of E_p is to the right and the positive direction of E_p'' is also to the right. The positive directions of the reflected E vectors are not the same as the actual directions of the reflected E vectors. These latter directions will depend on the refractive index of the material and may be either positive or negative. For example, if $n_1 > n_0$, at normal incidence E_s'' will be in the negative direction and E_p'' will be in the positive direction. Thus we say that there is a phase change on reflection of 180° for the s wave and a phase change of 0° for the p wave.

With this coordinate system, the Fresnel amplitude reflection coefficients for a single interface, obtained from Eq. (4) by setting up and solving the boundary-value problem, can be written

$$\frac{E_s''}{E_s} \equiv r_s = \frac{n_0 \cos \theta_0 - n_1 \cos \theta_1}{n_0 \cos \theta_0 + n_1 \cos \theta_1} \quad (11)$$

and

$$\frac{E_p''}{E_p} \equiv r_p = \frac{n_1 \cos \theta_0 - n_0 \cos \theta_1}{n_1 \cos \theta_0 + n_0 \cos \theta_1} \quad (12)$$

The amplitude transmission coefficients are

$$\frac{E_s'}{E_s} \equiv t_s = \frac{2n_0 \cos \theta_0}{n_0 \cos \theta_0 + n_1 \cos \theta_1} \quad (13)$$

and

$$\frac{E_p'}{E_p} \equiv t_p = \frac{2n_0 \cos \theta_0}{n_1 \cos \theta_0 + n_0 \cos \theta_1} \quad (14)$$

Other forms of the Fresnel amplitude reflection and transmission coefficients containing only the angles of incidence and refraction are somewhat more convenient. These relations can be derived using Snell's law

$$\frac{\sin \theta_0}{\sin \theta_1} = \frac{n_1}{n_0} \quad (15)$$

to eliminate n_0 and n_1 from Eqs. (1) to (14):

$$r_s = \frac{-\sin (\theta_0 - \theta_1)}{\sin (\theta_0 + \theta_1)} \quad (16)$$

$$r_p = \frac{\tan (\theta_0 - \theta_1)}{\tan (\theta_0 + \theta_1)} \quad (17)$$

$$t_s = \frac{2 \sin \theta_1 \cos \theta_0}{\sin (\theta_0 + \theta_1)} \quad (18)$$

$$t_p = \frac{2 \sin \theta_1 \cos \theta_1}{\sin (\theta_0 + \theta_1) \cos (\theta_0 - \theta_1)} \quad (19)$$

For nonabsorbing materials the intensity reflection coefficients R_s and R_p are simply the squares of Eqs. (16) and (17):

$$R_s = r_s^2 = \frac{\sin^2(\theta_0 - \theta_1)}{\sin^2(\theta_0 + \theta_1)} \quad (20)$$

$$R_p = r_p^2 = \frac{\tan^2(\theta_0 - \theta_1)}{\tan^2(\theta_0 + \theta_1)} \quad (21)$$

and, at normal incidence,

$$R_s = R_p = \frac{(n_0 - n_1)^2}{(n_0 + n_1)^2} \quad (22)$$

from Eqs. (11) and (12). In the lower part of Fig. 2, R_s and R_p are given as a function of angle of incidence for various values of the refractive-index ratio n_1/n_0 with k_1 for the material equal to zero. The curves for $n_1/n_0 = 1.3, 1.8,$ and 2.3 show that the normal-incidence reflectance increases as n_1 increases. The curves for $n_1/n_0 = 0.3$ and 0.8 and $k_1 = 0$ have no physical significance as long as the incident medium is air. However,

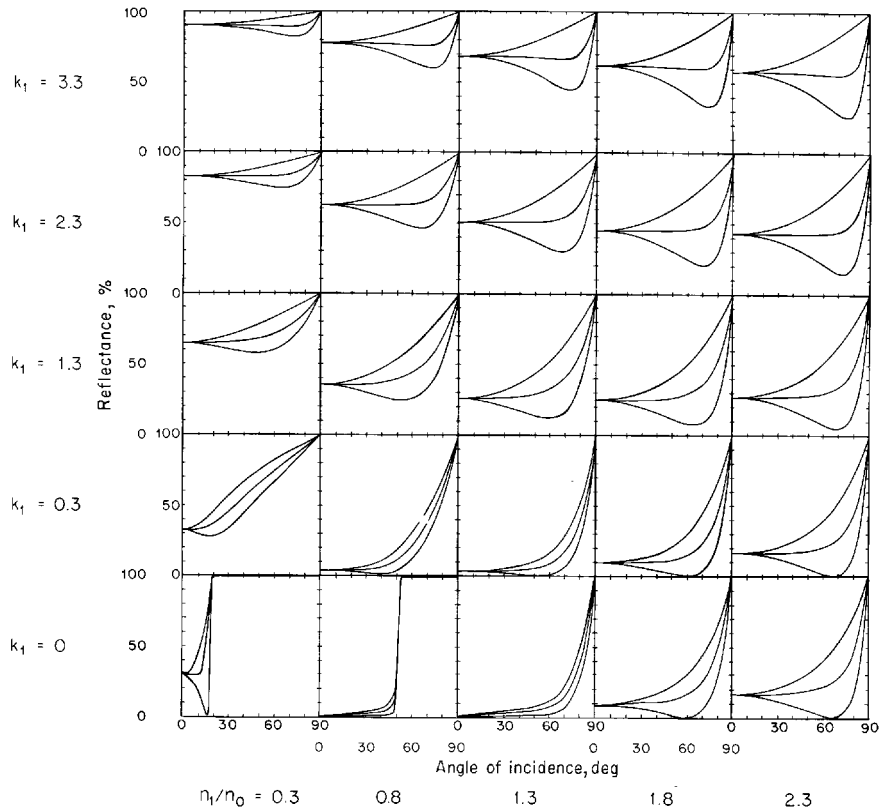


FIGURE 2 R_s (upper curves), R_p (lower curves), and $R_{av} = (R_s + R_p)/2$ as a function of angle of incidence for various values of the refractive-index ratio n_1/n_0 and k_1 . The incident medium, having refractive index n_0 , is assumed to be nonabsorbing. [Modified from Hunter, Ref. 4.]

they are representative of *internal reflections* in materials of refractive index $n_0 = 3.33$ and 1.25, respectively, when the *other* medium is air ($n_1 = 1$).

The intensity transmission coefficients T_s and T_p are obtained from the Poynting vector and for nonabsorbing materials are

$$T_s = 1 - R_s = \frac{n_1 \cos \theta_1}{n_0 \cos \theta_0} t_s^2 = \frac{4n_0 n_1 \cos \theta_0 \cos \theta_1}{(n_0 \cos \theta_0 + n_1 \cos \theta_1)^2} = \frac{\sin 2\theta_0 \sin 2\theta_1}{\sin^2 (\theta_0 + \theta_1)} \quad (23)$$

$$\begin{aligned} T_p = 1 - R_p &= \frac{n_1 \cos \theta_1}{n_0 \cos \theta_0} t_p^2 = \frac{4n_0 n_1 \cos \theta_0 \cos \theta_1}{(n_1 \cos \theta_0 + n_0 \cos \theta_1)^2} \\ &= \frac{\sin 2\theta_0 \sin 2\theta_1}{\sin^2 (\theta_0 + \theta_1) \cos^2 (\theta_0 - \theta_1)} \end{aligned} \quad (24)$$

These coefficients are for light passing through a single boundary and hence are of limited usefulness. In actual cases, the light is transmitted through a slab of material where there are two boundaries, generally multiple reflections within the material, and sometimes interference effects when the boundaries are smooth and plane-parallel.

The intensity transmission coefficient T_{sample} for a slab of transparent material in air is given by the well-known Airy equation⁵ when the sample has smooth, plane-parallel sides and coherent multiple reflections occur within it:

$$T_{\text{sample}} = \frac{1}{1 + [4R_{s,p}/(1 - R_{s,p})^2] \sin^2 \gamma} \quad (25)$$

where

$$\gamma = \frac{2\pi n_1 d \cos \theta_1}{\lambda} \quad (26)$$

The values of R_s and R_p can be determined from Eqs. (20) to (22); d is the sample thickness, λ the wavelength, n_1 the refractive index of the material, and θ_1 the angle of refraction. Equation (25) holds for all angles of incidence including the Brewster angle, where $R_p = 0$ [see Eq. (48)]. The Airy equation predicts that at a given angle of incidence the transmission of the sample will vary from a maximum value of 1 to a minimum value of $(1 - R_{s,p})^2 / (1 + R_{s,p})^2$ as the wavelength or the thickness is changed. If the sample is very thick, the oscillations in the transmittance will occur at wavelengths very close together and hence will be unresolved. A complete oscillation occurs every time γ changes by π , so that the wavelength interval $\Delta\lambda$ between oscillations is

$$\Delta\lambda \approx \frac{\lambda^2}{2n_1 d \cos \theta_1} \quad (27)$$

An example, a sample 1 mm thick with an index of 1.5 at 5000 Å will have transmission maxima separated by 0.83 Å when measured at normal incidence ($\cos \theta_1 = 1$). These maxima would not be resolved by most commercial spectrophotometers. In such a case, one would be measuring the average transmission $T_{\text{sample,av}}$:

$$T_{\text{sample,av}} = \frac{1 - R_{s,p}}{1 + R_{s,p}} \quad (28)$$

For nonabsorbing materials, this is the same value as that which would be obtained if the multiply reflected beams did not coherently interfere within the sample. If the sample is wedge-shaped, so that no multiply reflected beams contribute to the measured transmittance, T_{sample} is simply T_s^2 or T_p^2 and can be calculated from Eq. (23) or (24).

When the material is absorbing, i.e., has a complex refractive index, it is not so easy to

calculate the reflectance and transmittance since the angle of refraction is complex. However, Snell's law [Eq. (15)] and Fresnel's equations (11) and (12) are sometimes used with complex values of n_1 and θ_1 . The resulting amplitude reflection coefficients are written

$$r_s = |r_s| e^{i\delta_s} \quad (29)$$

and

$$r_p = |r_p| e^{i\delta_p} \quad (30)$$

where $|r_s|$ and $|r_p|$ are the magnitudes of the reflectances and δ_s and δ_p are the phase changes on reflection. The intensity reflection coefficients are

$$R_{s,p} = r_{s,p} r_{s,p}^* \quad (31)$$

An alternative approach is to use the method of effective indexes to calculate R_s and R_p . In the medium of incidence, which is assumed to be nonabsorbing, the effective indexes η_{0s} and η_{0p} for the s and p components are

$$\eta_{0s} = n_0 \cos \theta_0 \quad (32)$$

$$\eta_{0p} = \frac{n_0}{\cos \theta_0} \quad (33)$$

where n_0 generally equals 1 for air. In the absorbing material both η 's are complex and can be written, according to the Bernings,^{6,7}

$$\tilde{\eta}_{1s} = \tilde{n}_1 \cos \theta_1 \quad (34)$$

$$\tilde{\eta}_{1p} = \frac{\tilde{n}_1}{\cos \theta_1} \quad (35)$$

where $\tilde{n}_1 = n_1 - ik_1$ is the complex refractive index of the material, and

$$\cos \theta_1 = \left[\frac{(\alpha_1^2 + \beta_1^2)^{1/2} + \alpha_1}{2} \right]^{1/2} - i \left[\frac{(\alpha_1^2 + \beta_1^2)^{1/2} - \alpha_1}{2} \right]^{1/2} \quad (36)$$

$$\alpha_1 = 1 + \left(\frac{n_0 \sin \theta_0}{n_1^2 + k_1^2} \right)^2 (k_1^2 - n_1^2) \quad (37)$$

and

$$\beta_1 = -2n_1 k_1 \left(\frac{n_0 \sin \theta_0}{n_1^2 + k_1^2} \right)^2 \quad (38)$$

Abelès' method⁸ also uses effective indexes for the absorbing material, but they are calculated differently:

$$\tilde{\eta}_{1s} = a - ib \quad (39)$$

$$\tilde{\eta}_{1p} = c - id \quad (40)$$

where

$$a^2 - b^2 = n_1^2 - k_1^2 - n_0^2 \sin^2 \theta_0 \quad (41)$$

$$ab = n_1 k_1 \quad (42)$$

$$c = a \left(1 + \frac{n_0^2 \sin^2 \theta_0}{a^2 + b^2} \right) \quad (43)$$

$$d = b \left(1 - \frac{n_0^2 \sin^2 \theta_0}{a^2 + b^2} \right) \quad (44)$$

In both methods employing effective indexes, the amplitude reflection coefficients are

$$r_s = \frac{\eta_{0s} - \eta_{1s}}{\eta_{0s} + \eta_{1s}} \quad (45)$$

$$r_p = \frac{\eta_{1p} - \eta_{0p}}{\eta_{1p} + \eta_{0p}} \quad (46)$$

which are equivalent to Eqs. (29) and (30) and reduce to Eqs. (11) and (12) when $k_1 = 0$. The intensity reflection coefficients are given by Eq. (31), as before. At normal incidence,

$$R_s = R_p = \frac{(n_0 - n_1)^2 + k_1^2}{(n_0 + n_1)^2 + k_1^2} \quad (47)$$

Values of R_s and R_p are plotted as a function of angle of incidence in Fig. 2 for various values of n_1 and k_1 . (The incident medium is assumed to be air with $n_0 = 1$ unless otherwise noted.) As n_1 increases with $k_1 > 0$ held constant, the magnitudes of R_s and R_p at normal incidence both decrease. As k_1 increases with n_1 held constant, the magnitudes of R_s and R_p at normal incidence both increase. Tables of R_s and R_p for various values of n_1 and k_1 are given for angles of incidence from 0 to 85° by Holl.⁹

The absolute phase changes on reflection δ_s and δ_p are also of interest in problems involving polarization. When the material is nonabsorbing, the phase changes can be determined from the amplitude reflection coefficients, Eqs. (11) and (12); when $\theta_0 = 0$ and $n_1 > n_0$, $\delta_s = 180^\circ$ and $\delta_p = 360^\circ$.[†] This is an apparent contradiction since at normal incidence the s and p components should be indistinguishable. However, the problem is resolved by recalling that by convention we are always looking against the direction of propagation of the light (see Fig. 1). To avoid complications, the phase change on reflection at normal incidence (often defined as β) is identified with δ_s .

For a dielectric, if $n_0 < n_1$, δ_s remains 180° for all angles of incidence from 0 to 90° , as can be seen from the numerator of Eq. (11). However, there is an abrupt discontinuity in δ_p , as can be seen from Eq. (12). If $n_0 < n_1$, $\delta_p = 360^\circ$ at normal incidence and at larger angles for which the numerator of Eq. (12) is positive. Since $\cos \theta_0$ becomes increasingly less than $\cos \theta_1$ as θ_0 increases, and since $n_1 > n_0$, there will be an angle for which $n_1 \cos \theta_0 = n_0 \cos \theta_1$. At this angle δ_p undergoes an abrupt change from 360 to 180° , and it remains 180° for larger angles of incidence. At the transition value of θ_0 , which is called the *Brewster angle* θ_B since $R_p = 0$,

$$\tan \theta_B = \frac{n_1}{n_0} \quad (48)$$

(This angle is also called the *polarizing angle* since $\theta_0 + \theta_1 = 90^\circ$.)

The phase changes δ_s and δ_p are not simply 360 or 180° for an absorbing material. At normal incidence it follows from Eq. (45) that

$$\tan \delta_s = \frac{2n_0k_1}{n_0^2 - n_1^2 - k_1^2} \quad (49)$$

[†] Since 360° and 0° are indistinguishable, many optics books state that $\delta_p = 0^\circ$ for dielectrics at normal incidence, but this makes the ellipsometric parameter $\Delta = \delta_p - \delta_s < 0$, which is incompatible with ellipsometric conventions—see the section on Ellipsometry.

so that $\delta_s = 180^\circ$ only if $k_1 = 0$. As before, $\delta_p = \delta_s + 180^\circ$, as seen by comparing Eqs. (45) and (46). At nonnormal incidence

$$\tan \delta_s = \frac{2\eta_{0s}b}{\eta_{0s}^2 - a^2 - b^2} \quad (50)$$

and

$$\tan \delta_p = \frac{-2\eta_{0p}d}{c^2 + d^2 - \eta_{0p}^2} \quad (51)$$

where the relations for a , b , c , and d have been given in Eqs. (41) to (44). The following relations between these quantities may also prove helpful:

$$a^2 + b^2 = [(n_1^2 - k_1^2 - n_0^2 \sin^2 \theta_0)^2 + 4n_1^2 k_1^2]^{1/2} \quad (52)$$

$$c^2 + d^2 = \frac{(n_1^2 + k_1^2)^2}{a^2 + b^2} \quad (53)$$

$$b^2 = \frac{n_1^2 - k_1^2 - n_0^2 \sin^2 \theta_0}{2} + \frac{a^2 + b^2}{2} \quad (54)$$

Figure 3 shows how δ_s and δ_p change as a function of angle of incidence for an absorbing material. At normal incidence they are 180° apart because of the mirror-image effect, mentioned previously. As the angle of incidence increases, δ_p approaches δ_s , and at the *principal angle* $\bar{\theta}$ the two quantities differ by only 90° . At grazing incidence they coincide.

The reflectance R_p does not reach zero for an absorbing material as it does for a dielectric, but the angle for which it is a minimum is called the *pseudo Brewster angle* θ_B^* . Two other angles closely associated with the pseudo Brewster are also of interest. The angle for which the ratio R_p/R_s is a minimum is sometimes called the *second Brewster*

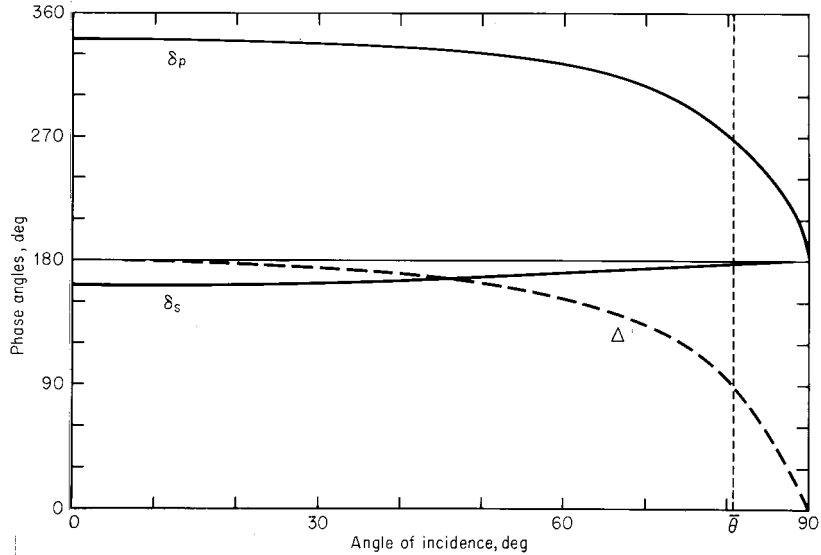


FIGURE 3 Phase changes on reflection δ_s and δ_p and phase difference $\Delta = \delta_p - \delta_s$ as a function of angle of incidence for an absorbing material. The principal angle, for which $\Delta = 90^\circ$, is also shown. [Bennett and Bennett, Ref. 10.]

angle. It is generally only slightly larger than θ'_B . The *principal angle* $\bar{\theta}$, at which $\delta_p - \delta_s = 90^\circ$, is always larger than the second Brewster angle and θ'_B . For most metals θ'_B and $\bar{\theta}$ are only a fraction of a degree apart, but it is possible for them to differ by as much as 45° .⁹ There is no polarizing angle as such for an absorbing material because the angle of refraction is complex.

3. Basic Relations for Polarizers A linear[†] polarizer is anything which when placed in an incident unpolarized beam produces a beam of light whose electric vector is vibrating primarily in one plane, with only a small component vibrating in the plane perpendicular to it. If a polarizer is placed in a plane-polarized beam and is rotated about an axis parallel to the beam direction, the transmittance T will vary between a maximum value T_1 and a minimum value T_2 according to the law

$$T = (T_1 - T_2) \cos^2 \theta + T_2 \quad (55)$$

Although the quantities T_1 and T_2 are called the *principal transmittances*, in general $T_1 \gg T_2$; θ is the angle between the plane of the principal transmittance T_1 and the plane of vibration (of the electric vector) of the incident beam. If the polarizer is placed in a beam of unpolarized light, its transmittance is

$$T = \frac{1}{2}(T_1 + T_2) \quad (56)$$

so that a perfect polarizer would transmit only 50 percent of an incident unpolarized beam.[‡]

When two identical polarizers are placed in an unpolarized beam, the resulting transmittance will be

$$T_{\parallel} = \frac{1}{2}(T_1^2 + T_2^2) \quad (57)$$

when their principal transmittance directions are parallel and will be

$$T_{\perp} = T_1 T_2 \quad (58)$$

when they are perpendicular. In general, if the directions of principal transmittance are inclined at an angle θ to each other, the transmittance of the pair will be

$$T_{\theta} = \frac{1}{2}(T_1^2 + T_2^2) \cos^2 \theta + T_1 T_2 \sin^2 \theta \quad (59)$$

The polarizing properties of a polarizer are generally defined in terms of its *degree of polarization* P ^{§,||}

$$P = \frac{T_1 - T_2}{T_1 + T_2} \quad (60)$$

or its *extinction ratio* ρ_p

$$\rho_p = \frac{T_2}{T_1} \quad (61)$$

When one deals with nonnormal-incidence reflection polarizers, one generally writes P

[†] Circular polarizers are discussed in Par. 6.

[‡] Jones¹¹ has pointed out that a perfect polarizer can transmit more than 50 percent of an incident unpolarized beam under certain conditions.

[§] Bird and Shurcliff¹² distinguish between *degree of polarization*, which is a constant of the light beam, and *polarizance*, which is a constant of the polarizer. The polarizance is defined as being equal to the degree of polarization the polarizer produces in an incident monochromatic beam that is unpolarized. In practice, incident beams are often slightly polarized, so that the polarizance values differ slightly from the ideal degree of polarization. Other authors have not followed this distinction.

^{||} Authors dealing with topics such as scattering from aerosols sometimes define *degree of polarization* (of the scattered light) in terms of the Stokes vectors (Par. 7) as $P = (S_1^2 + S_2^2 + S_3^2)^{1/2} / S_0$.

and ρ_p in terms of R_p and R_s , the reflectances of light polarized parallel and perpendicular to the plane of incidence, respectively. As will be shown in Par. 4, R_s can be equated to T_1 and R_p to T_2 , so that Eqs. (60) and (61) become $P = (R_s - R_p)/(R_s + R_p)$ and $\rho_p = R_p/R_s$. If either ρ_p or P is known, the other can be deduced since

$$P = \frac{1 - \rho_p}{1 + \rho_p} \quad (62)$$

and

$$\rho_p = \frac{1 - P}{1 + P} \quad (63)$$

If one is determining the degree of polarization or the extinction ratio of a polarizer, the ratio of T_\perp to T_\parallel can be measured for two identical polarizers in unpolarized light. From Eqs. (57) and (58),

$$\frac{T_\perp}{T_\parallel} = \frac{T_1 T_2}{(T_1^2 + T_2^2)/2} \approx \frac{2T_2}{T_1} = 2\rho_p \quad (64)$$

if $T_2^2 \ll T_1^2$. If a perfect polarizer or a source of perfectly plane-polarized light is available, T_2/T_1 can be determined directly by measuring the ratio of the minimum to the maximum transmittance of the polarizer. Other relations for two identical partial polarizers are given by West and Jones,¹³ as well as the transmittance $T_{\theta ab}$ of two dissimilar partial polarizers a and b whose principal axes are inclined at an angle θ with respect to each other. This latter expression is

$$T_{\theta ab} = \frac{1}{2}(T_{1a}T_{1b} + T_{2a}T_{2b}) \cos^2 \theta + \frac{1}{2}(T_{1a}T_{2b} + T_{1b}T_{2a}) \sin^2 \theta \quad (65)$$

where the subscripts 1 and 2 refer to the principal transmittances, as before.

Spectrophotometric measurements can involve polarizers and dichroic samples. Dichroic (optically anisotropic) materials are those which absorb light polarized in one direction more strongly than light polarized at right angles to that direction. (Dichroic materials are to be distinguished from birefringent materials, which may have different refractive indices for the two electric vectors vibrating at right angles to each other but similar, usually negligible, absorption coefficients.) When making spectrophotometric measurements, one should know the degree of polarization of the polarizer and how to correct for instrumental polarization. This latter quantity may arise from nonnormal-incidence reflections from a grating, dispersing prism, or mirrors. Light sources are also sometimes polarized. Simon,¹⁴ Charney,¹⁵ Gonatas *et al.*,¹⁶ and Wizinowich¹⁷ suggest methods for dealing with imperfect polarizers, dichroic samples, and instrumental polarization. In addition, when a dichroic sample is placed between a polarizer and a spectrophotometer which itself acts like an imperfect polarizer, one has effectively three polarizers in series. This situation has been treated by Jones,¹⁸ who showed that anomalies can arise when the phase retardation of the polarizers takes on certain values. Mielenz and Eckerle¹⁹ have discussed the accuracy of various types of polarization attenuators.

4. Polarization by Nonnormal-Incidence Reflection (Pile of Plates) Pile-of-plates polarizers make use of reflection or transmission of light at nonnormal incidence, frequently near the Brewster or polarizing angle [Eq. (48) in Par. 2]. The extinction ratio and "transmittance" of these polarizers can be calculated directly from the Fresnel equations. Some simplifications occur for nonabsorbing or slightly absorbing plates. Equations (20) and (21) give the values of the intensity reflection coefficients R_s and R_p for light vibrating perpendicular to the plane of incidence (s component) and parallel to the plane of incidence (p component). The angle of refraction θ_1 in those equations is related

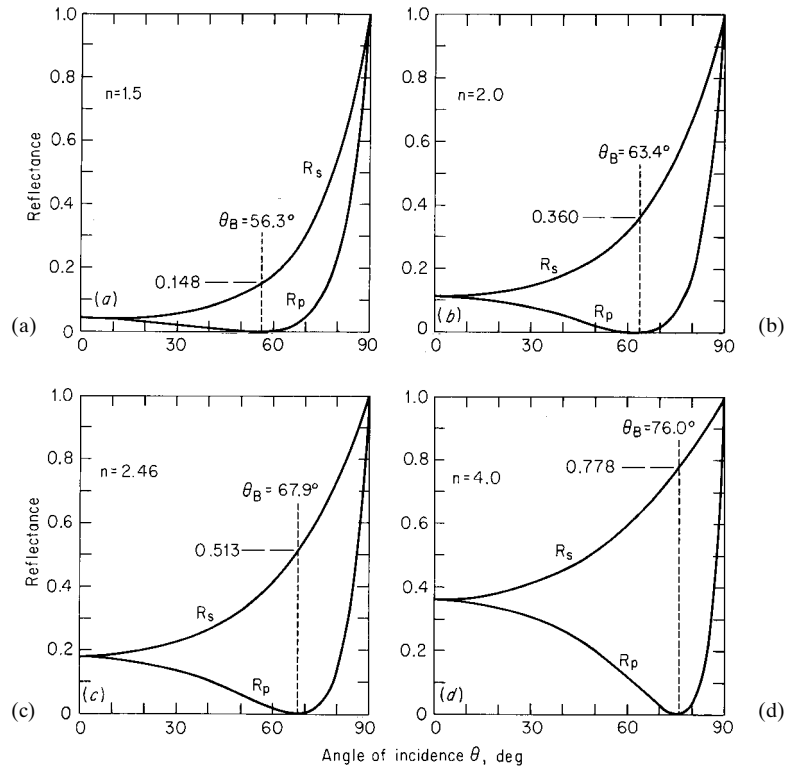


FIGURE 4 Reflectance of light polarized parallel R_p and perpendicular R_s to the plane of incidence from materials of different refractive index n as a function of angle of incidence: (a) $n = 1.5$ (alkali halides in ultraviolet and sheet plastics in infrared), (b) $n = 2.0$ (AgCl in infrared), (c) $n = 2.46$ (Se in infrared), and (d) $n = 4.0$ (Ge in infrared). The Brewster angle θ_B (at which R_p goes to 0) and the magnitude of R_s at θ_B are also indicated.

to the refractive index n of the material by Snell's law [Eq. (15)[†]]. At the Brewster angle $R_p = 0$, so that the reflected light is, in principle, completely plane-polarized. This is the basis for all Brewster angle reflection polarizers.

Let us now see how the characteristics of a reflection polarizer depend on its refractive index. In Fig. 4 the reflectances R_s and R_p have been plotted for different values of the refractive index, roughly representing alkali halides in the ultraviolet and sheet-plastic materials, silver chloride, selenium, and germanium in the infrared. The Brewster angle, given by Eq. (48), is also indicated, as well as the magnitude of R_s at the Brewster angle. We note from these graphs that if light is polarized by a single reflection from a nonabsorbing material, the polarizer with the highest refractive index will have the largest throughput. In reflection polarizers, the quantity R_s is essentially the principal "transmittance" of the polarizer [T_i in Eqs. (55) to (65)] except that it must be multiplied by the reflectance of any other mirrors used to return the beam to its axial position.

The reflectance R_p can be equated to T_2 , the minimum "transmittance" of the polarizer, so that the extinction ratio ρ_p of a reflection polarizer [Eq. (61)] is $\rho_p = R_p/R_s$. If R_p is

[†] Since we are assuming that the medium of incidence is air, $n_0 = 1$ and $n_1 = n$, the refractive index of the material.

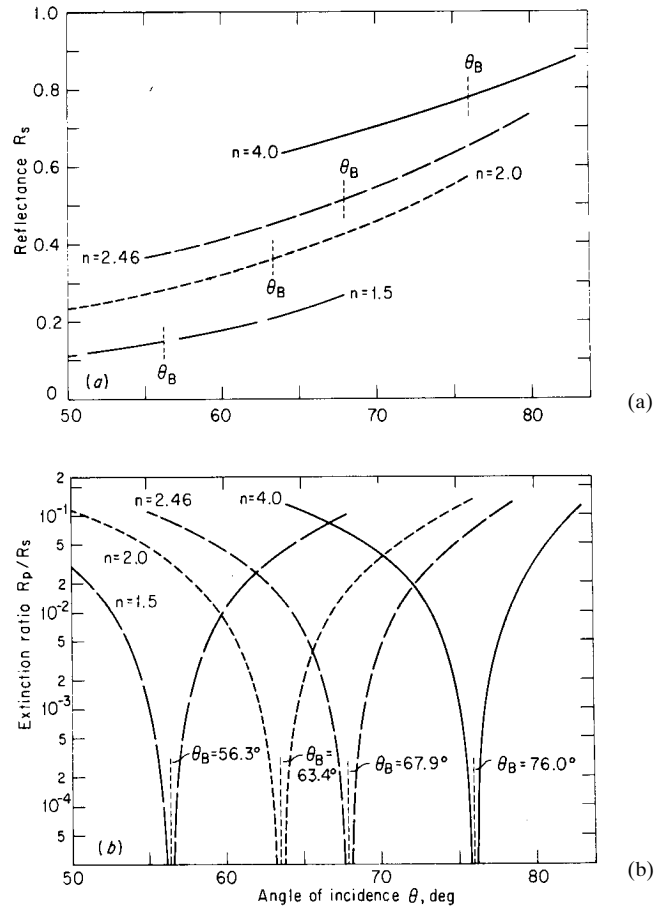


FIGURE 5 (a) Reflectance R_s and (b) extinction ratio R_p/R_s for materials of different refractive index at angles near the Brewster angle θ_B . A single surface of the material is assumed.

really zero at the Brewster angle, the extinction ratio will be zero for all materials independent of the value of n . If a given extinction ratio is desired, for example, 10^{-3} [corresponding to 99.8 percent polarization; see Eq. (62)], then the convergence angle of the light beam must be small so that all the angles of incidence lie within about $\pm 1^\circ$ of the Brewster angle. The convergence angle depends only weakly on the refractive index for this case, varying from $\pm 1.2^\circ$ for $n = 1.5$ to $\pm 0.8^\circ$ for $n = 4.0$.

If a good extinction ratio is required for a beam of larger convergence angle, two polarizing reflections may be used. Then all the exponents in Fig. 5b are doubled, and the convergence angles for a given extinction ratio are greatly increased. To obtain a value of 10^{-3} with two reflections, the angle of incidence must be within about $\pm 6^\circ$ of the Brewster angle for values of n less than 3.5; for $n = 4$ it is reduced slightly and becomes more asymmetric ($+4.0$ and -5.2°). A disadvantage of having two reflections from the polarizing materials is that the throughput is reduced. All the values of R_s in Fig. 5a are squared, so that for $n = 4$, $R_s = 0.78$ but $R_s^2 = 0.61$; for smaller refractive indexes the reduction in throughput is much greater.

The information shown graphically in Figs. 4 and 5 is given analytically in a paper by Azzam²⁰ who is concerned about the angular sensitivity of Brewster-angle reflection polarizers, particularly those made with silicon or germanium plates. Also, Murty and Shukla²¹ show analytically that the shadowy extinction patterns sometimes seen with a crossed Brewster angle reflection polarizer and analyzer are caused by light incident on the surfaces at angles different from the Brewster angle.

Although in many cases multiple reflections within a plate degrade its polarizing properties, this is not true for Brewster angle reflection polarizers. For multiple reflections within a plane-parallel plate of material

$$(R_{s,p})_{\text{plate}} = \frac{2R_{s,p}}{1 + R_{s,p}} \quad (66)$$

assuming no interference or absorption; R_s and R_p are given by Eqs. (20) and (21). Multiple reflections have a minor effect on the extinction ratio but the increase in R_s is appreciable. To fulfill the conditions of Eq. (66), the plate must have plane-parallel sides and be unbacked. We are also assuming that the plate is thick or nonuniform enough for interference effects within it to be neglected.

All the preceding discussion applies only to nonabsorbing materials. If a small amount of absorption is present, R_p will have a minimum that is very close to zero and the material will still make a good reflection polarizer. However, if the extinction coefficient k becomes appreciable, the minimum in R_p will increase and the polarizing efficiency will be degraded. By referring to Fig. 2 one can see roughly what the ratio of R_p to R_s will be for a given set of optical constants. Exact values of R_p and R_s can be calculated from n and k using Eqs. (45), (46), (31), and the other pertinent relations in Par. 2. When choosing materials for possible use as metallic reflection polarizers, one wants the largest difference between R_s and R_p and the smallest magnitude of R_p at the minimum. Thus, ideally n should be much larger than k .

The Abelès condition²² applies to the amplitude reflectances r_s and r_p for either dielectrics or metals at 45° angle of incidence. At this angle

$$r_s^2 = r_p \quad (67)$$

and

$$2\delta_s = \delta_p \quad (68)$$

where the δ 's are the absolute phase changes on reflection for the p and s components (see Par. 2). Relation (67) is frequently applied to the intensity reflectances R_s and R_p , which are directly related to the amplitude reflectances [Eqs. (20), (21), and (31)].

5. Polarization by Nonnormal-Incidence Transmission (Pile of Plates) The theory of Brewster angle transmission polarizers follows directly from that given for reflection polarizers. Table 1 lists the relations giving the s and p transmittances of the polarizers with various assumptions about multiple reflections, interference, absorption, etc.† All these relations contain R_s and R_p , the reflectances at a single interface, which are given at the bottom of the table.

At the Brewster angle, R_p at a single interface equals zero, and the transmittances of the plates can be expressed in terms of the refractive index of the material and the number of plates. The relations for the s and p transmittances at this angle are given in Table 2. Most references that contain an expression for the degree of polarization of a pile of plates give the formula of Provostaye and Desains,²³ which assumes an infinite series of multiple reflections between all surfaces, i.e., multiple reflections within and between plates. This

† Transmission polarizers in which the multiply internally reflected beams are coherent and produce interference effects are discussed in Chap. 3, "Polarizers," in Vol. II of this Handbook.

TABLE 1 Transmittances and Degree of Polarization for a Single Plate and Multiple Plates at any Angle of Incidence in Terms of R_s and R_p for a Single Surface†

	One plate (two surfaces) ($T_{s,p}$) _{sample}	m plates ($2m$ surfaces) ($T_{s,p}$) _{sample}
Single transmitted beam, no multiple reflections, no absorption	$(1 - R_{s,p})^2$	$(1 - R_{s,p})^{2m} \ddagger$
Multiple reflections within plate, no interference effects, no absorption	$\frac{1 - R_{s,p}}{1 + R_{s,p}} \parallel$	$\frac{1 - \left[\cos^2(\theta_0 - \theta_1) \frac{1 + R_p}{1 + R_s} \right]^{m \ddagger, \parallel}}{1 + \left[\cos^2(\theta_0 - \theta_1) \frac{1 + R_p}{1 + R_s} \right]^m}$
		$\frac{1 - \cos^2(\theta_0 - \theta_1) \frac{1 + (2m - 1)R_p}{1 + (2m - 1)R_s} \S}{1 + \cos^2(\theta_0 - \theta_1) \frac{1 + (2m - 1)R_p}{1 + (2m - 1)R_s}}$
Multiple reflections within plate, no interference effects, small absorption	$\frac{(1 - R_{s,p})^2 e^{-\alpha d}}{1 - R_{s,p}^2 e^{-2\alpha d}}$	$\frac{1 - \cos^{4m}(\theta_0 - \theta_1) \left(\frac{1 - R_p^2 e^{-2\alpha d}}{1 - R_s^2 e^{-2\alpha d}} \right)^{m \ddagger}}{1 + \cos^{4m}(\theta_0 - \theta_1) \left(\frac{1 - R_p^2 e^{-2\alpha d}}{1 - R_s^2 e^{-2\alpha d}} \right)^m}$
Multiple reflections within plate, interference within plate, no absorption	$\frac{1}{1 + \frac{4R_{s,p}}{(1 - R_{s,p})^2} \sin^2 \gamma}$	$\left[\frac{1}{1 + \frac{4R_{s,p}}{(1 - R_{s,p})^2} \sin^2 \gamma} \right]^{m \ddagger} \left\{ \max 1 \right. \left. \min \left(\frac{1 - R_{s,p}}{1 + R_{s,p}} \right)^{2m} \right\}$
Single surface	$R_s = \frac{\sin^2(\theta_0 - \theta_1)}{\sin^2(\theta_0 + \theta_1)}$ $T_s = 1 - R_s = \frac{\sin 2\theta_0 \sin 2\theta_1}{\sin^2(\theta_0 + \theta_1)}$	$R_p = \frac{\tan^2(\theta_0 - \theta_1)}{\tan^2(\theta_0 + \theta_1)}$ $T_p = 1 - R_p = \frac{\sin 2\theta_0 \sin 2\theta_1}{\sin^2(\theta_0 + \theta_1) \cos^2(\theta_0 - \theta_1)}$ $P = \frac{1 - \cos^2(\theta_0 - \theta_1)}{1 + \cos^2(\theta_0 - \theta_1)}$

† $\alpha = 4\pi k / (\lambda \cos \theta_1)$, $\gamma = 2\pi m d \cos \theta_1 / \lambda$, θ_0 = angle of incidence, θ_1 = refractive index = $(\sin \theta_0) / (\sin \theta_1)$, k = extinction coefficient, d = plate thickness, λ = wavelength.

‡ No multiple reflections between plates.

§ Multiple reflections between plates.

∥ Also holds for coherent multiple reflections averaged over one period of $\sin^2 \gamma$.

TABLE 2 Transmittances and Degree of Polarization for a Single Plate and Multiple Plates at the Brewster Angle θ_B , where $\tan \theta_B = n$ and $\theta_0 + \theta_1 = 90^\circ$ [†]

One plate (two surfaces)			m plates ($2m$ surfaces)	
$(T_p)_{\text{sample}}$	$(T_s)_{\text{sample}}$	$(T_p)_{\text{sample}}$	$(T_s)_{\text{sample}}$	$P = \frac{T_p - T_s}{T_p + T_s} = \frac{1 - T_s/T_p}{1 + T_s/T_p}$
Single transmitted beams, no multiple reflections, no absorption	1	$\left(\frac{2n}{n^2+1}\right)^4$	1	$\frac{1 - [2n/(n^2+1)]^{4m}}{1 + [2n/(n^2+1)]^{4m}}$
Multiple reflections within plate, no interference effects, no absorption	1	$\frac{2n^2}{n^4+1} \P$	1	$\frac{1 - [2n^2/(n^4+1)]^{m \ddagger, \P}}{1 + [2n^2/(n^4+1)]^m}$
Multiple reflections within plate, no interference effects, small absorption	$e^{-\alpha d}$	$\left(\frac{2n}{n^2+1}\right)^4 \frac{e^{-\alpha d}}{1 - \left(\frac{n^2-1}{n^2+1}\right)^4 e^{-2\alpha d}}$	1	$\frac{m \S, \parallel}{m + [2n/(n^2-1)]^2}$
Multiple reflections within plate, interference within plate, no absorption	1	$\frac{1}{1 + \frac{(n^4-1)^2}{4n^4} \sin^2 \gamma}$	1	$1 - \frac{[2n/(n^2+1)]^{4m \ddagger}}{\{1 - [n^2-1)/(n^2+1)]^4 e^{-2\alpha d} \gamma m}} + \frac{[2n/(n^2+1)]^{4m}}{\{1 - [(n^2-1)/(n^2+1)]^4 e^{-2\alpha d} \gamma m}}$
Single surface	$R_p = 0$	$R_s = \left(\frac{n^2-1}{n^2+1}\right)^2$	$T_s = 1 - R_s = \left(\frac{2n}{n^2+1}\right)^2$	$P = \frac{1 - [2n/(n^2+1)]^2}{1 + [2n/(n^2+1)]^2}$

[†] Where $\alpha = 4\pi k(n^2+1)^{1/2}/\lambda n$, $\gamma = 2\pi n^2 d/\lambda(n^2+1)^{1/2}$, n = refractive index, k = extinction coefficient, d = plate thickness, λ = wavelength.

[‡] No multiple reflections between plates.

[§] Multiple reflections between plates.

^{||} Formula of Provostaye and Desains, Ref. 23.

[¶] Also holds for coherent multiple reflections averaged over one period of $\sin^2 \gamma$.

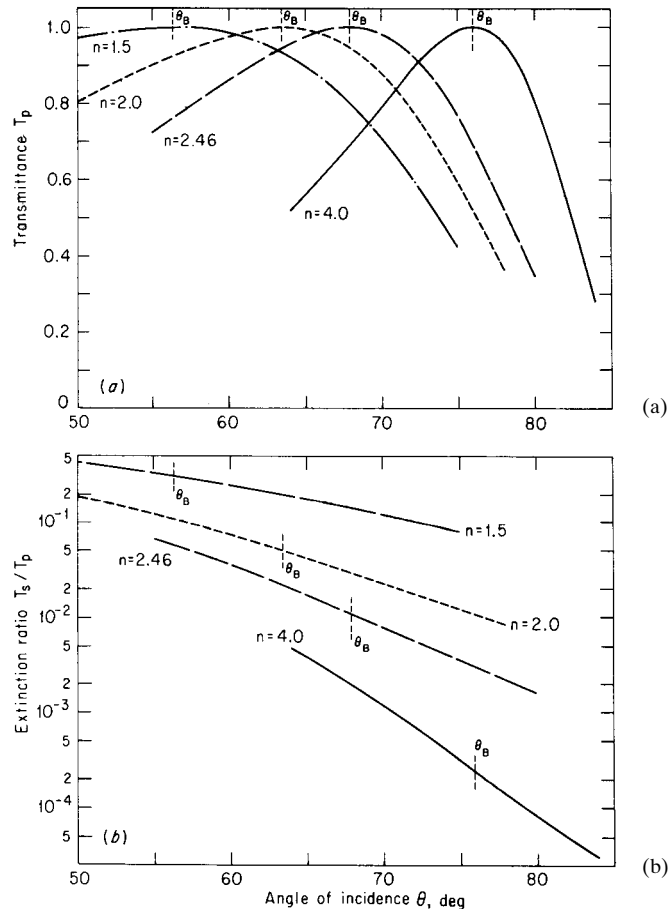


FIGURE 6 (a) Transmittance and (b) extinction ratio of four plane-parallel plates of refractive index n as a function of angle of incidence, for angles near the Brewster angle. Assumptions are multiple reflections but no interference within each plate and no reflections between plates.

assumption is not valid for most real transmission polarizers (see Chap. 3, “Polarizers,” in Vol. II of this Handbook, specifically Brewster Angle Transmission Polarizers).

For most parallel-plate polarizers it is reasonable to assume incoherent multiple reflections within each plate and no reflections between plates. Figure 6 shows the principal transmittance (p component) and extinction ratio for several four-plate polarizers having the refractive indexes indicated.[†] The extinction ratio improves considerably with increasing refractive index. It is also improved by using the plates at an angle of incidence slightly above the Brewster angle. This procedure, which is most helpful for high refractive index plates, reduces the transmission per plate so that a trade-off is required between losses resulting from absorption or scattering when many plates are used and the

[†] The extinction ratio of a pile of m plates (no multiple reflections between plates) is simply the product of the extinction ratios of the individual plates.

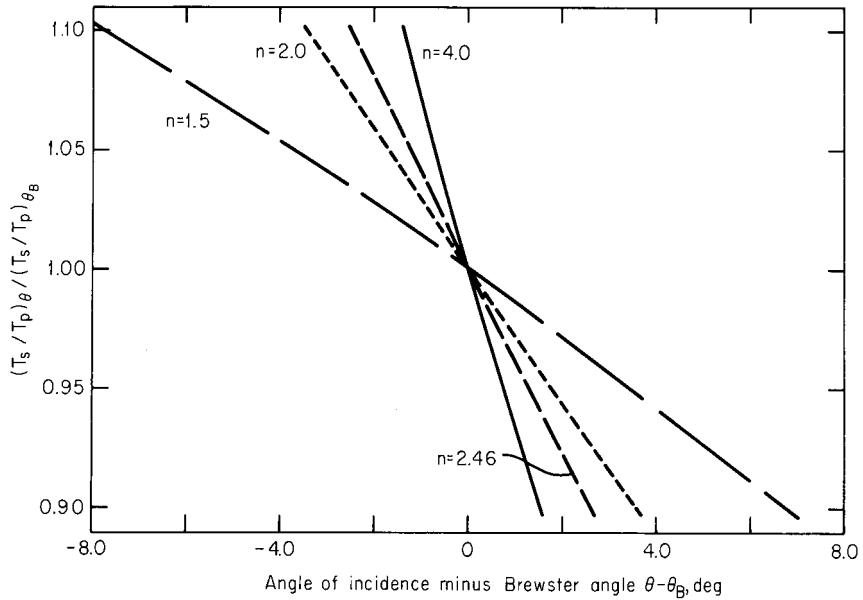


FIGURE 7 Variation of extinction ratio (per film) as a function of angle near the Brewster angle $\theta - \theta_B$. The ordinate is the extinction ratio at θ divided by the extinction ratio at θ_B .

reflectance loss per plate when only a few plates are used above the Brewster angle. In some cases significant improvements have been achieved by following the latter course.²⁴

When the number of plates of a given refractive index is increased, the transmittance is unaffected (in the absence of absorption) and the extinction ratio is greatly increased, as shown in the earlier Polarization chapter.¹ In the absence of absorption, comparable transmittances and extinction ratios are obtained with a large number of low-refractive-index plates or a small number of high refractive index plates. Small amounts of absorption decrease the transmittance, but have little effect on the extinction ratio.¹ Tuckerman²⁵ has derived exact expressions for light reflected from or transmitted through a pile of absorbing plates. He has also noted mistakes that have been perpetuated in some of the formulas for light reflected from or transmitted through a pile of nonabsorbing plates.

A figure of merit giving the variation of the extinction ratio with angle of incidence can be defined as in Fig. 7, where the ordinate is the extinction ratio at a given angle of incidence divided by the extinction ratio at the Brewster angle. The angles of incidence are referred to the Brewster angle, and curves for different values of the refractive index are shown. These curves are calculated from the ratio

$$\frac{\left(\frac{T_s}{T_p}\right)_\theta}{\left(\frac{T_s}{T_p}\right)_{\theta_B}} = \frac{\left(\frac{1-R_s}{1+R_s} \frac{1+R_p}{1-R_p}\right)_\theta}{\left(\frac{1-R_s}{1+R_s}\right)_{\theta_B}} \quad (69)$$

and are for a single transparent film or plate having multiple incoherent internal reflections within the material. As an example of how to use the graphs, consider an optical system having a two-plate germanium polarizer with a refractive index of 4.0. If the angles of incidence vary from -1.4 to $+1.5^\circ$ around the Brewster angle, the ratio of the extinction ratios will vary between $1.10^2 = 1.21$ and $0.90^2 = 0.81$, respectively. (For m plates it would

be 1.10^m and 0.90^m .) Thus, in order to restrict the percent variation of the extinction ratio to a given value, one must use a smaller acceptance angle when using more plates.

We have assumed that there are multiple incoherent reflections within each plate and no multiple reflections between plates. The difference in extinction ratios for a series of four plates with and without internal reflections is shown in Fig. 8. The principal transmittance is essentially the same as in Fig. 6 for values of T_p above 0.70 (and only about 0.025 lower when T_p drops to 0.30). However, the extinction ratio of high-refractive-index

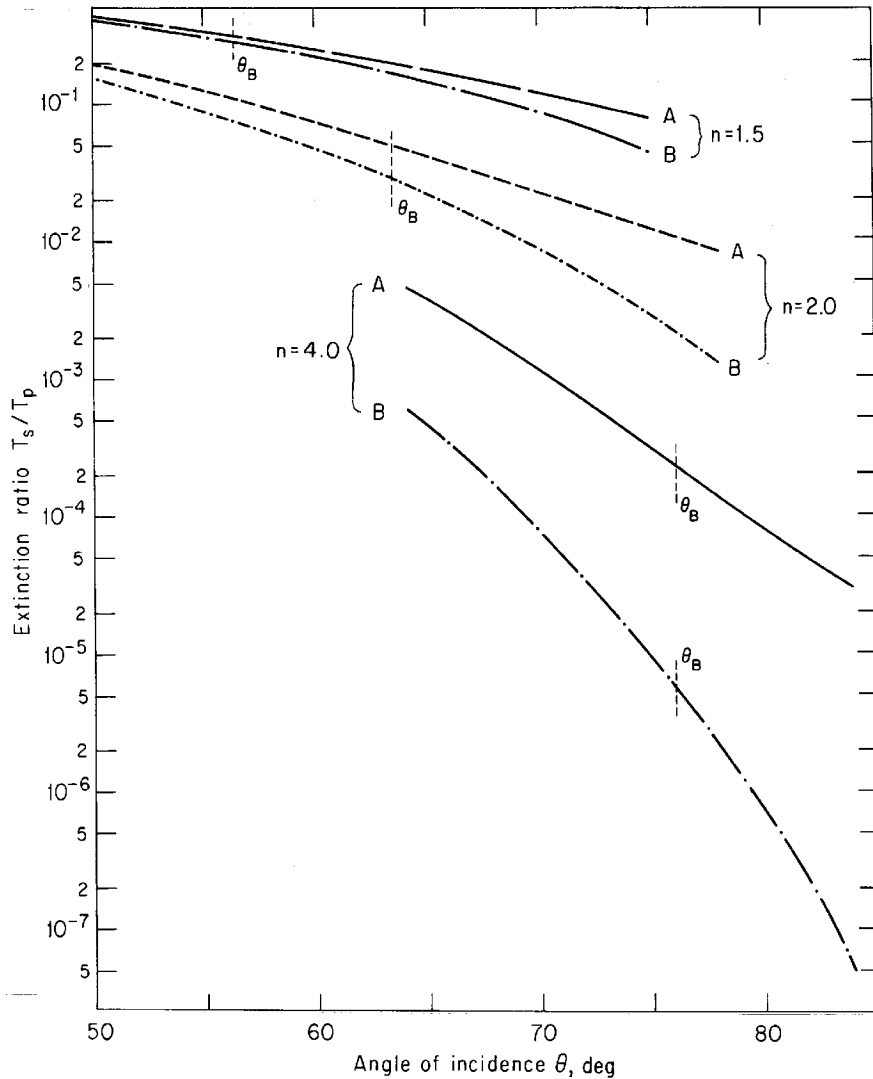


FIGURE 8 Extinction ratio of four plane-parallel plates of refractive index n as a function of angle of incidence for angles near the Brewster angle. Assumptions are A, multiple reflections but no interference within each plate and no reflections between plates; B, no multiple reflections within each plate or between plates. The transmittances for conditions A and B are essentially identical (see Fig. 6a).

materials is much better without multiple internal reflections; for low-refractive-index materials the difference in extinction ratios is small.

The effect of multiple reflections on the extinction ratio can readily be seen from the three relations for the transmittances of the p and s components:

No multiple reflections:

$$(T_{s,p})_{\text{sample}} = (1 - R_{s,p})^{2m} = 1 - 2mR_{s,p} + 2m^2R_{s,p}^2 - mR_{s,p}^2 + \dots \quad (70)$$

Multiple reflections within plates:

$$(T_{s,p})_{\text{sample}} = \left(\frac{1 - R_{s,p}}{1 + R_{s,p}} \right)^m = 1 - 2mR_{s,p} + 2m^2R_{s,p}^2 + \dots \quad (71)$$

Multiple reflections within and between plates:

$$(T_{s,p})_{\text{sample}} = \frac{1 - R_{s,p}}{1 + (2m - 1)R_{s,p}} = 1 - 2mR_{s,p} + 4m^2R_{s,p}^2 - 2mR_{s,p}^2 + \dots \quad (72)$$

At the Brewster angle, $R_p = 0$, $T_p = 1$, and the extinction ratio will be smallest, i.e., highest degree of polarization, for the smallest values of the s transmittance. The first three terms in Eqs. (70) and (71) are identical, but Eq. (70) has an additional negative term in R_s^2 and so it will give a slightly smaller value of the s transmittance. Equation (72), from which the formula of Provostaye and Desains was derived, has twice as large a third term as the other two equations, and the negative fourth term is only $1/2m$ of the third term, so that it does not reduce the overall value of the expression appreciably. Thus, Eq. (72) gives an appreciably larger value of the s transmittance, but fortunately it is a limiting case and is rarely encountered experimentally.

6. Quarter-Wave Plates and Other Phase Retardation Plates A retardation plate is a piece of birefringent, uniaxial (or uniaxial-appearing) material in which the ordinary and extraordinary rays travel at different velocities. Thus, one ray is retarded relative to the other, and the path $N\lambda$ between the two rays is given by

$$N\lambda = \pm d(n_e - n_o) \quad (73)$$

where n_o = refractive index of ordinary ray
 n_e = refractive index of extraordinary ray
 d = physical thickness of plate
 λ = wavelength

The positive sign is used when $n_e > n_o$, that is, a positive uniaxial crystal, and the negative sign is used for a negative uniaxial crystal, for which $n_e < n_o$. Since $N\lambda$ is the path difference between the two rays, N can be considered the retardation expressed in fractions of a wavelength. For example, $N = 1/4$ for a quarter-wave (or $\lambda/4$) plate, $1/2$ for a half-wave (or $\lambda/2$) plate, $3/4$ for a three-quarter-wave (or $3\lambda/4$) plate, etc.

The phase difference between two rays traveling through a birefringent material is $2\pi/\lambda$ times the path difference, so that the phase retardation δ is

$$\delta = 2\pi N = \pm \frac{2\pi d(n_e - n_o)}{\lambda} \quad (74)$$

Thus, phase differences of $\pi/2$, π , and $3\pi/2$ are introduced between the two beams in quarter-wave, half-wave, and three-quarter-wave plates, respectively.

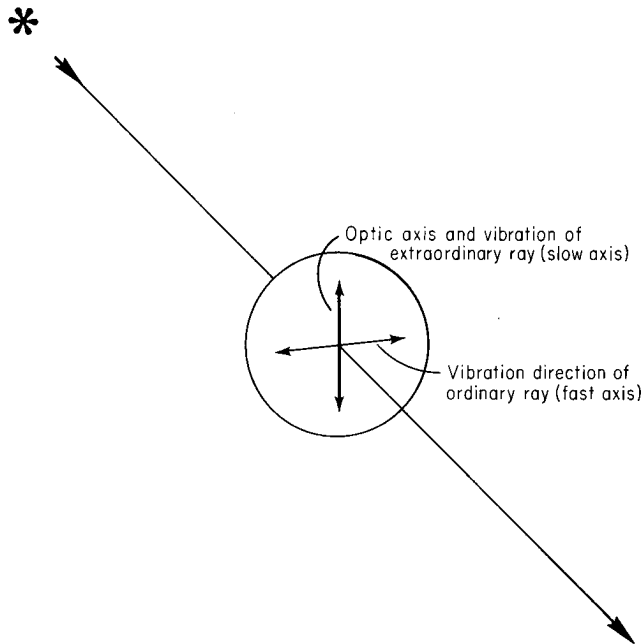


FIGURE 9 Light incident normally on the front surface of a retardation plate showing the vibration directions of the ordinary and extraordinary rays. In a positive uniaxial crystal, the fast and slow axes are as indicated in parentheses; in a negative uniaxial crystal, the two axes are interchanged.

A retardation plate can be made from a crystal which is cut so that the optic axis lies in a plane parallel to the face of the plate, as shown in Fig. 9. Consider a beam of unpolarized or plane-polarized light normally incident on the crystal. It can be resolved into two components traveling along the same path through the crystal but vibrating at right angles to each other. The ordinary ray vibrates in a direction perpendicular to the optic axis, while the extraordinary ray vibrates in a direction parallel to the optic axis. In a positive uniaxial crystal $n_e > n_o$, so that the extraordinary ray travels more slowly than the ordinary ray. The fast axis is defined as the direction in which the faster-moving ray vibrates; thus in a positive uniaxial crystal, the fast axis (ordinary ray) is perpendicular to the optic axis, while the slow axis (extraordinary ray) coincides with the optic axis. For a negative uniaxial crystal, the fast axis coincides with the optic axis.

Figure 10 shows how the state of polarization of a light wave changes after passing through retardation plates of various thicknesses when the incident light is plane-polarized at an azimuth of 45° to the fast axis of the plate. If the plate has a retardation of $\lambda/8$, which means that the ordinary and extraordinary waves are out of phase by $\pi/4$ with each other, the transmitted light will be elliptically polarized with the major axis of the ellipse coinciding with the axis of the original plane-polarized beam. As the retardation gradually increases (plate gets thicker for a given wavelength or wavelength gets shorter for a given plate thickness), the ellipse gradually turns into a circle, but its major axis remains at 45° to the fast axis of the retardation plate. For a retardation of $\lambda/4$, the emerging light is right circularly polarized. As the retardation continues to increase, the transmitted light becomes elliptically polarized with the major axis of the ellipse lying perpendicular to the plane of the incident polarized beam, and then the minor axis of the ellipse shrinks to zero

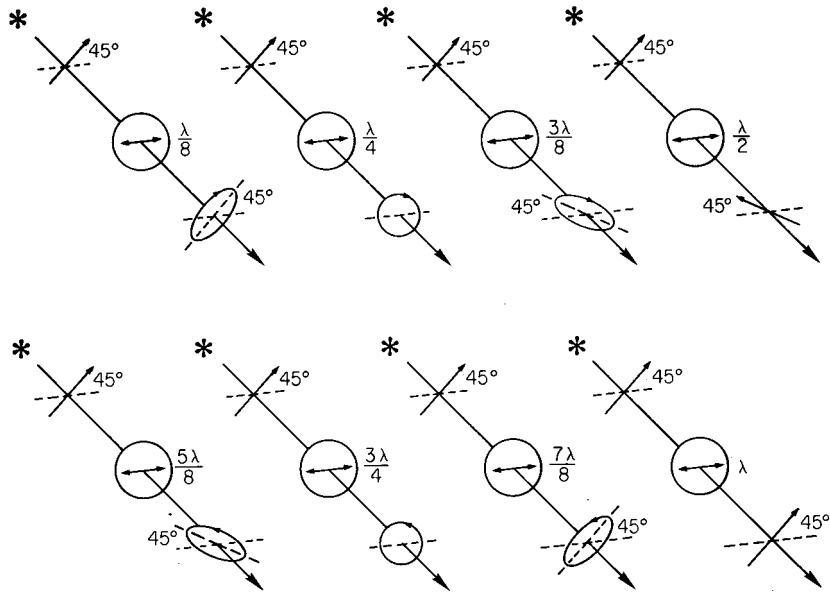


FIGURE 10 State of polarization of a light wave after passing through a crystal plate whose retardation is indicated in fractions of a wavelength (phase retardation $2\pi/\lambda$ times these values) and whose fast axis is indicated by the double arrow. In all cases the incident light is plane-polarized at an azimuth of 45° to the direction of the fast axis.

and plane-polarized light is produced when the retardation becomes $\lambda/2$. As the retardation increases further, the patterns change in opposite order and the polarized light is left circularly polarized when the retardation equals $3\lambda/4$. Finally, when the retardation is a full wave, the incident plane-polarized light is transmitted unchanged although the slow wave has now been retarded by a full wavelength relative to the fast wave.

The most common type of retardation plate is the quarter-wave plate. Figure 11 shows how this plate affects the state of polarization of light passing through it when the fast axis is positioned in the horizontal plane and the azimuth of the incident plane-polarized light is changed from $\theta = 0^\circ$ to $\theta = 90^\circ$. When $\theta = 0^\circ$, only the ordinary ray (for a positive birefringent material) passes through the plate, so that the state of polarization of the beam is unchanged. When θ starts increasing, the transmitted beam is elliptically polarized with the major axis of the ellipse lying along the fast axis of the $\lambda/4$ plate; $\tan \theta = b/a$, the

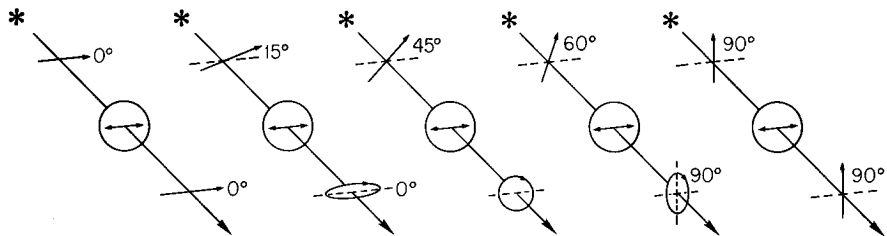


FIGURE 11 State of polarization of a light wave after passing through a $\lambda/4$ plate (whose fast axis is indicated by the double arrow) for different azimuths of the incident plane-polarized beam.

ratio of the minor to the major axis of the ellipse. In the next case, $\theta = 15^\circ$ and $\tan \theta = 0.268$, and so the ellipse is long and narrow. When the plane of vibration has rotated to an azimuth of 45° , the emerging beam is right circularly polarized (the same situation as that shown in the second part of Fig. 10). For values of θ between 45 and 90° , the light is again elliptically polarized, this time with the major axis of the ellipse lying along the direction of the slow axis of the $\lambda/4$ plate. The angle shown in the figure is 60° , and $\tan 60^\circ = 1.732$, so that b/a (referred to the fast axis) is greater than unity. When θ increases to 90° , the plane of vibration coincides with the slow axis and the transmitted light is again plane-polarized. As θ continues to increase, the transmitted patterns repeat those already described and are symmetric about the slow axis, but the direction of rotation in the ellipse changes from right-handed to left-handed, so that left-circularly polarized light is produced when $\theta = 135^\circ$.

The definition of right- and left-circularly polarized light should be clear from Figs. 10 and 11. When the rotation is *clockwise* with the observer looking *opposite to the direction of propagation*, the light is called *right-circularly polarized*; if the rotation is *counterclockwise*, the light is called *left-circularly polarized*.²⁶ When circularly polarized light is reflected from a mirror, the direction of propagation is reversed, so that the sense of the circular polarization changes; i.e., left-circularly polarized light changes on reflection into right-circularly polarized light and vice versa. Therefore, in experiments involving magnetic fields in which the sense of the circularly polarized light is important,^{27,28} it is important to know which kind one started with and how many mirror reflections occurred in the rest of the light path. Cyclotron resonance experiments can sometimes be used to determine the sense of the circular polarization.²⁸ Another method utilizing a polarizer and $\lambda/4$ plate has been described by Wood.²⁹

The behavior of a half-wave plate in a beam of plane-polarized light is completely different from that of a quarter-wave plate; the transmitted light is always plane-polarized. If the incident plane of vibration is at an azimuth θ with respect to the fast axis of the $\lambda/2$ plate, the transmitted beam will be rotated through an angle 2θ relative to the azimuth of the incident beam. The case showing $\theta = 45^\circ$ where the phase of vibration is rotated through 90° is illustrated in the fourth part of Fig. 10. In this situation the extraordinary beam is retarded by half a wavelength relative to the ordinary beam (for a positive birefringent material), hence the name, half-wave plate. If the polarizer is fixed and the $\lambda/2$ plate is rotated (or vice versa), the plane of vibration of the transmitted beam will rotate at twice the frequency of rotation of the $\lambda/2$ plate.

Quarter-wave plates are useful for analyzing all kinds of polarized light. In addition, they are widely employed in experiments using polarized light, e.g., measurements of the thickness and refractive index of thin films by ellipsometry or measurements of optical rotary dispersion, circular dichroism, or strain birefringence. Polarizing microscopes, interference microscopes, and petrographic microscopes are usually equipped with $\lambda/4$ plates. In some applications the $\lambda/4$ plate is needed only to produce circularly polarized light, e.g., for optical pumping in some laser experiments, or to convert a partially polarized light source into one which appears unpolarized, i.e., has equal amplitudes of vibration in all azimuths. For these and similar applications, one can sometimes use a circular polarizer which does not have all the other properties of a $\lambda/4$ plate (see Pars. 73 to 76 in Ref. 1).

The customary application for a $\lambda/2$ plate is to rotate the plane of polarization through an angle of 90° . In other applications the angle of rotation can be variable. Automatic setting ellipsometers or polarimeters sometimes employ rotating $\lambda/2$ plates in which the azimuth of the transmitted beam rotates at twice the frequency of the $\lambda/2$ plate.

7. Matrix Methods for Computing Polarization In dealing with problems involving polarized light, it is often necessary to determine the effect of various types of polarizers (linear, circular, elliptical, etc.), rotators, retardation plates, and other polarization-sensitive devices on the state of polarization of a light beam. The Poincaré sphere

construction is helpful for giving a qualitative understanding of the problem; for quantitative calculations, one of several forms of matrix calculus can be used. The matrix methods are based on the fact that the effect of a polarizer or retarder is to perform a linear transformation (represented by a matrix) on the vector representation of a polarized light beam. The advantage of these methods over conventional techniques is that problems are reduced to simple matrix operations; thus since one does not have to think through the physics of every problem, the probability of making an error is greatly reduced. The most common forms of matrix calculus are the Mueller calculus and the Jones calculus, but the coherency-matrix formulation is also gaining popularity for dealing with problems involving partially polarized light. We give here a brief description of the Poincaré sphere and the different matrix methods, indicating how they are used, the different types of problems for which they are helpful, and where complete descriptions of each may be found.

The *Poincaré sphere* is a useful device for visualizing the effects of polarizers and retarders on a beam of polarized light. The various states of polarization are represented on the sphere as follows. The equator represents various forms of linear polarization, the poles represent right- and left-circular polarization,[†] and other points on the sphere represent elliptically polarized light. Every point on the sphere corresponds to a different polarization form. The radius of the sphere indicates the intensity of the light beam (which is usually assumed to be unity). The effects of polarizers and retarders are determined by appropriate displacements on the sphere. Partially polarized light or absorption may be dealt with approximately by ignoring the intensity factor, since one is generally interested only in the state of polarization; however, the construction is most useful when dealing with nonabsorbing materials. Good introductory descriptions of the Poincaré sphere, including references, can be found in *Polarized Light* by Schurcliff,^{30‡} *Ellipsometry and Polarized Light* by Azzam and Bashara,³¹ and *Polarized Light in Optics and Spectroscopy* by Kliger, Lewis, and Randall;^{32§} illustrative examples and problems are given in Sutton and Panati.³³ More comprehensive treatments are given by Ramachandran and Ramaseshan^{34§} and Jerrard^{35§} and include numerous examples of applications to various types of problems. The new book *Polarized Light, Fundamentals and Applications* by Collett³⁶ has a comprehensive 35-page chapter on the mathematical aspects of the Poincaré sphere; this material can be best understood after reading some of the introductory descriptions of the Poincaré sphere. The main advantage of the Poincaré sphere, like other graphical methods, is to reveal by essentially a physical argument which terms in exceedingly complex equations are negligible or can be made negligible by modifying the experiment. It is characteristic of problems in polarized light that the trigonometric equations are opaque to inspection and yield useful results only after exact calculation with the aid of a computer or after complex manipulation and rather clever trigonometric identities. The Poincaré sphere thus serves as a guide to the physical interpretation of otherwise obscure polarization phenomena. It can be used for solving problems involving retarders or combinations of retarders,^{30,32,36–39} compensators, half-shade devices, and depolarizers,³⁴ and it has also been applied to ellipsometric problems⁴⁰ and stress-optical measurements.⁴¹

The Poincaré sphere is based on the Stokes vectors, which are sometimes designated S_0 , S_1 , S_2 , and S_3 . The physical interpretation of the vectors is as follows. S_0 is the intensity of the light beam, corresponding to the radius of the Poincaré sphere. S_1 is the difference in intensities between the horizontal and vertical polarization components of the beam; when S_1 is positive, the preference is for horizontal polarization, and when it is negative, the

[†] Right-circularly polarized light is defined as a *clockwise* rotation of the electric vector when the observer is looking *against the direction the wave is traveling*.

[‡] Schurcliff and Kliger, Lewis, and Randall have the S_3 axis pointing down, so that the upper pole represents left-circular polarization. The more logical convention, followed by most others, is for the upper pole to represent right-circular polarization.

[§] The notation is similar to that used by Schurcliff,³⁰ with the upper pole representing left-circular polarization.

preference is for vertical polarization.[†] S_2 indicates preference for $+45^\circ$ or -45° polarization, depending upon whether it is positive or negative, and S_3 gives the preference for right or left circular polarization. The Stokes vectors S_1 , S_2 , and S_3 are simply the three cartesian coordinates of a point on the Poincaré sphere: S_1 and S_2 are perpendicular to each other in the equatorial plane, and S_3 points toward the north pole of the sphere.[‡] Thus, any state of polarization of a light beam can be specified by these three Stokes vectors. The intensity vector S_0 is related to the other three by the relation $S_0^2 = S_1^2 + S_2^2 + S_3^2$ when the beam is completely polarized. If the beam is partially polarized, $S_0^2 > S_1^2 + S_2^2 + S_3^2$. Good introductory material on Stokes vectors is given by Shurcliff,³⁰ Azzam and Bashara,³¹ Kliger *et al.*,³² Sutton and Panati,³³ and Walker.⁴² A comprehensive discussion of the Stokes vectors has been given by Collett.³⁶ Rigorous definitions of the simple vectors and those for partially coherent light can be found in Born and Wolf;⁴³ other authors are cited by Shurcliff³⁰ and Collett.³⁶ Stokes vectors are generally used in conjunction with the Mueller calculus, and some examples of applications will be given there. We note here that Budde⁴⁴ has demonstrated a method for experimentally determining the Stokes vectors and other polarization parameters from a Fourier analysis of measured quantities. Ioshpa and Obridko⁴⁵ have proposed a photoelectric method for simultaneously and independently measuring the four Stokes parameters. Collett⁴⁶ has developed a method for measuring the four Stokes vectors using a single circular polarizer. Azzam and coworkers⁴⁷⁻⁵¹ have built, tested, analyzed, and calibrated a four-detector photopolarimeter for measuring normalized Stokes vectors of a large number of polarization states, and have given a physical meaning to the rows and columns in the instrument matrix. Other methods for measuring Stokes parameters are discussed by Collett.³⁶ Hauge⁵² has surveyed different types of methods for completely determining the state of polarization of a light beam using combinations of Stokes vectors.

The matrix methods for solving problems involving polarized light have certain properties in common. All use some type of representation for the original light beam (assumed to be a plane wave traveling in a given direction) that uniquely describes its state of polarization. Generally the beam is completely polarized, but for some of the matrix methods it can also be unpolarized or partially polarized or its phase may be specified. The beam encounters one or more devices which change its state of polarization. These are called *instruments* and are represented by appropriate matrices. After the instruments operate on the light beam, it emerges as an outgoing plane wave in an altered state of polarization. The basic problem for all the methods is to find a suitable representation for the incident plane wave (usually a two- or four-component column vector), and the correct matrices (2×2 or 4×4) to represent the instruments. Once the problem is set up, one can perform the appropriate matrix operations to obtain a representation for the outgoing plane wave. Its properties are interpreted in the same way as the properties of the incident plane wave.

An introduction to the Jones and Mueller calculus is given by Shurcliff,³⁰ Azzam and Bashara,³¹ and Kliger *et al.*,³² and an excellent systematic and rigorous discussion of all the matrix methods has been given by O'Neill⁵³ and Collett.³⁶ All references contain tables of vectors for the various types of polarized beams and tables of instrument matrices. More complete tables are given by Sutton and Panati.³³ In the Mueller calculus the beam is represented by the four-component Stokes vector, written as a column vector. This vector has all real elements and gives information about *intensity* properties of the beam. Thus it is not able to handle problems involving phase changes or combinations of two beams that are coherent. The instrument matrix is a 4×4 matrix with all real elements. In the Jones calculus, the Jones vector is a two-component column vector that generally has complex elements. It contains information about the *amplitude* properties of the beam and hence is

[†] Some authors dealing with light scattering from aerosols define S_1 as positive when the preference is for vertical polarization.

[‡] See Shurcliff and Kliger, Lewis, and Randall footnote, p. 5.26.

well suited for handling coherency problems. However, it cannot handle problems involving depolarization, as the Mueller calculus can. The Jones instrument matrix is a 2×2 matrix whose elements are generally complex.

Shurcliff³⁰ has noted some additional differences between Jones calculus and Mueller calculus. The Jones calculus is well suited to problems involving a large number of similar devices arranged in series in a regular manner and permits an investigator to arrive at an answer expressed explicitly in terms of the number of such devices. The Mueller calculus is not suited for this type of problem. The Jones instrument matrix of a train of transparent or absorbing nondepolarizing polarizers and retarders contains no redundant information. The matrix contains four elements each of which has two parts, so that there are a total of eight constants, none of which is a function of any other. The Mueller instrument matrix of such a train contains much redundancy; there are 16 constants but only 7 of them are independent.

In order to handle problems involving partially coherent polarized light, coherency-matrix formalism has been developed. In this system the beam is represented by a 4×4 matrix called a *coherency* or *density matrix*, which is the time average of the product of the Jones vector with its hermitian conjugate. The instrument matrices are the same as those used in the Jones calculus. O'Neill⁵³ and Born and Wolf⁴³ have good basic descriptions of coherency-matrix formalism; later extensions of the theory are given by Marathay.^{54,55}

There have been some modifications of the various matrix methods. Priebe⁵⁶ has introduced an operational notation for the Mueller matrices that facilitates the analysis by simplifying the functional description of a train of optical components. Collins and Steele⁵⁷ have suggested a modification of the Jones calculus in which the light vector is expressed as the sum of two circularly polarized (rather than linearly polarized) components. Schmieder⁵⁸ has given a unified treatment of Jones calculus and Mueller calculus including the coherency matrix and has shown that if the Stokes parameters are ordered in a different way from that customarily used, familiar relationships are preserved and the rotation matrix looks like a rotation matrix rather than like a rearranged one. Tewarson⁵⁹ presents a generalized reciprocity equation expressing an algebraic relationship between the parameters of an optical system and its reciprocal system and has verified the equation for both plane-polarized and circularly polarized light beams. Since his equation follows from the reciprocity law in the Mueller calculus, that law is verified also. Cernosek⁶⁰ presents a simple geometric method based on the properties of quaternions to give a quick, quantitative analysis of the effect of any combination of linear retarders and rotators on the state of polarization of a system.

Among the applications of Mueller calculus and Jones calculus to problems involving polarized light, McCrackin⁶¹ has used both matrix methods to analyze instrumental errors in ellipsometry, and Hellerstein⁶² has used Mueller calculus to study the passage of linearly, circularly, and elliptically polarized light through a Sénarmont polariscope. Azzam and Bashara⁶³ have used Jones calculus to give a unified analysis of errors in ellipsometry, including effects of birefringence in cell windows, imperfect components, and incorrect azimuth angles. Azzam⁶⁴ also describes a simple photopolarimeter with rotating polarizer and analyzer for measuring Jones and Mueller matrices.

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