On $|V,\lambda|_k$ Summability Factors of Fourier Series

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Abstract— In this paper a general theorem concerning the $|V,\lambda|_k$ summability factors of Fourier series has been proved.

Keywords— $|V,\lambda|_k$ Summability, Fourier series, Summability factors.

I. INTRODUCTION

Let $\sum u_n$ be a given series with the sequences of partial sums $\{s_n\}$ and let $\lambda = \{\lambda_n\}$ be a monotonic non-decreasing sequence of natural numbers with $\lambda_{n+1} - \lambda_n \le 1$ and $\lambda_1 = 1$.

The sequence-to-sequence transformation

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{v=n-\lambda_n+1}^{n} s_v$$

defines the generalised de-la Vallée Poussin means of the sequence $\{s_n\}$ generated by λ.

The series $\sum u_n$ is said to be summable $|V,\lambda|$, if the sequence $\{V_n(\lambda)\}$ is of bounded variation, i.e. to say (LEINDLER [03])

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda) - V_n(\lambda)| < \infty$$

We say that the series $\sum u_n$ is summable $|V,\lambda|_k$, $k \ge 1$, if

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} |V_{n+1}(\lambda) - V_n(\lambda)|^k < \infty$$

The series $\sum u_n$ is also summable $|V,\lambda;\gamma|_k$, $k \ge 1$, $\gamma \ge 0$ if

$$\sum_{n=1}^{\infty} \lambda_n^{\gamma k+k+-1} |V_{n+1}(\lambda) - V_n(\lambda)|^k < \infty$$

on taking $\lambda_n = n$ and $\gamma = 0$, this summability reduces $|C,1|_k$ and for k=1 and $\gamma = 0$ this is the same as summability $|V,\lambda|$.

Let f(t) be a 2π-periodic and L-integrable function over $(-\pi,\pi)$. We assume, as we may without any loss of generality that

$$\sum_{n=1}^{\infty} A_n(t) \equiv \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

is the Fourier series of f(t) .

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

and

$$S_n(x) = \sum_{v=1}^{n} A_v(x)$$

II. THEOREM

In this paper, we shall prove the following theorem.

If $\{u_n\}$ is a convex sequence such that

$$\sum \frac{n u_n (\log n)^{\frac{(1-2\delta)}{2}}}{\lambda_n^{2-\gamma k}} < \infty, \left(|\delta| \le \frac{1}{2}\right) \quad (1)$$

$$\sum \frac{n (\log n)^{\frac{(1-2\delta)}{2}} \Delta u_n}{\lambda_n^{1-\gamma k}} < \infty \quad (2)$$

and

$$\int_0^t |\phi(u)|^k du = O\left\{t\left(\log\frac{1}{t}\right)^{\beta}\right\}, \beta \ge 0,$$

and $1 \le k \le 2$ (3)

Then the series $\sum \frac{u_n A_n(t)}{\{\log(n+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}$,

at t=x is summable $|V,\lambda;\gamma|_k$, $0 \le \gamma \le \frac{1}{k}$.

On taking $\lambda_n = 1$, $k = 1$, $\delta = 0$ and $\gamma = 0$ in our theorem, we obtain the theorem of SINGH [08] which is a an extension of a well-known result of PATI [06].

III. LEMMAS

We need the following lemmas for the proof of our theorem.

.Lemma A : (JAIN, GANGULY and MADAN [02])

If (3) holds, then

$$\sum_{v=0}^{n} |S_v(x) - f(x)|^k =$$

$$= O\left(n(\log n)^{\left(\frac{k}{2}\right)+\beta}\right),$$

$\beta \ge 0$, and $1 \le k \le 2$ (4)

.Lemma B : (JAIN, GANGULY and MADAN [02])

If (3) holds and

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$$T_n(x)=\frac{1}{n+1}\sum_{v=1}^{n} vA_v(x),$$

then

$$\sum_{v=1}^{n}|T_v(x)|^k = O\left(n(\log n)^{\left(\frac{k}{2}\right)+\beta}\right).$$

.Lemma C : (JAIN, GANGULY and MADAN [02])

If $\{u_n\}$ is a convex sequence such that $\sum \frac{nu_n}{\lambda_n^2}<\infty$,

Then

(i) $\sum_{n=1}^{m}\log(n+1)\Delta u_n = O(1)$, as m $\to\infty$

(ii) $\sum_{n=1}^{m} n\log(n+1)\Delta^2 u_n = O(1)$, as m $\to\infty$

IV. PROOF OF THE THEOREM

For k=1 and $\gamma=0$ the theorem directly follows on taking the series $\sum \frac{u_n A_n(t)}{\{\log(n+1)\}^{\beta+\delta}}$ in place of $\sum u_n A_n(t)$ and applying the condition (1), (2) and (3) (with k=1 and $\gamma=0$) instead of the set of conditions used by SHARMA and JAIN [07] in the proof of their theorem.

Therefore, we prove our theorem for $1\le k\le 2$ only.

Let $C_n = V_{n+1}(\lambda;x)-V_n(\lambda;x)$,

Where $V_n(\lambda;x)$ is the n-th de-la Vallee Poussin mean of series $\sum \frac{u_n A_n(t)}{\{\log(n+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}$

By an easy computation, we have

$$C_n=\frac{1}{\lambda_n\lambda_{n+1}}\sum_{v=n-\lambda_n+2}^{n+1}\{(\lambda_{n+1}-\lambda_n)(v-n-1)+\lambda_n\}\times$$

$$\times\frac{u_v A_v(t)}{\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}$$

Therefore, in order to prove the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}\lambda_n^{\gamma k+k-1}|C_n|^k<\infty$$

Let $\sum_n^{(i)}$ be the summation over all n satisfying $\lambda_{n+1}=\lambda_n$ and $\sum_n^{(ii)}$ be the summation over all n where $\lambda_{n+1}>\lambda_n$, when $\lambda_{n+1}=\lambda_n$,

Abel's transformation gives that

$$C_n=$$

$$=\frac{1}{\lambda_{n+1}}\left[\sum_{v=n-\lambda_n+2}^{n}\left\{\sum_{r=1}^{v} rA_r(x)\right\}\times\right.$$

$$\times\Delta\left\{\frac{u_v}{r\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}+$$

$$-\frac{u_{n-\lambda_n+2}}{(n-\lambda_n+2)\{\log(n-\lambda_n+3)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\times$$

$$\times\left\{\sum_{r=1}^{n-\lambda_n+1} rA_r(x)\right\}+$$

$$\left.+\frac{u_{n+1}}{(n+1)\{\log(n+2)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\left\{\sum_{r=1}^{n+1} rA_r(x)\right\}\right]$$

$$=L_n^1+L_n^2+L_n^3,\ \text{say.}$$

By Minkowski's inequality, it is therefore, sufficient to prove that

$$\sum_n^{(i)}\lambda_n^{\gamma k+k-1}|L_n^r|^k<\infty,\quad \text{for r=1,2,3}$$

Now,

$$\sum_{n=1}^{(i)}\lambda_n^{\gamma k+k-1}|L_n^1|^k=$$

$$=O(1)\sum_n^{(i)}\frac{1}{\lambda_n^{1-\gamma k}}\times$$

$$\times\left[\sum_{v=n-\lambda_n+2}^{n} v|T_v(x)|\Delta\left\{\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}\right]^k$$

$$=O(1)\sum_n^{(i)}\frac{1}{\lambda_n^{1-\gamma k}}\times$$

$$\times\sum_{v=n-\lambda_n+2}^{n+1} v|T_v(x)|^k\Delta\left\{\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}$$

$$=O(1)\sum_{v=1}^{\infty} v|T_v(x)|^k\Delta\left\{\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}\times$$

$$\times \sum_{n=v}^{n+\lambda_v-1} \frac{1}{\lambda_n^{1-\gamma k}}$$

$$= O(1)\sum_{v=1}^{\infty} v|T_v(x)|^k \Delta\left\{\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}\lambda_v^{\gamma k}$$

Using Abel's transformation again, by Lemma B. We easily have

$$\sum_n^{(i)} \lambda_n^{\gamma k+k-1}|L_n^1|^k =$$

$$= O(1)\sum_{n=1}^{\infty} n^2(\log n)^{\left(\frac{k}{2}\right)+\beta} \times$$

$$\times \Delta^2\left\{\frac{u_v}{n\{\log(n+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}\lambda_n^{\gamma k}$$

$$= O(1)\sum_{n=1}^{\infty} n(\log n)^{\frac{(1-2\delta)}{2}} \Delta^2 u_n \lambda_n^{\gamma k} +$$

$$+ O(1)\sum_{n=1}^{\infty} (\log n)^{\frac{(1-2\delta)}{2}} \Delta u_n \lambda_n^{\gamma k} +$$

$$+ O(1)\sum_{n=1}^{\infty} \frac{u_n}{n}(\log n)^{\frac{(1-2\delta)}{2}} \lambda_n^{\gamma k}$$

=O(1) (5)

by Lemma C(i), C(ii) and hypothesis (1).

Further, Applying Abel's transformation and Lemma B, it is easy to see that

$$\sum_n^{(i)} \lambda_n^{\gamma k+k-1}|L_n^2|^k + \sum_n^{(i)} \lambda_n^{\gamma k+k-1}|L_n^3|^k =$$

$$= O(1)\sum_{n=1}^{\infty} |T_n(x)|^k \frac{u_n}{\lambda_n\{\log(n+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\lambda_n^{\gamma k}$$

$$= O(1)\sum_{n=1}^{\infty} \frac{n\Delta u_n(\log n)^{\frac{(1-2\delta)}{2}}}{\lambda_n^{1-\gamma k}} +$$

$$+ O(1)\sum_{n=1}^{\infty} \frac{nu_n(\log n)^{\frac{(1-2\delta)}{2}}}{\lambda_n^{2-\gamma k}} +$$

$$+ O(1)\sum_{n=1}^{\infty} \frac{u_n(\log n)^{\frac{(-1-2\delta)}{2}}}{\lambda_n^{1-\gamma k}}$$

=O(1) (6)

by hypothesis (1) and (2).

Now, in order to estimate $\sum_n^{(ii)}$ we have, with the aid of Abel's transformation, that

$$|C_n| \le \frac{1}{\lambda_n\lambda_{n+1}}\left[\sum_{v=n-\lambda_n+2}^{n} v|T_v(x)| \times\right.$$

$$\times\left|\Delta\{(\lambda_n+v-\lambda-1)\}\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right| +$$

$$+(n-\lambda_n+1)|T_{n-\lambda_{n+1}}(x)| \times$$

$$\times \frac{u_{n-\lambda_n+2}}{(n-\lambda_n+2)\{\log(n-\lambda_n+3)\}^{\frac{(2\beta+2\delta+k-1)}{2}}} +$$

$$\left. +(n+1)|T_{n+1}(x)|\frac{\lambda_n u_{n+1}}{(n+1)\{\log(n+2)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right]$$

$$= M_n^1 + M_n^2 + M_n^3, \text{ say.}$$

By Minkowski's inequality, it is therefore sufficient to prove that

$$\sum_n^{(ii)} \lambda_n^{\gamma k+k-1}|M_n^r|^k < \infty, \text{ for r=1,2,3}$$

Now,

$$\sum_n^{(ii)} \lambda_n^{\gamma k+k-1}|M_n^1|^k \le \sum_n^{(ii)} \frac{1}{\lambda_n^{k-\gamma k+1}}\left[\sum_{v=n-\lambda_n+2}^{n} v|T_v(x)| \times\right.$$

$$\times\left\{\lambda_v\Delta\left(\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right)+\right.$$

$$\left.\left. + \frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}\right]^k$$

$$\le \left(\left[\sum_n^{(ii)} \frac{1}{\lambda_n^{k-\gamma k+1}}\left\{\sum_{v=n-\lambda_n+2}^{n} v|T_v(x)|\lambda_v \times\right.\right.\right.$$

$$\left.\left.\left.\times\Delta\left(\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right)\right\}^k\right]^{\frac{1}{k}} +$$

$$+\left[\sum_n^{(ii)} \frac{1}{\lambda_n^{k-\gamma k+1}}\left\{\sum_{v=n-\lambda_n+2}^{n} \lambda_v|T_v(x)| \times\right.\right.$$

$$\times\left(\frac{u_v}{\lambda_v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right)\Bigg\}^k\Bigg]^{\frac{1}{k}}\Bigg)^k$$

$$=\left(N_1^{\frac{1}{k}}+N_2^{\frac{1}{k}}\right)^k \text{, say}$$

We observe that

N_1 =

$$=O(1)\sum_n^{(ii)}\Big[\frac{1}{\lambda_n^{k-\gamma k+1}}\sum_{v=n-\lambda_n+2}^{n} v|T_v(x)|^k \lambda_v^k \times$$

$$\times\Delta\left\{\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}$$

$$=O(1)\sum_{v=1}^{\infty} v|T_v(x)|^k \lambda_v^k \Delta\left\{\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}\times$$

$$\times\sum_{n\ge v}^{(ii)}\frac{1}{\lambda_n^{k-\gamma k+1}}$$

$$=O(1)\sum_{v=1}^{\infty} v|T_v(x)|^k \Delta\left\{\frac{u_v}{v\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}\right\}$$

= O (1). by (5)

And similarly,

N_2 =

$$=O(1)\sum_n^{(ii)}\frac{1}{\lambda_n^{k-\gamma k+1}}\sum_{v=n-\lambda_n+2}^{n}\frac{|T_v(x)|^k \lambda_v^{k-1} u_v}{\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}$$

$$=O(1)\sum_{v=1}^{\infty}\frac{|T_v(x)|^k u_v}{\lambda_v^{1-\gamma k}\{\log(v+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}$$

= O (1), by (6)

Therefore,

$$\sum_n^{(ii)}\lambda_n^{\gamma k+k-1}\left|M_n^1\right|^k = O(1).$$

Finally,

$$\sum_n^{(ii)}\lambda_n^{\gamma k+k-1}\left|M_n^2\right|^k + \sum_n^{(ii)}\lambda_n^{\gamma k+k-1}\left|M_n^3\right|^k =$$

$$=O(1)\sum_{n=1}^{\infty}\frac{|T_n(x)|^k u_n}{\lambda_n^{1-\gamma k}\{\log(n+1)\}^{\frac{(2\beta+2\delta+k-1)}{2}}}$$

= O (1). By (6)

This is complete proof of theorem.

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