INTEGRAL ESTIMATES FOR TRANSPORT DENSITIES

L. DE PASCALE, L. C. EVANS, AND A. PRATELLI

Abstract. We introduce some integration-by-parts methods that improve upon the L^p estimates on transport densitites from the recent paper by De Pascale–Pratelli [DP-P].

1. Introduction

This paper provides some PDE methods that improve upon the L^p estimates on the "transport densities" in certain Monge–Kantorovich mass transfer problems, as derived in the earlier paper [DP-P] by the first and third authors. Our main estimate provides the bound

$$\|\sigma_k\|_{L^q} \leq C\,(\|f\|_{L^q} + 1) \tag{1}$$

for each $2 \leq q < \infty$, when u solves the quasilinear elliptic equation

$$-\operatorname{div}(\sigma_k Du_k) = f \tag{2}$$

for

$$\sigma_k := e^{\frac{k}{2}\left(|Du_k|^2 - 1\right)} \tag{3}$$

and k sufficiently large. The constant C in (1) depends on q, but not on the parameter k.

This problem arises as an approximation of the fundamental *transport (*or *continuity) equation* for the Monge–Kantorovich mass transfer problem, as explained for instance in [E2]. In this interpretation, we seek an optimal rearrangement of the measure $\mu^+ := f^+ dx$ into $\mu^- := f^- dy$. In the limit $k \to \infty$, we have $u_k \to u, \sigma_k \to a$ and the potential u solves

$$\begin{cases} -\operatorname{div}(aDu) = f, \\ |Du| \leq 1, \\ |Du| = 1 \text{ where } a > 0. \end{cases} \tag{4}$$

We call a the *transport density*. It turns out that an optimal mass reallocation plan can be constructed using u and a.

The paper [DP-P] by De Pascale and Pratelli studied how the integrability properties of $f = f^+ - f^-$ affect those of the transport density. They showed that

(i) $a \in L^\infty$ if $f \in L^\infty$, and
(ii) $a \in L^{q-\epsilon}$ if $f \in L^q$, for $1 \leq q < \infty$ and each $\epsilon > 0$.

We introduce in this paper some PDE integration–by–parts methods to improve assertion (ii), by demonstrating

$$a \in L^q \quad \text{if } f \in L^q, \quad \text{for} \;\; 2 \leq q < \infty.$$

We have tried, and failed, to extend our methods to include $q = \infty$.

LCE's research is supported in part by NSF Grant DMS-0070480 and by the Miller Institute for Basic Research in Science, UC Berkeley.

A PDE like (4) comes up also in the general formulation of Bouchitté and Buttazzo [B-B] for finding a distribution of a given amount of conductor to best dissipate heat. Then f represents a heat source and u the temperature of the system. The survey [E2] describes several more applications.

2. Approximation

We will for simplicity take $U = B^0(0,R)$, the open ball with center 0 and radius $R > 0$. Hereafter we always suppose that $f \in L^1(U)$, with $\int_U f\,dx = 0$. Denote by u_k the solution of the nonlinear boundary–value problem

$$\begin{cases} -\operatorname{div}(\sigma_k Du_k) = f & \text{in } U \\ \qquad\qquad u_k = 0 & \text{on } \partial U, \end{cases} \tag{5}$$

where we write

$$\sigma_k := e^{\frac{k}{2}\left(|Du_k|^2-1\right)}. \tag{6}$$

Observe that u_k is the unique minimizer of the functional

$$F_k[v] := \int_U \frac{1}{k} e^{\frac{k}{2}(|Dv|^2-1)} - fv\,dx$$

in $W_0^{1,k}$. This approximation is suggested by the recent paper [E1]. Regularity theory (Cf. Marcellini [M]) implies that u_k is smooth, provided f is.

We want to study the limits of u_k and σ_k as $k \to \infty$, and begin with some uniform bounds.

Lemma 2.1. *Suppose that $f \in L^1(U)$. Then the sequence $\{u_k\}_{k=1}^\infty$ is bounded in $W_0^{1,q}(U)$, for each $1 \le q < \infty$.*

Proof. Observe first that $x \le e^{\frac{x^2-1}{2}}$ for $x \ge 0$, and therefore that $|Du_k| \le \sigma_k^{\frac{1}{k}}$. Recalling then (5), (6), we deduce for $k > n$ that

$$\int_U |Du_k|^{k+2}\ dx \le \int_U |Du_k|^2 \sigma_k\ dx = \int_U f u_k\,dx \le C\|u_k\|_{L^\infty} \le C\|Du_k\|_{L^k}.$$

Note that $\|Du_k\|_{L^k}^k \le \|Du_k\|_{L^{k+2}}^{k+2} + C$. Hence $\|Du_k\|_{L^k}^k \le C + C\|Du_k\|_{L^k}$, and so $\|Du_k\|_{L^k} \le C$. We deduce for each $k > q$ that

$$\|Du_k\|_{L^q} \le \|Du_k\|_{L^k}\ \|1\|_{L^{\frac{kq}{k-q}}} \le C.$$

$\square$

We next indentify the Γ–limit of problem (5), (6) as $k \to \infty$. For this, define

$$F[v] := \begin{cases} -\int_U fv\,dx & \text{if } v \in C_0^{0,1}(U),\ |Dv| \le 1 \ a.e. \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

Theorem 2.2. *As k goes to infinity, we have*

$$F_k \xrightarrow{\Gamma} F.$$

with respect to the uniform convergence of functions.

Proof. **1.** Since the mapping $u \mapsto \langle f, u\rangle = \int_U fu\,dx$ is linear, it is enough to prove

$$E_k[v] := \frac{1}{k}\int_U e^{\frac{k}{2}(|Dv|^2-1)}dx \xrightarrow{\Gamma} E[v], \tag{8}$$

for

$$E[v] := \begin{cases} 0 & \text{if } v \in C_0^{0,1}(U),\ |Dv| \le 1 \ a.e. \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

2. If $E[v] < \infty$, we clearly have

$$E[v] = 0 = \lim_{k\to\infty} E_k[v].$$

Suppose now that $v_k \to v$ uniformly, and $\limsup_{k\to\infty} E_k[v_k] \le C < \infty$. Fix an integer m and let $k > m$. Since $e^{\frac{x^2-1}{2}} \ge x$, we have for each open set $V \subseteq U$ that

$$\begin{aligned}\left(\int_V |Dv_k|^m\ dx\right)^{1/m} &\le |V|^{1/m-1/k}\left(\int_V |Dv_k|^k dx\right)^{1/k} \\ &\le |V|^{1/m-1/k}\,k^{1/k}\,E_k(v_k)^{1/k} \le |V|^{1/m-1/k}\,k^{1/k}\,C^{1/k}.\end{aligned}$$

Passing to limits in k and recalling the lower semicontinuity of the L^m norm of the gradient, we discover

$$\left(\int_V |Dv|^m dx\right)^{1/m} \le |V|^{1/m}.$$

This inequality, valid for all V as above, implies that Dv is in L^∞, with $|Dv| \le 1$ almost everywhere. Consequently,

$$E[v] = 0 \le \liminf_{k\to\infty} E_k[v_k].$$

□

Introduce next the vector fields

$$\mathbf{G}_k := \sigma_k Du_k \qquad (k = 1, \dots).$$

Theorem 2.3. *Suppose that for some $1 < q < \infty$ we have the uniform bounds*

$$\sup_k \|\mathbf{G}_k\|_{L^q(U;\mathbb{R}^n)} < \infty.$$

Define

$$f_k := -\operatorname{div}(\mathbf{G}_k),$$

and assume

$$\begin{cases} f_k \rightharpoonup f & \textit{weakly in } L^q(U) \\ \mathbf{G}_k \rightharpoonup \mathbf{G} & \textit{weakly in } L^q(U;\mathbb{R}^n), \\ u_k \to u & \textit{uniformly.} \end{cases}$$

Then there exists a positive function $a \in L^q$ such that

$$\begin{cases} \mathbf{G} = aDu, \\ |Du| = 1 \ \textit{a.e. on } \{a > 0\}, \ \textit{and} \\ \sigma_k \rightharpoonup a & \textit{weakly in } L^q(U). \end{cases}$$

In particular, $a = |\mathbf{G}|$.

Proof. **1.** First of all, note that $-\operatorname{div}\mathbf{G} = f$; that is,

$$\int_U \mathbf{G}\cdot D\psi\,dx = \int_U f\psi\,dx$$

for all $\psi \in C^1$, $\psi = 0$ on ∂U.

Let us now fix $0<\lambda<1$ and calculate:

$$\begin{aligned}\int_U |\mathbf{G}|\,dx \le \liminf_{k\to\infty}\int_U |\mathbf{G}_k|\,dx &= \liminf_{k\to\infty}\left(\int_{\{|Du_k|^2>1-\lambda\}}|\mathbf{G}_k|\,dx + \int_{\{|Du_k|^2\le 1-\lambda\}}|\mathbf{G}_k|\,dx\right)\\ &\le \liminf_{k\to\infty}\left(\frac{1}{\sqrt{1-\lambda}}\int_U |\mathbf{G}_k||Du_k|\,dx + \int_U e^{-\frac{k}{2}\lambda}\sqrt{1-\lambda}\,dx\right).\end{aligned}$$

When k goes to infinity, the last integral goes to 0. Notice also that

$$\int_U |\mathbf{G}_k||Du_k|\,dx = \int_U \sigma_k|Du_k|^2\,dx = \int_U f_k u_k\,dx.$$

Therefore

$$\sqrt{1-\lambda}\int_U |\mathbf{G}|\,dx \le \liminf_{k\to\infty}\int_U f_k u_k\,dx = \int_U fu\,dx = \int_U \mathbf{G}\cdot Du\,dx$$

for each $0<\lambda<1$, and consequently

$$\int_U |\mathbf{G}|\,dx \le \int_U \mathbf{G}\cdot Du\,dx. \tag{10}$$

2. Reasoning now as in the proof of Theorem 2.2, we fix an integer m and let $k>m$. Then for each open set $V\subseteq U$

$$\begin{aligned}\left(\int_V |Dv_k|^m\ dx\right)^{1/m} &\le |V|^{1/m-1/k+1}\left(\int_V |Dv_k|^{k+1}dx\right)^{1/k+1}\\ &\le |V|^{1/m-1/k+1}||\mathbf{G}_k||_{L^1}^{1/k+1} \le |V|^{1/m-1/k+1}C^{1/k+1}.\end{aligned}$$

Pass to limits in k to find

$$\left(\int_V |Dv|^m dx\right)^{1/m} \le |V|^{1/m},$$

and therefore $|Du|\le 1$ almost everywhere. The first two assertions of the Theorem now follow from (10).

3. To show also that $\sigma_k \rightharpoonup a$, let us fix $\psi\in C_0^\infty$ and prove

$$\int_U \sigma_k\psi\,dx \to \int_U a\psi\,dx.$$

We write

$$\int_U \sigma_k\psi\,dx = \int_U \sigma_k|Du_k|^2\psi\,dx + \int_U \sigma_k(1-|Du_k|^2)\psi\,dx =: A_1 + A_2.$$

Notice now that

$$\begin{aligned}A_1 = \int_U \psi\,\mathbf{G}_k\cdot Du_k\,dx &= \int_U \mathbf{G}_k\cdot D(u_k\psi)\,dx - \int_U u_k\,\mathbf{G}_k\cdot D\psi\,dx\\ &= \int_U f_k u_k\psi\,dx - \int_U u_k\mathbf{G}_k\cdot D\psi\,dx.\end{aligned}$$

This expression converges as $k \to \infty$ to

$$\begin{aligned}\int_U fu\psi\,dx - \int_U u\,\mathbf{G}\cdot D\psi\,dx &= \int_U \mathbf{G}\cdot D(u\psi)\,dx - \int_U u\,\mathbf{G}\cdot D\psi\,dx\\ &= \int_U \psi\,\mathbf{G}\cdot Du\,dx = \int_U \psi a|Du|^2\,dx = \int_U a\psi\,dx.\end{aligned}$$

4. It remains to show that $A_2 \to 0$. If we write $\varphi_k := |Du_k|^2 - 1$, then

$$|A_2| \le \|\psi\|_{L^\infty} \int_U e^{\frac{k}{2}\varphi_k}|\varphi_k|\,dx.$$

Since $xe^{-\frac{x}{2}} \le 1$ for each $x > 0$, we have

$$\int_{\{\varphi_k<0\}} e^{\frac{k}{2}\varphi_k}|\varphi_k|\,dx = \frac{1}{k}\int_U k|\varphi_k|e^{-\frac{k|\varphi_k|}{2}}\,dx \le \frac{|U|}{k} \to 0.$$

Finally, since $q > 1$ there exists a constant $c_q > 0$ such that

$$\frac{e^{x(q-1)}}{x} \ge c_q > 0$$

for all $x > 0$. Consequently,

$$\begin{aligned}\int_{\{\varphi_k>0\}} e^{\frac{k}{2}\varphi_k}|\varphi_k|\,dx &= \frac{2}{k}\int_{\{\varphi_k>0\}} e^{\frac{k}{2}\varphi_k}\ \frac{k}{2}\varphi_k\,dx\\ &\le \frac{2}{c_q k}\int_{\{\varphi_k>0\}} e^{\frac{qk}{2}\varphi_k}\,dx = \frac{2}{c_q k}\int_U \sigma_k^q\,dx \le \frac{2C^q}{c_q k} \to 0.\end{aligned}$$

This completes the proof that $A_2 \to 0$. ☐

3. Estimates I

The full calculations for our main estimate in §4 are fairly involved, and so for the reader's convenience we provide in this section a simpler computation illustrating the main ideas. Suppose $2 \le q < \infty$.

Theorem 3.1. *There exists a constant C, depending on q, but independent of k, such that*

$$\int_U \sigma_k^q\,dx \le C\left(\int_U |f|^q\,dx + 1\right). \tag{11}$$

Proof. **1.** To simplify notation, we hereafter in the proof do not write the subscripts k. Observe that since Du is bounded in each space L^q and $u = 0$ on ∂U, we have the bound

$$|u| \le C$$

for some constant C.

2. Multiply (5) by $\sigma^{q-1}u$ and integrate by parts:

$$\begin{aligned}\int_U \sigma^q|Du|^2 + (q-1)\sigma^{q-1}Du\cdot D\sigma\,u\,dx &= \int_U \sigma u_i(\sigma^{q-1}u)_i\,dx = \int_U f\sigma^{q-1}u\,dx\\ &\le C\left(\int_U |f|^q\,dx\right)^{\frac{1}{q}}\left(\int_U \sigma^q\,dx\right)^{1-\frac{1}{q}}.\end{aligned} \tag{12}$$

Here and afterwards we write the subscript i to denote the partial derivative with respect to the variable x_i.

Notice that $|Du|^2 \geq 1$ if $\sigma \geq 1$. Therefore

$$\int_U \sigma^q \, dx \leq C \left(\int_U |f|^q \, dx + \int_U \sigma^{q-1} |Du \cdot D\sigma| \, dx + 1 \right). \tag{13}$$

3. Next, multiply (5) by $-(\sigma^{q-1} u_j)_j$:

$$\begin{aligned} \int_U (\sigma u_i)_i (\sigma^{q-1} u_j)_j \, dx &= - \int_U f (\sigma^{q-1} u_j)_j \, dx \\ &= \int_U f \sigma^{q-2} (-(\sigma u_j)_j) \, dx - \int_U f (q-2) \sigma^{q-2} \sigma_j u_j \, dx \\ &\leq C \int_U f^2 \sigma^{q-2} + |f| \sigma^{q-2} |Du \cdot D\sigma| \, dx. \end{aligned} \tag{14}$$

The term on the left is

$$\begin{aligned} A :&= - \int_U \sigma u_i (\sigma^{q-1} u_j)_{ij} \, dx + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1} u_j)_j \, d\mathcal{H}^{n-1} \\ &= \int_U (\sigma u_i)_j (\sigma^{q-1} u_j)_i \, dx + \int_{\partial U} \sigma u_i \nu^i (\sigma^{q-1} u_j)_j - \sigma u_i \nu^j (\sigma^{q-1} u_j)_i \, d\mathcal{H}^{n-1}, \end{aligned} \tag{15}$$

where $\nu = (\nu^1, \dots, \nu^n)$ is the unit outer normal to ∂U. The boundary integral is

$$\begin{aligned} B := &\int_{\partial U} \sigma^q (u_i \nu^i u_{jj} - u_i \nu^j u_{ij}) \, d\mathcal{H}^{n-1} \\ &+ \int_{\partial U} (q-1) \sigma^{q-1} (u_i \nu^i u_j \sigma_j - u_i \nu^j \sigma_i u_j) \, d\mathcal{H}^{n-1}. \end{aligned} \tag{16}$$

The integrand of the last term equals 0, since $\sigma = e^{\frac{k}{2}(|Du|^2 - 1)}$ and so $\sigma_j = k u_l u_{lj} \sigma$.

Consider a point $x_0 \in \partial U$; without loss, we can take $x_0 = (0, \dots, R)$. Then $\nu = (0, \dots, 1)$ and $Du = (0, \dots, u_n)$, since $u = 0$ on ∂U. The integrand of the first term on the right hand side of (16) at x_0 therefore equals

$$\sigma^q (\Delta u - u_{nn}) u_n. \tag{17}$$

Lastly, write $x' = (x_1, \dots, x_{n-1})$ and observe that $u(x', \sqrt{R^2 - |x'|^2}) \equiv 0$ for small x'. We differentiate this identity twice and set $x' = 0$, to compute $\Delta u - u_{nn} = \frac{n-1}{R} u_n$ at x_0. Hence

$$B = \frac{n-1}{R} \int_{\partial U} \sigma^q |Du|^2 \, d\mathcal{H}^{n-1} \geq 0.$$

4. Therefore

$$\begin{aligned} A &= \int_U (\sigma u_i)_i \left(\sigma^{q-1} u_j \right)_j \, dx \geq \int_U (\sigma u_i)_j (\sigma^{q-1} u_j)_i \, dx \\ &= \int_U (\sigma u_{ij} + \sigma_j u_i)(\sigma^{q-1} u_{ij} + (q-1) \sigma^{q-2} \sigma_i u_j) \, dx \\ &= \int_U \sigma^q |D^2 u|^2 + (q-1) \sigma^{q-2} |Du \cdot D\sigma|^2 + q \sigma^{q-1} \sigma_j u_i u_{ij} \, dx. \end{aligned} \tag{18}$$

Recall that $\sigma_j = k u_l u_{lj} \sigma$. Hence (14) and (18) imply

$$\begin{aligned}\int_U \sigma^q |D^2 u|^2 + (q-1)\sigma^{q-2}|Du\cdot D\sigma|^2 + \frac{q}{k}\sigma^{q-2}|D\sigma|^2 dx \\ \leq C\int_U f^2\sigma^{q-2} + |f|\sigma^{q-2}|Du\cdot D\sigma|\,dx \\ \leq \frac{q-1}{2}\int_U \sigma^{q-2}|Du\cdot D\sigma|^2 + C\int_U |f|^2\sigma^{q-2}\,dx;\end{aligned} \tag{19}$$

and consequently

$$\int_U \sigma^{q-2}|Du\cdot D\sigma|^2\,dx \leq C\int_U |f|^2\sigma^{q-2}\,dx. \tag{20}$$

5. Combine (13),(20):

$$\begin{aligned}\int_U \sigma^q\,dx &\leq C\int_U |f|^q\,dx + C\int_U \sigma^{q-1}|Du\cdot D\sigma|\,dx + C \\ &\leq C\int_U |f|^q\,dx + \frac{1}{3}\int_U \sigma^q\,dx + C\int_U \sigma^{q-2}|Du\cdot D\sigma|^2\,dx + C \\ &\leq C\int_U |f|^q\,dx + \frac{1}{3}\int_U \sigma^q\,dx + C\int_U |f|^2\sigma^{q-2}\,dx + C \\ &\leq C\int_U |f|^q\,dx + \frac{2}{3}\int_U \sigma^q\,dx + C\end{aligned} \tag{21}$$

This gives (11). □

Remark. The boundary integral term B is in fact nonnegative for any convex, smooth domain replacing $U = B(0,R)$: see for instance the similar calculations in §1.5 of Ladyzhenskaja [L]. □

4. Estimates II

In this section we derive our main integral estimate.

Theorem 4.1. *Assume that $2 \leq q < \infty$ and that $f \in C^\infty(\bar{U})$. Then there exist a constant C, depending only on q, and a constant K, depending only on $\|f\|_{L^\infty}$, such that*

$$\int_U \sigma_k^q |Du_k|^q\,dx \leq C\left(\int_U |f|^q\,dx + 1\right) \tag{22}$$

for all $k \geq K$.

The proof is similar to that of Theorem 3.1, except that we must handle the additional term $|Du_k|^q$ on the left. This makes our multipliers and estimates more intricate.

Proof. **1.** For notational simplicity we hereafter write σ and u in place of σ_k and u_k.

Since f is smooth, the same is true for u and σ. Observe also the bound

$$|u| \leq C.$$

We record for later reference these consequences of (6):

$$|Du|_i = \frac{\sigma_i}{k\sigma|Du|},\ u_i u_{ij} = \frac{\sigma_j}{k\sigma}. \tag{23}$$

2. We multiply the PDE (5) by $\sigma^{q-1}|Du|^{q+1}u$ and integrate by parts, to find

$$\int_U \sigma Du \cdot D\left(\sigma^{q-1}|Du|^{q+1}u\right)\, dx = \int_U \sigma^{q-1}|Du|^{q+1}uf\, dx. \tag{24}$$

The right hand term in (24) is less than or equal to

$$C\int_U \sigma^{q-1}|Du|^{q+1}|f|\, dx \le \frac{1}{2}\int_{\{|f|\le\frac{\sigma|Du|}{2C}\}} \sigma^q|Du|^{q+2}\, dx + 2^{q-1}C^q\int_{\{|f|>\frac{\sigma|Du|}{2C}\}} |Du|^2|f|^q\, dx.$$

But if $\sigma|Du| < 2C|f|$, then obviously $\sigma|Du| \le 2C\|f\|_{L^\infty}$. Recalling (6), we see that this implies $|Du| \le 2$ provided $k \ge K$, for some constant K depending only upon $\|f\|_{L^\infty}$. Therefore

$$\int_U \sigma^{q-1}|Du|^{q+1}uf\, dx \le \frac{1}{2}\int_U \sigma^q|Du|^{q+2}\, dx + C\int_U |f|^q\, dx. \tag{25}$$

3. We use (23) to evaluate the left hand term in (24):

$$\begin{aligned}
\int_U \sigma Du \cdot D\left(\sigma^{q-1}|Du|^{q+1}u\right)\, dx &= \int_U \sigma^q|Du|^{q+3}\, dx \\
+(q-1)\int_U \sigma^{q-1}|Du|^{q+1}u\, D\sigma\cdot Du\, dx &+ (q+1)\int_U \sigma^q u|Du|^q Du\cdot(D|Du|)\, dx \\
&= \int_U \sigma^q|Du|^{q+3}\, dx \\
+(q-1)\int_U \sigma^{q-1}|Du|^{q+1}u\, D\sigma\cdot Du\, dx &+ \frac{q+1}{k}\int_U \sigma^{q-1}u|Du|^{q-1}Du\cdot D\sigma\, dx.
\end{aligned} \tag{26}$$

But $\sigma \ge 1$ only if $|Du| \ge 1$; and hence

$$\int_U \sigma^q|Du|^{q+2}\, dx \le \int_U \sigma^q|Du|^{q+3}\, dx + C, \tag{27}$$

since U is bounded.

Combining (27), (26), (24) and (25), we deduce the inequality

$$\begin{aligned}
\int_U \sigma^q|Du|^{q+2}\, dx \le C &+ \frac{1}{2}\int_U \sigma^q|Du|^{q+2}\, dx + C\int_U |f|^q\, dx \\
&+ C\int_U \sigma^{q-1}|Du|^{q+1}|D\sigma\cdot Du|\, dx + \frac{C}{k}\int_U \sigma^{q-1}|Du|^{q-1}|D\sigma\cdot Du|\, dx.
\end{aligned}$$

Arguing as before (this means dividing the integrals in the set where $|D\sigma\cdot Du| \le \epsilon\,\sigma|Du|/C$ and in the rest of U), we see that therefore

$$\begin{aligned}
\int_U \sigma^q|Du|^{q+2}\, dx \le C + C\int_U |f|^q\, dx &+ \epsilon\int_U \sigma^q|Du|^{q+2}\, dx + \frac{\epsilon}{k}\int_U \sigma^q|Du|^q\, dx \\
&+ \frac{C^2}{\epsilon}\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\, dx + \frac{C^2}{k\epsilon}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\, dx
\end{aligned} \tag{28}$$

for any $\epsilon > 0$. Since $\int_U \sigma^q|Du|^q\, dx \le \int_U \sigma^q|Du|^{q+2}\, dx + C$, this implies our first main estimate:

$$\begin{aligned}
\int_U \sigma^q|Du|^{q+2}\, dx \le C + C\int_U |f|^q\, dx &+ C\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\, dx \\
&+ \frac{C}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\, dx.
\end{aligned} \tag{29}$$

4. The last two terms in (29) involving $D\sigma \cdot Du$ are dangerous, since $D\sigma$ is of order k: we need another estimate to control them.

Let us therefore continue by multiplying the PDE (5) by $-\operatorname{div}\left(\sigma^{q-1}|Du|^q Du\right)$ and thereby deriving the identity

$$\int_U \operatorname{div}(\sigma Du)\,\operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx = -\int_U f\,\operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx. \tag{30}$$

The term on the right equals

$$\begin{aligned}\int_U f\,\sigma^{q-2}|Du|^q\,(-\operatorname{div}(\sigma Du))\,dx - \int_U f\sigma Du\cdot D\left(\sigma^{q-2}|Du|^q\right)\,dx = \int_U |f|^2\sigma^{q-2}|Du|^q\,dx \\ -(q-2)\int_U f\sigma^{q-2}|Du|^q Du\cdot D\sigma\,dx - q\int_U f\sigma^{q-1}|Du|^{q-1}Du\cdot(D|Du|)\,dx.\end{aligned}$$

We again recall (23) and deduce

$$\begin{aligned}-\int_U f\,\operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx &\le \int_U |f|^2\sigma^{q-2}|Du|^q\,dx \\ &+(q-2)\int_U |f|\sigma^{q-2}|Du|^q|Du\cdot D\sigma|\,dx + \frac{q}{k}\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|\,dx.\end{aligned} \tag{31}$$

The left hand term of (30) is

$$\begin{aligned}A := &-\int_U \sigma u_i(\sigma^{q-1}|Du|^q u_j)_{ij}\,dx + \int_{\partial U}\sigma u_i\nu^i(\sigma^{q-1}|Du|^q u_j)_j\,d\mathcal{H}^{n-1} \\ &= \int_U (\sigma u_i)_j(\sigma^{q-1}|Du|^q u_j)_i\,dx \\ &\quad + \int_{\partial U}\sigma u_i\nu^i(\sigma^{q-1}|Du|^q u_j)_j - \sigma u_i\nu^j(\sigma^{q-1}|Du|^q u_j)_i\,d\mathcal{H}^{n-1}.\end{aligned} \tag{32}$$

Call the boundary term B. Then, almost exactly as in step 3 of the previous proof, we can show that

$$B = \frac{n-1}{R}\int_{\partial U}\sigma^q|Du|^{q+2}\,d\mathcal{H}^{n-1} \ge 0.$$

Consequently,

$$\begin{aligned}A &= \int_U (\sigma u_i)_i\left(\sigma^{q-1}|Du|^q u_j\right)_j\,dx \ge \int_U (\sigma u_i)_j\left(\sigma^{q-1}|Du|^q u_j\right)_i\,dx \\ &= \int_U (\sigma u_{ij}+\sigma_j u_i)\left(\sigma^{q-1}|Du|^q u_{ij} + (q-1)\sigma^{q-2}|Du|^q\sigma_i u_j + \frac{q}{k}\sigma^{q-2}|Du|^{q-2}\sigma_i u_j\right)\,dx \\ &= \int_U \sigma^q|Du|^q|D^2u|^2 + \frac{q}{k}\sigma^{q-2}|Du|^q|D\sigma|^2 + (q-1)\sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2 + \\ &\quad + \frac{q}{k^2}\sigma^{q-2}|Du|^{q-2}|D\sigma|^2 + \frac{q}{k}\sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx.\end{aligned}$$

The first, the second and the fourth terms in the last expression are positive, and so we deduce

$$\begin{aligned}(q-1)\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx + \frac{q}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\ \le \int_U \operatorname{div}(\sigma Du)\,\operatorname{div}(\sigma^{q-1}|Du|^q Du)\,dx.\end{aligned} \tag{33}$$

Collecting (33), (30) and (31), we find

$$
\begin{aligned}
\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx &+ \frac{1}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\
&\le C\int_U |f|^2\sigma^{q-2}|Du|^q\,dx + C\int_U |f|\sigma^{q-2}|Du|^q|Du\cdot D\sigma|\,dx \\
&\quad + \frac{C}{k}\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|\,dx.
\end{aligned}
\tag{34}
$$

Take $\epsilon > 0$ to be a small constant, which will be fixed later on. Then

$$
\begin{aligned}
\int_U f^2\sigma^{q-2}|Du|^q\,dx &\le \epsilon\int_U \sigma^q|Du|^{q+2}\,dx + C\int_{\{|f|>\sigma|Du|\sqrt{\epsilon}\}}|f|^q|Du|^2\,dx \\
&\le \epsilon\int_U \sigma^q|Du|^{q+2}\,dx + C\int_U |f|^q\,dx,
\end{aligned}
\tag{35}
$$

since $|Du| \le 2$ wherever $|f| > \sigma|Du|\sqrt{\epsilon}$, provided $k \ge K$ and K is large.

Recalling (35), we can likewise estimate for each $\delta > 0$ that

$$
\begin{aligned}
\int_U |f|\sigma^{q-2}|Du|^q|D\sigma\cdot Du|\,dx &\le \delta\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx + C\int_U f^2\sigma^{q-2}|Du|^q\,dx \\
&\le \delta\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx + C\left(\epsilon\int_U \sigma^q|Du|^{q+2}\,dx + \int_U |f|^q\,dx\right).
\end{aligned}
\tag{36}
$$

Similarly,

$$
\begin{aligned}
\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|dx &\le \delta\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2dx + C\int_U f^2\sigma^{q-2}|Du|^{q-2}dx \\
&\le \delta\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx + C\left(\epsilon\int_U \sigma^q|Du|^q\,dx + C\int_U |f|^q\,dx\right).
\end{aligned}
\tag{37}
$$

Since $\sigma \ge 1$ only if $|Du| \ge 1$, we have

$$
\int_U \sigma^q|Du|^q\,dx \le \int_U \sigma^q|Du|^{q+2}\,dx + C,
$$

and therefore

$$
\begin{aligned}
\int_U |f|\sigma^{q-2}|Du|^{q-2}|Du\cdot D\sigma|\,dx &\le \delta\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\
&\quad + C\left(\epsilon\int_U \sigma^q|Du|^{q+2}\,dx + \int_U |f|^q\,dx + 1\right).
\end{aligned}
\tag{38}
$$

Taking $\delta > 0$ small, we then derive from (34), (35), (36) and (38) our second main inequality

$$
\begin{aligned}
\int_U \sigma^{q-2}|Du|^q|D\sigma\cdot Du|^2\,dx &+ \frac{1}{k}\int_U \sigma^{q-2}|Du|^{q-2}|D\sigma\cdot Du|^2\,dx \\
&\le \epsilon\int_U \sigma^q|Du|^{q+2}\,dx + C\int_U |f|^q + C.
\end{aligned}
\tag{39}
$$

5. Putting together inequalities (29) and (39) and fixing $\epsilon > 0$ small, we finally discover

$$
\int_U \sigma^q|Du|^{q+2}\,dx \le C + C\int_U |f|^q\,dx.
$$

As $\sigma \ge 1$ only if $|Du| \ge 1$, estimate (22) follows. $\square$

Theorem 4.1 concerns only smooth functions f. However, since the bound for the L^q norm of the transport density depends only upon the L^q norm of f, we can approximate:

Theorem 4.2. *For each $2 \leq q < \infty$ and $f \in L^q(U)$ the associated transport density a belongs to $L^q(U)$. Furthermore, there is a constant C, depending only upon n and U, such that*

$$\|a\|_{L^q} \leq C\left(\|f\|_{L^q} + 1\right). \tag{40}$$

Proof. **1.** Let us first define

$$f_j := f * \eta_{1/j},$$

the convolution of f with a standard mollifier. For each integer j, we then solve

$$\begin{cases} -\operatorname{div}(\sigma_{k,j} Du_{k,j}) = f_j & \text{in } U \\ u_{k,j} = 0 & \text{on } \partial U, \end{cases} \tag{41}$$

for

$$\sigma_{k,j} := e^{\frac{k}{2}\left(|Du_{k,j}|^2 - 1\right)}. \tag{42}$$

2. According to (22), we have the estimate

$$\int_U \sigma_{k,j}^q |Du_{k,j}|^q \, dx \leq C\left(\int_U |f_j|^q \, dx + 1\right) \leq C\left(\int_U |f|^q \, dx + 1\right) \tag{43}$$

for all k greater than or equal to some constant $K = k(j)$, depending only on the L^∞ norm of f_j. Now define

$$\sigma_j := \sigma_{k(j),j}, \quad u_j := u_{k(j),j}, \quad \mathbf{G}_j := \sigma_j Du_j.$$

Clearly $f_j \to f$ in L^q. Furthermore, (43) implies that $\mathbf{G}_j$ is bounded in L^q. We may therefore assume upon reindexing that

$$\mathbf{G}_j \rightharpoonup \mathbf{G} \quad \text{weakly in } L^q(U;\mathbb{R}^n).$$

Finally we may pass as necessary to a further subsequence to ensure u_j converges uniformly to a limit u. Apply Theorem 2.3. □

References

[B-B] G. Bouchitté and G. Buttazzo, Characterization of optimal shapes and masses through Monge–Kantorovich equation, Journal European Math. Soc., **3** (2001), 139–168.

[B-B-S] G. Bouchitté, G. Buttazzo and P. Seppecher, Shape optimization solutions via Monge–Kantorovich equation, C. R. Acad. Sci. Paris, **324–I** (1997), 1185–1191.

[B-B-DP] G. Bouchitté, G. Buttazzo and L. De Pascale, A p-Laplacian approximation for some mass optimization problems, to appear on J.O.T.A. (Journal of Optimization Theory and Applications), (2002).

[DP-P] L. De Pascale and A. Pratelli, Regularity properties for Monge transport density and for solutions of some shape optimization problems, Calculus of Variations and Partial Differential Equations, **14** (2002), 249–274.

[E1] L. C. Evans, Some new PDE methods for weak KAM theory, to appear in Calculus of Variations and Partial Differential Equations.

[E2] L. C. Evans, Partial differential equations and Monge-Kantorovich mass transfer (survey paper), *Current Developments in Mathematics, 1997*, International Press (1999), edited by S. T. Yau.

[L] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, trans. by Richard Silverman, Gordon and Breach, (1969).

[M] P. Marcellini, General growth conditions and regularity, in *Variational methods for discontinuous structures*, edited by Serapioni, Raul et al., Progress in Nonlinear Differential Equations and Applications #25, Birkhauser (1996), 111-118.

(L. De Pascale) Dipartimento di Matematica Applicata, Università di Pisa, 56126 Pisa, Italy

(L. C. Evans) Department of Mathematics, University of California, Berkeley, CA 94720, USA

(A. Pratelli) Scuola Normale Superiore, 56126 Pisa, Italy