NOTES ON KOSZUL ALGEBRAS

ULRICH KRÄHMER

Abstract. This is a self-contained and elementary survey of some well-known material on connected and especially Koszul algebras. We discuss in particular the global dimension of connected algebras and the equivalence of various characterisations of Koszulness.

Contents

1. Introduction

A first version of this text arose when I wanted to learn how to show that the algebra of quantum matrices (see Example 5 below) has finite global dimension. By browsing papers by Manin, by Smith and by Van den Bergh I found out that one way is to apply the following theorem which is well-known to experts:

Theorem 1. *Let $A = \bigoplus_{i \geq 0} A_i$ be a Noetherian graded algebra over a field $k = A_0$. Then we have*

$$\mathrm{gl.dim}(A) = \mathrm{pd}(k) = \sup\{i \mid \mathrm{Ext}^i_A(k,k) \neq 0\} = \sup\{i \mid \mathrm{Tor}^A_i(k,k) \neq 0\}.$$

This was not at all obvious to me, and I decided to collect together for my own benefit everything needed to understand both statement and proof of this theorem. A workout of this material forms now Sections 2 and 3 of the present note which hopefully make Theorem 1 and its surrounding theory of minimal resolutions accessible for anyone who knows some basic ring theory and homological algebra.

One year later I had the pleasure to attend some lectures on cyclic homology by J.-L. Loday in Warsaw in which he also mentioned Koszul algebras, a class of algebras as in Theorem 1 with particularly good homological properties. When I replaced him in one lecture I decided to speak about their Hochschild homology, and all this fitted well to what I had written up about Theorem 1. So I added Section 4 where I explain some of the standard characterisations of Koszulness:

Theorem 2. *For A as in Theorem 1, the following are equivalent:*

1. *A is a Koszul algebra.*
2. *Both A and its Yoneda algebra $\mathrm{Ext}_A(k,k)$ are generated in degree 1.*
3. *A is quadratic and its Koszul dual $A^!$ is Koszul.*
4. *The canonical map $A^! \to \mathrm{Ext}_A(k,k)$ is an isomorphism.*
5. *The Koszul complex $K(A,k)$ is acyclic.*
6. *The Koszul complex $K(A^e,A)$ is acyclic.*

Finally I added Section 5 where I sketch without proofs the interplay of Koszulness with Poincaré-Birkhoff-Witt bases.

I would like to thank Brad Shelton and Paul Smith for prompt and detailed answers to questions that I asked while working on the first half of this text (parts of it are just an elaboration of these answers). Similarly, I thank Jean-Louis Loday and Christian Voigt for our discussions on the material of the second half and for several comments about a draft of these notes. Finally, Matthew Tucker-Simmons has pointed out some typos that I have corrected now.

2. Preliminaries and terminology

2.1. **Vector spaces.** Throughout, k denotes a field and essentially all appearing mathematical objects have an underlying structure of a $\mathbb{Z}$-graded k-vector space

$$V = \bigoplus_{i \in \mathbb{Z}} V_i.$$

The existence of a grading will sometimes be stressed by writing $V_\bullet$ instead of just V. The elements of V_i are called ***homogeneous*** of ***degree*** i. Usually we tacitly assume by linear decomposition that we deal with homogeneous elements e.g. when defining linear maps. Then we write $|v| = i$ when $v \in V_i$.

A graded vector space V is said to be ***locally finite-dimensional*** if $\dim_k V_i < \infty$ for all i and ***connected*** if $V_i = 0$ for $i < 0$ and $V_0 = k$.

We define

$$\mathrm{Hom}_k^j(V, W) := \{\varphi \in \mathrm{Hom}_k(V, W) \mid \varphi(V_i) \subset W_{i-j}\}$$

and

$$\underline{\mathrm{Hom}}_k(V, W) := \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_k^j(V, W) \subset \mathrm{Hom}_k(V, W). \tag{1}$$

A morphism of graded vector spaces is an element of $\mathrm{Hom}_k^0(V, W)$, and we denote the resulting category by k-$\underline{\mathbf{Mod}}$. The full subcategory of locally finite-dimensional connected vector spaces is denoted by **Conn**.

Example 1. The ***graded dual*** of $V = \bigoplus_{i \in \mathbb{Z}} V_i$ is

$$V^{\underline{*}} := \underline{\mathrm{Hom}}_k(V, k) = \bigoplus_{i \in \mathbb{Z}} V_i^* \subset V^* = \mathrm{Hom}_k(V, k).$$

If V is infinite-dimensional, then V^* has strictly greater dimension than V. But we have $V^{\underline{*}} \simeq V$ in k-$\underline{\mathbf{Mod}}$ as long as V is locally finite-dimensional. Hence $\underline{*}$ defines an internal duality functor on **Conn**.

2.2. **Tensor products.** An unadorned $\otimes$ denotes a tensor product of k-vector spaces. For $V, W \in k$-$\underline{\mathbf{Mod}}$ we consider $V \otimes W$ as an object of k-$\underline{\mathbf{Mod}}$ with grading defined by

$$(V \otimes W)_i := \bigoplus_{r+s=i} V_r \otimes W_s,$$

that is, $|v \otimes w| = |v| + |w|$, and we denote the tensor product then by $V \underline{\otimes} W$. While this is not so important on objects, there will be a real difference on morphisms: for the tensor product of linear maps between graded vector spaces we adopt ***Koszul's sign convention***

$$(\varphi \underline{\otimes} \psi)(v \underline{\otimes} w) := (-1)^{|\psi||v|} \varphi(v) \underline{\otimes} \psi(w),$$

where the ***degree*** $|\psi|$ of ψ is j if $\psi \in \mathrm{Hom}_k^j(V,W)$. A more abstract way to deal with all this is to work in suitable braided monoidal categories such as graded vector spaces with $\underline{\mathrm{Hom}}$ as morphisms and with a braiding that introduces signs according to Koszul's convention. Then the dual of $V \otimes W$ should be $W^* \otimes V^*$ and the signs would come in through the braiding which is used to identify this with $V^* \otimes W^*$. But I finally decided that I want to avoid this abstraction to keep the preliminary section shorter than the main text, and there will be [4] where this material is going to be presented.

Example 2. There are isomorphisms $(V \underline{\otimes} W)^{\underline{*}} \simeq V^{\underline{*}} \underline{\otimes} W^{\underline{*}}$ for any $V, W \in \mathbf{Conn}$, but when identifying $\varphi \underline{\otimes} \psi \in V^{\underline{*}} \underline{\otimes} W^{\underline{*}}$ with a linear functional on $V \underline{\otimes} W$ we will take into account the sign convention.

2.3. **Algebras.** By "***algebra***" we mean "unital associative algebra over k", and in fact an algebra in **Conn** if not explicitly stated otherwise. That is, an algebra is a connected vector space $A = \bigoplus_{i \geq 0} A_i$, $A_0 = k$, with a product for which $A_i A_j \subset A_{i+j}$. The symbol A_+ will denote the ***augmentation ideal*** $\bigoplus_{i>0} A_i$. We usually also assume that all algebras are left and right Noetherian.

Example 3. An algebra that we will use in several constructions is the ***tensor algebra*** of a vector space V, that is, the graded vector space $T(V) = \bigoplus_{i \geq 0} V^{\otimes i}$, $V^{\otimes 0} := k$, with product defined by concatenation of elementary tensors. This is connected and locally finite-dimensional if $\dim_k V < \infty$, but it is not Noetherian.

2.4. **Quadratic algebras.** For us, the most important examples of algebras are ***quadratic algebras***, that is, algebras defined in terms of a finite number of generators satisfying homogeneous quadratic relations. In other words, a quadratic algebra is a quotient of a tensor algebra $T(V)$ of a finite-dimensional vector space V by the two-sided ideal generated in $T(V)$ by a vector subspace $R \subset V \otimes V$. Any basis of V defines a (minimal) set of generators of A and R encodes the relations between them. We denote the corresponding quadratic algebra by $A(V,R)$.

Example 4. The algebra $k[x_1, \ldots, x_n]$ of polynomials in n indeterminates with coefficients in k is quadratic with

$$V = k^n = \mathrm{span}_k\{x_1, \ldots, x_n\}$$

and

$$R = \mathrm{span}_k\{x_i \otimes x_j - x_j \otimes x_i \mid i,j = 1, \ldots, n\}.$$

Changing the sign in the relations to

$$R = \mathrm{span}_k\{x_i \otimes x_j + x_j \otimes x_i \mid i,j = 1, \ldots, n\}$$

yields the exterior algebra ΛV.

Example 5. [6] A genuine noncommutative examples are the ***quantum plane*** which has two generators x, y fulfilling $xy = qyx$ for some nonzero parameter $q \in k$, so here we have

$$V = k^2 = \operatorname{span}_k\{x, y\}$$

and

$$R = \operatorname{span}_k\{x \otimes y - qy \otimes x\}.$$

A slightly more complicated one are the ***quantum matrices*** which have generators a, b, c, d with relations

$$ab = qba,\ ac = qca,\ ad - da = (q - q^{-1})bc,\ bc = cb,\ bd = qdb,\ cd = qdc.$$

Both these algebras are Noetherian (use [7] Theorem 1.2.9).

Example 6. The algebra with two generators x, y having relations $x^2 = yx = 0$ is right, but not left Noetherian.

2.5. **Modules.** By A***-module*** we mean left A-module with $1 \in A$ acting as identity, and their category (with A-linear maps as morphisms) is denoted by A-**Mod**. We will deal almost exclusively with finitely generated modules whose category is denoted by A-**mod**. The category of (finitely generated) graded modules

$$M = \bigoplus_{j \in \mathbb{Z}} M_j, \quad A_i M_j \subset M_{i+j},$$

is denoted by A-$\underline{\mathbf{Mod}}$ (A-$\underline{\mathbf{mod}}$). As for vector spaces, morphisms are understood to preserve the degree, that is, we consider $\operatorname{Hom}^0_A(M, N)$ and not $\underline{\operatorname{Hom}}_A(M, N)$ in the notation analogous to (1). Note that any $M \in A$-$\underline{\mathbf{mod}}$ is ***bounded below***, $M = \bigoplus_{j \geq s} M_j$ for some s.

Example 7. The canonical ***augmentation*** $\varepsilon : A \to A/A_+ \simeq k$ gives k the structure of a left and a right module over A. The module k is called the ***trivial*** module. It is the only simple object in A-$\underline{\mathbf{mod}}$.

2.6. **Complexes.** A ***(chain) complex*** of vector spaces (analogously of A-modules) is a pair (C, d) where $C \in k$-$\underline{\mathbf{Mod}}$ and $\mathrm{d} \in \operatorname{Hom}^1_k(C, C)$ satisfies $\mathrm{d} \circ \mathrm{d} = 0$. A complex is ***exact*** or ***acyclic*** if its ***homology***

$$H(C) := \ker \mathrm{d} / \operatorname{im} \mathrm{d} \in k\text{-}\underline{\mathbf{Mod}}$$

vanishes. Often we write d_i for the component $C_i \to C_{i-1}$ of d and the whole complex as a sequence

$$\cdots \xrightarrow{\mathrm{d}_{i+2}} C_{i+1} \xrightarrow{\mathrm{d}_{i+1}} C_i \xrightarrow{\mathrm{d}_i} C_{i-1} \xrightarrow{\mathrm{d}_{i-1}} \cdots$$

With slight ambiguity we also speak of an acylic complex when $C_i = 0$ for $i < 0$ and $H_i(C) = 0$ for $i > 0$, because then one can simply add $C_{-1} := H_0(C) = C_0/\operatorname{im} \mathrm{d}_1$ to C to obtain an honestly acyclic complex.

Cochain complexes are a variation of chain complexes in which we have $\mathrm{d} \in \operatorname{Hom}_k^{-1}(C, C)$ (degree -1 rather than $+1$). Usually the grading is then written upstairs, $C^\bullet = \bigoplus_i C^i$. From the theoretical point of view there is no need to introduce them, since any cochain complex can be turned into a chain complex by setting $C_i := C^{-i}$. However, it is handy to use both variants.

2.7. **Derived functors.** A module $P \in A$-**Mod** is ***projective*** if for any epimorphism $M \to N$ in A-**Mod** the induced morphism

$$\operatorname{Hom}_A(P, M) \to \operatorname{Hom}_A(P, N)$$

is also surjective. If (P, φ) is an exact complex in A-**Mod** with $P_i = 0$ for $i \leq -2$ and P_i projective for $i \geq 0$, then the truncated complex

$$\cdots \xrightarrow{\varphi_4} P_3 \xrightarrow{\varphi_3} P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \longrightarrow 0 \tag{2}$$

is called a ***projective resolution*** of $P_{-1} \in A$-**Mod**. Any $M \in A$-**Mod** admits such resolutions which might differ in their ***lengths***

$$\ell(P) := \sup\{i \mid P_i \neq 0\},$$

but are all related by algebraic analogues of homotopies. The ***projective dimension*** of M is

$$\operatorname{pd}(M) := \inf\{\ell(P) \mid P \text{ is a projective resolution of } M\}.$$

Thus $\operatorname{pd}(M) = 0$ if and only if M is projective, and in some sense pd measures the nonprojectivity of M.

If one applies the functor $\operatorname{Hom}_A(\cdot, N)$, $N \in A$-**Mod**, to a projective resolution $P_\bullet$ of M, then the cohomology of the resulting cochain complex $\operatorname{Hom}_A(P_\bullet, N)$ of abelian groups is (as a consequence of the homotopy equivalence of resolutions) independent of the chosen one, and it equals $\operatorname{Ext}_A^\bullet(M, N) = \bigoplus_{i \geq 0} \operatorname{Ext}_A^i(M, N)$. These Ext-groups measure the failure of $\operatorname{Hom}_A(\cdot, N)$ to be exact: one has

$$\operatorname{Ext}_A^0(M, N) = \operatorname{Hom}_A(M, N),$$

and

$$\operatorname{Ext}_A^i(M, N) = 0, \quad i < 0,$$

and if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of modules, then there is a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_A(M_3, N) \longrightarrow \mathrm{Ext}^i_A(M_2, N) \longrightarrow \mathrm{Ext}^i_A(M_1, N) \longrightarrow \mathrm{Ext}^{i+1}_A(M_3, N) \longrightarrow \mathrm{Ext}^{i+1}_A(M_2, N) \longrightarrow \mathrm{Ext}^{i+1}_A(M_1, N) \longrightarrow \cdots$$

From this one deduces that

$$\mathrm{pd}(M) = \sup\{d \mid \exists N \in A\text{-}\mathbf{Mod} : \mathrm{Ext}^d_A(M, N) \neq 0\}.$$

If instead of $\mathrm{Hom}_A(\cdot, N)$ one applies $L \otimes_A \cdot$ (L a right module) to $P_\bullet$, then the homology of $L \otimes_A P_\bullet$ is $\mathrm{Tor}^A_\bullet(L, M)$ which analogously measures the nonexactness of $L \otimes_A \cdot$. In fact, the resolution needs to be only by ***flat*** modules in this case, that is, for any monomorphism $L \to N$ of right modules the induced morphism $L \otimes_A P \to N \otimes_A P$ must be again injective. Any projective module is flat, and for finitely generated modules over Noetherian rings the converse holds as well.

Note: When talking about projectiveness, flatness, Tor, Ext etc. one must specify precisely which category one is working in. A graded module $M \in A$-$\underline{\mathbf{mod}}$ can be considered in A-$\underline{\mathbf{mod}}$, A-$\underline{\mathbf{Mod}}$, A-$\mathbf{mod}$ or in A-$\mathbf{Mod}$, and a priori this might make a difference. However, Noetherianity ensures that nothing will go wrong in our applications so that we suppress this question whenever possible.

2.8. **Spectral sequences.** Let $(C_\bullet, \mathrm{d})$ be a chain complex (say of vector spaces, $C_n = 0$ for $n < 0$ and $\mathrm{d}(C_n) \subset C_{n-1}$). Then a ***filtration*** of $C_\bullet$, that is, a filtration

$$\ldots \subset F_p C_n \subset F_{p+1} C_n \subset \ldots \subset C_n$$

of each C_n with $\mathrm{d}(F_p C_n) \subset F_p C_{n-1}$, gives rise to a ***spectral sequence***. This is a standard yet somewhat advanced topic, so I give at least some more explanation (see e.g. [10] for a relatively compact presentation).

The spectral sequence consists of filtered complexes (E^r, d^r) (its "***pages***") each of which is the homology of the previous one,

$$E^{r+1} = H(E^r),$$

and these approximate in some sense the homology $H(C)$ of the original complex. Usually, one applies a reshuffling of gradings as in (7) below and uses two indices p, q on each page: p is the filtration degree and $n = p + q$ is the homological degree that is lowered by the differential.

The sequence starts with the graded complex associated to the filtered one:

$$E^0_{pq} := F_pC_{p+q}(A)/F_{p-1}C_{p+q}(A),$$

and $\mathrm{d}^0 : E^0_{pq} \to E^0_{pq-1}$ is the map induced by d. Then one puts

$$E^1_{pq} := H_{p+q}(E^0_{p\bullet}).$$

Thus classes in E^1_{pq} are represented by elements $c \in F_pC_{p+q}$ that satisfy $\mathrm{d}c \in F_{p-1}C_{p+q-1}$. More generally, classes in E^r_{pq} are represented by elements in

$$Z^r_{pq} := \{c \in F_pC_{p+q} \mid \mathrm{d}c \in F_{p-r}C_{p+q-1}\},$$

and $\mathrm{d}^r : E^r_{pq} \to E^r_{p-rq+r-1}$ takes into account more and more parts of d that were cut off when passing from C to E^0. If B^r is the image of d^r, then one has

$$B^0 \subset B^1 \subset B^2 \subset \ldots \subset \ldots \subset Z^2 \subset Z^1 \subset Z^0, \quad E^{r+1} \simeq Z^r/B^r.$$

Intuitively, the E^r thus approach the ***limit***

$$E^\infty := (\bigcap_{r\in\mathbb{N}} Z^r)/(\bigcup_{r\in\mathbb{N}} B^r)$$

of the spectral sequence. The precise statement is this: the filtration of $C_\bullet$ yields a filtration of $H(C)$, and in good cases the spectral sequence ***converges*** in the sense that

$$E^\infty_{pq} \simeq F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C),$$

that is, E^∞ is the graded vector space associated to the filtered $H(C)$. In particular, $\bigoplus_{p+q=n} E^\infty_{pq}$ is then isomorphic to $H_n(C)$ as vector space.

In very good cases, the spectral sequence even stabilises for fixed $p+q$ at some term, $E^r_{pq} = E^{r+1}_{pq} = \ldots = E^\infty_{pq}$ (with r depending on $p+q$). For example, this is the case if the filtration is ***bounded***, i.e. if

$$F_{p^{\min}_n}C_n = 0, \quad F_{p^{\max}_n}C_n = C_n$$

for suitable $p^{\min}_n \leq p^{\max}_n$. And in extremely good but in fact not rare cases, spectral sequences stabilise independently of p and q on E^2. Assume for example that $E^1_{pq} = 0$ for all q except one, say q_0. Then of course also $E^2_{pq} = 0$ for $q \neq q_0$. The differentials $\mathrm{d}^r : E^r_{pq} \to E^r_{p-rq+r-1}$ are therefore all zero, so $E^2_{pq} = E^\infty_{pq}$.

For cochain complexes we will use the convention that filtrations are decreasing, $F^pC^n \supset F^{p+1}C^n$ - via $F_pC_n := F^{-p}C^{-n}$ one obtains then filtered chain complexes as before. The same recipe is used to translate spectral sequences to $\{E^{pq}_r, \mathrm{d}_r\}$.

3. The global dimension of connected algebras

3.1. **Graded projective modules.** Our first main topic will be the theory of shortest possible ("minimal") resolutions of graded modules. We start with some auxiliary results, showing in particular that in the graded case flat and projective just means free. All of this relies on the following version of ***Nakayama's lemma***:

Proposition 1. *For* $M \in A$-$\underline{\textbf{mod}}$ *we have*

$$M = 0 \quad \Leftrightarrow \quad k \otimes_A M = M/A_+M = 0.$$

Proof. $M = \bigoplus_{j \geq s} M_j$ gives $A_+M \subset \bigoplus_{j>s} M_j \subsetneq M$ unless $M = 0$. ☐

Example 8. This might be not true for $M \in A$-$\underline{\textbf{Mod}}$: Consider for example the polynomial ring $A = k[x]$ and take as M the Laurent polynomials $k[x, x^{-1}]$ with the multiplication of polynomials as action. Then $k \otimes_A M = 0$, since for all $\lambda \in k, m \in M$ we have

$$\lambda \otimes_A m = \lambda \otimes_A xx^{-1}m = \lambda x \otimes_A x^{-1}m = 0.$$

Proposition 2. *For every* $M \in A$-$\underline{\textbf{mod}}$ *there exists* $V \in k$-$\underline{\textbf{mod}}$ *and an epimorphism* $\varphi : A \underline{\otimes} V \to M$ *in* A-$\underline{\textbf{mod}}$ *such that the induced morphism* $V = k \underline{\otimes}_A A \underline{\otimes} V \to k \underline{\otimes}_A M$ *in* k-$\underline{\textbf{mod}}$ *is an isomorphism.*

Proof. As graded k-vector space, M can be written as $V \oplus A_+M$ for some complement V of A_+M, and then we have an epimorphism

$$\varphi : A \underline{\otimes} V \to AV \subset M, \quad a \underline{\otimes} x \mapsto ax.$$

Now Nakayama applied to M/AV gives $M = AV$. ☐

Proposition 3. $M \in A$-$\underline{\textbf{mod}}$ *is a free module iff* $\operatorname{Tor}_1^A(k, M) = 0$.

Proof. Proposition 2 yields a short exact sequence

$$0 \to \ker \varphi \to A \otimes V \to M \to 0.$$

By the definition of Tor this leads after tensoring over A with k to an exact sequence

$$\ldots \to \operatorname{Tor}_1^A(k, M) \to k \otimes_A \ker \varphi \to k \otimes_A A \otimes V \to k \otimes_A M \to 0$$

with all the other Tor's coming in on the left. If $\operatorname{Tor}_1^A(k, M) = 0$, then this becomes a short exact sequence telling that $k \otimes_A \ker \varphi$ is the kernel of the map $k \otimes_A A \otimes V = V \to k \otimes_A M$. But by Proposition 2 this is an isomorphism having trivial kernel. Proposition 1 then gives $\ker \varphi = 0$, so φ itself is an iso, and $A \otimes V$ is free. The other direction is clear by the definition of Tor. ☐

Corollary 1. *For* $M \in A$-$\underline{\textbf{mod}}$, *the following are equivalent:*

(1) *M is flat.*
(2) *M is projective.*
(3) *M is free.*

Here we should clarify in which category we are in: $M \in A$-$\underline{\mathbf{mod}}$ is free in A-$\underline{\mathbf{mod}}$ if it is of the form $A \underline{\otimes} V \simeq A^n$ for a graded k-vector space V of finite-dimension n. Thus if we write A^n as row vectors, then $(1, 0, \dots, 0)$ can be of arbitrary degree $s_1 \in \mathbb{Z}$, $(0, 1, 0, \dots, 0)$ can be of degree s_2 and so on. Of course M is then also free in A-$\underline{\mathbf{Mod}}$, A-$\mathbf{mod}$ and A-$\mathbf{Mod}$. As the proofs show, the equivalences in the above corollary hold for $M \in A$-$\underline{\mathbf{mod}}$ independently of the category we are considering it in (but the corollary might be wrong for general $M \in A$-$\mathbf{Mod}$). The point is that passing from $\underline{\otimes}$ to $\otimes$ means just forgetting the grading, and this carries over to derived functors.

The same holds for Hom's:

Proposition 4. *For $M, N \in A$-$\underline{\mathbf{mod}}$ we have*

$$\underline{\mathrm{Hom}}_A(M, N) = \mathrm{Hom}_A(M, N).$$

Proof. Fix homogeneous generators $e_1, \dots, e_n \in M$. If $\varphi \in \mathrm{Hom}_A(M, N)$, then $\varphi(e_i) \in \bigoplus_{j=r_i}^{s_i} N_j$ for some finite r_i, s_i. Hence we have

$$\varphi \in \bigoplus_{j=s}^{r} \mathrm{Hom}_A^j(M, N) \subset \underline{\mathrm{Hom}}_A(M, N),$$

where $r := \max_i\{|e_i| - r_i\}$ and $s := \min_i\{|e_i| - s_i\}$. □

In slightly more fancy language: we have an isomorphism of bifunctors $\underline{\mathrm{Hom}}_A(\cdot, \cdot) \simeq \mathrm{Hom}_A(\cdot, \cdot)$ on A-$\underline{\mathbf{mod}}$, and therefore we also do not have to distinguish between their derived functors $\underline{\mathrm{Ext}}_A(\cdot, \cdot)$ and $\mathrm{Ext}_A(\cdot, \cdot)$ as long as we work in A-$\underline{\mathbf{mod}}$.

3.2. **Minimal resolutions.** By Corollary 1 any projective resolution in A-$\underline{\mathbf{mod}}$ is free, so there are $b_i \in \mathbb{N}$ with $P_i = A^{b_i}$ in (2). A morphism $A^{b_i} \to A^{b_{i-1}}$ is given by multiplication of a row vector from the right by some $b_i \times b_{i-1}$-matrix $T_i \in M_{b_i \times b_{i-1}}(A)$ with entries in A, so the full information about the resolution is encoded in these matrices.

Definition 1. *A resolution is **minimal** if $T_i \in M_{b_i \times b_{i-1}}(A_+)$ for all i.*

The most natural resolutions one constructs have this property:

Proposition 5. *If A is Noetherian, then any $M \in A$-$\underline{\mathbf{mod}}$ admits a minimal projective resolution in A-$\underline{\mathbf{mod}}$.*

Proof. Given M, construct an epimorphism $\varphi_0 : A^{b_0} \to M$ as in Proposition 2. By Noetherianity, $\ker \varphi_0$ is again an object in A-**mod**, so we can construct an epimorphism $\varphi_1 : A^{b_1} \to \ker \varphi_0 \subset A^{b_0}$ and so on. In this way we obtain a resolution. Since the maps φ_i tensored with k are isomorphisms (this is part of Proposition 2), their kernels belong to $A_+^{b_i}$. Hence the resolution is minimal. □

Now comes the big clue to everything: given a minimal resolution

$$P_\bullet^{\min} = \ldots \to A^{b_1} \to A^{b_0} \to 0$$

of $M \in A$-**mod** one already knows $\mathrm{Ext}_A(M,k)$ and $\mathrm{Tor}^A(k,M)$: minimality means nothing but the fact that the boundary maps of the complexes

$$\mathrm{Hom}_A(P_\bullet^{\min}, k) = 0 \to \mathrm{Hom}_A(A^{b_0}, k) \to \mathrm{Hom}_A(A^{b_1}, k) \to \ldots$$

and

$$k \otimes_A P_\bullet^{\min} = \ldots \to k \otimes_A A^{b_1} \to k \otimes_A A^{b_0} \to 0$$

are all zero! Hence the (co)homology of the complexes is equal to the complexes themselves. Furthermore, we have

$$\mathrm{Hom}_A(A^{b_i}, k) \simeq k \otimes_A A^{b_i} \simeq k^{b_i}$$

as k-vector spaces, so that we deduce

$$\mathrm{Ext}^i_A(M,k) \simeq \mathrm{Tor}_i^A(k,M) \simeq k^{b_i}. \tag{3}$$

This tells us several things. First of all, minimal resolutions are more or less unique:

Proposition 6. *The ranks b_i occuring in a minimal resolution are uniquely determined by M. In particular, the length of a minimal resolution is unique.*

In particular, b_0 is the minimal number of generators of M. For example, a minimal resolution of k has to start with $P_0^{\min} = A$ with $\varepsilon : A \to k$ as $\varphi_0 : P_0^{\min} \to k$.

More crucially, we now can justify terminology "minimal resolution":

Proposition 7. *One has $\ell(P_\bullet^{\min}) = \mathrm{pd}(M)$.*

Proof. A minimal resolution is in particular a projective resolution in A-**Mod**, so $\mathrm{pd}(M) \leq d$. But if there would be a resolution of length less than d, then we could use it to compute $\mathrm{Ext}^d_A(M,k) = \mathrm{Tor}_d^A(k,M) = 0$ in contradiction to (3). □

For general rings there is no notion of minimal projective resolution. However, there are other types of rings (e.g. commutative local rings) with similar theories of minimal resolutions. Furthermore, there is for arbitrary rings a theory of minimal *injective* resolutions based on the concept of injective hulls, see e.g. [10].

As an obvious consequence of Proposition 7 we have:

Corollary 2. *For $M \in A$-$\underline{\mathbf{mod}}$ we have*

$$\mathrm{pd}(M) = \sup\{i \mid \mathrm{Ext}^i_A(M,k) \neq 0\} = \sup\{i \mid \mathrm{Tor}^A_i(k,M) \neq 0\}.$$

Let us now reflect on the case $M = k$. So far we worked entirely with left modules, but of course all we did can be done with right modules as well. Now $\mathrm{Tor}^A(L,M)$ can be computed either using a projective resolution of M to which we apply $L \otimes_A \cdot$ or by a projective resolution of L to which we apply $\cdot \otimes_A M$ (and take homology). In combination with the above results this gives:

Corollary 3. *The projective dimensions of k as left and as right module coincide.*

So there is a minimal resolution of the right module k of length $\mathrm{pd}(k)$. Using this to compute $\mathrm{Tor}^A(k,M)$ we get from Corollary 2 the final result of this paragraph:

Proposition 8. *For all $M \in A$-$\underline{\mathbf{mod}}$ we have* $\mathrm{pd}(M) \leq \mathrm{pd}(k)$.

Example 9. We construct a minimal resolution of k for the quantum plane from Example 5. First we set $P^{\min}_0 := A$.

The kernel of the augmentation $\varepsilon : A \to k$ is generated as left A-module by the generators x, y of A, so we put $P^{\min}_1 := A^2$ and

$$\varphi_1 : A^2 \to A, \quad (f,g) \mapsto fx + gy.$$

Now we need to determine the kernel of φ_1. It is immediate that $e_{ij} := x^i y^j$ form a vector space basis of A, and we have

$$\begin{aligned} & \varphi_1\Big(\sum_{ij} \lambda_{ij} e_{ij}, \sum_{rs} \mu_{rs} e_{rs}\Big) \\ = & \sum_{ij} \lambda_{ij} q^{-j} e_{i+1j} + \sum_{rs} \mu_{rs} e_{rs+1} \\ = & \sum_{i \geq 0} \lambda_{i0} x^{i+1} + \mu_{0i} y^{i+1} + \sum_{i>0,j>0} (\lambda_{i-1j} q^{-j} + \mu_{ij-1}) e_{ij}. \end{aligned}$$

From this one deduces that $\ker \varphi_1$ is spanned over k by the elements

$$(-q e_{ij+1}, q^{-j} e_{i+1j}) = e_{ij}(-qy, x), \quad i,j \geq 0,$$

that is, generated as A-module by $(-qy, x)$. Hence $b_2 = 1$ with

$$\varphi_2 : A \to A^2, \quad f \mapsto (-qfy, fx).$$

Using e.g. grading arguments one easily checks that A has no zero divisors, so $\ker \varphi_2 = 0$ and the resolution ends here. In other words, we have $\mathrm{pd}(k) = 2$. The combinatorics behind this resolution will be the heart of the notion of Koszul algebra that we will study in Section 4.

3.3. **Ungraded modules.** Obviously, any graded algebra is also ***filtered*** by

$$F_j A := \bigoplus_{i \leq j} A_i, \quad (F_j A)(F_k A) \subset F_{j+k} A.$$

As banal as this is, it becomes interesting when passing to modules: let $M \in A$-**mod** be arbitrary and choose generators $e_1, \ldots, e_n \in M$,

$$M = Ae_1 + \ldots + Ae_n.$$

Then we can turn M into a ***filtered module***:

$$F_j M := \{a_1 e_1 + \ldots + a_n e_n \mid a_i \in F_j A\}, \quad (F_j A)(F_k M) \subset F_{j+k} M.$$

In general, the graded module $\underline{M} = \bigoplus_j F_j M / F_{j-1} M$ associated to a filtered module over a filtered ring A is a module over the associated graded ring $\underline{A} = \bigoplus_j F_j A / F_{j-1} A$, but in our case the latter is just A itself. In [7], Section 7.6 one can find a detailed discussion how filtered resolutions of filtered modules over filtered rings are related to graded resolutions of the associated graded modules over the associated graded rings. The main result is Theorem 7.6.17 therein: A graded free resolution of an associated graded module can always be lifted to a projective resolution of the original module. Given any $M \in A$-**mod** the minimal resolution of the associated graded module thus lifts to a projective resolution of M of the same length, and we arrive at (cf. Corollary 7.6.18 ibidem):

Proposition 9. *For any* $M \in A$-**mod** *we have* $\mathrm{pd}(M) \leq \mathrm{pd}(k)$.

In other words: No module, graded or not, is less projective than k.

3.4. **Global dimension.** The ***global dimension*** of A is

$$\begin{aligned} \mathrm{gl.dim}(A) &:= \sup\{\mathrm{pd}(M) \mid M \in A\text{-}\mathbf{mod}\} \\ &= \sup\{d \mid \exists M, N \in A\text{-}\mathbf{mod} : \mathrm{Ext}^d_A(M, N) \neq 0\}. \end{aligned}$$

We are not entirely precise here: usually $\mathrm{gl.dim}(A)$ is defined taking into account as well not finitely generated modules. It turns out that it is sufficient to consider $\mathrm{pd}(A/I)$ for left ideals, but in the resolutions modules still might be infinitely generated. Anyhow, for Noetherian rings this problem disappears. Furthermore, there might be in general a difference between left and right modules. But we are again on the safe side in the Noetherian case, where we can replace Ext's by Tor's which implies that left and right global dimension coincide.

Theorem 1 is now a direct consequence of Proposition 9. Many theorems in ring theory require finite global dimension as a technical condition which is often difficult to check. For Noetherian $A \in \mathbf{Conn}$ we now know an algorithm solving this problem: construct a minimal resolution of k, and its length will be the number we are looking for. But still this construction might be tedious for concretely given A. In the remainder of this text we now discuss Koszul algebras, a class of algebras for which also this problem admits an efficient and elegant general solution.

4. Koszul algebras

4.1. **Definition.** Throughout, we let A be an algebra (locally finite-dimensional and connected) and $P_\bullet^{\min}$ be a minimal resolution of $k \in A$-$\underline{\mathbf{mod}}$ determined by matrices T_i.

Definition 2. *A is* ***Koszul*** *if the entries of the T_i all belong to A_1.*

One calls the resolution in this case ***linear***. Note that this property is independent of the choice of minimal resolution.

Example 10. The quantum plane is Koszul, since the matrices in the minimal resolution of k (see Example 9) were

$$T_2 = (-qy, x), \quad T_1 = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The Koszul condition singles out a class of in many respects well-behaved connected algebras. We do neither want to speak about the history nor about the vast literature on Koszul algebras (let us only mention [5, 6, 9] and the recent monograph [8]), and we warn that there are more general notions of Koszul algebras which might be not connected or might be only filtered.

4.2. **The internal grading of** $\mathrm{Ext}_A^i(k,k)$**.** Proposition 4 leads to the decomposition of $\mathrm{Hom}_A(M,N)$ into graded components, and this also induces a decomposition

$$\mathrm{Ext}_A^i(k,k) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}_A^{ij}(k,k).$$

Koszulness can be reformulated in terms of this internal grading:

Proposition 10. *We have* $\mathrm{Ext}_A^{ij}(k,k) = 0$ *for* $j < i$, *and* A *is Koszul if and only if* $\mathrm{Ext}_A^{ij}(k,k) = 0$ *also for* $j > i$, *that is, if we have*

$$\mathrm{Ext}_A(k,k) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^{ii}(k,k).$$

Proof. Use a minimal resolution $P_\bullet^{\min}$ to compute $\mathrm{Ext}_A(k,k)$. The grading of $P_i^{\min} = A^{b_i}$ is fixed in such a way that $\varphi_i : A^{b_i} \to A^{b_{i-1}}$ is of degree zero. The generator $1 \in A = P_0^{\min}$ has degree 0, and the φ_i are given by matrices with entries in A_+. Thus the generators of $P_i^{\min}$ are of degree $\geq i$. Hence $\mathrm{Hom}_A^j(P_i^{\min}, k) = 0$ for $j < i$ (k is nonzero only in degree 0). The second part is now also obvious. □

4.3. **Koszul algebras are quadratic.** Here is another observation that follows by inspection of a minimal resolution of k:

Proposition 11. *Any Koszul algebra is a quadratic algebra.*

Proof. Consider the first three terms of a minimal resolution of k:

$$\cdots \longrightarrow P_2^{\min} = A^{b_2} \xrightarrow{\varphi_2} P_1^{\min} = A^{b_1} \xrightarrow{\varphi_1} P_0^{\min} = A \longrightarrow 0.$$

Since φ_1 is given by $T_1 \in M_{b_1\times 1}(A_1)$, $A_+ = \ker\varepsilon$ is generated as A-module by A_1. Hence A is generated as an algebra by A_1 (and $b_1 = \dim_k A_1$). Considering $\mathrm{im}\,\varphi_2 = \ker\varphi_1$ in the light of Proposition 10 one similarly sees that the relations between a minimal set of generators (a vector space basis of A_1) are quadratic. □

But not any quadratic algebra is Koszul, going on with the considerations as in the preceding proof in higher degrees $i > 2$ leads to further restrictions.

4.4. **The Koszul dual $A^!$.** A new actor enters the scene: for a quadratic algebra $A = A(V,R)$ we define a new quadratic algebra $A^!$ to be $A(V^*, R^\perp)$, where

$$R^\perp := \{r \in V^* \otimes V^* \simeq (V\otimes V)^* \mid r(R) = 0\}.$$

Obviously, we have $(A^!)^! \simeq A$ so the following definition makes sense:

Definition 3. *One calls* $A^!$ *the* ***Koszul dual*** *of* A.

Example 11. Let A be the quantum plane from Example 5. Here $V = k^2$ and $\dim_k R = 1$. Hence $\dim_k R^\perp = 3$. If $\{\hat{x},\hat{y}\} \subset V^*$ is the basis dual to the one given by the generators $x, y \in V$ with $xy - qyx = 0$, then a basis for $R^\perp$ is given by

$$\{\hat{x}\otimes\hat{x}, \hat{y}\otimes\hat{y}, \hat{x}\otimes\hat{y} + q^{-1}\hat{y}\otimes\hat{x}\}.$$

That is, $A^!$ has generators $\hat{x}, \hat{y}$ with relations

$$\hat{x}^2 = \hat{y}^2 = 0, \quad \hat{x}\hat{y} = -q^{-1}\hat{y}\hat{x}.$$

Thus it is a quantised exterior algebra. This is easily generalised to more generators. In particular, the Koszul dual of $k[x_1, \ldots, x_n]$ is the exterior algebra in n generators.

We will elaborate in a moment a close relation between $A^!$ and Koszulness of A. But for this we have to recall some prerequisites, so we now make an excursion about the bar resolution of $k \in A$-$\underline{\mathbf{mod}}$ and the Yoneda product on $\mathrm{Ext}_A(k,k)$.

4.5. **The Yoneda algebra** $\mathrm{Ext}_A(k,k)$**.** The Yoneda product is defined between Ext-groups over general rings, but for algebras over fields the presentation can be simplified using canonical resolutions. Throughout, $A \in$ **Conn** denotes an algebra.

Proposition 12. *Consider* $C^{\mathrm{bar}}_i := A \underline{\otimes} A_+^{\otimes i}$ *as graded* A*-modules via multiplication in* A*. Then* $\mathrm{b}' : C^{\mathrm{bar}}_i \to C^{\mathrm{bar}}_{i-1}$ *given by*

$$\begin{aligned} a_0 \underline{\otimes} \ldots \underline{\otimes} a_i \quad \mapsto \quad & a_0 a_1 \underline{\otimes} a_2 \underline{\otimes} \ldots \underline{\otimes} a_i - a_0 \underline{\otimes} a_1 a_2 \underline{\otimes} \ldots \underline{\otimes} a_i \\ & + a_0 \underline{\otimes} a_1 \underline{\otimes} a_2 a_3 \underline{\otimes} \ldots \underline{\otimes} a_i - \ldots \\ & + (-1)^{i-1} a_0 \underline{\otimes} \ldots \underline{\otimes} a_{i-1} a_i \end{aligned} \tag{4}$$

turns $C^{\mathrm{bar}}_\bullet$ *into a graded free resolution of* $k =: C^{\mathrm{bar}}_{-1}$*.*

Proof. C^{bar}_i is free since k is a field. It is obvious that $\mathrm{b}' \circ \mathrm{b}' = 0$ and that b' respects the total grading. Finally, $\mathrm{b}'(a_0 \underline{\otimes} \ldots \underline{\otimes} a_i) = 0$ implies

$$\mathrm{b}'(1 \underline{\otimes} (a_0 - \varepsilon(a_0)) \underline{\otimes} \ldots \underline{\otimes} a_i) = a_0 \underline{\otimes} \ldots \underline{\otimes} a_i,$$

and clearly we have $C^{\mathrm{bar}}_0 / \mathrm{im}\, \mathrm{b}' \simeq k$. □

Definition 4. $C^{\mathrm{bar}}_\bullet$ *is the* ***normalised bar resolution*** *of* $k \in A$-$\underline{\mathbf{mod}}$.

"***Normalised***" refers to the fact that we kicked out $k \subset A$ in all middle tensor components (search the literature for key words like simplicial modules, (co)monads or (co)triples if you want to understand the abstract theory behind this resolution).

Forget for a moment graded modules and consider $\mathrm{Ext}_A(k,k)$ as the cohomology of the cochain complex $\mathrm{Hom}_A(C^{\mathrm{bar}}_\bullet, k)$. Under the isomorphism of vector spaces

$$C^i := (A_+^{\otimes i})^* \to \mathrm{Hom}_A(C^{\mathrm{bar}}_i, k), \quad \varphi \mapsto \varepsilon \otimes \varphi,$$

the coboundary of $\mathrm{Hom}_A(C_\bullet^{\mathrm{bar}}, k)$becomes $\mathrm{d} : C^i \mapsto C^{i+1}$ given by

$$\begin{aligned} \mathrm{d}(\varphi)(a_1 \otimes \ldots \otimes a_i) &= -\varphi(a_1 a_2 \otimes a_3 \otimes \ldots \otimes a_i) \\ &\quad +\varphi(a_1 \otimes a_2 a_3 \otimes \ldots \otimes a_i) - \ldots \\ &\quad +(-1)^{i-1}\varphi(a_1 \otimes \ldots \otimes a_{i-1}a_i). \end{aligned} \tag{5}$$

This map satisfies

$$\mathrm{d}(\varphi \otimes \psi) = \mathrm{d}\varphi \otimes \psi + (-1)^{|\varphi|}\varphi \otimes \mathrm{d}\psi, \quad \varphi \in C^i, \psi \in C^j,$$

where

$$C^{i+j} \ni \varphi \otimes \psi : a_1 \otimes \ldots \otimes a_{i+j} \mapsto \varphi(a_1 \otimes \ldots \otimes a_i)\psi(a_{i+1} \otimes \ldots \otimes a_{i+j}), \tag{6}$$

so $(C^\bullet, \mathrm{d})$ is a ***differential graded algebra***. As a consequence,

$$[\varphi] \otimes [\psi] := [\varphi \otimes \psi]$$

gives a well-defined product of cohomology classes. Hence we have:

Proposition 13. *The concatenation of cocycles yields a product*

$$\smile : \mathrm{Ext}^i_A(k,k) \otimes \mathrm{Ext}^j_A(k,k) \to \mathrm{Ext}^{i+j}_A(k,k)$$

called the ***Yoneda*** *or* ***cup product****. This product turns* $\mathrm{Ext}_A(k,k)$ *into a connected algebra called the* ***Yoneda algebra*** *of* A.

Now we take into account gradings. The internal grading

$$(C_i^{\mathrm{bar}})_j := \bigoplus_{\substack{j_0+\ldots+j_i=j \\ j_1,\ldots,j_i \geq 1}} A_{j_0} \otimes \ldots \otimes A_{j_i}.$$

of C_i^{bar} gives rise to the internal grading of $\mathrm{Ext}^i_A(k,k)$ discussed in Section 4.2. On C^i it gives a decomposition as a direct product

$$C^i = (A_+^{\otimes i})^* = (\bigoplus_{j\geq 0}(A_+^{\otimes i})_j)^* = \prod_{j \geq 0}(A_+^{\otimes i})_j^*, \quad (A_+^{\otimes i})_j = \bigoplus_{\substack{j_1+\ldots+j_i=j \\ j_1,\ldots,j_i \geq 1}} A_{j_1} \otimes \ldots \otimes A_{j_i},$$

and d respects this decomposition: $\mathrm{d}(C^{ij}) \subset C^{i+1j}$ where $C^{ij} := (A_+^{\otimes i})_j^*$.

However, we know from the discussion after Proposition 4 that we can safely replace the product by a direct sum here: we have $\mathrm{Ext}_A(k,k) = \underline{\mathrm{Ext}}_A(k,k)$, so we could consider from the beginning $\underline{\mathrm{Hom}}_A(C_\bullet^{\mathrm{bar}}, k)$, and then we arrive through our identifications at $\underline{C}^i := \bigoplus_{j \geq 0} C^{ij} \subset C^i$, knowing that this inclusion of complexes is a ***quasi-isomorphism*** (becomes an isomorphism after passing to cohomology).

Proposition 14. *One has* $\mathrm{Ext}^{ij}_A(k,k) \smile \mathrm{Ext}^{rs}_A(k,k) \subset \mathrm{Ext}^{i+rj+s}_A(k,k)$.

This is clear by construction.

Corollary 4. $\bigoplus_{i \geq 0} \mathrm{Ext}^{ii}_A(k,k)$ *is a subalgebra of the Yoneda algebra.*

4.6. $A^! \subset \mathrm{Ext}_A(k,k)$ **and Koszulness.** We return to our main road:

Proposition 15. *If A is a quadratic algebra, then we have*

$$\bigoplus_{i\geq 0} \mathrm{Ext}_A^{ii}(k,k) \simeq A^!$$

and this equals the subalgebra of $\mathrm{Ext}_A(k,k)$ generated by $\mathrm{Ext}_A^1(k,k)$.

Proof. Since we assume A to be locally finite-dimensional, we have

$$C^{ij} = (A_+^{\otimes i})_j^* = (\bigoplus_{\substack{j_1+\ldots+j_i=j\\ j_1,\ldots,j_i\geq 1}} A_{j_1}\otimes\ldots\otimes A_{j_i})^* \simeq \bigoplus_{\substack{j_1+\ldots+j_i=j\\ j_1,\ldots,j_i\geq 1}} A_{j_1}^*\otimes\ldots\otimes A_{j_i}^*.$$

In particular,

$$C^{i-1i} \simeq \bigoplus_{j=1}^{i-1} (A_1^*)^{\otimes j-1}\otimes A_2^*\otimes (A_1^*)^{\otimes i-j-1}, \quad C^{ii}\simeq (A_1^*)^{\otimes i}, \quad C^{i+1i}=0.$$

Hence $\mathrm{d}(C^{ii})=0$ and $\mathrm{Ext}_A^{ii}(k,k)$ is simply the quotient of C^{ii} by the image of the incoming map $\mathrm{d}: C^{i-1i}\to C^{ii}$. As an algebra, $\underline{C}^\bullet$ is obviously generated by C^1, so this implies that $\bigoplus_{i\geq 0}\mathrm{Ext}_A^{ii}(k,k)$ is generated as an algebra by $\mathrm{Ext}_A^{11}(k,k)$. Considering the first terms

$$\begin{aligned}
&\mathrm{Ext}_A^{00}(k,k) = A_0^* \simeq k,\\
&\mathrm{Ext}_A^{11}(k,k) = A_1^*,\\
&\mathrm{Ext}_A^{22}(k,k) \simeq (A_1^*\otimes A_1^*)/\mathrm{d}(A_2^*),
\end{aligned}$$

we obtain the identification with $A^!$. Finally, we have $\mathrm{Ext}_A^1(k,k) = \mathrm{Ext}_A^{11}(k,k)$ since A_1 generates A (see the proof of Proposition 11). □

Proposition 10 can now be reformulated as follows:

Corollary 5. *A quadratic algebra A is Koszul iff $\mathrm{Ext}_A(k,k)\simeq A^!$.*

Here "$\simeq$" means "isomorphic as graded algebras". In particular, Theorem 1 implies for Koszul algebras:

Corollary 6. *A Koszul algebra A has finite global dimension if and only if its Koszul dual $A^!$ is finite-dimensional as k-vector space.*

Note that we fudged a bit on the signs: if we stick to the Koszul sign convention, then the product in (6) should carry a sign $(-1)^{|\psi||a_1\underline{\otimes}\cdots\underline{\otimes} a_i|}$ when taking into account the grading, we should rather work with $\varphi\underline{\otimes}\psi = (-1)^{jr}\varphi\otimes\psi$ for $\varphi\in C^{ij}$ and $\psi\in C^{rs}$. This ruins the signs in our differential graded algebra condition, but in fact it will later be useful to reshuffle the gradings a bit anyway: if we define

$$B_n := \bigoplus_{j-i=n} C^{ij}, \tag{7}$$

then $(B_\bullet, \mathrm{d})$ is again a differential graded algebra with respect to $\underline{\otimes}$, but now $|\mathrm{d}| = +1$. The new grading allows us to formulate things in a totally new way. For example, Propositions 10 and 15 become an acyclicity statement:

Corollary 7. *$H_0(B) \simeq A^!$, and A is Koszul $\Leftrightarrow$ $H_n(B) = 0$ for $n > 0$.*

This directly leads to our last big topic which gave Koszul algebras their name.

4.7. **The Koszul complex $K(A, k)$.** Now we describe the abstract construction of the minimal resolution of k that we promised in the end of Section 3. We will introduce for any quadratic algebra A a small complex $K_\bullet(A, k)$ of A-modules with $H_0(K(A, k)) \simeq k$ which is acyclic if and only if A is Koszul. It is an analogue of the Koszul complexes used so heavily in local algebra (see e.g. [1]), and was one of Priddy's main motivations for introducing Koszul algebras [9]. As a vector space, its degree n part is defined as

$$K_n(A, k) := A \otimes (A^!)^*_n.$$

The differential d_K can be introduced as follows: consider A as right A-module via multiplication and $(A^!)^*$ as right $A^!$-module via $(\varphi b)(c) := \varphi(bc)$. Here $b \in A^!$ acts on $\varphi \in (A^!)^*$ and $\varphi b \in (A^!)^*$ is evaluated on $c \in (A^!)^*$. These actions turn $K_\bullet(A, k)$ into a right $A \otimes A^!$-module, and d_K is given by the action of $d := \sum_i x_i \otimes \hat{x}_i$, where $\{x_i\}, \{\hat{x}_i\}$ are dual bases of $V = A_1$ and $V^* = A^!_1$, respectively:

$$\mathrm{d}_K : K_n(A, k) \to K_{n-1}(A, k), \quad a \otimes \varphi \mapsto \sum_i a x_i \otimes \varphi \hat{x}_i.$$

Note that d and hence d_K are independent of the chosen basis of V.

Proposition 16. *One has $d^2 = 0$ and hence $\mathrm{d}_K \circ \mathrm{d}_K = 0$.*

Proof. One way to see this clearly is to identify

$$\begin{aligned}(A_1 \otimes A^!_1) \otimes (A_1 \otimes A^!_1) &\simeq V \otimes V^* \otimes V \otimes V^* \\ &\simeq (V \otimes V) \otimes (V \otimes V)^* \\ &\simeq \mathrm{End}_k(V \otimes V)\end{aligned}$$

and

$$\begin{aligned}(A_2 \otimes A^!_2) &\simeq (V \otimes V)/R \otimes (V \otimes V)^*/R^\perp \\ &\simeq (V \otimes V)/R \otimes R^* \\ &\simeq \mathrm{Hom}_k(R, (V \otimes V)/R).\end{aligned}$$

This leads to a commutative diagram

$$\begin{array}{ccc} (A_1 \otimes A_1^!) \otimes (A_1 \otimes A_1^!) & \longrightarrow & \mathrm{End}_k(V \otimes V) \\ \mu \downarrow & & \downarrow \mathrm{m} \\ (A_2 \otimes A_2^!) & \longrightarrow & \mathrm{Hom}_k(R, (V \otimes V)/R) \end{array}$$

where μ is the multiplication map of $A \otimes A^!$ and m is the canonical map that restricts $\varphi \in \mathrm{End}_k(V \otimes V)$ to R and composes with the canonical projection to $(V \otimes V)/R$. Then $d \otimes d$ corresponds to to $\mathrm{id}_{V \otimes V}$, but clearly $\mathrm{m}(\mathrm{id}_{V \otimes V}) = 0$. □

Definition 5. *One calls* $(K_\bullet(A,k), \mathrm{d}_K)$ *the* ***Koszul complex*** *of* A.

Example 12. For the quantum plane, $K(A,k)$ is the minimal resolution of k that we constructed by bare hands in Example 9.

We now relate the Koszul complex to the bar resolution. We apply to $C_\bullet^{\mathrm{bar}}$ the same reshuffling of gradings as in the pasage from $\underline{C}^\bullet$ to $B_\bullet$ in (7) and denote the resulting cochain complex by $(B_{\mathrm{bar}}^\bullet, \mathrm{b}')$. The really new step is to introduce a filtration on $(B_{\mathrm{bar}}^\bullet, \mathrm{b}')$ using the grading of the zeroth tensor component,

$$F^p B_{\mathrm{bar}}^n := \bigoplus_{\substack{j_0 + \ldots + j_i = i+n \\ i \geq 0, j_1, \ldots, j_i \geq 1, j_0 \geq p}} A_{j_0} \otimes \cdots \otimes A_{j_i}.$$

Consider the resulting spectral sequence E_r^{pq}. The filtration comes from a grading, so the passage from B_{bar} to E_0 reduces to yet another reshuffling of degrees:

$$E_0^{pq} = \bigoplus_{\substack{j_1 + \ldots + j_i = i+q \\ i \geq 0, j_1, \ldots, j_i \geq 1}} A_p \otimes A_{j_1} \otimes \cdots \otimes A_{j_i} = A_p \otimes B_q^*,$$

where B_q is as in (7). In the differential d_0 the zeroth term $a_0 a_1 \otimes \ldots \otimes a_i$ of b' gets cut away since it shifts the degree in A_p up at least by one ($a_1 \in A_+$). Hence d_0 is precisely $\mathrm{id}_A \otimes \mathrm{d}$ where d is dual to the differential (5) of $B_\bullet$. So we have

$$E_1^{pq} = A_p \otimes H^q(B^*).$$

If A is quadratic, then we know from Corollary 7 that $H_0(B)$ can be identified with $A^!$, so $H^0(B^*)$ is by what they call the universal coefficient theorem $(A^!)^*$. Therefore, the differential $\mathrm{d}_1 : E_1^{pq} \to E_1^{p+1q}$ of the spectral sequence turns

$$E_1^{p0} \simeq A_p \otimes (A^!)^*$$

into a cochain complex. It is given by the zeroth term $a_0 a_1 \otimes \ldots \otimes a_i$ of b' that got lost in d_0, but d_1 still strips off every contribution in which the degree in A is raised by more than one. A moment of reflection will tell you that this means that d_1 is precisely d_K, the differential of the Koszul complex $K(A,k)$.

Thus $(E_1^{\bullet 0}, \mathrm{d}_1)$ is essentially $(K_\bullet(A,k), \mathrm{d}_K)$, only the grading is chosen differently (the same game throughout...). However, we have:

Proposition 17. *$(E_1^{\bullet 0}, \mathrm{d}_1)$ is acyclic iff $(K_\bullet(A,k), \mathrm{d}_K)$ is acyclic.*

Indeed, $H^0(E_1^{\bullet 0}) \simeq H_0(K(A,k)) \simeq k$ is concentrated in degree 0 with respect to both the degree in A and the one in $(A^!)^*$. Note that similarly $B^\bullet_{\mathrm{bar}}$ is like $C^{\mathrm{bar}}_\bullet$ acyclic since the homology of $C^{\mathrm{bar}}_\bullet$ is concentrated in degree 0 with respect to both the homological and the internal grading i and j. From all this we now conclude the main result of this section:

Proposition 18. *$K(A,k)$ is acyclic if and only if A is Koszul.*

Proof. If A is Koszul, then $B_\bullet$ and hence $B_\bullet^*$ are acyclic by Corollary 7. Thus $E_2^{pq} = 0$ for $q \neq 0$, and as explained at the end of Section 2.8 the spectral sequence stabilises at E_2. Therefore $E_1^{\bullet 0}$ is acyclic becuase $B^\bullet_{\mathrm{bar}}$ is so:

$$H^n(E_1^{\bullet 0}) = E_2^{n0} = \bigoplus_{p+q=n} E_2^{pq} = \bigoplus_{p+q=n} E_\infty^{pq} \simeq H^n(B_{\mathrm{bar}}) = 0 \text{ for } n > 0.$$

If conversely $K(A,k)$ is acyclic, then it is a minimal resolution which we can use to compute $\mathrm{Ext}_A(k,k) \simeq A^!$, so A is Koszul by Corollary 5. □

4.8. **The Koszul complex $K(A^e, A)$.** Finally we add some results from [11] about the Hochschild homology of a Koszul algebra. The role of A will be played by its ***enveloping algebra*** $A^e := A \otimes A^{\mathrm{op}}$, where A^{op} is the ***opposite algebra*** (same vector space, opoosite product $a \cdot_{\mathrm{op}} b := ba$). Thus left A^{op}-modules are right A-modules and vice versa, and A^e-modules are A-bimodules (with symmetric action of k). Similarly, the module $k \in A$-**mod** becomes replaced by $A \in A^e$-**mod** on which A^e acts by multiplication,

$$(a \otimes b)c := acb, \quad a \otimes b \in A^e, c \in A.$$

The Hochschild homology of A is $HH_\bullet(A) := \mathrm{Tor}_\bullet^{A^e}(A,A)$ and is important in many contexts. I do not want to spend time on motivation here, see e.g. [3]. To compute it, we need a projective resolution of A as A^e-module, and such a resolution can be obtained as a slight variation of $K(A,k)$: define free A^e-modules

$$K_n(A^e, A) := A \otimes (A^!)_n^* \otimes A,$$

where the action of A^e is given by multiplication in the first and third tensor components. Let the differential d_K of $K(A,k)$ act on the first two tensor components and define an analogous second differential d^K which acts on the second and third component, but from the other side:

$$\mathrm{d}^K : a \otimes \varphi \otimes b \mapsto a \otimes \sum_i \hat{x}_i \varphi \otimes x_i b.$$

These two differentials obviously commute, so their weighted sum

$$\mathrm{d}_K^K := \mathrm{d}_K + (-1)^n \mathrm{d}^K : K_n(A^e, A) \to K_{n-1}(A^e, A)$$

turns $K(A^e, A)$ into a complex, and one easily checks that

$$H_0(K(A^e, A)) \simeq A.$$

Proposition 19. *A is Koszul if and only if $K(A^e, A)$ is acyclic.*

Proof. Assume A is Koszul. When applying $\cdot \otimes_A k$ to $K(A^e, A)$ one obtains $K(A,k)$ which is acyclic. Using the Künneth formula in combination with the Nakayama lemma (Proposition 1) one now shows inductively that for all $n > 1$

$$\begin{aligned} & H_n(K(A^e, A)) \otimes_A k \simeq H_n(K_n(A^e, A) \otimes_A k) \simeq H_n(K(A, e)) = 0 \\ \Rightarrow \quad & H_n(K(A^e, A)) = 0. \end{aligned}$$

Conversely, $K(A^e, A)$ consists of free A-modules and $K_n(A^e, A) = 0$ for $n < 0$. Hence it is split exact if it is acylic, that is, it admits a contracting homotopy

$$h : K_n(A^e, A) \to K_{n+1}(A^e, A), \quad \mathrm{d}_K^K \circ h + h \circ \mathrm{d}_K^K = \mathrm{id}.$$

Applying the functor $\cdot \otimes_A k$ yields a contracting homotopy of $K(A,k)$, so this is acyclic and A is Koszul. □

Thus if A is Koszul, then $H_\bullet(A \otimes_{A^e} K(A^e, A)) \simeq HH_\bullet(A)$. Note that as a vector space, we have $A \otimes_{A^e} K_n(A^e, A) \simeq K_n(A,k)$, and the differential $\mathrm{d} = \mathrm{id} \otimes \mathrm{d}_K^K$ looks like d_K^K only that both d_K and d^K now act on the same tensor components.

Using a resolution analogous to the bar resolution one usually defines $HH_\bullet(A)$ in terms of an explicit chain complex which is $C_\bullet^{\mathrm{bar}}$ but with the differential

$$\begin{aligned} \mathrm{b} : a_0 \otimes \ldots \otimes a_i \quad \mapsto \quad & a_0 a_1 \otimes a_2 \otimes \ldots \otimes a_i - a_0 \otimes a_1 a_2 \otimes \ldots \otimes a_i \\ & + a_0 \otimes a_1 \otimes a_2 a_3 \otimes \ldots \otimes a_i - \ldots \\ & + (-1)^{i-1} a_0 \otimes \ldots \otimes a_{i-1} a_i \\ & + (-1)^i a_i a_0 \otimes \ldots \otimes a_{i-1} \end{aligned}$$

Van den Bergh has points out in [11], Proposition 3.3, that the quasi-isomorphism between the small complex $A \otimes_{A^e} K(A^e, A)$ and $(C_\bullet^{\mathrm{bar}}, \mathrm{b})$ is simply the restriction of the canonical inclusion

$$A \otimes V^{\otimes i} \to A \otimes A_+^{\otimes i}, \quad V = A_1$$

to $A \otimes (A^!)^*$, where $(A^!)^*$ is embedded into $\bigoplus_i V^{\otimes i}$ using the dual of the surjective multiplication map $\bigoplus_i (V^*)^{\otimes i} = \bigoplus_i (A_1^!)^{\otimes i} \mapsto A^!$. In this way one can easily translate cycles from the small complex to the standard one which is needed e.g. when one wants to go on and compute the cyclic homology of A.

Let us remark two further consequences of Proposition 19: first, the roles of A and A^{op} become completely symmetric in $K(A^e, A)$. Therefore, we have:

Corollary 8. *A is Koszul if and only if A^{op} is Koszul.*

Now consider the dual $(K(A,k))^*$ of the Koszul complex. By the universal coefficient theorem, this is again acyclic when A is Koszul. But observe that this is just $K(A^!, k)$ with a change from left to right actions (the differential is d^K rather than d_K). In other words, we have $(K(A,k))^* \simeq K((A^!)^{\mathrm{op}}, k)$ as chain complexes. In combination with the previous corollary we obtain:

Corollary 9. *A quadratic algebra A is Koszul iff $A^!$ is Koszul.*

Example 13. We compute the Hochschild homology of the quantum plane. Since $\dim A_0^! = \dim A_2^! = 1$, $\dim A_1^! = 2$, the complex we have to consider is

$$0 \to A \to A^2 \to A \to 0,$$

and if I computed everything correctly, the two nontrivial differentials are given by

$$a \mapsto (ya - qay, -qxa + ax), \quad (a, b) \mapsto xa - ax + yb - by,$$

where x, y are the generators of A. Using the vector space basis $e_{ij} = x^i y^j$ one now easily proves that the following classes form a vector space basis of the homology:

$$[(x^i, 0)], [(0, y^j)], [x^i], [y^j], \quad i, j \geq 0$$

as long as q is not a root of unity. The identification with cycles in the standard complex $(C_\bullet^{\mathrm{bar}}, \mathrm{b})$ is given by the identity in degree 0 and in degree 1 by

$$(a, b) \mapsto a \otimes x + b \otimes y.$$

For $q = 1$ the differential of our complex is zero and we recover the theorem of Hochschild, Kostant and Rosenberg that the Hochschild homology of the coordinate ring of a smooth affine algebraic variety (and $k[x, y]$ is the coordinate ring of the plane k^2 which is as smooth as one can only be) can be identified with algebraic differential forms on that variety. The identification is given in our case by

$$a \mapsto a\mathrm{d}x \wedge \mathrm{d}y, \quad (b, c) \mapsto b\mathrm{d}x + c\mathrm{d}y$$

in degrees 2 and 1 and by the identity in degree 0, where $\mathrm{d}x, \mathrm{d}y$ are the usual differentials of the coordinates x, y as in differential geometry. The fact that $HH_2(A) = 0$ in the quantum case is the beginning of a long story about a "dimension drop" that can be observed in the Hochschild homology of quantised Poisson varieties and can be cured by studying Hochschild homology with noncanonical coefficients.

5. Postludium: PBW algebras

The presentation of $A^!$ in terms of generators and relations often enables one to check $\dim_k A^! < \infty$ easily. In this way, Theorem 1 becomes a valuable tool for showing that $\mathrm{gl.dim} A < \infty$ provided that one can prove Koszulness of A. One helpful criterion for that is the existence of Poincaré-Birkhoff-Witt type vector space bases of A. Due to limitation in space and time we sketch this only without proofs, see e.g. [8] or the original reference [9] for the details.

Suppose $A = A(V, R)$ is a quadratic algebra. We fix a vector space basis $\{x_1, \ldots, x_n\}$ of V and write the relations as

$$x_i x_j = \sum_{S \ni (r,s) < (i,j)} c_{ij}^{rs} x_r x_s, \quad (i, j) \notin S,$$

where $<$ denotes lexicographical ordering and $S \subset S_1 \times S_1$, $S_1 := \{1, \ldots, n\}$, is the set of those (i, j) for which the class of $x_i \otimes x_j$ in $(V \otimes V)/R$ is not in the span of the classes of $x_r \otimes x_s$ with $(r, s) < (i, j)$. Define further $S_0 = \emptyset$ and for $i \geq 2$

$$S_i := \{(j_1, \ldots, j_i) \in S_1^i \mid (j_m, j_{m+1}) \in S, m = 1, \ldots, i-1\}$$

and consider finally the monomials

$$\{x_{j_1} \cdots x_{j_i} \in A_i \mid (j_1, \ldots, j_i) \in S_i\}. \tag{8}$$

Note that these monomials always span A_i as a vector space.

Definition 6. *One calls (8) the PBW generators of A, and A is called a PBW algebra if they are linearly independent and hence altogether form a k-linear basis.*

The big theorem is:

Theorem 3. *A PBW algebra is Koszul.*

See e.g. [8], Theorem 4.3.1 for a proof.

Example 14. Consider the quantum plane (Example 5). Here $S_1 = \{1, 2\}$ and $S = S_2 = \{(1, 1), (1, 2), (2, 2)\}$. More generally, S_i consists of tuples of the form $(1, \ldots, 1, 2, \ldots, 2)$, and the corresponding PBW monomials are $e_{ij} = x^i y^j$ used already in Example 9. They form a vector space basis, so Theorem 3 gives a confirmation of our direct proof that the quantum plane is Koszul.

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School of Mathematics and Statistics, University of Glasgow, 15 University Gardens, Glasgow G12 8QW, Scotland
E-mail address: ulrich.kraehmer@glasgow.ac.uk