AN INTRODUCTION TO DELIGNE-LUSZTIG THEORY

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ABSTRACT. We give an informal introduction to the theory of Deligne-Lusztig which gives all the irreducible representations of reductive groups over finite fields [DL], with an emphasis on the geometry of the Deligne-Lusztig variety. **Disclaimer: by no means complete! Comments welcome / use at your own risk!**

CONTENTS

1. INTRODUCTION

The aim of this paper is to give an informal introduction to the theory of Deligne-Lusztig [DL], which laid an important foundation to the representation theory of reductive groups over finite fields. Their method succeeded to give all the irreducible representations of the reductive groups over finite fields inside virtual representations obtained from the compact support ℓ-adic étale cohomology of certain smooth algebraic varieties (which are affine in most cases) over the algebraic closure of the finite field. This theory is a striking application of the ℓ-adic étale cohomology theory of varieties over finite fields, which was developed by Grothendieck and his co-workers, and gives a prototype of the linkage between the algebraic geometry and the representation theory.

For example, if we take a look at the character table of $GL_2(\mathbb{F}_p)$ for a prime p, we see three kinds of irreducible representations (other than the one-dimensional characters which factor through the determinant $\det : GL_2(\mathbb{F}_p) \to \mathbb{F}_p^\times$) ; *principal series* representations $I(\chi_1, \chi_2)$ which is obtained by inducing a pair of characters χ_1, χ_2 of

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$\mathbb{F}_p^\times$ regarded as a character of the split torus of diagonal matrices $\left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$ via the Borel subgroup of upper-triangular matrices $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$, and the *Steinberg* or *special* representations Sp (and its twists Sp_χ by one-dimensional characters $\chi \circ \det$) which is obtained from the action of $GL_2(\mathbb{F}_p)$ on the $\mathbb{F}_p$-rational points $\mathbb{P}^1(\mathbb{F}_p)$ of the projective line, and the *cuspidal* or *discrete series* representations $\Theta(\chi)$ corresponding to the character χ of the non-split torus isomorphic to $\mathbb{F}_{p^2}^\times$. This was known from the old days, but the construction of the cuspidal representations remained somewhat mysterious, until Drinfeld realized that these are obtained from the first ℓ-adic cohomology group of the affine curve $(XY^p - X^pY)^{p-1} = 1$, on which the groups $GL_2(\mathbb{F}_p)$ and the non-split torus act, and the actions commute with each other to give the correspondence of the representations of two groups on the cohomology group. The Deligne-Lusztig theory generalizes this fact to the vast general setting of the general reductive groups over finite fields.

Let G be a reductive group defined over a finite field $\mathbb{F}_q$, and $F : G \to G$ be the q-th power Frobenius morphism. The fixed point set G^F is a finite group consisting of all the $\mathbb{F}_q$-rational points, and we are interested in the representation theory of G^F. The conjecture of MacDonald in the 1960's generalizes the above observation for GL_2, and states that there should be a well-defined correspondence between the irreducible representations of G^F and the pairs (T, θ) where T is an F-stable maximal torus and θ is a character of T^F in general position (see Def. 3.11), which generalizes the usual "parabolic induction" in the case where T^F is contained in a Borel subgroup B^F. In particular, the representation should be cuspidal when the corresponding T modulo the center of G is anisotropic over $\mathbb{F}_q$. This conjecture was solved affirmatively by the Deligne-Lusztig theory, which constructs a virtual representation R_T^θ of G^F corresponding to (T, θ), inside the ℓ-adic compact support cohomology groups of a certain smooth variety. The characters of these representations can be computed via Lefschetz fixed point formula for the ℓ-adic cohomology, which in turn is used to prove the various properties that the representations R_T^θ have.

Here we summarize some of the main theorems of this theory :

Theorem 1.1. *For each F-stable maximal torus T and a character θ of T^F, there is a virtual representation R_T^θ in the Grothendieck group $\mathcal{R}(G^F)$ of G^F, satisfying the following :*

(i) (Cor. 7.7 of [DL]) *Every irreducible representation ρ of G^F occurs in some R_T^θ, i.e. $\langle \rho, R_T^\theta \rangle \neq 0$ where $\langle , \rangle$ is the natural inner product on $\mathcal{R}(G^F)$.*
(ii) (Cor. 6.3 of [DL]) *If (T, θ) and (T', θ') are not geometrically conjugate* (Def. 3.7)*, no irreducible representation of G^F occurs in both R_T^θ and $R_{T'}^{\theta'}$.*
(iii) (Th. 6.8 of [DL]) *If (T, θ) is in general position* (Def. 3.11)*, one of $\pm R_T^\theta$ is an irreducible representation of G^F.*

This paper is organized as follows. In section 2, we give a fairly detailed description of the construction of Deligne-Lusztig variety and the virtual representations R_T^θ of G^F, and try to show that the idea of construction can be naturally understood in terms of the rational structure of the flag variety of G. In sections 3 and 4 we try to give a sketch of proofs of the main theorems and some important further result. The emphasis is on the geometric ideas concerning Deligne-Lusztig variety, and all the detailed computations of the characters are omitted.

Also, there is an excellent summary of this theory in J.-P. Serre's Bourbaki Seminar talk [Serre].

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Notations. Throughout this paper, K denotes an algebraically closed field of characteristic p, $p > 0$ a prime number. G is a reductive algebraic group (always connected and smooth) over K, defined over some finite field $\mathbb{F}_q$, q a power of p. Namely G/K is obtained by the base extension from an algebraic group G_0 over $\mathbb{F}_q$. We identify G with the set of its K-rational points, which should not cause any confusion.

For any scheme X over K which is defined over $\mathbb{F}_q$, F denotes the Frobenius endomorphism $F : X \to X$. For any endomorphism T of the X, G^T denotes the fixed point subscheme of the scheme X, for example $G^F = G_0(\mathbb{F}_q)$ is the finite group consisting of the $\mathbb{F}_q$-rational points of G. ℓ is a prime different from p and $\overline{\mathbb{Q}}_\ell$ is an algebraic closure of $\mathbb{Q}_\ell$, the field of ℓ-adic numbers. $\mathbb{Z}_\ell = \varprojlim_n \mathbb{Z}/\ell^n$ is the ring of ℓ-adic integers. We use freely the terminology of the schemes and the ℓ-adic cohomology of schemes developed in [SGA]. For any scheme X over K, $H^i(X)$ (resp. $H_c^i(X)$) denotes the usual (resp. compact support) ℓ-adic cohomology groups $H^i(X, \overline{\mathbb{Q}}_\ell)$ (resp. $H_c^i(X, \overline{\mathbb{Q}}_\ell)$).

For any finite group H, $\mathcal{R}(H)$ denotes the Grothendieck group of the finite dimensional representation of H over $\overline{\mathbb{Q}}_\ell$. The characters of H takes values in the maximal cyclotomic field $\mathbb{Q}(\zeta_\infty) = \bigcup_n \mathbb{Q}(\zeta_n) \subset \overline{\mathbb{Q}}_\ell$, which has a well defined "complex conjugate" automorphism defined by $\zeta_n \mapsto \zeta_n^{-1}$ for any n, denoted by $x \mapsto \overline{x}$. Therefore a natural inner product on $\mathcal{R}(H) \otimes \mathbb{Q}$ is defined as ;

$$\langle f, f' \rangle_H = \frac{1}{|H|} \sum_{x \in H} f(x)\overline{f'(x)}$$

where f, f' are the characters which are identified with the elements of $\mathcal{R}(H)$. Note that $\overline{f(x)} = f(x^{-1})$. The set of the irreducible representations of H gives an orthonormal basis of $\mathcal{R}(H) \otimes \mathbb{Q}$ with respect to this inner product.

Return to the reductive group G/K. For a maximal torus T and a Borel subgroup B containing T, the Weyl group of (T, B) is $W = N(T)/T$ where $N(T)$ is the normalizer $\{x \in G \mid xT = Tx\}$ of T. Any other pair (T', B') is a conjugate of (T, B), which is the image of (T, B) by $\operatorname{ad} g : x \mapsto gxg^{-1}$ for some $g \in G$, and $\operatorname{ad} g$ gives the isomorphism between the Weyl groups W, W'. Therefore we can speak of the *canonical* maximal torus $\mathbb{T}$ and the *canonical* Weyl group $\mathbb{W}$, provided with the isomorphism $\mathbb{T} \to T$ and $\mathbb{W} \to W$ for any pair (T, B), which are compatible with the isomorphisms $\operatorname{ad} g$ between anyy pairs (T, B). $\mathbb{W}$ is a Coxeter group generated by the elements $s_1, \ldots, s_n$ of order 2, and the length function is denoted by $w \mapsto l(w)$. $w \in \mathbb{W}$ can be expressed in the form $w = s_{i_1} \ldots s_{i_k}$ for $k = l(w)$ (*reduced expression*).

2. The construction of the Deligne-Lusztig variety

2.1. **The variety X_w.** Let $X = X_G$ be the flag variety of G, i.e. set of all Borel subgroups of G which is a smooth projective variety; G acts on X from the left by conjugation. When we refer to a Borel subgroup B as a point $x \in X$, we denote this left action by $x \mapsto gx$. When we fix a Borel subgroup B, the stabilizer of $B \in X$ for this action is B itself, which gives the isomorphism $G/B \ni \overline{g} \mapsto gBg^{-1} \in X$.

Recall the *Bruhat decompostion* of $X \times X$ into the orbits of G ; the set of orbits are identified with the Weyl group $\mathbb{W}$ of G, and we denote by O_w the orbit corresponding to $w \in \mathbb{W}$;

$$X \times X = \coprod_{w \in \mathbb{W}} O_w, \qquad O_w = G \cdot (B, \tilde{w} B \tilde{w}^{-1})$$

Here $\tilde{w} \in N(T)$ is the representative for $w \in \mathbb{W} \cong N(T)/T$, and O_w does not depend on the choice of the fixed Borel subgroup B. We say that two Borel subgroups B', B'' are in *relative position* w if $(B', B'') \in O_w$.

Definition 2.1. For any $w \in \mathbb{W}$, X_w is the locally closed subscheme of X consisting of all Borel subgroups B' of G such that B' and $F(B')$ are in relative position w, i.e. for any fixed Borel subgroup B there exists $g \in G$ such that $B' = gBg^{-1}$, $F(B') = (g\tilde{w})B(g\tilde{w})^{-1}$.

As the first projection $(B, F(B)) \longmapsto B$ of the graph of Frobenius onto X is an isomorphism, X_w can be defined as the (transverse) intersection of O_w with the graph of the Frobenius map in $X \times X$. The orbit O_w being smooth of dimension $\dim X + l(w)$, X_w is smooth and purely of dimension $l(w)$. We get a partition $X = \coprod_{w \in \mathbb{W}} X_w$. By definition, each X_w is stable under the action of G^F.

Example 2.2. X_e is a zero dimensional variety, namely the set of all F-stable Borel subgroups. As they are all G^F-conjugate, $X_e \cong G^F/B^F$ for any F-stable Borel subgroup B.

2.2. **The variety Y_w.** Now choose a maximal torus T and a Borel subgroup B containing T, and denote the unipotent radical by U with $B = TU$. Then the quotient $Y = G/U$ has a natural left action of G, and is a right T-torsor (right principal homogeneous space of T) over $X = G/B$, where the fiber of $x \in X$ of the left G-equivariant covering $Y \to X$ is

$$Y(x) = \{g \in G \mid ge = x\}/U$$

where $e \in X$ is the point corresponding to B, and the left action of G on X is the conjugation as usual, i.e. ge corresponds to gBg^{-1}. Note that the set $\{g \in G \mid ge = x\}$ is a right B-torsor, and $Y(x)$ is a right T-torsor as it should be.

Now for any $w \in \mathbb{W}$, choose a lifting $\tilde{w} \in N(T)$ of w, and let $x, y \in X$ be any pair which is in relative position w, i.e. there exists $g \in G$ such that $x = ge,\ y = g\tilde{w}e$ (cf. Def. 2.1). The set $A(x,y) = \{g \in G \mid ge = x,\ g\tilde{w}e = y\}$ is a torsor under the action of $B \cap \tilde{w}B\tilde{w}^{-1} = T(U \cap \tilde{w}U\tilde{w}^{-1})$, therefore $A(x,y)$ surjects to $Y(x)$ by the natural projection mod U. Therefore we can define a map :

$$\cdot\tilde{w} : Y(x) \ni \overline{g} \longmapsto \overline{g\tilde{w}} \in Y(y)$$

by choosing the representative g inside $A(x,y)$, as the class $\overline{g\tilde{w}}$ of $g\tilde{w} \bmod U$ depends only on the class of g.

Now assume that T, B are F-stable. Note that the identifications of T and $N(T)/T$ with $\mathbb{T}$ and $\mathbb{W}$ are compatible with F, and as for any $\overline{g} \in Y(x)$ we have $F(\overline{g})e = F(ge) = F(x)$, we have a map $F : Y(x) \to Y(F(x))$. The subspace $Y_{\tilde{w}}$ of Y is defined as :

$$Y_{\tilde{w}} = \{\overline{g} \in Y \mid F(\overline{g}) = \overline{g} \cdot \tilde{w}\}$$

$Y_{\tilde{w}}$ is stable under the action of G^F. As the image $x = ge$ of $\overline{g} \in Y_{\tilde{w}}$ under the projection $Y \to X$ satisfies $F(x) = F(ge) = \overline{g}\tilde{w}e = g\tilde{w}e$, x and $F(x)$ are in relative position w, i.e. $x \in X_w$. Therefore this variety is a G^F-equivariant covering of X_w.

As the fiber of $x \in X_w$ for the projection $Y_{\tilde{w}} \to X_w$ is :

$$Y_{\tilde{w}}(x) = \{\overline{g} \in Y(x) \mid F(\overline{g}) = \overline{g} \cdot \tilde{w}\}$$

and for $\overline{g} \in Y_{\tilde{w}}(x)$, $\overline{g}t$ for $t \in T$ lies in $Y_{\tilde{w}}(x)$ if and only if t satisfies $F(\overline{g}t) = \overline{g}t \cdot \tilde{w}$, i.e. $F(\overline{g})F(t) = (\overline{g} \cdot \tilde{w})(\tilde{w}^{-1}t\tilde{w}) = (\overline{g} \cdot \tilde{w})(\operatorname{ad} w^{-1}(t))$, the structure group of the covering is

$$\mathbb{T}_w^F = \{t \in T \mid F(t) = \operatorname{ad} w^{-1}(t)\}$$

which is the fixed set of the Frobenius of $\mathbb{T}_w$, which is defined as the torus $\mathbb{T}$ provided with the rational structure for which the Frobenius is $\operatorname{ad} w \circ F$. We denote this G^F-equivariant $\mathbb{T}_w^F$-torsor by $\pi : Y_{\tilde{w}} \to X_w$. Also we can show that this G^F-equivariant $\mathbb{T}_w^F$-torsor is independent of the choice of the lifting $\tilde{w} \in N(T)$ of $w \in \mathbb{W}$, because if we had $\tilde{w}' = \tilde{w}t$, $\overline{g} \mapsto \overline{g}t'$ with $t' \in T$ satisfying $t = F(t')(\operatorname{ad} w^{-1}(t'))^{-1}$ would give an G^F-equivariant isomorphism $Y_{\tilde{w}} \to Y_{\tilde{w}'}$ of $\mathbb{T}_w^F$-torsors over X_w.

Definition 2.3. For any $w \in \mathbb{W}$, we denote the above G^F-equivariant $\mathbb{T}_w^F$-torsor by $\pi : Y_w \to X_w$.

The above construction is independent of the choice of the fixed F-stable (T, B) (up to isomorphism), as they are all G^F-conjugate.

2.3. **The representations R_w^θ.** The groups G^F and $\mathbb{T}_w^F$ act on the variety Y_w, therefore these two groups act on the compact support cohomology groups $H_c^*(Y_w) = H_c^*(Y_w, \overline{\mathbb{Q}}_\ell)$. And as these actions commute with each other, if for each character

$\theta \in \mathrm{Hom}(\mathbb{T}_w^F, \overline{\mathbb{Q}}_\ell^\times)$ of $\mathbb{T}_w^F$, we denote by $H_c^*(Y_w)_\theta$ the subspace of $H_c^*(Y_w)$ where $\mathbb{T}_w^F$ acts by θ, $H_c^*(Y_w)_\theta$ is a representation of G^F.

Definition 2.4. For any $w \in \mathbb{W}$ and any character θ of $\mathbb{T}_w^F$, we denote by R_w^θ the alternating sum virtual representation $\sum_i (-1)^i H_c^i(Y_w)_\theta$ of G^F inside the Grothendieck group $\mathcal{R}(G^F)$.

Omitting ℓ from the notation is justified by the fact that for any variety X and an automorphism σ of X, the alternating sum of the trace of the endomorphism σ^* on the graded vector space $H_c^*(X)$ is an integer independent of ℓ (Prop. 3.3 of [DL]).

Also, if we decompose the direct image sheaf $\pi_*\overline{\mathbb{Q}}_\ell$ on X_w according to the action of $\mathbb{T}_w^F$, we get the decomposition :

$$\pi_*\overline{\mathbb{Q}}_\ell = \bigoplus_\theta \mathcal{F}_\theta$$

into the smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_\theta$ of rank one, where $\mathcal{F}_\theta$ is the subsheaf of $\pi_*\overline{\mathbb{Q}}_\ell$ where $\mathbb{T}_w^F$ acts by θ, and the decompostion of the cohomology groups :

$$H_c^*(Y_w) = H_c^*(Y_w, \overline{\mathbb{Q}}_\ell) = H_c^*(X_w, \pi_*\overline{\mathbb{Q}}_\ell) = \bigoplus_\theta H_c^*(X_w, \mathcal{F}_\theta)$$

and in particular $H_c^*(Y_w)_\theta = H_c^*(X_w, \mathcal{F}_\theta)$.

Example 2.5. For $\theta = 1$, $R_w^1 = \sum_i (-1)^i H_c^i(X_w)$.

Example 2.6. For $w = e$, $\pi : Y_e \to X_e$ is no other than the projection $G^F/U^F \to G^F/B^F$, and R_e^θ is the representation of G^F on the space of functions $f : G^F \to \overline{\mathbb{Q}}_\ell$ satisfying $f(gtu) = \theta(t)^{-1}f(g)$, on which G^F acts by $(g\cdot f)(x) = f(g^{-1}x)$. This coincides with the usual definition of the induced representation induced from the characters θ of T^F which is contained in B^F, the rational points of the F-stable Borel subgroup B.

2.4. **An alternative description :** $X_{T\subset B},\ Y_{T\subset B},\ R_{T\subset B}^\theta$**.** Now we would want to represent the above representations in terms of the characters of the F-fixed sets of actual F-stable maximal tori, rather than the characters of $\mathbb{T}_w^F$. For this, we introduce more concrete descriptions of the variety X_w, Y_w and the representations R_w^θ.

First we give concrete models of the varieties X_w, Y_w. In this subsection, we denote the fixed F-stable maximal torus, the fixed F-stable Borel subgroup and its unipotent radical by T^*, B^* and U^*.

We observe that for any $g \in G$, $x = ge$ lies in X_w if and only if gB^*g^{-1} and $F(gB^*g^{-1}) = F(g)B^*F(g)^{-1}$ are in the relative position w, i.e. $F(g)B^*F(g)^{-1} = (g\tilde{w})B^*(g\tilde{w})^{-1}$ or $g^{-1}F(g) \in \tilde{w}B^*$. Here the set $\{g \in G \mid x = ge,\ F(x) = g\tilde{w}e\}$ is a $B^* \cap \tilde{w}B^*\tilde{w}^{-1}$-torsor :

$$X_w = \{g \in G \mid g^{-1}F(g) \in \tilde{w}B^*\}/(B^* \cap \tilde{w}B^*\tilde{w}^{-1})$$

Here $B^* \cap \tilde{w}B^*\tilde{w}^{-1} = T^*(U^* \cap \tilde{w}U^*\tilde{w}^{-1})$, and we normalize $g^{-1}F(g) \in \tilde{w}B^*$ to $g^{-1}F(g) \in \tilde{w}U^*$ by changing g to gt where $t \in T^*$. Then for $g \in \{g \in G \mid g^{-1}F(g) \in$

$\tilde{w}U^*\}$, gt for $t \in T^*$ would be in the same set if and only if

$$(gt)^{-1}F(gt) = t^{-1}(g^{-1}F(g)\tilde{w}^{-1})(\tilde{w}F(t)\tilde{w}^{-1}) \in \tilde{w}U^*\tilde{w}^{-1},$$

and as $T^* = \tilde{w}T^*\tilde{w}^{-1}$ commutes with $\tilde{w}U^*\tilde{w}^{-1}$, this is equivalent to $t^{-1}(\tilde{w}F(t)\tilde{w}^{-1}) = 1$, i.e. $t \in \mathbb{T}_w^F$. Therefore this set is the $\mathbb{T}_w^F(U^* \cap \tilde{w}U^*\tilde{w}^{-1})$-torsor:

$$X_w = \{g \in G \mid g^{-1}F(g) \in \tilde{w}U^*\}/\mathbb{T}_w^F(U^* \cap \tilde{w}U^*\tilde{w}^{-1})$$

Now if we trace back the definition of Y_w, a point of Y_w is defined by a point $x \in X_w$, i.e. a Borel subgroup B, and $g \in G$ such that $ge = x,\ g\tilde{w}e = F(x),\ g\tilde{w} = F(g) \mod U^*$. Therefore we conclude that :

$$Y_w = \{g \in G \mid g^{-1}F(g) \in \tilde{w}U^*\}/(U^* \cap \tilde{w}U^*\tilde{w}^{-1})$$

Now we would like to change the coordinate to give another model isomorphic to the above $Y_w \to X_w$, which would be a G^F-equivariatnt T^F-torsor for a particular F-stable maximal torus T instead of $\mathbb{T}_w^F$-torsor.

Definition 2.7. Let T be a F-stable maximal torus and B be a Borel subgroup containing T, with unipotent radical U. Define $X_{T\subset B}$ by :

$$\begin{aligned} X_{T\subset B} &= \{g \in G \mid g^{-1}F(g) \in F(B)\}/(B \cap F(B)) \\ &= \{g \in G \mid g^{-1}F(g) \in F(U)\}/T^F(U \cap F(U)) \end{aligned}$$

and $Y_{T\subset B}$ by $Y_{T\subset B} = \{g \in G \mid g^{-1}F(g) \in F(U)\}/(U \cap F(U))$.

Proposition 2.8. *For (T,B) as above, let w be the relative position of B and $F(B)$. We can choose $h \in G$ such that $h(T^*,B^*)h^{-1} = (T,B)$, so that the map $g \mapsto gh^{-1}$ gives an isomorphism from the G^F-equivariant $\mathbb{T}_w^F$-torsor $Y_w \to X_w$ to the G^F-equivariant T^F-torsor $Y_{T\subset B} \to X_{T\subset B}$.*

Definition 2.9. For a character $\theta : T^F \to \overline{\mathbb{Q}}_\ell$, we denote by $R^\theta_{T\subset B}$ the alternating sum virtual representation $\sum_i (-1)^i H^i_c(Y_{T\subset B})_\theta$ of G^F in the Grothendieck group $\mathcal{R}(G^F)$.

For the h in Prop. 2.8, we have $R^\theta_{T\subset B} = R_w^{\theta\circ\mathrm{ad}\, w}$.

3. Main theorems

3.1. **Independence of $R^\theta_{T\subset B}$ on B and the character formula.** The virtual representation $R^\theta_{T\subset B}$ which we defined in the last section turns out to be independent of the choice of B. To prove this, we must analyze the representation using the traces of the elements of G^F.

The proof starts from the case $\theta = 1$ case :

Proposition 3.1. (Prop. 1.6, Cor. 1.14 of [DL]) *The virtual representation $R^1_{T\subset B} = R^1_w \in \mathcal{R}(G^F)$ depends only on the F-conjugacy class of the relative position w of B and $F(B)$, which in turn depends only on the G^F-conjugacy class of the maximal tori T.*

The proof of the first part depends on the rather detailed investigation of the geometry of the variety X_w, which is described using the partition and fibration according to the reduced expression of w. The second part is a lemma in the theory of algebraic groups. This proposition proves the next general theorem when $\theta = 1$:

Theorem 3.2. (Cor. 4.3 of [DL]) *$R^\theta_{T\subset B}$ is independent of the choice of a Borel subgroup B containing T.*

The general case follows from the following *character formula*, which is in turn proved using the $\theta = 1$ case :

Proposition 3.3. *If $x = su$ is the Jordan decomposition of $x \in G^F$, then :*

$$\mathrm{Tr}(x, R^\theta_{T\subset B}) = \frac{1}{|Z^0(s)^F|} \sum_{s\in G^F,\ gTg^{-1}\subset Z^0(s)} Q_{gTg^{-1},Z^0(s)}(u)\,\mathrm{ad}\,g(\theta)(s)$$

where Q is defined below.

Definition 3.4. For any reductive group G and a F-stable maximal torus T of G, the character of $R^1_{T\subset B}$ (which doesn't depend on B by Prop. 3.1) on the unipotent elements u of G^F is called the *Green function* and denoted by $Q_{T,G}(u)$.

The above character formula comes from the calculation of the trace of $x = su$ on $H^*_c(Y_{T\subset B})_\theta$, where the next *fixed point formula* plays a vital role :

Proposition 3.5. *Let X be a variety over K, and let $\sigma : X \to X$ be an automorphism of finite order. If we decompose σ as $\sigma = su$ where s, u are powers of σ of orders respectively prime to p and a power of p, we have :*

$$\mathrm{Tr}(\sigma^*, H^*_c(X)) = \mathrm{Tr}(u^*, H^*_c(X^s))$$

where X^s is the fixed point set of s, and the traces are the alternating sums of the traces on each cohomology group, namely $\mathrm{Tr}(f, V^*) = \sum_i (-1)^i \mathrm{Tr}(f, V^*)$ *for any endomorphism f on a graded vector space V.*

This is a purely algebro-geometric lemma which is proved in the section 3 of [DL] by a standard [SGA]-type argument. Now if we apply this fixed point formula to our G^F-equivariant T^F-torsor $Y_{T\subset B} \to X_{T\subset B}$, we can write the trace $\mathrm{Tr}(g^*, H^*_c(Y_{T\subset B})_\theta)$ of $g = su$ in terms of the character θ and the trace $\mathrm{Tr}(u^*, H^*_c(X^s))$ of u on the cohomology of the fixed point set $X^s_{T\subset B}$. By carefully working out the fixed point set in this case, we obtain the above character formula.

Definition 3.6. We denote the virtual representation $R^\theta_{T\subset B}$ simply by R^θ_T.

Note that the isomorphism between the (virtual) representations in the cohomology is proved, but the varieties $X_{T\subset B}, Y_{T\subset B}$ and $X_{T\subset B'}, Y_{T\subset B'}$ might not be isomorphic.

3.2. **Disjointness theorem.**

Definition 3.7. Let T, T' be two F-stable maximal tori of G, and θ, θ' be characters of T^F, T'^F. The pairs $(T, \theta), (T', \theta')$ are said to be *geometrically conjugate* when the pairs $(T, \theta \circ N), (T', \theta' \circ N)$ where N is the norm from T^{F^n} to T^F (resp. T'^{F^n} to T'^F) are G^{F^n}-conjugate for some integer n. Here the *norm* N for T is the map $\sum_{i=0}^{n-1} F^i : T^{F^n} \to T^F$.

Theorem 3.8. (Cor. 6.3 of [DL]) *If (T, θ) and (T', θ') are not geometrically conjugate, $\langle R_T^\theta, R_{T'}^{\theta'} \rangle = 0$.*

This theorem is the direct consequence of the following geomteric fact :

Proposition 3.9. *If θ^{-1} is not geometrically conjugate to θ', then :*

$$[H_c^*(Y_{T\subset B}) \otimes H_c^*(Y_{T'\subset B'})]^{G^F} = 0$$

By Künneth formula, we are reduced to show that $H_c^*(Y_{T\subset B} \times Y_{T'\subset B'})_{\theta,\theta'}^{G^F} = 0$, where the subscript $_{\theta,\theta'}$ denotes the part where $T^F \times T'^F$ acts by $\theta \otimes \theta'$. If we introduce the varieties

$$S_{T\subset B} = \{g \in G \mid g^{-1}F(g) \subset F(U)\}, \quad S_{T'\subset B'} = \{g' \in G \mid g'^{-1}F(g') \subset F(U')\}$$

where U, U' is the corresponding unipotent radicals, the cohomology of $Y_{T\subset B} \times Y_{T'\subset B'}$ is just the shift of the cohomology of the covering space $S_{T\subset B} \times S_{T'\subset B'}$, as the former is the free quotient of the latter by the unipotent group $(U \cap F(U)) \times (U' \cap F(U'))$. Now the G^F-invariant part of the cohomology of $S_{T\subset B} \times S_{T'\subset B'}$ is the cohomology of the quotient $S_{T\subset B} \times S_{T'\subset B'}/G^F$, and this quotient is shown to be isomorphic to the variety $\overline{S} = \{(x, x', y) \in F(U) \times F(U') \times G \mid xF(y) = yx'\}$. This variety has a finite partition (stratification) into the locally closed subschemes $\overline{S} = \coprod_{w\in W(T,T')} \overline{S}_w$ indexed by the set $W(T, T') = T\backslash N(T, T') = N(T, T')/T'$ where $N(T, T') = \{g \in G \mid Tg = gT'\}$, corresponding to the Bruhat cells. Therefore we are reduced to proving $H_c^*(\overline{S}_w)_{\theta,\theta'} = 0$, and in order to show this, we extend the action of $T^F \times T'^F$ to an action of a closed subgroup :

$$H_w = \{(t, t') \in T \times T' \mid t'F(t')^{-1} = F(\tilde{w})^{-1}tF(t)^{-1}F(\tilde{w})\}$$

of $T \times T'$, where $\tilde{w} \in N(T, T')$ is a representative of w. Now as a connected algebraic group cannot act on the cohomology of a variety non-trivially (a simple homotopy argument), in order to have $H_c^*(\overline{S}_w)_{\theta,\theta'} \neq 0$, $\theta \otimes \theta'$ must be trivial on $H_w^0 \cap (T^F \times T'^F)$ where H_w^0 is the connected component of H_w, and this would show that $\theta'^{-1} = \theta \circ \operatorname{ad} F(w)$, i.e. θ^{-1}, θ' is geometrically conjugate.

By combining this theorem with the character formula Prop. 3.3, we can actually determine the intertwining numbers between $R_T^\theta, R_{T'}^{\theta'}$ when θ, θ' *are* geometrically conjugate as follows :

Proposition 3.10. *In general, $\langle R_T^\theta, R_{T'}^{\theta'} \rangle_{G^F} = |\{w \in W(T, T')^F \mid \operatorname{ad} w(\theta') = \theta\}|$.*

This proposition gives a criterion for the irreducibility of R_T^θ :

Definition 3.11. The character θ of T^F is called in *general position* if it is not fixed by any non-trivial element of $(N(T)/T)^F$.

Corollary 3.12. *If θ is in general position, on of $\pm R_T^\theta$ is an irreducible representation of G^F.*

To determine the sign, we have to compute some of the characters on the semisimple elements of G^F.

3.3. **Calculation on semisimple elements.** Recall that the *Steinberg representation* St_G is the irreducible representation of G occuring in the induced representation $\mathrm{Ind}_{B^F}^{G^F}(1) = R_T^1$ where (T, B) is F-stable. The character of this representation vanishes outside the semisimple elements :

$$\mathrm{St}_G(g) = \begin{cases} (-1)^{\sigma(G)-\sigma(Z^0(g))}\mathrm{St}_{Z^0(g)}(e) & (g \in G^F \text{ is semisimple}) \\ 0 & (g \in G^F \text{ is not semisimple}) \end{cases}$$

where $\sigma(G)$ denotes the $\mathbb{F}_q$-rank of G for any reductive group G.

Now consider R_T^θ for general (T, θ). By the disjointness theorem Th. 3.8 we have $\langle R_T^\theta, \mathrm{St}_G\rangle = 0$ if $\theta \neq 1$, and by working out the character formula from this fact we find the following equalities :

Proposition 3.13. (Th. 7.1, Prop. 7.3 of [DL]) *Let $\sigma_T = \sigma(G) - \sigma(T)$.*

(i) $Q_{T,G}(e)\mathrm{St}_G(e) = (-1)^{\sigma_T}|G^F|/|T^F|$.
(ii) $(-1)^{\sigma_T} R_T^\theta \otimes \mathrm{St}_G = \mathrm{Ind}_{T^F}^{G^F}(\theta)$.
(iii) *For any semisimple $s \in G^F$, we have :*

$$\sum_{T \ni s} (-1)^{\sigma_T} \sum_\theta \theta(s)^{-1} R_T^\theta(g) = \begin{cases} \mathrm{St}_G(s)|Z(s)^F| & (g \in G^F \text{ is conjugate to } s) \\ 0 & (\text{otherwise}) \end{cases}$$

where θ runs through all the characters of T^F.

As a corollary of Prop. 3.13 (ii), we can determine the sign occurring in Cor. 3.12 :

Theorem 3.14. (Prop. 7.4 of [DL]) *If θ is in general position, $(-1)^{\sigma(G)-\sigma(T)} R_T^\theta$ is irreducible.*

As a corollary of Prop. 3.13 (iii), we have for any $\rho \in \mathcal{R}(G^F)$:

$$\dim \rho = \rho(e) = \frac{1}{\mathrm{St}_G(e)} \sum_T (-1)^{\sigma_T} \sum_\theta \langle \rho, R_T^\theta \rangle$$

where T runs through the set of all the F-stable maximal tori, and θ runs through all the characters of T^F. This gives :

Theorem 3.15. (Cor. 7.7 of [DL]) *Any irreducible representation ρ of G^F occurs in a R_T^θ for some (T,θ), i.e. $\langle \rho, R_T^\theta \rangle \neq 0$.*

Also we remark that applying Prop. 3.13 (iii) to $s = e$ gives the following decomposition of the regular representation $\mathrm{Ind}_e^{G^F}(1)$ of G^F :

$$\mathrm{Ind}_e^{G^F}(1) = \frac{1}{\mathrm{St}_G(e)} \sum_{(T,\theta)} (-1)^{\sigma_T} R_T^\theta$$

where (T,θ) runs through the set of all pairs consisting of the F-stable maximal torus T of G and a character θ of T^F.

4. Further results

4.1. **Induced and cuspidal representations.** Here we want to examine when the (virtual) representations R_T^θ of G_F are induced from the group of rational points P^F of a F-stable proper parabolic subgroup P, and when they are cuspidal.

Let P be a F-stable parabolic subgroup P of G, and T a F-stable maximal torus T of P and U the unipotent radical of P. The quotient group $L = P/U$ is a connected reductive algebraic group with the action of F, which is isomorphic to the Levi subgroup of P. If we denote the natural projection by $\pi : P \to L$, it induces an isomorphism $T \cong \pi(T)$, and therefore $T^F \cong \pi(T)^F$. This allows us to identify the character θ of T^F with a corresponding character of $\pi(T)^F$, and we denote by $R_{T,P}^\theta$ the image of the virtual representation $R_{\pi(T)}^\theta$ of L^F under the canonical embedding $\mathcal{R}(L^F) \subset \mathcal{R}(P^F)$.

Then, as one naturally expects, we have :

Proposition 4.1. (Prop. 8.2 of [DL]) $R_T^\theta = \mathrm{Ind}_{P^F}^{G^F}(R_{T,P}^\theta)$.

This is shown by the partition of the Deligne-Lusztig variety as follows. Let us choose a Borel subgroup B which satisfies $T \subset B \subset P$, and let $\mathcal{P}$ denote the set of all parabolic subgroups P' of G which are G^F-conjugate to P (which implies that they are all F-stable). Then we have a partition :

$$Y_{T\subset B} = \coprod_{P' \in \mathcal{P}} Y_{P'}$$

where we put :

$$Y_{P'} = \{g \in G \mid g^{-1}F(g) = F(U),\ gPg^{-1} = P'\}/(U \cap F(U))$$

Then by choosing $h \in G^F$ such that $hPh^{-1} = P'$, whe get an isomorphism :

$$Y_{P'} \ni g \longmapsto \pi(h^{-1}g) \in Y_{\pi(T)\subset\pi(B)}$$

Therefore we have an isomorphism of G^F-modules :

$$H_c^*(Y_{T\subset B}) \cong \mathrm{Ind}_{P^F}^{G^F}\left(H_c^*(Y_{\pi(T)\subset\pi(B)})\right)$$

which is also compatible with the action of T^F, which proves the proposition.

Proposition 4.2. (Th. 8.3 of [DL]) *If an F-stable maximal torus T is not contained in any F-stable proper parabolic subgroup of G and θ is a character of T^F in general position, the irreducible representation $(-1)^{\sigma_T} R_T^\theta$ of G^F is cuspidal.*

To show this proposition, we want $\langle R_T^\theta, \mathrm{Ind}_{U^F}^{G^F}(1)\rangle = 0$. This follows from the disjointness theorem and the decomposition of the virtual representation $\mathrm{Ind}_{U^F}^{G^F}(1)$ into R_T^θ's with T contained in proper F-stable parabolic subgroups P, which is readily obtained by observing that $\mathrm{Ind}_{U^F}^{G^F}(1) = \mathrm{Ind}_{P^F}^{G^F}(\mathrm{Ind}_{U^F}^{P^F}(1))$ and writing $\mathrm{Ind}_{U^F}^{P^F}(1)$ in terms of the regular representation of L^F where $L = P/U$.

4.2. **Vanishing theorems.** By constructing a natural compactification of the variety $X_{T\subset B}$, we can actually determine exactly where the irreducible representations $(-1)^{\sigma_T} R_T^\theta$ (for θ in general position) appear inside the cohomology $H_c^*(Y_{T\subset B})$, namely they only appear in the $l(w)$-th cohomology $H_c^{l(w)}(Y_{T\subset B})$ where w is the relative position of B and $F(B)$.

In this subsection we use the notations X_w, Y_w of the subsections 2.1 – 2.3. The main result is :

Theorem 4.3. (Cor. 9.9 of [DL]) *If X_w is affine and the character θ of $\mathbb{T}_w^F$ is in general position, we have $H_c^i(Y_w)_\theta = 0$ for $i \neq l(w)$.*

First we remark that the condition that X_w is affine is always satisfied for classical groups, and is satisfied in general as soon as q is sufficiently large (Th. 9.7 of [DL]).

Now this theorem is deduced from the following proposition :

Proposition 4.4. (Th. 9.8 of [DL]) *For a character θ of $\mathbb{T}_w^F$ in general position, $H_c^*(Y_w)_\theta \cong H^*(Y_w)$, or equivalently $H_c^*(X_w, \mathcal{F}_\theta) \cong H^*(X_w, \mathcal{F}_\theta)$.*

First we describe how to deduce Th. 4.3 from Prop. 4.4. As we know that $H^i(X_w, \mathcal{F}_\theta) = 0$ for $i > \dim X_w = l(w)$ when X_w is affine ([SGA] 4-XIV). To show that $H^i(X_w, \mathcal{F}_\theta) = 0$ for $i < l(w)$, observe that by Poincaré duality $H^i(X_w, \mathcal{F}_\theta)$ is the dual of $H^{2l(w)-i}(X_w, \mathcal{F}_{\theta^{-1}})$ up to twist as $\mathcal{F}_{\theta^{-1}}$ is the dual sheaf of $\mathcal{F}_\theta$, hence vanishes for $2l(w) - i > l(w)$, i.e. $i < l(w)$.

To prove Th. 4.3, it is enough to construct a compactification $j : X_w \to \overline{X_w}$ of X_w, and show that $j_*\mathcal{F}_\theta = j_!\mathcal{F}_\theta$ and $R^k j_*\mathcal{F}_\theta = 0$ for every $k > 0$, as then we have $H_c^*(X_w, \mathcal{F}_\theta) = H^*(X_w, j_!\mathcal{F}_\theta) \cong H^*(X_w, \mathcal{F}_\theta)$ by the degeneration of the Leray spectral sequence for j.

First we give the construction of a compactification as follows.

Definition 4.5. Let $w = s_1 \cdots s_n$, $n = l(w)$ be a reduced expression of $w \in \mathbb{W}$. We define $\overline{O_w} = \overline{O(s_1, \ldots, s_n)}$ (depends on the reduced expression) as the space of sequences $(B_0, \ldots, B_n)$ of Borel subgroups of G, where B_{i-1} and B_i are in relative position s_i or e for every $1 \le i \le n$.

We have a sequence of natural maps :

$$\overline{O_w} = \overline{O(s_1, \dots, s_n)} \to \overline{O(s_1, \dots, s_{n-1})} \to \cdots \to \overline{O(s_1)} \to X$$

where each map is a $\mathbb{P}^1$-bundle, therefore $\overline{O_w}$ is proper. For each i, we have a natural section of the above map given by :

$$\overline{O(s_1, \dots, s_{i-1})} \ni (B_0, \dots, B_{i-1}) \longmapsto (B_0, \dots, B_{i-1}, B_{i-1}) \in \overline{O(s_1, \dots, s_i)}$$

The inverse image of this section under $\overline{O_w} \to \overline{O(s_1, \dots, s_i)}$ is a divisor D_i of $\overline{O_w}$ consisting of all sequences $(B_0, \dots, B_n)$ with $B_{i-1} = B_i$, and the union $D = \bigcup_i D_i$, consisting of all sequences $(B_0, \dots, B_n)$ where $B_{i-1} = B_i$ for at least one i, is a divisor of $\overline{O_w}$ with normal crossings. As B_0, B_n are in relative position w outside D, and we obtain a natural isomorphism $\overline{O_w} - D \ni (B_0, \dots, B_n) \longmapsto (B_0, B_n) \in O_w$ to see that $\overline{O_w}$ is a compactification of O_w.

Definition 4.6. We define $\overline{X_w}$ (depends on the reduced expression but we suppress the notation) as the space of sequences $(B_0, \dots, B_n)$ where $B_n = F(B_0)$ and B_{i-1} and B_i in relative position s_i or e for every $1 \leq i \leq n$.

$\overline{X_w}$ is a subvariety of $\overline{O_w}$, namely the inverse image of the graph of Frobenius map under the map $\overline{O_w} \ni (B_0, \dots, B_n) \longmapsto (B_0, B_n) \in X \times X$, and it is shown to be a compactification of X_w with a divisor with normal crossings $\overline{D} = \bigcup_i \overline{D_i}$, where $\overline{D_i} = D_i \cap \overline{X_w}$ (Lemma 9.11 of [DL]).

Now the proof of $j_* \mathcal{F}_\theta = j_! \mathcal{F}_\theta$ and $R^k j_* \mathcal{F}_\theta = 0$ for $k > 0$ is accomplished by analyzing the ramification of the smooth sheaf $\mathcal{F}_\theta$ along $\overline{D_i}$, which is equivalent to the ramification of the covering $Y_w \to X_w$ at the infinity $\overline{D_i}$. As the structure group $\mathbb{T}_w^F$ of this covering is of order prime to p, the ramification is tame, and it turn out to be ramifying in the same way as the pull back of the Lang covering $F - 1 : \mathbb{T} \to \mathbb{T}$ under a coroot of $\mathbb{T}$ depending on i. From this description, we deduce that when θ is in general position, $\mathcal{F}_\theta$ would ramify along every $\overline{D_i}$ and gives $R^k j_* \mathcal{F}_\theta|_{\overline{D_i}} = 0$ for every $k \geq 0$, as desired.

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