A Variable Transformation Method for Stabilizing Abstract Delay Systems on Banach Lattices

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Abstract

In this study, we introduce the concept of an abstract delay system that can be used to characterize the behavior of a wide class of mathematical models that include partial differential equations and delay differential equations. We examine the stabilization problem of an abstract delay system on a Banach lattice using semigroup theory. To tackle this problem, we take advantage of the properties of a non-negative C_0 semigroup on a Banach lattice. The objective of this paper is to propose a stabilization method for an abstract delay system on a Banach lattice. We derive a sufficient condition under which an abstract delay system is uniformly exponentially stabilizable. Furthermore, we provide illustrative examples to verify the effectiveness of the proposed method.

Keywords: Stabilization, C_0 semigroup, Banach lattice, Nonnegativity

1. Introduction

Partial differential equations arise from many physical, chemical, biological, thermal, and fluid systems which are characterized by both spatial and temporal variables. Time delays also arise in many dynamical systems because, in most instances, physical, chemical, biological, and economic phenomena naturally depend not only on the present state but also on some past occurrences. The importance of the control of partial differential equations and delay differential equations is well recognized in a wide range of applications. Hence, this paper examines the stabilization problem of partial differential equations with time delays.

Partial differential equations and delay differential equations are known to be infinite-dimensional systems, while ordinary differential equations are finite-dimensional systems. The control of infinite-dimensional systems is a challenging problem attracting considerable attention in many research fields. Semigroups have become important tools in infinite-dimensional control theory over the past several decades. The semigroup method is a unified approach to addressing systems that include ordinary differential equations, partial differential equations, and delay differential equations. The behaviors of many dynamical systems including infinite-dimensional systems and finite-dimensional systems can be characterized by semigroup theory. The recent well-developed theory in such a framework has been accumulated in several books (Nagel, 1986; Curtain & Zwart, 1995; Engel & Nagel, 2000; Bátkai & Piazzera, 2005). In this paper, using semigroup theory, we introduce the concept of an abstract delay system that can be used to describe the behavior of a wide class of dynamical systems.

The linear quadratic control problem for an abstract delay system has been studied in (Pritchard & Salamon, 1985). Furthermore, the H^∞ control problem for such a system has been examined in (Kojima & Ishijima, 2006). The problems addressed in those papers have been reduced to finding a solution of the corresponding operator Riccati equation in Hilbert spaces. The feedback stabilizability of an abstract delay system on a Banach space has been investigated in (Hadd & Zhong, 2009). The analytic approach in (Hadd & Zhong, 2009) is based on the compactness of Banach spaces, while the problem in (Pritchard & Salamon, 1985; Kojima & Ishijima, 2006) is formulated in Hilbert spaces to make use of the properties of the inner product.

In this paper, we study the stabilization problem of an abstract delay system on a Banach lattice, which is a Banach space supplied with an order relation (Schaefer, 1974). To tackle this problem, we take advantage of the properties of a non-negative C_0 semigroup on a Banach lattice. The objective of this paper is to propose a stabilization method for an abstract delay system on a Banach lattice. We derive a sufficient condition under which an abstract delay system is uniformly exponentially stabilizable. Furthermore, we provide illustrative examples to verify the effectiveness of the proposed method.

This paper is organized as follows. Some notation and terminology are given in Sec. 2. The system considered here is defined in Sec. 3. Moreover, Sec. 3 is devoted to the introduction of a stability criterion for an abstract delay system on a Banach lattice. In Sec. 4, we derive a sufficient condition for the stabilization of an abstract delay system under the assumption that the system has a non-negative delay operator. To remove such a restrictive assumption from the obtained result in Sec. 4, we propose a variable transformation method in Sec. 5. Using this method, we can construct a stabilizing controller for an abstract delay system that might not satisfy the non-negativity assumption. Furthermore, we provide illustrative examples in both Sec. 4 and Sec. 5 to verify the effectiveness of the proposed method. Finally, some concluding remarks are presented in Sec. 6.

2. Notation and Terminology

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real numbers and non-negative real numbers, respectively. Let $\mathbb{N}_+$ denote the set of positive integers. Let X be a Banach space endowed with the operator norm $\|\cdot\|$. Let $\mathcal{L}(X, Y)$ denote the set of all bounded linear operators from a Banach space X to another Banach space Y. Let $\mathcal{L}(X)$ be defined by $\mathcal{L}(X, X)$. Let $I_d \in \mathcal{L}(X)$ denote the identity operator on X.

Definition 1 A family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a C_0 semigroup if all the following properties hold:

(i) $T(0) = I_d$.

(ii) $T(t + s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_+$.

(iii) The orbit maps $t \mapsto T(t)x$ are continuous maps from $\mathbb{R}_+$ into X for every $x \in X$.

Definition 2 Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on a Banach space X and let $D(\mathcal{A})$ be the subspace of X defined as

$$D(\mathcal{A}) := \left\{ x \in X : \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x) \text{ exists} \right\}.$$

For every $x \in D(\mathcal{A})$, we define

$$\mathcal{A}x := \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x).$$

The operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ is called the generator of the semigroup $(T(t))_{t\geq 0}$. In the following, let $(\mathcal{A}, D(\mathcal{A}))$ denote the operator $\mathcal{A}$ with domain $D(\mathcal{A})$.

Definition 3 Let $(\mathcal{A}, D(\mathcal{A}))$ be the generator of a C_0 semigroup $(T(t))_{t\geq 0}$.

$$\omega_0(\mathcal{A}) := \inf\{\omega \in \mathbb{R} : \exists M > 0 \text{ such that } \|T(t)\| \leq Me^{\omega t}, \forall t \in \mathbb{R}_+\}$$

is called the semigroup's growth bound.

Definition 4 Let $(\mathcal{A}, D(\mathcal{A}))$ be a closed operator on a Banach space X. The set

$$\rho(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda I_d - \mathcal{A} \text{ is bijective}\}$$

is called the resolvent set of $\mathcal{A}$, and the set

$$\sigma(\mathcal{A}) := \mathbb{C}\backslash\rho(\mathcal{A})$$

is called the spectrum of $\mathcal{A}$. For $\lambda \in \rho(\mathcal{A})$,

$$R(\lambda, \mathcal{A}) := (\lambda I_d - \mathcal{A})^{-1}$$

is called the resolvent of $\mathcal{A}$ at λ.

$$s(\mathcal{A}) := \sup\{\text{Real part of } \lambda : \lambda \in \sigma(\mathcal{A})\}$$

is called the spectral bound of $\mathcal{A}$.

Definition 5 A C_0 semigroup $(T(t))_{t\geq 0}$ with generator $(\mathcal{A}, D(\mathcal{A}))$ is said to be uniformly exponentially stable if $\omega_0(\mathcal{A}) < 0$.

Definition 6 A Banach space X is called a Banach lattice if X is supplied with an order relation such that all the following conditions hold:

(i) $f \geq g \Rightarrow f + h \geq g + h$ for all $f, g, h \in X$.

(ii) $f \geq 0 \Rightarrow \lambda f \geq 0$ for all $f \in X$ and $\lambda \in \mathbb{R}_+$.

(iii) $|f| \geq |g| \Rightarrow \|f\| \geq \|g\|$ for all $f, g \in X$.

Definition 7 A C_0 semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X is said to be non-negative if

$$0 \leq x \in X \;\Rightarrow\; 0 \leq T(t)x, \quad \text{for all } t \geq 0.$$

An operator $T(x) \in \mathcal{L}(X)$ on a Banach lattice X is also said to be non-negative if $T(x) \geq 0$ whenever $0 \leq x \in X$.

3. Preliminaries

In this section, we introduce the concept of an abstract delay system (Engel & Nagel, 2000) that can be used to describe the behavior of a wide class of dynamical systems. For a Banach space Y and a constant $\tau \in \mathbb{R}_+$, let $C([-\tau, 0], Y)$ denote the set of all continuous functions with domain $[-\tau, 0]$ and range Y. For a Banach space $X := C([-\tau, 0], Y)$, let $\Phi \in \mathcal{L}(X, Y)$ be a delay operator, and let $(\mathcal{B}, D(\mathcal{B}))$ be the generator of a C_0 semigroup on Y. With these notations, an abstract delay system is described by the following equation with an initial function $\varphi : [-\tau, 0] \to Y$:

$$\begin{cases} \dot{x}(t) = \mathcal{B}x(t) + \Phi(x(t-\tau)) & \text{for } t \geq 0, \\ x_0 = \varphi \in X. \end{cases} \tag{1}$$

A continuous function $x : [-\tau, \infty) \to Y$ is called a solution of (1) if all the following properties hold:

(i) $x(t)$ is right-sided differentiable at $t = 0$ and continuously differentiable for all $t > 0$.

(ii) $x(t) \in D(\mathcal{B})$ for all $t \geq 0$.

(iii) $x(t)$ satisfies (1).

Let C^r be the set of all r-times continuously differentiable functions. Let $(\mathcal{A}, D(\mathcal{A}))$ be the corresponding delay differential operator on X defined by

$$\begin{aligned} &\mathcal{A}f := \dot{f}, \\ &D(\mathcal{A}) := \{f \in C^1([-\tau, 0], Y) : f(0) \in D(\mathcal{B}) \text{ and } \dot{f}(0) = \mathcal{B}f(0) + \Phi(f(-\tau))\} \end{aligned} \tag{2}$$

Lemma 1 (Engel & Nagel, 2000) *The operator $(\mathcal{A}, D(\mathcal{A}))$ in (2) generates a C_0 semigroup $(T(t))_{t\geq 0}$ on X.*

Lemma 2 (Engel & Nagel, 2000) *If $\varphi \in D(\mathcal{A})$, then the function $x : [-\tau, \infty) \to Y$ defined by*

$$x(t) := \begin{cases} \varphi(t) & \text{if } -\tau \leq t \leq 0, \\ [T(t)\varphi](0) & \text{if } 0 < t, \end{cases} \tag{3}$$

is the unique solution of (1).

In the subsequent discussion, we assume that each Banach space X, Y in (1) is a Banach lattice.

Lemma 3 (Engel & Nagel, 2000) *If $\mathcal{B}$ generates a non-negative C_0 semigroup on Y and the delay operator $\Phi \in \mathcal{L}(X, Y)$ is non-negative, then the C_0 semigroup $(T(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ in (2) is also non-negative, and the following equivalence holds:*

$$s(\mathcal{A}) < 0 \Leftrightarrow s(\mathcal{B} + \Phi) < 0.$$

Lemma 4 (Engel & Nagel, 2000) *Assume that $(T(t))_{t\geq 0}$ is a non-negative C_0 semigroup with generator $(\mathcal{A}, D(\mathcal{A}))$ on X. Then,*

$$s(\mathcal{A}) = \omega_0(\mathcal{A}).$$

The following proposition directly follows from Lemmas 3 and 4.

Proposition 1 *Under the assumption that $\mathcal{B}$ generates a non-negative C_0 semigroup on Y and the delay operator $\Phi \in \mathcal{L}(X, Y)$ is non-negative, the C_0 semigroup $(T(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ in (2) is uniformly exponentially stable if and only if the spectral bound $s(\mathcal{B} + \Phi) < 0$.*

Note that the equality in Lemma 4 might not hold in general. This means that a C_0 semigroup $(T(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ is not necessarily uniformly exponentially stable even if the spectral bound is negative, i.e., $s(\mathcal{A}) < 0$. It can be seen from Proposition 1 that the non-negativity assumption enables us to determine the stability of an abstract delay system simply by examining the spectral bound.

4. Stabilization of Abstract Delay Systems

Let $(C, D(C))$ be the generator of a C_0 semigroup on a Banach lattice Y. For a Banach lattice $X := C([-\tau, 0], Y)$, let $\Phi \in \mathcal{L}(X, Y)$ be a delay operator. In this section, we consider the stabilization problem of an abstract delay system described by

$$\begin{cases} \dot{x}(t) = Cx(t) + \Phi(x(t-\tau)) + \mathcal{D}u(t), \\ x_0 = \varphi \in X, \end{cases} \tag{4}$$

where $u(t) : t \in \mathbb{R}_+ \to Y$ is the control input, and $(\mathcal{D}, D(\mathcal{D}))$ is the generator of a C_0 semigroup on Y.

Assumption 1 Φ is assumed to be non-negative.

Next, we consider the feedback stabilization problem of (4). Let $u(t)$ be given by

$$u(t) = \mathcal{K}x(t), \tag{5}$$

where $(\mathcal{K}, D(\mathcal{K}))$ is the generator of a C_0 semigroup on Y. Substituting (5) into (4) yields

$$\dot{x}(t) = (C + \mathcal{DK})x(t) + \Phi(x(t-\tau)). \tag{6}$$

Considering

$$\mathcal{B} = (C + \mathcal{DK}), \tag{7}$$

we see that the resulting closed-loop system (6) can be rewritten as (1).

Definition 8 System (4) is said to be uniformly exponentially stabilizable if there exists $u(t)$ in (5) such that the equilibrium point $x = 0$ of the resulting closed-loop system (6) is uniformly exponentially stable.

Now, we state the following theorem.

Theorem 1 *If there exists $\mathcal{K}$ such that $(C + \mathcal{DK})$ generates a non-negative C_0 semigroup and*

$$s(C + \mathcal{DK} + \Phi) < 0 \tag{8}$$

is satisfied, then system (4) is uniformly exponentially stabilizable.

Proof: Under the assumption that Φ is non-negative and $(C + \mathcal{DK})$ generates a non-negative C_0 semigroup, we see from Proposition 1 that the resulting closed-loop system (6) is uniformly exponentially stable if $s(C + \mathcal{DK} + \Phi) < 0$ holds.

An illustrative example is shown below. Let ℓ be a constant. We consider the following partial differential equation with a time delay, defined for $t \geq 0$, $x \in [0, \ell]$, $s \in [-\tau, 0]$, as

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} - d(x)z(x,t) + b(x)z(x,t-\tau) + u(x,t), \tag{9}$$

with the Dirichlet boundary condition

$$z(0,t) = z(\ell,t) = 0 \quad \text{for all } t \geq 0, \tag{10}$$

and with the initial condition

$$z(x,s) = h(x,s). \tag{11}$$

This equation can be interpreted as a model for the growth of a population in $[0, \ell]$. $z(x,t)$ is the population density at time t and space x. The term $\partial^2 z(x,t)/\partial x^2$ describes the internal migration. Moreover, the continuous functions $d(x)$ and $b(x)$ represent space-dependent death and birth rates, respectively. τ is the delay due to pregnancy. Let $d(x)$ and $b(x)$ be given as follows:

$$d(x) = 1 + \cos(8\pi x/\ell), \tag{12}$$

$$b(x) = 1 + 2\sin(\pi x/\ell). \tag{13}$$

Let $u(x,t)$ be given by

$$u(x,t) = -k(x)z(x,t). \tag{14}$$

To rewrite system (9) as an abstract delay system, we introduce the spaces $Y := C[0, \ell]$ and $X := C([-\tau, 0], Y)$. Moreover, we define the following operators:

$$\Delta := \frac{d^2}{dx^2}, \tag{15}$$

$$D(\Delta) := \left\{ f \in C^2[0,\ell] : f(0) = f(\ell) = 0 \right\}, \tag{16}$$

$$\mathcal{B} := \Delta - M_d - M_k, \quad D(\mathcal{B}) := D(\Delta), \tag{17}$$

$$\Phi := M_b \phi_\tau \in \mathcal{L}(X, Y), \tag{18}$$

where M_d, M_k, and M_b are the multiplication operators induced by $d(x)$, $k(x)$, and $b(x)$, respectively. $\phi_\tau : X \to Y$ denotes the point evaluation in $t \in [-\tau, 0]$. Considering that $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{K}$ in (7) are given by $\mathcal{C} = \Delta - M_d$, $\mathcal{D} = I_d$, and $\mathcal{K} = -M_k$, respectively, we see that system (9) can be rewritten as an abstract delay system (1). Hence, the solution to (9) can be given by (3).

The following lemma is known as the Trotter product formula.

Lemma 5 (Engel & Nagel, 2000) *Let $(S(t))_{t\geq 0}$ be a C_0 semigroup with generator $\mathcal{C}$ on a Banach space X. If $\mathcal{K} \in \mathcal{L}(X)$, then the C_0 semigroup $(T(t))_{t\geq 0}$ generated by $\mathcal{C} + \mathcal{K}$ is given by*

$$T(t)x = \lim_{n\to\infty} \left[S\left(\frac{t}{n}\right) e^{\frac{t\mathcal{K}}{n}} \right]^n x. \tag{19}$$

It is shown in (Engel & Nagel, 2000) that Δ generates a non-negative C_0 semigroup. Since $e^{-t(M_d+M_k)}$ is non-negative, we see from the Trotter product formula that $\mathcal{B}$ in (17) generates a non-negative C_0 semigroup. Moreover, we see from (13) and (18) that Φ is a non-negative operator. Consequently, it is seen that system (9), which can be reformulated as an abstract delay system (1), satisfies the non-negativity assumption that $\mathcal{B}$ generates a non-negative C_0 semigroup and Φ is non-negative. Therefore, it follows from Lemmas 3 and 4 that

$$\omega_0(\Delta + M_b - M_d - M_k) = s(\Delta + M_b - M_d - M_k).$$

In the following, we design $\mathcal{K}$ such that

$$s(\mathcal{B} + \Phi) < 0$$

is satisfied. Let δ be defined by

$$\delta := \inf_{x\in[0,\ell]} \left(d(x) + k(x) - b(x)\right).$$

If $\delta > 0$, then the operator $(\Delta + M_b - M_d - M_k + \delta)$ is dissipative. Hence, we obtain

$$\omega_0(\Delta + M_b - M_d - M_k) < -\delta.$$

This condition shows that if

$$b(x) - d(x) - k(x) < 0, \quad \text{for all } x \in [0, \ell],$$

then a solution of (9) is uniformly exponentially stable. For example, if we design

$$k(x) = 1 - d(x) + b(x), \tag{20}$$

then system (9) is uniformly exponentially stable.

5. Variable Transformation Method

In the previous section, we examined the stabilization problem of an abstract delay system (4) under Assumption 1. In many systems arising from physics, biology, chemistry, and economics, a solution with a non-negative initial value should remain non-negative. In fact, there are many systems that satisfy the non-negativity assumption. Nevertheless, the applicability of the stabilization method proposed in Sec. 4 is restricted to a class of systems that satisfy Assumption 1. To remove such a restrictive assumption, we propose a variable transformation method in this section. We also consider system (4) here, but Φ is not assumed to be non-negative in this section. In the subsequent discussion, we investigate the stabilization problem of system (4) whose $\mathcal{C}$ and Φ are not necessarily non-negative. To tackle this problem, we introduce the variable transformation

$$w = \mathcal{V}^{-1}(x), \tag{21}$$

where $\mathcal{V}$ is bijective, i.e., $w = \mathcal{V}^{-1}(x)$ uniquely exists, and $w = 0$ whenever $x = 0$, i.e., $\mathcal{V}^{-1}(0) = 0$.

Substituting (5) and (21) into (4), we have

$$\dot{w}(t) = \mathcal{V}^{-1}(\mathcal{C} + \mathcal{D}\mathcal{K})\mathcal{V}w(t) + \mathcal{V}^{-1}\Phi\mathcal{V}w(t-\tau). \tag{22}$$

Note that the stabilization problem of system (4) at $x = 0$ has been reduced to stabilizing system (22) at $w = 0$. Therefore, we see that the following statement directly follows from Theorem 1.

Theorem 2 *If there exist $\mathcal{K}$ and $\mathcal{V}$ such that all the following conditions are satisfied, then system (4) is uniformly exponentially stabilizable.*

(i) $\mathcal{V}^{-1}(\mathcal{C} + \mathcal{D}\mathcal{K})\mathcal{V}$ generates a non-negative C_0 semigroup.

(ii) $\mathcal{V}^{-1}\Phi\mathcal{V}$ is non-negative.

(iii) $s(\mathcal{V}^{-1}(C+\mathcal{DK})\mathcal{V}+\mathcal{V}^{-1}\Phi\mathcal{V})<0$.

An illustrative example is shown below. For $A, B \in \mathbb{R}^{n\times n}$, every inequality between A and B, such as $A > B$, indicates that it is satisfied componentwise. The transpose of $A \in \mathbb{R}^{n\times n}$ is denoted by A'. Let diag.$\{\cdots\}$ denote a diagonal matrix. Let I denote the identity matrix.

In the subsequent discussion, we consider the following delay system defined in X^3, where X is a real Banach lattice.

$$\dot{x}(t) = Cx(t) + \Phi x(t-\tau) + Du(t), \tag{23}$$

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & \xi_1 & 0 \end{bmatrix},$$

$$\Phi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \xi_2 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where $x(t) \in X^3$ is the state, $u(t) \in X$ is the control input and $\xi_1, \xi_2 \in X$ are uncertain parameters but bounded as $|\xi_1| < 1$, $|\xi_2| < 1$.

The following lemma enables the verification of whether a given matrix generates a non-negative C_0 semigroup.

Lemma 6 *A real matrix $A \in \mathbb{R}^{n\times n}$ generates a non-negative C_0 semigroup $(T(t))_{t\geq 0} := e^{tA}$ if and only if every off-diagonal entry of A is non-negative.*

Proof: (Necessity) Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the generator of $(T(t))_{t\geq 0}$, then

$$A = \lim_{h\searrow 0} \frac{e^{hA} - I}{h} \tag{24}$$

holds. Let f_{ij} denote the (i, j)-th entry of e^{hA}. Then, it follows from (24) that

$$a_{ij} = \begin{cases} \lim_{h\searrow 0} \frac{f_{ij}}{h} & \text{for } i \neq j, \\ \lim_{h\searrow 0} \frac{f_{ij}-1}{h} & \text{for } i = j. \end{cases} \tag{25}$$

Assume that $f_{ij} \geq 0$ for all i, j, then we obtain

$$\begin{array}{ll} a_{ij} \geq 0 & \text{for } i \neq j, \\ a_{ij} \in \mathbb{R} & \text{for } i = j. \end{array} \tag{26}$$

Therefore, we see that if A generates a non-negative C_0 semigroup $(T(t))_{t\geq 0}$, then $a_{ij} \geq 0$ for all $i \neq j$.

(Sufficiency) Suppose that every off-diagonal entry of A is non-negative, then we can find $\delta \in R$ such that

$$P_\delta := A + \delta I \geq 0. \tag{27}$$

Note that if $P_\delta \geq 0$, then

$$e^{tP_\delta} = \sum_{k=0}^{\infty} \frac{tP_\delta^k}{k!} \geq 0 \ \text{ for all } \ t \geq 0.$$

Considering

$$e^{-t\delta I} = \text{diag.}\{e^{-t\delta}, \cdots, e^{-t\delta}\},$$

we obtain

$$\begin{aligned} e^{tA} &= e^{\{t(A+\delta I)-t\delta I\}} = e^{tP_\delta} e^{-t\delta I} \\ &= e^{tP_\delta} e^{-t\delta} \geq 0 \ \text{ for all } \ t \geq 0. \end{aligned} \tag{28}$$

Therefore, we see that if every off-diagonal entry of A is non-negative, then A generates a non-negative C_0 semigroup $(T(t))_{t\geq 0}$.

It is apparent from Lemma 6 that system (23) cannot satisfy Assumption 1 for all $t \in \mathbb{R}_+$. Hence, here we apply the variable transformation method.

Our objective in the subsequent discussion is to find the controller (5) and the variable transformation (21) such that conditions (i)-(iii) in Theorem 2 are satisfied. As possible $\mathcal{K}$ and $\mathcal{V}$, we focus on the following $K' \in \mathbb{R}^3$ and $V \in \mathbb{R}^{3\times 3}$:

$$K := \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}, \tag{29}$$

$$V := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & v \end{bmatrix}. \tag{30}$$

Considering that

$$V^{-1}\Phi V = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ (\xi_2 - 1)/v & 0 & 0 \end{bmatrix}, \tag{31}$$

we see that if we choose $v < 0$, then $V^{-1}\Phi V$ is non-negative for all $t \in \mathbb{R}_+$.

In the following, we fix v as $v = -1$. It follows from careful calculation that $V^{-1}(C + DK)V$ is given as follows:

$$V^{-1}(C + DK)V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -k_1 & -1 - \xi_1 - k_2 - k_3 & k_3 \end{bmatrix}. \tag{32}$$

It can be seen from (32) that if we choose K so as to satisfy

$$k_1 < 0 \text{ and } k_2 + k_3 < -2,$$

then every off-diagonal entry of $V^{-1}(C + DK)V$ is non-negative for all $t \in \mathbb{R}_+$. Therefore, we see from Lemma 6 that $V^{-1}(C + DK)V$ generates a non-negative C_0 semigroup.

In the following, we examine the possibility of choosing k_1, k_2, and k_3 such that condition (iii) of Theorem 2 is satisfied.

Lemma 7 (Engel & Nagel, 2000) *For a non-negative C_0 semigroup with generator $\mathcal{A}$, the following properties are equivalent for $\mu \in \rho(\mathcal{A})$:*

(i) $s(\mathcal{A}) < \mu$.

(ii) $R(\mu, \mathcal{A}) \geq 0$.

Using Lemma 7, we can state the following lemma that is useful for evaluating whether condition (iii) of Theorem 2 is satisfied.

Lemma 8 *Assume that $A \in \mathbb{R}^{n\times n}$ generates a non-negative C_0 semigroup, then the following assertions are equivalent.*

(i) $s(A) < 0$.

(ii) $(-A)^{-1} \geq 0$.

Proof: Taking $\mu = 0$ in (i) and (ii) of Lemma 7, we see that $s(A) < 0 \Leftrightarrow R(0, A) \geq 0$. Considering that $R(0, A) = (0-A)^{-1} = (-A)^{-1}$, we see from Lemma 7 that

$$s(A) < 0 \Leftrightarrow (-A)^{-1} \geq 0.$$

This completes the proof.

From a careful calculation, we have the following:

$$\left(-V^{-1}(C + DK + \Phi)V\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{\xi_1+\xi_2+k_1+k_2+k_3}{k_3} & \frac{1+\xi_1+k_2+k_3}{k_3} & \frac{-1}{k_3} \end{bmatrix}. \tag{33}$$

Then, we see that if

$$k_1 < 0, \quad k_2 + k_3 < -2, \quad k_3 < 0, \tag{34}$$

then every entry of $(-V^{-1}(C + DK + \Phi)V)^{-1}$ is non-negative for all $t \in \mathbb{R}_+$. At the same time, it is clear that if (34) is satisfied, then conditions (i)-(iii) in Theorem 2 are satisfied. Therefore, we see that system (23) is uniformly exponentially stabilizable using a controller such that (34) is satisfied.

6. Conclusion

In this study, we examined the stabilization problem of a partial differential equation with a time delay using semigroup theory. We first introduced the concept of an abstract delay system that can be used to characterize the behavior of a wide class of dynamical systems. Next, we investigated the stabilization problem of an abstract delay system on a Banach lattice on the basis of the properties of a non-negative C_0 semigroup. In Sec. 4, we derived a sufficient condition for the stabilization of an abstract delay system under the assumption that the system has a non-negative delay operator. An illustrative example revealed that the stabilization method proposed in Sec. 4 is useful for designing a controller to stabilize an abstract delay system whose delay operator satisfies the non-negativity assumption. However, the applicability of the stabilization method proposed in Sec. 4 is restricted to a class of systems that satisfy the non-negativity assumption. To remove such a restrictive assumption, in Sec. 5 we proposed a variable transformation method for stabilizing an abstract delay system without the non-negativity assumption. It was shown that the variable transformation method proposed in Sec. 5 is applicable to an abstract delay system whose delay operator is not necessarily non-negative.

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