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METRISABILITY OF TWO-DIMENSIONAL PROJECTIVE STRUCTURES

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Abstract

We carry out the programme of R. Liouville [**19**] to construct an explicit local obstruction to the existence of a Levi–Civita connection within a given projective structure $[\Gamma]$ on a surface. The obstruction is of order 5 in the components of a connection in a projective class. It can be expressed as a point invariant for a second order ODE whose integral curves are the geodesics of $[\Gamma]$ or as a weighted scalar projective invariant of the projective class. If the obstruction vanishes we find the sufficient conditions for the existence of a metric in the real analytic case. In the generic case they are expressed by the vanishing of two invariants of order 6 in the connection. In degenerate cases the sufficient obstruction is of order at most 8.

1. Introduction

Recall that a *projective structure* [**7**, **22**, **12**] on an open set $U \subset \mathbb{R}^n$ is an equivalence class of torsion free connections $[\Gamma]$. Two connections Γ and $\hat{\Gamma}$ are projectively equivalent if they share the same unparametrised geodesics. This means that the geodesic flows project to the same foliation of $\mathbb{P}(TU)$. The analytic expression for this equivalence class is

$$\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + \delta^c_a\omega_b + \delta^c_b\omega_a, \qquad a, b, c = 1, 2, ..., n \tag{1.1}$$

for some one-form $\omega = \omega_a dx^a$. A basic unsolved problem in projective differential geometry is to determine the explicit criterion for the *metrisability* of projective structure, i.e. answer the following question:

- What are the necessary and sufficient local conditions on a connection Γ^c_{ab} for the existence of a one form ω_a and a symmetric non-degenerate tensor g_{ab} such that the projectively equivalent connection

$$\Gamma^c_{ab} + \delta^c_a\omega_b + \delta^c_b\omega_a$$

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is the Levi-Civita connection for g_{ab}. (We are allowing Lorentzian metrics.)

We shall focus on local metrisability, i.e. the pair (g, ω) with $\det(g)$ nowhere vanishing is required to exist in a neighbourhood of a point $p \in U$. This problem leads to a vastly overdetermined system of partial differential equations for g and ω. There are $n^2(n+1)/2$ components in a connection, and $(n + n(n+1)/2)$ components in a pair (ω, g). One could therefore naively expect $n(n^2-3)/2$ conditions on Γ.

In this paper we shall carry out the algorithm laid out by R. Liouville [**19**] to solve this problem when $n = 2$ and U is a surface. (Let us stress that the 'solution' here means an explicit criterion, given by vanishing of a set of invariants, which can be verified on any representative of $[\Gamma]$.) In the two-dimensional case the projective structures are equivalent to second order ODEs which are cubic in the first derivatives. To see it consider the geodesic equations for $x^a(t) = (x(t), y(t))$ and eliminate the parameter t between the two equations

$$\ddot{x}^c + \Gamma_{ab}^c \dot{x}^a \dot{x}^b = v\dot{x}^c.$$

This yields the desired ODE for y as a function of x

$$\frac{d^2y}{dx^2} = \Gamma_{22}^1\Big(\frac{dy}{dx}\Big)^3 + (2\Gamma_{12}^1 - \Gamma_{22}^2)\Big(\frac{dy}{dx}\Big)^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2)\Big(\frac{dy}{dx}\Big) - \Gamma_{11}^2. \tag{1.2}$$

Conversely, any second order ODEs cubic in the first derivatives

$$\frac{d^2y}{dx^2} = A_3(x,y)\Big(\frac{dy}{dx}\Big)^3 + A_2(x,y)\Big(\frac{dy}{dx}\Big)^2 + A_1(x,y)\Big(\frac{dy}{dx}\Big) + A_0(x,y) \tag{1.3}$$

gives rise to some projective structure as the independent components of Γ_{ab}^c can be read off from the As up to the equivalence (1.1). The advantage of this formulation is that the projective ambiguity (1.1) has been removed from the problem as the combinations of the connection symbols in the ODE (1.2) are independent of the choice of the one form ω. There are 6 components in Γ_{ab}^c and 2 components in ω_a, but only $4 = 6 - 2$ coefficients $A_\alpha(x,y), \alpha = 0, ..., 3$. The diffeomorphisms of U can be used to further eliminate 2 out of these 4 functions (for example to make the equation (1.3) linear in the first derivatives) so one can say that a general projective structure in two dimensions depends on two arbitrary functions of two variables. We are looking for invariant conditions, so we shall not make use of this diffeomorphism freedom.

We shall state our first result. Consider the 6 by 6 matrix given in terms of its row vectors

$$\mathcal{M}([\Gamma]) = (\mathbf{V}, D_a\mathbf{V}, D_{(b}D_{a)}\mathbf{V}) \tag{1.4}$$

which depends on the functions A_α and their derivatives up to order five. The vector field $\mathbf{V} : U \to \mathbb{R}^6$ is given by (3.21), the expressions $D_a\mathbf{V} =$

$\partial_a \mathbf{V} - \mathbf{V}\mathbf{\Omega}_a$ are computed using the right multiplication by 6 by 6 matrices $\mathbf{\Omega}_1, \mathbf{\Omega}_2$ given by (A53) and $\partial_a = \partial/\partial x^a$. We also make a recursive definition $D_a D_b D_c ... D_d \mathbf{V} = \partial_a(D_b D_c ... D_d \mathbf{V}) - (D_b D_c ... D_d \mathbf{V})\mathbf{\Omega}_a$.

Theorem 1.1. *If the projective structure* $[\Gamma]$ *is metrisable then*

$$\det(\mathcal{M}([\Gamma])) = 0. \tag{1.5}$$

There is an immediate corollary

Corollary 1.2. *If the integral curves of a second order ODE*

$$\frac{d^2y}{dx^2} = \Lambda\Big(x, y, \frac{dy}{dx}\Big), \tag{1.6}$$

are geodesics of a Levi-Civita connection then Λ *is at most cubic in* dy/dx *and* (1.5) *holds.*

The expression (1.5) is written in a relatively compact form using $(\mathbf{V}, \mathbf{\Omega}_1, \mathbf{\Omega}_2)$. All the algebraic manipulations which are required in expanding the determinant have been done using MAPLE code which can be obtained from us on request.

We shall prove Theorem 1.1 in three steps. The first step, already taken by Liouville [**19**], is to associate a linear system of four PDEs for three unknown functions with each metrisable connection. This will be done in the next Section. The second step will be prolonging this linear system. This point was also understood by Liouville although he did not carry out the explicit computations. Geometrically this will come down to constructing a connection on a certain rank six real vector bundle over U. The non-degenerate parallel sections of this bundle are in one to one correspondence the metrics whose geodesics are the geodesics of the given projective structure. In the generic case, the bundle has no parallel sections and hence the projective structure does not come from metric. In the real analytic case the projective structure for which there is a single parallel section depends on one arbitrary function of two variables, up to diffeomorphism. Finally we shall obtain (1.5) as the integrability conditions for the existence of a parallel section of this bundle. This will be done in Section 3.

In Section 4 we shall present some sufficient conditions for metrisability. All considerations here will be in the real analytic category. The point is that even if $[\Gamma]$ is locally metrisable around every point in U, the global metric on U may not exist in the smooth category even in the simply-connected case. Thus no set of local obstructions can guarantee metrisability of the whole surface U.

Theorem 1.3. *Let* $[\Gamma]$ *be a real analytic projective structure such that rank* $(\mathcal{M}([\Gamma])) < 6$ *on* U *and there exist* $p \in U$ *such that rank* $(\mathcal{M}([\Gamma])) = 5$ *and* $W_1 W_3 - W_2^2 \neq 0$ *at* p*, where* $(W_1, W_2, ..., W_6)$ *spans* *the kernel of* $\mathcal{M}([\Gamma])$. *Then* $[\Gamma]$ *is metrisable in a sufficiently small neighbourhood of* p *if the rank of a* 10 *by* 6 *matrix with the rows*

$$(\mathbf{V}, D_a\mathbf{V}, D_{(a}D_{b)}\mathbf{V}, D_{(a}D_bD_{c)}\mathbf{V})$$

is equal to 5. *Moreover this rank condition holds if and only if two relative invariants* E_1, E_2 *of order 6 constructed from the projective structure vanish.*

We shall explain how to construct these two additional invariants and show that the resulting set of conditions, a single 5th order equation (1.5) and two 6th order equations $E_1 = E_2 = 0$ form an involutive system whose general solution depends on three functions of two variables. In the degenerate cases when $\text{rank}(\mathcal{M}([\Gamma])) < 5$ higher order obstructions will arise (We shall always assume that the rank of $\mathcal{M}([\Gamma])$ is constant in a sufficiently small neighbourhood of some $p \in U$.): one condition of order 8 in the rank 3 case and one condition of order 7 in the rank 4 case. If rank $(\mathcal{M}([\Gamma])) = 2$ there is always a four parameter family of metrics. If rank $(\mathcal{M}([\Gamma])) < 2$ then $[\Gamma]$ is projectively flat in agreement with a theorem of Koenigs [**16**]. In general we have

Theorem 1.4. *A real analytic projective structure* $[\Gamma]$ *is metrisable in a sufficiently small neighbourhood of* $p \in U$ *if and only if the rank of a* 21 *by* 6 *matrix with the rows*

$$\mathcal{M}_{\text{max}} =$$
$$\big(\mathbf{V}, D_a\mathbf{V}, D_{(a}D_{b)}\mathbf{V}, D_{(a}D_bD_{c)}\mathbf{V}, D_{(a}D_bD_cD_{d)}\mathbf{V}, D_{(a}D_bD_cD_dD_{e)}\mathbf{V}\big)$$

is smaller than 6 and there exists a vector $\mathbf{W}$ *in the kernel of this matrix such that* $W_1W_3 - W_2^2$ *does not vanish at* p.

The signature of the metric underlying a projective structure can be Riemannian or Lorentzian depending on the sign of $W_1W_3 - W_2^2$. In the generic case described by Theorem 1.3 this sign can be found by evaluating the polynomial (4.28) of degree 10 in the entries of $\mathcal{M}([\Gamma])$ at p.

In Section 5 we shall construct various examples illustrating the necessity for the genericity assumptions that we have made. In Section 6 we shall discuss the twistor approach to the problem. In this approach a real analytic projective structure on U corresponds to a complex surface Z having a family of rational curves with self-intersection number one. The metrisability condition and the associated linear system are both deduced from the existence of a certain anti-canonical divisor on Z. In Section 7 we shall present an alternative tensorial expression for (1.5) in terms of the curvature of the projective connection and its covariant derivatives. In particular we will shall show that a section of the 14th power of the canonical bundle of U

$$\det(\mathcal{M})([\Gamma])\,(dx \wedge dy)^{\otimes 14}$$

is a projective invariant. The approach will be that of tractor calculus [**10**].

In the derivation of the necessary condition (1.5) we assume that the projective structure $[\Gamma]$ admits continuous fifth derivatives. The discussion of the sufficient conditions and considerations in Section 6 require $[\Gamma]$ to be real analytic. We relegate some long formulae to the Appendix.

We shall finish this introduction with a comment about the formalism used in the paper. The linear system governing the metrisability problem and its prolongation are constructed in elementary way in Sections 2–3 and in tensorial *tractor* formalism in Section 7. The resulting obstructions are always given by invariant expressions. The machinery of the Cartan connection could of course be applied to do the calculations invariantly from the very beginning. This is in fact how some of the results have been obtained [**2**]. The readers familiar with the Cartan approach will realise that the rank six vector bundles used in our paper are associated to the $SL(3,\mathbb{R})$ principal bundle of Cartan. Such readers should beware, however, that the connection D_a that we naturally obtain on such a vector bundle is not induced by the Cartan connection of the underlying projective structure but is a minor modification thereof, as detailed for example in [**11**]. Various weighted invariants on U, like (1.5), are pull-backs of functions from the total space of Cartan's bundle.

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2. Linear System

Let us assume that the projective structure $[\Gamma]$ is metrisable. Therefore there exist a symmetric bi-linear form

$$g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2 \tag{2.7}$$

such that the unparametrised geodesics of g coincide with the integral curves of (1.3). The diffeomorphisms can be used to eliminate two arbitrary functions from g (for example to express g in isothermal coordinates) but we shall not use this freedom.

We want to determine whether the four functions $(A_0, ..., A_3)$ arise from three functions (E, F, G) so one might expect only one condition on the As. This heuristic numerology is wrong and we shall demonstrate in Section 4 that three conditions are needed to establish sufficiency in the generic case. (Additional conditions would arise if we demanded that there be more than one metric with the same unparametrised geodesics.

In our approach this situation corresponds to the existence of two independent parallel sections of the rank six bundle over U. The corresponding metrics were, in the positive definite case, found by J. Liouville (the more famous of the two Liouvilles) and characterised by Dini. They are of the form (2.7) where $F = 0, E = G = u(x) + v(y)$ up to diffeomorphism. Roger Liouville whose steps we follow in this paper was a younger relative of Joseph and attended his lectures at the Ecole Polytechnique.)

We choose a direct route and express the equation for non-parametrised geodesics of g in the form (1.3). Using the Levi-Civita relation

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\Big(\frac{\partial g_{ad}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d}\Big)$$

and formulae (1.2), (1.3) yields the following expressions

$$\begin{aligned} A_0 &= \frac{1}{2}\frac{E\partial_y E - 2E\partial_x F + F\partial_x E}{EG - F^2}, \\ A_1 &= \frac{1}{2}\frac{3F\partial_y E + G\partial_x E - 2F\partial_x F - 2E\partial_x G}{EG - F^2}, \\ A_2 &= \frac{1}{2}\frac{2F\partial_y F + 2G\partial_y E - 3F\partial_x G - E\partial_y G}{EG - F^2}, \\ A_3 &= \frac{1}{2}\frac{2G\partial_y F - G\partial_x G - F\partial_y G}{EG - F^2}. \end{aligned} \tag{2.8}$$

This gives a first order nonlinear differential operator

$$\sigma^0 : J^1(S^2(T^*U)) \longrightarrow J^0(\mathrm{Pr}(U)) \tag{2.9}$$

which carries the metric to its associated projective structure. This operator is defined on the first jet space of symmetric two-forms as it depends on the metric and its derivatives. It takes its values in the affine rank 4 bundle $\mathrm{Pr}(U)$ of projective structures whose associated vector bundle $\Lambda^2(TU) \otimes S^3(T^*U)$ arises as a quotient in the exact sequence

$$0 \longrightarrow T^*U \longrightarrow TU \otimes S^2(T^*U) \longrightarrow \Lambda^2(TU) \otimes S^3(T^*U) \longrightarrow 0.$$

This is a more abstract way of defining the equivalence relation (1.1). We will return to it in Section 4. The operator σ^0 is homogeneous of degree zero so rescaling a metric by a constant does not change the resulting projective structure.

Following Liouville [**19**] we set

$$E = \psi_1/\Delta^2, \quad F = \psi_2/\Delta^2, \quad G = \psi_3/\Delta^2, \qquad \Delta = \psi_1\psi_3 - {\psi_2}^2$$

and substitute into (2.8). This yields an overdetermined system of four linear first order PDEs for three functions (ψ_1, ψ_2, ψ_3) and proves the following

Lemma 2.1 (Liouville [**19**])**.** *A projective structure* $[\Gamma]$ *corresponding to the second order ODE* (1.3) *is metrisable on a neighbourhood of a* *point $p \in U$ iff there exists functions $\psi_i(x,y), i = 1,2,3$ defined on a neighbourhood of p such that*

$$\psi_1\psi_3 - {\psi_2}^2$$

does not vanish at p and such that the equations

$$\begin{aligned}
\frac{\partial \psi_1}{\partial x} &= \frac{2}{3}A_1\psi_1 - 2A_0\psi_2,\\
\frac{\partial \psi_3}{\partial y} &= 2A_3\psi_2 - \frac{2}{3}A_2\psi_3,\\
\frac{\partial \psi_1}{\partial y} + 2\frac{\partial \psi_2}{\partial x} &= \frac{4}{3}A_2\psi_1 - \frac{2}{3}A_1\psi_2 - 2A_0\psi_3,\\
\frac{\partial \psi_3}{\partial x} + 2\frac{\partial \psi_2}{\partial y} &= 2A_3\psi_1 - \frac{4}{3}A_1\psi_3 + \frac{2}{3}A_2\psi_2
\end{aligned} \tag{2.10}$$

hold on the domain of definition.

This linear system forms a basis of our discussion of the metrisability condition. It has recently been used in [**5**] to construct a list of metrics on a two-dimensional surface admitting a two-dimensional group of projective transformations. Its equivalent tensorial form, applicable in higher dimensions, is presented for example in [**11**]. We shall use this form in Section 7.

Here is a way to 'remember' (2.10). Introduce the symmetric projective connection ∇^{Π} with connection symbols

$$\Pi^c_{ab} = \Gamma^c_{ab} - \frac{1}{n+1}\Gamma^d_{da}\delta^c_b - \frac{1}{n+1}\Gamma^d_{db}\delta^c_a \tag{2.11}$$

where in our case $n = 2$. Formula (1.1) implies that the symbols Π^c_{ab} do not depend on a choice of Γ is a projective class. They are related to the second order ODE (1.3) by

$$\Pi^1_{11} = \frac{1}{3}A_1, \quad \Pi^1_{12} = \frac{1}{3}A_2, \quad \Pi^1_{22} = A_3,$$

$$\Pi^2_{11} = -A_0, \quad \Pi^2_{21} = -\frac{1}{3}A_1, \quad \Pi^2_{22} = -\frac{1}{3}A_2.$$

The projective covariant derivative is defined on one-forms by ${\nabla^{\Pi}}_a\phi_b = \partial_a\phi_b - \Pi^c_{ab}\phi_c$ with natural extension to other tensor bundles. The Liouville system (2.10) is then equivalent to

$$\nabla^{\Pi}{}_{(a}\sigma_{bc)} = 0, \tag{2.12}$$

where the round brackets denote symmetrisation and σ_{bc} is a rank 2 symmetric tensor with components $\sigma_{11} = \psi_1, \sigma_{12} = \psi_2, \sigma_{22} = \psi_3$.

We shall end this Section with a historical digression. The solution to the metrisability problem has been reduced to finding differential relations between (A_0, A_1, A_2, A_3) when (2.8), or equivalently (2.10), holds. These relations are required to be diffeomorphism invariant conditions, so we are searching for *invariants* of the ODE (1.3) under the point transformations

$$(x,y) \longrightarrow (\bar{x}(x,y), \bar{y}(x,y)). \tag{2.13}$$

The point invariants of 2nd order ODEs have been extensively studied by the classical differential geometers in late 19th and early 20th century. The earliest reference we are aware of is the work of Liouville [**18**, **19**], who constructed point invariants of 2nd order ODEs cubic in the first derivatives (it is easy to verify that the 'cubic in the first derivative' condition is itself invariant under (2.13)). The most complete work was produced by Tresse (who was a student of Sophus Lie) in his dissertation [**23**]. Tresse studied the general case (1.6) and classified all point invariants of a given differential order. The first two invariants are of order four

$$I_0 = \Lambda_{1111}, \quad I_1 = D_x^2\Lambda_{11} - 4D_x\Lambda_{01} - \Lambda_1 D_x\Lambda_{11} + 4\Lambda_1\Lambda_{01} - 3\Lambda_0\Lambda_{11} + 6\Lambda_{00},$$

where

$$\Lambda_0 = \frac{\partial \Lambda}{\partial y}, \qquad \Lambda_1 = \frac{\partial \Lambda}{\partial y'}, \qquad D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \Lambda\frac{\partial}{\partial y'}.$$

Strictly speaking these are only relative invariants as they transform with a certain weight under (2.13). Their vanishing is however invariant. Tresse showed that if $I_0 = 0$, then I_1 is linear in y'. This is the case considered by Liouville. To make contact with the work of Liouville we note that $I_1 = -6L_1 - 6L_2y'$ where the expressions

$$\begin{aligned} L_1 &= \frac{2}{3}\frac{\partial^2 A_1}{\partial x \partial y} - \frac{1}{3}\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_0}{\partial y^2} + A_0\frac{\partial A_2}{\partial y} + A_2\frac{\partial A_0}{\partial y} \\ &\quad - A_3\frac{\partial A_0}{\partial x} - 2A_0\frac{\partial A_3}{\partial x} - \frac{2}{3}A_1\frac{\partial A_1}{\partial y} + \frac{1}{3}A_1\frac{\partial A_2}{\partial x}, \\ L_2 &= \frac{2}{3}\frac{\partial^2 A_2}{\partial x \partial y} - \frac{1}{3}\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_3}{\partial x^2} - A_3\frac{\partial A_1}{\partial x} - A_1\frac{\partial A_3}{\partial x} \\ &\quad + A_0\frac{\partial A_3}{\partial y} + 2A_3\frac{\partial A_0}{\partial y} + \frac{2}{3}A_2\frac{\partial A_2}{\partial x} - \frac{1}{3}A_2\frac{\partial A_1}{\partial y} \end{aligned} \tag{2.14}$$

were constructed by Liouville who has also proved that

$$Y = (L_1 dx + L_2 dy) \otimes (dx \wedge dy)$$

is a projectively invariant tensor.

The following result was known to both Tresse and Liouville

Theorem 2.2 (Liouville [**18**], Tresse [**23**])**.** *The 2nd order ODE* (1.6) *is trivialisable by point transformation (i.e. equivalent to $y'' = 0$) iff $I_0 = I_1 = 0$, or, equivalently, if Λ is at most cubic in y' and $Y = 0$.*

We note that the separate vanishing of L_1 or L_2 is not invariant. If both L_1 and L_2 vanish the projective structure is flat is the sense described in Section 7.

3. Prolongation and Consistency

Proof of Theorem 1.1. The obstruction (1.5) will arise as the compatibility condition for the system (2.10). This system is overdetermined, as there are more equations than unknowns. We shall use the method of *prolongation* and make (2.10) even more overdetermined by specifying the derivatives of $\psi_i, i = 1, 2, 3$ at any given point $(x, y, \psi_i) \in \mathbb{R}^5$, thus determining a tangent plane to a solution surface (if one exists)

$$(x, y) \longrightarrow (x, y, \psi_1(x, y), \psi_2(x, y), \psi_3(x, y)).$$

(Another approach more in the spirit of Liouville [**19**] would be to eliminate ψ_2 and ψ_3 from (2.10) to obtain a system of two 3rd order PDEs for one function $f := \psi_1$

$$(\partial_x^3)f = F_1, \quad \partial_y(\partial_x^2)f = F_2,$$

where F_1, F_2 are linear in f and its first and second derivatives with coefficients depending on $A_\alpha(x, y)$ and their derivatives (the coefficient of $(\partial_y^2)f$ in F_1 is zero). The consistency $\partial_y(\partial_x)^3 f = \partial_x\partial_y(\partial_x)^2 f$ gives a linear equation for $\partial_x(\partial_y)^2 f$. Then $\partial_x(\partial_y)^2\partial_x f = (\partial_y)^2(\partial_x)^2 f$ gives an equation for $(\partial_y)^3 f$. After this step the system is closed: all third order derivatives are expressed in terms of lower order derivatives. To work out further consistencies impose $\partial_x(\partial_y)^3 f = (\partial_y)^3\partial_x f$ which gives (when all 3rd order equations are used) a second order linear PDE for f. We carry on differentiating this second order relation to produce the remaining second order relations (because we know all third order derivatives), then the first order relations and finally an algebraic relation which will constrain the initial data unless (1.5) holds.)

For this we need six conditions, and the system (2.10) consist of four equations. We need to add two conditions and we choose

$$\frac{\partial \psi_2}{\partial x} = \frac{1}{2}\mu, \qquad \frac{\partial \psi_2}{\partial y} = \frac{1}{2}\nu, \tag{3.15}$$

where μ, ν depend on (x, y). The integrability conditions $\partial_x\partial_y\psi_i = \partial_y\partial_x\psi_i$ give three PDEs for (μ, ν) of the form

$$\frac{\partial \mu}{\partial x} = P, \qquad \frac{\partial \nu}{\partial y} = Q, \qquad \frac{\partial \nu}{\partial x} - \frac{\partial \mu}{\partial y} = 0, \tag{3.16}$$

where (P, Q) given by (A55) are expressions linear in (ψ_i, μ, ν) with coefficients depending on A_α and their (x, y) derivatives.

The system (3.16) is again overdetermined but we still need to prolong it to specify the values of all first derivatives. It is immediate that the complex characteristic variety of the system (2.10) is empty, so the general theory (see Chapter 5 of [**3**]) implies that, after a finite number of differentiations of these equations (i.e., prolongations), all of the partials of the ψ_i above a certain order can be written in terms of lower order partials, i.e., the prolonged system will be complete. Alternatively, the Liouville system written in the form (2.12) is one of the simplest examples covered by [**1**] in which the form of the prolongation is easily predicted. In any case no appeal to the general theory is needed as it is easy to see that completion is reached by adding one further equation

$$\frac{\partial \mu}{\partial y} = \rho, \tag{3.17}$$

where $\rho = \rho(x, y)$ and imposing the consistency conditions on the system of four PDEs (3.16, 3.17). This leads to

$$\frac{\partial \rho}{\partial x} = R, \qquad \frac{\partial \rho}{\partial y} = S, \tag{3.18}$$

where R, S given by (A55) are functions of $(\rho, \mu, \nu, \psi_i, x, y)$ which are linear in (ρ, μ, ν, ψ_i). After this step the prolongation process is finished and all the first derivatives have been determined. The final compatibility condition $\partial_x \partial_y \rho = \partial_y \partial_x \rho$ for the system (3.18) yields

$$\frac{\partial R}{\partial y} - \frac{\partial S}{\partial x} + S\frac{\partial R}{\partial \rho} - R\frac{\partial S}{\partial \rho} = 0. \tag{3.19}$$

All the first derivatives are now determined, so (3.19) is an algebraic linear condition of the form

$$\mathbf{V} \cdot \mathbf{\Psi} := \sum_{p=1}^{6} V_p \Psi_p = 0, \tag{3.20}$$

where

$$\mathbf{\Psi} = (\psi_1, \psi_2, \psi_3, \mu, \nu, \rho)^T$$

is a vector in $\mathbb{R}^6$, and $\mathbf{V} = (V_1, ..., V_6)$ where

$$\begin{aligned} V_1 &= 2\frac{\partial L_2}{\partial y} + 4A_2 L_2 + 8A_3 L_1, \\ V_2 &= -2\frac{\partial L_1}{\partial y} - 2\frac{\partial L_2}{\partial x} - \frac{4}{3}A_1 L_2 + \frac{4}{3}A_2 L_1, \\ V_3 &= 2\frac{\partial L_1}{\partial x} - 8A_0 L_2 - 4A_1 L_1, \\ V_4 &= -5L_2, \quad V_5 = -5L_1, \quad V_6 = 0 \end{aligned} \tag{3.21}$$

and L_1, L_2 are given by (2.14) . We collect the linear PDEs (2.10, 3.15 3.16, 3.17, 3.18) as

$$d\mathbf{\Psi} + \mathbf{\Omega}\, \mathbf{\Psi} = 0, \tag{3.22}$$

where

$$\mathbf{\Omega} = \mathbf{\Omega}_1\, dx + \mathbf{\Omega}_2\, dy$$

and $(\mathbf{\Omega}_1, \mathbf{\Omega}_2)$ are 6 by 6 matrices with coefficients depending on A_α and their first and second derivatives (A53). Now differentiate (3.20) twice with respect to $x^a = (x, y)$, and use (3.22). This yields six linear conditions

$$\begin{aligned} (3.23)\quad & \mathbf{V}\cdot\boldsymbol{\Psi} = 0, \\ & (D_a\mathbf{V})\cdot\boldsymbol{\Psi} := (\partial_a\mathbf{V} - \mathbf{V}\,\boldsymbol{\Omega}_a)\cdot\boldsymbol{\Psi} = 0, \\ & (D_bD_a\mathbf{V})\cdot\boldsymbol{\Psi} := \\ & (\partial_b\partial_a\mathbf{V} - (\partial_b\mathbf{V})\,\boldsymbol{\Omega}_a - (\partial_a\mathbf{V})\,\boldsymbol{\Omega}_b - \mathbf{V}\,(\partial_b\boldsymbol{\Omega}_a - \boldsymbol{\Omega}_a\boldsymbol{\Omega}_b))\cdot\boldsymbol{\Psi} = 0 \end{aligned}$$

which must hold, or there are no solutions to (2.10). Therefore the determinant of the associated 6 by 6 matrix (1.4) must vanish, thus giving our first desired metrisability condition (1.5). We note that the expression $(D_bD_a\mathbf{V})\cdot\boldsymbol{\Psi}$ in (3.23) is symmetric in its indices. This symmetry condition reduces to $\mathbf{V}F = 0$ (where F is given by (A54)) and holds identically.

The expression $\det(\mathcal{M}([\Gamma]))$ is 5th order in the derivatives of connection coefficients. It does not vanish on a generic projective structure, but vanishes on metrisable connections (2.8) by construction. This ends the proof of Theorem 1.1. q.e.d.

In the next Section we shall need the following generalisation of the symmetry properties of (3.23). Let $D_a\mathbf{W} = \partial_a\mathbf{W} - \mathbf{W}\boldsymbol{\Omega}_a$, where $\mathbf{W} : U \to \mathbb{R}^6$. Then

$$[D_a, D_b]\mathbf{W} = (\mathbf{W}F)\varepsilon_{ab} = W_6\mathbf{V}\varepsilon_{ab},$$

where $\varepsilon_{00} = \varepsilon_{11} = 0, \varepsilon_{01} = -\varepsilon_{10} = 1$. Thus

$$D_aD_b\mathbf{V} = D_{(a}D_{b)}\mathbf{V}, \quad D_aD_bD_c\mathbf{V} = D_{(a}D_bD_{c)}\mathbf{V} + \varepsilon_{ab}L_c\mathbf{V}, \quad \dots,$$

$$(3.24)\quad D_{a_1}D_{a_2}\dots D_{a_k}\mathbf{V} = D_{(a_1}D_{a_2}\dots D_{a_k)}\mathbf{V} + o(k-2)$$

where $o(k-2)$ denotes terms linear in $D_{(a_1}D_{a_2}\dots D_{a_m)}$ where $m \leq k-2$. Thus we can restrict ourselves to the symmetrised expressions as the antisymmetrisations do not add any new conditions.

4. Sufficiency conditions

It is clear from the discussion in the preceding Section that the condition (1.5) is necessary for the existence of a metric in a given projective class. It is however not sufficient and in this Section we shall establish some sufficiency conditions in the real analytic case. We require the real analyticity in order to be able to apply the Cauchy–Kowalewski Theorem to the prolonged system of PDEs. In particular Theorem 4.1 which underlies our approach in this Section builds on the Cauchy–Kowalewski Theorem.

Let us start off by rephrasing the construction presented in the last Section in the geometric language. The exterior differential ideal $\mathcal{I}$ associated to the prolonged system (3.22) consists of six one-forms

$$(4.25)\qquad \theta_p = d\boldsymbol{\Psi}_p + ((\boldsymbol{\Omega}_a)_{pq}\boldsymbol{\Psi}_q)\,dx^a, \qquad p, q = 1, \dots, 6 \quad a = 1, 2.$$

Two vector fields annihilating the one-forms θ_p span the solution surface in $\mathbb{R}^8$. The closure of this ideal comes down to one compatibility (3.20). We now want to find one parallel section $\mathbf{\Psi} : U \to \mathbb{E}$ of a rank six vector bundle $\mathbb{E} \to U$ with a connection $D = d + \mathbf{\Omega}$. Locally the total space of this bundle is an open set in $\mathbb{R}^8$.

Differentiating (3.22) and eliminating $d\mathbf{\Psi}$ yields $\mathbf{F}\mathbf{\Psi} = 0$, where

$$\begin{aligned} \mathbf{F} &= d\mathbf{\Omega} + \mathbf{\Omega} \wedge \mathbf{\Omega} = (\partial_x \mathbf{\Omega}_2 - \partial_y \mathbf{\Omega}_1 + [\mathbf{\Omega}_1, \mathbf{\Omega}_2]) dx \wedge dy \\ &= F dx \wedge dy \end{aligned}$$

is the curvature of D. Thus we need

$$F\mathbf{\Psi} = 0, \tag{4.26}$$

where $F = F(x, y)$ is a 6 by 6 matrix given by (A54). We find that this matrix is of rank one and in the chosen basis its first five rows vanish and its bottom row is given by the vector $\mathbf{V}$ with components given by (3.21). Therefore (4.26) is equivalent to (3.20). We differentiate the condition (4.26) and use (3.22) to produce algebraic matrix equations

$$F\mathbf{\Psi} = 0, \quad (D_a F)\mathbf{\Psi} = 0, \quad (D_a D_b F)\mathbf{\Psi} = 0, \quad (D_a D_b D_c F)\mathbf{\Psi}, \quad ...$$

where $D_a F = \partial_a F + [\mathbf{\Omega}_a, F]$. Using the symmetry argument (3.24) shows that after K differentiations this leads to $n(K) = 1 + 2 + 3 + ... + (K + 1)$ linear equations which we write as

$$\mathcal{F}_K \mathbf{\Psi} = 0,$$

where $\mathcal{F}_K$ is a $n(K)$ by 6 matrix depending on As and their derivatives. We also set $\mathcal{F}_0 = F$.

We continue differentiating and adjoining the equations. The Frobenius Theorem adapted to (4.26) and (3.22) tells us when we can stop the process.

Theorem 4.1. *Assume that the ranks of the matrices* $\mathcal{F}_K, K = 0, 1, 2, ...$ *are maximal and constant. (This can always be achieved by restricting to a sufficiently small neighbourhood of some point* $p \in U$*.) Let* K_0 *be the smallest natural number such that*

$$\text{rank}\ (\mathcal{F}_{K_0}) = \text{rank}\ (\mathcal{F}_{K_0+1}). \tag{4.27}$$

If K_0 *exists then* $\text{rank}(\mathcal{F}_{K_0}) = \text{rank}(\mathcal{F}_{K_0+k})$ *for* $k \in \mathbb{N}$ *and the space of parallel sections* (3.22) *of* $d + \mathbf{\Omega}$ *has dimension*

$$\mathcal{S}([\Gamma]) = 6 - \text{rank}(\mathcal{F}_{K_0}).$$

The first and second derivatives of (4.26) will produce six independent conditions on $\mathbf{\Psi}$, and these conditions are precisely (3.23). Thus the necessary metrisability condition (1.5) comes down to restricting the holonomy of the connection D on the rank six vector bundle $\mathbb{E}$.

We shall now assume that (1.5) holds and use Theorem 4.1 to construct the sufficient conditions for the existence of a Levi–Civita connection in a given projective class. First of all there must exist a vector $\mathbf{W} = (W_1, ..., W_6)^T$ in the kernel of $\mathcal{M}([\Gamma])$, such that $W_1W_3 - (W_2)^2 \neq 0$. This will guarantee that the corresponding quadratic form (if one exists) on U is non-degenerate. It is straightforward to verify in the case when $\mathcal{M}([\Gamma])$ has rank 5 as then $\text{kernel}\,(\mathcal{M}([\Gamma]))$ is spanned by any non-zero column of $\text{adj}\,(\mathcal{M}([\Gamma]))$ where the adjoint of a matrix $\mathcal{M}$ is defined by $\mathcal{M}\,\text{adj}(\mathcal{M}) = \det(\mathcal{M})\,I$. The entries of $\text{adj}\,(\mathcal{M}([\Gamma]))$ are determinants of the co-factors of $\mathcal{M}([\Gamma])$ and thus are polynomials of degree 5 in the entries of $\mathcal{M}([\Gamma])$ so

$$P([\Gamma]) = W_1W_3 - (W_2)^2 \tag{4.28}$$

is a polynomial of degree 10 in the entries of $\mathcal{M}([\Gamma])$.

Definition 4.2. A projective structure for which *(*1.5*)* holds is called generic in a neighbourhood of $p \in U$ if rank $\mathcal{M}([\Gamma])$ is maximal and equal to 5 and $P([\Gamma]) \neq 0$ in this neighbourhood.

In this generic case Theorem 4.1 and Lemma 2.1 imply that there will exist a Levi–Civita connection in the projective class if the rank of the next derived matrix $\mathcal{F}_3$ does not go up and is equal to five. We shall see that this can be guaranteed by imposing two more 6th order conditions on $[\Gamma]$.

Proof of Theorem 1.3. First note that, in the generic case, the three vectors

$$\mathbf{V}, \quad \mathbf{V}_a := \partial_a\mathbf{V} - \mathbf{V}\,\mathbf{\Omega}_a, \qquad a = 1, 2$$

must be linearly independent or otherwise the rank of $\mathcal{M}([\Gamma])$ would be at most 3. Now pick two independent vectors from the set

$$\mathbf{V}_{ab} := (\partial_b\partial_a\mathbf{V} - (\partial_b\mathbf{V})\,\mathbf{\Omega}_a - (\partial_a\mathbf{V})\,\mathbf{\Omega}_b - \mathbf{V}\,(\partial_b\mathbf{\Omega}_a - \mathbf{\Omega}_a\mathbf{\Omega}_b))$$

such that the resulting set of five vectors is independent. Say we have picked $\mathbf{V}_{00}$ and $\mathbf{V}_{11}$. We now take the third derivatives of (3.20) with respect to x^a and use (3.22) to eliminate derivatives of $\mathbf{\Psi}$. This adds four vectors to our set of five and so a priori we need to satisfy four six order equations to ensure that the rank does not go up. However only two of these are new and the other two are derivatives of the 5th order condition (1.5). Before we shall prove this statement examining the images of linear operators induced from (2.9) on jet spaces let us indicate why this counting works. Let $\mathbf{V}_{ab...c}$ denote the vector in $\mathbb{R}^6$ annihilating $\mathbf{\Psi}$ (in the sense of (3.20)) which is obtained by eliminating the derivatives of $\mathbf{\Psi}$ from $\partial_a\partial_b...\partial_c(\mathbf{V}\cdot\mathbf{\Psi}) = 0$. We have already argued in (3.24) that the antisymmetrising over any pair of indices in $\mathbf{V}_{ab...c}$ only adds lower order conditions. Thus we shall always assume that these expressions are symmetric. We shall also set $\mathbf{V}_0 = \mathbf{V}_x, \mathbf{V}_1 = \mathbf{V}_y$.

Our assumptions imply that

$$\mathbf{V}_{xy} = c_1\mathbf{V} + c_2\mathbf{V}_x + c_3\mathbf{V}_y + c_4\mathbf{V}_{xx} + c_5\mathbf{V}_{yy} \tag{4.29}$$

for some functions $c_1, ..., c_5$ on U. The two six order conditions

$$E_1 := \det\begin{pmatrix} \mathbf{V} \\ \mathbf{V}_x \\ \mathbf{V}_y \\ \mathbf{V}_{xx} \\ \mathbf{V}_{yy} \\ \mathbf{V}_{xxx} \end{pmatrix}, \quad E_2 := \det\begin{pmatrix} \mathbf{V} \\ \mathbf{V}_x \\ \mathbf{V}_y \\ \mathbf{V}_{xx} \\ \mathbf{V}_{yy} \\ \mathbf{V}_{yyy} \end{pmatrix}, \tag{4.30}$$

have to be added for sufficiency. Now differentiating (4.29) w.r.t x, y and using $\mathbf{V}_{xyy} = \mathbf{V}_{yyx}, \mathbf{V}_{xyx} = \mathbf{V}_{xxy}$ (which hold modulo lower order terms), implies that $\mathbf{V}_{xyy}$ and $\mathbf{V}_{xyx}$ are in the span of $\{\mathbf{V}, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_{xx}, \mathbf{V}_{yy}, \mathbf{V}_{xxx}, \mathbf{V}_{yyy}\}$ and no additional conditions need to be added. This procedure can be repeated if instead $\mathbf{V}_{yy}$ belongs to the span of $\{\mathbf{V}, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_{xx}, \mathbf{V}_{xy}\}$.

Now we shall present the general argument. Consider the homogeneous differential operator (2.9). It maps the 1st jets of metrics on U to the 0th jets of projective structures. Differentiating the relations (2.8) prolongs this operator to bundle maps

$$\sigma^k : J^{k+1}(S^2(T^*U)) \longrightarrow J^k(\mathrm{Pr}(U)) \tag{4.31}$$

from $(k+1)$-jets of metrics to k-jets of projective structures. It has at least one dimensional fibre because of the homogeneity of σ^0. The rank of σ^k is not constant as we already know that the system (2.8) (or its equivalent linear form (2.10)) does not have to admit any solutions in general but will admit at least one solution if the projective structure is metrisable. The table below gives the ranks of the jet bundles of metrics and projective structures, the dimensions of the fibres of σ^k and finally the image codimension. The number of new conditions on $[\Gamma]$ arising at each step is denoted by a bold figure in the column co-rank(kerσ^k).

k	rank($J^{k+1}(S^2(T^*U))$)	rank($J^k(\mathrm{Pr}(U))$)	rank(kerσ^k)	co-rank(kerσ^k)
-1	3	–	–	–
0	9	4	5	0
1	18	12	6	0
2	30	24	6	0
3	45	40	5	0
4	63	60	3	0
5	84	84	1	$1 = \mathbf{1}$
6	108	112	1	$5 = 3 + \mathbf{2}$
7	135	144	1	$10 = 6 + 6 - 2$

There is no obstruction on a projective structure before the order 5 so σ^k are onto and generically submersive for $k < 4$. At $k = 5$ there has to be at least a 1-dimensional fiber, so the image of the derived map can at most be 83-dimensional at its smooth points. In fact, we have shown that there is a condition there, given by (1.5), so it must define a codimension 1 variety that is generically smooth. When the matrix $\mathcal{M}([\Gamma])$ has rank 5, the equation (1.5) is regular, so it follows that, outside the region where (1.5) ceases to be a regular 5th order PDE the solutions of this PDE will have their k-jets constrained by the derivatives of (1.5) of order $k-5$ or less. This shows that, at $k = 6$, the 6-jets of the regular solutions of (1.5) will have codimension 3 in all 6-jets of projective structures, i.e., they will have dimension $112-3 = 109$. However, we know that the image of the 7-jets of metric structures can have only dimension $108 - 1 = 107$. Thus, the 6-jets of regular metric structures have codimension 2 in the 6-jets of regular solutions of (1.5). That is why there have to be two more 6th order equations

$$E_1 = 0, \qquad E_2 = 0. \tag{4.32}$$

The image in 6-jets has total codimension 5, i.e., it is cut out by a 5th order equation and four 6th order equations. However, two of the 6th order equations are obviously the derivatives of the 5th order equation. The next line shows that, at 7th order, the image has only codimension 10, which means that there must be 2 relations between the first derivatives of the 6th order equations and the second derivatives of the 5th order equation which implies that the resulting system of three equations is involutive. This ends the proof of Theorem 1.3. q.e.d.

The analysis of the non-generic cases where the rank of $\mathcal{M}([\Gamma]) < 5$ is slightly more complicated. The argument based on the dimensionality of jet bundles associated to (4.31) breaks down as the PDE $\det \mathcal{M}([\Gamma]) = 0$ is not regular and does not define a smooth co-dimension 1 variety in $J^5(\mathrm{Pr}(U))$.

Let $\mathcal{S}([\Gamma])$ be the dimension of the vector space of solutions to the linear system (2.10). Some of these solutions may correspond to degenerate quadratic forms on U but nevertheless we have

Lemma 4.3. *If $\mathcal{S}([\Gamma]) > 1$ then there are $\mathcal{S}([\Gamma])$ independent non-degenerate quadratic forms among the solutions to* (2.10)*.*

Proof. Let us assume that at least one solution of (2.10) gives rise to a quadratic form which is degenerate (rank 1) everywhere. We can choose coordinates such that this solution is of the form $(\psi_1, 0, 0)$. The statement of the Lemma will follow if we can show that there is no other solution of the form $(\phi(x, y)\psi_1, 0, 0)$ where $\phi(x, y)$ is a non-constant function. The Liouville system (2.10) is readily solved in this case to give

$$A_1 = \frac{3}{2}\frac{1}{\psi_1}\frac{\partial \psi_1}{\partial x}, \quad A_2 = \frac{3}{4}\frac{1}{\psi_1}\frac{\partial \psi_1}{\partial y}, \quad A_3(x,y) = 0$$

with A_0 unspecified. Thus for a given projective class the only freedom in this solution is to rescale ψ_1 by a constant. q.e.d.

Proof of Theorem 1.4. We shall list the number and the order of obstructions one can expect depending on the rank $\mathcal{M}([\Gamma])$.

- If rank $\mathcal{M}([\Gamma]) < 2$ the projective structure is projectively flat as $L_1 = L_2 = 0$, and the second order ODE is equivalent to $y'' = 0$ by Theorem 2.2. This is obvious if rank $\mathcal{M}([\Gamma]) = 0$ as then $\mathbf{V} = 0$ and formula (3.21) gives $L_1 = L_2 = 0$. If rank $\mathcal{M}([\Gamma]) = 1$ then

$$\partial_a \mathbf{V} - \mathbf{V}\mathbf{\Omega}_a = \gamma_a \mathbf{V} \tag{4.33}$$

  for some γ_a. Using the expressions (A53) for $\mathbf{\Omega}_a$ and the formula (3.21) yields

$$\mathbf{V}\mathbf{\Omega}_a = (*, *, *, *, *, 5L_a)$$

  where $*$ are some terms which need not concern us and L_a are the Liouville expressions (2.14). Combining this with (4.33) yields $L_1 = L_2 = 0$.
- If rank $\mathcal{M}([\Gamma]) = 2$ then

$$\mathbf{V} + c_1\mathbf{V}_x + c_2\mathbf{V}_y = 0 \tag{4.34}$$

  for functions c_1, c_2 at least one of which does not identically vanish. Differentiating this relation and using the fact that $\mathbf{V}_{ab} \in \text{span}\{\mathbf{V}, \mathbf{V}_a\}$ we see that no new relations arise and so the system is closed at this level. In this case there exists a four dimensional family of metrics compatible with the given projective structure.
- If rank $\mathcal{M}([\Gamma]) = 3$ we have to consider two cases. If $\{\mathbf{V}, \mathbf{V}_x, \mathbf{V}_y\}$ are linearly independent then reasoning as above shows that further differentiations do not add any new conditions. The other possibility is that $\{\mathbf{V}, \mathbf{V}_x, \mathbf{V}_{xx}\}$ or $\{\mathbf{V}, \mathbf{V}_y, \mathbf{V}_{yy}\}$ are linearly independent. Let us concentrate on the first case (or swap x with y if necessary). Taking further x derivatives may increase the rank of the resulting system, but the y derivatives will not yield any new conditions as can be seen by mixing the partial derivatives and using

$$c_0\mathbf{V} + c_1\mathbf{V}_x + c_2\mathbf{V}_y = 0,$$

  which is a consequence of the rank 3 condition.

  Let us assume that the rank increases to 5 by adding two vectors $\mathbf{V}_{xxx}, \mathbf{V}_{xxxx}$ (otherwise the system is closed with rank 3 or 4). The rank will stay 5 if one further differentiation does not add new conditions. Thus the first and only obstruction in this case is of order 8 in the projective structure

$$\det\begin{pmatrix}\mathbf{V}\\ \mathbf{V}_x\\ \mathbf{V}_{xx}\\ \mathbf{V}_{xxx}\\ \mathbf{V}_{xxxx}\\ \mathbf{V}_{xxxxx}\end{pmatrix}=0. \tag{4.35}$$

- The analogous procedure can be carried over if $\mathrm{rank}(\mathcal{M}([\Gamma]))=4$. Assuming that the four linearly independent vectors are $\{\mathbf{V},\mathbf{V}_x,\mathbf{V}_y,\mathbf{V}_{xx}\}$ leads to one obstruction of order 7

$$\det\begin{pmatrix}\mathbf{V}\\ \mathbf{V}_x\\ \mathbf{V}_{xy}\\ \mathbf{V}_{xx}\\ \mathbf{V}_{xxx}\\ \mathbf{V}_{xxxx}\end{pmatrix}=0.$$

This completes the proof of Theorem 1.4. q.e.d.

As a corollary from this analysis we deduce the result of Koenigs [**16**]

Theorem 4.4. [**16**] *The space of metrics compatible with a given projective structures can have dimensions* $0,1,2,3,4$ *or* 6.

Our approach to the Koenigs's theorem is similar to that of Kruglikov's [**17**] who has however constructed an additional set of invariants determining whether a metrisable projective structure admits more than one metric in its projective class.

5. Examples

It is possible that the determinant (1.4) vanishes and the projective structure $[\Gamma]$ is non metrisable either because the further higher order obstructions do not vanish, or because a solution to the Liouville system (2.10) is degenerate as a quadratic form on TU. It can also happen when the projective structure fails to be real analytic.

In this section we shall give four examples illustrating this.

5.1. The importance of 6th order conditions. Consider a one parameter family of homogeneous projective structures corresponding to the second order ODE

$$\frac{d^2y}{dx^2}=c\,e^x+e^{-x}\Big(\frac{dy}{dx}\Big)^2.$$

For generic c the matrix $\mathcal{M}([\Gamma])$ has rank six and the 5th order condition (1.5) holds if $\hat{c} = 48c - 11$ is a root of a quartic

$$\hat{c}^4 - 11286\,\hat{c}^2 - 850968\,\hat{c} - 19529683 = 0. \tag{5.36}$$

The 6th order conditions (4.32) are satisfied iff

$$3\,\hat{c}^5 + 529\,\hat{c}^4 + 222\,\hat{c}^3 - 2131102\,\hat{c}^2 - 103196849\,\hat{c} - 1977900451 = 0,$$

$$\hat{c}^3 - 213\,\hat{c}^2 - 7849\,\hat{c} - 19235 = 0.$$

It is easy to verify that these three polynomials do not have a common root. Choosing $\hat{c}$ to be a real root of (5.36) we can make the 5th order obstruction (1.5) vanish, but the two 6th order obstructions E_1, E_2 do not vanish.

5.2. The importance of the non-degenerate kernel. This example illustrates why we cannot hope to characterise the metrisability condition purely by vanishing of any set of invariants.

Let f be a smooth function on an open set $U \subset \mathbb{R}^2$. Consider a one-parameter family of metrics

$$g_c = c\exp\left(f(x,y)\right)dx^2 + dy^2, \qquad \text{where} \quad c \in \mathbb{R}^+.$$

The corresponding one-parameter family of projective structures $[\Gamma_c]$ is given by the ODE

$$\frac{d^2y}{dx^2} = \frac{c}{2}\frac{\partial f}{\partial y}\exp\left(f\right) + \frac{1}{2}\frac{\partial f}{\partial x}\Big(\frac{dy}{dx}\Big) + \frac{\partial f}{\partial y}\Big(\frac{dy}{dx}\Big)^2.$$

The 5th order obstruction (1.5) and 6th order conditions E_1, E_2 of course vanish. Moreover rank $\mathcal{M}([\Gamma_c]) = 5$ for generic $f(x,y)$.

Now take the limit $c = 0$. The obstructions still vanish and rank $\mathcal{M}([\Gamma_0]) = 5$ but $[\Gamma_0]$ is not metrisable. This is because one can select a 3 by 3 linear subsystem $\widetilde{\mathcal{M}}_0\,\phi = 0$, where $\phi = (\psi_1, \psi_2, \mu)^T$, from the 6 by 6 system (3.22). The 3 by 3 matrix $\widetilde{\mathcal{M}}_0$ can be read off (3.22). For generic f the determinant of $\widetilde{\mathcal{M}}_0$ does not vanish and so there does not exist a parallel section $\mathbf{\Psi}$ of (3.22) such that $\psi_1\psi_3 - \psi_2^2 \neq 0$. For example $f = xy$ gives rank $\mathcal{M}([\Gamma_0]) = 5$ and

$$\det(\widetilde{\mathcal{M}}_0) = \frac{3xy}{4} - \frac{9}{2}.$$

This non-metrisable example fails the genericity assumption $P([\Gamma]) \neq 0$ where $P([\Gamma])$ is given by (4.28). The kernel of $\mathcal{M}([\Gamma_0])$ is spanned by a vector $(0,0,1,0,0,0)^T$ and the corresponding quadratic form on TU is degenerate.

5.3. The importance of real analyticity. This example illustrates why we need to work in the real analytic case to get sufficient conditions. We shall construct a simply connected projective surface in which every point has a neighbourhood on which there is a metric compatible with the given projective structure, but there is no metric defined on the whole surface that is compatible with the projective structure.

Consider a plane $U = \mathbb{R}^2$ with cartesian coordinates (x, y). Take two constant coefficient metrics on the plane that are linearly independent, say, g_+ and g_-. Now consider a modification of g_- in the half-plane $x < -1$ such that the modified g_- is the only global metric that is compatible with its underlying projective structure. Similarly, consider a modification of g_+ on the half-plane $x > 1$ such that the modified g_+ is the only global metric that is compatible with its underlying projective structure. The two projective structures agree (with the flat one) in the strip $-1 < x < 1$, so let the new projective structure be the one that agrees with that of modified g_- when $x < 1$ and with the modified g_+ when $x > -1$. This final projective structure will have compatible metrics locally near each point (sometimes, more than one, up to multiples), but will not have a compatible metric globally. Thus, metrisability cannot be detected locally in the smooth category.

5.4. One more degenerate example. Take $\Gamma_{11}^2 = A(x, y)$ and set all other components of Γ_{bc}^a to zero. Equivalently, take $A_1 = A_2 = A_3 = 0, A_0 = -A(x, y)$ (the case $A_0 = A_1 = A_2 = 0$ is also degenerate and can be obtained by reversing the role of x and y). For this degenerate case the Liouville relative invariant [**19**]

$$\begin{aligned} \nu_5 \;\; = \;\; & L_2(L_1\partial_x L_2 - L_2\partial_x L_1) + L_1(L_2\partial_y L_1 - L_1\partial_y L_2) \\ & + A_3(L_1)^3 - A_2(L_1)^2 L_2 + A_1 L_1 (L_2)^2 - A_0 (L_2)^3 \end{aligned}$$

vanishes.

The matrix $\mathcal{M}([\Gamma])$ in (1.4) has rank five and its determinant vanishes identically. In this case we can nevertheless analyse the linear system (2.10) directly without even prolonging it. We solve for

$$\psi_2 = -(1/2)y\alpha(x) + \beta(x), \quad \psi_3 = \alpha(x),$$

where α and β are some arbitrary functions of x, and cross-differentiate the remaining equations to find

$$2\beta'' - y\alpha''' + 2(\partial_x A)\alpha - 2(\partial_y A)\beta + (3A + y\partial_y A)\alpha' = 0. \tag{5.37}$$

Now assume further that $5\partial_y^2 A + y\partial_y^3 A \neq 0, \partial_y^3 A \neq 0$ and perform further differentiations to eliminate α, β from (5.37) and to find the necessary metrisable condition for $A(x, y)$

$$
\begin{aligned}
&7(\partial_y^3 A)\,(\partial_y^4 A)\,(\partial_x\partial_y^3 A) - 5(\partial_x\partial_y^3 A)\,(\partial_y^5 A)\,(\partial_y^2 A) - 6(\partial_x\partial_y^4 A)\,(\partial_y^3 A)^2 \\
&\quad +6(\partial_y^5 A)\,(\partial_x\partial_y^2 A)\,(\partial_y^3 A) - 7(\partial_y^4 A)^2\,(\partial_x\partial_y^2 A) \\
&\quad\quad +5(\partial_x\partial_y^4 A)\,(\partial_y^4 A)\,(\partial_y^2 A) = 0. \qquad (5.38)
\end{aligned}
$$

The obstruction (5.38) is of the same differential order as the 6 by 6 matrix (1.4), and we checked that it arises as a vanishing of a determinant of some 5 by 5 minors of (1.4) (which factorise in this case with (5.38) as a common factor).

We have pointed out that further genericity assumptions for A were needed to arrive at (5.38). To construct an example of non-metrisable projective connection where these assumptions do not hold consider the first Painlevé equation [**15**]

$$\frac{d^2y}{dx^2} = 6y^2 + x,$$

for which both (1.5) and (5.38) vanish. However equation (5.37) implies that $\alpha(x) = \beta(x) = 0$ so no metric exists in this case. We would have reached the same conclusion by observing that in the Painlevé I case $\mathrm{rank}(\mathcal{M})([\Gamma]) = 3$ and verifying that the 6 by 6 matrix in the 8th order obstruction (4.35) has rank 5. This obstruction therefore vanishes but the corresponding one-dimensional kernel is spanned by $(1, 0, 0, 0, 0, 0)^T$ and the corresponding solution to the linear system (2.10) is degenerate.

In [**13**] it was shown that the Liouville invariant ν_5 vanishes for all six Painlevé equations, and we have verified that our invariant (1.5) also vanishes. The metrisability analysis would need to be done on a case by case basis in a way analogous to our treatment of Painlevé I.

6. Twistor Theory

In this Section we shall give a twistorial treatment of the problem, which clarifies the rather mysterious linearisation (2.10) of the non-linear system (2.8).

In the real analytic case one complexifies the projective structure, and establishes a one-to-one correspondence between holomorphic projective structures $(U, [\Gamma])$ and complex surfaces Z with rational curves with self-intersection number one [**14**]. The points in Z correspond to geodesics in U, and all geodesics in U passing through a point $u \in U$ form a rational curve $\hat{u} \subset Z$ with normal bundle $N(\hat{u}) = \mathcal{O}(1)$. Here $\mathcal{O}(n)$ denotes the nth tensor power of the dual of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(TU)$ which arises as a quotient of $TU - \{0\}$ by the Euler vector field. Restricting the canonical line bundle κ_Z of Z to a twistor line $\hat{u} = \mathbb{CP}^1$ gives

$$\kappa_Z = T^*(\hat{u}) \otimes N^*(\hat{u}) = \mathcal{O}(-3)$$

since the holomorphic tangent bundle to $\mathbb{CP}^1$ is $\mathcal{O}(2)$. If U is a complex surface with a holomorphic projective structure, then its twistor space Z is $\mathbb{P}(TU)/D_x$, where D_x is the geodesic spray of the projective connection (2.11)

$$D_x = z^a \frac{\partial}{\partial x^a} - \Pi^c_{ab} z^a z^b \frac{\partial}{\partial z^c} = \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + (A_0 + \zeta A_1 + \zeta^2 A_2 + \zeta^3 A_3) \frac{\partial}{\partial \zeta}. \tag{6.39}$$

Here (x^a, z^a) are coordinates on TU and the second line uses projective coordinate $\zeta = z^2/z^1$. This leads to the double fibration

$$U \longleftarrow \mathbb{P}(TU) \longrightarrow Z.$$

All these structures should be invariant under an anti-holomorphic involution of Z to recover a real structure on U. This works in the real analytic case, but can in principle be extended to the smooth case using the holomorphic discs of LeBrun-Mason [**20**].

Now if the projective structure is metrisable, Z is equipped with a preferred section of the anti-canonical divisor line bundle ${\kappa_Z}^{-2/3}$ [**6, 20**]. The zero set of this section intersects each rational curve in Z at two points. The pullback of this section to TU is a homogeneous function of degree two $\sigma = \sigma_{ab} z^a z^b$, where z^a are homogeneous coordinates on the fibres of $\mathbb{P}(TU) \to U$, and σ_{ab} with $a, b = 1, 2$ is a symmetric 2-tensor on U.

This function Lie derives along the spray (6.39) and this gives the overdetermined linear system as the vanishing of a polynomial homogeneous of degree 3 in z^a: The condition $D_x(\sigma) = 0$ implies the equation (2.12) which is equivalent to (2.10).

We can understand the equation (2.12) using any connection Γ in a projective class instead of the projective connection ∇^Π. To see it we need to introduce a concept of projective weight [**10**]. First recall that the covariant derivative of the projective connection acting on vector fields is given by $\nabla_a X^c = \partial_a X^c + \Gamma^c_{ab} X^b$ and on 1-forms by $\nabla_a \phi_b = \partial_a \phi_b - \Gamma^c_{ab} \phi_c$. Let $\epsilon_{ab} = \epsilon_{[ab]}$ be a volume form on U. Changing a representative of the projective class yields

$$\hat{\nabla}_a \epsilon_{bc} = \nabla_a \epsilon_{bc} - 3\, \omega_a \epsilon_{bc}. \tag{6.40}$$

Let $\mathcal{E}(1)$ be a line bundle over U such that the 3rd power of its dual bundle is the canonical bundle of U. The bundles $\mathcal{E}(w) = \mathcal{E}(1)^{\otimes w}$ have a flat connection induced from $[\Gamma]$. It changes according to

$$\hat{\nabla}_a h = \nabla_a h + w\, \omega_a h$$

under (1.1), where h is a section of $\mathcal{E}(w)$.

Definition 6.1. The weighted vector field with projective weight w is a section of a bundle $TU \otimes \mathcal{E}(w)$.

This definition naturally extends to other tensor bundles. Now we shall choose a convenient normalisation of $[\Gamma]$. For any choice of ϵ_{ab} we must have $\nabla_a \epsilon_{bc} = \theta_a \epsilon_{bc}$ for some θ_a. We can change the projective representative with $\omega_a = \theta_a/3$ and use (6.40) to set $\theta_a = 0$ so that ϵ_{ab} is parallel. Let us assume that such a choice has been made. We shall use the volume forms to raise and lower indices according to $z_a = \epsilon_{ba} z^b$, $z^a = z_b \epsilon^{ba}$. The residual freedom in (1.1) is to use $\omega_a = \nabla_a f$ where f is any function on U. If $\nabla_a \epsilon_{bc} = 0$ then

$$\hat{\nabla}_a \hat{\epsilon}_{bc} = 0, \qquad \text{if} \quad \hat{\epsilon}_{ab} = e^{3f} \epsilon_{ab}. \tag{6.41}$$

Thus if $h \in \mathcal{E}(w)$ is a scalar of weight w and we change the volume form as in (6.41) then we must rescale

$$h \longrightarrow \hat{h} = e^{wf} h$$

with natural extension to other tensor bundles. Thus ϵ^{ab} has weight -3.

Let us now come back to equation (2.12) where the Πs are replaced by components of some connection in $[\Gamma]$

$$\nabla_{(a} \sigma_{bc)} = 0.$$

If we change the representative of the projective class by (1.1) with $\omega_a = \nabla_a f$ the equation $D_x(\sigma) = 0$ stays invariant if

$$\sigma_{ab} \longrightarrow \hat{\sigma}_{ab} = e^{4f} \sigma_{ab}.$$

This argument shows that the linear operator

$$\sigma_{ab} \longrightarrow \nabla_{(a} \sigma_{bc)}$$

is projectively invariant on symmetric two-tensors with weight 4. Now $\sigma^{ab} := \epsilon^{ac} \epsilon^{bd} \sigma_{cd}$ is a section of $S^2(TU) \otimes \mathcal{E}(-2)$ and satisfies

$$\nabla_a \sigma^{bc} = \delta_a^b \mu^c + \delta_a^c \mu^b \tag{6.42}$$

for some μ^b. The Liouville lemma 2.1 implies that if σ^{ab} satisfies this equation then $g^{ab} = (\det \sigma) \sigma^{ab}$ is a metric in the projective class.

The expression (6.42) is the tensor version of the first prolongation of the linear system (2.10). In the next section we shall carry over the prolongation in the invariant manner and express the 5th order obstruction (1.5) as a weighted projective scalar invariant.

7. An Alternative Derivation

In this section, we use the approach of [**11**] to derive the obstruction $\det \mathcal{M}([\Gamma])$ of Theorem 1.1. One advantage of this approach is that (1.5) may then be written in terms of the curvature of the connection and its covariant derivatives for any connection in the given projective class. The symmetric form σ^{ab} used in this Section is proportional to the quadratic form (2.7) and the objects (μ^a, ρ) are related but not equal to (μ, ν, ρ) defined by (3.15) and (3.17) from Section 3. Similarly the 6 by 6 matrix (7.47) is related but not equal to $\mathcal{M}([\Gamma])$ given by (1.4). This is because the choices made in the prolongation procedure leading (7.44) are different than those made in Section 3. The resulting obstructions (1.5) and (7.48) do not depend on these choices and are the same up to a non-zero exponential factor.

Let $\Gamma \in [\Gamma]$ be a connection in the projective class. Its curvature is defined by

$$[\nabla_a, \nabla_b]X^c = R_{abd}^c X^d$$

and can be uniquely decomposed as

$$R_{abd}^c = \delta_a^c \mathrm{P}_{bd} - \delta_b^c \mathrm{P}_{ad} + \beta_{ab}\delta_d^c \tag{7.43}$$

where β_{ab} is skew. In dimensions higher than 2 there would be another term (the Weyl tensor) in this curvature but dimension in 2 it vanishes identically.

If we change the connection in the projective class using (1.1) then

$$\hat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\omega_b + \omega_a\omega_b, \quad \hat{\beta}_{ab} = \beta_{ab} + 2\nabla_{[a}\omega_{b]}.$$

The Bianchi identity implies that β_{ab} is closed and so locally it is clear that we can always choose a connection in our projective class with $\beta_{ab} = 0$ (in fact, this also true globally on an oriented manifold). The residual freedom in changing the representative of the equivalence class (1.1) is given by gradients $\omega_a = \nabla_a f$, where f is a function on U.

Now $\mathrm{P}_{ab} = \mathrm{P}_{ba}$ and the Ricci tensor of Γ is symmetric. The Bianchi identity implies that Γ is flat on a bundle of volume forms on U. Thus the normalisation of ∇_a may, equivalently, be stated as requiring the existence of a volume form ϵ^{ab} such that

$$\nabla_a\epsilon^{bc} = 0.$$

Locally, such a volume form is unique up to scale: let us fix one. This is the normalisation used in the previous Section.

The linear system and its prolongation developed in §2 and §3 is assembled in [**11**] into a single connection on a rank 6 vector bundle over U. Specifically, sections of this bundle comprise triples of contravariant tensors $(\sigma^{ab}, \mu^a, \rho)$ with σ^{ab} being symmetric. The connection is given by

$$\begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} \xmapsto{\nabla_a} \begin{pmatrix} \nabla_a \sigma^{bc} - \delta_a^b \mu^c - \delta_a^c \mu^b \\ \nabla_a \mu^b - \delta_a^b \rho + \mathrm{P}_{ac}\sigma^{bc} \\ \nabla_a \rho + 2\mathrm{P}_{ab}\mu^b - 2Y_{abc}\sigma^{bc} \end{pmatrix}, \tag{7.44}$$

where $Y_{abc} = \frac{1}{2}(\nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac})$, the Cotton tensor. The following is proved in [**11**].

Theorem 7.1. *The connection* ∇_a *is projectively equivalent to a Levi–Civita connection if and only if there is a covariantly constant section* $(\sigma^{ab}, \mu^a, \rho)$ *of the bundle with connection* (7.44) *for which* σ^{ab} *is non-degenerate.*

It is also shown in [**11**] how the rank 6 bundle itself and its connection (7.44) may be viewed as projectively invariant. In any case, obstructions to the existence of a covariantly constant section may be obtained from the curvature of this connection, which we now compute.

$$\nabla_a \nabla_b \begin{pmatrix} \sigma^{cd} \\ \mu^c \\ \rho \end{pmatrix} = \nabla_a \begin{pmatrix} \nabla_b \sigma^{cd} - \delta_b^c \mu^d - \delta_b^d \mu^c \\ \nabla_b \mu^c - \delta_b^c \rho + \mathrm{P}_{bd}\sigma^{cd} \\ \nabla_b \rho + 2\mathrm{P}_{bc}\mu^c - 2Y_{bcd}\sigma^{cd} \end{pmatrix} =$$

$$\begin{pmatrix} \nabla_a(\nabla_b \sigma^{cd} - \delta_b^c \mu^d - \delta_b^d \mu^c) - \delta_a^c(\nabla_b \mu^d - \delta_b^d \rho + \mathrm{P}_{be}\sigma^{de}) \\ \nabla_a(\nabla_b \mu^c - \delta_b^c \rho + \mathrm{P}_{bd}\sigma^{cd}) - \delta_a^c(\nabla_b \rho + 2\mathrm{P}_{bd}\mu^d - 2Y_{bde}\sigma^{de}) \\ \nabla_a(\nabla_b \rho + 2\mathrm{P}_{bc}\mu^c - 2Y_{bcd}\sigma^{cd}) + 2\mathrm{P}_{ac}(\nabla_b \mu^c - \delta_b^c \rho + \mathrm{P}_{bd}\sigma^{cd}) \end{pmatrix}$$

$$+ \begin{pmatrix} -\delta_a^d(\nabla_b \mu^c - \delta_b^c \rho + \mathrm{P}_{be}\sigma^{ce}) \\ \mathrm{P}_{ad}(\nabla_b \sigma^{cd} - \delta_b^c \mu^d - \delta_b^d \mu^c) \\ -2Y_{acd}(\nabla_b \sigma^{cd} - \delta_b^c \mu^d - \delta_b^d \mu^c) \end{pmatrix}$$

$$= \begin{pmatrix} \nabla_a \nabla_b \sigma^{cd} - \delta_a^c \mathrm{P}_{be}\sigma^{de} - \delta_a^d \mathrm{P}_{be}\sigma^{ce} + \star\star \\ \nabla_a \nabla_b \mu^c - \delta_a^c \mathrm{P}_{bd}\mu^d + (\nabla_a \mathrm{P}_{bd})\sigma^{cd} + 2\delta_a^c Y_{bde}\sigma^{de} + \star\star \\ \nabla_a \nabla_b \rho + 2(\nabla_a \mathrm{P}_{bc})\mu^c - 2(\nabla_a Y_{bcd})\sigma^{cd} + 2Y_{abd}\mu^d + 2Y_{acb}\mu^c + \star\star \end{pmatrix}$$

where $\star\star$ denotes expressions that are manifestly symmetric in ab. Also notice that

$$(\nabla_{[a}\mathrm{P}_{b]d})\sigma^{cd} + 2\delta_{[a}^c Y_{b]de}\sigma^{de} = \delta_d^c Y_{abe}\sigma^{de} + 2\delta_{[a}^c Y_{b]de}\sigma^{de} = 3\delta_{[a}^c Y_{bd]e}\sigma^{de} = 0,$$

and that

$$Y_{[abc]} = 0 \implies Y_{acb} - Y_{bca} = Y_{abc}.$$

Therefore,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \begin{pmatrix} \sigma^{cd} \\ \mu^c \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10Y_{abc}\mu^c - 4(\nabla_{[a}Y_{b]cd})\sigma^{cd} \end{pmatrix}. \tag{7.45}$$

Denoting the triple $(\sigma^{ab}, \mu^b, \rho)$ by Σ^α, we are seeking a section Σ^α of our rank 6 bundle so that $\nabla_a \Sigma^\alpha = 0$ and have found the explicit form of the evident necessary condition $(\nabla_a \nabla_b - \nabla_b \nabla_a)\Sigma^\alpha = 0$. We may rewrite our necessary condition as $\epsilon^{ab}\nabla_a\nabla_b\Sigma^\alpha = 0$. Notice, however, that there is only one non-zero entry on the right hand side of (7.45). Our necessary condition analogous to (3.20) becomes

$$\Xi_\alpha \Sigma^\alpha = 0 \tag{7.46}$$

for

$$\Xi_\alpha \equiv \begin{pmatrix} 0 \\ 5Y_a \\ Z_{ab} \end{pmatrix}, \text{ where } Y_c \equiv \epsilon^{ab} Y_{abc} \text{ and } Z_{cd} \equiv -2\epsilon^{ab}\nabla_a Y_{b(cd)} = \nabla_{(c} Y_{d)}.$$

Evidently, the quantity Ξ_α is a section of a rank 6 bundle dual to our previous one. Its sections consist of triples of covariant tensors $(\kappa, \lambda_a, \tau_{ab})$ with τ_{ab} being symmetric and it inherits a connection dual to the previous one. Specifically,

$$\begin{pmatrix} \kappa \\ \lambda_b \\ \tau_{bc} \end{pmatrix} \overset{\nabla_a}{\longmapsto} \begin{pmatrix} \nabla_a \kappa + \lambda_a \\ \nabla_a \lambda_b + 2\tau_{ab} - 2\mathrm{P}_{ab}\kappa \\ \nabla_a \tau_{bc} - \mathrm{P}_{a(b}\lambda_{c)} + 2Y_{a(bc)}\kappa \end{pmatrix},$$

where

$$\begin{pmatrix} \kappa \\ \lambda_b \\ \tau_{bc} \end{pmatrix} \lrcorner \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} \equiv \kappa\rho + \lambda_b \mu^b + \tau_{bc}\sigma^{bc}$$

is the dual pairing. By differentiating our necessary condition for $\nabla_a \Sigma^\gamma = 0$ we obtain

$$\Xi_\gamma \Sigma^\gamma = 0 \qquad (\nabla_a \Xi_\gamma)\Sigma^\gamma = 0 \qquad (\nabla_{(a}\nabla_{b)}\Xi_\gamma)\Sigma^\gamma = 0.$$

Since Σ^α is supposed to be a non-zero section, it follows that the 6×6 matrix

$$\left(\begin{pmatrix} 0 \\ 5Y_c \\ Z_{cd} \end{pmatrix}, \nabla_a \begin{pmatrix} 0 \\ 5Y_c \\ Z_{cd} \end{pmatrix}, \nabla_{(a}\nabla_{b)} \begin{pmatrix} 0 \\ 5Y_c \\ Z_{cd} \end{pmatrix} \right) \tag{7.47}$$

must be singular. Its determinant is the obstruction from Theorem 1.1. We compute

$$\nabla_a \begin{pmatrix} 0 \\ 5Y_c \\ Z_{cd} \end{pmatrix} = \begin{pmatrix} 5Y_a \\ 5\nabla_a Y_c + 2Z_{ac} \\ \nabla_a Z_{cd} - 5\mathrm{P}_{a(c}Y_{d)} \end{pmatrix}$$

and

$$\nabla_a\nabla_b \begin{pmatrix} 0 \\ 5Y_c \\ Z_{cd} \end{pmatrix} =$$

$$\begin{pmatrix} 5\nabla_a Y_b + 5\nabla_b Y_a + 2Z_{ba} \\ \nabla_a(5\nabla_b Y_c + 2Z_{bc}) + 2\nabla_b Z_{ac} - 10\mathrm{P}_{b(a}Y_{c)} - 10\mathrm{P}_{ac}Y_b \\ \nabla_a(\nabla_b Z_{cd} - 5\mathrm{P}_{b(c}Y_{d)}) - 5\mathrm{P}_{a(c}\nabla_{|b|}Y_{d)} - 2\mathrm{P}_{a(c}Z_{|b|d)} + 10Y_{a(cd)}Y_b \end{pmatrix}$$

so

$$\nabla_{(a}\nabla_{b)} \begin{pmatrix} 0 \\ 5Y_c \\ Z_{cd} \end{pmatrix} =$$

$$\begin{pmatrix} 12Z_{ab} \\ 5\nabla_{(a}\nabla_{b)}Y_c + 4\nabla_{(a}Z_{b)c} - 5\mathrm{P}_{ab}Y_c - 15\mathrm{P}_{c(a}Y_{b)} \\ \nabla_{(a}\nabla_{b)}Z_{cd} - 5(\nabla_{(a}\mathrm{P}_{b)(c})Y_{d)} - 5\mathrm{P}_{c(a}\nabla_{b)}Y_d - 5\mathrm{P}_{d(a}\nabla_{b)}Y_c \\ \qquad -\mathrm{P}_{c(a}Z_{b)d} - \mathrm{P}_{d(a}Z_{b)c} + 10Y_{(a}Y_{b)(cd)} \end{pmatrix}.$$

To compute the determinant of the 6×6 matrix (7.47) we may use the following.

Lemma 7.2. *Let ϵ^{ab} denote the skew form in two dimensions normalised as*

$$\epsilon^{00} = 0 \quad \epsilon^{01} = 1 \quad \epsilon^{10} = -1 \quad \epsilon^{11} = 0.$$

Then the determinant of the 6×6 matrix

$$\begin{pmatrix} 0 & P_0 & P_1 & Q_{00} & Q_{01} & Q_{11} \\ R_0 & S_{00} & S_{01} & T_{000} & T_{001} & T_{011} \\ R_1 & S_{10} & S_{11} & T_{100} & T_{101} & T_{111} \\ U_{00} & V_{000} & V_{001} & X_{0000} & X_{0001} & X_{0011} \\ U_{01} & V_{010} & V_{011} & X_{0100} & X_{0101} & X_{0111} \\ U_{11} & V_{110} & V_{111} & X_{1100} & X_{1101} & X_{1111} \end{pmatrix}$$

is $\epsilon^{ab}\epsilon^{cd}\epsilon^{ef}\epsilon^{gh}\epsilon^{ij}\epsilon^{kl}\epsilon^{mn}\epsilon^{pq}$

$$\begin{bmatrix} Q_{gi}S_{mp}T_{njk}U_{ac}V_{deq}X_{bfhl} - \frac{1}{6}P_pR_mS_{nq}X_{acgi}X_{behk}X_{dfjl} \\ - \frac{1}{2}P_pS_{mq}T_{njl}U_{ce}X_{adgk}X_{bfhi} - \frac{1}{2}P_pT_{mgi}T_{njk}U_{ac}V_{deq}X_{bfhl} \\ + \frac{1}{2}P_pR_mT_{ngi}V_{acq}X_{dejk}X_{bfhl} - \frac{1}{2}Q_{gi}R_mS_{np}V_{acq}X_{dejk}X_{bfhl} \\ - \frac{1}{2}Q_{gi}R_mT_{njk}V_{acp}V_{deq}X_{bfhl} - \frac{1}{4}Q_{gi}S_{mp}S_{nq}U_{ac}X_{dejk}X_{bfhl} \\ - \frac{1}{4}Q_{gi}T_{mjk}T_{nhl}U_{ac}V_{dep}V_{bfq} \end{bmatrix} \tag{7.48}$$

where $Q_{ab} = Q_{(ab)}$, $T_{cab} = T_{c(ab)}$, $U_{cd} = U_{(cd)}$, $V_{cda} = V_{(cd)a}$, and $X_{cdab} = X_{(cd)(ab)}$.

Proof. A tedious computation. q.e.d.

Every tensor $Q, S, T, ...$ in this expression is constructed using one ϵ^{ab}. Thus counting the total number of ϵ^{ab}s shows that the determinant has a total projective weight -42 in a sense of Definition 6.1 which also means that it represents a section of the 14th power of the canonical bundle of U.

To summarise, we have proved the following alternative formulation of Theorem 1.1.

Theorem 7.3. *Suppose that* ∇_a *is a torsion-free connection in two-dimensions and that* ϵ^{bc} *is a volume form such that* $\nabla_a\epsilon^{bc} = 0$. *Define the Schouten tensor* P_{ab} *by* (7.43) *and*

$$Y_{abc} \equiv \tfrac{1}{2}(\nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac}) \qquad Y_c \equiv \epsilon^{ab}\nabla_a \mathrm{P}_{bc} \qquad Z_{ab} \equiv \nabla_{(a}Y_{b)}.$$

Let

$$\begin{aligned}
&P_a \equiv 5Y_a \qquad Q_{ab} \equiv 12Z_{ab} \qquad R_c \equiv 5Y_c \qquad S_{ca} \equiv 5\nabla_a Y_c + 2Z_{ac}\\
&T_{cab} \equiv 5\nabla_{(a}\nabla_{b)}Y_c + 4\nabla_{(a}Z_{b)c} - 5\mathrm{P}_{ab}Y_c - 15\mathrm{P}_{c(a}Y_{b)}\\
&U_{cd} \equiv Z_{cd} \qquad V_{cda} \equiv \nabla_a Z_{cd} - 5\mathrm{P}_{a(c}Y_{d)}\\
&X_{cdab} \equiv \ \nabla_{(a}\nabla_{b)}Z_{cd} - 5(\nabla_{(a}\mathrm{P}_{b)(c})Y_{d)} - 5\mathrm{P}_{c(a}\nabla_{b)}Y_d - 5\mathrm{P}_{d(a}\nabla_{b)}Y_c\\
&\qquad\qquad - \mathrm{P}_{c(a}Z_{b)d} - \mathrm{P}_{d(a}Z_{b)c} + 10Y_{(a}Y_{b)(cd)}
\end{aligned}$$

and define $\mathcal{D}(\Gamma)$ *by the formula* (7.48). *If* ∇_a *is projectively equivalent to a Levi–Civita connection, then* $\mathcal{D}(\Gamma) = 0$.

In addition to giving an explicit formula for $\mathcal{D}(\Gamma)$, there are several other consequences of this theorem, which we shall now discuss. We have found that

$$\mathcal{D}(\Gamma) = \det \begin{pmatrix} 0 & P_a & Q_{ab} \\ R_c & S_{ca} & T_{cab} \\ U_{cd} & V_{cda} & X_{cdab} \end{pmatrix}$$

where

$$Q_{ab} = Q_{(ab)},\ T_{cab} = T_{c(ab)},\ U_{cd} = U_{(cd)},\ V_{cda} = V_{(cd)a},\ X_{cdab} = X_{(cd)(ab)}$$

and the precise meaning of determinant is given by Lemma 7.2. Though it makes no difference to the determinant and seemingly gives a more complicated expression, it is more convenient to write

$$\mathcal{D}(\Gamma) = \frac{1}{4320}\det\bar{\Theta} \text{ where } \bar{\Theta} \equiv \begin{pmatrix} 0 & 12P_a & Q_{ab} \\ 30R_c & 12S_{ca} & T_{cab} - 5\mathrm{P}_{ab}R_c \\ 30U_{cd} & 12V_{cda} & X_{cdab} - 5\mathrm{P}_{ab}U_{cd} \end{pmatrix},$$

where the underlying matrix is evidently obtained by column operations from the previous one. The reason is that this matrix better transforms under projective change of connection. Specifically, if we write

$$\bar\Theta = \left(\begin{array}{ccc} 0 & \bar P_a & \bar Q_{ab} \\ \bar R_c & \bar S_{ca} & \bar T_{cab} \\ \bar U_{cd} & \bar V_{cda} & \bar X_{cdab} \end{array}\right),$$

then under the change in connection

$$\widehat\nabla_a\phi_b = \nabla_a\phi_b - \omega_a\phi_b - \omega_b\phi_a$$

induced by (1.1), we find

$$\widehat{\bar\Theta} = \left(\begin{array}{ccc} 0 & \tilde P_a & \tilde Q_{ab} - \tilde P_{(a}\omega_{b)} \\ \tilde R_c & \tilde S_{ca} - 2\tilde R_c\omega_a & \tilde T_{cab} - \tilde S_{c(a}\omega_{b)} + \tilde R_c\omega_a\omega_b \\ \tilde U_{cd} & \tilde V_{cda} - 2\tilde U_{cd}\omega_a & \tilde X_{cdab} - \tilde V_{cd(a}\omega_{b)} + \tilde U_{cd}\omega_a\omega_b \end{array}\right) \tag{7.49}$$

where

(7.50)

$$\widetilde\Theta = \left(\begin{array}{ccc} 0 & \tilde P_a & \tilde Q_{ab} \\ \tilde R_c & \tilde S_{ca} & \tilde T_{cab} \\ \tilde U_{cd} & \tilde V_{cda} & \tilde X_{cdab} \end{array}\right) =$$

$$\left(\begin{array}{ccc} 0 & \bar P_a & \bar Q_{ab} \\ \bar R_c & \bar S_{ca} - 2\omega_c\bar P_a & \bar T_{cab} - 2\omega_c\bar Q_{ab} \\ \bar U_{cd} - \omega_{(c}\bar R_{d)} & \bar V_{cda} - \omega_{(c}\bar S_{d)a} + \omega_c\omega_d\bar P_a & \bar X_{cdab} - \omega_{(c}\bar T_{d)ab} + \omega_c\omega_c\bar Q_{ab} \end{array}\right).$$

Notice that $\widetilde\Theta$ is obtained from $\bar\Theta$ by column operations and then $\widehat{\bar\Theta}$ is obtained from $\widetilde\Theta$ by row operations. It follows that determinant does not change, i.e. $\mathcal{D}(\widehat\Gamma) = \mathcal{D}(\Gamma)$ is a projective invariant (from the formula (7.48) it is already apparent that $\mathcal{D}(\Gamma)$ is independent of choice of coördinates). Thus we use the notation $\mathcal{D}([\Gamma])$.

The argument following the formula (4.31) shows that there is only one obstruction to the metrisability at order 5 so $\det\mathcal{M}([\Gamma]) = 0$ iff $\mathcal{D}([\Gamma]) = 0$. Thus

$$\det\mathcal{M}([\Gamma])(dx\wedge dy)^{\otimes 14}$$

is indeed a projective invariant as claimed in the Introduction.

A more invariant viewpoint on these matters is as follows. The formula (7.44) is for a connection on an invariantly defined vector bundle, denoted by $\mathcal{E}^{(BC)}$ in [**11**]. It arises from a representation of $\mathrm{SL}(3,\mathbb{R})$ and the connection (7.44) is closely related (but not equal to) the projective Cartan connection induced on bundles so arising. The bundle is canonically filtered with composition series

$$\mathcal{E}^{AB} = \mathcal{E}^{bc}(-2) + \mathcal{E}^{b}(-2) + \mathcal{E}(-2)$$

as detailed in [**11**]. Strictly speaking the quantity Ξ_γ is not a section of the dual bundle $\mathcal{E}_{(CD)}$ but rather the projectively weighted bundle $\mathcal{E}_{(CD)}(-5)$ with composition series

$$\mathcal{E}_{(CD)}(-5) = \mathcal{E}(-3) + \mathcal{E}_c(-3) + \mathcal{E}_{(cd)}(-3)$$

and $\bar{\Theta}$ is then obtained by applying the invariantly defined 'splitting operator'

$$\mathcal{E}(-5) \ni \xi \mapsto \left(\begin{array}{c} 30\xi \\ 12\nabla_a \xi \\ \nabla_{(a}\nabla_{b)}\xi - 5\mathrm{P}_{ab}\xi \end{array} \right) \in \mathcal{E}_{(AB)}(-7)$$

coupled to the projectively invariant connection (7.44). The upshot is that $\bar{\Theta}$ is an invariantly defined section of $\mathcal{E}_{(CD)(AB)}(-7)$. Indeed, the formulae (7.49) and (7.50) giving $\widehat{\bar{\Theta}}$ in terms of $\bar{\Theta}$ are precisely how sections of $\mathcal{E}_{(CD)(AB)}$ or $\mathcal{E}_{(CD)(AB)}(-7)$ transform under projective change. Consequently, the obstruction $\mathcal{D}(\Gamma)$ is an invariant of projective weight -42.

8. Outlook

In the language of Cartan [**7**, **4**], the general 2nd order ODE (1.6) defines a *path geometry*, and the paths are geodesics of projective connection if the ODE is of the form (1.3). In this paper we have shown under what conditions the paths in this geometry are unparametrised geodesics of some metric. In case of higher dimensional projective structures the link with ODEs is lost, but nevertheless one could search for conditions obstructing the metrisability in a way analogous to what we did in Section (7). The results will have a different character, however, owing to the presence of the Weyl curvature which will modify the connection (7.44) as explained in [**11**]. The first necessary condition analogous to (1.5) occurs already at order 2. Specifically, it is shown in [**11**] that the curvature of the relevant connection in n dimensions is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a) \left(\begin{array}{c} \sigma^{cd} \\ \mu^c \\ \rho \end{array} \right)$$
$$= \left(\begin{array}{c} W_{abe}{}^c\sigma^{de} + W_{abe}{}^d\sigma^{ce} + \frac{2}{n}\delta_{[a}^c W_{b]ef}{}^d\sigma^{ef} + \frac{2}{n}\delta_{[a}^d W_{b]ef}{}^c\sigma^{ef} \\ * \\ * \end{array} \right)$$

where $W_{abd}{}^c$ is the Weyl curvature and $*$ denotes expressions that we shall not need. Since we are searching for covariant constant sections with non-degenerate σ^{cd}, in particular it follows that the linear transformation $\sigma^{ef} \mapsto \Xi_{abef}^{cd}\sigma^{ef}$ where

$$\Xi_{abef}^{cd} := W_{ab(e}{}^c\delta_{f)}^d + W_{ab(e}{}^d\delta_{f)}^c + \tfrac{2}{n}\delta_{[a}^c W_{b](ef)}{}^d + \tfrac{2}{n}\delta_{[a}^d W_{b](ef)}{}^c \tag{8.51}$$

is obliged to have a non-trivial kernel. Regarding Ξ_{abef}^{cd} as a matrix representing this linear transformation, it should have $n(n+1)/2$ columns accounting for the symmetric indices ef. In its remaining indices it is skew in ab, symmetric cd, and trace-free. These symmetries specify an irreducible representation of $\mathrm{GL}(n,\mathbb{R})$ of dimension $(n^2-1)(n^2-4)/4$, which we may regard as the number of rows of the matrix Ξ_{abef}^{cd}. Notice that when $n=2$ this matrix is zero but as soon as $n\geq 3$ it has more rows than columns (for example, it is a 10×6 matrix in dimension 3). We claim that having a non-trivial kernel is a genuine condition and therefore an obstruction to metrisability. For this, we need to show that Ξ_{abef}^{cd} can have maximal rank even when it is of the special form (8.51) for some W_{abd}^{c} having the symmetries of a Weyl tensor, namely

$$W_{abd}^{c}=-W_{bad}^{c},\qquad W_{[abd]}^{c}=0,\qquad W_{abd}^{a}=0. \tag{8.52}$$

Choose a frame and, for $n\geq 3$, consider the particular tensor W_{abd}^{c} having as its only non-zero components (no summation)

$$\begin{array}{ll}
W_{121}^{1}=-W_{211}^{1}=3(n^2-n-1) & W_{122}^{2}=-W_{212}^{2}=3\\
W_{123}^{3}=-W_{213}^{3}=-(n-1)(2n+3) & W_{12c}^{c}=-W_{21c}^{c}=-(n-1),\ \forall c\geq 4\\
W_{132}^{3}=-W_{312}^{3}=-(n^2-n-3) & W_{1c2}^{c}=-W_{c12}^{c}=n+2,\ \forall c\geq 4\\
W_{231}^{3}=-W_{321}^{3}=n(n+2) & W_{2c1}^{c}=-W_{c21}^{c}=2n+1,\ \forall c\geq 4.
\end{array}$$

It is readily verified that the symmetries (8.52) are satisfied. Form the corresponding Ξ_{abef}^{cd} according to (8.51) and consider Ξ_{12ef}^{cd}. Being symmetric in cd and ef, we may regard it as a square matrix of size $n(n+1)/2$ and it suffices to show that this matrix is invertible. In fact, it is easy to check that it is diagonal with non- zero entries along its diagonal.

The twistor analysis of Section (6) suggest that there is some analogy between the metrisability problem we studied in two dimensions and existence of (possibly indefinite) Kähler structure in a given anti-self-dual (ASD) conformal class $\mathbf{c}$ in on a four-manifold M. A Kähler structure corresponds to a preferred section of anti-canonical divisor ${\kappa_B}^{-1/2}$, where κ_B is the canonical bundle of the twistor space [**21**] B (a complex three-fold with an embedded rational curve with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$). Not all ASD structures are Kähler and the existence of the divisor should lead to vanishing of some conformal invariants constructed out of the ASD Weyl tensor (to the best of our knowledge they have never been written down. If one adds the Ricci flat condition, some of the invariants are known and can be expressed in terms of the Bach tensor).

These two constructions (ASD+Kähler in four dimensions and projective + metrisable in two dimensions) are linked in the following way: every ASD structure in $(2,2)$ signature with a conformal null Killing vector induces a projective structure on a two-dimensional space U of the β surfaces (null ASD surfaces) in M. Conversely any two-dimensional projective structure gives rise to (a class of) ASD structures with null conformal symmetry [**8**, **6**]. Consider B to be a holomorphic fibre bundle over Z with one dimensional fibres, where Z is the twistor space of $(U,[\Gamma])$ introduced in Section 6.

Let $\hat{u} \subset Z$ be rational curve in Z corresponding to $u \in U$. The three-fold B will be a twistor space of an ASD conformal structure if B restricts to $\mathcal{O}(1)$ on each twistor line $\hat{u} \subset Z$. If Z corresponds to a metrisable projective structure then the divisor σ lifts to a section of ${\kappa_B}^{-1/2}$, thus giving a $(2,2)$ Kähler class. If the conformal Killing vector is not hyper-surface orthogonal the local expression for the conformal class is

$$\mathbf{c} = dz_a \otimes dx^a - \Pi^c_{ab}\, z_c\, dx^a \otimes dx^b,$$

where Π^c_{ab} are components of the projective connection (2.11). The conformal Killing vector is a homothety $z_a/\partial z_a$. This formula for $\mathbf{c}$ is equivalent to a special case of expression (1.3) in [**8**] after a change of coordinates and a conformal rescaling (set $z_a = (-ze^t, e^t)$ and take $G = z^2/2 + \gamma(x,y)z + \delta(x,y)$ for certain γ, δ in [**8**]). It is a projectively invariant modification of the Riemannian extensions of spaces with affine connection studied by Walker [**24**]. The conformal class $\mathbf{c}$ is conformally flat iff $[\Gamma]$ is projectively flat, i.e. its Cotton tensor vanishes. This in turn is equivalent to the vanishing of the Liouville expressions (2.14).

The metrisable projective structures will therefore give rise to (2, 2) ASD Kähler metric with conformal null symmetry. Ultimately, the metrisability invariant (1.5) in two dimensions will have its counterpart: a conformal invariant in four dimensions. Some progress in this direction has been made in [**9**].

Appendix

The connection $D = d + \mathbf{\Omega}_1 dx + \mathbf{\Omega}_2 dy$ on the rank six vector bundle $\mathbb{E} \to U$ is

$$\mathbf{\Omega}_1 = \begin{pmatrix} -\frac{2}{3}A_1 & 2A_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -2A_3 & -\frac{2}{3}A_2 & \frac{4}{3}A_1 & 0 & 1 & 0 \\ (\mathbf{\Omega}_1)_{41} & (\mathbf{\Omega}_1)_{42} & (\mathbf{\Omega}_1)_{43} & -\frac{1}{3}A_1 & -3A_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ (\mathbf{\Omega}_1)_{61} & (\mathbf{\Omega}_1)_{62} & (\mathbf{\Omega}_1)_{63} & (\mathbf{\Omega}_1)_{64} & (\mathbf{\Omega}_1)_{65} & (\mathbf{\Omega}_1)_{66} \end{pmatrix},$$

$$\text{(A53)}\quad \mathbf{\Omega}_2 = \begin{pmatrix} -\frac{4}{3}A_2 & \frac{2}{3}A_1 & 2A_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -2A_3 & \frac{2}{3}A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ (\mathbf{\Omega}_2)_{51} & (\mathbf{\Omega}_2)_{52} & (\mathbf{\Omega}_2)_{53} & 3A_3 & \frac{1}{3}A_2 & 0 \\ (\mathbf{\Omega}_2)_{61} & (\mathbf{\Omega}_2)_{62} & (\mathbf{\Omega}_2)_{63} & (\mathbf{\Omega}_2)_{64} & (\mathbf{\Omega}_2)_{65} & (\mathbf{\Omega}_2)_{66} \end{pmatrix}.$$

Let $V_1, ..., V_6$ be given by (3.21). The curvature of D is

$$\mathbf{F} = d\mathbf{\Omega} + \mathbf{\Omega} \wedge \mathbf{\Omega} = F\, dx \wedge dy =$$

$$\text{(A54)}\quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{pmatrix} dx \wedge dy.$$

The matrix elements of the connection are

$$\begin{aligned}
(\mathbf{\Omega}_1)_{41} &= -\frac{4}{3}\partial_x A_2 + 4A_0A_3 + \frac{2}{3}\partial_y A_1, \\
(\mathbf{\Omega}_1)_{42} &= -2\partial_y A_0 + \frac{2}{3}\partial_x A_1 + 4A_2A_0 - \frac{4}{9}(A_1)^2, \\
(\mathbf{\Omega}_1)_{43} &= 2\partial_x A_0 - 4A_0A_1, \\
(\mathbf{\Omega}_1)_{61} &= -\frac{4}{3}\partial_x\partial_y A_2 - \frac{20}{3}A_0A_2A_3 + \frac{2}{3}(\partial_y)^2 A_1 + 4A_3\partial_y A_0 \\
&\quad -2A_0\partial_y A_3 - \frac{16}{9}A_2\partial_x A_2 + \frac{8}{9}A_2\partial_y A_1, \\
(\mathbf{\Omega}_1)_{62} &= \frac{2}{3}\partial_x\partial_y A_1 - \frac{4}{3}A_1\partial_y A_1 + 2A_0\partial_y A_2 - 2(\partial_y)^2 A_0 \\
&\quad +4A_2\partial_y A_0 + 4A_3\partial_x A_0 + 6A_0\partial_x A_3 \\
&\quad +\frac{8}{9}A_1\partial_x A_2 + \frac{4}{3}A_0A_1A_3 - \frac{4}{3}A_0(A_2)^2, \\
(\mathbf{\Omega}_1)_{63} &= 2\partial_x\partial_y A_0 + \frac{2}{3}A_0\partial_x A_2 - \frac{4}{3}A_2\partial_x A_0 - 4A_1\partial_y A_0 \\
&\quad -\frac{4}{3}A_0\partial_y A_1 + \frac{8}{3}A_0A_1A_2 + 4A_3(A_0)^2, \\
(\mathbf{\Omega}_1)_{64} &= \frac{4}{3}\partial_x A_2 - \partial_y A_1 + 5A_0A_3, \\
(\mathbf{\Omega}_1)_{65} &= \frac{1}{3}\partial_x A_1 - 4\partial_y A_0 + 3A_2A_0 - \frac{2}{9}(A_1)^2, \\
(\mathbf{\Omega}_1)_{66} &= -\frac{1}{3}A_1,
\end{aligned}$$

$$
\begin{aligned}
(\mathbf{\Omega}_2)_{51} &= -2\partial_y A_3 - 4A_3A_2,\\
(\mathbf{\Omega}_2)_{52} &= 2\partial_x A_3 - \frac{2}{3}\partial_y A_2 + 4A_1A_3 - \frac{4}{9}(A_2)^2,\\
(\mathbf{\Omega}_2)_{53} &= \frac{4}{3}\partial_y A_1 - \frac{2}{3}\partial_x A_2 + 4A_0A_3,\\
(\mathbf{\Omega}_2)_{61} &= -2\partial_x\partial_y A_3 - 4A_2\partial_x A_3 - \frac{4}{3}A_3\partial_x A_2 + \frac{2}{3}A_3\partial_y A_1\\
&\quad -\frac{4}{3}A_1\partial_y A_3 - 4A_0(A_3)^2 - \frac{8}{3}A_1A_2A_3,\\
(\mathbf{\Omega}_2)_{62} &= 2(\partial_x)^2 A_3 - \frac{4}{3}A_2\partial_x A_2 - \frac{4}{3}A_0A_2A_3 - \frac{2}{3}\partial_x\partial_y A_2\\
&\quad +4A_1\partial_x A_3 + 4A_0\partial_y A_3 + 6A_3\partial_y A_0\\
&\quad +\frac{4}{3}A_3(A_1)^2 + 2A_3\partial_x A_1 + \frac{8}{9}A_2\partial_y A_1,\\
(\mathbf{\Omega}_2)_{63} &= -\frac{2}{3}(\partial_x)^2 A_2 + \frac{8}{9}A_1\partial_x A_2 + 4A_0\partial_x A_3 - 2A_3\partial_x A_0\\
&\quad -\frac{16}{9}A_1\partial_y A_1 + \frac{4}{3}\partial_x\partial_y A_1 + \frac{20}{3}A_0A_1A_3,\\
(\mathbf{\Omega}_2)_{64} &= 4\partial_x A_3 - \frac{1}{3}\partial_y A_2 + 3A_1A_3 - \frac{2}{9}(A_2)^2,\\
(\mathbf{\Omega}_2)_{65} &= \partial_x A_2 - \frac{4}{3}\partial_y A_1 + 5A_0A_3,\\
(\mathbf{\Omega}_2)_{66} &= \frac{1}{3}A_2.
\end{aligned}
$$

The prolongation formulae are

$$
\begin{aligned}
P &= -(\mathbf{\Omega}_1)_{41}\,\psi_1 - (\mathbf{\Omega}_1)_{42}\,\psi_2 - (\mathbf{\Omega}_1)_{43}\,\psi_3 - (\mathbf{\Omega}_1)_{44}\,\mu - (\mathbf{\Omega}_1)_{45}\,\nu,\\
Q &= -(\mathbf{\Omega}_2)_{51}\,\psi_1 - (\mathbf{\Omega}_2)_{52}\,\psi_2 - (\mathbf{\Omega}_2)_{53}\,\psi_3 - (\mathbf{\Omega}_2)_{54}\,\mu - (\mathbf{\Omega}_2)_{55}\,\nu,\\
R &= -(\mathbf{\Omega}_1)_{61}\,\psi_1 - (\mathbf{\Omega}_1)_{62}\,\psi_2 - (\mathbf{\Omega}_1)_{63}\,\psi_3\\
&\quad -(\mathbf{\Omega}_1)_{64}\,\mu - (\mathbf{\Omega}_1)_{65}\,\nu - (\mathbf{\Omega}_1)_{66}\,\rho,\\
S &= -(\mathbf{\Omega}_2)_{61}\,\psi_1 - (\mathbf{\Omega}_2)_{62}\,\psi_2 - (\mathbf{\Omega}_2)_{63}\,\psi_3\\
&\quad -(\mathbf{\Omega}_2)_{64}\,\mu - (\mathbf{\Omega}_2)_{65}\,\nu - (\mathbf{\Omega}_2)_{66}\,\rho.
\end{aligned}
\tag{A55}
$$

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