Journal of Mathematics Research; Vol. 5, No. 1; 2013
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Adjoint n-point Boundary Value Problem for the Linear Differential Equation and Green's Function

Kazbek A. Khasseinov[1]

[1] Department of Mathematics, Kazakh National Technical University, Almaty, Kazakhstan

Correspondence: Kazbek A. Khasseinov, Department of Mathematics, Kazakh National Technical University named after K. I. Satpayev, Satpayev St. 22, P. O. 50013, Almaty, Kazakhstan. E-mail: dorteh77@mail.ru

Received: October 9, 2012 Accepted: November 28, 2012 Online Published: January 21, 2013

doi:10.5539/jmr.v5n1p83 URL: http://dx.doi.org/10.5539/jmr.v5n1p83

Abstract

The n-point problem with linear boundary conditions of general type is studied in this work. We have found the boundary conditions for the adjoint differential operator. The Green's function has been constructed where we have used solutions of the adjoint differential equation and studied its new properties. Through the Green's function and saltus of its derivatives, we have solved the nonhomogeneous n-point boundary value problem for the linear differential equation with variable coefficients.

Keywords: Green's function, new properties, adjoint differential equation, n-point, value problem

Classification: 39-02

1. Introduction

Multipoint Problem have been studied least of all because the interim points included into the boundary conditions cause a range of such serious hardships as breach of smoothness of the Green's function, absence of the adjoint boundary value Problem, etc. Redefined multipoint Problem where boundary conditions in the intermediate nodes are "unnecessary", important in the application context, have been turned out to be poorly studied. These tasks directly relate to the theory of spline (Coddington & Levinson, 1955; Yerugin, 1974; Pokornyi, 1980; Householder, 1956). While some strong developments in the linear two-point problems are determined by the modern analysis technique, such methods are not effective enough for the multipoint boundary value Problem. Difficulties arising in the multipoint Problems are overcome due to applying the Green's function which reflects the entire specificity of the boundary Problem and is a complicated object studied very poorly. Therefore, construction of the Green's function for the multipoint Problem with general type boundary conditions and research of its properties are still topical (Kiguradze, 1987; Klokov, 1967; Maksimov & Rakhmatullina, 1977; Liu, 2011; Peterson, 1979; Jackson, 1977).

2. Research

Let us consider a linear differential operator

$$Ly = y^{(n)} + \sum_{\nu=1}^{n} b_{\nu-1}(x) y^{(\nu-1)}, \tag{1}$$

with coefficients $b_{\nu-1}(x) \in C^{\nu-1}[x_1, x_n], \quad \nu = 1, 2, ..., n.$

Let us introduce an auxiliary linear differentiation operator of the boundary conditions

$$(Ty)(x) = \sum_{\nu=1}^{n} \rho_\nu(x) y^{(\nu-1)}(x), \tag{2}$$

where $\rho_\nu(x) \in C[x_1, x_n], \quad \nu = 1, 2, ..., n.$

Assume that the domain of existence of the operator L consists of functions $y \in C^{n-1}[x_1, x_n]$, complying with their boundary conditions

$$(Ty)(x_i) = \rho_n(x_i) y^{(n-1)}(x_i) + ... + \rho_2(x_i) y'(x_i) + \rho_1(x_i) y(x_i) = 0, \tag{3}$$

where $x_1 < x_2 < \cdots < x_n$ are known points and coefficients comply with a condition of degeneracy absence in the points

$$\sum_{v=1}^{n} | \rho_v(x_i) | \neq 0, \quad i = 1, 2, ..., n.$$

Let us introduce an adjointed operator

$$L^+z = (-1)^n z^{(n)} + (-1)^{n-1}(b_{n-1}(x)z)^{(n-1)} + ... - (b_1(x)z)' + b_0(x)z. \tag{4}$$

Its domain of existence $D(L^+)$ is described below.

Let us consider Lagrange identity (Coddington & Levinson, 1955)

$$zLy - yL^+z = \frac{d}{dx}\Phi(y, z),$$

where a bilinear form is set by the equality

$$\Phi(y, z) = \sum_{v=1}^{n} \sum_{\substack{p+q=v-1 \\ p\geq 0, q\geq 0}} (-1)^p y^{(q)}(b_v(x)z)^{(p)}, \quad b_n(x) \equiv 1.$$

Let us put $p = v - 1 - q$ into the internal sum and specify q at $p = 0$ and present a bilinear form as follows

$$\Phi(y, z) = \sum_{v=1}^{n} \sum_{q=0}^{v-1} (-1)^{v-1-q} y^{(q)}[b_v(x)z]^{(v-1-q)}, \quad b_n(x) \equiv 1.$$

Since $p \geq 0$, $q \geq 0$, we can produce $v \geq 1 + q$ from $p = v - 1 - q \geq 0$. Let us allocate the sums from the bilinear form at $q = 0$, $q = v - 1$, then, we should take $v \geq 2 + q$ in the other sums:

$$\Phi(y, z) = \sum_{v=1}^{n} y^{(v-1)}[b_v(x)z] + \sum_{v=2}^{n}(-1)^{v-1}y[b_v(x)z]^{(v-1)} + \sum_{v=3}^{n}\sum_{q=1}^{v-2}(-1)^{v-1-q}y^{(q)}[b_v(x)z]^{(v-1-q)}, \quad b_n(x) \equiv 1.$$

Having set integral values for q, let us write the bilinear form with the help of derivatives $y'(x), y''(x), ..., y^{(n-2)}(x)$:

$$\Phi[y(x), z(x)] = z(x)\sum_{v=1}^{n} b_v(x)y^{(v-1)}(x) + y(x)\sum_{v=2}^{n}(-1)^{v-1}[b_v(x)z(x)]^{(v-1)} + y'(x)\sum_{v=3}^{n}(-1)^{v-2}[b_v(x)z(x)]^{(v-2)} +$$

$$y''(x)\sum_{v=4}^{n}(-1)^{v-3}[b_v(x)z(x)]^{(v-3)} + ... + y^{(n-3)}(x)\sum_{v=n-1}^{n}(-1)^{v-n+2}[b_v(x)z(x)]^{(v-n+2)} + y^{(n-2)}(x)z'(x), \quad b_n(x) \equiv 1. \tag{5}$$

Let us consider a differential equation $\Phi(y, z) = 0$. It connects the adjoint family functions $\{y(x)\}, \{z(x)\}$ and their derivatives to $(n - 1)$ order included. Let us identify dependence of solutions of the adjoint equation $L^+z = 0$ on the solutions of the equation $Ly = 0$ and also find the adjoint boundary condition $D(L^+)$.

Lemma 1 *Let us assume that for the points* $\{x_i\}_1^n$ *and fundamental system of solutions* $\{y_i(x)\}_1^n$ *of the equation* $Ly = 0$

$$\Delta = \det \|(Ty_i)(x_i)\| = \begin{vmatrix} (Ty_1)(x_1) & (Ty_2)(x_1) & \dots & (Ty_n)(x_1) \\ (Ty_1)(x_2) & (Ty_2)(x_2) & \dots & (Ty_n)(x_2) \\ \vdots & \vdots & & \vdots \\ (Ty_1)(x_n) & (Ty_2)(x_n) & \dots & (Ty_n)(x_n) \end{vmatrix} \neq 0, (*)$$

then there is a fundamental system of solutions $\{\varphi_j(x)\}_1^n$ *of the homogenous equation* $Ly = 0$

$$(T\varphi_j)(x_i) = \delta_{ji}; \quad i, j = 1, 2, ..., n, \tag{6}$$

where δ_{ji} *is the Kronecker symbol.*

Proof. Since $b_{v-1}(x)$, $v = 1, 2, ..., n$ continuous, there is a fundamental system of solutions $y_1, y_2, \dots, y_n$ of the homogenous equation $Ly = 0$. Let us find a solution as follows

$$\varphi_j(x) = C_{j1}y_1(x) + C_{j2}y_2(x) + ... + C_{jn}y_n(x). \tag{7}$$

To make it clear, we will prove the lemma for ℓ, $\ell = 1, 2, .., n$. Let us apply the operator T to the function (7) and transfer all the members to the right part

$$C_{\ell 1}(Ty_1)(x) + C_{\ell 2}(Ty_2)(x) + ... + C_{\ell n}(Ty_n)(x) - (T\varphi_\ell)(x) = 0.$$

Having written this expression in the points $\{x_i\}_1^n$ having taken into account the boundary conditions (6) and having considered it jointly with (7), we have a system of the homogenous linear algebraic equations with respect to $C_{\ell 1}, C_{\ell 2}, ..., C_{\ell n}, -1$:

$$\begin{cases} C_{\ell 1}(Ty_1)(x_1) + C_{\ell 2}(Ty_2)(x_1) + ... + C_{\ell n}(Ty_n)(x_1) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_{\ell 1}(Ty_1)(x_\ell) + C_{\ell 2}(Ty_2)(x_\ell) + ... + C_{\ell n}(Ty_n)(x_\ell) - 1 = 0; \ (T\varphi_\ell)(x_\ell) = 1, \\ C_{\ell 1}y_1(x) + C_{\ell 2}y_2(x) + ... + C_{\ell n}y_n(x) - \varphi_\ell(x) = 0, \\ C_{\ell 1}(Ty_1)(x_{\ell+1}) + C_{\ell 2}(Ty_2)(x_{\ell+1}) + ... + C_{\ell n}(Ty_n)(x_{\ell+1}) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ C_{\ell 1}(Ty_1)(x_n) + C_{\ell 2}(Ty_2)(x_n) + ... + C_{\ell n}(Ty_n)(x_n) = 0. \end{cases} \qquad (**)$$

A determinant should be equal to zero for the nontrivial solution of the homogenous system, i.e.

$$\begin{vmatrix} (Ty_1)(x_1) & (Ty_2)(x_1) & \dots & (Ty_n)(x_1) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ (Ty_1)(x_\ell) & (Ty_2)(x_\ell) & \dots & (Ty_n)(x_\ell) & 1 \\ y_1(x) & y_2(x) & \dots & y_n(x) & \varphi_\ell(x) \\ (Ty_1)(x_{\ell+1}) & (Ty_2)(x_{\ell+1}) & \dots & (Ty_n)(x_{\ell+1}) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ (Ty_1)(x_n) & (Ty_2)(x_n) & \dots & (Ty_n)(x_n) & 0 \end{vmatrix} = 0, \forall x \in [x_1, x_n].$$

Let us decompose the determinant by the last column elements

$$\varphi_\ell(x) \cdot \det \left\| (Ty_j)(x_i) \right\| = \begin{vmatrix} (Ty_1)(x_1) & (Ty_2)(x_1) & \dots & (Ty_n)(x_1) \\ \vdots & \vdots & & \vdots \\ (Ty_1)(x_{\ell-1}) & (Ty_2)(x_{\ell-1}) & \dots & (Ty_n)(x_{\ell-1}) \\ y_1(x) & y_2(x) & \dots & y_n(x) \\ (Ty_1)(x_{\ell+1}) & (Ty_2)(x_{\ell+1}) & \dots & (Ty_n)(x_{\ell+1}) \\ \vdots & \vdots & & \vdots \\ (Ty_1)(x_n) & (Ty_2)(x_n) & \dots & (Ty_n)(x_n) \end{vmatrix}.$$

Since a determinant is different from zero under the lemma condition, the solution exists $\varphi_\ell(x) = \frac{\Delta_\ell(x)}{\Delta}$, $\Delta \neq 0$. Thus, we have found solutions of the homogenous equation $Ly = 0$ complying with the boundary conditions (6),

$$\varphi_j(x) = \frac{\Delta_j(x)}{\Delta}, \ \Delta \neq 0, \quad j = 1, 2, ..., n. \qquad (8)$$

Determinants $\Delta_j(x)$ are produced from Δ by replacement of elements of the j-line by the fundamental system of solutions $y_1, y_2, \dots, y_n$. Let us prove linear independence of the function (8). If we write a system (**) without proportions (7) for $\ell = 1, 2, ..., n$, we then have a system consisting of n^2 equations, that can be presented in the matrix form

$$\begin{pmatrix} (Ty_1)(x_1) & (Ty_2)(x_1) & \dots & (Ty_n)(x_1) \\ (Ty_1)(x_2) & (Ty_2)(x_2) & \dots & (Ty_n)(x_2) \\ \vdots & \vdots & & \vdots \\ (Ty_1)(x_n) & (Ty_2)(x_n) & \dots & (Ty_n)(x_n) \end{pmatrix} \cdot \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix} = E,$$

where E is an identity matrix.

Due to the condition at Δ (*), we have $\det \left\| C_{ji} \right\| \neq 0, \quad i, j = 1, 2, ..., n$. Let us differentiate (7) by $(n-1)$ times and write the produced proportions for $j = 1, 2, ..., n$ in the following way

$$\begin{pmatrix} \varphi_1(x) & \varphi_2(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & \dots & \varphi_n'(x) \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & \dots & \varphi_n^{(n-1)}(x) \end{pmatrix} = \begin{pmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{pmatrix} \times \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}.$$

It is obvious that Wronskian for $\{\varphi_j(x)\}_1^n$ is different from zero and it proves their linear independence.

Now, let us identify functions $\{z_i(x)\}_1^n$ as a solution to the system of algebraic equations

$$\begin{cases} \varphi_1(x)z_1(x) + \varphi_2(x)z_2(x) + ... + \varphi_n(x)z_n(x) = 0, \\ \varphi_1'(x)z_1(x) + \varphi_2'(x)z_2(x) + ... + \varphi_n'(x)z_n(x) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \varphi_1^{(n-2)}(x)z_1(x) + \varphi_2^{(n-2)}(x)z_2(x) + ... + \varphi_n^{(n-2)}(x)z_n(x) = 0, \\ \varphi_1^{(n-1)}(x)z_1(x) + \varphi_2^{(n-1)}(x)z_2(x) + ... + \varphi_n^{(n-1)}(x)z_n(x) \ = 1. \end{cases} \tag{9}$$

Solving this system by the Cramer method and writing the determinant with the help of elements of the i-column, we have

$$z_i(x) = (-1)^{n+i} \begin{vmatrix} \varphi_1(x) & \dots & \varphi_{i-1}(x) & \varphi_{i+1}(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \dots & \varphi_{i-1}'(x) & \varphi_{i+1}'(x) & \dots & \varphi_n'(x) \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-3)}(x) & \dots & \varphi_{i-1}^{(n-3)}(x) & \varphi_{i+1}^{(n-3)}(x) & \dots & \varphi_n^{(n-3)}(x) \\ \varphi_1^{(n-2)}(x) & \dots & \varphi_{i-1}^{(n-2)}(x) & \varphi_{i+1}^{(n-2)}(x) & \dots & \varphi_n^{(n-2)}(x) \end{vmatrix} e^{\int_{x_i}^{x} b_{n-1}(t)dt}. \tag{10}$$

Lemma 2 *Let $\{\varphi_j(x)\}_1^n$ be a fundamental system of solutions to the equation $Ly = 0$ in Lemma 1, and $\{z_i(x)\}_1^n$ be a system of functions specified by the formula (10). Then:*

a) $\{z_i(x)\}_1^n$ is a fundamental system of solutions of the adjoint differential equation $L^+z = 0 \ \ \forall(x_\mu, x_{\mu+1}), \ \ \mu = 1, 2, ..., n-1$,

b) functions $z_i(x)$comply with the adjoint boundary conditions

$$(T_k^+ z)(x_i) = \sum_{v=0}^{k} (-1)^{k-v} [b_{n-v}(s)z(s)]^{k-v} - \rho_{n-k}(s) \Big|_{s=x_i} = 0, \quad k = 0, 1, ..., n-1, \tag{11}$$

c) the following proportions are true

$$\Phi[\varphi_j(x), z_i(x)] = \delta_{ji} \ \ \forall x \in [x_1, x_n], \quad i, j = 1, 2, ..., n. \tag{12}$$

Proof. Let us show that $L^+z_i(s) = 0$. Applying the rules of product differentiation and determinant by lines, we can find from (10).

$$z_i'(s) = \begin{vmatrix} \varphi_1(s) & \dots & \varphi_{i-1}(s) & \varphi_{i+1}(s) & \dots & \varphi_n(s) \\ \varphi_1'(s) & \dots & \varphi_{i-1}'(s) & \varphi_{i+1}'(s) & \dots & \varphi_n'(s) \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)}(s) & \dots & \varphi_{i-1}^{(n-4)}(s) & \varphi_{i+1}^{(n-4)}(s) & \dots & \varphi_n^{(n-4)}(s) \\ \varphi_1^{(n-3)}(s) & \dots & \varphi_{i-1}^{(n-3)}(s) & \varphi_{i+1}^{(n-3)}(s) & \dots & \varphi_n^{(n-3)}(s) \\ \varphi_1^{(n-1)}(s) & \dots & \varphi_{i-1}^{(n-1)}(s) & \varphi_{i+1}^{(n-1)}(s) & \dots & \varphi_n^{(n-1)}(s) \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt} + b_{n-1}(s)z_i(s),$$

because there will be two similar lines in the other determinants. Let us differentiate again

$$z_i''(s) = \begin{vmatrix} \varphi_1 & \dots & \bullet & \bullet & \dots & \varphi_n \\ \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-4)} \\ \varphi_1^{(n-2)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-2)} \\ \varphi_1^{(n-1)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-1)} \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt} + \begin{vmatrix} \varphi_1 & \dots & \bullet & \bullet & \dots & \varphi_n \\ \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-4)} \\ \varphi_1^{(n-3)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-3)} \\ \varphi_1^{(n)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n)} \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt}$$

$$+b_{n-1}(s)\begin{vmatrix} \varphi_1 & \dots & \bullet & \bullet & \dots & \varphi_n \\ \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-4)} \\ \varphi_1^{(n-3)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-3)} \\ \varphi_1^{(n-1)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-1)} \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt} + [b_{n-1}(s)z_i(s)]'.$$

Let us express the derivatives $\varphi_j^{(n)}(s)$ in the second determinant from $L\varphi_j(s) = 0$:

$$\varphi_j^{(n)}(s) = -b_{n-1}(s)\varphi_j^{(n-1)}(s) - b_{n-2}(s)\varphi_j^{(n-2)}(s) - ... - b_1(s)\varphi_j'(s) - b_0(s)\varphi_j(s). \tag{13}$$

Decomposing the second determinant for the sum and taking into account (10), we have

$$z_i''(s) = \begin{vmatrix} \varphi_1 & \dots & \bullet & \bullet & \dots & \varphi_n \\ \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-4)} \\ \varphi_1^{(n-2)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-2)} \\ \varphi_1^{(n-1)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-1)} \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt} - b_{n-1}(s)\begin{vmatrix} \varphi_1 & \dots & \bullet & \bullet & \dots & \varphi_n \\ \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-4)} \\ \varphi_1^{(n-3)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-3)} \\ \varphi_1^{(n-1)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-1)} \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt}$$

$$-b_{n-2}(s)z_i(s) + b_{n-1}(s)\begin{vmatrix} \varphi_1 & \dots & \bullet & \bullet & \dots & \varphi_n \\ \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-4)} \\ \varphi_1^{(n-3)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-3)} \\ \varphi_1^{(n-1)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-1)} \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt} + [b_{n-1}(s)z_i(s)]'.$$

The second derivative is equal to

$$z_i''(s) = \begin{vmatrix} \varphi_1(s) & \dots & \varphi_{i-1}(s) & \varphi_{i+1}(s) & \dots & \varphi_n(s) \\ \varphi_1'(s) & \dots & \varphi_{i-1}'(s) & \varphi_{i+1}'(s) & \dots & \varphi_n'(s) \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-4)}(s) & \dots & \varphi_{i-1}^{(n-4)}(s) & \varphi_{i+1}^{(n-4)}(s) & \dots & \varphi_n^{(n-4)}(s) \\ \varphi_1^{(n-2)}(s) & \dots & \varphi_{i-1}^{(n-2)}(s) & \varphi_{i+1}^{(n-2)}(s) & \dots & \varphi_n^{(n-2)}(s) \\ \varphi_1^{(n-1)}(s) & \dots & \varphi_{i-1}^{(n-1)}(s) & \varphi_{i+1}^{(n-1)}(s) & \dots & \varphi_n^{(n-1)}(s) \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt} - [b_{n-2}(s)z_i(s)] + [b_{n-1}(s)z_i(s)]'.$$

To avoid cumbersome records, let us specify: $W_{m,n-1}(s)$ is Wronskian for the function $\varphi_1(s), ..., \varphi_{i-1}(s), \varphi_{i+1}(s), ..., \varphi_n(s)$, which elements of the last line are $(n-1)$-derivatives and there is no line with m-derivative; $W_{m,n}(s)$ is the same determinant but elements of the last line are $\varphi_1^{(n)}(s), ..., \varphi_{i-1}^{(n)}(s), \varphi_{i+1}^{(n)}(s), ..., \varphi_n^{(n)}(s)$, and there is no line with $(n-1)$ and m-derivatives. It is not difficult to determine by direct differentiation of the determinants that

$$\frac{d}{ds}W_{m,n-1}(s) = W_{m-1,n-1}(s) + W_{m,n}(s), \quad m = 1, 2, ..., n-2. \tag{a}$$

In the last line $W_{m,n}(s)$, expressing elements $\varphi_j^{(n)}(s)$ from (13) and decomposing it for the sum of determinants, due to absence of the lines with m- and $(n$-1$)$-derivatives, we have

$$W_{m,n}(s) = -b_{n-1}(s)W_{m,n-1}(s) - b_m(s)W_{m,m}(s), \quad m = 0, 1, ..., n-2. \tag{b}$$

Let us displace the elements $\varphi_j^{(m)}(s)$ to the $(m$+1$)$-line by $(n$-2-$m)$ changing of the adjoining lines, then taking into account the proportion (10), we have

$$W_{m,m}(s)e^{\int_{x_i}^{s} b_{n-1}(t)dt} = (-1)^{n-2-m}z_i(s), \quad m = 0, 1, ..., n-2. \tag{c}$$

These derivatives can be represented through the introduced indications as follows

$$z_i'(s) = W_{n-2,n-1}(s)e^{\int_{x_i}^{s} b_{n-1}(t)dt} + [b_{n-1}z_i],$$

$$z_i''(s) = W_{n-3,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} - [b_{n-2}z_i] + [b_{n-1}z_i]'.$$

Let us differentiate the last expression, and, taking into account (a), we have

$$z_i'''(s) = W_{n-4,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + W_{n-3,n}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + b_{n-1}(s)W_{n-3,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} - [b_{n-2}z_i]' + [b_{n-1}z_i]''.$$

Let us express the second determinant from (b)

$$\begin{aligned} z_i'''(s) \quad = \quad & W_{n-4,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} - b_{n-1}(s)W_{n-3,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} - [b_{n-2}z_i]' - \\ & -b_{n-3}(s)W_{n-3,n-3}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + b_{n-1}(s)W_{n-3,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + [b_{n-1}z_i]''. \end{aligned}$$

Annihilating the second and the fifth terms and taking into account (c), we obtain

$$z_i'''(s) = W_{n-4,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + [b_{n-3}z_i] - [b_{n-2}z_i]' + [b_{n-1}z_i]''.$$

Finding z_i^{IV} and so on, let us suppose that the k-derivative is defined by the formula

$$\begin{aligned} z_i^{(k)}(s) = W_{n-k-1,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + (-1)^{k-1}[b_{n-k}z_i] + (-1)^{k-2}[b_{n-k+1}z_i]' + \\ +... - [b_{n-2}z_i]^{(k-2)} + [b_{n-1}z_i]^{(k-1)},\ k = 1, 2, ..., n-1. \end{aligned} \tag{14}$$

Let us show that the formula is true for the (k+1)-derivative. Differentiating (14) and taking into account (a), we have

$$\begin{aligned} z_i^{(k+1)}(s) \quad = \quad & W_{n-k-2,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + W_{n-k-1,n}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + b_{n-1}(s)W_{n-k-1,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} \\ & +(-1)^{k-1}[b_{n-k}z_i]' + (-1)^{k-2}[b_{n-k+1}z_i]'' + +... - [b_{n-2}z_i]^{(k-1)} + [b_{n-1}z_i]^{(k)}. \end{aligned}$$

Let us express the second determinant from (b)

$$\begin{aligned} z_i^{(k+1)}(s) \quad = \quad & W_{n-k-2,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} - b_{n-1}(s)W_{n-k-1,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} - b_{n-k+1}(s)W_{n-k-1,n-k-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + \\ & b_{n-1}(s)W_{n-k-1,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + (-1)^{k-1}[b_{n-k}z_i]' + (-1)^{k-2}[b_{n-k+1}z_i]'' + ... - [b_{n-2}z_i]^{(k-1)} + [b_{n-1}z_i]^{(k)}. \end{aligned}$$

Annihilating the second and the fourth terms and taking into account (c), we obtain that

$$z_i^{(k+1)}(s) = W_{n-k-2,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + (-1)^k[b_{n-k-1}z_i] + (-1)^{k-1}[b_{n-k}z_i]' + (-1)^{k-2}[b_{n-k+1}z_i]'' + ... - [b_{n-2}z_i]^{(k-1)} + [b_{n-1}z_i]^{(k)}.$$

It is easy to notice that this derivative has the same form as the k-derivative. Therefore, we have proved that the formula of the k-derivative is true (14).

Let us write the formula (14) at $k = n-1$:

$$z_i^{(n-1)}(s) = W_{0,n-1}(s)e^{\int_{x_i}^s b_{n-1}(t)dt} + (-1)^{n-2}[b_1z_i] + (-1)^{n-3}[b_2z_i]' + ... - [b_{n-2}z_i]^{(n-3)} + [b_{n-1}z_i]^{(n-2)}$$

or

$$\begin{aligned} z_i^{(n-1)}(s) = & \begin{vmatrix} \varphi_1'(s) & \dots & \varphi_{i-1}'(s) & \varphi_{i+1}'(s) & \dots & \varphi_n'(s) \\ \varphi_1''(s) & \dots & \varphi_{i-1}''(s) & \varphi_{i+1}''(s) & \dots & \varphi_n''(s) \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)}(s) & \dots & \varphi_{i-1}^{(n-2)}(s) & \varphi_{i+1}^{(n-2)}(s) & \dots & \varphi_n^{(n-2)}(s) \\ \varphi_1^{(n-1)}(s) & \dots & \varphi_{i-1}^{(n-1)}(s) & \varphi_{i+1}^{(n-1)}(s) & \dots & \varphi_n^{(n-1)}(s) \end{vmatrix} e^{\int_{x_i}^s b_{n-1}(t)dt} \\ & +(-1)^{n-2}[b_1z_i] + (-1)^{n-3}[b_2z_i]' + ... - [b_{n-2}z_i]^{(n-3)} + [b_{n-1}z_i]^{(n-2)}. \end{aligned} \tag{15}$$

Let us differentiate the expression, so,

$$z_i^{(n)}(s) = \begin{vmatrix} \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \varphi_1'' & \dots & \bullet & \bullet & \dots & \varphi_n'' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-2)} \\ \varphi_1^{(n)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n)} \end{vmatrix} e^{\int_{x_i}^s b_{n-1}(t)dt} + b_{n-1}(s) \begin{vmatrix} \varphi_1' & \dots & \bullet & \bullet & \dots & \varphi_n' \\ \varphi_1'' & \dots & \bullet & \bullet & \dots & \varphi_n'' \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-2)} \\ \varphi_1^{(n-1)} & \dots & \bullet & \bullet & \dots & \varphi_n^{(n-1)} \end{vmatrix} e^{\int_{x_i}^s b_{n-1}(t)dt}$$

$$+(-1)^{n-2}[b_1 z_i]' + (-1)^{n-3}[b_2 z_i]'' + ... - [b_{n-2} z_i]^{(n-2)} + [b_{n-1} z_i]^{(n-1)}.$$

Expressing elements $\varphi_j^{(n)}(s)$ from the homogenous equation (13) and decomposing the first determinant for the sum, due to absence of the lines with elements $\varphi_j^{(n-1)}(s)$ and $\varphi_j(s)$, we have

$$z_i^{(n)}(s) = -b_0(s) \begin{vmatrix} \varphi_1'(s) & \dots & \varphi_{i-1}'(s) & \varphi_{i+1}'(s) & \dots & \varphi_n'(s) \\ \varphi_1''(s) & \dots & \varphi_{i-1}''(s) & \varphi_{i+1}''(s) & \dots & \varphi_n''(s) \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)}(s) & \dots & \varphi_{i-1}^{(n-2)}(s) & \varphi_{i+1}^{(n-2)}(s) & \dots & \varphi_n^{(n-2)}(s) \\ \varphi_1(s) & \dots & \varphi_{i-1}(s) & \varphi_{i+1}(s) & \dots & \varphi_n(s) \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt} +$$

$$+(-1)^{n-2}[b_1 z_i]' + (-1)^{n-3}[b_2 z_i]'' + ... - [b_{n-2} z_i]^{(n-2)} + [b_{n-1} z_i]^{(n-1)}.$$

The first and the third determinants had the opposite observations. Moving the adjoining lines $(n-2)$ times, let us put the last line to the place of the first line and, taking into account (10), we finally find that

$$z_i^{(n)}(s) = (-1)^{n-1}[b_0 z_i] + (-1)^{n-2}[b_1 z_i]' + ... - [b_{n-2} z_i]^{(n-2)} + [b_{n-1} z_i]^{(n-1)}.$$

To prove that $z_i(s)$ are solutions of the adjoint equation, let us multiply the derivative $z_i^{(n)}(s)$ by (-1)^n by terms and transpose terms on the decreasing derivatives, so,

$$(-1)^n z_i^{(n)}(s) = (-1)^n[b_{n-1} z_i]^{(n-1)} + (-1)^{n-1}[b_{n-2} z_i]^{(n-2)} + ... + [b_1 z_i]' - [b_0 z_i].$$

Having set the found expression instead of $(-1)^n z_i^{(n)}(s)$ in the left part of the adjoint operator we see that functions $z_i(s)$ comply with the equation

$$L^+ z(s) = 0.$$

Now, let us identify the expressions regarding the values $z_i(x_i), z_i'(x_i), ..., z_i^{(n-2)}(x_i)$ and find the adjoint boundary conditions for the operatorL^+. Let us represent the function (10) in a form of the n-order determinant

$$z_i(x_i) = \begin{vmatrix} \varphi_1(x_i) & \dots & \varphi_{i-1}(x_i) & \varphi_i(x_i) & \varphi_{i+1}(x_i) & \dots & \varphi_n(x_i) \\ \varphi_1'(x_i) & \dots & \varphi_{i-1}'(x_i) & \varphi_i'(x_i) & \varphi_{i+1}'(x_i) & \dots & \varphi_n'(x_i) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)}(x_i) & \dots & \varphi_{i-1}^{(n-2)}(x_i) & \varphi_i^{(n-2)}(x_i) & \varphi_{i+1}^{(n-2)}(x_i) & \dots & \varphi_n^{(n-2)}(x_i) \\ (T\varphi_1)(x_i) & \dots & (T\varphi_{i-1})(x_i) & (T\varphi_i)(x_i) & (T\varphi_{i+1})(x_i) & \dots & (T\varphi_n)(x_i) \end{vmatrix} e^{\int_{x_i}^{x_i} b_{n-1}(t)dt},$$

where

$$(T\varphi_i)(x_i) = \sum_{\nu=1}^{n} \rho_\nu(x_i)\varphi_i^{(\nu-1)}(x_i) = 1, \quad i = 1, 2, ..., n,$$

$$(T\varphi_j)(x_i) = \sum_{\nu=1}^{n} \rho_\nu(x_i)\varphi_j^{(\nu-1)}(x_i) = 0, \quad i \neq j, \quad j = 1, 2, ..., n$$

due to the boundary conditions (6). It is possible because the elements of the last line consist of zeros, except one that is equal to 1 and located at the intersection the n-line and i-column. Since the form of each element of the last line is congruent, let us decompose $z_i(x_i)$ for the sum of n determinants where (n-1) have the proportional lines. Since the determinant is the Wronskian in point x_i, based on the Lioville-Ostrogradsky formula we have

$$z_i(x_i) = \begin{vmatrix} \varphi_1(x_i) & \dots & \varphi_n(x_i) \\ \varphi_1'(x_i) & \dots & \varphi_n'(x_i) \\ \vdots & & \vdots \\ \varphi_1^{(n-2)}(x_i) & \dots & \varphi_n^{(n-2)}(x_i) \\ \rho_n(x_i)\varphi_1^{(n-1)}(x_i) & \dots & \rho_n(x_i)\varphi_n^{(n-1)}(x_i) \end{vmatrix} e^{\int_{x_i}^{x_i} b_{n-1}(t)dt} = \rho_n(x_i). \qquad (16)$$

Let us find the derivative $z_i'(s)$ at $s = x_i$ and represent it in a form of the n-determinant adding the determinant with elements of the (n-1)-line $\{(T\varphi_j)(x_i)\}_1^n$ and i-column-$\{\varphi_i^{(\chi-1)}(x_i)\}_1^n$.

$$z_i'(x_i) = (-1)\cdot \begin{vmatrix} \varphi_1(x_i) & \dots & \varphi_i(x_i) & \dots & \varphi_n(x_i) \\ \varphi_1'(x_i) & \dots & \varphi_i'(x_i) & \dots & \varphi_n'(x_i) \\ \vdots & & \vdots & & \vdots \\ \varphi_1^{(n-3)}(x_i) & \dots & \varphi_i^{(n-3)}(x_i) & \dots & \varphi_n^{(n-3)}(x_i) \\ (T\varphi_1)(x_i) & \dots & (T\varphi_i)(x_i) & \dots & (T\varphi_n)(x_i) \\ \varphi_1^{(n-1)}(x_i) & \dots & \varphi_i^{(n-1)}(x_i) & \dots & \varphi_n^{(n-1)}(x_i) \end{vmatrix} e^{\int_{x_i}^{x_i} b_{n-1}(t)dt} + b_{n-1}(x_i)z_i(x_i),$$

where

$$(T\varphi_j)(x_i) = \delta_{ji};\ \ i,j = 1,2,...,n.$$

It is possible because elements of the last but one line consist of zero, except one that is equal to 1 and located at the intersection of the (n-1)-line and i-column. Decomposing the determinant for the sum and taking into account absence of the line with derivatives $\varphi_j^{(n-2)}(x_i)$, we produce the following from the formula (10) taking into account a sigh at z_i

$$z_i'(x_i) = -\begin{vmatrix} \varphi_1(x_i) & \dots & \varphi_n(x_i) \\ \varphi_1'(x_i) & \dots & \varphi_n'(x_i) \\ \vdots & & \vdots \\ \rho_{n-1}(x_i)\varphi_1^{(n-2)}(x_i) & \dots & \rho_{n-1}(x_i)\varphi_n^{(n-2)}(x_i) \\ \varphi_1^{(n-1)}(x_i) & \dots & \varphi_n^{(n-1)}(x_i) \end{vmatrix} e^{\int_{x_i}^{x_i} b_{n-1}(t)dt} + b_{n-1}(x_i)z_i(x_i).$$

Factoring out the coefficient $\rho_{n-1}(x_i)$, we obtain

$$z_i'(s) - [b_{n-1}z_i] + \rho_{n-1}(s)\Big|_{s=x_i} = 0.$$

Similarly, let us represent $z_i''(x_i)$ in a form of the n-determinant adding the i-column and the (n−2)-line which elements are the boundary conditions $(T\varphi_j)(x_i)$. Decomposing it for the sum of the determinants and taking into account absence of the line with $\varphi_j^{(n-3)}(x_i)$, we have

$$z_i''(x_i) = (-1)^2 \begin{vmatrix} \varphi_1(x_i) & \dots & \varphi_n(x_i) \\ \vdots & & \vdots \\ \rho_{n-2}(x_i)\varphi_1^{(n-3)}(x_i) & \dots & \rho_{n-2}(x_i)\varphi_n^{(n-3)}(x_i) \\ \varphi_1^{(n-2)}(x_i) & \dots & \varphi_n^{(n-2)}(x_i) \\ \varphi_1^{(n-1)}(x_i) & \dots & \varphi_n^{(n-1)}(x_i) \end{vmatrix} e^{\int_{x_i}^{x_i} b_{n-1}(t)dt} - [b_{n-2}z_i]\Big|_{s=x_i} + [b_{n-1}z_i]'\Big|_{s=x_i}$$

or

$$z_i''(s) - [b_{n-1}z_i]' + [b_{n-2}z_i] - (-1)^2\rho_{n-2}(s)\Big|_{s=x_i} = 0.$$

Let us consider the value of the (n−1)-determinant $W_{n-k-1,n-1}(s)$ in the point x_i, there is no line with derivatives $\varphi_j^{(n-k-1)}(x_i)$. Let us represent the boundary conditions $(T\varphi_j)(x_i)$ and $\{\varphi_i^{(\chi-1)}(x_i)\}_1^n$ as the elements of the $(n-k)$-line and the i-column correspondingly completing the determinant to the n-degree. Let us decompose this determinant for the sum of the determinants where all of them except one have the proportional lines. Since there is no line with elements $\varphi_j^{(n-k-1)}(x_i)$, we have the following factoring the coefficient $\rho_{n-k}(x_i)$ out of the determinant and taking into account the sign of the formula (10)

$$W_{n-k-1,n-1}(x_i) = (-1)^k\rho_{n-k}(x_i)W(x_i), \tag{d}$$

where $W(x_i)$is the Wronskian value in the point x_i.

Let us find $z_i^{(k)}(x_i)$ from the formula (14) at $s = x_i$, taking into account the produced proportion (d) and the Lioville-Ostrogradsky formula:

$$z_i^{(k)}(s) - [b_{n-1}z_i]^{(k-1)} + [b_{n-2}z_i]^{(k-2)} + ... + (-1)^{k-1}[b_{n-k+1}z_i]' + (-1)^k[b_{n-k}z_i] - (-1)^k\rho_{n-k}(s)\Big|_{s=x_i} = 0,$$

$$k = 1,2,...,n-1.$$

To find the adjoint boundary conditions $(T^+z)(x_i)$, let us set $s = x_i$ in the formula (15), as the determinant with the first line and the i-column which elements are $(T\varphi_j)(x_i)$ and $\{\varphi_i^{(\chi-1)}(x_i)\}_1^n$ respectively.

Then we have

$$z_i^{(n-1)}(x_i) = (-1)^{n-1} \begin{vmatrix} (T\varphi_1)(x_i) & \dots & (T\varphi_i)(x_i) & \dots & (T\varphi_n)(x_i) \\ \varphi_1'(x_i) & \dots & \varphi_i'(x_i) & \dots & \varphi_n'(x_i) \\ \vdots & & \vdots & & \vdots \\ \varphi_1^{(n-1)}(x_i) & \dots & \varphi_i^{(n-1)}(x_i) & \dots & \varphi_n^{(n-1)}(x_i) \end{vmatrix} e^{\int_{x_i}^{x_i} b_{n-1}(t)dt} +$$

$$+ \left\{ (-1)^{n-2}[b_1 z_i] + (-1)^{n-3}[b_2 z_i]' + ... - [b_{n-2} z_i]^{(n-3)} + [b_{n-1} z_i]^{(n-2)} \right\} \Big|_{s=x_i} .$$

Decomposing the determinant for the sum of n determinants, due to absence of the line with elements $\varphi_j(x_i)$ and based on the known Lioville-Ostrogradsky formula, we finally obtain

$$z_i^{(n-1)}(s) - [b_{n-1} z_i]^{(n-2)} + ... + (-1)^{n-2}[b_2 z_i]' + (-1)^{n-1}[b_1 z_i] - (-1)^{n-1}\rho_1(s) \Big|_{s=x_i} = 0.$$

Let us multiply by terms the derivatives $z_i^{(k)}(x_i)$ and $z_i^{(n-1)}(x_i)$ by $(-1)^k$ and $(-1)^{n-1}$ respectively. Then considering evenness and oddness of the degrees, we have

$$(-1)^k z_i^{(k)}(s) + (-1)^{k-1}[b_{n-1} z_i]^{(k-1)} + ... - [b_{n-k+1} z_i]' + [b_{n-k} z_i] - \rho_{n-k}(s) \Big|_{s=x_i} = 0, \tag{17}$$

$k = 0, 1, \dots, n-1$, or in a form of the sum (11) at $z = z_i(s)$

$$\sum_{\nu=0}^{k} (-1)^{k-\nu}[b_{n-\nu}(s) z_i(s)]^{(k-\nu)} - \rho_{n-k}(s) \Big|_{s=x_i} = 0, \quad (11*)$$

$k = 0, 1, .., n-1$; $b_n(x) \equiv 1$, at $k = n-1$

$$(-1)^{n-1} z^{(n-1)}(s) + (-1)^{n-2}[b_{n-1} z]^{(n-2)} + ... - [b_2 z]' + [b_1 z] - \rho_1(s) \Big|_{s=x_i} = 0. \tag{18}$$

Proportion (11) is the adjoint boundary conditions $(T^+ z)(x_i)$ which the solutions of the differential equation $L^+ z = 0$ comply with.

Let us prove the second part of lemma. Being a bilinear form with respect to the derivatives $\varphi_j(x), \varphi_j'(x), ..., \varphi_j^{(n-1)}(x)$ and $z_i(x), z_i'(x), ..., z_i^{(n-1)}(x)$, expression (5) is equal to the constant value

$$\Phi[\varphi_j(x), z_i(x)] = const \quad \forall x \in [x_1, x_n].$$

Actually, as we can see from the Lagrange identity, its derivative is equal to zero

$$\frac{d}{dx}\Phi[\varphi_j(x), z_i(x)] = z_i(x) L\varphi_j - \varphi_j(x) L^+ z_i = 0,$$

Because $L\varphi_j(x) = 0$ and $L^+ z_i(x) = 0$. Therefore, to prove the proportion (12) it is sufficient to show that the following is true at the point $x_i \in [x_1, x_n]$

$$\Phi[\varphi_j(x_i), z_i(x_i)] = \delta_{ji}; \quad i, j = 1, 2, ..., n. (12*)$$

Let us write the bilinear form (5) for the functions $\varphi_j(x)$ and $z_i(x)$ at the point $x = x_i$

$$\Phi[\varphi_j(x_i), z_i(x_i)] = z(x_i) \sum_{\nu=1}^{n} b_\nu(x_i) \varphi_j^{(\nu-1)}(x_i) + \varphi_j(x_i) \left(\sum_{\nu=2}^{n} (-1)^{\nu-1}[b_\nu z_i]^{(\nu-1)} \Big|_{x=x_i} \right)$$

$$+\varphi_j'(x_i) \left(\sum_{\nu=3}^{n} (-1)^{\nu-2}[b_\nu z_i]^{(\nu-2)} \Big|_{x=x_i} \right) + ... + \varphi_j^{(n-3)}(x_i) \left(z_i''(x_i) + [b_{n-1} z_i]' \Big|_{x=x_i} \right) - \varphi_j^{(n-2)}(x_i) z_i(x_i).$$

Let us substitute the expressions in brackets for $\varphi_j(x_i), \varphi_j'(x_i), ..., \varphi_j^{(n-2)}(x_i)$ from the adjoint condition (18) and corresponding point proportions (17) for $k = n-2, n-3, ..., 2, 1$, so,

$$\begin{aligned} \Phi[\varphi_j(x_i), z_i(x_i)] &= z_i(x_i) \sum_{\nu=1}^{n} b_\nu(x_i) \varphi_j^{(\nu-1)}(x_i) + \varphi_j(x_i)[\rho_1(x_i) - b_1(x_i) z_i(x_i)] + \varphi_j'(x_i)[\rho_2(x_i) - b_2(x_i) z_i(x_i)] \\ &+ ... + \varphi_j^{(n-3)}(x_i)[\rho_{n-2}(x_i) - b_{n-2}(x_i) z_i(x_i)] + \varphi_j^{(n-2)}(x_i)[\rho_{n-1}(x_i) - b_{n-1}(x_i) z_i(x_i)]. \end{aligned}$$

Factoring $z_i(x_i)$ out the bracket, collecting similar terms and taking into account that $z_i(x_i) = \rho_n(x_i)$, we have

$$\Phi[\varphi_j(x_i), z_i(x_i)] = \rho_n(x_i)\varphi_j^{(n-1)}(x_i) + \rho_{n-1}(x_i)\varphi_j^{(n-2)}(x_i) + \rho_{n-2}(x_i)\varphi_j^{(n-3)}(x_i) + ... + \rho_2(x_i)\varphi_j'(x_i) + \rho_1(x_i)z_i(x_i).$$

Or due to the boundary conditions we get

$$\Phi[\varphi_j(x_i), z_i(x_i)] = (T\varphi_j)(x_i) = \delta_{ji},$$

which demonstrates that (12^*) is true, and thereby, the proportion (12) is true as well. Lemma is completely proved.

It results from execution of the equality

$$\Phi[\varphi_j(x_i), z_i(x_i)] = 0 \quad \forall x \in [x_1, x_n] \text{ at } j \neq i$$

that family of functions $\{\varphi_j(x)\}_1^n$ and $\{z_i(x)\}_1^n$ at $j \neq i$ are adjoint.

Note 1. It is interesting when coefficients of the operator T comply with the proportions

$$\rho_\nu(x) = \alpha_\nu(x)b_\nu(x), \quad b_n(x) = 1 \ \forall x \in [x_1, x_n], \ \alpha_\nu(x) \in C[x_1, x_n]. \tag{19}$$

Hence, the boundary conditions (3) are

$$(Ty)(x_i) = \sum_{\nu=1}^{n} \alpha_\nu(x_i)b_\nu(x_i)y^{(\nu-1)}(x_i) = 0, \tag{4'}$$

and

$$\sum_{\nu=1}^{n} |\alpha_\nu(x_i)|\,|b_\nu(x_i)| \neq 0.$$

And the adjoint boundary conditions (11) are specified

$$(-1)^k z^{(k)}(s) + (-1)^{k-1}[b_{n-1}z]^{(k-1)} + ... + [b_{n-k}z] - \alpha_{n-k}(s)b_{n-k}(s)\big|_{s=x_i} = 0. \tag{11'}$$

Let us consider special cases.

A. $\alpha_\nu(x) = const, \quad \nu = 1, 2, ..., n$. Then the boundary conditions (3) are

$$(Ty)(x_i) = \sum_{\nu=1}^{n} \alpha_\nu b_\nu(x_i)y^{(\nu-1)}(x_i) = 0, \tag{3^a}$$

and

$$\sum_{\nu=1}^{n} |\alpha_\nu|\,|b_\nu(x_i)| \neq 0, \quad b_n(x) \equiv 1.$$

And the adjoint boundary conditions are specified

$$\sum_{\nu=0}^{k}(-1)^{k-\nu}[b_{n-\nu}(s)z(s)]^{(k-\nu)} - \alpha_{n-k}b_{n-k}(s)\big|_{s=x_i} = 0. \tag{11^a}$$

In the boundary conditions (3^a), the constants $\alpha_\nu$can be represented as equal to zero or to 1.

B. Let $\rho_\nu(x_i) = \alpha_{\nu i}b_\nu(x_i)$. Then the operator T represents the point operator T_i, i.e. the separate operator is set in each point and they can be congruent in some points. In this case, the boundary conditions (3) are met

$$(T_i y)(x_i) = \sum_{\nu=1}^{n} \alpha_{\nu i}b_\nu(x_i)y^{(\nu-1)}(x_i) = 0, \tag{3'}$$

and $\sum_{\nu=1}^{n} |\alpha_{\nu i}|\,|b_\nu(x_i)| \neq 0, \quad b_n(x) \equiv 1$.

It should be noted, that these conditions are similar to the conditions (3) but the difference is that in the first case the functions $\alpha_\nu(x)$ are set, but here the $n \times n$ matrix of the constant numbers $\|\alpha_{\nu i}\|$ is set. So, the adjoint boundary conditions are specified

$$\sum_{\nu=0}^{k}(-1)^{k-\nu}[b_{n-\nu}(s)z(s)]^{(k-\nu)} - \alpha_{n-k,i}b_{n-k}(s)\big|_{s=x_i} = 0. \tag{11'}$$

All specified above statements and connections are true for the described cases as well.

Note 2. Let us find the corresponding nonlinear differential equation of the Riccati type for the adjoint differential equation

$$L^{+}z = \sum_{i=0}^{n}(-1)^{i}[b_i(x)z]^{(i)} = 0,$$

where $b_i(x) \in C^i[x_1, x_n]$, $i = 0, 1, 2, ..., n-1$; $b_n(x) \equiv 1$.

Similarly to the linear differential equation $Ly = 0$, we represent the solution as follows $z(x) = e^{\int_{x_0}^{x} u(t)dt}$.

Let us use a formula of the n-derivative of the exponent and the product function, then we have

$$L^{+}z = \sum_{i=0}^{n}(-1)^{i}p^{i}[z(x)b_i(x)] = \sum_{i=0}^{n}(-1)^{i}p^{i}e^{\int_{x_0}^{x} u(t)dt}b_i(x) = e^{\int_{x_0}^{x} u(t)dt} \cdot \sum_{i=0}^{n}(-1)^{i}[p + u(x)]^{i}b_i(x).$$

Therefore, we will produce the adjoint characteristic of the $(n-1)$-th order equation of the Riccati type

$$\sum_{i=0}^{n}(-1)^{i}[p + u(x)]^{i}b_i(x) = 0, \tag{20}$$

here

$$(p + u)b_i(x) = b_i(x)u + b_i'(x), (p + u)b_n(x) = (p + u)1 = u(x).$$

If $u_i(x)$ are the impaired solutions around points Σ of the adjoint characteristic equation of the Riccati type (20) and $D[u(x)] \neq 0$ $\forall x \in \Omega = [x_1, x_n] \setminus \sum$, then the functions

$$z_i(x) = e^{\int_{x_0}^{x} u_i(t)dt}, x_0 \in \Omega$$

form a fundamental system of solutions of the adjoint differential equation (4), and the inverse statement is true as well.

Thus, to find a fundamental system of solutions $L^{+}z = 0$, it is sufficient to find the impaired solutions to the adjoint characteristic Equation (20). But to avoid cumbersome records, we express solutions of the n-point boundary problem through the fundamental system of solutions of the linear differential equation $\{\varphi_j(x)\}_1^n$ and $\{z_i(x)\}_1^n$ [13].

3. Solution to the n-point Boundary Value Problem

Let us assume, we need to solve the nonhomogenous boundary value n-point problem

$$Ly = f(x), \tag{21}$$

where $f(x) \in C[x_1, x_n]$, $b_{\nu-1}(x) \in C^{\nu-1}[x_1, x_n], \nu = 1, 2, ..., n,$

$$(Ty)(x_i) = \sum_{\nu=1}^{n}\rho_\nu(x_i)y^{(\nu-1)}(x_i) = a_i. \tag{22}$$

Regarding the problem it is true.

Theorem 1 *Let $\{\varphi_j(x)\}_1^n$ and $\{z_i(x)\}_1^n$ be the systems of functions specified in Lemmas 1 and 2. Then there is the only solution to the nonhomogenous problem (21), (22)*

$$y(x) = \sum_{i=1}^{n} a_i\varphi_i(x) + \sum_{i=1}^{n}\varphi_i(x)\int_{x_i}^{x} f(s)z_i(s)ds. \tag{23}$$

Proof. Let us find the derivative

$$y' = \sum_{i=1}^{n} a_i\varphi_i'(x) + \sum_{i=1}^{n}\varphi_i'(x)\int_{x_i}^{x} f(s)z_i(s)ds + f(x)\sum_{i=1}^{n}\varphi_i(x)z_i(x).$$

Since functions $\{z_i(x)\}_1^n$ are defined as solutions to the system of the algebraic Equations (9), the last sum is equal to zero due to the first equation of this system, so

$$y' = \sum_{i=1}^{n} a_i\varphi_i'(x) + \sum_{i=1}^{n} \varphi_i'(x) \int_{x_i}^{x} f(s)z_i(s)ds,$$

Taking into account the second equation, the second derivative considering the second Equation (9) is equal to

$$y'' = \sum_{i=1}^{n} a_i\varphi_i''(x) + \sum_{i=1}^{n} \varphi_i''(x) \int_{x_i}^{x} f(s)z_i(s)ds.$$

Similarly, at repeated differentiation $(n-2)$ times and considering the last but one Equation (9), we have

$$y^{(n-1)} = \sum_{i=1}^{n} a_i\varphi_i^{(n-1)}(x) + \sum_{i=1}^{n} \varphi_i^{(n-1)}(x) \int_{x_i}^{x} f(s)z_i(s)ds.$$

Let us differentiate again

$$y^{(n)} = \sum_{i=1}^{n} a_i\varphi_i^{(n)}(x) + \sum_{i=1}^{n} \varphi_i^{(n)}(x) \int_{x_i}^{x} f(s)z_i(s)ds + f(x)\sum_{i=1}^{n} \varphi_i^{(n-1)}(x)z_i(x).$$

Here the third sum is the last equation of the system (9) and is equal to 1, therefore,

$$y^{(n)} = \sum_{i=1}^{n} a_i\varphi_i^{(n)}(x) + \sum_{i=1}^{n} \varphi_i^{(n)}(x) \int_{x_i}^{x} f(s)z_i(s)ds + f(x).$$

Setting the derivatives in the left part of the nonhomogenous Equation (21), we have

$$Ly = y^{(n)} + b_{n-1}(x)y^{(n-1)} + ... + b_2(x)y'' + b_1(x)y' + b_0(x)y = \sum_{i=1}^{n} a_iL\varphi_i(x) + \sum_{i=1}^{n} L\varphi_i(x) \int_{x_i}^{x} f(s)z_i(s)ds + f(x) \equiv f(x),$$

because $\{\varphi_i(x)\}_1^n$ are solutions of the homogenous equation and $L\varphi_i(x) = 0$. Therefore, the function $y(x)$ assigned by the formula (23) is a solution of the nonhomogenous differential equation (21).

Now, let us show that the solution (23) complies with the boundary conditions (22). Let us apply the operator of the boundary conditions to the function (23)

$$(Ty)(x) = \rho_n(x)y^{(n-1)} + \rho_{n-1}(x)y^{(n-2)} + ... + \rho_2(x)y' + \rho_1(x)y = \sum_{i=1}^{n} a_i(T\varphi_i)(x) + \sum_{i=1}^{n} (T\varphi_i)(x) \int_{x_i}^{x} f(s)z_i(s)ds.$$

Let us consider this expression in points x_j, $j = 1, 2, ..., n$:

$$(Ty)(x_i) = \sum_{i=1}^{n} a_i(T\varphi_i)(x_j) + \sum_{i=1}^{n} (T\varphi_i)(x_j) \int_{x_i}^{x_j} f(s)z_i(s)ds. \tag{24}$$

When $j \neq i$, $(T\varphi_i)(x_j) = 0$, under the boundary conditions (6) and consequently the terms of the sum will remain in the right part only at $j = i$, i.e.

$$(Ty)(x_i) = a_i(T\varphi_i)(x_i) + (T\varphi_i)(x_i) \int_{x_i}^{x_i} f(s)z_i(s)ds.$$

Since $(T\varphi_i)(x_i) = 1$ under (6) and an integral with identical integration limits is equal to zero, the second addend is transformed to zero as well. So, we have $(Ty)(x_i) = a_i$, that is congruent with the boundary conditions (22). Theorem is proved.

Now, let us produce a formula of the solution to the nonhomogenous n-point problem through the preset fundamental system of solutions $y_1(x), y_2(x), ..., y_n(x)$ of the homogenous equation $Ly = 0$. This formula can be useful for practical use.

It is well-known that a partial solution of the nonhomogenous Equation (21) is represented through the Cauchy kernel (Yerugin, 1974; Beesack, 1962):

$$y_*(x) = \int_{x_i}^{x} f(s)\frac{W_n(x,s)}{W(s)}ds,$$

where the determinant $W_n(x,s)$ is produced from the Wronskian $W(s)$ by substituting the last line for the functions $y_1(x), y_2(x), ..., y_n(x)$. Similarly to the Householder method (Householder, 1956), let us take in the last line instead of solutions $\{y_i(x)\}_1^n$ their linear combination, namely functions $\{\varphi_i(x)\}_1^n$. So, decomposing the determinant by elements of the n-line, we get

$$\frac{W_n(x,s)}{W(s)} = \sum_{i=1}^{n} A_{ni}(s)\varphi_i(x), \tag{25}$$

where $A_{ni}(s)$ are the algebraic supplements for the i-element of the i-line. Let us apply an operator of the boundary conditions T to the both parts (25) on the variable x

$$\frac{(TW_n)(x,s)}{W(s)} = \sum_{i=1}^{n} A_{ni}(s)(T\varphi_i)(x).$$

As conditions are met

$$(T\varphi_i)(x_j) = \delta_{ij}$$

Let us find values of the Cauchy kernel in the points x_j

$$\frac{(TW_n)(x_j,s)}{W(s)} = \sum_{i=1}^{n} A_{ni}(s)(T\varphi_i)(x_j) = A_{nj}(s).$$

According to the determinant and linearity of the operator T, substituting the index j for i, gives us

$$A_{ni}(s) = \frac{W_n[(Ty_k)(x_i), s]}{W(s)}; \quad k, i = 1, 2, ..., n.$$

Setting the algebraic supplements in (25), we have

$$\frac{W_n(x,s)}{W(s)} = \sum_{i=1}^{n} \varphi_i(x)\frac{W_n[(Ty_k)(x_i), s]}{W(s)}.$$

It means that

$$y_*(x) = \sum_{i=1}^{n} \varphi_i(x)\int_{x_i}^{x} f(s)\frac{W_n[(Ty_k)(x_i), s]}{W(s)}ds,$$

which complies with zero boundary conditions $(Ty_*)(x_i) = 0$.

Thus, comparing with the second addend (23), we can find that

$$z_i(s) = \frac{W_n[(Ty_k)(x_i), s]}{W(s)}; \quad k, i = 1, 2, ..., n, \tag{26}$$

and values of the points $(Ty_k)(x_i)$ for the fundamental system of solutions $y_1(x), y_2(x), ..., y_n(x)$ are located in the numerator's last line instead of $\{y_k(x)\}_1^n$. It should be noted that these functions $z_i(s)$ have properties proved in Lemma 2. Therefore, the formula of solution (23) of the nonhomogenous problem taking into account (26) can be represented as follows

$$y(x) = \sum_{i=1}^{n} a_i\varphi_i(x) + \sum_{i=1}^{n} \varphi_i(x)\int_{x_i}^{x} f(s)\frac{W_n[(Ty_k)(x_i), s]}{W(s)}ds. \tag{27}$$

4. Green's Function and Its New Properties

There is the following definition of the Green's function (Yerugin, 1974; Kiguradze, 1987; Peterson, 1979; Jackson, 1977).

Definition. Green's function for the operator Ly and nonhomogenous boundary conditions

$$(Ty)(x_i) = a_i, \quad i = 1, 2, ..., n$$

is a function of two variables $G(x, s)$, complying with the following conditions

1) Derivatives $\dfrac{\partial^r G(x, s)}{\partial x^r}$, $\quad r = 0, 1, 2, ..., n-2$ are continuous on the strength of all the variables x, s for the entire domain $x_1 \le x, s \le x_n$ except the lines $s = x_\mu, \quad \mu = 2, 3, ..., n-1$.

2) Derivative $\dfrac{\partial^{n-1} G(x, s)}{\partial x^{n-1}}$ is continuous on the variables x, s at $s \neq x_\mu$. Besides, function $G(x, s)$ and its derivatives on x to $(n-2)$-order at $x = s$ are continuous , and $(n-1)$-derivative has a saltus equal to 1, i.e.

$$\begin{gathered}
G(s+0, s) - G(s-0, s) = 0,\\
\frac{\partial G(s+0, s)}{\partial x} - \frac{\partial G(s-0, s)}{\partial x} = 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{\partial^{n-2} G(s+0, s)}{\partial x^{n-2}} - \frac{\partial^{n-2} G(s-0, s)}{\partial x^{n-2}} = 0,\\
\frac{\partial^{n-1} G(s+0, s)}{\partial x^{n-1}} - \frac{\partial^{n-1} G(s-0, s)}{\partial x^{n-1}} = 1,
\end{gathered} \tag{28}$$

3) $G(x, s)$ at the variable x complies with the homogenous equation $LG(x, s) = 0$ at $s \neq x_\mu$, $s \neq x$.

4) At $s \neq x_\mu$, $G(x, s)$ complies with the boundary conditions $(TG)(x_i, s) = 0, \quad i = 1, 2, ..., n$.

Let us show that the Green's function for n-point boundary value problem exists and it can be helpful to solve the nonhomogenous problem (21), (22).

Let us consider the function on $x_\mu \le x \le x_{\mu+1}, \quad \mu = 1, 2, ..., n-1$

$$G(x, s) = \begin{cases}
\varphi_1(x) z_1(s), & x_1 \le s < x_2,\\
\cdots\cdots\cdots & \cdots\cdots\\
\sum_{l=1}^{\mu-1} \varphi_l(x) z_l(s), & x_{\mu-1} \le s < x_\mu,\\
\sum_{l=1}^{\mu} \varphi_l(x) z_l(s), & x_\mu \le s \le x,\\
-\sum_{l=\mu+1}^{n} \varphi_l(x) z_l(s), & x \le s \le x_{\mu+1},\\
-\sum_{l=\mu+2}^{n} \varphi_l(x) z_l(s), & x_{\mu+1} < s \le x_{\mu+2},\\
\cdots\cdots\cdots & \cdots\cdots\\
-\varphi_n(x) z_n(s), & x_{n-1} < s \le x_n,
\end{cases} \tag{29}$$

where the linearly independent functions $\varphi_j(x)$ comply with the homogenous equation $Ly = 0$ and boundary conditions $(T\varphi_j)(x_i) = \delta_{ji}$, and $z_i(s)$ comply with the adjoint equation $L^+ z(s) = 0$ and adjoint boundary conditions (11). On the issue of the problem, it is true

Theorem 1 *Let coefficients of the Equation (21) $b_{\nu-1}(x) \in C^{\nu-1}[x_1, x_n]$, $\nu = 1, 2, ..., n$, $f(x)$ be uninterrupted on $[x_1, x_n]$ and $\{\varphi_j(x)\}_1^n, \{z_i(x)\}_1^n$ are systems of functions specified in Lemma 1 and 2. Then $G(x, s)$ (29) is a Green's function for the n-point problem (21), (22).*

Proof. We will try to find the Green's function for the n-point boundary value problem in a form for $x_\mu \le x \le x_{\mu+1}, \mu = 1, 2, ..., n-1$:

$$G(x, s) = \begin{cases}
\varphi_1(x) \psi_1(s), & x_1 \le s < x_2,\\
\cdots\cdots\cdots & \cdots\cdots\\
\sum_{l=1}^{\mu-1} \varphi_l(x) \psi_l(s), & x_{\mu-1} \le s < x_\mu,\\
\sum_{l=1}^{\mu} \varphi_l(x) \psi_l(s), & x_\mu \le s \le x,\\
-\sum_{l=\mu+1}^{n} \varphi_l(x) \psi_l(s), & x \le s \le x_{\mu+1},\\
-\sum_{l=\mu+2}^{n} \varphi_l(x) \psi_l(s), & x_{\mu+1} \le s \le x_{\mu+2},\\
\cdots\cdots\cdots & \cdots\cdots\\
-\varphi_n(x) \psi_n(s), & x_{n-1} \le s \le x_n,
\end{cases} \tag{30}$$

where $\varphi_j(x)$ are the linearly independent functions complying with the theorem conditions.

Let us select the unknown functions $\psi_i(s)$ to fulfill the second requirement of the Green's function definition. It is easy to notice that the first proportion for $x_\mu \leq x \leq x_{\mu+1}$ will be as follows

$$G(s+0,s) - G(s-0,s) = \varphi_1(s)\psi_1(s) + \varphi_2(s)\psi_2(s) + ... + \varphi_n(s)\psi_n(s) = 0.$$

We have the similar expression for the other vertical strips$x_\nu \leq x \leq x_{\nu+1}$. Thus, having written the second requirement of the Green's function definition (28) for all strips $x_\mu \leq x \leq x_{\mu+1}$, we have the same system of the linear equations regarding the unknown functions $\psi_i(s)$:

$$\begin{cases} \varphi_1(s)\psi_1(s) + \varphi_2(s)\psi_2(s) + ... + \varphi_n(s)\psi_n(s) = 0, \\ \varphi_1'(s)\psi_1(s) + \varphi_2'(s)\psi_2(s) + ... + \varphi_n'(s)\psi_n(s) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \varphi_1^{(n-2)}(s)\psi_1(s) + \varphi_2^{(n-2)}(s)\psi_2(s) + ... + \varphi_n^{(n-2)}(s)\psi_n(s) = 0, \\ \varphi_1^{(n-1)}(s)\psi_1(s) + \varphi_2^{(n-1)}(s)\psi_2(s) + ... + \varphi_n^{(n-1)}(s)\psi_n(s) = 1. \end{cases}$$

By solving a system of the algebraic equations by the Cramer method, we can find that

$$\psi_i(s) = (-1)^{n+i} \begin{vmatrix} \varphi_1(s) & \dots & \varphi_{i-1}(s) & \varphi_{i+1}(s) & \dots & \varphi_n(s) \\ \varphi_1'(s) & \dots & \varphi_{i-1}'(s) & \varphi_{i+1}'(s) & \dots & \varphi_n'(s) \\ \vdots & & \vdots & \vdots & & \vdots \\ \varphi_1^{(n-2)}(s) & \dots & \varphi_{i-1}^{(n-2)}(s) & \varphi_{i+1}^{(n-2)}(s) & \dots & \varphi_n^{(n-2)}(s) \end{vmatrix} e^{\int_{x_i}^{s} b_{n-1}(t)dt}.$$

Comparing these functions with (10), we can see that $\psi_i(s) = z_i(s)$.

Fulfillment of the third requirement of the definition is obvious from the formula (30). Let us show that the function (30) complies with the fourth requirement as well. Let us apply the operator of the boundary conditions T to the function $G(x,s)$ and set $x = x_i$(set $\mu = i$ in (30). So, the sum at $x_i < s \leq x$will disappear in representation of the function (30) because an interval comes to the point $x_i < s \leq x_i$.

Taking into account the linearity of the operator T, we have

$$(TG)(x_i,s) = \begin{cases} z_1(s)(T\varphi_1)(x_i), & x_1 \leq s < x_2, \\ \dots\dots\dots & \dots\dots \\ \sum_{l=1}^{i-1} z_l(s)(T\varphi_l)(x_i), & x_{i-1} \leq s \leq x_i, \\ -\sum_{l=i+1}^{n} z_l(s)(T\varphi_l)(x_i), & x_i \leq s \leq x_{i+1}, \\ \dots\dots\dots & \dots\dots \\ -z_n(s)(T\varphi_n)(x_i), & x_{n-1} < s \leq x_n. \end{cases}$$

Since $l \neq i$, all sums in the right part transform to zero based on (6). Thus, $(TG)(x_i,s) = 0,\ s \neq x_\mu$. Therefore, the function created in a form (29) is a Green's function for the n-point boundary value problem (21), (22).

Let us note that $G(x,s)$ is congruent on structure with the Green's function created in (Kiguradze, 1987; Jackson, 1977; Levin, 1961). The difference is that it is represented in the horizontal stripes $x_\mu \leq s \leq x_{\mu+1}$, but we do it in vertical strips - $x_\mu \leq x \leq x_{\mu+1}$. Solutions to the adjoint problems are used in a form (29) as well.

In our opinion, it is interesting to look at the diagram of the Green's function assignment.

A picture of the area at $n = 3$ is provided in (Davidson & Rynne, 2007), we then provide it for $n = 5$.

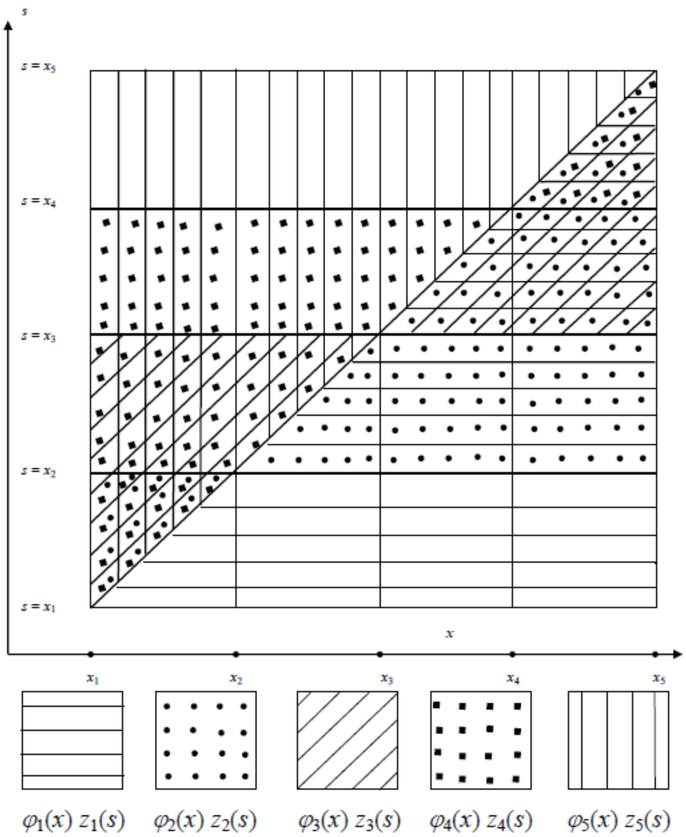

Figure 1. Green's function area for $n = 5$

Green's function on the vertical strips $E_\mu \le E \le E_{\mu+1}, \mu = 1, 2, 3, 4$:

$x_1 \le x \le x_2$,

$$G(x, s) = \begin{cases} \varphi_1(x)z_1(s), & x_1 \le s \le x, \\ -\varphi_2(x)z_2(s) - \varphi_3(x)z_3(s) - \varphi_4(x)z_4(s) - \varphi_5(x)z_5(s), & x \le s \le x_2, \\ -\varphi_3(x)z_3(s) - \varphi_4(x)z_4(s) - \varphi_5(x)z_5(s), & x_2 < s \le x_3, \\ -\varphi_4(x)z_4(s) - \varphi_5(x)z_5(s), & x_3 < s \le x_4, \\ -\varphi_5(x)z_5(s), & x_4 < s \le x_5. \end{cases}$$

$x_2 \le x \le x_3$,

$$G(x, s) = \begin{cases} \varphi_1(x)z_1(s), & x_1 \le s < x_2, \\ \varphi_1(x)z_1(s) + \varphi_2(x)z_2(s), & x_2 \le s \le x, \\ -\varphi_3(x)z_3(s) - \varphi_4(x)z_4(s) - \varphi_5(x)z_5(s), & x \le s \le x_3, \\ -\varphi_4(x)z_4(s) - \varphi_5(x)z_5(s), & x_3 < s \le x_4, \\ -\varphi_5(x)z_5(s), & x_4 < s \le x_5. \end{cases}$$

$x_3 \le x \le x_4$,

$$G(x, s) = \begin{cases} \varphi_1(x)z_1(s), & x_1 \le s < x_2, \\ \varphi_1(x)z_1(s) + \varphi_2(x)z_2(s), & x_2 \le s < x_3, \\ \varphi_1(x)z_1(s) + \varphi_2(x)z_2(s) + \varphi_3(x)z_3(s), & x_3 \le s \le x, \\ -\varphi_4(x)z_4(s) - \varphi_5(x)z_5(s), & x \le s \le x_4, \\ -\varphi_5(x)z_5(s), & x_4 < s \le x_5. \end{cases}$$

$x_4 \le x \le x_5$,

$$G(x, s) = \begin{cases} \varphi_1(x)z_1(s), & x_1 \le s < x_2, \\ \varphi_1(x)z_1(s) + \varphi_2(x)z_2(s), & x_2 \le s < x_3, \\ \varphi_1(x)z_1(s) + \varphi_2(x)z_2(s) + \varphi_3(x)z_3(s), & x_3 \le s < x_4, \\ \varphi_1(x)z_1(s) + \varphi_2(x)z_2(s) + \varphi_3(x)z_3(s) + \varphi_4(x)z_4(s), & x_4 \le s \le x, \\ -\varphi_5(x)z_5(s), & x \le s \le x_5. \end{cases}$$

An issue of the Green's function attributes for the multipoint boundary task was deeply studied in a range of works of Beesack, Krall, Peterson, Kiguradze, Levin, Pokorniy and others (Kiguradze, 1987; Maksimov & Rakhmatullina, 1977; Liu, 2011; Peterson, 1979; Jackson, 1977; Levin, 1961; Eloe & Grimm, 1980). As far as we know, solutions of the adjoint differential equation were not used previously in the course of creating the Green's function and the adjoint boundary conditions were not produced. Thereby, some new attributes of the Green's function can be determined.

1) Green's function $G(x, s)$ on variable s in the rectangle $x_1 \le s, x \le x_n$, except lines $s = x_\mu$, $s = x$, complies with the homogenous adjoint equation $L^+G(x, s) = 0$.

Proof. Function $G(x, s)$ represents the sum of products $\forall x \in [x_\mu, x_{\mu+1}]$, $\mu = 1, 2, ..., n-1$, $G(x, s) = \pm \sum_{k=1\ (\mu+1)}^{\mu(n)} \varphi_k(x) z_k(s)$ for $x_\mu < s < x$ $(x < s < x_{\mu+1})$.

Applying to it the adjoint operator L^+ on the variable s and considering its linearity, we will have

$$L^+G(x, s) = \pm \sum_{k=1\ (\mu+1)}^{\mu(n)} \varphi_k(x) L^+ z_k(s) = 0, x_\mu < s < x \ \ (x < s < x_{\mu+1}),$$

because functions $z_k(x)$ are solutions of the adjoint equation due to Lemma 2.

2) Green's function $G(x, s)$ on the lines $s = x_\mu$, $\mu = 2, 3, ..., n-1$ has a break of the first kind, and the proportion is fulfilled for $\forall x \in [x_1, x_n]$

$$G(x, x_\mu + 0) - G(x, x_\mu - 0) = \varphi_\mu(x) z_\mu(s)\big|_{s=x_\mu} = \rho_n(x_\mu)\varphi_\mu(x). \tag{31}$$

Proof. We can see from the structure of the Green's function assignment (29) that it is sufficient to subtract the expression of the previous line at $s < x_\mu$ from the expression $G(x, s)$ at $x_\mu < s$ for definition of the difference, i.e.

$$G(x, x_\mu + 0) - G(x, x_\mu - 0) = \left(\sum_{l=1}^{\mu} \varphi_l(x) z_l(s) - \sum_{l=1}^{\mu-1} \varphi_l(x) z_l(s) \right)\Big|_{s=x_\mu} = \varphi_\mu(x) z_\mu(s)\big|_{s=x_\mu} = \rho_n(x_\mu)\varphi_\mu(x), \forall x \in [x_1, x_n],$$

based on values $z_i(x_i)$ from (10). Similarly it is also set at $s \ge x$.

Function $G(x, s)$ and, correspondingly, its derivatives on x to $(n-2)$ order is uninterrupted on the lines $s = x_\mu$, $\mu = 2, 3, ..., n-1$ where the coefficient $\rho_n(x_\mu)$ at the higher derivative in the boundary conditions (22) transforms to zero. This feature simply results from the proportions (31). The same feature was produced by Levin (1961) under the other considerations.

Let us specify the Green's function saltus on the lines $s = x_\mu$, $\mu = 2, 3, ..., n-1$:

$$\delta G(x, x_\mu) = G(x, x_\mu + 0) - G(x, x_\mu - 0) = \varphi_\mu(x) z_\mu(s)\big|_{s=x_\mu}.$$

To make it comfortable, having assumed $G(x, x_1 - 0) = 0$, we find that

$$\delta G(x, x_1) = G(x, x_1 + 0) = G(x, s)\big|_{s=x_1} = \varphi_1(x) z_1(s)\big|_{s=x_1}.$$

Similarly, if $G(x, x_n + 0) = 0$, so

$$\delta G(x, x_n) = -G(x, x_n - 0) = -G(x, s)\big|_{s=x_n} = \varphi_n(x) z_n(s)\big|_{s=x_n}.$$

Now, we can take i=1, 2, ..., n instead of the index μ, i.e.

$$\delta G(x, x_i) = G(x, x_i + 0) - G(x, x_i - 0) = \varphi_i(x) z_i(s)\big|_{s=x_i}. \tag{32}$$

Let us specify the difference of derivatives as well

$$\delta G^{(k)}(x, x_i) = \frac{\partial^k G(x, x_i + 0)}{\partial s^k} - \frac{\partial^k G(x, x_i - 0)}{\partial s^k} = \varphi_i(x) \frac{d^k z_i(s)}{ds^k}\Big|_{s=x_i}. \tag{33}$$

Assuming in the boundary conditions (22), $\rho_{n-k_i}(x_i)$ are the first different from zero corresponding coefficients at the higher derivatives. So, it is easy to determine from the adjoint boundary conditions (17) that

$$\left.\frac{d^{k_i} z_i(s)}{ds^{k_i}}\right|_{s=x_i} = (-1)^{k_i} \rho_{n-k_i}(x_i) \neq 0, \tag{34}$$

where k_i is a natural number corresponding to each point x_i.

Let us consider a saltus k_i of the Green's function derivatives on the corresponding lines $s = x_i$:

$$\delta G^{(k_i)}(x, x_i) = \frac{\partial^{k_i} G(x, x_i + 0)}{\partial s^{k_i}} - \frac{\partial^{k_i} G(x, x_i - 0)}{\partial s^{k_i}} = \varphi_i(x) \frac{d^{k_i} z_i(s)}{ds^{k_i}} \Big|_{s=x_i} = \varphi_i(x)(-1)^{k_i} \rho_{n-k_i}(x_i).$$

So, we can identify that

$$\varphi_i(x) = (-1)^{k_i} \frac{1}{\rho_{n-k_i}(x_i)} \delta G^{(k_i)}(x, x_i). \tag{35}$$

Using this expression and taking into account (23), we can produce a solution of the homogenous equation $Ly = 0$ at the boundary conditions (22) for the case (34) as follows:

$$y(x) = \sum_{i=1}^{n} (-1)^{k_i} \frac{a_i}{\rho_{n-k_i}(x_i)} \delta G^{(k_i)}(x, x_i), \quad \rho_{n-k_i}(x_i) \neq 0. \tag{36}$$

Theorem 2 *Let $G(x, s)$ be a Green's function of the n-point boundary value problem (21), (22). So, the only solution to the nonhomogenous boundary problem (21), (22) is specified by the formula*

$$y(x) = \sum_{i=1}^{n} (-1)^{k_i} \frac{a_i}{\rho_{n-k_i}(x_i)} \delta G^{(k_i)}(x, x_i) + \int_{x_1}^{x_n} f(s) G(x, s) ds. \tag{37}$$

Proof. It is obvious from Theorem 1 that the sum is a solution of the homogenous equation $Ly = 0$ at condition (22). Therefore, it is sufficient to prove that an integral is a solution of the nonhomogeneous equation (21) at zero boundary conditions $(Ty)(x_i) = 0$. Let us take a random vertical strip $x_\mu \leq x \leq x_{\mu+1}$ and decompose the integral for the sum of n integrals taking into account the Green function assignment (29):

$$\int_{x_1}^{x_n} f(s) G(x, s) ds = \int_{x_1}^{x_2} f(s) \varphi_1(x) z_1(s) ds + \int_{x_2}^{x_3} f(s) \left[\varphi_1(x) z_1(s) + \varphi_2(x) z_2(s) \right] ds + ... + \int_{x_{\mu-1}}^{x_\mu} f(s) \sum_{l=1}^{\mu-1} \varphi_l(x) z_l(s) ds$$

$$+ \int_{x_\mu}^{x} f(s) \sum_{l=1}^{\mu} \varphi_l(x) z_l(s) ds - \int_{x}^{x_{\mu+1}} f(s) \sum_{l=\mu+1}^{n} \varphi_l(x) z_l(s) ds - \int_{x_{\mu+1}}^{x_{\mu+2}} f(s) \sum_{l=\mu+2}^{n} \varphi_l(x) z_l(s) ds - ... - \int_{x_{n-1}}^{x_n} f(s) \varphi_n(x) z_n(s) ds.$$

Since the addend $\varphi_1(x) z_1(s)$ is contained in all the integrals beginning with the first and finishing with the integral with the variable upper limit, and $\varphi_2(x) z_2(s)$ is contained there beginning with the second integral and so on, we have

$$\int_{x_1}^{x_n} f(s) G(x, s) ds = \sum_{i=1}^{\mu} \varphi_i(x) \int_{x_i}^{x} f(s) z_i(s) ds - \int_{x}^{x_{\mu+1}} f(s) \sum_{l=\mu+1}^{n} \varphi_l(x) z_l(s) ds$$

$$- \int_{x_{\mu+1}}^{x_{\mu+2}} f(s) \sum_{l=\mu+2}^{n} \varphi_l(x) z_l(s) ds - ... - \int_{x_{n-1}}^{x_n} f(s) \varphi_n(x) z_n(s) ds.$$

It is true for the negative integrals as well. Let us use addition of the integrals moving from the last one to the integral with the variable lower limit. Changing the integration limits, we finally have

$$\int_{x_1}^{x_n} f(s) G(x, s) ds = \sum_{i=1}^{n} \varphi_i(x) \int_{x_i}^{x} f(s) z_i(s) ds.$$

This integral is congruent with the second sum of the proportion (23), therefore, theorem 1 proves that the formula (37) is true.

We note that it is advisable to take functions $z_i(s)$ under the established proportion (26) instead of the formula (10) in the Green function and solutions of the n-point boundary value Problem.

References

Beesack, P. R. (1962). On the Green's function of an n-point Boundary Value Problem. *Pac. J. of Math., 12*(3), 801-812.

Coddington, E. A., & Levinson, N. (1955). *Theory of Ordinary Differential Equations* (p. 443). New York.

Davidson, F. A., & Rynne, B. P. (2007). Self-adjoint boundary-value problems on time-scales. *Electronic Journal of Differential Equations, 2007*(2007), 1-10.

Eloe, P. W., & Grimm, L. J. (1980). Monotone Iteration and Green's Function for Boundary Value Problems. *Proc. Amer. Math. Soc., 78*, 533-538.

Householder, A. (1956). *The basis of numerical analysis* (p. 321). Moscow, IL.

Jackson, L. K. (1977). Boundary Value problems for ordinary differential Equations in Studies in Ordinary Differential Equations. In J. K. Hale (Ed.), *MAA Studies in Mathematics, 14*, 93-127. Washington: Math. Assoc. of Amer.

Khasseinov, K. A. (1984). *Initial and multipoint problems for LDE and characteristic equations of Riccati type*. Synopsis of thesis for degree of a candidate of physic-mathematical sciences. Moscow, p. 114.

Kiguradze, I. T. (1987). *Boundary value problem of Ordinary Differential Equations* (p. 100). Moscow.

Klokov, Y. A. (1967). On a boundary value problem for ordinary differential equations of n-th order. *Reports of the USSR Academy of Sciences, 176*(3), 512-514.

Levin, A. Y. (1961). The differential properties of the Green's function of a multipoint Boundary value problem. *Reports of the USSR Academy of Sciences, 136*(5), 1022-1025.

Liu, Y. (2011). Studies on Nonhomogeneous Multi-point BVPs of Difference Equations with one-Dimensional p-Laplacian. *Mediterranean Journal of Mathematics, 8*(4), 577-602.

Maksimov, V. P., & Rakhmatullina, L. F. (1977). Adjoint equation for the general linear boundary value problem. *Differential Equations, 13*(11), 1966-1973.

Peterson, A. C. (1979). Green's functions for focal type boundary value problems. *Rocky Mountain J. of Math., 9*(4), 721-732. http://dx.doi.org/10.1216/RMJ-1979-9-4-721

Pokornyi, Y. V. (1980). *Problems of the qualitive theory of boundary value problem de la Valee Poussin*. Thesis of Doctor of Physical and Mathematical Sciences, Leningrad.

Yerugin, N. P. (1974). *Course of ordinary differential equations* (p. 471). Kiev: Higher School.