

Discrete Differential Manifolds and Dynamics on Networks

Aristophanes Dimakis

Department of Mathematics, University of Crete, GR-71409 Iraklion, Greece

Folkert Müller-Hoissen

Institut für Theoretische Physik, Bunsenstr. 9, D-37073 Göttingen, Germany

Francois Vanderseypen[‡]

Instituut voor Theoretische Fysica, Katholieke Universiteit, B-3001 Leuven, Belgium

[‡] *Aspirant NFWO*

Abstract

A *discrete differential manifold* we call a countable set together with an algebraic differential calculus on it. This structure has already been explored in previous work and provides us with a convenient framework for the formulation of dynamical models on networks and physical theories with discrete space and time. We present several examples and introduce a notion of differentiability of maps between discrete differential manifolds. Particular attention is given to differentiable curves in such spaces. Every discrete differentiable manifold carries a topology and we show that differentiability of a map implies continuity.

02.10.+w, 02.40.+m, 03.20.+i, 04.20.Cv

I. INTRODUCTION

The aim of our present work is to develop a mathematical formalism which allows an *intrinsic* formulation of dynamics and field theory on networks. By a network we mean a directed graph (digraph) which consists of a set of points (vertices) and a set of arrows connecting pairs of points. The formalism should give us a natural way to guarantee, for example, that a ‘particle’ hopping in discrete time steps on the set of vertices of a network respects the network structure in the sense that the motion can only take place in the direction of existing arrows of the digraph.

In [1] it was found that a digraph with at most two antiparallel arrows between each pair of vertices determines an ‘algebraic differential calculus’ on the set \mathcal{M} of vertices of the digraph (respectively, on the algebra of functions on this set). This observation was crucial for our solution of the problem mentioned above.

An algebraic differential calculus is an analogue of the calculus of differential forms on a manifold. It should be regarded as a basic structure for the formulation of dynamical systems and field theories. In the present case, the differential calculus is ‘noncommutative’ in the sense that differential forms and functions do not commute, in general. It fits into the more general framework of noncommutative geometry (see [2]). Indeed, some of the constructions used in this work are defined on arbitrary (not necessarily commutative) associative algebras. In this sense our choice of the commutative algebra of functions on a discrete set is just an example. But in the latter case we are able to associate a physical picture with the formalism. A comparable understanding is lacking in the case of noncommutative algebras.

The physical motivations for our work are ‘manifold’. In particular, ideas about a discrete space (-time) structure and space-time as a network stimulated our interest. There is already a vast literature in this field, but especially close to our work seems to be [3–5]. In the present work we are not going beyond the classical level. Ideas about quantization of topology and space-time can be pursued in this framework (see also [1]) and we should then expect relations, e.g., with the work by Isham on ‘quantum topology’ [6] and Finkelstein’s work on ‘quantum space-time networks’ [7]. In addition we should mention the work on ‘pregeometry’ in the sense of [8] and references given there. Of special interest is also ‘t Hooft’s work [9] suggesting that a theory which at large distance scales behaves like a quantum field theory may be deterministic and discrete at a small length scale (which may be the Planck scale).

The usefulness of algebraic differential calculus was demonstrated in [10] for lattice field theories. More generally, differential calculus on discrete sets was then developed in [11,1] (see also [12] for the case of discrete groups). A special example is the 2-point space which appeared in particle physics models of noncommutative geometry [13].

We shall now make more precise with what kind of mathematical framework we are working. Let \mathcal{M} be a discrete (in the sense of countable) set. An algebraic *differential calculus* on \mathcal{M} is an extension of the algebra \mathcal{A} of \mathbb{C} -valued functions on \mathcal{M} to a differential algebra $(\Omega(\mathcal{M}), d)$. Here $\Omega(\mathcal{M}) = \bigoplus_{r=0}^{\infty} \Omega^r(\mathcal{M})$ is a \mathbb{Z} -graded associative algebra where $\Omega^0(\mathcal{M}) = \mathcal{A}$ and $\Omega^{r+1}(\mathcal{M})$ is generated as an \mathcal{A} -bimodule via the action of a linear operator $d : \Omega^r(\mathcal{M}) \rightarrow \Omega^{r+1}(\mathcal{M})$. This operator is assumed to satisfy $d^2 = 0$ and the graded

Leibniz rule $d(\omega\omega') = (d\omega)\omega' + (-1)^r\omega d\omega'$ where $\omega \in \Omega^r(\mathcal{M})$. It has been shown in [1] how this structure supplies \mathcal{M} with a topology and assigns a (local) notion of dimension to it. Each differential algebra on \mathcal{M} can be obtained as the quotient of the so-called universal differential algebra by some differential ideal. A systematic construction of such ‘reductions’ of the universal differential algebra has been given in [1].

We take the point of view that a discrete set \mathcal{M} supplied with a differential calculus may be regarded as a kind of analogue of a (continuous) differentiable manifold. This is justified by the results in [1] and suggests the following definition.

Definition. A *discrete differential manifold* is a discrete set \mathcal{M} together with a differential calculus on it.

This is the basic structure which we explore in the following. After recalling differential calculus on discrete sets in section II we collect some examples in section III. A notion of a ‘differentiable map’ between discrete differential manifolds is the subject of section IV. Special cases are ‘diffeomorphisms’ (section V) and ‘differentiable curves’ (section VI). A natural further step is to consider the set of all differentiable curves in a given discrete differential manifold (section VII). Every discrete differential manifold carries the structure of a topological space. More precisely, it determines a larger set $\hat{\mathcal{M}}$ with a topology on it. This is the subject of section VIII. In section IX we show that differentiability of a map implies continuity. Section X contains some conclusions and additional remarks.

The formalism as presented in this paper basically applies to the case of a finite set. For an infinite set some of the calculations are formal and more efforts have to be invested to put things on a rigorous footing.

II. DIFFERENTIAL CALCULUS ON DISCRETE SETS

With each element $i \in \mathcal{M}$ we associate a function $e_i \in \mathcal{A}$ via

$$e_i(j) = \delta_{ij} . \quad (2.1)$$

Then

$$e_i e_j = \delta_{ij} e_j \quad \sum_i e_i = \mathbb{I} \quad (2.2)$$

where $\mathbb{I}(i) = 1 \forall i \in \mathcal{M}$. Acting with d on these relations and using the Leibniz rule yields

$$e_i de_j = -de_i e_j + \delta_{ij} de_j \quad \sum_i de_i = 0 . \quad (2.3)$$

The special 1-forms

$$e_{ij} := e_i de_j \quad (i \neq j) \quad (2.4)$$

($e_{ii} := 0$) satisfy

$$e_i e_{jk} = \delta_{ij} e_{jk} \quad e_{ij} e_k = \delta_{jk} e_{ij} \quad e_{ij} e_{kl} = \delta_{jk} e_{ij} e_{kl} . \quad (2.5)$$

As products of these 1-forms only the $(r - 1)$ -forms

$$e_{i_1 \dots i_r} := e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{r-1} i_r} \quad (2.6)$$

are therefore allowed not to vanish. On these forms the operator d acts as follows,

$$de_{i_1 \dots i_r} = \sum_j \sum_{k=1}^{r+1} (-1)^{k+1} e_{i_1 \dots i_{k-1} j i_k \dots i_r} . \quad (2.7)$$

The 1-form

$$p := \sum_{k, \ell} e_{k \ell} \quad (2.8)$$

satisfies

$$p^r = \sum_{i_1, \dots, i_{r+1}} e_{i_1 \dots i_{r+1}} . \quad (2.9)$$

For $r = 1, \dots, 4$ the formula (2.7) can be rewritten as

$$de_i = [p, e_i] \quad (2.10)$$

$$de_{ij} = \{p, e_{ij}\} - e_i p^2 e_j \quad (2.11)$$

$$de_{ijk} = [p, e_{ijk}] + e_i (p e_j p^2 - p^2 e_j p) e_k \quad (2.12)$$

and so forth. These formulae are easily obtained by using $e_{ij} = e_i p e_j$ and

$$dp = p^2 + \sum_i e_i p^2 e_i . \quad (2.13)$$

They display the deviation of d from a graded commutator. If no further relations are imposed, we are dealing with the universal differential calculus which we denote as $\tilde{\Omega}(\mathcal{M})$. In this case the $e_{i_1 \dots i_r}$ constitute a basis over \mathbb{C} of $\tilde{\Omega}^{r-1}(\mathcal{M})$ for $r > 1$.

A systematic way of constructing smaller differential algebras from the universal one is given by setting linear combinations of ‘basic’ forms to zero (*reduction*). Setting a linear combination of the $e_{i_1 \dots i_{r+1}}$ to zero does not influence those r -forms which do not appear in this equation and also not the forms with grade $< r$. It leads, however, to constraints for forms of grade $> r$ via the action of d . For 1-forms, the vanishing of a linear combination implies the vanishing of each basic 1-form which appears in the sum. We will be mainly concerned with reductions on the level of 1-forms.

It is convenient to associate a diagram with a differential calculus on \mathcal{M} as follows. On horizontal levels we draw vertices corresponding to all the basic r -forms $e_{i_1 \dots i_{r+1}} \neq 0$ in such a way that vertices representing $(r + 1)$ -forms are below those representing r -forms ($r \geq 0$). An arrow is drawn between two vertices on neighboring levels if the corresponding basic forms appear in (2.7) whereby the relative sign determines the orientation of the arrow. The result is an *oriented Hasse diagram* which completely specifies the differential calculus. Several examples can be found in [1]. If in the differential calculus a linear combination of

basic r -forms ($r > 1$) vanishes, this determines one of the r -forms in terms of the others. That one should then be discarded in the diagram. But one has to add a corresponding note to the diagram in order to be able to reconstruct the differential calculus from the diagram.

When we replace arrows by edges we obtain a *Hasse diagram* which determines a topology on \mathcal{M} or, more precisely, on a certain extension of \mathcal{M} (cf also [3]). More details are given in section VIII.

We also associate a *digraph* with a differential calculus on \mathcal{M} in the following way. If $e_{ij} \neq 0$ we draw an arrow from i to j . If reductions are only considered on the level of 1-forms, this digraph already contains all the information about the differential calculus and the oriented Hasse diagram can be derived from it (cf [1]).

A homomorphism of differential algebras $\Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$ is an algebra homomorphism which intertwines the respective d 's. According to a general result (see [15], Corollary 1.9), each differential algebra $\Omega(\mathcal{M})$ is the image of a homomorphism of differential algebras $\pi : \tilde{\Omega}(\mathcal{M}) \rightarrow \Omega(\mathcal{M})$ where $\tilde{\Omega}(\mathcal{M})$ is the universal differential algebra on \mathcal{M} . It is therefore the quotient $\Omega(\mathcal{M}) = \tilde{\Omega}(\mathcal{M})/\mathcal{I}$ by some two-sided differential ideal \mathcal{I} (the kernel of the homomorphism) in $\tilde{\Omega}(\mathcal{M})$. A ‘differential ideal’ is an ideal which is mapped by d into itself. This alternative description of differential calculi on \mathcal{M} will be helpful in the following sections. The ideal \mathcal{I} is generated by those linear combinations of basic forms viewed as elements of $\tilde{\Omega}(\mathcal{M})$ which vanish in the reduced differential calculus $\Omega(\mathcal{M})$.

Remark. We have seen that there are different differential calculi on \mathcal{M} and thus different d 's. As a consequence, e_{ij} also depends on the choice of the calculus. For the sake of notational simplicity we do not indicate this dependence in the hope that the latter will be clear from the respective context in which these symbols appear. \square

III. EXAMPLES OF DISCRETE DIFFERENTIAL MANIFOLDS

In this section we collect some examples of discrete differential manifolds in the sense of the definition given in the introduction. Examples 2-4 are taken from [1,10] where the reader can find ample discussions. Here we concentrate on those formulae which are needed in particular in section VI. Example 5 is new and therefore presented in some more detail. We also point out some ways to construct discrete differential manifolds from given ones.

Example 1. Let \mathcal{M} be a discrete set. If we regard it as a subset of \mathbb{Z}^n , then

$$x^\mu := \sum_{a \in \mathcal{M}} a^\mu e_a \quad (\mu = 1, \dots, n) \quad (3.1)$$

are natural coordinate functions on \mathcal{M} . With the help of (2.3) and (2.7) one obtains

$$[dx^\mu, x^\nu] = \tau^{\mu\nu} \quad \text{where} \quad \tau^{\mu\nu} = \sum_{a,b} (a^\mu - b^\mu)(a^\nu - b^\nu) e_{ab} . \quad (3.2)$$

Furthermore, one finds

$$[\tau^{\mu\nu}, x^\lambda] = \sum_{a,b} (a^\mu - b^\mu)(a^\nu - b^\nu)(a^\lambda - b^\lambda) e_{ab} . \quad (3.3)$$

So far we did not specify the differential calculus. This will be done in the following examples. \square

Example 2. Let $\mathcal{M} = \mathbb{Z}^n$ and $\Omega(\mathcal{M})$ the differential calculus determined by

$$e_{ab} \neq 0 \quad \Leftrightarrow \quad b = a + \hat{\mu} \quad \text{for some } \mu \quad (3.4)$$

where $\hat{\mu} = (\hat{\mu}^\nu) := (\delta_\mu^\nu)$. This is the (oriented) lattice calculus first considered in [10] (see also [1]). We obtain in this case

$$\tau^{\mu\nu} = \delta^{\mu\nu} \sum_a e_{a, a+\hat{\mu}} = \delta^{\mu\nu} dx^\mu. \quad (3.5)$$

In terms of the coordinate functions (3.1) the reduction condition thus reads

$$[dx^\mu, x^\nu] = \delta^{\mu\nu} dx^\nu. \quad (3.6)$$

We refer to [10] for applications of this calculus to lattice field theories. \square

Example 3. Let $\mathcal{M} = \mathbb{Z}$ and $\Omega(\mathcal{M})$ be the differential calculus on \mathcal{M} determined by the condition

$$[dt, t] = dt \quad (3.7)$$

in terms of the natural coordinate function $t(k) = k \in \mathbb{Z}$. This is a special case ($n = 1$) of the previous example and corresponds to the reduction

$$e_{ij} \neq 0 \quad \Leftrightarrow \quad j = i + 1 \quad (3.8)$$

of the universal calculus (see also [10,1]). It assigns a 1-dimensional structure to \mathbb{Z} . Now

$$df(t) = dt (\partial_+ f)(t) = (\partial_- f)(t) dt \quad (3.9)$$

define functions $\partial_\pm f$ on \mathcal{M} . A simple calculation (cf [10]) shows that

$$(\partial_+ f)(t) = f(t+1) - f(t) \quad (\partial_- f)(t) = f(t) - f(t-1). \quad (3.10)$$

We will also use the notation

$$\dot{f}(t) := f(t+1) - f(t) \quad (3.11)$$

instead of $(\partial_+ f)(t)$. For functions $f(t), h(t)$ the relation (3.7) now generalizes to

$$[df(t), h(t)] = \dot{f}(t) [dt, h(t)] = \dot{f}(t) [dh(t), t] = \dot{f}(t) \dot{h}(t) [dt, t] = \dot{f}(t) \dot{h}(t) dt. \quad (3.12)$$

We will take $(\mathbb{Z}, \Omega(\mathcal{M}))$ as a mathematical model for the parameter space of discrete time. The notion of discrete time in physics has been explored by many authors (see [7,9,16], for example). Instead of \mathbb{Z} we may consider a subset of \mathbb{Z} like $\mathbf{N} := \{0, \dots, N-1\}$ with the induced differential calculus (a ‘submanifold’ of $(\mathbb{Z}, \Omega(\mathcal{M}))$, see below). In this case one has to pay attention to the fact that $f dt = 0$ with a function $f = \sum_{i=0}^{N-1} f(i) e_i$ does not imply

that f vanishes. The equation does not determine $f(N-1)$. In the same way $dt f = 0$ leaves $f(0)$ undetermined. \square

Example 4. We choose $\mathcal{M} = \mathbb{Z}^n$ with the reduction of the universal calculus determined by the conditions

$$e_{ab} \neq 0 \quad \Leftrightarrow \quad b = a + \hat{\mu} \quad \text{or} \quad b = a - \hat{\mu} \quad \text{for some } \mu \quad (3.13)$$

where $\hat{\mu} = (\delta_\mu^\nu)$. This is the ‘symmetric lattice calculus’ discussed in [1]. One finds

$$\tau^{\mu\nu} = \delta^{\mu\nu} \tau^\mu \quad \text{where} \quad \tau^\mu := \sum_{a, \epsilon = \pm 1} e_{a, a + \epsilon \hat{\mu}} \quad (3.14)$$

so that

$$[dx^\mu, x^\nu] = \delta^{\mu\nu} \tau^\mu \quad [\tau^\mu, x^\nu] = \delta^{\mu\nu} dx^\nu. \quad (3.15)$$

\square

Example 5. Let $\mathcal{M} = \mathbb{Z}_N$. A reduction of the universal differential calculus on \mathbb{Z}_N is given by

$$e_{ij} \neq 0 \quad \Leftrightarrow \quad j = i + \epsilon \bmod N \quad (3.16)$$

where $\epsilon = \pm 1$. The associated digraph assigns to \mathbb{Z}_N the structure of a closed (i.e. periodic) lattice which is ‘symmetric’ in the sense that any two neighboring sites are connected by a pair of antiparallel arrows. Let $q \in \mathbb{C}$ be a primitive N th root of unity, i.e. $q^N = 1$, and define

$$y := \sum_{i=0}^{N-1} q^i e_i. \quad (3.17)$$

Then

$$dy = y [(q-1)e^+ + (q^{-1}-1)e^-] \quad (3.18)$$

where

$$e^\epsilon := \sum_k e_{k, k+\epsilon}. \quad (3.19)$$

On the lhs, ϵ stands for \pm (instead of ± 1). Using (2.5) we find

$$[dy, y] = y^2 \sum_\epsilon (q^\epsilon - 1)^2 e^\epsilon = (q^{1/2} - q^{-1/2})^2 y dy + \tau \quad (3.20)$$

with

$$\tau := (q^{1/2} - q^{-1/2})^2 y^2 \sum_\epsilon e^\epsilon. \quad (3.21)$$

Furthermore,

$$[\tau, y] = (q^{1/2} - q^{-1/2})^2 y^2 dy . \quad (3.22)$$

Each function on \mathcal{M} can be regarded as a function of (the function) y . Then $f(y) = \sum_k f(y) e_k = \sum_k f(q^k) e_k$. Applying d to this expression leads to

$$\begin{aligned} df(y) &= \sum_{k,\epsilon} f(q^k) (e_{k-\epsilon} - e_k) e^\epsilon = \sum_{k,\epsilon} [f(q^{k+\epsilon}) - f(q^k)] e_k e^\epsilon = \sum_{\epsilon} [f(q^\epsilon y) - f(y)] e^\epsilon \\ &=: \sum_{\epsilon} (q^\epsilon - 1) y \partial_\epsilon f(y) e^\epsilon \end{aligned} \quad (3.23)$$

where the ‘partial derivatives’ defined via the last equality are q -derivatives. From (3.18) and (3.21) we obtain

$$e^\epsilon = (q^\epsilon - q^{-\epsilon})^{-1} y^{-1} dy + (1 + q^\epsilon)^{-1} (q^{1/2} - q^{-1/2})^{-2} y^{-2} \tau . \quad (3.24)$$

Inserting this into the above expression for df we find

$$\begin{aligned} df(y) &= \sum_{\epsilon} \partial_\epsilon f(y) \left[\frac{q^\epsilon - 1}{q^\epsilon - q^{-\epsilon}} dy + \frac{q^\epsilon - 1}{q^\epsilon + 1} (q^{1/2} - q^{-1/2})^{-2} y^{-1} \tau \right] \\ &= \bar{\partial} f(y) dy + \frac{1}{2} \Delta f(y) \tau \end{aligned} \quad (3.25)$$

where in the last step we have introduced the symmetric q -derivative and the q -Laplacian,

$$\bar{\partial} f(y) := \frac{f(qy) - f(q^{-1}y)}{(q - q^{-1}) y} \quad (3.26)$$

$$\begin{aligned} \Delta f(y) &:= 2 (q^{1/2} + q^{-1/2})^{-1} (q^{1/2} - q^{-1/2})^{-2} y^{-2} [q^{-1/2} f(qy) \\ &\quad + q^{1/2} f(q^{-1}y) - (q^{1/2} + q^{-1/2}) f(y)] . \end{aligned} \quad (3.27)$$

With the help of (3.25), (3.20) and (3.22) one can now calculate, e.g., the commutator $[dy, f(y)] = [df(y), y]$. Furthermore, (3.22) obviously generalizes to

$$df(y) = [(q^{1/2} - q^{-1/2})^{-2} y^{-2} \tau, f(y)] . \quad (3.28)$$

□

Let $(\mathcal{M}, \Omega(\mathcal{M}))$ be a discrete differential manifold and \mathcal{M}' a subset of \mathcal{M} . To \mathcal{M}' corresponds a subdiagram of the oriented Hasse diagram for $\Omega(\mathcal{M})$ which then defines a differential calculus $\Omega(\mathcal{M}')$ on \mathcal{M}' . $(\mathcal{M}', \Omega(\mathcal{M}'))$ is a *discrete differential submanifold* of $(\mathcal{M}, \Omega(\mathcal{M}))$.

Example 6. Fig. 1 shows the digraph of a differential calculus on a 3-point set. It generates the oriented Hasse diagram drawn to the right of the digraph. The black points in the digraph select a 2-point subset. In the depicted oriented Hasse diagram the corresponding subgraph is emphasized. It is the oriented Hasse diagram generated by the subdigraph with two points and one arrow.

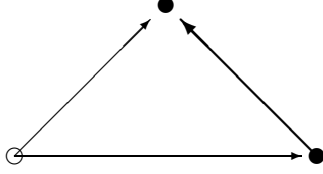
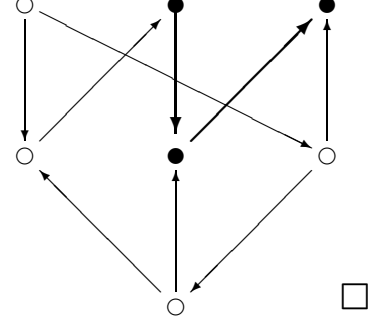


Fig. 1

A (sub)digraph corresponding to a differential calculus on a 3-point (2-point) set and the corresponding oriented (sub-) Hasse diagram.



Let $(\mathcal{M}, \Omega(\mathcal{M}))$ and $(\mathcal{M}', \Omega(\mathcal{M}'))$ be two discrete differential manifolds. From these one can build the *product manifold* $(\mathcal{M} \times \mathcal{M}', \Omega(\mathcal{M} \times \mathcal{M}'))$ where $\mathcal{M} \times \mathcal{M}'$ is the cartesian product of the two sets $\mathcal{M}, \mathcal{M}'$ and

$$\Omega(\mathcal{M} \times \mathcal{M}') := \Omega(\mathcal{M}) \hat{\otimes} \Omega(\mathcal{M}') \quad (3.29)$$

is the skew tensor product of the two differential algebras (cf [15], Appendix A). The product in $\Omega(\mathcal{M} \times \mathcal{M}')$ is

$$(\omega \hat{\otimes} \omega')(\rho \hat{\otimes} \rho') = (-1)^{\partial \omega' \cdot \partial \rho} (\omega \rho \hat{\otimes} \omega' \rho') \quad (3.30)$$

and the operator d on $\Omega(\mathcal{M} \times \mathcal{M}')$ is given by

$$d(\omega \hat{\otimes} \omega') = (d\omega) \hat{\otimes} \omega' + (-1)^{\partial \omega} \omega \hat{\otimes} d\omega' . \quad (3.31)$$

Here $\partial \omega$ denotes the grade of the form ω . The discrete differential manifold $(\mathbb{Z}^n, \Omega(\mathbb{Z}^n))$ in example 2 is the n -fold product of $(\mathbb{Z}, \Omega(\mathbb{Z}))$ from example 3. Also in example 4 we have an n -fold product manifold.

The Euler-Poincaré theorem (see [14], for example) suggests the following definition.

$$\chi(\Omega(\mathcal{M})) := \sum_{r \geq 0} (-1)^r \dim_{\mathbb{C}} \Omega^r(\mathcal{M}) \quad (3.32)$$

is the *Euler characteristic* of the discrete differential manifold $(\mathcal{M}, \Omega(\mathcal{M}))$.

IV. DIFFERENTIABLE MAPS BETWEEN DISCRETE DIFFERENTIAL MANIFOLDS

Let ϕ be a map from a discrete differential manifold $(\mathcal{N}, \Omega(\mathcal{N}), d_{\mathcal{N}})$ to another one, $(\mathcal{M}, \Omega(\mathcal{M}), d_{\mathcal{M}})$.¹ We define

¹The notation $d_{\mathcal{M}}$ is actually a bit misleading. For different differential calculi on \mathcal{M} also the associated operators $d_{\mathcal{M}}$ are different.

$$\phi^* f := f \circ \phi \quad (4.1)$$

where $f \in \mathcal{A}_{\mathcal{M}}$ and \circ denotes composition of maps.

Definition. The map ϕ is called *differentiable* if ϕ^* can be consistently extended to a homomorphism of the corresponding differential algebras, i.e. a linear map $\Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$ such that

$$\phi^*(\omega\omega') = \phi^*(\omega) \phi^*(\omega') \quad (4.2)$$

$$\phi^* d_{\mathcal{M}} = d_{\mathcal{N}} \phi^* . \quad (4.3)$$

We refer to ϕ^* as the *pull-back* map. Clearly, if ϕ^* exists, then it is unique.

Lemma. The composition of differentiable maps is differentiable.

Proof: Let $\phi = \rho \circ \gamma$ be the composition of two differentiable maps $\mathcal{N} \xrightarrow{\gamma} \mathcal{M} \xrightarrow{\rho} \mathcal{P}$. With

$$\phi^* := \gamma^* \circ \rho^*$$

it is easy to verify that (4.1), (4.2) and (4.3) hold. \square

Lemma. Any map ϕ into a set \mathcal{M} supplied with the universal differential calculus $\tilde{\Omega}(\mathcal{M})$ is differentiable.

Proof: On the algebra $\mathcal{A}_{\mathcal{M}}$ of functions on \mathcal{M} the pull-back ϕ^* is a homomorphism to the algebra $\mathcal{A}_{\mathcal{N}}$ of functions on \mathcal{N} . A general result (see [15], Proposition 1.8) then tells us that there is a unique extension to a homomorphism $\tilde{\Omega}(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$. More concretely, one defines

$$\phi^*(f_0 d_{\mathcal{M}} f_1 \cdots d_{\mathcal{M}} f_r) := (\phi^* f_0) (d_{\mathcal{N}} \phi^* f_1) \cdots (d_{\mathcal{N}} \phi^* f_r)$$

($\forall r \in \mathbb{N}$, $f_s \in \mathcal{A}_{\mathcal{M}}$). By linearity and the Leibniz rule $(df)h = d(fh) - f dh$ for $f, h \in \mathcal{A}_{\mathcal{M}}$, ϕ^* is then defined on arbitrary forms. It can now be shown that (4.2) and (4.3) are satisfied for arbitrary forms (see [15], Proposition 1.8, for details). \square

We recall that a differential algebra $\Omega(\mathcal{M})$ is the image of a homomorphism $\pi : \tilde{\Omega}(\mathcal{M}) \rightarrow \Omega(\mathcal{M})$ and therefore the quotient of the universal differential algebra $\tilde{\Omega}(\mathcal{M})$ by some differential ideal $\mathcal{I}_{\mathcal{M}}$. A useful characterization of differentiability is now given in the next Lemma.

Lemma. A map $\phi : \mathcal{N} \rightarrow \mathcal{M}$ is differentiable with respect to differential calculi $\Omega(\mathcal{N})$ and $\Omega(\mathcal{M}) = \tilde{\Omega}(\mathcal{M})/\mathcal{I}_{\mathcal{M}}$ iff

$$\tilde{\phi}^*(\mathcal{I}_{\mathcal{M}}) = 0 \quad (4.4)$$

where $\tilde{\phi}^*$ is the homomorphism $\tilde{\Omega}(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$ which exists according to the previous Lemma.

Proof: “ \Rightarrow ”: If ϕ is differentiable so that ϕ^* extends to a homomorphism $\Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$, then $\tilde{\phi}^* := \phi^* \circ \pi$ extends $\phi^* : \mathcal{A}_{\mathcal{M}} \rightarrow \mathcal{A}_{\mathcal{N}}$ to a homomorphism $\tilde{\Omega}(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$. It must then coincide with the map $\tilde{\Omega}(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$ which always exists according to the last Lemma. By definition of π we have $\tilde{\phi}^*(\mathcal{I}_{\mathcal{M}}) = 0$.

“ \Leftarrow ”: According to the previous Lemma the homomorphism $\phi^* : \mathcal{A}_{\mathcal{M}} \rightarrow \mathcal{A}_{\mathcal{N}}$ lifts to a homomorphism $\tilde{\phi}^* : \tilde{\Omega}(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$ of differential algebras. Using a general result in algebra (see [17], for example), this map induces a homomorphism $\phi^* : \Omega(\mathcal{M}) \rightarrow \Omega(\mathcal{N})$ of differential algebras if (4.4) holds. \square

Let $\Omega(\mathcal{M})$ be a reduction of $\tilde{\Omega}(\mathcal{M})$ such that $e_{ab} = 0$ (respectively $\pi(e_{ab}) = 0$ referring to $e_{ab} \in \tilde{\Omega}(\mathcal{M})$). For differentiable ϕ we then have

$$0 = \tilde{\phi}^*(e_{ab}) = (\phi^* e_a) d_{\mathcal{N}}(\phi^* e_b) = \sum_{i \in \phi^{-1}(a)} e_i d_{\mathcal{N}} \sum_{j \in \phi^{-1}(b)} e_j = \sum_{\substack{i \in \phi^{-1}(a) \\ j \in \phi^{-1}(b)}} e_{ij} \quad (4.5)$$

where we have used (4.2), (4.3) and

$$e_a \circ \phi = \sum_{i \in \phi^{-1}(a)} e_i \quad (4.6)$$

(i, j, \dots denote elements of \mathcal{N}). For a given map ϕ we may regard (4.5) as a constraint on the differential calculi on \mathcal{N} and \mathcal{M} which are needed to render a map ϕ differentiable. It has a simple interpretation in terms of the digraphs associated with the differential calculi. If two points of \mathcal{N} connected by an arrow are mapped into two different points, then – in order for ϕ to be differentiable – there must be an arrow between the image points with the same orientation. If we fix differential calculi on \mathcal{N} and \mathcal{M} , respectively, the differentiability restricts the allowed class of maps, of course. Corresponding examples are given in the following section.

More generally we may have reductions on the level of r -forms. Differentiability conditions for ϕ are then obtained by using the general formula

$$\tilde{\phi}^*(e_{a_1 \dots a_{r+1}}) = \sum_{\substack{i_1 \in \phi^{-1}(a_1) \\ \vdots \\ i_{r+1} \in \phi^{-1}(a_{r+1})}} e_{i_1 \dots i_{r+1}} \quad (4.7)$$

V. DIFFEOMORPHISMS OF A DISCRETE DIFFERENTIAL MANIFOLD

Let $\mathcal{N} = \mathcal{M}$ and ϕ a bijection. If ϕ is differentiable as a map $(\mathcal{M}, \Omega'(\mathcal{M})) \rightarrow (\mathcal{M}, \Omega(\mathcal{M}))$, then ϕ^{-1} is not differentiable, in general. If we want both, ϕ and ϕ^{-1} , differentiable then we must have

$$e_{\phi(a)\phi(b)} \neq 0 \quad \Leftrightarrow \quad e_{ab} \neq 0 \quad (5.1)$$

and corresponding conditions in case of higher order reductions. This is only possible if $\Omega'(\mathcal{M}) = \Omega(\mathcal{M})$. We call a bijection ϕ a *diffeomorphism* of a discrete differential manifold $(\mathcal{M}, \Omega(\mathcal{M}))$ if ϕ and ϕ^{-1} are differentiable with respect to $\Omega(\mathcal{M})$.

Lemma. Let \mathcal{M} be a finite set and ϕ a bijection which is differentiable with respect to a first order reduction $\Omega(\mathcal{M})$ of the universal differential calculus on \mathcal{M} . Then ϕ^{-1} is also differentiable (and therefore a diffeomorphism).

Proof: The differentiability of ϕ implies that $e_{ij} \neq 0 \Rightarrow e_{\phi(i)\phi(j)} \neq 0$ so that an arrow in the digraph of the differential calculus which points from i to j is mapped into an arrow from $\phi(i)$ to $\phi(j)$. But also the inverse implication $e_{\phi(i)\phi(j)} \neq 0 \Rightarrow e_{ij} \neq 0$ holds since otherwise the map would ‘create’ an arrow and thus change the differential calculus. Hence $e_{ij} = 0 \Leftrightarrow e_{\phi(i)\phi(j)} = 0$. Now the statement follows using the last Lemma of the previous section. \square

The statement in the Lemma is not true for infinite sets, in general.

The *adjacency matrix* $A = (A_{ij})$ of a digraph G is defined by

$$A_{ij} = \begin{cases} 1 & \text{if there is an arrow from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases} \quad (5.2)$$

From graph theory we recall the following characterization of an automorphism of a digraph G [18]. A bijection ϕ is an element of the automorphism group $\text{Aut}(G)$ of G iff the associated matrix

$$(M_\phi)_{ij} = \delta_{\phi^{-1}(i)j} \quad (5.3)$$

commutes with the adjacency matrix of G , i.e.

$$[M_\phi, A] = 0. \quad (5.4)$$

Proposition. Let $\Omega(\mathcal{M})$ be a first order reduction of the universal differential calculus on \mathcal{M} with adjacency matrix A . A bijection ϕ is a diffeomorphism iff it is an automorphism of the corresponding digraph.

Proof: Since

$$(M_\phi A)_{ij} = A_{\phi^{-1}(i)j}, \quad (A M_\phi)_{ij} = A_{i\phi(j)}$$

the condition $[M_\phi, A] = 0$ is equivalent to

$$A_{ij} = A_{\phi(i)\phi(j)}$$

respectively,

$$A_{ij} = 0 \Leftrightarrow A_{\phi(i)\phi(j)} = 0.$$

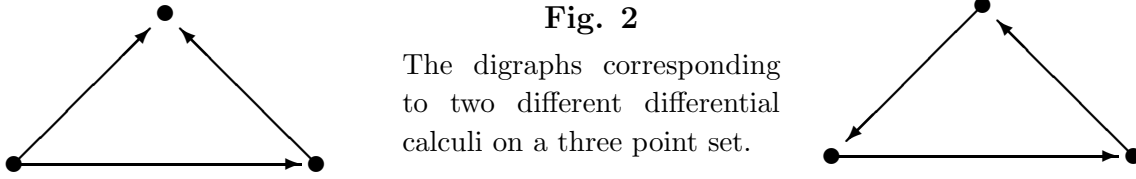
But this in turn is equivalent to

$$e_{ij} \in \mathcal{I} \Leftrightarrow e_{\phi(i)\phi(j)} \in \mathcal{I}$$

if we express $\Omega(\mathcal{M}) = \tilde{\Omega}(\mathcal{M})/\mathcal{I}$. The last statement is equivalent to the differentiability of ϕ and ϕ^{-1} . \square

For a digraph with a finite number N of vertices, the automorphism group $\text{Aut}(G)$ is a subgroup of the symmetric group S_N (the group of permutations). For the first graph in

Fig. 2 the automorphism group consists of the identity only. The second graph obviously has a (discrete) rotational symmetry. In this case we have $\text{Aut}(G) \cong \mathbb{Z}_3$.



VI. DIFFERENTIABLE CURVES IN DISCRETE DIFFERENTIAL MANIFOLDS

A 1-dimensional discrete differential manifold has been described in example 3 in section III. Its differential operator will be denoted by d in the following. A *differentiable curve* in a discrete differential manifold \mathcal{M} should then be a differentiable map γ from \mathbb{Z} (with the differential calculus of example 3 in section III) to \mathcal{M} (with its differential calculus). Instead of \mathbb{Z} we may take as well $\mathbf{N} = \{0, \dots, N-1\}$ (with the induced differential calculus).

Example 1. Let $\mathcal{M} = \mathbb{Z}^n$ with the differential calculus of example 2 in section III, i.e. $\tilde{\Omega}(\mathcal{M})/\mathcal{I}_{\mathcal{M}}$ where the ideal $\mathcal{I}_{\mathcal{M}}$ is generated by

$$[d_{\mathcal{M}}x^{\mu}, x^{\nu}] - \delta^{\mu\nu} d_{\mathcal{M}}x^{\nu} \quad (\mu, \nu = 1, \dots, n) . \quad (6.1)$$

According to the last Lemma in section IV, $\gamma : \mathbb{Z} \rightarrow \mathcal{M}$ is differentiable if $\tilde{\gamma}^*$ maps the last expression to the zero in $\Omega(\mathbb{Z})$ so that

$$0 = \tilde{\gamma}^*([d_{\mathcal{M}}x^{\mu}, x^{\nu}] - \delta^{\mu\nu} d_{\mathcal{M}}x^{\nu}) = [dx^{\mu}(t), x^{\nu}(t)] - \delta^{\mu\nu} dx^{\nu}(t) \quad (6.2)$$

where $x^{\mu}(t) := x^{\mu} \circ \gamma(t)$. Now (3.12) leads to the differentiability condition

$$\dot{x}^{\mu}(t) [\dot{x}^{\nu}(t) - \delta^{\mu\nu}] = 0 . \quad (6.3)$$

If we regard γ as a curve describing the motion of a particle on the lattice \mathbb{Z}^n , the last condition restricts the motion such that the particle can either rest at a given site or hop to a neighboring site. Furthermore, only a motion to a site with increasing values of x^{μ} is allowed. This apparently unplausible restriction is absent in our next example. It reminds us, however, of right (and left) movers in 2-dimensional chiral field theories. If we had taken the differential calculus on \mathbb{Z}^n with the opposite orientation of arrows in the corresponding digraph, then only motion to a site with decreasing values of x^{μ} would be allowed. There are, of course, $2n$ ways of choosing the direction of arrows along the n axes of \mathbb{Z}^n and thus 2^n ‘chiral sectors’ in the lattice. In order to reach (in principle) all lattice sites, we would need 2^n particles, each moving in a separate chiral sector of the lattice (directed by a separate

differential calculus). This reminds us of the fermion doubling problem in lattice theories.² The problem is solved if we take the symmetric lattice (see next example) on which a single particle can move in each lattice direction. \square

Example 2. Again we choose $\mathcal{M} = \mathbb{Z}^n$, but now with the differential calculus determined by (3.15), i.e. we are dealing with the ‘symmetric lattice’ calculus of [1]. Differentiability of $\gamma : \mathbb{Z} \rightarrow \mathcal{M}$ requires

$$\dot{x}^\mu(t) [dt, x^\nu(t)] = \delta^{\mu\nu} w^\nu(t) dt \quad w^\mu(t) [dt, x^\nu(t)] = \delta^{\mu\nu} \dot{x}^\nu(t) dt \quad (6.4)$$

where $w^\mu(t)$ is given by $\gamma^* \tau^\mu = w^\mu(t) dt$. From these equations one derives

$$\dot{x}^\mu \dot{x}^\nu = \delta^{\mu\nu} w^\nu \quad w^\mu \dot{x}^\nu = \delta^{\mu\nu} \dot{x}^\nu . \quad (6.5)$$

The first equation implies $\dot{x}^\mu \dot{x}^\nu = 0$ for $\mu \neq \nu$ so that at most one \dot{x}^κ (with fixed κ) can be different from zero (at a given value of t). The above equations then reduce to $\dot{x}^\kappa = \pm 1$. Hence, the ‘particle’ is allowed to jump to an arbitrary neighboring site on the lattice. The remaining solution of the above conditions, namely $\dot{x}^\mu = 0$ for all μ , allows the particle to remain at a site. \square

Example 3. Now we choose $\mathcal{M} = \mathbb{Z}_N$ with the differential calculus of example 5 in section III. If $\gamma : \mathbb{Z} \rightarrow \mathbb{Z}_N$ is a differentiable curve, then the pull-back of (3.20) and (3.22) yields

$$\dot{y}(t)^2 = (q^{1/2} - q^{-1/2})^2 y(t) \dot{y}(t) + w(t) \quad w(t) \dot{y}(t) = (q^{1/2} - q^{-1/2})^2 y(t)^2 \dot{y}(t) \quad (6.6)$$

where $dy(t) =: \dot{y}(t) dt$ and $\gamma^* \tau =: w dt$. If \dot{y} vanishes at a certain ‘time’, then $w = 0$ at that time. If $\dot{y} \neq 0$ (at some time), the above equations imply $w = (q^{1/2} - q^{-1/2})^2 y(t)^2$ and restrict \dot{y} to the values $(q^\epsilon - 1) y$ where $\epsilon = \pm 1$. All these conditions can now be summarized in the formula

$$\dot{y} [\dot{y} - (q - 1) y] [\dot{y} - (q^{-1} - 1) y] = 0 . \quad (6.7)$$

Using $\dot{y}(t) = y(t+1) - y(t)$, this means

$$y(t+1) = y(t) \quad \text{or} \quad q y(t) \quad \text{or} \quad q^{-1} y(t) . \quad (6.8)$$

The particle thus either remains at a site or moves one step on the periodic lattice (which the coordinate y describes as a q -lattice). \square

In all these examples the particle can only move one lattice spacing in at least one time step. This means that there is a maximal velocity which we may identify with the vacuum velocity of light. It should be noticed, however, that such an interpretation presumes that there is a time and a space metric. A natural choice is indeed suggested by our description of discrete time as \mathbb{Z} (or \mathbb{N}) and discrete space as (a subset of) \mathbb{Z}^n . But these are extra structures which we still have to introduce on discrete sets and to discuss in more generality.

²We refer to [19] for a discussion of this problem. See also [9] for the relation between left and right movers on a one-dimensional lattice and the one-dimensional free massless fermion gas.

Example 4. Let γ be a curve in \mathbb{Z}^n subject to the equation of motion

$$\partial_- \partial_+ x^\mu(t) = 0 . \quad (6.9)$$

With (3.10) this becomes

$$x(t+1) - 2x(t) + x(t-1) = 0 \quad (6.10)$$

which implies

$$x(t+1) - x(t) = x(t) - x(t-1) . \quad (6.11)$$

Hence, in each time step the distance traversed on \mathbb{Z}^n (in each of the n canonical lattice directions) is constant. This corresponds to our intuitive conception of a free motion on a lattice. Imposing differentiability of γ with respect to some choice of differential calculus on \mathbb{Z}^n now further restricts these motions. A motion on such a discrete differentiable manifold is then only allowed if the associated digraph has an arrow between every two adjacent points along the traversed path which points in the direction of the motion. A particular consequence is the existence of a maximal velocity (as already pointed out). In case of the oriented lattice calculus (example 1) there are only two free differentiable motions. Either the particle remains forever at one site or it moves steadily (in one time step) to the neighboring site in one of the canonical lattice directions. For motion on the one-dimensional oriented lattice this is illustrated in Fig. 3.

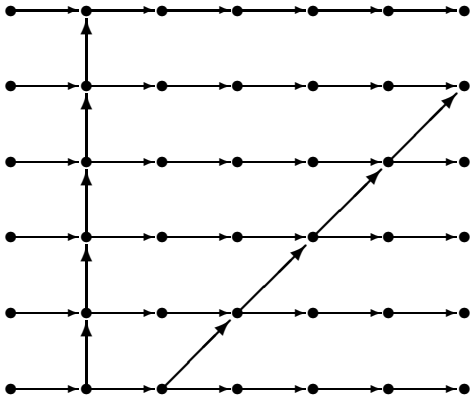


Fig. 3

Free differentiable motions on the one-dimensional oriented lattice in a space-time picture.

□

Again, let γ be a differentiable curve in a discrete differential manifold $(\mathcal{M}, \Omega(\mathcal{M}))$. Acting with $\tilde{\gamma}^*$ (cf (4.5)) on $e_a e_b = \delta_{ab} e_b$ and $\sum_a e_a = \mathbb{I}$ we obtain

$$e_a(t) e_b(t) = \delta_{ab} e_b(t) \quad \sum_a e_a(t) = \mathbb{I}(t) = 1 \quad (a, b \in \mathcal{M}) \quad (6.12)$$

where $e_a(t) := e_a(\gamma(t))$. If an arrow from a to a different point b is missing in the digraph associated with a differential calculus on \mathcal{M} , then $\tilde{\gamma}^*(e_{ab}) = 0$ (cf (4.5)) and the homomorphism property of $\tilde{\gamma}$ leads to

$$0 = e_a(t) \dot{e}_b(t) = e_a(t) [e_b(t+1) - e_b(t)] = e_a(t) e_b(t+1) \quad (6.13)$$

where we used (6.12).

As a *Lagrangian* for a dynamical system on a discrete differential manifold we may regard a function

$$L[\gamma](t) = L'(e_a(t), \dot{e}_b(t), \ddot{e}_c(t), \dots) \quad (6.14)$$

with suitably defined second and higher order derivatives of $e_a(t)$. If there are no higher than first order derivatives, it can be rewritten as

$$L[\gamma](t) = L(e_a(t), e_b(t+1)) = \sum_{a,b} L_{ab} e_a(t) e_b(t+1) . \quad (6.15)$$

The last expression gives the most general form of a (first order) Lagrangian. Note that it also allows terms linear in $e_a(t)$ (or $e_b(t+1)$) since we have the relation $\sum_a e_a(t) = \mathbf{1}$.

VII. THE SPACE OF DIFFERENTIABLE CURVES

We recall that a digraph $(\mathcal{M}_0, \mathcal{M}_1)$ consists of a set $\mathcal{M} := \mathcal{M}_0$ of vertices and a set \mathcal{M}_1 of arrows. In the present context only a subclass of digraphs is considered. No multiple arrows are allowed between the same pair of vertices. Also no loops are allowed which forbids arrows originating and ending at the same vertex. We already know that any digraph defines a differential calculus on \mathcal{M} and vice versa. From the set \mathcal{M}_1 of arrows one can construct the set \mathcal{M}_r of paths $(a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_r)$ of length r with $(a_{k-1} \rightarrow a_k) \in \mathcal{M}_1$. The vertices are paths of length 0, arrows (elements of \mathcal{M}_1) are paths of length 1. For $a, b \in \mathcal{M}_0$, let ℓ_{ab} denote the minimal length of paths from a to b . In terms of the adjacency matrix,

$$\ell_{ab} = \min\{\ell \mid (A^\ell)_{ab} \neq 0\} \quad (7.1)$$

since $(A^\ell)_{ab}$ is the number of paths of length ℓ from a to b . If there is no path from a to b , we set $\ell_{ab} = \infty$.³

The formulae (6.12) and (6.13) motivate the following algebraic construction. Let $\tilde{\mathcal{A}}(\mathbf{N}, \mathcal{M})$ be the commutative and associative algebra generated by elements $e_a(i)$, $a \in \mathcal{M}$, $i \in \mathbf{N} = \{0, \dots, N-1\}$ subject to the relations

$$e_a(i) e_b(i) = \delta_{ab} e_b(i) \quad \sum_a e_a(i) = 1 \quad (a, b \in \mathcal{M}, i \in \mathbf{N}) . \quad (7.2)$$

As we will see in the following, $\tilde{\mathcal{A}}(\mathbf{N}, \mathcal{M})$ may be regarded as the algebra of functions on the space of curves from \mathbf{N} to \mathcal{M} with the universal differential calculus corresponding to the complete digraph (which has exactly two antiparallel arrows between all pairs of vertices representing elements of \mathcal{M}). More generally, we will associate an algebra $\mathcal{A}(\mathbf{N}, \mathcal{M})$ with any digraph (with set of vertices \mathcal{M}).

³Note that ℓ_{ab} is not a distance function on \mathcal{M} (as considered, for example, in [8,20] in a context related to our work) since it is directed, i.e. $\ell_{ab} \neq \ell_{ba}$, in general.

Let \mathcal{J} denote the two-sided ideal in $\tilde{\mathcal{A}}(\mathbf{N}, \mathcal{M})$ generated by those products $e_a(i)e_b(j)$ for which $0 \leq j - i < \ell_{ab}$ for a given digraph. We define

$$\mathcal{A}(\mathbf{N}, \mathcal{M}) := \tilde{\mathcal{A}}(\mathbf{N}, \mathcal{M}) / \mathcal{J} . \quad (7.3)$$

In case of the complete digraph we have $\ell_{ab} \in \{0, 1\}$ ($\forall a, b \in \mathcal{M}, a \neq b$) and thus $\mathcal{J} = 0$.

From the last section we infer that any differentiable curve $\gamma : \mathbf{N} \rightarrow \mathcal{M}$ defines an irreducible representation ρ_γ of $\mathcal{A}(\mathbf{N}, \mathcal{M})$ via

$$\rho_\gamma(e_a(i)) = e_a(\gamma(i)) . \quad (7.4)$$

Conversely, any irreducible representation of $\mathcal{A}(\mathbf{N}, \mathcal{M})$ is a differentiable curve. Indeed, since the algebra is commutative, all irreducible representations are one-dimensional. (7.2) then implies $\rho(e_a(i)) \in \{0, 1\}$ and that for each i there is precisely one $a \in \mathcal{M}$ for which $\rho(e_a(i)) = 1$. Hence we have a curve $\gamma : \mathbf{N} \rightarrow \mathcal{M}$. The relations defining the ideal \mathcal{J} now restrict γ to be differentiable.

Let $\Gamma := \{\gamma : \mathbf{N} \rightarrow \mathcal{M} \mid \gamma \text{ differentiable with respect to } \Omega(\mathcal{M})\}$ be the space of differentiable curves (with respect to some differential calculus $\Omega(\mathcal{M})$). This is again a discrete set to which the formalism of section II applies. The algebra $\mathcal{A}(\Gamma)$ of \mathbb{C} -valued functions on Γ is then generated by elements e_γ such that

$$e_\alpha(\beta) = \delta_{\alpha\beta} , \quad e_\alpha e_\beta = \delta_{\alpha\beta} e_\beta , \quad \sum_{\gamma \in \Gamma} e_\gamma = \mathbb{1}_\Gamma . \quad (7.5)$$

Via (7.4) we may regard $e_a(i)$ as a function on Γ ,

$$e_a(i) = \sum_{\gamma} e_a(\gamma(i)) e_\gamma = \sum_{\gamma} \delta_{a\gamma(i)} e_\gamma . \quad (7.6)$$

Now

$$e_\gamma = \prod_{i \in \mathbf{N}} e_{\gamma(i)}(i) \quad (7.7)$$

shows that $\mathcal{A}(\mathbf{N}, \mathcal{M}) = \mathcal{A}(\Gamma)$.

A (first order) *action* is a function on Γ . It can always be written in the form

$$S = \sum_i \sum_{a,b} L_{ab}(i) e_a(i) e_b(i+1) \quad (7.8)$$

with real (or complex) coefficients $L_{ab}(i)$. ‘Classical motions’ should correspond to (local) extrema of the action. To find a local extremum a formalism of variations should be helpful. The latter may be realized as a differential calculus on the space of curves.

A. Differential calculus on the space of curves

Let $(\tilde{\Omega}(\Gamma), \tilde{d})$ be the universal differential calculus on Γ . For $a \neq b$ we define

$$e_{ab}(i) := e_a(i) \tilde{d}e_b(i) \quad (7.9)$$

and $e_{aa}(i) := 0$. The differential calculus $\Omega(\mathcal{M}) = \tilde{\Omega}(\mathcal{M})/\mathcal{I}_{\mathcal{M}}$ which enters the definition of Γ induces a reduction of $\tilde{\Omega}(\Gamma)$ in the following way. Let \mathcal{I}_{Γ} be the two-sided differential ideal of $\tilde{\Omega}(\Gamma)$ generated by those $e_{ab}(i)$ for which $e_{ab} \in \mathcal{I}_{\mathcal{M}}$. We define

$$\Omega(\Gamma) := \tilde{\Omega}(\Gamma)/\mathcal{I}_{\Gamma}. \quad (7.10)$$

In the associated digraph there is an arrow from α to β (regarded as vertices) iff for all $i \in \mathbf{N}$ there is an arrow from $\alpha(i)$ to $\beta(i)$ in the digraph corresponding to $\Omega(\mathcal{M})$. Whereas the basic 1-forms $e_{\alpha\beta}$ constitute a basis of $\Omega^1(\Gamma)$ over \mathbb{C} , this is not so for the set of 1-forms $e_{ab}(i)$. The latter is in general not even a basis of $\Omega^1(\Gamma)$ as a left $\mathcal{A}(\Gamma)$ -module. There are not enough commutation relations between $e_a(i)$ and $de_b(j)$ for $i \neq j$.

If there is no arrow from a to $c \neq a$ in the digraph for $\Omega(\mathcal{M})$, but a path from a to c via one further vertex b , then $e_a(i) e_c(i+1) = 0$ (cf the definition of $\mathcal{A}(\mathbf{N}, \mathcal{M})$). Acting with d on this equation one finds

$$\sum_b [e_a(i) e_{bc}(i+1) - e_{ab}(i) e_c(i+1)] = 0. \quad (7.11)$$

Here we have used

$$de_a(i) = \sum_{b \in \mathcal{M}} [e_{ba}(i) - e_{ab}(i)] \quad (7.12)$$

which follows from (7.6) and the general formula (2.7).

The problem to determine a local minimum of an action S can now be formulated as follows. One has to find a curve $\gamma \in \Gamma$ such that

$$dS(\gamma, \alpha) \geq 0 \quad \text{and} \quad dS(\alpha, \gamma) \leq 0 \quad \forall \alpha \in \Gamma. \quad (7.13)$$

Here we make use of the representation $e_{ab}(i) = e_a(i) \otimes e_b(i)$ for $e_{ab} \neq 0$. Note that dS can only be nonvanishing on pairs of neighboring curves.

Remark. Let us call two curves α, β neighbors if for all $i \in \mathbf{N}$ either $\alpha(i) = \beta(i)$ or there is an arrow between $\alpha(i)$ and $\beta(i)$ in the digraph for $\Omega(\mathcal{M})$. Then, for $\alpha \in \Gamma$ there are, in general, neighboring curves β which are not in Γ (i.e., not differentiable). A corresponding extended space of curves could be of relevance for a calculus of variations which should determine discrete dynamics from an action. \square

B. A simple example

Let $\mathcal{M} = \mathbb{Z}$ with the oriented lattice calculus (example 3 in section III). In this case we have

$$e_a(i) e_{a+2}(i+1) = 0 \quad (7.14)$$

which implies

$$e_{a,a+1}(i) e_{a+2}(i+1) = e_a(i) e_{a+1,a+2}(i+1) \quad (7.15)$$

and

$$d(e_a(i) e_{a+1}(i+1)) = e_{a,a+1}(i+1) - e_{a,a+1}(i) . \quad (7.16)$$

In the case under consideration, the most general first order action takes the form

$$S = \sum_{a \in \mathbb{Z}} \left(\sum_{i=0}^{N-2} K_a(i) e_a(i) e_{a+1}(i+1) - \sum_{i=0}^{N-1} V_a(i) e_a(i) \right) . \quad (7.17)$$

With the help of (7.16) we can calculate its differential,

$$\begin{aligned} dS &= \sum_{a \in \mathbb{Z}} \left(\sum_{i=0}^{N-2} K_a(i) [e_{a,a+1}(i+1) - e_{a,a+1}(i)] - \sum_{i=0}^{N-1} [V_{a+1}(i) - V_a(i)] e_{a,a+1}(i) \right) \\ &= \sum_{a \in \mathbb{Z}} \left(- \sum_{i=1}^{N-2} [K_a(i) - K_a(i-1) + V_{a+1}(i) - V_a(i)] e_{a,a+1}(i) \right. \\ &\quad \left. - [K_a(0) + V_{a+1}(0) - V_a(0)] e_{a,a+1}(0) \right. \\ &\quad \left. + [K_a(N-2) - V_{a+1}(N-1) + V_a(N-1)] e_{a,a+1}(N-1) \right) \\ &=: \sum_{a \in \mathbb{Z}} \sum_{i=0}^{N-1} S_a(i) e_{a,a+1}(i) . \end{aligned} \quad (7.18)$$

The inequalities (7.13) now read

$$\sum_{i=0}^{N-1} S_{\gamma(i)}(i) \delta_{\gamma(i)+1, \alpha(i)} \geq 0 \quad (7.19)$$

$$\sum_{i=0}^{N-1} S_{\gamma(i)-1}(i) \delta_{\gamma(i)-1, \alpha(i)} \leq 0 \quad (7.20)$$

for all curves $\alpha \in \Gamma$ which are neighbors of γ . In the case under consideration there are not enough neighbors in Γ so that we could convert the sums into ‘local’ inequalities in the sense that they involve at most two time steps.

Let us specify the action by choosing $V_a(i) = 0$ and $K_a(i) = 1$ ($\forall a \in \mathbb{Z}, i \in \mathbb{N}$), so that

$$S = \sum_{i=0}^{N-2} \sum_{a \in \mathbb{Z}} e_a(i) e_{a+1}(i+1) . \quad (7.21)$$

Obviously, $0 \leq S(\gamma) \leq N-1$ and $S(\gamma)$ is the length of the path corresponding to the curve γ . The curve given by $\gamma_{\min}(i) := a$ (for some fixed $a \in \mathbb{Z}$) is a minimum, $\gamma_{\max}(i) := i + a$ is a maximum of S . We find

$$dS = \sum_{a \in \mathbb{Z}} [e_{a,a+1}(N-1) - e_{a,a+1}(0)] \quad (7.22)$$

and therefore

$$dS(\gamma, \alpha) = \delta_{\alpha(N-1), \gamma(N-1)+1} - \delta_{\alpha(0), \gamma(0)+1} \quad (7.23)$$

$$dS(\alpha, \gamma) = \delta_{\alpha(N-1), \gamma(N-1)-1} - \delta_{\alpha(0), \gamma(0)-1} . \quad (7.24)$$

For γ_{min} this leads indeed to $dS(\gamma_{min}, \alpha) \geq 0$ and $dS(\alpha, \gamma_{min}) \leq 0$.

Admittedly, this example is too simple to be of real interest. It nicely demonstrates, however, how our calculus works. \square

VIII. DISCRETE DIFFERENTIAL MANIFOLDS AND TOPOLOGICAL SPACES

Let $\Omega(\mathcal{M})$ be a differential calculus on a discrete set \mathcal{M} with elements i, j, \dots and $\{e_I\}_{I=(i_1 i_2 \dots)}$ a basis of $\Omega(\mathcal{M})$ (as a \mathbb{C} -vector space) consisting of basic r -forms. We represent these forms as vertices of a digraph in such a way that vertices corresponding to $(r+1)$ -forms are below those corresponding to r -forms. If in the differential calculus some e_J appears in the expression (2.7) for de_I , then we draw an edge between the vertices representing e_J and e_I . The result is a *Hasse diagram* which determines a topology in the following way [3]. Any vertex together with all lower lying vertices which are connected to it forms an open set. Together with the empty and the whole set, the open sets obtained in this way define a topology.

Although for each $i \in \mathcal{M}$ we obtain an open set containing i , we do not have points in \mathcal{M} lying in intersections of these sets. This suggests to consider the *extended set* $\hat{\mathcal{M}}$, the points of which correspond to the vertices of the Hasse diagram. With the topology determined by the Hasse diagram, $\hat{\mathcal{M}}$ becomes a topological space, the *extended space*.

Let X be a countable set and τ a locally finite topology on it, i.e. a collection of open sets such that

$$U(a) := \bigcap_{a \in U \in \tau} U \quad (8.1)$$

is open for each $a \in \hat{\mathcal{M}}$. Then

$$a \hookrightarrow b \iff a \in U(b) \quad (8.2)$$

defines a *preorder* (a transitive and reflexive relation) on X (cf [3]). This order relation is displayed in a Hasse diagram in such a way that $a \hookrightarrow b$ iff the vertex a is connected from below to the vertex b .

Now the following question arises. Is every finite topological space (or, more generally, countable space with locally finite topology) (X, τ) the extended space of some discrete differential manifold ?

Definition. A topological space (X, τ) is *generated* by a discrete differential manifold $(\mathcal{M}, \Omega(\mathcal{M}))$ if

- (1) X is the extended set $\hat{\mathcal{M}}$ of \mathcal{M} ,
- (2) $\Omega(\mathcal{M})$ induces the topology τ on X .

Our construction of topological (extended) spaces from differential calculi on countable sets reaches many examples. Trivially, any set with the discrete topology is ‘generated’ (with $\Omega^r(\mathcal{M}) = 0$ for $r > 0$). In general, the answer to the above question is ‘no’, however. A simple counterexample is the 2-point set with the indiscrete topology (consisting of the empty and the whole set only). This space is not of much interest, however. It is excluded if we confine our considerations to T_0 -spaces. (X, τ) is a T_0 -space if for each pair of distinct points in X there is an open set containing one point but not the other. This is the case if and only if \hookrightarrow is a *partial order* in which case X receives the structure of a *poset* (partially ordered set) [3]. But there are also counterexamples which are T_0 -spaces. On the 2-point set with elements a and b we may choose as open sets $\{a\}$, $\{a, b\}$ (together with the empty set). A T_0 -counterexample with a 3-point set is $\bullet \hookrightarrow \bullet \hookleftarrow \bullet$. One might think of imposing a stronger condition than T_0 . T_1 requires that each set which consists of a single point is closed. This is too strong since the lowest points in a Hasse diagram form open sets. The Hausdorff property T_2 is obviously too strong.

Example. Let (X, τ) be the topological 3-point space determined by $\bullet \hookrightarrow \bullet \hookleftarrow \bullet$. Let \mathcal{M} be the 2-point set consisting of the first and the last point. With the differential calculus on \mathcal{M} corresponding to the digraph $\bullet \rightarrow \bullet$ (or $\bullet \leftarrow \bullet$) one finds that (X, τ) is ‘generated’. The topology is T_0 but not T_1 . \square

The condition for a topological space to be ‘generated’ seems to eliminate less useful topologies. It has still to be explored how restrictive this condition actually is. The most interesting aspect of a generated topological space is that all the information about this space is already contained in a subset with a digraph structure. In some cases this subset is much smaller than the original set. It may be finite even when the original set is infinite.

IX. DIFFERENTIABILITY IMPLIES CONTINUITY

We have seen that a discrete differential manifold generates a topological space. One should then expect that a differentiable map between discrete differential manifolds extends to a continuous map between the corresponding topological spaces.

In this section we inessentially depart from our definition of the Hasse diagram in the previous section. If there is a form which is annihilated by d , then we draw a line to an additional lower lying vertex which represents $0 \in \Omega(\mathcal{M})$. In the topology determined by the Hasse diagram this vertex stands for the empty set. Instead of labeling the vertices of the Hasse diagram by the elements of a basis of $\Omega(\mathcal{M})$ consisting of basic forms, it seems to be more appropriate to use the dual basis. The reason is the following.

The elements of \mathcal{M} may be identified with linear maps dual to the functions e_i . This suggests the construction of an extended space $\hat{\mathcal{M}}$, the points of which are objects dual to the forms $\{e_I\}$. In any case, the points of $\hat{\mathcal{M}}$ correspond to the vertices of the Hasse diagram (as in section VIII).

If a map $\phi : \mathcal{M} \rightarrow \mathcal{M}'$ is differentiable (with respect to differential calculi $\Omega(\mathcal{M})$ and $\Omega(\mathcal{M}')$), we will see that there is a natural extension to a map $\phi_\star : \hat{\mathcal{M}} \rightarrow \hat{\mathcal{M}}'$ and this map is then continuous with respect to the topologies defined by the Hasse diagrams (derived from $\Omega(\mathcal{M})$ and $\Omega(\mathcal{M}')$, respectively).

Let $\Omega(\mathcal{M})^*$ be the dual of $\Omega(\mathcal{M})$ as a \mathbb{C} -vector space. A basis $\{e^J\}$ is defined by

$$\delta_I^J = e^J(e_I) =: \langle e^J, e_I \rangle . \quad (9.1)$$

These basis elements together with 0 constitute the points of $\hat{\mathcal{M}}$.

A boundary operator $\partial : \Omega^r(\mathcal{M})^* \rightarrow \Omega^{r-1}(\mathcal{M})^*$ dual to d is now determined via

$$\langle \partial e^J, e_I \rangle = \langle e^J, de_I \rangle . \quad (9.2)$$

If $\phi : \mathcal{M} \rightarrow \mathcal{M}'$ is a differentiable map with respect to differential calculi $\Omega(\mathcal{M})$ and $\Omega(\mathcal{M}')$, so that $\phi^\star : \Omega(\mathcal{M}') \rightarrow \Omega(\mathcal{M})$ is a homomorphism of differential algebras, then

$$\langle e^J, \phi^\star e_{I'} \rangle = \langle \phi_\star e^J, e_{I'} \rangle \quad (9.3)$$

defines a linear map $\phi_\star : \Omega(\mathcal{M})^* \rightarrow \Omega(\mathcal{M}')^*$.

Lemma. If $\phi : \mathcal{M} \rightarrow \mathcal{M}'$ is differentiable, then

$$\partial' \phi_\star = \phi_\star \partial . \quad (9.4)$$

Proof:

$$\langle \phi_\star \partial e^J, e_{I'} \rangle = \langle \partial e^J, \phi^\star e_{I'} \rangle = \langle e^J, d \phi^\star e_{I'} \rangle = \langle e^J, \phi^\star d' e_{I'} \rangle = \langle \phi_\star e^J, d' e_{I'} \rangle = \langle \partial' \phi_\star e^J, e_{I'} \rangle .$$

□

Let us write $e_J \in de_I$ if e_J appears in the expression (2.7) for de_I . We also write $e^I \in \partial e^J$ if e^I appears in the corresponding expression for ∂e^J . From (9.2) we infer that

$$e_J \in de_I \quad \Leftrightarrow \quad e^I \in \partial e^J . \quad (9.5)$$

A simple consequence of the last Lemma is then

$$e^I \in \partial e^J \quad \Rightarrow \quad \phi_\star e^I \in \partial' \phi_\star e^J . \quad (9.6)$$

It follows from the next Lemma that ϕ_\star induces a map from $\hat{\mathcal{M}}$ to $\hat{\mathcal{M}}'$. The presence of the auxiliary point in $\hat{\mathcal{M}}'$ which represents $0 \in \Omega(\mathcal{M}')^*$ and stands for the empty set in $\hat{\mathcal{M}}'$ is necessary, since in general ϕ_\star maps some dual forms to 0. If 0 were not represented by a point in $\hat{\mathcal{M}}'$ then ϕ_\star would not define a map $\hat{\mathcal{M}} \rightarrow \hat{\mathcal{M}}'$.

Lemma. If $\phi : \mathcal{M} \rightarrow \mathcal{M}'$ is differentiable, then

$$\phi_\star e^J = e^{\phi(J)} \quad (9.7)$$

where $\phi(J) = \phi(j_1 \dots j_r) := (\phi(j_1) \dots \phi(j_r))$.

Proof:

$$\langle \phi_* e^J, e_{I'} \rangle = \langle e^J, \phi^* e_{I'} \rangle = \langle e^J, \sum_{I \in \phi^{-1}(I')} e_I \rangle = \sum_{I \in \phi^{-1}(I')} \delta_I^J = \delta_{I'}^{\phi(J)} = \langle e^{\phi(J)}, e_{I'} \rangle .$$

□

Proposition. If $\phi : \mathcal{M} \rightarrow \mathcal{M}'$ is differentiable, then ϕ_* is continuous as a map from $\hat{\mathcal{M}}$ to $\hat{\mathcal{M}}'$ (with respect to the topologies derived from the differential calculi on $\Omega(\mathcal{M})$ and $\Omega(\mathcal{M}')$, respectively).

Proof: Let U be a nonempty open set in $\hat{\mathcal{M}}'$. Let us assume that $\phi_*^{-1}(U)$ is not open in $\hat{\mathcal{M}}$. In the Hasse diagram defining the topology of $\hat{\mathcal{M}}$ there must then be a vertex corresponding to a point in $\phi_*^{-1}(U)$ which is connected below to a vertex which does not correspond to a point in $\phi_*^{-1}(U)$. The latter vertex cannot be the one which stands for the empty set in $\hat{\mathcal{M}}$ since ϕ_* maps it to the empty set in $\hat{\mathcal{M}}'$ which belongs to every open set, so also to U . Hence there are dual forms e^I, e^J such that

$$\phi_* e^I \in U, \quad \phi_* e^J \notin U, \quad e^I \in \partial e^J .$$

But (9.6) then implies

$$\phi_* e^I \in \partial' \phi_* e^J .$$

Now we have a contradiction since U is open. □

X. CONCLUSIONS

Motivated by the results in [1] we have introduced the notion of a ‘discrete differential manifold’ and suggested to regard it as an analogue of (continuous) differentiable manifolds.

This structure has already been shown to be useful for lattice field theories [10]. As we have demonstrated in the present work, it is also a convenient mathematical framework to study mechanics on discrete spaces. We are, however, not bound to an interpretation of the underlying discrete set as an analogue of space or space-time. Let us give some other examples.

The set $\mathcal{T}(\mathcal{M})$ of all topologies on a given finite set \mathcal{M} is partially ordered by set inclusion. For $\alpha, \beta \in \mathcal{T}(\mathcal{M})$, the relation $\alpha \subset \beta$ means that the topology α is coarser (weaker) than β or, equivalently, β is finer (stronger) than α . There is a natural way to associate a graph with $\mathcal{T}(\mathcal{M})$. We represent each topology by a vertex. If $\alpha \subset \beta$ such that $\alpha \neq \beta$ and there is no $\gamma \neq \alpha, \beta$ with $\alpha \subset \gamma \subset \beta$, then we draw an edge between the vertex α and the vertex β . Turning edges into arrows (or pairs of antiparallel arrows) yields a digraph which defines a differential calculus on $\mathcal{T}(\mathcal{M})$. A (differentiable) curve in $\mathcal{T}(\mathcal{M})$ describes topology change on \mathcal{M} . If, for example, we choose the orientation of the arrow from α to β according to the relation $\alpha \subset \beta$, a differentiable curve describes topology change only from coarser to finer topology. It goes the other way if we reverse all the arrows. See also [6] in this context.

Any differential calculus on a discrete set \mathcal{M} is the quotient of the universal differential calculus by some differential ideal \mathcal{I} . The inclusion of ideals then partially orders the set

$\mathcal{DC}(\mathcal{M})$ of all differential calculi on \mathcal{M} , i.e. the set of all discrete differential manifolds with point space \mathcal{M} . $\mathcal{DC}(\mathcal{M})$ naturally carries the structure of a graph. If the relation $\mathcal{I} \subset \mathcal{I}'$ holds with $\mathcal{I} \neq \mathcal{I}'$ and there is no $\mathcal{I}'' \neq \mathcal{I}, \mathcal{I}'$ such that $\mathcal{I} \subset \mathcal{I}'' \subset \mathcal{I}'$, then we draw an edge between the vertices representing the two differential calculi $\tilde{\Omega}(\mathcal{M})/\mathcal{I}$ and $\tilde{\Omega}(\mathcal{M})/\mathcal{I}'$. Turning edges into arrows (or pairs), we obtain a digraph which then defines a differential calculus on $\mathcal{DC}(\mathcal{M})$. A curve in $\mathcal{DC}(\mathcal{M})$ describes a change of the differential calculus on \mathcal{M} .

An algebraic approach to discrete mechanics appeared recently in [22]. The authors of that paper considered a commutative algebra \mathcal{A} over a commutative ring k . It is assumed that \mathcal{A} is freely generated, say, by elements x_1, \dots, x_n . It is then possible to have functions commuting with differentials ('Kähler differentials'). Choosing $k = \mathbb{Z}$, for example, the formalism is able to describe motion on the lattice \mathbb{Z}^n . In contrast, we consider algebras of functions over $k = \mathbb{C}$. On a discrete set these are subject to the constraints (2.2) which force us to work with a 'noncommutative differential calculus'.

A next step in our programme should be a formulation of discrete quantum mechanics (see [21] and references given there) on discrete differential manifolds. Here a path integral approach is preferred.

The association of a digraph with a discrete differential manifold suggests a natural way how to quantize it, namely to turn it into a 'quantum network' (e.g., in the sense of Finkelstein [7], see also [8]). This is a further interesting route to proceed.

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