POINCARÉ DUALITY IN HOCHSCHILD (CO)HOMOLOGY

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Abstract

These are notes on van den Bergh's analogue of Poincaré duality in Hochschild (co)homology [VdB98]. They are based on survey talks that I gave in 2006 in Göttingen, Cambridge and Warsaw and consist of an elementary explanation of the proof in terms of Ischebeck's spectral sequence [Isch69] and a detailed discussion of the commutative case, plus some motivating background material. The reader is assumed to be familiar with standard homological algebra, but the commutative algebra and algebraic geometry needed to understand the commutative case is recalled. For more preliminaries see e.g. [Ei77, Se00] (commutative algebra and algebraic geometry), [MR01] (noncommutative rings) and [Bou87, CE56, Wei95] (homological algebra).

I would like to thank Shahn Majid, Andreas Thom and Jan Dereziński for the invitations to give the talks on which this note is based and for discussions on its subject. Thanks for discussions go also to Tomasz Maszczyk and Yorck Sommerhäuser.

CONTENTS

1. POINCARÉ DUALITY IN TERMS OF TOR AND EXT

1.1. THE FUNCTOR $M \mapsto M^*$

Let R be a unital, associative ring and consider the functor that sends a (left) module to its linear dual (which is a right module with action $(\phi x)(m) := \phi(m)x$, $\phi \in M^*, x \in R, m \in M$),

$$R\text{-}\mathbf{Mod} \to \mathbf{Mod}\text{-}R, \quad M \mapsto M^* = \mathrm{Hom}_R(M,R). \tag{1}$$

Except when R is quasi-Frobenius (injective as R-module), this is not an exact functor, and its derived functors $\mathrm{Ext}^n_R(\cdot, R)$ define important invariants of M such as

$$\mathrm{grade}(M) := \inf\{n \,|\, \mathrm{Ext}^n_R(M,R) \neq 0\} \in \mathbb{N} \cup \{\infty\}. \tag{2}$$

As in the case of vector spaces over a field, its properties are also related to the size of M. If M is for example projective, then M^* needs not to be projective (e.g. $\mathbf{Mod}\text{-}\mathbb{Z} \ni \prod_{\mathbb{N}} \mathbb{Z} \simeq (\bigoplus_{\mathbb{N}} \mathbb{Z})^*$ is not, see [La99] for a nice proof). But if M is finitely generated projective, then it is not difficult to see that so is M^*, that $M^{**} \simeq M$, and that for all $N \in R\text{-}\mathbf{Mod}$ the canonical morphism

$$M^* \otimes_R N \to \mathrm{Hom}_R(M,N), \quad \phi \otimes n \mapsto (m \mapsto \phi(m)n) \tag{3}$$

is bijective. For arbitrary M this is in general neither injective nor surjective.

1.2. THE ISCHEBECK SPECTRAL SEQUENCE

Now we study (3) for modules M which are not finitely generated projective but not too far away from being so. Viewing (3) as a morphism of functors (leave M, N open) and taking derived functors one obtains the following classical result [Isch69]:

Theorem 1..1. *Assume that $M \in R$-**Mod** admits a finite resolution*

$$0 \to P_d \to P_{d-1} \to \ldots \to P_1 \to P_0 \to M \to 0 \tag{4}$$

*by finitely generated projective modules. Then for any $N \in R$-**Mod** there is a convergent spectral sequence*

$$E^2_{-pq} = \mathrm{Tor}^R_q(\mathrm{Ext}^p_R(M,R), N) \Rightarrow \mathrm{Ext}^{p-q}_R(M,N), \quad p,q \geq 0. \tag{5}$$

Proof. Some people would say this is obvious, but we include the details to see where the assumptions precisely enter. We fix a projective resolution $Q_\bullet$ of N and define the bicomplex

$$C_{pq} := \mathrm{Hom}_R(P_{-p}, Q_q) \simeq P^*_{-p} \otimes_R Q_q, \quad p \leq 0, q \geq 0. \tag{6}$$

The minus sign at the p is just to turn the cochain complex $\mathrm{Hom}_R(P_\bullet, Q_q)$ (fixed q) into a chain complex (negatively graded). The isomorphism $\simeq$ from (3) holds since P_{-p} is finitely generated projective, so here this assumption is used.
Now one computes the homology of the total complex $\mathrm{Tot}_n := \bigoplus_{p+q=n} C_{pq}$ using the two spectral sequences arising from its filtration by rows and by columns. Here the finite length d of $P_\bullet$ becomes crucial. It implies that after a shift $+d$ in degree p our bicomplex is in the first quadrant and hence both spectral sequences converge and converge to the same object (since $\bigoplus_{p+q=n} C_{pq} = \prod_{p+q=n} C_{pq}$). Convergence alone would be automatic for example when R is

Gorenstein (has finite injective dimension as R-module), but even if both spectral sequences stabilise on the second page the result can be wrong (this led to the erratum to [VdB98]).
The first spectral sequence starts with computation of homologies of $C_{p\bullet}$ for fixed p, the boundary lowering q. Since $Q_\bullet$ is a projective resolution of N, this gives $\mathrm{Tor}^R_\bullet(P^*_{-p}, N)$. But since P_{-p} and hence P^*_{-p} is finitely generated projective, these Tor's vanish for $q > 0$. For $q = 0$ we have $\mathrm{Tor}^R_0(P^*_{-p}, N) = P^*_{-p} \otimes_R N \simeq \mathrm{Hom}_R(P_{-p}, N)$. Thus the first page of the spectral sequence is

$$
{}^{I}E^1_{pq} = \begin{cases} \mathrm{Hom}_R(P_{-p}, N) & q = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}
$$

In the next step of the spectral sequence one continues with the boundary on ${}^{I}E^1$ that lowers p. Since $P_\bullet$ is a projective resolution of M, this gives

$$
{}^{I}E^2_{pq} = \begin{cases} \mathrm{Ext}^{-p}_R(M, N) & q = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8}
$$

Since all terms of this page vanish except for $q = 0$, the spectral sequence becomes stable and we obtain the total homology of our bicomplex

$$
\mathrm{H}_n(\mathrm{Tot}(C)) \simeq \mathrm{Ext}^{-n}_R(M, N). \tag{9}
$$

The other spectral sequence is the one whose existence is the claim of the theorem. Here one fixes conversely first q. Since P^*_{-p} and Q_q are projective and hence flat, the universal coefficient theorem gives

$$
\mathrm{H}_{-p}(\mathrm{Hom}_R(P_\bullet, R) \otimes_R Q_q) \simeq \mathrm{H}_{-p}(\mathrm{Hom}_R(P_\bullet, R)) \otimes_R Q_q, \tag{10}
$$

so the first page of this spectral sequence is

$$
{}^{II}E^1_{pq} = \mathrm{Ext}^{-p}_R(M, R) \otimes_R Q_q. \tag{11}
$$

In the second step of this spectral sequence one now clearly gets

$$
{}^{II}E^2_{pq} = \mathrm{Tor}^R_q(\mathrm{Ext}^{-p}_R(M, R), N) \tag{12}
$$

since $Q_\bullet$ is a projective resolution of N. ☐

1.3. POINCARÉ DUALITY

There are two simple cases in which Ischebeck's spectral sequence stabilises on its second page. The first one is when all E^2-terms are zero for $q \neq 0$:

Corollary 1..2. *Suppose M is as in Theorem 1..1 and N is flat. Then*

$$
\mathrm{Ext}^p_R(M, R) \otimes_R N \simeq \mathrm{Ext}^p_R(M, N) \tag{13}
$$

for all $p \geq 0$. In particular, (3) is an isomorphism.

However, we are even more interested in the orthogonal case:

Definition 1..3. $M \in R$-**Mod** satisfies Poincaré duality in dimension d with dualising module $\omega_M := \mathrm{Ext}^d_R(M, R)$ if it satisfies the assumptions in Theorem 1..1 and $\mathrm{Ext}^n_R(M, R) = 0$ for $n \neq d$.

In this case the E^2-terms of Ischebeck's spectral sequence are zero for all $p \neq d$ and the sequence again stabilises on its second page. Thus Theorem 1..1 yields:

Corollary 1..4. *A module $M \in R$-**Mod** satisfies Poincaré duality if and only if*

$$\mathrm{Tor}_n^R(\omega_M, N) \simeq \mathrm{Ext}_R^{d-n}(M,N) \tag{14}$$

*for all $N \in R$-**Mod**. In particular, one has*

$$\mathrm{proj.dim}_R(M) := \sup\{n \in \mathbb{N} \mid \exists N \in R\text{-}\mathbf{Mod} : \mathrm{Ext}_R^n(M,N) \neq 0\} = d. \tag{15}$$

This is the algebraic mechanism underlying the phenomenon of Poincaré duality well-known in geometry and topology: The homology in degree n say of a compact smooth manifold X can be identified with its cohomology in degree $\dim(X) - n$. Starting from Corollary 1..4 one can derive such identifications in all kinds of (co)homology theories that can be expressed in terms of Tor and Ext over suitable rings. Our main topic is a particularly nice one for Hochschild (co)homology, but before specialising to this, let us make the final general remark that there is also a dual spectral sequence (described as well in [Isch69])

$$\mathrm{Ext}_R^p(\mathrm{Ext}_R^q(M,R),N) \Rightarrow \mathrm{Tor}_{q-p}^R(N,M) \tag{16}$$

in which the roles of Tor and Ext are exchanged. Taking here $N = R$ shows in particular that if M satisfies Poincaré duality in dimension d, then so does ω_M (with everything now developed for right modules), and that $\omega_{\omega_M} \simeq M$.

2. APPLICATION TO HOCHSCHILD (CO)HOMOLOGY

2.1. HOCHSCHILD (CO)HOMOLOGY

In [Ho45] Hochschild introduced the (co)homology groups of a unital associative k-algebra A with coefficients in an A-bimodule N (we assume for simplicity that k is a field). To define Hochschild's theory, let us introduce the opposite algebra A^{op} (same k-vector space, opposite product $a \cdot_{\mathrm{op}} b = ba$) and the enveloping algebra $A^e := A \otimes_k A^{\mathrm{op}}$ of A. Left A-modules are the same as right A^{op}-modules and vice versa. Since $a \otimes b \mapsto b \otimes a$ is an algebra isomorphism $A^e \to (A^e)^{\mathrm{op}}$, left and right A^e-modules become identified, and they are also the same as A-bimodules with symmetric action of k. Thus there are equivalences of categories

$$A\text{-}\mathbf{Mod} \simeq \mathbf{Mod}\text{-}A^{\mathrm{op}}, \quad A^e\text{-}\mathbf{Mod} \simeq \mathbf{Mod}\text{-}A^e \simeq A\text{-}\mathbf{Mod}_k\text{-}A. \tag{17}$$

Definition 2..1. The Hochschild (co)homology groups of A with coefficients in N are

$$\mathrm{H}_n(A,N) := \mathrm{Tor}_n^{A^e}(N,A), \quad \mathrm{H}^n(A,N) := \mathrm{Ext}_{A^e}^n(A,N). \tag{18}$$

If the ground ring k is not assumed to be a field, then one should rather consider k-relative Tor and Ext here, see e.g. [Lo92, Wei95]. Conversely there are k-vector space isomorphisms

$$\mathrm{Tor}_n^A(L,M) \simeq \mathrm{H}_n(A, M \otimes_k L), \quad \mathrm{Ext}_A^n(M',M'') \simeq \mathrm{H}^n(A, \mathrm{Hom}_k(M',M'')) \tag{19}$$

for all $L \in \mathbf{Mod}\text{-}A, M, M', M'' \in A\text{-}\mathbf{Mod}$ (see e.g. [CE56], Chapter IX). Using this, most of the standard (co)homology theories (e.g. group and Lie algebra (co)homology) can be viewed as special cases of Hochschild (co)homology. We refer to [Lo92, Wei95] for explicit descriptions of the Hochschild (co)homology groups in low degrees, but mention only the following one that will be used below and follows immediately from the definition:

Proposition 2..2. *There is a canonical isomorphism of vector spaces*

$$\mathrm{H}^0(A,N) \simeq Z(N) := \{n \in N \mid an = na \ \forall a \in A\}. \tag{20}$$

2.2. SMOOTHNESS AND $\mathrm{DIM}(A)$

Recall that the (left) global dimension of a ring R is

$$\mathrm{gl.dim}(R) := \sup\{\mathrm{proj.dim}_R(M) \,|\, M \in R\text{-}\mathbf{Mod}\}, \tag{21}$$

and that a ring whose (left) global dimension is finite is called (left) regular (the geometric motivation will be reviewed below). In view of (18) and (19), we have:

Proposition 2..3. *There are inequalities* $\mathrm{gl.dim}(A) \leq \mathrm{proj.dim}_{A^e}(A) \leq \mathrm{gl.dim}(A^e)$.

Following [CE56] we call $\mathrm{proj.dim}_{A^e}(A)$ simply the dimension

$$\dim(A) := \mathrm{proj.dim}_{A^e}(A) = \sup\{n \in \mathbb{N} \,|\, \exists N \in A\text{-}\mathbf{Mod}_k\text{-}A : \mathrm{H}^n(A,N) \neq 0\} \tag{22}$$

of A, although it must not be confused in general with the Krull dimension. Unlike the latter or $\mathrm{gl.dim}(A)$ which only see the ring structure of A, $\dim(A)$ can depend heavily on k. For example, A might be a field in which case it is Noetherian and Krull and global dimension vanish, but $A^e = A \otimes_k A$ can be quite wild if k is sufficiently small and A is sufficiently big. While gl.dim is thus quite ill-behaved on tensor products of algebras, we have [CE56], Proposition IX.7.4:

Proposition 2..4. *One has* $\dim(A \otimes_k B) \leq \dim(A) + \dim(B)$.

Since obviously $\dim(A^{\mathrm{op}}) = \dim(A)$, this implies together with Proposition 2..3:

Corollary 2..5. *One has* $\dim(A) < \infty$ *if and only if* $\mathrm{gl.dim}(A^e) < \infty$. *In this case, A is both left and right regular.*

Thus finiteness of $\dim(A)$ is a sharpened form of regularity, and van den Bergh suggested to call algebras with this property smooth:

Definition 2..6. A is called smooth if $\dim(A) < \infty$.

As we remarked already, the converse of Corollary 2..5 is in general not true even for commutative A. However, as we will discuss below, smoothness and regularity actually agree for coordinate rings $A = k[X]$ of affine varieties over perfect fields and then correspond precisely to the nonsingularity of X. Therefore, the terminology is in our opinion well motivated, although probably slightly nonstandard. We warn the reader that there is also a much stronger notion of smoothness ("quasi-freeness") which means $\dim(A) \leq 1$ and is studied for example in [Sch86, CQ95]. Note that even $\mathbb{C}[x,y]$ is not smooth in this sense (but it is of course in ours). Yet another notion of smoothness especially of commutative algebras ("geometric regularity") means regularity of $A \otimes_k K$ for any algebraic field extension $k \subset K$.

2.3. VAN DEN BERGH'S THEOREM

In the setting of Hochschild (co)homology, Corollary 1..4 can be restated as follows:

Corollary 2..7. *If an algebra A satisfies Poincaré duality as an A^e-module, then A is smooth and there are k-vector space isomorphisms*

$$\mathrm{H}_n(A,\omega_A) \simeq \mathrm{H}^{d-n}(A,A), \quad d := \dim(A). \tag{23}$$

In particular, $\mathrm{H}_{\dim(A)}(A,\omega_A) \simeq \mathrm{H}^0(A,A) \simeq Z(A) \neq 0$.

In [VdB98], M. van den Bergh pointed out a nice refinement of the above. To state his result, we recall that an A-bimodule N is invertible provided that there exists another bimodule N^{-1} such that $N \otimes_A N^{-1} \simeq N^{-1} \otimes_A N \simeq A$ as bimodules. Recall also that this means that $N \otimes_A \cdot$ is an equivalence from A^e-**Mod** to A^e-**Mod** itself (cf. Section 9.5 in [Wei95]). The result of van den Bergh is the following:

Theorem 2..8. *Suppose that $A \in A^e$-**Mod** satisfies Poincaré duality and that $\omega_A \in$ **Mod**-$A^e \simeq A$-**Mod**$_k$-A is invertible. Then A is smooth and*

$$\mathrm{H}_n(A,N) \simeq \mathrm{H}^{d-n}(A, \omega_A^{-1} \otimes_A N), \quad N \in A^e\text{-}\mathbf{Mod}, d := \dim(A). \tag{24}$$

Proof. Look back into the proof of Theorem 1..1 where we computed $E^1_{pq} = {}^{II}E^1_{pq}$ (equation (11) at the very end of the proof). Under our assumptions this is zero except for $p = -d$ where we have $E^2_{-dq} = \omega_A \otimes_{A^e} Q_q$. By plain definition of the tensor product we have

$$\omega_A \otimes_{A^e} Q_q \simeq A \otimes_{A^e} (\omega_A \otimes_A Q_q), \tag{25}$$

and if ω_A is invertible, the functor $\omega_A \otimes_A \cdot$ is an equivalence A^e-**Mod** $\to A^e$-**Mod**, so it sends the projective resolution $Q_\bullet$ of N to the projective resolution $\omega_A \otimes_A Q_\bullet$ of $\omega_A \otimes_A N$. Hence the homology computed in the second step of the spectral sequence E is also the same as $\mathrm{Tor}^{A^e}_\bullet(A, \omega_A \otimes_A N) \simeq \mathrm{H}_\bullet(A, \omega_A \otimes_A N)$ (here we used the canonical identification A^e-**Mod** $\simeq$ **Mod**-A^e). The claim follows. ☐

See e.g. [VdB98, Fa05, BZ06, HK06] for various applications of this theorem. What we will explain in the remainder of this text is its meaning in the setting of affine algebraic geometry: A coordinate ring of an affine variety satisfies duality if and only if the variety is smooth.

3. THE COMMUTATIVE CASE

3.1. PRELIMINARIES

This text is written both for and by someone who is working mainly on noncommutative rings, and therefore I decided to include here a lot of definitions, explanations and proofs concerning commutative algebra and algebraic geometry. I apologise to experts for the blow up.
So let R be now commutative. We identify left and right modules and symmetric bimodules, but note that there are bimodules which are not symmetric.

Definition 3..1. A regular sequence in R is a sequence of elements $x_1, \ldots, x_d \in R$ such that each x_n is not a zero divisor of $R/(x_1, \ldots, x_{n-1})$.

Here $(x_1, \ldots, x_{n-1})$ is the ideal generated by the x_i. The length of maximal regular sequences contained in an ideal $I \subset R$ is equal to $\mathrm{depth}(I,R) := \mathrm{grade}(R/I)$. In algebraic geometry, regular sequences play the role of coordinates transversal to the subspace $V(I) = \{\mathfrak{p} \in \mathrm{Spec}\, R \,|\, I \subset \mathfrak{p}\}$ of the prime ideal space $\mathrm{Spec}\, R$. This elucidates the relation $\mathrm{grade}(R/I) \leq \mathrm{codim}(V(I))$ between the grade and a geometrically defined codimension of $V(I)$. By definition, one has equality for all I when R is Cohen-Macaulay, so then the grade serves as a homologically defined codimension. Being Cohen-Macaulay is a weak notion of regularity since

$$R \text{ regular } \Rightarrow R \text{ Gorenstein } \Rightarrow R \text{ Cohen-Macaulay}, \tag{26}$$

but the inverse implications do not hold in general.
All the attributes in (26) are local properties, that is, R is regular, Gorenstein or Cohen-Macaulay iff all its localisations $R_{\mathfrak{p}}$ are so. And as in differential geometry, coordinate systems are helpful often only locally. For example, one has for local rings the following theorem [Va67]:

Theorem 3..2. *A proper ideal I in a Noetherian local ring R with $\mathrm{proj.dim}_R(I) < \infty$ is generated by a regular sequence of length d if and only if $I/I^2 \in R/I$-**Mod** is free of rank d.*

If R is the Noetherian local ring of a variety X in $x \in X$ (to be interpreted as the ring of rational functions on X that are regular in x), then R is regular iff X has no singularity in x. Its maximal ideal $\mathfrak{m}$ consists of the functions vanishing in x, the canonical map $R \to R/\mathfrak{m} =: k$ corresponds to the evaluation of a function in x, and $\mathfrak{m}/\mathfrak{m}^2$ is geometrically the cotangent space of X in x. Then the above theorem links regular sequences generating $\mathfrak{m}$ (local coordinates on X around x) to k-vector space bases of $\mathfrak{m}/\mathfrak{m}^2$ (formed by the differentials of the coordinates).

3.2. LOCAL POINCARÉ DUALITY

In this section we prove a general result that establishes Poincaré duality for quotients of commutative rings by ideals generated by regular sequences. We will later apply this to the local rings of smooth affine varieties, hence the title of this paragraph. Throughout, we assume that R a commutative regular Noetherian ring. In particular, any finitely generated module admits a finite resolution by finitely generated projective modules.

Theorem 3..3. *Suppose $x_1, \dots, x_d \in R$ form a regular sequence. Then $M := R/(x_1, \dots, x_d) \in R$-**Mod** satisfies Poincaré duality in dimension d with $\omega_M \simeq M$.*

Proof. This follows from some standard results in commutative algebra. First, we need:

Proposition 3..4. *Suppose $x \in R$ is not a zero divisor of R and $N \in R$-**Mod**. Then there are isomorphisms of R-modules*

$$\mathrm{Ext}^n_R(R/(x), N) \simeq \begin{cases} N/(x)N & n = 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{27}$$

Proof. By the assumptions, there is a short exact sequence of R-modules

$$0 \longrightarrow R \xrightarrow{x\cdot} R \longrightarrow R/(x) \longrightarrow 0\,. \tag{28}$$

This provides a free resolution of $R/(x)$ which one can use to compute $\mathrm{Ext}_R(R/(x), N)$ as the cohomology of the complex

$$0 \longrightarrow \mathrm{Hom}_R(R, N) \xrightarrow{\phi \mapsto \phi(x\cdot)} \mathrm{Hom}_R(R, N) \longrightarrow 0\,. \tag{29}$$

Finally, $\mathrm{Hom}_R(R,N) \to N$, $\phi \mapsto \phi(1)$ induces an isomorphism with the complex

$$0 \longrightarrow N \xrightarrow{x\cdot} N \longrightarrow 0\,. \tag{30}$$

The claim follows. □

Recall next that the injective envelope (or injective hull) $I(N)$ of $N \in R$-**Mod** is the unique injective left R-module containing N as an essential submodule (that is, $N \cap M = 0$ for $M \subset I(N)$ implies $M = 0$). See e.g. [Bou87] for more information.

Proposition 3..5. *Let $x \in R$ be not a zero divisor of $N \in R$-**Mod** but act trivially on $L \in R$-**Mod**. Then* $\mathrm{Hom}_R(L, I(N)) = 0$.

Proof. If $\phi \in \mathrm{Hom}_R(L, I(N))$, then $0 = \phi(0) = \phi(xy) = x\phi(y)$ for all $y \in L$, so $\mathrm{im}\,\phi \cap N = 0$ since x is not a zero divisor of N. But N is an essential submodule of $I(N)$, so $\mathrm{im}\,\phi = 0$. ☐

As one of its main applications, the concept of injective envelope allows to construct a unique minimal injective resolution of any $N \in R$-**Mod**,

$$0 \xrightarrow{i_{-1}=0} N \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} \ldots, \quad I_n := I(\mathrm{coker}\, i_{n-1}), \quad n \geq 0. \tag{31}$$

Proposition 3..6. *Let $x \in R$ be not a zero divisor of $R, N \in R$-**Mod** and $I_\bullet$ be the minimal injective resolution of N. Then*

$$\mathrm{Hom}_R(R/(x), I_1) \to \mathrm{Hom}_R(R/(x), I_2) \to \ldots \tag{32}$$

*is an injective resolution of $N/(x)N \in R/(x)$-**Mod**.*

Proof. Since $I_\bullet$ is an injective resolution, the cohomology of

$$0 \to \mathrm{Hom}_R(R/(x), I_0) \to \mathrm{Hom}_R(R/(x), I_1) \to \mathrm{Hom}_R(R/(x), I_2) \to \ldots \tag{33}$$

is $\mathrm{Ext}_R(R/(x), N)$. But $\mathrm{Hom}_R(R/(x), I_0) = 0$ by Proposition 3..5. Therefore, the computation of $\mathrm{Ext}_R(R/(x), N)$ in Proposition 3..4 shows that the terms in degree ≥ 1 form a resolution of $N/(x)N$. For the injectivity of $\mathrm{Hom}_R(R/(x), I_n) \in R/(x)$-**Mod** see e.g. [Ei77], Lemma A3.8. ☐

Since for any R-module N and any $R/(x)$-module L we have

$$\mathrm{Hom}_{R/(x)}(L, \mathrm{Hom}_R(R/(x), N)) \simeq \mathrm{Hom}_R(L \otimes_{R/(x)} R/(x), N) \simeq \mathrm{Hom}_R(L, N), \tag{34}$$

the above implies immediately:

Corollary 3..7. *Let $x \in R$ be not a zero divisor of $R, N \in R$-**Mod** but act trivially on $L \in R$-**Mod**. Then there is an isomorphism of R-modules* $\mathrm{Ext}^n_R(L, N) \simeq \mathrm{Ext}^{n-1}_{R/(x)}(L, N/(x)N)$.

Assume in particular that $x_1, \ldots, x_d \in R$ form a regular sequence. Then Theorem 3..3 follows by repeated application of this corollary with $x = x_1, \ldots, x_d$, $L = R/(x_1, \ldots, x_d)$, $N = R$. ☐

3.3. THE HOCHSCHILD-KOSTANT-ROSENBERG THEOREM

From now on we focus on the setting of affine algebraic geometry and assume that A is the coordinate ring $k[X]$ of an (irreducible) affine variety over a perfect field k. That is, A is a quotient of a polynomial ring $k[x_1, \ldots, x_n]$ without zero divisors, and every finite field extension $k \subset K$ is separable (this includes of course algebraically closed fields, but also fields of characteristic 0 and finite fields). One could work in greater generality, but we want to avoid any technicality (see e.g. [Lo92], Section 3.4 and Appendix E and [Wei95], Sections 9.3.1 and 9.3.2[1]).
Recall first that the formal (Kähler) differentials over A are defined by

$$\Omega^n(A) := \Lambda^n_A \Omega^1(A), \quad \Omega^1(A) := \ker \mu / (\ker \mu)^2, \tag{35}$$

where $\mu : A^e = A \otimes_k A =: k[X \times X] \to A$ denotes the multiplication map. Note that $(\Omega^1(A))^*$ can be identified with $\mathrm{Der}_k(A)$, the k-linear derivations of A.
The fundamental paper on the Hochschild (co)homology of $k[X]$ is [HKR62] where amongst other things the following results were obtained:

[1] Be aware that there were some serious mistakes in these sections in the first edition.

Theorem 3..8. *1. If* $A = k[X], B = k[Y]$ *as above are regular, then so is* $A \otimes_k B =: k[X \times Y]$*. In particular, A is smooth iff it is regular.*
2. A is smooth iff $\Omega^1(A)$ *is finitely generated projective.*
3. There are isomorphisms of A-modules

$$\Omega^1(A) \simeq \mathrm{H}_1(A,A), \quad \mathrm{Der}_k(A) \simeq \mathrm{H}^1(A,A). \tag{36}$$

4. If A is smooth, then there are isomorphisms of A-modules

$$\Omega^n(A) \simeq \mathrm{H}_n(A,A), \quad \Lambda^n_A \mathrm{Der}_k(A) \simeq \mathrm{H}^n(A,A) \simeq (\Omega^n(A))^*. \tag{37}$$

3.4. GLOBAL POINCARÉ DUALITY

Now we prove that Poincaré duality as in Theorem 2..8 is not an exotic phenomenon in the commutative case:

Theorem 3..9. $A = k[X]$ *is smooth iff it satisfies the assumptions of Theorem 2..8.*

Proof. Poincaré duality implies $\mathrm{proj.dim}_{A^e}(A) < \infty$, so $\Leftarrow$ is obvious.
For the other direction we consider the localisations of the right A^e-modules $\mathrm{H}^n(A,A^e)$ at $\mathfrak{q} \in \mathrm{Spec}\, A^e$. Since $A^e = k[X \times X]$ is Noetherian, we can use the compatibility of Ext with localisation (see [Wei95], Proposition 3.3.10):

Proposition 3..10. *If R is a commutative Noetherian ring and* $M, N \in R$-**Mod** *are finitely generated, then for all* $\mathfrak{p} \in \mathrm{Spec}\, R$ *one has* $(\mathrm{Ext}^n_R(M,N))_{\mathfrak{p}} \simeq \mathrm{Ext}^n_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.

This implies in particular that $(\mathrm{H}^n(A,A^e))_{\mathfrak{q}} = 0$ unless $\ker \mu \subset \mathfrak{q}$, since otherwise $A_{\mathfrak{q}} = 0$ (this is the localisation of the A^e-module A with module structure induced by μ). Geometrically speaking, this means that $(\mathrm{H}^n(A,A^e))_{\mathfrak{q}}$ is supported only on X, embedded into $X \times X$ as the diagonal, or in terms of prime ideals on the image of the homeomorphism

$$\mu^* : \mathrm{Spec}\, A \to V(\ker \mu) \subset \mathrm{Spec}\, A^e, \quad \mathfrak{p} \mapsto \mu^{-1}(\mathfrak{p}). \tag{38}$$

By Theorem 3..8, $\Omega^1(A) = \ker \mu / (\ker \mu)^2$ is a finitely generated projective A-module and hence locally free over $\mathrm{Spec}\, A$ (this is the algebraic version of the Serre-Swan theorem that characterises vector bundles as finitely generated projective modules, see e.g. [Se00], p. 73, Corollary 2 and Proposition 20). Hence Theorem 3..2 implies together with Theorem 3..3 that $A_{\mathfrak{q}}$ satisfies for $\mathfrak{q} \supset \ker \mu$ Poincaré duality as an $A^e_{\mathfrak{q}}$-module in dimension d equal to the rank of $\Omega^1(A)$ which is $\dim(X)$, and that the dualising module is $\omega_{A_{\mathfrak{q}}} \simeq A_{\mathfrak{q}}$ itself.
In other words, $\mathrm{H}^n(A,A^e) = 0$ for all n except $n = \dim(X)$ (since a module is zero iff all its localisations are), and as an A-module, $\omega_A = \mathrm{H}^{\dim(X)}(A,A^e)$ is locally free of rank 1, that is, it is the module of sections of an algebraic line bundle over X.
Finally, Proposition 3..10 applied to $\mathrm{H}^0(A, \omega_A)$ implies in view of (20) that ω_A is a symmetric bimodule, so we obtain the identification of ω_A with the sections of our line bundle as A^e-module. Hence it is an invertible bimodule with $\omega_A^{-1} \simeq \mathrm{Hom}_A(\omega_A, A)$, the sections of the dual line bundle (this must not be confused with $\mathrm{Hom}_{A^e}(\omega_A, A^e) = 0$ for $\dim(X) > 0$). ☐

At the end we merge the above result with the Hochschild-Kostant-Rosenberg theorem. It is not difficult to extend (37) to coefficients in finitely generated projective $N \in A$-**Mod** and to identify thus Hochschild homology of A with algebraic differential forms on X with coefficients in the vector bundle whose module of sections is N,

$$\mathrm{H}_n(A,N) \simeq \Omega^n(A,N) := \Omega^n(A) \otimes_A N. \tag{39}$$

Therefore, Theorem 3..9 and Theorem 2..8 (and (20)) allow us to specify the line bundle corresponding to ω_A explicitly and to reformulate Theorem 3..9 as follows:

Theorem 3..11. *If A is the coordinate ring $k[X]$ of a smooth affine variety X over a perfect field k, then for all $N \in A^e$-**Mod** we have*

$$\mathrm{H}^n(A,N) \simeq \mathrm{H}_{\dim(X)-n}(A, (\Omega^{\dim(X)}(A))^{-1} \otimes_A N). \tag{40}$$

Proof. Indeed, we have $\omega_A \otimes_A \Omega^{\dim(X)}(A) \simeq \mathrm{H}^{\dim(X)}(A,\omega_A) \simeq \mathrm{H}^0(A,A) \simeq A$, and both bimodules are symmetric, so $\omega_A \simeq (\Omega^{\dim(X)}(A))^{-1} \simeq \Lambda^{\dim(X)}\mathrm{Der}_k(A)$. ☐

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