TOPOLOGICAL TYPES OF ALGEBRAIC STACKS(FINAL DRAFT)

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Abstract. In developing homotopy theory in algebraic geometry, Michael Artin and Barry Mazur studied the étale homotopy types of schemes. Later, Eric Friedlander generalized them to the étale topological types of simplicial schemes. The aim of this paper is to extend further these theories to algebraic stacks. To achieve this goal, we exploit the derived functor approach of étale homotopy types by Ilan Barnea and Tomer Schlank, and use Daniel Isaksen's model category structure on pro-simplicial sets.

Contents

1. Introduction

1.1. Motivation.

1.1.1. The étale homotopy theory was invented by Michael Artin and Barry Mazur [1] in 1969. Associated to a scheme is their étale homotopy type which is a pro-object in the homotopy category of simplicial sets. This object not only recovers the étale cohomology and Grothendieck's étale fundamental group of the scheme, but also enables one to define homotopical invariants, like higher homotopy groups, of the scheme.

Artin-Mazur's étale homotopy theory has many important applications. They include étale K-theory and the proofs of Adam's conjecture by Quillen-Friedlander and by Sullivan. More recently, people including Kęstutis Česnavičius, Yonatan Harpaz, Ambrus Pal, Tomer M. Schlank, Alexei N. Sokorobogatov use the étale homotopy theory to study rational points of algebraic varieties.

1.1.2. The main goal of this paper is to modify and extend Artin-Mazur's étale homotopy theory. In fact, their theory has two drawbacks. One is that it can be only applied to schemes and more generally to Deligne-Mumford stacks. This is due to the use of the small étale topology, which is not suitable for algebraic stacks. In 1982, Eric Friedlander [8] extended the theory to simplicial schemes, not just schemes. Moreover, his work lifts the étale homotopy types, pro-objects in the homotopy category of simplicial sets, to étale topological types, pro-objects in the category of simplicial sets. However, his theory still cannot not be applied to algebraic stacks (sometimes called Artin stacks). The main issue is again the use of the small étale topology. In this paper, we show that the big étale topology can replace the small étale topology in order to recover Artin-Mazur's and Friedlander's theories. In fact, the big étale topology behaves better and enables us to develop a homotopy theory of algebraic stacks. For instance, we can discuss the étale homotopy type of the classifying stack $\mathcal{B}\mathbb{G}_m$ where $\mathbb{G}_m$ is the multiplicative group scheme over the complex numbers $\mathbb{C}$.

Of course, one can define a homotopy type of an algebraic stack to be the étale topological type of any hypercover which is a simplicial algebraic space. In this way, one can discuss the homotopy type of algebraic stacks. However, we seek for an own definition of homotopy types of algebraic stacks, not depending on hypercovers. As a result, this new approach provides a general frame work for a homotopy theory of algebraic stacks. Furthermore, this general frame work puts the homotopy theory of schemes, algebraic spaces, and algebraic stacks altogether in one place.

1.1.3. On the other hand, the subtlety in the notion of weak equivalence of étale homotopy types results in another drawback of Artin-Mazur's theory. Indeed, étale homotopy types are objects of the pro-category associated to the homotopy category of simplicial sets, rather than the homotopy category associated to the pro-category of simplicial sets, which is more natural. In order to remedy this issue, we put Daniel Isaksen's model category structure [12] on the category of pro-simplicial sets.

1.1.4. In order to develop homotopy theory of algebraic stacks, we use the recent machinery developed by Ilan Barnea and Tomer Schlank [2]. Let $X_{\text{ét}}$ be the small étale topos of a scheme X. Consider the category $X_{\text{ét}}^{\Delta^{\text{op}}}$ of simplicial objects in the topos, and then the pro-category $\text{pro}-X_{\text{ét}}^{\Delta^{\text{op}}}$ associated to it. They defined model category structures to the pro-category and to the category of pro-simplicial sets so that the connected component functor induces a left Quillen functor:

$$\Pi : \text{pro} - X_{\text{ét}}^{\Delta^{\text{op}}} \to \text{pro} - \mathbf{SSet}$$

They proved that one can recover Artin-Mazur's étale homotopy type of X by deriving a final object in $X_{\text{ét}}$ along the left Quillen functor.

Using Barnea-Schlank machinery, we extend the scope of étale homotopy theory from schemes or simplicial schemes to algebraic stacks. The main strength of our approach is that one can systematically deal with étale homotopy types of algebraic stacks thanks to the use of model category theory.

1.1.5. After setting up fundamentals, we provide two main applications of our theory. One is the generalization of Artin-Mazur's comparison theorem. The classical comparison theorem [1, 12.9] says that for a connected finite type scheme X over $\mathbb{C}$, its étale topological type is isomorphic to the underlying complex topological space $X(\mathbb{C})$ of its analytification, after profinite completion. For example, for the multiplicative group scheme $\mathbb{G}_m$ over $\mathbb{C}$, its étale homotopy type is the profinite completion of the unit circle S^1. The comparison theorem is useful because in general étale homotopy types are hard to compute. We generalize t the comparison theorem to the case of algebraic stacks (see 5.2.17 and 5.2.23). For instance, the étale homotopy type of $\mathcal{B}\mathbb{G}_m$ is the profinite completion of $K(\mathbb{Z}, 2)$ (see 5.2.18).

Another main application of our work is the study of étale homotopy types with respect to group actions. David Cox [5, 6.1] showed that for a variety X over $\mathbb{R}$, its étale homotopy type is the homotopy orbit space of the étale homotopy type of $\bar{X} := X \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ with respect to the Galois action $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$. This theorem is useful at it provides a cohomological criterion for the existence of $\mathbb{R}$-points of X; The set of $\mathbb{R}$-points is non-empty if and only if $H^i_{\text{ét}}(X, \mathbb{Z}/2)$ is non-zero for some $i > 2n$ where $n = \dim X$. Later Gereon Quick [28, 5.3] generalized the theorem to an arbitrary base field but with some subtle issue on the Galois action. That is, for a variety X over a field k, it is not clear whether the étale homotopy type of X admits a continuous action from the profinite group $\operatorname{Gal}(k^{\text{sep}}/k)$ where k^{sep} is a separable closure of k. Later he showed that for a geometrically connected quasi-projective variety X over k, one can use Friedlander's rigid Čech étale topological types ([7, 3.1]) to avoid the continuity issue ([31, p.13]). In this paper, we take an alternative approach which can be applied to every scheme over k. For this, we first recover Quick's result at the level of pro-simplicial sets and then show that it generalizes the theorems by Cox and Quick (see 5.1.26).

1.2. **Statement of the main results.**

1.2.1. Fix a locally noetherian base scheme S. Define a site $\text{LFÉ}(S)$ to be the full subcategory of the category of schemes over S, whose objects are locally of finite type morphisms to S with coverings induced by coverings in the big étale topology on S. Note from 3.2.5 that its associated topos $\text{LFÉ}(S)^{\sim}$ is locally connected in a sense that the left adjoint Γ^* of the global section functor admits a left adjoint denoted by Π_S. By the work of Ilan Barnea and Tomer Schlank [2], the category of simplicial objects in the topos $\text{LFÉ}(S)^{\sim}$ has a weak fibration category structure [2, 7.11] and thus induces a model category structure on its pro-category [2, 4.8]. Moreover, there is a Quillen adjunction [2, 8.1] (see also 2.3.5)

$$(\Pi_S, \Gamma^*) : \text{pro} - \mathbf{SSet} \to \text{pro} - (\text{LFÉ}(S)^{\sim})^{\Delta^{\text{op}}}$$

where $\text{pro} - (\text{LFÉ}(S)^{\sim})^{\Delta^{\text{op}}}$ is endowed with Barnea-Schlank's model category structure and the category of pro-simplicial sets, $\text{pro} - \mathbf{SSet}$, is equipped with Isaksen's model category structure [12, 6.4]. Now consider the left derived functor

$$\mathbf{L}\Pi_S : \mathbf{Ho}(\text{pro} - (\text{LFÉ}(S)^{\sim})^{\Delta^{\text{op}}}) \to \mathbf{Ho}(\text{pro} - \mathbf{SSet})$$

between the homotopy categories. We define topological types of simplicial algebraic spaces as follows:

Definition 1.2.2 (Definition 3.4.2)**.** The *topological type of a simplicial algebraic space* $X_\bullet$ over S is the pro-simplicial set

$$h(X_\bullet/S) := \mathbf{L}\Pi_S(X_\bullet)$$

Remark 1.2.3. Our definition is compatible with that of Artin-Mazur for schemes (see [2, 8.3] and 3.3.5) and of Friedlander for simplicial schemes (see 3.3.7).

1.2.4. Let $\mathcal{X}$ be an algebraic stacks. Note that one cannot use the topos $\text{LFÉ}(S)^{\sim}$ because algebraic stacks cannot be viewed as sheaves. Nonetheless, we can still apply the same machinery to the big étale topos on $\mathcal{X}$. More precisely, we apply the Barnea-Schlank machinery to the topos associated to the site $\text{LFÉ}(\mathcal{X})$ which is the full subcategory of the big étale site of the algebraic stack $\mathcal{X}$ with the induced topology (see 3.5.2 for more detail).

Definition 1.2.5 (Definition 3.5.5)**.** The *topological type of an algebraic stack* $\mathcal{X}$ over S is the pro-simplicial set

$$h(\mathcal{X}/S) := \mathbf{L}\Pi_{\mathcal{X}}(*_{\text{LFÉ}(\mathcal{X})^{\sim}})$$

where $\Pi_{\mathcal{X}} : \text{LFÉ}(\mathcal{X})^{\sim} \to \mathbf{Set}$ is the connected component functor.

1.2.6. We can compute the topological types of algebraic stacks via smooth coverings by schemes:

Theorem 1.2.7 (Definition 3.5.9)**.** *Let $\mathcal{X}/S$ be an algebraic stack. For any smooth surjection $X \to \mathcal{X}$ with X a scheme, there is an isomorphism*

$$h(\text{cosk}_0(X/\mathcal{X}))\xrightarrow{\sim} h(\mathcal{X})$$

between pro-simplicial sets in the homotopy category of pro-simplicial sets. Furthermore, these pro-simplicial sets are strictly weakly equivalent.

1.2.8. After building basics on topological types in our own language, we provide a computational tool for topological types. That is, we generalize Artin-Mazur's comparison theorem [1, 12.9] from schemes to algebraic stacks.

Theorem 1.2.9. (Simplicial Comparison)[Theorem 5.2.17] *Let $X_\bullet$ be a pointed finite type simplicial scheme over $\mathbb{C}$. Then the map*

$$\widehat{X_\bullet(\mathbb{C})} \to \widehat{h(X_\bullet)}$$

of the profinite completions of topological types is a weak equivalence of profinite spaces.

Theorem 1.2.10. (Stacky Comparison)[Theorem 5.2.23] *Let $\mathcal{X}$ be a finite type algebraic stack over $\mathbb{C}$. Then the map*

$$\widehat{h(\mathcal{X}^{\text{top}})} \to \widehat{h(\mathcal{X})}$$

of the profinite completions of topological types is a weak equivalence of profinite spaces.

1.2.11. For example, we can compute the topological type of the classifying stack $\mathcal{B}\mathbb{G}_m$ where $\mathbb{G}_m$ is the multiplicative group scheme over $\mathbb{C}$. After profinite completion, the topological type $h(\mathcal{B}\mathbb{G}_m)$ is weakly equivalent to the classifying space BS^1 of the unit circle, which is in turn weakly equivalent to $\mathbb{C}P^\infty$ which is well-known as $K(\mathbb{Z}, 2)$.

1.2.12. One of main contents of this paper is the study of topological types with respect to group actions. Recall the notion of relative topological types by Barnea-Schlank [2, 8.5], which encodes the action of Galois group $G = \text{Gal}(k^{\text{sep}}/k)$ where k^{sep} is a separable closure of the field k. The following shows the relationship between the relative topological type and the usual topological type:

Proposition 1.2.13 (Proposition 4.1.6)**.** *The pro-simplicial set $h_k(X)/G$ which is the relative topological type of X over k taken quotient by G, and the topological type $h(X/k)$ of X are strictly weakly equivalent (see 2.3.16).*

1.2.14. Note that the underlying pro-simplicial set of the relative topological type is the usual topological type of the base change $X^{\text{sep}} = X \times_k k^{\text{sep}}$ (see 4.1.10). So the above proposition leads to a generalization, at the level of pro-simplicial sets, of Gereon Quick's result [28, 5.3] which in turn generalizes David Cox's result [5, 1.1] :

Theorem 1.2.15 (Theorem 5.1.26)**.** *Let X be a scheme over a field k. Then the completion $\widehat{h(X)}$ of the topological type $h(X)$ of X is weakly equivalent to the Borel construction*

$$\widehat{h_k(X)}_G \times_G EG$$

of the G-equivariant completion (5.1.12) of the relative topological type $h_k(X)$ with respect to the Galois group $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

1.3. **Connection to earlier works.**

1.3.1. As mentioned earlier, we develop a homotopy theory of algebraic stacks by using Barnea-Schlank's derived functor definition of étale topological types. They defined *topological realizations* for topoi [2, 8.2] which we refer to *topological types* in this paper. We exploit their approach by studying relationship among various topoi. Especially, localized topoi play a pivotal role.

While we develop a theory of homotopy types, we use topoi rather than sites. Sites are more restricted than topoi which enjoy various categorical properties like the existence of limits and colimits. On top of that, we can deal with more objects like algebraic spaces when working with topoi.

1.3.2. Fix a topos T. For simplicity, assume it has enough points. Remark that a local weak equivalence (resp. a local fibration) between simplicial objects in T is a weak equivalence (resp. a Kan fibration) at stalks. The category $T^{\Delta^{\mathrm{op}}}$ of simplicial objects in T forms a model category structure whose class of weak equivalences (resp. cofibrations) is the class of local weak equivalences (resp. monomorphisms). The class of fibrations is automatically determined by lifting property and is called *global fibrations*. This model category structure is due to Joyal [23], and later generalized to the category of simplicial presheaves by Jardine. One problem lying in the Joyal-Jardine's model category is that the class of fibrations is not equal to that of local fibrations. In fact, a global fibration is a local fibration, but not vice versa (see 2.1.14). However, it is a morphism which is simultaneously a local fibration and a local weak equivalence that generalizes Artin-Mazur's notion of hypercovers [1, 8.4]. Indeed, a morphism $X_\bullet \to Y_\bullet$ of simplicial sheaves is both a local fibration and a local weak equivalence if and only if the following morphisms are epimorphisms (see 2.1.10):

(i) $X_0 \to Y_0$
(ii) $X_{n+1} \to (\mathrm{cosk}_n \,\mathrm{sk}_n\, X_\bullet)_{n+1} \times_{(\mathrm{cosk}_n \,\mathrm{sk}_n\, Y_\bullet)_{n+1}} Y_{n+1}$ for each $n \geq 0$.

Consequently it will become much easier to deal with hypercovers if we have all local fibrations in the class of fibrations in a model category. Unfortunately, this is impossible in most cases (see 2.2.12). However, Barnea-Schlank showed [2, p.55] that one can remedy this issue after enlarging the category to the associated pro-category. Indeed, they proved that there is a model category structure on the pro-category of simplicial objects in T where a local weak equivalence (resp. a local fibration) of simplicial sheaves is a weak equivalence (resp. a fibration) as a morphism in the pro-category.

1.3.3. The key feature of Barnea-Schlank's model category structure is that one can understand Artin-Mazur's étale homotopy types as derived objects. To be more concrete, recall their definition of *topological realization* [2, 8.2]. Assume the topos T is locally connected. i.e., the pull-back of the 2-categorical unique morphism $\Gamma : T \to \mathbf{Set}$ admits a left adjoint. Denote it by Π and call it the connected component functor. In geometric situations, this functor plays the role of the connected component functor. Barnea-Schlank proved [2, p.59] that the adjoint pair (Π, Γ^*) induces a Quillen adjunction with respect to their model category structures. Note that the Barnea-Schlank model category structure in the case of pro-simplicial sets is simply the strict model category structure [15, 4.15] on the pro-category induced by the classical model category on **SSet** (see 2.3.1). We refer to weak equivalences in the strict model category structure as *strict weak equivalences*. However, what we want is a model category on pro-simplicial sets whose weak equivalences are equivalent to those which induce isomorphisms on all homotopy groups. Since there are not enough weak equivalences in the strict model category structure, we should enlarge the class of weak equivalences. This is accomplished by Isaksen [12, 6.4]. Hence we make a variant of Barnea-Schlank's *topological realization* by adopting Isaksen's model category structure on pro-simplicial sets. This does no harm in using Barnea-Schlank's method and gives us a notion of *topological type* of the topos T:

Definition 1.3.4 (Definition 2.3.9)**.** A *topological type* $h(T)$ of a topos T is the pro-simplicial set

$$\mathbf{L}L_{\Gamma^*}(*)$$

where $*$ is a final object of $T^{\Delta^{\mathrm{op}}}$ and $\mathbf{L}L_{\Gamma^*} : \mathbf{Ho}(\mathrm{pro} - T^{\Delta^{\mathrm{op}}}) \to \mathbf{Ho}(\mathrm{pro} - \mathbf{SSet})$ is the left derived functor of L_{Γ^*} between the homotopy categories associated to model categories. More generally, a *topological type* $h(F_\bullet)$ (or $h_T(F_\bullet)$ if we wish to make the reference to T explicit) of a simplicial object $F_\bullet$ in T is the pro-simplicial set

$$\mathbf{L}L_{\Gamma^*}(*)(F_\bullet)$$

Remark 1.3.5. The main improvement compared to Barnea-Schlank's original definition is the definition of the topological types of simplicial objects in T, which comes from the weak equivalence between the topological type $h(F)$ of F in T and the topological type $h(T/F)$ of the localized topos T/F (see 2.3.30).

1.3.6. Isaksen [14] also showed that one can take the derived functor approach for étale topological types. Let S be a noetherian scheme. Consider the category $\mathbf{Sm/S}$ of schemes of finite type over S. By endowing the étale local or the Nisnevich local projective model category structure to the category of simplicial presheaves on $\mathbf{Sm/S}$ (see [14, §2] for details), he proved [14, 2.2] that there is a left Quillen functor from the category of simplicial presheaves on $\mathbf{Sm/S}$ to the category of pro-simplicial sets

$$\text{Ét} : (\widehat{\mathbf{Sm/S}})^{\Delta^{\mathrm{op}}} \to \mathrm{pro} - \mathbf{SSet}$$

where the category of pro-simplicial sets is equipped with Isaksen's model category structure [12, 6.4]. Furthermore, for any scheme $X \in \mathbf{Sm/S}$, the pro-simplicial set $\mathbf{L}\text{Ét}(X)$ is the usual topological type of X in the sense of Friedlander ([14, 2.4]).

Since the class of weak equivalence in the local model category coincides with the class of local weak equivalences, it follows immediately that Isaksen's approach is compatible with ours. Indeed, there is a factorization

$$\begin{array}{ccc}
\mathbf{Ho}(\mathrm{pro}-\widetilde{\mathrm{LF\acute{E}}(S)}^{\Delta^{\mathrm{op}}}) & \xrightarrow{\mathbf{L}\Pi} & \mathbf{Ho}(\mathrm{pro}-\mathbf{SSet}) \\
\uparrow & \nearrow_{\mathbf{L}\acute{\mathrm{E}}\mathrm{t}} & \\
\mathbf{Ho}((\widehat{\mathbf{Sm}/\mathbf{S}})^{\Delta^{\mathrm{op}}}) & &
\end{array}$$

Therefore, our topological type functor can be viewed as a generalization of Isaksen.

1.4. **Outline of the paper.**

1.4.1. In Section 2 we develop a basic theory of topological types. We first review a variety of homotopy theoretical ingredients for pro-simplicial sheaves. Then define topological types in a general context of topoi and provide elementary properties of them. Especially, we obtain a series of descent results (see 2.3.41, 2.3.51, and 2.3.53). In the last subsection, we discuss a connection to cohomology theory.

1.4.2. In Section 3 we apply the general construction to algebro-geometric objects like schemes, algebraic spaces, and algebraic stacks. Also, we prove the compatibility with the classical theories by Artin-Mazur and by Friedlander.

1.4.3. In Section 4 we study topological types with respect to group actions. For this we revisit Barnea-Schlank's notion of relative topological types [2, 8.5] and see its behavior with respect to our topological types 4.1.10.

1.4.4. In Section 5 we have concrete computations on topological types of algebraic stacks over $\mathbb{C}$ via their associated topological stacks. For this we establish a relationship between topological types and profinite completion introduced by Gereon Quick [27]. Finally we prove that our theorem that relates topological types to relative topological types (see 4.1.10) is a generalization of earlier results by David Cox and by Gereon Quick (see 5.1.26).

1.5. **Convention.**

1.5.1. In this paper, an *algebraic space* X over a scheme S is a functor $X : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ such that the following holds:

(i) X is a sheaf with respect to the big étale topology.
(ii) The diagonal
$$\Delta : X \to X \times_S X$$
is representable by schemes.
(iii) There exists a S-scheme U and an étale surjection $U \to X$.

1.5.2. An *algebraic stack* $\mathcal{X}$ over a scheme S is a stack in groupoids over the big étale site $(\mathbf{Sch}/S)_{\acute{e}t}$ of S-schemes such that the following holds:

(i) The diagonal
$$\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$
is representable by algebraic spaces.
(ii) There exists a S-scheme X and a smooth surjection $\pi : X \to \mathcal{X}$.

Remark 1.5.3. These two definitions only assume minimum conditions compared to those in the literature. For example, we do not assume quasi-compactness of the diagonal.

1.5.4. In what follows, for schemes, algebraic spaces, and algebraic stacks, we work over a fixed base scheme S unless stated otherwise. Moreover, we assume that S is locally noetherian throughout the paper.

1.5.5. Most interesting model categories are equipped with a functorial cofibrant replacement functor [11, 8.1.15]. In this paper, however, we work with those model categories not necessarily admitting functorial cofibrant replacement. So when we say a *cofibrant replacement* of X, it only means a chosen trivial fibration $C(X) \to X$ with H cofibrant. On the other hand, a *cofibrant approximation* [11, 8.1.2] of X is any weak equivalence $C \to X$ with C cofibrant.

1.5.6. Ilan Barnea pointed out a set-theoretical issue on the site LFÉ(S) (cf. 3.2.1). For example, the Barnea-Schlank model category structure 2.2.17 is applied to small sites. Whenever this issue arises, we invoke [22, Tag 020M] so that we can assume the smallness on the site LFÉ(S).

1.5.7. Throughout this paper T is a topos and, if necessary, $\mathcal{C}$ is a site whose associated topos is T unless otherwise specified. Also, we assume that $\mathcal{C}$ is small.

1.6. **Acknowledgements.** TO BE ADDED LATER

2. Topological types

In this section we develop a general theory of topological types of topoi.

2.1. Review on simplicial (pre)sheaves.

In their paper, Barnea-Schlank defined *weak fibration categories* [2, 1.2] and used it to construct étale homotopy types as derived functors in the sense of Quillen. In this subsection, we recall basic model categories that are necessary to define topological types of topoi. The main references are [2], [19], and [21].

2.1.1. Recall that for $n \geq 1$ and $0 \leq k \leq n$, the k th horn Λ_n^k of the standard n-simplex $\Delta[n]$ is the sub-simplicial set generated by the image of the face maps $d_i : \Delta[n-1] \to \Delta[n]$ where $0 \leq i \leq n$ and $i \neq k$.

2.1.2. (*The classical model category structure on* ***SSet*** [32, II.§3]) There is a model category structure on the category **SSet** of simplicial sets; if $f : X_\bullet \to Y_\bullet$ is a morphism of simplicial sets,

(i) f is a *weak equivalence* if the induced map on geometric realizations

$$|f| : |X_\bullet| \to |Y_\bullet|$$

is a weak equivalence of topological spaces,

(ii) f is a *cofibration* if it is a monomorphism, and

(iii) f is a *fibration* if it has the right lifting property with respect to all horn inclusions; for all k th horn $\Lambda_n^k \to \Delta[n]$ for $n \geq 1, 0 \leq k \leq n$ and for every commutative diagram

$$\begin{array}{ccc} \Lambda_n^k & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y_\bullet \end{array}$$

there exists a dotted arrow that fills in the diagram. These fibrations are called *Kan fibrations.*

2.1.3. For a simplicial set $X_\bullet$, we can consider homotopy groups with all base points at once:

$$\pi_n(X_\bullet) := \coprod_{x \in X_0} \pi_n(|X_\bullet|, x)$$

Note that a map $X_\bullet \to Y_\bullet$ of simplicial sets is a weak equivalence if and only if the following holds:

(i) $\pi_0(X_\bullet) \to \pi_0(Y_\bullet)$ is a bijection,
(ii) For each $n \geq 1$, the commutative diagram

$$\begin{array}{ccc} \pi_n(X_\bullet) & \longrightarrow & \pi_n(Y_\bullet) \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

is cartesian.

2.1.4. This re-interpretation of weak equivalences of simplicial sets enables us to generalize the notion of weak equivalences from simplicial sets to simplicial presheaves. Indeed, let $X_\bullet$ be a simplicial presheaf on $\mathcal{C}$, or equivalently, a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{SSet}$. For each $n \geq 0$, one can associate a presheaf

$$\widehat{\pi_n}(X_\bullet) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set} : U \mapsto \pi_n(X_\bullet(U))$$

Definition 2.1.5. ([20, p.64]) A morphism $f : X_\bullet \to Y_\bullet$ of simplicial (pre)sheaves is a *local weak equivalence* if the following holds:

(i) The morphism $\widehat{\pi}_0 X_\bullet \to \widehat{\pi}_0 Y_\bullet$ induces an isomorphism of associated sheaves,
(ii) For each $n \geq 1$, the commutative diagram

$$\begin{array}{ccc} \widehat{\pi_n} X_\bullet & \longrightarrow & \widehat{\pi_n} Y_\bullet \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

induces a cartesian diagram of associated sheaves.

Definition 2.1.6. A morphism $f : X_\bullet \to Y_\bullet$ of simplicial (pre)sheaves is a *global fibration* if it has the right lifting property with respect to morphisms which are both local weak equivalence and monomorphism.

Definition 2.1.7. A morphism $f : X_\bullet \to Y_\bullet$ of simplicial (pre)sheaves is a *local fibration* if it has the local right lifting property with respect to all horn inclusions; for all k th horn $\Lambda_n^k \to \Delta[n]$ for $n \geq 1, 0 \leq k \leq n$ and for every commutative diagram

$$\begin{array}{ccc} \Lambda_n^k & \longrightarrow & X_\bullet(U) \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & Y_\bullet(U) \end{array}$$

with $U \in \mathcal{C}$, there exists a covering $\{U_i \to U\}$ such that for each i there exists a dotted arrow that fills in the diagram below:

$$\begin{array}{ccccc} \Lambda_n^k & \longrightarrow & X_\bullet(U) & \longrightarrow & X_\bullet(U_i) \\ \downarrow & & & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y_\bullet(U) & \longrightarrow & Y_\bullet(U_i) \end{array}$$

Remark 2.1.8. If T has enough points, then local weak equivalences (resp. local fibrations) can be checked at stalks.

2.1.9. The following proposition shows the meaning of having local trivial fibrations rather than global fibrations in a model category structure on simplicial sheaves. Indeed, the equivalent conditions in the statement is a generalization of the classical notion of hypercovers (see [1, 8.4]).

Proposition 2.1.10. *A morphism $X_\bullet \to Y_\bullet$ of simplicial sheaves is both a local weak equivalence and a local fibration if and only if the following morphisms of sheaves are epimorphisms;*

(i) $X_0 \to Y_0$
(ii) $X_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n X_\bullet)_{n+1} \times_{(\operatorname{cosk}_n \operatorname{sk}_n Y_\bullet)_{n+1}} Y_{n+1}$ *for each* $n \geq 0$.

Proof. The morphisms above are epimorphisms if and only if the local lifting property of $X_\bullet \to Y_\bullet$ with respect to the n-boundary inclusion $\partial\Delta[n] \to \Delta[n]$ for all $n \geq 0$ holds. See [20, 4.32] for further details. □

2.1.11. The following model category structure on the category of simplicial sheaves is due to Joyal from his letter to Alexander Grothendieck, and to Jardine in the case of presheaves.

2.1.12. (*The Joyal-Jardine's model category structure on the category of simplicial (pre)sheaves [23], [21, 2.3]*) There is a model category structure on the category $\hat{\mathcal{C}}^{\Delta^{\mathrm{op}}}$ of simplicial presheaves on $\mathcal{C}$; if $f : X_\bullet \to Y_\bullet$ is a morphism of simplicial presheaves,

(i) f is a *weak equivalence* if it is a local weak equivalence,
(ii) f is a *cofibration* if it is a monomorphism, and
(iii) f is a *fibration* if it is a global fibration.

Remark 2.1.13. The classical model category structure **SSet** is a simplicial model category [11, 9.1.6]. In particular, it is equipped with these notions; a *tensoring* $S_\bullet \otimes (-)$ with a simplicial set $S_\bullet$ which is a product of simplicial sets, and for two simplicial sets $X_\bullet$ and $Y_\bullet$, a *simplicial mapping space* $\mathrm{Map}(X_\bullet, Y_\bullet)$ which is a simplicial set whose degree n is given by $\mathrm{Mor}_{\mathbf{SSet}}(X_\bullet \times \Delta[n], Y_\bullet)$. There is an adjunction

$$\mathrm{Map}(S_\bullet \otimes X_\bullet, Y_\bullet) = \mathrm{Map}(S_\bullet, \mathrm{Map}(X_\bullet, Y_\bullet))$$

This simplicial model category structure naturally induces a simplicial model category structure on Joyal-Jadine's model category structure on the category of simplicial (pre)sheaves. Indeed, a tensoring $S_\bullet \otimes X_\bullet$ of a simplicial presheaf $X_\bullet$ with a simplicial set $S_\bullet$ is defined by section-wise tensoring: $(U \in \mathcal{C}) \mapsto X_\bullet(U) \otimes S_\bullet$. A mapping space $\mathrm{Map}(X_\bullet, Y_\bullet)$ between two simplicial presheaves is the simplicial set defined by $\mathrm{Mor}_{\hat{\mathcal{C}}^{\Delta^{\mathrm{op}}}}(\Delta[n] \otimes X_\bullet, Y_\bullet)$ in degree n. There is an induced adjunction

$$\mathrm{Mor}_{\hat{\mathcal{C}}^{\Delta^{\mathrm{op}}}}(S_\bullet \otimes X_\bullet, Y_\bullet) = \mathrm{Mor}_{\mathbf{SSet}}(S_\bullet, \mathrm{Map}(X_\bullet, Y_\bullet))$$

Proposition 2.1.14. *A global fibration $X_\bullet \to Y_\bullet$ of simplicial (pre)sheaves on a site $\mathcal{C}$ is a section-wise fibration of simplicial sets (i.e., $X_\bullet(U) \to Y_\bullet(U)$ is a fibration of simplicial sets for all $U \in \mathcal{C}$) and thus is a local fibration.*

Proof. That a section-wise fibration is a local fibration follows from the definitions of Kan fibrations and local fibrations. To prove the first assertion, we may assume $X_\bullet$ and $Y_\bullet$ are simplicial presheaves because the same argument with sheafification does the job. So consider a lifting problem

$$\begin{array}{ccc} \Lambda_n^k & \longrightarrow & X_\bullet(U) \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y_\bullet(U) \end{array}$$

for $U \in \mathcal{C}$. By 2.1.13, for a simplicial set $S_\bullet$ we have an adjunction

$$\mathrm{Mor}_{\hat{\mathcal{C}}^{\Delta^{\mathrm{op}}}}(S_\bullet \otimes h_U, X_\bullet) = \mathrm{Mor}_{\mathbf{SSet}}(S_\bullet, \mathrm{Map}(h_U, X_\bullet))$$

Note also that there is a bijection of simplicial sets

$$\mathrm{Map}(h_U, X_\bullet) \to X_\bullet(U)$$

So the lifting problem is equivalent to the lifting problem

$$\begin{array}{ccc} \Lambda_n^k \otimes h_U & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] \otimes h_U & \longrightarrow & Y_\bullet \end{array}$$

The left vertical arrow is a section-wise weak equivalence of simplicial sets and thus is a weak equivalence of simplicial presheaves. Moreover, it is a monomorphism. Therefore, the lift exists by the definition of global fibrations. □

2.2. **Review on pro-simplicial (pre)sheaves.** In this subsection we recall basic background materials about pro-categories and model category structures on them. Then review two different model category structures on the pro-category of simplicial (pre)sheaves: Edwards-Hastings and Barnea-Schlank. We compare these two model category structures and discuss why Barnea-Schlank's model category structure is more suitable when it comes to *topological types* 2.3.9.

Definition 2.2.1. ([1, A.§1]) A category I is *cofiltered* if it satisfies the following conditions:

(i) For every two objects i and j, there exists an object k and two morphisms $k \to i$ and $k \to j$,
(ii) For every two morphisms $a, b : i \rightrightarrows j$, there exists an object k and a morphism $c : k \to i$ such that $a \circ c = b \circ c$.

More generally, a functor $\phi : I \to J$ between categories is cofinal if the following conditions hold:

(i) For every two objects $j_1, j_2 \in J$, there exists an object $i \in I$ and two morphisms $\Phi(i) \to j_1$ and $\phi(i) \to j_2$,
(ii) For every two morphisms $a, b : \phi(i) \rightrightarrows j$, there exists an object $i' \in I$ and a morphism $c : i' \to i$ in I such that $a \circ \phi(c) = b \circ \phi(c)$.

Definition 2.2.2. ([1, A.§2]) Let $\mathcal{C}$ be a category. A *pro-object* in $\mathcal{C}$ is a functor

$$I \to \mathcal{C}$$

from a cofiltered category I to $\mathcal{C}$. The *pro-category* associated to $\mathcal{C}$, denoted by $\mathrm{pro}-\mathcal{C}$, is the category whose objects are pro-objects in $\mathcal{C}$ and whose morphisms are defined by

$$\mathrm{Mor}_{\mathrm{pro}-\mathcal{C}}(X, Y) = \lim_{j \in J} \operatorname*{colim}_{i \in I} \mathrm{Mor}_{\mathcal{C}}(X_i, Y_j)$$

2.2.3. Let F and G be pro-object in $\mathcal{C}$ indexed by the same category I. If there is a morphism of functors $\alpha : F \to G$, then there is an induced morphism of pro-objects. A *level presentation* of morphisms in $\mathrm{pro}-\mathcal{C}$ replaces a morphism of pro-objects by a morphism of pro-objects induced by a morphism between functors:

Definition 2.2.4. A *level presentation* of a morphism $X \to Y$ in $\mathrm{pro}-\mathcal{C}$ is a cofiltered category K, pro-objects $\tilde{X}$ and $\tilde{Y}$ indexed by K, a morphism $\tilde{X} \to \tilde{Y}$ of functors, and isomorphisms $X \to \tilde{X}$ and $Y \to \tilde{Y}$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & \tilde{Y} \end{array}$$

commutes.

Lemma 2.2.5 ([1, A.3.2])**.** *Every morphism in* $\mathrm{pro}-\mathcal{C}$ *admits a level presentation.*

Definition 2.2.6. ([15, p.4]) A directed set $(I, \leq)$ is *cofinite* if for every t in I, the set

$$\{s \in I : s \leq t\}$$

is finite. Note that a direct set can be regarded as a cofiltered category.

Lemma 2.2.7 ([34, Expose I.8.1.6])**.** *Let I be a cofiltered category. Then there exists a cofinite directed set J and a cofinal functor $J \to I$. In particular, every morphism in a pro-category has a cofinite directed level representation in a sense that it has a level representation with the index category cofinite directed.*

Definition 2.2.8 ([15, 2.3])**.** Let $f : X \to Y$ be a cofinite directed, indexed by I, level representation of a morphism in the pro-category of simplicial (pre)sheaves. For each $t \in I$, the *relative matching map* $M_t f$ is the canonical map

$$X_t \to \varprojlim_{s<t} X_s \times_{\varprojlim_{s<t} Y_t} Y_s$$

Definition 2.2.9. A morphism $f : X \to Y$ of pro-simplicial (pre)sheaves is a *special global fibration* (resp. *special local fibration*) if it has a cofinite directed level representation for which every relative matching map is a global fibration (resp. local fibration).

2.2.10. (*Edwards-Hastings' model category structure on the pro-category of simplicial (pre)sheaves* [6, §3.5], [19, 14]) There is a model category structure on the pro-category of simplicial (pre)sheaves on $\mathcal{C}$; if $f : X \to Y$ is a morphism of pro-simplicial (pre)sheaves,

- (i) f is a *weak equivalence* if it is isomorphic to a level-wise weak equivalence,
- (ii) f is a *cofibration* if it is a monomorphism, and
- (iii) f is a *fibration* if it is a retract of a special global fibration.

Denote this model category structure by $\text{pro}^{\text{E-H}} - \hat{\mathcal{C}}^{\Delta^{\text{op}}}$ (resp. $\text{pro}^{\text{E-H}} - T^{\Delta^{\text{op}}}$).

Remark 2.2.11.

- (i) As pointed out in [19, 20], Edwards-Hastings' model category structure is the strict model category structure [15, 4.15] on the pro-category of simplicial (pre)sheaves induced by Joyal-Jardine's model category structure on the category of simplicial (pre)sheaves.
- (ii) Originally, Edwards-Hastings constructed [**EH**] a model category on the category of pro-simplicial sets. After then Jardine generalized [19, 14] it to the category of pro-simplicial sheaves. He also introduced the terminology Edwards-Hastings' model category structure, which we follow in this paper.

2.2.12. Barnea-Schlank noticed that local weak equivalences and local fibrations are enough to define a model category structure on the pro-category of simplicial sheaves [2, p.55]. They define *weak fibration categories* [2, 1.2] which is roughly a category with classes of weak equivalences and fibrations that are enough to define model category structure on the associated pro-category:

Definition 2.2.13. (*Barnea-Schlank's weak fibration category* [2, 1.2]) A *weak fibration category* is a category $\mathcal{C}$ equipped with two subcategories $\mathcal{F}, \mathcal{W}$ satisfying the following conditions:

- (i) $\mathcal{F}$ and $\mathcal{W}$ contain all isomorphisms,
- (ii) $\mathcal{C}$ has all finite limits,
- (iii) $\mathcal{F}$ has the 2-out-of-3 property,
- (iv) The subcategories $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under base change,
- (v) Every morphism $A \to B$ in $\mathcal{C}$ can be factored as $A \xrightarrow{f} C \xrightarrow{g} B$, where f is in $\mathcal{W}$ and g is in $\mathcal{F}$.

The morphisms in $\mathcal{F}$ (resp. $\mathcal{W}$) are called *fibrations* (resp. *weak equivalences*).

2.2.14. The main example of weak fibration category is:

Definition 2.2.15. (*Barnea-Schlank's weak fibration category structure on the category of simplicial (pre)sheaves* [2, 7.7], [2, 7.11]) There is a weak fibration category structure [2, 1.2] on the category of simplicial (pre)sheaves on $\mathcal{C}$; if $f : X \to Y$ is a morphism of pro-simplicial presheaves,

(i) f is a *weak equivalence* if it is a local weak equivalence, and
(ii) f is a *fibration* if it is a local fibration.

2.2.16. Here is a justification for introducing weak fibration categories and passing to pro-categories. Beginning with local fibrations and local weak equivalences to endow a model category structure on simplicial (pre)sheaves does not work in general. Assume for convenience that T has enough points. If there is a model category structure on the category of simplicial sheaves with local fibrations and local weak equivalences, the class of cofibrations ought to be the class of monomorphisms. Indeed, as every morphism of simplicial sheaves is factored by a cofibration followed by a trivial fibration, by looking at stalks, cofibration at each point must be a monomorphism of sets. However, such a model category structure is precisely Joyal-Jardine's one 2.1.12 in which fibrations are global fibrations, but not local fibrations. (see 2.3.20). Nevertheless, the classes of local fibrations and local weak equivalences still define a model category structure on the associated pro-category:

Definition 2.2.17. (*Barnea-Schlank's model category structure on the pro-category of simplicial (pre)sheaves* [2, p.55]) There is a model category structure on the pro-category of simplicial (pre)sheaves on $\mathcal{C}$; if $f : X \to Y$ is a morphism of pro-simplicial presheaves,

(i) f is a *weak equivalence* if it is isomorphic to a level-wise weak equivalence,
(ii) f is a *fibration* if it is a retract of a special local fibration, and
(iii) f is a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

Denote this model category structure by $\text{pro} - \hat{\mathcal{C}}^{\Delta^{\text{op}}}$ (resp. $\text{pro} - T^{\Delta^{\text{op}}}$).

2.2.18. At this point, we have two different model category structures on the pro-category of simplicial (pre)sheaves. Note that they have the same class of weak equivalences, but not for fibrations and cofibrations.

Remark 2.2.19.

(i) In what follows, we use Barnea-Schlank's model category structure on the category of pro-simplicial (pre)sheaves unless otherwise stated. Barnea-Schlank called it the *projective* model category structure since every local fibration of simplicial sheaves is a fibration as a morphism of pro-simplicial sheaves.
(ii) Joyal-Jardine's model category structure on simplicial (pre)sheaves induces Edwards-Hastings' model category structure on the pro-category of simplicial (pre)sheaves. Barnea-Schlank called it the *injective* model category structure since every cofibration of simplicial sheaves is a cofibration as a morphism of pro-simplicial sheaves.
(iii) We do not follow Barnea-Schlank's terminologies to avoid any confusions. There is a definition of the *projective* model category structure for a given indexed category. The case of pro-simplicial (pre)sheaves is different from this because there is no uniform index category.

2.2.20. Of course, two model category structures on the pro-category of simplicial (pre)sheaves are closely related to each other:

Proposition 2.2.21 ([2, §7.4]). *The adjunction*

$$(\mathrm{id}, \mathrm{id}) : \mathrm{pro}^{E\text{-}H} - T^{\Delta^{\mathrm{op}}} \to \mathrm{pro} - T^{\Delta^{\mathrm{op}}}$$

between the Edwards-Hastings' and Barnea-Schlank's model category structures is a Quillen equivalence.

Proof. Every fibration in Edwards-Hastings' model category structure is a fibration in the Barnea-Schlank model category structure because every global fibration is a local fibration of simplicial (pre)sheaves 2.1.14. Since those two model category structures share the same class of weak equivalences, the adjunction induces a Quillen adjunction. This turns out to be a Quillen equivalence as the adjunction is given by the identities. □

2.3. **Definition and Properties of Topological types.**

2.3.1. In case of the punctual topos, the model category structure on pro-simplicial sets induced by Barnea-Schlank is nothing but the strict model category structure [15, 4.15] induced by the classical model category structure on simplicial sets. In the strict model category structure, weak equivalences (resp. cofibrations) are those isomorphic to level-wise weak equivalences (resp. monomorphisms) of simplicial sets. In particular, this structure coincides with Edwards-Hastings' model structure 2.2.11. Consequently, the three model category structures-Edwards-Hastings', Barnea-Schlank's, and the strict model category structure-on the pro-category of simplicial sets all coincide.

However, these equivalent ones are not adequate for developing a theory of topological types. For a morphism between connected pointed pro-simplicial sets, we hope it to be a weak equivalence if and only if it induces an isomorphism of homotopy groups in every degree. Although every weak equivalence in the strict model category structure on pro-simplicial sets induces an isomorphism of homotopy groups, but not vice versa. In other words, the class of weak equivalence is not big enough, which is why Artin-Mazur introduced ♮-isomorphisms [1, §4]. The desired model category structure was achieved by Isaksen [12]. The class of weak equivalence is a bit involved and it does have the property that a map of connected pointed pro-spaces is a weak equivalence if and only if it induces an isomorphism of all homotopy pro-groups [12, 7.5].

2.3.2. (*Isaksen's model category structure on pro-simplicial sets* [12, 6.4]) There is a model category structure on the pro-category of simplicial sets; if $f : X \to Y$ is a morphism of pro-simplicial sets,

(i) f is a *weak equivalence* if $\pi_0 f$ is an isomorphism of pro-sets and $\Pi_n X \to f^*\Pi_n Y$ is an isomorphism in $\mathrm{pro} - \mathcal{LS}(X)$ for all $n \geq 1$ (see [12, 6.1] for details),
(ii) f is a *cofibration* if it is isomorphic to a level-wise cofibration, and
(iii) f is a *fibration* if it has the right lifting property with respect to all trivial cofibrations.

Denote this model category structure by $\mathrm{pro} - \mathbf{SSet}$ and the strict one by $\mathrm{pro}^{\mathrm{str}} - \mathbf{SSet}$

Remark 2.3.3.

(i) Both the strict and Isaksen's model category structures have the same class of cofibrations which are exactly the class of monomorphisms.

(ii) A weak equivalence in the strict model category structure, called a strict weak equivalence, is also a weak equivalence in Isaksen.

2.3.4. In what follows, the model category structure on pro-simplicial sets is always that of Isaksen unless otherwise specified.

2.3.5. Consider the (2-categorical) unique morphism of topoi:

$$\Gamma = (\Gamma^*, \Gamma_*) : T \to \mathbf{Set}$$

According to Barnea-Schlank [2, 8.2], the pull-back functor Γ^* admits a left adjoint L_{Γ^*} for the associated pro-categories, and moreover there is a Quillen adjunction

$$(L_{\Gamma^*}, \Gamma^*) : \mathrm{pro}^{\mathrm{str}} - \mathbf{SSet} \to \mathrm{pro} - T^{\Delta^{\mathrm{op}}}$$

where the pro-category of simplicial sets is endowed with the strict model category structure. In particular, L_{Γ^*} preserves cofibrations and trivial cofibrations. It then follows from 2.3.3 that there is still a Quillen adjunction

$$(L_{\Gamma^*}, \Gamma^*) : \mathrm{pro} - \mathbf{SSet} \to \mathrm{pro} - T^{\Delta^{\mathrm{op}}}$$

Remark 2.3.6. If T is locally connected in a sense that Γ^* admits a left adjoint, denote the left adjoint by Π and call the connected component functor. In this case, L_{Γ^*} is simply the connected functor induced on the pro-categories.

2.3.7. The locally connectedness condition is a topos-theoretic generalization of the connected components of a topological space. For instance, the small étale topos $X_{\text{ét}}$ of a locally noetherian scheme X is locally connected. If Y is étale over X, then Π applied to the sheaf represented by Y over X is the set of connected components of the underlying topological space of the scheme Y.

2.3.8. In what follows, we do not assume the locally connectedness condition on T. Even if the condition is necessary for the comparison with Artin-Mazur's étale homotopy types (see 3.3.5), we can still develop a general theory of topological types without it:

Definition 2.3.9. A *topological type* $h(T)$ of a topos T is the pro-simplicial set

$$\mathbf{L}L_{\Gamma^*}(*)$$

where $*$ is a final object of $T^{\Delta^{\mathrm{op}}}$ and $\mathbf{L}L_{\Gamma^*} : \mathbf{Ho}(\mathrm{pro} - T^{\Delta^{\mathrm{op}}}) \to \mathbf{Ho}(\mathrm{pro} - \mathbf{SSet})$ is the left derived functor of L_{Γ^*}. More generally, a *topological type* $h(F_\bullet)$ (or $h_T(F_\bullet)$ if we wish to make the reference to T explicit) of a simplicial object $F_\bullet$ in T is the pro-simplicial set

$$\mathbf{L}L_{\Gamma^*}(F_\bullet)$$

The topological type $h(T)$ of T is the topological type $h(*)$ of a final object in $T^{\Delta^{\mathrm{op}}}$.

Remark 2.3.10.

(i) Though we use Isaksen's model category structure on pro-simplicial sets, most weak equivalences of topological types in this paper are strict weak equivalences. Indeed, most weak equivalence are induced by weak equivalences in $\mathrm{pro} - T^{\Delta^{\mathrm{op}}}$ and so they are strict weak equivalences of pro-simplicial sets(see 2.3.5). In particular, a choice of cofibrant approximation makes a difference only up to strict weak equivalence.

(ii) We could have developed the theory of topological types with the strict model category structure on pro-simplicial sets. However, we allow more weak equivalences to compare our definition with Friedlander's étale topological types 3.3.7.

2.3.11. The main goal of this paper is to build a theory of topological types for algebraic stacks. Since stacks cannot be thought as sheaves, extra cares are needed.

Definition 2.3.12. Let $\mathcal{X}$ be a stack over $\mathcal{C}$. A site $\mathcal{C}/\mathcal{X}$ is defined as following. An object is a pair (U, u), where $u : U \to \mathcal{X}$ is a morphism of fibered categories over $\mathcal{C}$. A morphism $(V, v) \to (U, u)$ is a pair (h, h^b) where $f : V \to U$ is a morphism in $\mathcal{C}$ and $h^b : v \to u \circ f$ is a 2-morphism of functors. A collection of maps

$$\{(h_i, h_i^b) : (U_i, u_i) \to (U, u)\}$$

is a covering if the underlying collection $\{h_i : U_i \to U\}$ of morphisms in $\mathcal{C}$ is a covering of U. Denote by $T/\mathcal{X}$ the associated topos.

Remark 2.3.13. For a fibered category in groupoids $p : F \to \mathcal{C}$, there is an inherited topology on F from $\mathcal{C}$. Namely, a family $\{x_i \to x\}$ of morphisms in F with fixed target is defined to be the covering if the family $\{p(x_i) \to p(x)\}$ is a covering in $\mathcal{C}$ (see [22, Tag 06NU] for details). Under the 2-Yoneda lemma, the site applied to the fibered category $\mathcal{X}/\mathcal{C}$ is equivalent to the site $\mathcal{C}/\mathcal{X}$ defined above.

Definition 2.3.14. A *topological type of a stack* $\mathcal{X}$ over $\mathcal{C}$ is the pro-simplicial set

$$h(\mathcal{X}) := \mathbf{L}L_{\Gamma^*_{T/\mathcal{X}}}(*_{T/\mathcal{X}})$$

where $\Gamma^*_{T/\mathcal{X}} : \mathbf{Set} \to T/\mathcal{X}$ is the constant sheaf functor and $*_{T/\mathcal{X}}$ is a final object in the topos $T/\mathcal{X}$.

2.3.15. After developing a general theory of topological types of simplicial sheaves, we will related the topological types of stacks to the topological types of simplicial schemes/algebraic spaces. Before that, we provide a couple of examples of topological types of topoi.

Definition 2.3.16.

(i) Two objects in a model category are *weakly equivalent* if there is a zig-zag of weak equivalences between them [11, 7.9.2].
(ii) Two pro-simplicial sets are *strictly weakly equivalent* if there is a zig-zag of strict weak equivalences between them.

Example 2.3.17. Let BG be the classifying topos of a discrete group G. Recall that it is the category of presheaves on the category $\mathcal{C}$ with one object $*$ and

$$\mathrm{Hom}(*, *) = G$$

Let us endow $\mathcal{C}$ the trivial topology so that BG is also the category of sheaves on $\mathcal{C}$. Note that BG is locally connected with its connected component functor the colimit functor. In fact, BG is equivalent to the category of G-sets.

To compute the topological type of BG, we follow the same argument in Barnea-Schlank [2, Example 6]. Their argument works verbatim despite the different choice of model category structures on pro-simplicial sets. Let us work in general. For any given category $\mathcal{C}$, endowed the trivial topology on it. Denote by T the associated category of sheaves, or equivalently, presheaves. It is locally connected with the colimit functor as the connected component functor. In the weak fibration category structure on the category of simplicial sheaves, weak equivalences (resp. fibrations) are exactly section-wise weak equivalences (resp. section-wise Kan fibrations) of simplicial sets. Notice that these two classes are exactly those classes in the model category structure on simplicial sheaves viewed as the the category $\mathbf{SSet}^{\mathcal{C}^{\text{op}}}$ of $\mathcal{C}^{\text{op}}$-diagrams of simplicial sets, induced by the model category structure on simplicial sets [11, 11.6.1]. So in this case, Barnea-Schlank's model category structure on $\text{pro}-T^{\Delta^{\text{op}}}$ is nothing but the strict model category structure [15, 10.4] induced by the model category structure on $\mathbf{SSet}^{\mathcal{C}^{\text{op}}}$. This enables us to compute the topological type because cofibrations in the strict model category structure are exactly morphisms isomorphic to level-wise cofibrations of simplicial sheaves. Now consider the $\mathcal{C}^{\text{op}}$-diagram of simplicial sets

$$N(-/\mathcal{C}) : \mathcal{C}^{\text{op}} \to \mathbf{SSet} : U \mapsto N(U/\mathcal{C})$$

where $N(U/\mathcal{C})$ is the nerve of the undercategory $U/\mathcal{C}$. It follows from [11, 14.8.9] that

$$N(-/\mathcal{C}) \to *$$

is a cofibrant approximation of a final object $*$ in $T^{\Delta^{\text{op}}}$, which then becomes a cofibrant approximation in $\text{pro}-T^{\Delta^{\text{op}}}$. Therefore, we have a strict weak equivalence

$$\text{colim}_{\mathcal{C}^{\text{op}}} N(-/\mathcal{C}) = \Pi(N(-/\mathcal{C})) \to h(T)$$

of pro-simplicial sets. Again, it follows from [11, 14.7.5] that $\text{colim}_{\mathcal{C}^{\text{op}}} N(-/\mathcal{C})$ is isomorphic to the nerve $N\mathcal{C}$ of the category $\mathcal{C}$. To sum up, the topological type $h(T)$ is strictly weakly equivalent to the simplicial set $N\mathcal{C}$ viewed as a pro-simplicial set. In particular, the topological type of BG is the classifying space $K(G,1)$.

2.3.18. Let us take a close look at the projective model category structure on $(BG)^{\Delta^{\text{op}}}$ which is the category of simplicial G-sets. A morphism of simplicial G-sets is a weak equivalence (resp. fibration) if and only if its underlying morphism of simplicial sets is a weak equivalence (resp. fibration). On the other hand, it follows from [10, 5.2.10] that a simplicial G-set $X_\bullet$ is cofibrant if and only if X_n is a free G-set for each n. Also, the map $G \to *$ is an epimorphism in BG and hence $\text{cosk}_0(G) \to *$ is a trivial fibration. Now let $X_\bullet$ be a simplicial G-set. Then there is an induced trivial fibration

$$\text{cosk}_0(G) \times X_\bullet \to X_\bullet$$

in $(BG)^{\Delta^{\text{op}}}$. Moreover, $\text{cosk}_0(G) \times X_\bullet$ is a cofibrant object in $(BG)^{\Delta^{\text{op}}}$ as G-action on $\text{cosk}_0(G)$ is free. Recall from the previous example that the Barnea-Schlank model category structure on $\text{pro}-(BG)^{\Delta^{\text{op}}}$ is the strict model category structure induced by the model category structure on $(BG)^{\Delta^{\text{op}}}$. Therefore, $\text{cosk}_0(G) \times X_\bullet \to X_\bullet$ is a cofibrant approximation of $X_\bullet$ in $\text{pro}-(BG)^{\Delta^{\text{op}}}$ as well. Consequently, there is a strict weak equivalence

$$(\text{cosk}_0(G) \times X_\bullet)/G = \Pi(\text{cosk}_0(G) \times X_\bullet) \to h(X_\bullet)$$

As a result,

$$\mathbf{L}\Pi : \text{pro}-(BG)^{\Delta^{\text{op}}} \to \text{pro}-\mathbf{SSet}$$

can be understood as a generalization of the Borel construction. Finally, note that $\text{cosk}_0(G)/G$ is the classifying space $K(G,1)$ of the discrete group G and hence we can recover the previous result that $h(BG)$ is $K(G,1)$.

Example 2.3.19. Let us extend the result to profinite groups. So let $G = \{G_i\}_{i \in I}$ be an inverse system of finite groups with surjective transition maps. Denote by K_i the kernel of $\varprojlim_{i \in I} G_i \to G_i$. Consider the classifying topos BG which is the category of discrete G-sets. As in the discrete case, BG is locally connected. Remark that for each $i \leq j$, there is a commutative diagram of topoi

$$\begin{array}{ccc} BG_j & \longrightarrow & BG_i \\ & {}_{p_j}\nwarrow \quad \nearrow_{p_i} & \\ & BG & \end{array}$$

To compute the topological type $h(BG)$, we use its compatibility with Artin-Mazur. Consider a cofibrant replacement $H \to *$ of $*$ in $\text{pro}-(BG)^{\Delta^{\text{op}}}$. Say

$$H : A \to (BG)^{\Delta^{\text{op}}}$$

It then follows from [2, 8.3] that there is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{H} & \text{HR}(BG) & \xrightarrow{\Pi} & \mathbf{SSet} \\ & \searrow & \downarrow & & \downarrow \\ & & \pi\text{HR}(BG) & \longrightarrow & \mathbf{Ho}(\mathbf{SSet}) \end{array}$$

where $\pi\text{HR}(BG)$ is the simplicial homotopy category [2, 6.15] of the category $\text{HR}(BG)$ of objects $X_\bullet$ with $X_\bullet \to *$ a trivial fibration in $(BG)^{\Delta^{\text{op}}}$. Since $\pi\text{HR}(BG)$ is cofiltered and the composition $A \to \pi\text{HR}(BG) \to \mathbf{Ho}(\mathbf{SSet})$ is cofinal [2, §6.2],

$$\pi\text{HR}(BG) \to \mathbf{Ho}(\mathbf{SSet})$$

computes $h(BG)$ as an object in $\text{pro}-\mathbf{Ho}(\mathbf{SSet})$. Further, we can restrict the index category to the full subcategory $\pi\text{HR}^{\text{repn}}(BG)$ consisting of representable objects by using a similar argument as in 3.2.8. Note that BG is equivalent to the topos associated to the discrete finite G-sets. So $X_\bullet$ being representable means that each X_n is a finite discrete G-set. Fix a non-negative integer n. A morphism between cofiltered categories

$$I \to \pi\text{HR}^{\text{repn}}(BG) : i \mapsto p_i^* EG_i$$

induces a morphism of nth homotopy pro-groups. We claim that it is an isomorphism. We need to prove that for any given representable simplicial discrete G-set $X_\bullet$, there exists some $i \in I$ such that one can choose a morphism from $p_i^* EG_i$ to $X_\bullet$. Since we are looking at nth homotopy groups, we may assume $X_\bullet$ is isomorphic to $\text{cosk}_{n+1} \text{sk}_{n+1} X_\bullet$. Consider the finite intersection of open stabilizers:

$$\bigcap_{x \in \bigcup_{m=0}^{n+1} X_m} G_x$$

Since $\{K_i\}_{i \in I}$ forms a fundamental system of neighborhood of the identity, K_i is a subgroup of the finite intersection for some i. This implies that $p_{i*} X_\bullet = X_\bullet^{K_i}$ is equal to $X_\bullet$ itself and thus p_i^* applied to a trivial fibration $X_\bullet \to *$ in $(BG)^{\Delta^{\text{op}}}$ is still a trivial fibration in $(BG_i)^{\Delta^{\text{op}}}$.

Then via adjunction we can solve the lifting problem

$$\begin{array}{ccc} \emptyset & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \downarrow \\ p_i^* EG_i & \longrightarrow & * \end{array}$$

in $(BG_i)^{\Delta^{\mathrm{op}}}$ where the lift exists because EG_i is cofibrant. Therefore, there is a $\natural$-isomorphism between $I \to \mathbf{SSet} : i \mapsto \Pi(p_i^* EG_i) = BG_i$ and $h(BG)$ in pro $-\, \mathbf{Ho}(\mathbf{SSet})$. Consequently, $h(BG)$ is $K(G,1)$ which is the inverse system $\{K(G_i,1)\}_{i\in I}$ of classifying spaces.

Remark 2.3.20. There is no model category structure on $(BG)^{\Delta^{\mathrm{op}}}$ with local weak equivalences as weak equivalences and local fibrations as fibrations. Recall from 2.1.8 that local weak equivalences and local fibrations can be checked at stalks. The topos BG has enough points; the point $\mathbf{Set} \to BG$ whose pull-back assigning underlying sets does the job. Therefore, a morphism of simplicial discrete G-sets is a local weak equivalence (resp. a local fibration) if and only if the underlying map of simplicial sets is a weak equivalence (resp. a Kan fibration). For the sake of contradiction, assume there is such a model category structure. We claim that there is no cofibrant object. If $X_\bullet$ is cofibrant, we can find a dotted arrow in the following diagram for each i:

$$\begin{array}{ccc} \emptyset & \longrightarrow & p_i^* EG_i \\ \downarrow & \nearrow & \downarrow \\ X_\bullet & \longrightarrow & * \end{array}$$

This is because the right vertical morphism is a trivial fibration as the pull-back of the trivial fibration $EG_i \to *$ in $(BG_i)^{\Delta^{\mathrm{op}}}$. Having a dotted arrow implies that the stabilizer G_x is a subgroup of K_i for each n and each $x \in X_n$. Since this is true for every i, each X_n has a free G-action. However a discrete G-set cannot have a free G-action unless G is discrete.

2.3.21. For the rest of the subsection, we study basic properties of topological types.

2.3.22. The notion of topological types is functorial up to strictly weakly equivalent object. This slight annoyance will be resolved when defining topological types of schemes. A morphism $f : T' \to T$ between topoi induces a morphism of pro-simplicial sets:

$$h(T') = \mathbf{L}L_{\Gamma'^*}(*') \to \mathbf{L}L_{\Gamma^*}(*) = h(T)$$

Indeed, consider a cofibrant replacement $H' \to *'$ of $*'$ in pro$-T'^{\Delta^{\mathrm{op}}}$ and choose a factorization

$$\begin{array}{ccc} L_{f^*}(H') & \overset{d}{\dashrightarrow} & K \\ \downarrow & & \vdots \\ L_{f^*}(*') & \longrightarrow & * \end{array}$$

of the composition $L_{f^*}(H') \to L_{f^*}(*') \to *$ as a cofibration followed by a trivial fibration. Here L_{f^*} is a left adjoint of $f^* : \text{pro} - T^{\Delta^{\mathrm{op}}} \to \text{pro} - T'^{\Delta^{\mathrm{op}}}$ which is a right Quillen functor [2, 8.1]. Since L_{f^*} preserves cofibrations, $K \to *$ is a cofibrant approximation of $*$. Then there is a weak equivalence $K \to H$ over $*$ where $H \to *$ is a cofibrant replacement of $*$. Since L_{Γ^*}

preserves weak equivalence between cofibrant objects, we get a morphism

$$h(T') = L_{\Gamma'^*}(H') = L_{\Gamma^*}(L_{f^*}(H')) \xrightarrow{d} L_{\Gamma^*}(K) \to L_{\Gamma^*}(H) = \mathbf{L}L_{\Gamma^*}(*) = h(T)$$

where the map $L_{\Gamma^*}(K) \to L_{\Gamma^*}(H)$ is a strict weak equivalence. So the map $h(T') \to h(T)$ is obtained by applying L_{Γ^*} to the composition $L_{f^*}H' \to K \to H$ and thus can be regarded as a morphism in the category of pro-simplicial sets rather than in its homotopy category.

2.3.23. If T is locally connected, then every object is isomorphic to a disjoint union of connected objects. The behavior of topological types with respect to coproduct is simple:

Proposition 2.3.24. *Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of objects in $T^{\Delta^{\mathrm{op}}}$. Then there is a strict weak equivalence*

$$\coprod_\alpha h(F_\alpha) \to h(\coprod_\alpha F_\alpha)$$

Proof. For each α, choose a cofibrant replacement $C_\alpha \to F_\alpha$ of F_α in pro$-T^{\Delta^{\mathrm{op}}}$. Since $\coprod C_\alpha$ is cofibrant, the morphism $\coprod_\alpha C_\alpha \to \coprod_\alpha F_\alpha$ is a cofibrant approximation by the lemma below. The statement follows from the fact that L_{Γ^*} commutes with coproducts. □

Lemma 2.3.25. *Let $\{X^\alpha \to Y^\alpha\}_{\alpha\in A}$ be a collection of weak equivalences in* pro$-T^{\Delta^{\mathrm{op}}}$. *Then the induced morphism*

$$\coprod_\alpha X^\alpha \to \coprod_\alpha Y^\alpha$$

is also a weak equivalence.

Proof. Say X^α is indexed by I_α. Note that the coproduct $\coprod_\alpha X^\alpha$ is simply

$$\coprod_\alpha I_\alpha \to T^{\Delta^{\mathrm{op}}} : (i_\alpha) \mapsto \coprod_\alpha X^\alpha_{i_\alpha}$$

By replacing morphisms by their level presentations 2.2.5, it suffices to prove the statement for morphisms in $T^{\Delta^{\mathrm{op}}}$ where the result follows from [20, 4.42]. In case T has enough points, the statement is reduced further to the case of simplicial sets where the result is also well-known. □

2.3.26. Different topoi can induce the same topological type up to strict weak equivalences. The following is an important case which plays a crucial role in replacing étale homotopy types by smooth topological types.

Proposition 2.3.27. *Let $f : \mathcal{C}' \to \mathcal{C}$ be a cocontinuous functor between sites with the associated morphism of topoi $f : T' \to T$. Assume that the functor is continuous and commutes with finite limits. For a simplicial object $F'_\bullet$ in T', the morphism*

$$h_{T'}(F'_\bullet) \to h_T(f_!(F'_\bullet))$$

between topological types is a strict weak equivalence of simplicial sets where $f_!$ is a left adjoint to f^*. In particular, the morphism

$$h(T') \to h(T)$$

between topological types is a strict weak equivalence of pro-simplicial sets.

Proof. By the assumption on continuity, there is also a morphism of topoi

$$(f_!, f^*) : T \to T'$$

whose push-forward is the pull-back of f. Consider a cofibrant replacement $H' \to F'_\bullet$ (resp. $H \to f_!(F'_\bullet)$) of $F'_\bullet$ (resp. $f_!(F'_\bullet)$) in pro $- T'^{\Delta^{\mathrm{op}}}$ (resp. pro $- T^{\Delta^{\mathrm{op}}}$). As a left Quillen adjoint of f^*, cofibrations are preserved under $f_!$. Whereas $f_!$, as a pull-back of $T \to T'$, preserves trivial fibrations. Therefore we can fill in the dotted arrow in the diagram below:

$$\begin{array}{ccc} \emptyset & \longrightarrow & H \\ \downarrow & \overset{d}{\nearrow} & \downarrow \\ f_!(H') & \longrightarrow & f_!(F'_\bullet) \end{array}$$

The dotted arrow is a weak equivalence by the 2-out-of-3 property of weak equivalences. Finally, L_{Γ^*} sends a weak equivalence between cofibrant objects to a strict weak equivalence:

$$h_{T'}(F'_\bullet) = L_{\Gamma'^*}(H') = L_{\Gamma^*}(f_!(H')) \xrightarrow{d} L_{\Gamma^*}(H) = h_T(f_!(F'_\bullet))$$

The last statement on topological types on topoi immediately follows from the fact that $f_!$ preserves a final object. $\square$

2.3.28. One of the key aspects of topological types of topoi is their behavior with respect to localizations:

Lemma 2.3.29. *Let F be an object in T and $j : T/F \to T$ be the localization morphism of topoi. Then the functor*

$$j_! : (T/F)^{\Delta^{\mathrm{op}}} \to T^{\Delta^{\mathrm{op}}} : X_\bullet/F \mapsto X_\bullet$$

preserves weak equivalences.

Proof. For a simplicial object $X_\bullet/F$ in the localized topos T/F, the presheaf $\widehat{\pi_n}(X_\bullet/F)$ is simply the presheaf $\widehat{\pi_n}X_\bullet$ over F. Since both the sheafification and $j_!$ commute with fiber products, $j_!$ preserves weak equivalences. $\square$

Proposition 2.3.30. *Let $G_\bullet \to F$ be a morphism of simplicial objects in T where F is a constant object. Then there is a strict weak equivalence*

$$h_{T/F}(G_\bullet/F) \to h_T(G_\bullet)$$

of pro-simplicial sets. In particular, for any object F in T there is a strict weak equivalence

$$h(T/F) \to h_T(F)$$

of pro-simplicial sets.

Proof. Denote by $j : T/F \to T$ the localization morphism of topoi. The pull-back functor j^* admits a left adjoint $j_!$ that forgets the structure morphisms to F. Consider a cofibrant replacement

$$H' \to G_\bullet/F$$

of $G_\bullet/F$ in pro $- (T/F)^{\Delta^{\mathrm{op}}}$. Apply $j_!$ to get

$$j_!H' \to j_!(G_\bullet/F) = G_\bullet$$

As a left Quillen adjoint, $j_!$ preserves cofibration and so $j_!H$ is cofibrant. Now by 2.3.29, $j_!$ preserves weak equivalences and thus $j_!H' \to G_\bullet$ is a cofibrant approximation of $G_\bullet$ in $\text{pro} - T^{\Delta^{\text{op}}}$. □

Remark 2.3.31. The last statement in 2.3.30 tells us that we could have defined the topological type of F as the topological type of the localized topos T/F.

2.3.32. Topological types behave well with respect to morphisms of topoi. Let $f : T' \to T$ be a morphism of topoi. For an object F in T, there is a map of topological types

$$h(f^*F) \to h(F)$$

This follows from the definition of topological types. Or one can apply 2.3.30 to the morphisms of localized topoi $T'/f^*F \to T/F$.

2.3.33. Next goal is to understand the topological types of simplicial objects via the topological types of objects in each degree. We first recall basic notions related to the realization and the homotopy colimit functors.

2.3.34. *(The Reedy model category structure)* For a model category M, there is a model category structure on the category $M^{\Delta^{\text{op}}}$ of simplicial objects in M; if $f : X_\bullet \to Y_\bullet$ is a morphism of simplicial objects,

(i) f is a *Reedy weak equivalence* if $X_n \to Y_n$ is a weak equivalence in M for each n,

(ii) f is a *Reedy cofibration* if the relative latching map

$$X_n \coprod_{L_n X_\bullet} Y_n \to Y_n$$

is a cofibration in M for each n where the latching object $L_n X_\bullet$ of $X_\bullet$ at $[n]$ is defined by

$$L_n X_\bullet := \varinjlim_{[m] \in \partial(\Delta^{\text{op}}/[n])} X_m$$

where the latching category $\partial(\Delta^{\text{op}}/[n])$ of Δ^{op} at $[n]$ is the full subcategory of the comma category $\Delta^{\text{op}}/[n]$ containing all objects except the the identity morphism on $[n]$.

(iii) f is a *Reedy fibration* if the relative matching map

$$X_n \to Y_n \times_{M_n Y_\bullet} M_n X_\bullet$$

is a fibration in M for each n.

We omit the details for Reedy fibrations (see [11, 15.3.3]).

Remark 2.3.35. Throughout this paper, whenever we consider the category of simplicial objects in a model category, we endow the Reedy model category structure on it.

2.3.36. A Reedy cofibration $X_\bullet \to Y_\bullet$ induces a cofibration $X_n \to Y_n$ for each n, but not vice versa. In particular, a level-wise cofibrant object $X_\bullet$ is not necessarily Reedy cofibrant. However, most model categories we study in this paper have the property that every simplicial object is Reedy cofibrant. These include:

(i) **SSet** in 2.1.2

(ii) $\text{pro}^{\text{str}} -$ **SSet** with the strict model category structure in 2.3.1

(iii) $\text{pro} -$ **SSet** with Isaksen's model category structure in 2.3.2

(iv) $\hat{\mathcal{C}}^{\Delta^{\mathrm{op}}}$ with Jardine's model category structure in 2.1.12
(v) $T^{\Delta^{\mathrm{op}}}$ with Joyal's model category structure in 2.1.12
(vi) $\mathrm{pro}^{\text{E-H}} - T^{\Delta^{\mathrm{op}}}$ with Edwards-Hastings' model category structure in 2.2.10
(vii) $\widehat{\mathbf{SSet}}$ with Quick's model category structure in 5.1.5

2.3.37. Let M be a simplicial model category. The *realization* functor

$$|-|: M^{\Delta^{\mathrm{op}}} \to M$$

is defined as following. For a simplicial object $X_\bullet \in M^{\Delta^{\mathrm{op}}}$, its realization $|X_\bullet|$ is the coequalizer of the diagram

$$\coprod_{([n]\to[m])\in\Delta} X_m \otimes \Delta[n] \rightrightarrows \coprod_{[n]\in\Delta} X_n \otimes \Delta[n]$$

The realization functor admits a right adjoint that sends X to $([n] \mapsto X^{\Delta[n]})$ and they are Quillen adjoint.

2.3.38. A *homotopy colimit*, $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} X_n$, of a simplicial pro-simplicial set $X_\bullet$ is defined by the coequalizer of the diagram

$$\coprod_{([n]\to[m])\in\Delta} X_m \otimes N([n]/\Delta^{\mathrm{op}})^{\mathrm{op}} \rightrightarrows \coprod_{[n]\in\Delta} X_n \otimes N([n]/\Delta^{\mathrm{op}})^{\mathrm{op}}$$

where $N : \mathbf{Cat} \to \mathbf{SSet}$ is the nerve functor. As a consequence of 2.3.36, for any simplicial pro-simplicial sets $X_\bullet$, the Bousfield-Kan map

$$\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, X_n \to |[n] \mapsto X_n|$$

is a weak equivalence of pro-simplicial sets in the strict model category structure. Particularly, it is a weak equivalence in the sense of Isaksen.

2.3.39. For a simplicial object $F_\bullet$ in T, one can associate a simplicial object in $T^{\Delta^{\mathrm{op}}}$:

$$\Delta^{\mathrm{op}} \to T^{\Delta^{\mathrm{op}}} : [n] \mapsto F_n$$

where F_n is viewed as a constant simplicial object. In turn, this gives rise to a simplicial object in $\mathrm{pro} - T^{\Delta^{\mathrm{op}}}$ by embedding $T^{\Delta^{\mathrm{op}}}$ into the associated pro-category:

$$\Delta^{\mathrm{op}} \to \mathrm{pro} - T^{\Delta^{\mathrm{op}}} : [n] \mapsto F_n$$

This will be used in the proof of 2.3.41 and in 2.3.43).

Remark 2.3.40. Eventually we would like to consider a simplicial object

$$[n] \mapsto h(F_n)$$

in the category of pro-simplicial sets. However, we should be careful because the cofibrant replacement in $\mathrm{pro} - T^{\Delta^{\mathrm{op}}}$ is not functorial and thus we may have trouble in getting the simplicial object. This caveat can be resolved once we choose a cofibrant replacement of

$$[n] \mapsto F_n$$

in $(\mathrm{pro} - T^{\Delta^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$. Of course, up to weak equivalence, this process is independent of the choice of cofibrant replacements.

Theorem 2.3.41. (Simplicial descent) *Let $F_\bullet$ be a simplicial object in T. There is an isomorphism*

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} h(F_n) \xrightarrow{\sim} h(F_\bullet)$$

of pro-simplicial sets in the homotopy category of pro-simplicial sets. Furthermore, these pro-simplicial sets are strictly weakly equivalent.

Proof. Recall from 2.3.38 that there is a strict weak equivalence

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} h(F_n) \to |[n] \mapsto h(F_n)|$$

So we can replace the hocolim by the realization. Since L_{Γ^*} commutes with colimits and tensoring with simplicial sets, we have a (2-categorical) commutative diagram

$$\begin{array}{ccc} \mathbf{Ho}((\mathrm{pro}-T^{\Delta^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}) & \xrightarrow{\mathbf{L}L_{\Gamma^*}} & \mathbf{Ho}((\mathrm{pro}-\mathbf{SSet})^{\Delta^{\mathrm{op}}}) \\ \downarrow{\scriptstyle \mathbf{L}|-|_{\mathrm{pro}-T^{\Delta^{\mathrm{op}}}}} & & \downarrow{\scriptstyle \mathbf{L}|-|} \\ \mathbf{Ho}(\mathrm{pro}-T^{\Delta^{\mathrm{op}}}) & \xrightarrow{\mathbf{L}L_{\Gamma^*}} & \mathbf{Ho}(\mathrm{pro}-\mathbf{SSet}) \end{array}$$

By applying the simplicial object $[n] \mapsto F_n$ in $\mathrm{pro}-T^{\Delta^{\mathrm{op}}}$ (see 2.3.39) to the diagram, we obtain a zig-zag strict weak equivalence between $\mathbf{L}L_{\Gamma^*}(\mathbf{L}|[n] \mapsto F_n|_{\mathrm{pro}-T^{\Delta^{\mathrm{op}}}})$ and $\mathbf{L}|\mathbf{L}L_{\Gamma^*}([n] \mapsto F_n)|$. Consider a cofibrant replacement $H_\bullet \to ([n] \mapsto F_n)$ of $([n] \mapsto F_n)$ in $(\mathrm{pro}-T^{\Delta^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$. By 2.3.42, the former is strictly weakly equivalent to $\mathbf{L}L_{\Gamma^*}(|[n] \mapsto F_n|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{\mathrm{op}}}})$ which is isomorphic to $\mathbf{L}L_{\Gamma^*}(F_\bullet) = h(F_\bullet)$ by 2.3.43.

On the other hand, we have a strict weak equivalence

$$\mathbf{L}|\mathbf{L}L_{\Gamma^*}([n] \mapsto F_n)| \to |L_{\Gamma^*}(H_\bullet)|$$

So far, we have shown that $\mathbf{L}L_{\Gamma^*}(F_\bullet)$ and $|L_{\Gamma^*}(H_\bullet)|$ are strictly weakly equivalent.

Say $K_\bullet \to ([n] \mapsto F_n)$ is the cofibrant replacement which is used to get the functoriality of $([n] \mapsto h(F_n))$. Then we can find a dotted arrow d in the diagram

$$\begin{array}{ccc} H_\bullet & \overset{d}{\dashrightarrow} & K_\bullet \\ & \searrow \quad \swarrow & \\ & ([n] \mapsto F_n) & \end{array}$$

Since $K_\bullet$ and $H_\bullet$ are Reedy cofibrant, $H_n \to K_n$ is a weak equivalence between cofibrant object and hence

$$L_{\Gamma^*} H_n \to L_{\Gamma^*} K_n$$

is a weak equivalence. Finally, every object in $(\mathrm{pro}-\mathbf{SSet})^{\Delta^{\mathrm{op}}}$ is Reedy cofibrant and thus

$$|L_{\Gamma^*}(H_\bullet)| = |[n] \mapsto L_{\Gamma^*}(H_n)| \to |[n] \mapsto L_{\Gamma^*}(K_n)| = |[n] \mapsto h(F_n)|$$

is a weak equivalence. □

Lemma 2.3.42. *Let $F_\bullet$ be a simplicial pro-object in $T^{\Delta^{\mathrm{op}}}$ and $H_\bullet \to F_\bullet$ be a cofibrant replacement of $F_\bullet$ in $(\mathrm{pro}-T^{\Delta^{\mathrm{op}}})^{\Delta^{\mathrm{op}}}$. Then two objects $|H_\bullet|_{\mathrm{pro}-T^{\Delta^{\mathrm{op}}}}$ and $|F_\bullet|_{\mathrm{pro}^{E\text{-}H}-T^{\Delta^{\mathrm{op}}}}$ are weakly equivalent in $\mathrm{pro}-T^{\Delta^{\mathrm{op}}}$.*

Proof. Recall from 2.2.21 that we have a Quillen equivalence

$$\mathrm{id} : \mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}} \to \mathrm{pro} - T^{\Delta^{op}}$$

Since the identity functor commutes with colimits, we have a (2-categorical) commutative diagram

$$\begin{array}{ccc} \mathbf{Ho}((\mathrm{pro} - T^{\Delta^{op}})^{\Delta^{op}}) & \xrightarrow{\mathbf{L}\mathrm{id}} & \mathbf{Ho}((\mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}})^{\Delta^{op}}) \\ {\scriptstyle \mathbf{L}|-|_{\mathrm{pro}-T^{\Delta^{op}}}} \downarrow & & \downarrow {\scriptstyle \mathbf{L}|-|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}}} \\ \mathbf{Ho}(\mathrm{pro} - T^{\Delta^{op}}) & \xrightarrow{\mathbf{L}\mathrm{id}} & \mathbf{Ho}(\mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}}) \end{array}$$

So we have a zig-zag weak equivalence between $\mathbf{L}\mathrm{id}(\mathbf{L}|F_\bullet|_{\mathrm{pro}-T^{\Delta^{op}}})$ and $\mathbf{L}|\mathbf{L}\mathrm{id}(F_\bullet)|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}}$. For the former, since the classes of weak equivalences are the same for both $\mathrm{pro} - T^{\Delta^{op}}$ and $\mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}}$, there is a weak equivalence

$$\mathbf{L}\mathrm{id}(\mathbf{L}|F_\bullet|_{\mathrm{pro}-T^{\Delta^{op}}}) \to |H_\bullet|_{\mathrm{pro}-T^{\Delta^{op}}}$$

On the other hand, by definition,

$$\mathbf{L}|\mathbf{L}\mathrm{id}(F_\bullet)|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}} = \mathbf{L}|H_\bullet|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}}$$

Also, there is a weak equivalence

$$\mathbf{L}|H_\bullet|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}} \to |H_\bullet|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}}$$

because every object in $\mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}}$ is Reedy cofibrant. Furthermore, there is a weak equivalence

$$|H_\bullet|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}} \to |F_\bullet|_{\mathrm{pro}^{\text{E-H}}-T^{\Delta^{op}}}$$

because $H_\bullet \to F_\bullet$ can be viewed as a Reedy weak equivalence between Reedy cofibrant objects in $(\mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}})^{\Delta^{op}}$. All things considered, we have proved the statement with respect to $\mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}}$. The same result holds in $\mathrm{pro} - T^{\Delta^{op}}$ because $\mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}}$ and $\mathrm{pro} - T^{\Delta^{op}}$ have the same class of weak equivalences. ☐

Lemma 2.3.43. *Let $F_\bullet$ be a simplicial object in T. Then $F_\bullet$ viewed as a pro-object in $T^{\Delta^{op}}$ and $|[n] \mapsto F_n|_{\mathrm{pro}^{E\text{-}H}-T^{\Delta^{op}}}$ are isomorphic pro-objects of $T^{\Delta^{op}}$.*

Proof. Let $\mathcal{C}$ be a site whose associated topos is T. In the category of simplicial presheaves on $\mathcal{C}$, the realization $|[n] \mapsto F_n|_{\hat{\mathcal{C}}^{\Delta^{op}}}$ is isomorphic to $F_\bullet$. That is, there is a natural isomorphism

$$\begin{array}{ccccc} \coprod_{([n]\to[m])\in\Delta} F_m \otimes \Delta[n] & \rightrightarrows & \coprod_{[n]\in\Delta} F_n \otimes \Delta[n] & \longrightarrow & |[n] \mapsto F_n| \\ & & & \searrow & \vdots\, d \\ & & & & F_\bullet \end{array}$$

where the top row is a coequalizer diagram. After sheafification, this induces an isomorphism

$$|[n] \mapsto F_n|_{T^{\Delta^{op}}} \longrightarrow F_\bullet$$

of simplicial sheaves on $\mathcal{C}$ because the realization taken as simplicial sheaves is isomorphic to the sheafification of the realization taken as simplicial presheaves.

The inclusion functor

$$i : T^{\Delta^{op}} \to \mathrm{pro}^{\text{E-H}} - T^{\Delta^{op}}$$

commutes with colimits and thus we have a commutative diagram

$$\begin{array}{ccc} (T^{\Delta^{\mathrm{op}}})^{\Delta^{\mathrm{op}}} & \xrightarrow{i} & (\mathrm{pro}^{\text{E-H}} - T^{\Delta^{\mathrm{op}}})^{\Delta^{\mathrm{op}}} \\ {\scriptstyle |-|_{T^{\Delta^{\mathrm{op}}}}}\downarrow & & \downarrow{\scriptstyle |-|_{\mathrm{pro}^{\text{E-H}} - T^{\Delta^{\mathrm{op}}}}} \\ T^{\Delta^{\mathrm{op}}} & \xrightarrow{i} & \mathrm{pro}^{\text{E-H}} - T^{\Delta^{\mathrm{op}}} \end{array}$$

Then the result follows immediately from diagram chasing. □

2.3.44. Let $\mathcal{P}$ be a class of morphism in T that is stable under base change and composition, and that contains all isomorphisms. Assume further that every morphism in $\mathcal{P}$ is an epimorphism.

Definition 2.3.45. A simplicial object $G_\bullet$ over an object F in T is a *$\mathcal{P}$-hypercover* of F if the unique morphism $G_\bullet/F \to *_{T/F}$ to a final object $*_{T/F}$ in the localized topos T/F satisfies the following conditions:

(i) $G_0/F \to *_{T/F}$,
(ii) $G_{n+1}/F \to (\mathrm{cosk}_n \, \mathrm{sk}_n \, G_\bullet/F)_{n+1}$ for each $n \geq 0$

are in $\mathcal{P}$.

Lemma 2.3.46. *Let $G_\bullet$ be a $\mathcal{P}$-hypercover of F in T. Then the structure morphism $G_\bullet \to F$ is a trivial fibration in the weak fibration structure (see 2.2.15) on $T^{\Delta^{\mathrm{op}}}$. In particular, it can be viewed as a trivial fibration in* $\mathrm{pro} - T^{\Delta^{\mathrm{op}}}$.

Proof. For $n \geq 1$, the functor $j_! : T/F \to T$ commutes with the coskeleton functors. For $n = 0$, $(\mathrm{cosk}_0^F \, \mathrm{sk}_0^F \, G_\bullet/F)_1$ is nothing but $G_0 \times_F G_0$ over F. As we have

$$G_0 \times_F G_0 = (\mathrm{cosk}_0 \, \mathrm{sk}_0 \, G_\bullet)_1 \times_{(\mathrm{cosk}_0 \, \mathrm{sk}_0 \, F)_1} F$$

the statement follows from that every morphism in $\mathcal{P}$ is an epimorphism and 2.1.10. □

2.3.47. For a simplicial object $[n] \mapsto F^n_\bullet$ in the category of simplicial objects in a category $\mathcal{C}$, we view it as a bi-simplicial object $F_{\bullet\bullet}$ in $\mathcal{C}$ sending $([n],[m])$ to F^n_m. When we say a bi-simplilcial object $G_{\bullet\bullet}$ over a simplicial object $F_\bullet$, the simplicial object sends (n,m) into F_n. So for each n, we have a simplicial object $G_{n\bullet}$ over a constant simplicial object F_n.

Definition 2.3.48. A bi-simplicial object $G_{\bullet\bullet}$ over a simplicial object $F_\bullet$ in T is a *$\mathcal{P}$-hypercover* of $F_\bullet$ if

$$G_{n\bullet} \to F_n$$

is a $\mathcal{P}$-hypercover for each n.

2.3.49. Thanks to our approach to the topological types, we easily obtain the hypercover descent theorems 2.3.51 and 2.3.53. The following proposition is due to Misamore [24, 2.1].

Proposition 2.3.50. *If a morphism $G_{\bullet\bullet} \to F_{\bullet\bullet}$ of simplicial objects in $T^{\Delta^{\mathrm{op}}}$ is a degree-wise trivial fibration in a sense that $G_{n\bullet} \to F_{n\bullet}$ is a trivial fibration of the weak fibration structure on $T^{\Delta^{\mathrm{op}}}$ for each n, then its diagonal $\Delta G_{\bullet\bullet} \to \Delta F_{\bullet\bullet}$ is a weak equivalence in $T^{\Delta^{\mathrm{op}}}$.*

Proof. Fix a Boolean localization $p : S \to T$. Then a morphism in $T^{\Delta^{\mathrm{op}}}$ is weak equivalence (resp. trivial fibration) if and only if its pull-back in $S^{\Delta^{\mathrm{op}}}$ is a weak equivalence (resp. section-wise trivial fibration). So we may assume that $G_{n\bullet} \to F_{n\bullet}$ is a section-wise trivial fibration. From the corresponding result for simplicial sets, we then have a section-wise weak equivalence of simplicial sets, which implies local weak equivalence. □

Theorem 2.3.51. (Hypercover descent) *Let $G_{\bullet\bullet} \to F_\bullet$ be a $\mathcal{P}$-hypercover of $F_\bullet$ in T. Then it induces a strict weak equivalence*

$$h(\Delta G_{\bullet\bullet}) \to h(F_\bullet)$$

of topological types.

Proof. By the definition of $\mathcal{P}$-hypercovers and 2.3.46, $G_{\bullet\bullet} \to F_\bullet$ is a degree-wise trivial fibration in $T^{\Delta^{\mathrm{op}}}$. So the result follows from 2.3.50. □

Lemma 2.3.52. *Let $G_\bullet \to F$ (resp. $H_\bullet \to F$) be a $\mathcal{P}$-hypercover of F. Then two topological types $h(G_\bullet/S)$ and $h(H_\bullet/S)$ are strictly weakly equivalent.*

Proof. Define a bi-simplicial object $K_{\bullet\bullet}$ by $K_{mn} = G_m \times_F H_n$. Then the projection $K_{\bullet\bullet} \to G_\bullet$ (resp. $K_{\bullet\bullet} \to H_\bullet$) is a $\mathcal{P}$-hypercover of $G_\bullet$ (resp. $H_\bullet$). Apply the bi-simplicial hypercover descent 2.3.51 to obtain strict weak equivalences

$$h(\Delta K_{\bullet\bullet}/S) \to h(G_\bullet/S)$$

and

$$h(\Delta K_{\bullet\bullet}/S) \to h(H_\bullet/S)$$

□

Theorem 2.3.53. (Simplicial hypercover descent) *Let $G_\bullet$ be a $\mathcal{P}$-hypercover of F in T. Then there is an isomorphism*

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} h(G_n) \xrightarrow{\sim} h(F)$$

of pro-simplicial sets in the homotopy category of pro-simplicial sets. Furthermore, these two pro-simplicial sets are strictly weakly equivalent.

Proof. This is a combination of the simplicial descent 2.3.41 and the bi-simplicial hypercover descent 2.3.51. □

2.3.54. We now study topological types of stacks and its relationship to topological types of simplicial sheaves.

Definition 2.3.55.

(i) A fibered category in groupoid $\mathcal{X}$ over $\mathcal{C}$ is *representable by sheaves* if there exists a sheaf F on $\mathcal{C}$ and that $\mathcal{X}$ is equivalent to the fibered category associated to the sheaf F in the 2-category of fibered categories over $\mathcal{C}$.

(ii) A morphism $\mathcal{X} \to \mathcal{Y}$ of fibered categories is *representable by sheaves* if for every sheaf Y on $\mathcal{C}$ and every morphism $Y \to \mathcal{Y}$, the base change $\mathcal{X} \times_{\mathcal{Y}} Y$ is representable by sheaves.

2.3.56. Whenever we discuss topological types of stacks, we assume that the site $\mathcal{C}$ has a subcanonical topology. Moreover, for the topological type of a stack $\mathcal{X}$, we assume further that the site $\mathcal{C}/\mathcal{X}$ is subcanonical, too. Note that these conditions are satisfied by algebraic stacks.

Theorem 2.3.57. *Let $\mathcal{X}/\mathcal{C}$ be a stack. Assume there exists an object $X \in \mathcal{C}$ and a morphism $X \to \mathcal{X}$ such that $X \to \mathcal{X}$ is representable by sheaves and the morphism*

$$h_{X\to\mathcal{X}} \to *_{T/\mathcal{X}}$$

of sheaves on $\mathcal{C}/\mathcal{X}$ is an epimorphism. Then there is an isomorphism

$$h(\mathrm{cosk}_0(X/\mathcal{X})) \xrightarrow{\sim} h(\mathcal{X})$$

of pro-simplicial sets in the homotopy category of pro-simplicial sets. Furthermore, these pro-simplicial sets are strictly weakly equivalent.

Proof. Note that $\mathrm{cosk}_0(X/\mathcal{X})$ is a simplicial sheaves since $X \to \mathcal{X}$ is representable by sheaves. By applying 2.3.51 to the morphism $h_{X\to\mathcal{X}} \to *_{T/\mathcal{X}}$, there is a strict weak equivalence

$$h(\mathrm{cosk}_0(h_{X\to\mathcal{X}}/*_{T/\mathcal{X}})) \to h(*_{T/\mathcal{X}}) = h(\mathcal{X})$$

Consider the following commutative diagram

$$\begin{array}{ccc}
\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, h((\mathrm{cosk}_0(h_{X\to\mathcal{X}}/*_{T/\mathcal{X}}))_n) & \longrightarrow & h(\mathrm{cosk}_0(h_{X\to\mathcal{X}}/*_{T/\mathcal{X}})) \\
\downarrow & & \downarrow \\
\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, h((\mathrm{cosk}_0(X/\mathcal{X}))_n) & \longrightarrow & h(\mathrm{cosk}_0(X/\mathcal{X}))
\end{array}$$

By 2.3.41, the two horizontal arrows are isomorphisms in $\mathbf{Ho}(\mathrm{pro}-\mathbf{SSet})$. On the other hand, there is a strict weak equivalence

$$h((\mathrm{cosk}_0(X/\mathcal{X}))_n) \to h((\mathrm{cosk}_0(h_{X\to\mathcal{X}}/*_{T/\mathcal{X}}))_n)$$

for each n because both topological types are the topological type of the topos $T/(\mathrm{cosk}_0(h_{X\to\mathcal{X}}/*_{T/\mathcal{X}}))_n$ by 2.3.30. Thus the left vertical arrow is an isomorphism in the homotopy category of pro-simplicial sets. From the commutative diagram above, the right vertical map is also an isomorphism. Since all these isomorphisms are induced by strictly weakly equivalences, so is the right vertical isomorphism. ☐

2.4. **Connected components, Fundamental groups, and Cohomology of Topological types.**

2.4.1. Let $X_\bullet$ be a simplicial set. Recall that the set of *connected component* $\pi_0(X_\bullet)$ of $X_\bullet$ is the coequalizer

$$X_1 \rightrightarrows X_0 \longrightarrow \pi_0(X_\bullet)$$

For a pro-simplicial set $X : I \to \mathbf{SSet} : i \mapsto X_i$, its *connected component* $\pi_0(X)$ is the pro-set

$$I \to \mathbf{Set} : i \mapsto \pi_0(X_i)$$

Proposition 2.4.2. *Let $F_\bullet$ be a simplicial object in T. Then there is a canonical bijection of pro-sets*

$$\pi_0(h(F_\bullet)) \xrightarrow{\sim} \pi_0(L_{\Gamma^*}(F_\bullet))$$

Proof. It suffices to show that for any given set S, the canonical map

$$\mathrm{Mor}_{\text{pro-sets}}(\pi_0(L_{\Gamma^*}(F_\bullet)), S) \to \mathrm{Mor}_{\text{pro-sets}}(\pi_0(h(F_\bullet)), S)$$

is a bijection. Consider a cofibrant replacement $H \to F_\bullet$ of $F_\bullet$ in pro$-T^{\Delta^{\mathrm{op}}}$. Say $H : I \to T^{\Delta^{\mathrm{op}}}$ with I the cofiltered index category.

$$\begin{aligned}
&(2.4.2.1) & \mathrm{Mor}_{\text{pro}-\mathbf{Set}}(\pi_0(L_{\Gamma^*}(F_\bullet)), S) &= \mathrm{Mor}_{\text{pro}-\mathbf{SSet}}(L_{\Gamma^*}(F_\bullet), S)\\
&(2.4.2.2) & &= \mathrm{Mor}_{\text{pro}-T^{\Delta^{\mathrm{op}}}}(F_\bullet, \underline{S})\\
&(2.4.2.3) & &= \mathrm{Mor}_{T^{\Delta^{\mathrm{op}}}}(F_\bullet, \underline{S})\\
&(2.4.2.4) & &= \mathrm{Mor}_{T}(\pi_0(F_\bullet), \underline{S})\\
&(2.4.2.5) & &= \mathrm{Mor}_{\text{pro}-T}(\pi_0(F_\bullet), \underline{S})\\
&(2.4.2.6) & &= \mathrm{Mor}_{\text{pro}-T}(\pi_0(H), \underline{S})\\
&(2.4.2.7) & &= \mathrm{Mor}_{\text{pro}-\mathbf{Set}}(\pi_0(h(F_\bullet)), S)
\end{aligned}$$

where $\underline{S} = \Gamma^*(S)$. Note that (2.4.2.6) follows from that $H \to F_\bullet$ is a weak equivalence. □

Remark 2.4.3. In a geometric situation, say for schemes, the proposition above will mean that the number of connected components of a scheme X is equal to the number of connected components of its topological type.

2.4.4. Let G be a discrete group. For a simplicial set $X_\bullet$ and a point $x \in X_0$, its fundamental group classifies G-torsors in a sense that there is a bijection

$$\mathrm{Mor}_{\mathbf{Gps}}(\pi_1(X_\bullet, x), G) = H^1(X_\bullet, G)$$

where $H^1(X_\bullet, G)$ is the set of isomorphism classes of G-torsors. We state and prove the corresponding result for topological types. We begin with a homotopy theoretic reinterpretation of the result.

2.4.5. Let G be a group object in T. The non-abelian cohomology group $H^1(T, G)$ is defined to be the set of isomorphism classes of G-torsors. Then it has been known from [17] that there is an identification

$$H^1(T, G) = \mathbf{Ho}_{T^{\Delta^{\mathrm{op}}}}(*, BG)$$

This behaves well with respect to the localization:

Lemma 2.4.6. *Let F be an object in T. For a group object G in T, there is a canonical bijection*

$$H^1(T, G) \xrightarrow{\sim} H^1(T/F, j^*G)$$

where $j^* : T \to T/F$ is the pull-back.

Proof. From [17] or [20, 9.8], the statement is equivalent to

$$\mathbf{Ho}_{T^{\Delta^{\mathrm{op}}}}(F, BG) = \mathbf{Ho}_{(T/F)^{\Delta^{\mathrm{op}}}}(*_{T/F}, B(j^*G))$$

which is a consequence of the Quillen adjunction

$$(j_!, j^*) : T^{\Delta^{\mathrm{op}}} \to (T/F)^{\Delta^{\mathrm{op}}}$$

with respect to Joyal-Jardine's model category structures 2.1.12. Indeed, $j_!$ preserves local weak equivalences by 2.3.29 and cofibrations which are exactly monomorphisms. □

Lemma 2.4.7. *Let $F_\bullet$ be a simplicial object in T and G be a discrete group. Then for each $n \geq 0$ there is a canonical bijection*

$$\boldsymbol{Ho}_{T^{\Delta^{\rm op}}}(F_\bullet, K(\underline{G}, n)) \xrightarrow{\sim} \boldsymbol{Ho}_{\text{pro}-\boldsymbol{SSet}}(h(F_\bullet), K(G, n))$$

Proof.

$$\begin{aligned}
&(2.4.7.1) && \mathbf{Ho}_{T^{\Delta^{\rm op}}}(F_\bullet, K(\underline{G}, n)) = \mathbf{Ho}_{\text{pro}^{\text{E-H}}-T^{\Delta^{\rm op}}}(F_\bullet, K(\underline{G}, n))\\
&(2.4.7.2) && = \mathbf{Ho}_{\text{pro}-T^{\Delta^{\rm op}}}(F_\bullet, K(\underline{G}, n))\\
&(2.4.7.3) && = \mathbf{Ho}_{\text{pro}-T^{\Delta^{\rm op}}}(F_\bullet, \mathbf{R}^{\text{Str}}\Gamma^* K(G, n))\\
&(2.4.7.4) && = \mathbf{Ho}_{\text{pro}^{\text{Str}}-\mathbf{SSet}}(\mathbf{L}^{\text{Str}} L_{\Gamma^*}(F_\bullet), K(G, n))\\
&(2.4.7.5) && = \mathbf{Ho}_{\text{pro}^{\text{Str}}-\mathbf{SSet}}(h(F_\bullet), K(G, n))\\
&(2.4.7.6) && = \mathbf{Ho}_{\text{pro}-\mathbf{SSet}}(h(F_\bullet), K(G, n))
\end{aligned}$$

That (2.4.7.1) is a consequence of the Quillen adjunction

$$(i : \varprojlim) : \text{pro}^{\text{E-H}} - T^{\Delta^{\rm op}} \to T^{\Delta^{\rm op}}$$

Similarly, (2.4.7.2) is a consequence of the Quillen adjunction

$$(\text{id}, \text{id}) : \text{pro}^{\text{E-H}} - T^{\Delta^{\rm op}} \to \text{pro} - T^{\Delta^{\rm op}}$$

For (2.4.7.3), $K(G, n)$ is a not just a simplicial set but also a simplicial group. So it is fibrant simplicial set and thus a fibrant object in $\text{pro}^{\text{Str}} - \mathbf{SSet}$. Hence there is a weak equivalence between $\mathbf{R}^{\text{Str}}\Gamma^* K(G, n)$ and $\Gamma^* K(G, n) = K(\underline{G}, n)$. Also, (2.4.7.4) follows from the Quillen adjunction

$$(L_{\Gamma^*}, \Gamma^*) : \text{pro}^{\text{str}} - \mathbf{SSet} \to \text{pro} - T^{\Delta^{\rm op}}$$

Finally, (2.4.7.6) follows from [12, 10.9]. ☐

Theorem 2.4.8. *Let F be an object in T. Fix a point $x \in h(F)$. For every discrete group G, there is a bijection*

$$\text{Mor}_{\textit{pro-Gps}}(\pi_1(h(F), x), \hat{G}) \longrightarrow H^1(F, \underline{G})$$

where $\underline{G}$ is the constant sheaf of groups associated to G.

Proof. Consider a cofibrant replacement $H \to F$ in $\text{pro} - T^{\Delta^{\rm op}}$. Say $H : I \to T^{\Delta^{\rm op}}$ with I cofiltered index category.

$$\begin{aligned}
&(2.4.8.1) && \text{Mor}_{\text{pro-Gps}}(\pi_1(h(F)), G) = \varinjlim_{i \in I^{\rm op}} \text{Mor}_{\text{Gps}}(\pi_1(L_{\Gamma^*}(H_i)), G)\\
&(2.4.8.2) && = \varinjlim_{i \in I^{\rm op}} \mathbf{Ho}_{\mathbf{SSet}}(L_{\Gamma^*}(H_i), BG)\\
&(2.4.8.3) && = \mathbf{Ho}_{\text{pro}-\mathbf{SSet}}(L_{\Gamma^*}(H), BG)\\
&(2.4.8.4) && = \mathbf{Ho}_{T^{\Delta^{\rm op}}}(F, B\underline{G})\\
&(2.4.8.5) && = H^1(F, \underline{G})
\end{aligned}$$

where (2.4.8.3) follows from [12, 8.1] and (2.4.8.4) from 2.4.7. ☐

2.4.9. Fix an abelian group Λ throughout the rest of the subsection.

Definition 2.4.10. For the topological type $h(F_\bullet)$ of a simplicial object $F_\bullet$ in T, the *cohomology, homology, and homotopy groups* are those of $h(F_\bullet)$ as a pro-simplicial set.

2.4.11. In more detail, consider a cofibrant replacement $H \to F_\bullet$ of $F_\bullet$ in $\text{pro} - T^{\Delta^{\text{op}}}$ so that $h(F_\bullet) = L_{\Gamma^*}(H)$. Say $H : I \to T^{\Delta^{\text{op}}}$ with I cofiltered. Then nth cohomology group with coefficient Λ is

$$H^n(h(F_\bullet), \Lambda) := \varinjlim_{i \in I^{\text{op}}} H^n(L_{\Gamma^*}(H(i)), \Lambda)$$

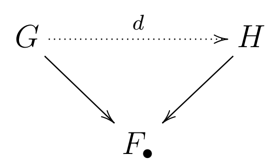
which is the filtered colimit of nth cohomology groups of simplicial sets with coefficient Λ.

Unlike cohomology groups, homology and homotopy groups are pro-groups. The nth homology group with coefficient Λ is a pro-group

$$H_n(h(F_\bullet), \Lambda) := \{H_n(L_{\Gamma^*}(H(i)), \Lambda)\}_{i \in I} = \ I \xrightarrow{L_{\Gamma^*} \circ H} \mathbf{SSet} \xrightarrow{H_n} \mathbf{Group}$$

which is obtained by degree-wise application of the homotopy groups to simplicial sets. The nth homotopy groups are defined in a similar way.

All those groups are independent of choice of cofibrant approximations, up to isomorphism. To see, let $G \to F_\bullet$ be a cofibrant approximation of $F_\bullet$ in $\text{pro} - T^{\Delta^{\text{op}}}$. We can find a dotted arrow filling in the diagram



The dotted arrow d is a weak equivalence by the 2-out-of-3 property of weak equivalences. Since both G and H are cofibrant, the morphism d induces a weak equivalence $L_{\Gamma^*}(G) \to L_{\Gamma^*}(H)$ in $\text{pro} - \mathbf{SSet}$. Not only that, it is a strict weak equivalence. Now we may assume $d : L_{\Gamma^*}(G) \to L_{\Gamma^*}(H)$ is a level-wise weak equivalence of simplicial sets because every morphism of pro-objects has a level presentation. It then follows from the fact that weak equivalences of simplicial sets induce isomorphisms of homology and cohomology groups with any abelian coefficient, and homotopy groups when they are pointed.

Remark 2.4.12. The strictly weakly equivalent pro-simplicial sets induce isomorphic cohomology, homology, and homotopy groups. Therefore, the nth cohomology group $H^n(h(F), \Lambda)$ could have defined by the nth cohomology group $H^n(h(T/F), \Lambda)$ of the topological type $h(T/F)$ by 2.3.30.

2.4.13. There is a notion of cohomology of T. On the other hand, we have defined cohomology groups of the topological type $h(T)$ (see 2.4.10) as cohomology groups of the pro-simplicial sets. Of course, these two coincide:

Proposition 2.4.14. *There is a canonical isomorphism*

$$H^n(T, \underline{\Lambda}) \xrightarrow{\sim} H^n(h(T), \Lambda)$$

of cohomology groups for each $n \geq 0$.

Proof. Consider a cofibrant replacement $H \to *$ in $\text{pro} - T^{\Delta^{\text{op}}}$.

(2.4.14.1) $$H^n(T, \underline{\Lambda}) = \mathbf{Ho}_{T^{\Delta^{\text{op}}}}(*, K(\underline{\Lambda}, n))$$

$$= \mathbf{Ho}_{\text{pro}-\mathbf{SSet}}(L_{\Gamma^*}(H), K(\Lambda, n)) \tag{2.4.14.2}$$

$$= \varinjlim_{i \in I^{\text{op}}} \mathbf{Ho}_{\mathbf{SSet}}(L_{\Gamma^*}(H(i)), K(\Lambda, n)) \tag{2.4.14.3}$$

$$= \varinjlim_{i \in I^{\text{op}}} H^n(L_{\Gamma^*}(H(i)), \Lambda) \tag{2.4.14.4}$$

$$= H^n(h(T), \Lambda) \tag{2.4.14.5}$$

That (2.4.14.2) is from 2.4.7. Also, (2.4.14.3) follows from [12, 8.1] as in (2.4.8.3). □

2.4.15. A simplicial object $F_\bullet$ in T induces a simplicial topos $[n] \mapsto T/F_n$. The cohomology of the total topos $T/F_\bullet$ associated to the simplicial topos agrees with the cohomology the topological type $h(F_\bullet)$:

Proposition 2.4.16. *Let $F_\bullet$ be a simplicial object in T. Then there is an isomorphism*

$$H^n(T/F_\bullet, \underline{\Lambda}) \xrightarrow{\sim} H^n(h(F_\bullet), \Lambda)$$

of cohomology groups for each $n \geq 0$.

Proof. Once we know that $H^n(T/F_\bullet, \underline{\Lambda})$ can be identified with $\mathbf{Ho}_{T^{\Delta^{\text{op}}}}(F_\bullet, K(\underline{\Lambda}, n))$, we can apply the proof of 2.4.14. Yet such an identification follows from [20, 8.34]. □

2.4.17. Recall the set-up in 2.3.27 where the pull-back f^* of morphism of topoi $f : T' \to T$ admits a left adjoint $f_!$ that commutes with finite limits. We have from 2.3.27 a strict weak equivalence between $h_{T'}(F'_\bullet)$ and $h_T(f_!(F'_\bullet))$. So they have isomorphic cohomology groups by 2.4.12. We give an alternative proof of this result with more general coefficient groups:

Lemma 2.4.18. *Let $f : \mathcal{C}' \to \mathcal{C}$ be a cocontinuous functor between sites with the associated morphism of topoi*

$$f : T' \to T$$

Assume that the functor is continuous and commutes with finite limits. For a sheaf of abelian groups G in T and for each $n \geq 0$, there is a canonical isomorphism

$$H^n(f_!(F'_\bullet), G) \xrightarrow{\sim} H^n(F'_\bullet, f^*G)$$

of cohomology groups.

Proof. Denote by $K(G,n)$ the simplicial abelian sheaf in T that corresponds to the chain complex $G[-n]$ under the Dold-Kan correspondence.

$$
\begin{aligned}
&(2.4.18.1) \quad & H^n(f_!(F'_\bullet), G) &= \mathbf{Ho}_{T^{\Delta^{op}}}(f_!(F'_\bullet), K(G,n)) \\
&(2.4.18.2) \quad & &= \mathbf{Ho}_{T^{\Delta^{op}}}(f_!(F'_\bullet), \mathbf{R}\varprojlim(K(G,n))) \\
&(2.4.18.3) \quad & &= \mathbf{Ho}_{\text{pro}^{\text{E-H}}-T^{\Delta^{op}}}(\mathbf{L}i(f_!(F'_\bullet)), K(G,n)) \\
&(2.4.18.4) \quad & &= \mathbf{Ho}_{\text{pro}^{\text{E-H}}-T^{\Delta^{op}}}(f_!(F'_\bullet), K(G,n)) \\
&(2.4.18.5) \quad & &= \mathbf{Ho}_{\text{pro}-T^{\Delta^{op}}}(f_!(F'_\bullet), K(G,n)) \\
&(2.4.18.6) \quad & &= \mathbf{Ho}_{\text{pro}-T^{\Delta^{op}}}(\mathbf{L}f_!((F'_\bullet)), K(G,n)) \\
&(2.4.18.7) \quad & &= \mathbf{Ho}_{\text{pro}-T'^{\Delta^{op}}}(F'_\bullet, \mathbf{R}f^*(K(G,n))) \\
&(2.4.18.8) \quad & &= \mathbf{Ho}_{\text{pro}-T'^{\Delta^{op}}}(F'_\bullet, f^*(K(G,n))) \\
&(2.4.18.9) \quad & &= \mathbf{Ho}_{T'^{\Delta^{op}}}(F'_\bullet, f^*(K(G,n))) \\
&(2.4.18.10) \quad & &= H^n(F'_\bullet, f^*G)
\end{aligned}
$$

Consider the Quillen adjunction

$$(i, \varprojlim) : \text{pro}^{\text{E-H}} - T^{\Delta^{op}} \to T^{\Delta^{op}}$$

Choose a fibrant replacement $K(G,n) \to F_{K(G,n)}$ of $K(G,n)$ in $T'^{\Delta^{op}}$. It then becomes a fibrant approximation of $K(G,n)$ in $\text{pro}^{\text{E-H}} - T^{\Delta^{op}}$ and so (2.4.18.2) follows. (2.4.18.4) follows because every object in $T^{\Delta^{op}}$ is cofibrant. That (2.4.18.5) follows from the Quillen equivalence between Barnea-Schlank's and Edwards-Hastings' model categories. For (2.4.18.6), by the assumption on $f_!$, it can be regarded as a pull-back of morphism of topoi and thus preserves trivial fibrations. So we have a weak equivalence $\mathbf{L}f_!(F'_\bullet) \to F'_\bullet$, which shows (2.4.18.6). That (2.4.18.8) follows from the fact that f^* preserves weak equivalences. Also, (2.4.18.9) is the application of the same argument from (2.4.18.2) to (2.4.18.5). □

2.4.19. In fact, this is a generalization of Jardine's lemma [16, 3.6]. In his proof, he applied Brown's adjoint functor lemma [3, p.426] to a *category of fibrant objects for a homotopy theory* [3, p.420]. A *category of fibrant objects* is similar to a model category without cofibrations just like Barnea-Schlank's weak fibration category, and is a sufficient structure for developing a homotopy theory. It is equipped with two classes of morphisms that are called *weak equivalences* and *fibrations*. In the proof, the category of sheaves on the big étale site on Spec k for an algebraically closed field k is equipped with local weak equivalences and local fibrations. By the time Jardine proved the lemma, it was before having the model category structure on the category of sheaves with local weak equivalences as weak equivalences and local fibrations as fibrations, which does not always exist. However, we now have such a model category by enlarging the category to its pro-category, thanks to Barnea-Schlank. Therefore, our proof of the lemma above can be understood as a reinterpretation of Jardine's proof by replacing the *category of fibrant objects* and Brown's adjoint functor lemma by the Barnea-Schlank model category and Quillen adjoint functors respectively. As a byproduct, we can remove the fibrant assumption on Jardine's lemma.

3. Topological types of algebraic stacks

In this section we apply the general discussion about topological types of topoi to algebro-geometric objects. We replace the small étale topology used by all previous theories by the big étale topology in order to define topological types of algebraic stacks.

3.1. **Motivation.**

3.1.1. Let us take a look at Artin-Mazur's étale homotopy type and see what can be improved on it. Let X be a locally noetherian scheme. The homotopy category $\mathrm{HR}(X)$ of hypercovers of X is cofiltered and gives rise to a pro-object, Artin-Mazur's étale homotopy type of X, in the homotopy category of simplicial sets by applying the connected component functor Π:

$$\mathrm{HR}(X) \to \mathbf{Ho}(\mathbf{SSet}) : U_\bullet \mapsto \Pi(U_\bullet)$$

Observe that this is only a pro-object in the homotopy category of simplicial sets, which does not fit into the model category theory. It would be better if this is an object in some homotopy category of pro-simplicial sets so as to utilize model category theory. This goal was partially accomplished by Friedlander with the introduction of rigid hypercovers. Indeed, his étale topological type of the scheme X is a pro-simplicial set. However, the use of model category theory was not a part of his theory as there was no appropriate model category structure on pro-simplicial sets at the time he developed the theory.

Another improvement was made by Barnea-Schlank. After introducing weak fibration categories and eventually putting their model category structure on the pro-category of simplicial sheaves, they applied the machinery to the small étale topos on X and recovered Artin-Mazur's étale homotopy type with the derived functor approach.

3.1.2. All these previous theories can be applied to Deligne-Mumford stacks as one can still use the small étale topology. However, none of them can be directly applied to general algebraic stacks as one cannot use the small étale topology. Recall that algebraic stacks only admit smooth covers, not étale covers, from schemes or algebraic spaces. Nonetheless, one can define étale homotopy type of algebraic stacks by using simplicial hypercovers. Namely, for an algebraic stack $\mathcal{X}$, choose a smooth cover $X \to \mathcal{X}$ with X a scheme. Take the 0th coskeleton $\mathrm{cosk}_0(X/\mathcal{X})$ which is a simplicial algebraic space. One defines the homotopy type of $\mathcal{X}$ to be the étale topological type of the simplicial algebraic space in the sense of Friedlander, and then verify that this definition is independent of choice of smooth covers.

The main aim of this paper is to give its own definition of topological types of algebraic stacks, not depending on smooth covers. We then prove that this definition coincides with the previous definition using smooth covers. This new approach provides a general frame work for a homotopy theory of algebraic stacks. We exploit Barnea-Schlank's model categorical approach for étale homotopy types of schemes and generalize their approach to define topological types of algebraic stacks. On the way to it, we actually modify their approach for schemes, which paves the way for developing a homotopy theory of algebraic stacks. That is, we use the big étale topology unlike the small étale topology used in all the previous theories including Barnea-Schlank. As a result, we can define topological types of algebraic stacks. Our theory of topological types of algebro-geometric objects is established under model category theory so that we can utilize the power of model category theory. Consequently, compared to the previous theories, we can provide more systematic approach for topological types of algebro-geometric objects.

View toward algebraic stacks, we may apply the big smooth topology or the big étale topology to the theory of Friedlander. Unfortunately, it would not work as the use of small étale topology is crucial. Namely, in Friedlander's definition of étale topological type, having the small étale topology guarantees that there is at most one morphism between two rigid hypercovers [8, 4.1] and thus one can obtain étale topological types as pro-objects in the category of simplicial sets, not in its homotopy category. We resolve this issue by taking the topos-theoretical approach. In particular, it does not matter which big topology, étale or smooth, we work as they induce equivalent topoi.

3.2. **A setup for topological types of schemes.** In this subsection we explore various properties of the big étale topology that are necessary for developing our theory of topological types.

Definition 3.2.1. Let X be a scheme. A site *LFÉ(X)* (resp. *LFS(X)*) is the full subcategory of the category of schemes over X, whose objects are locally of finite type morphisms to X with coverings induced by coverings in the big étale (resp. smooth) topology on X.

3.2.2. The following lemma enables us to replace the small étale topology by the big étale topology:

Lemma 3.2.3. *Let X be a scheme. Then the inclusion functors*

$$j : \textit{Ét}(X) \to \textit{LFÉ}(X)$$

and

$$i : \textit{LFÉ}(X) \to \textit{LFS}(X)$$

are cocontinuous, continuous, and commute with finite limits.

Proof. It follows immediately that the inclusion functor j has all the properties and i is continuous that commutes with finite limits. So it only remains to show that i is cocontinuous. Given a locally of finite type morphism $Y \to X$ and any smooth covering $\{Y_i \to Y\}$ of Y over X, the smooth surjection

$$\coprod Y_i \to Y$$

étale locally admits a section. So there exists an étale surjection $Z \to Y$

$$\begin{array}{ccc} & & Z \\ & \swarrow & \downarrow \\ \coprod Y_i & \longrightarrow & Y \end{array}$$

Let Z_i be the fiber product $Y_i \times_{\coprod Y_i} Z$. Then $\{Z_i \to Y\}$ is an étale covering of Y over X that refines $\{Y_i \to Y\}$ over X. □

3.2.4. In order to compare our topological types with Artin-Mazur, we study locally connectedness of topoi. We show that $\mathrm{LFS}(X)^{\sim}$ is locally connected provided that X is locally noetherian. In particular, two topoi $X_{\text{ét}}$ and $\mathrm{LFÉ}(X)^{\sim}$ are locally connected by 3.2.3.

Lemma 3.2.5. *Let X be a locally noetherian scheme. Then the topos LFÉ$(X)^\sim$ is locally connected.*

Proof. The proof is identical to [33, 3.7]. □

3.2.6. Let X be a locally noetherian scheme. For a locally of finite type morphism of schemes $Y \to X$ with Y connected. Then the representable sheaf $h_{Y\to X}$ is a connected object in the topos LFÉ$(X)^\sim$.

3.2.7. One of the importance of 3.2.3 is that one can use the big smooth or étale topology to study étale homotopy theory. Contrary to our topos-theoretical approach, we give a direct site-theoretical approach that recovers Artin-Mazur's étale homotopy type with the big smooth topology. The following proof is a slight modification of Jardine's argument [18, 2.2].

Theorem 3.2.8. *Let $U_\bullet \to X$ be a smooth hypercover of a scheme X. Then there exists an étale hypercover $V_\bullet \to X$ that factors through $U_\bullet$.*

Proof. Since every smooth morphism étale locally admits a section, there exists an étale surjection $V_0 \to X$ which lifts $U_0 \to X$. For $n \geq 0$, assume we have a n-truncated simplicial scheme

$$V_\bullet : \Delta^{op}_{\leq n} \to \mathbf{Sch}/X$$

satisfying the following conditions:

(i) $V_\bullet \to X$ factors through Ét(X)
(ii) $V_\bullet$ splits,
(iii) there is a commutative diagram

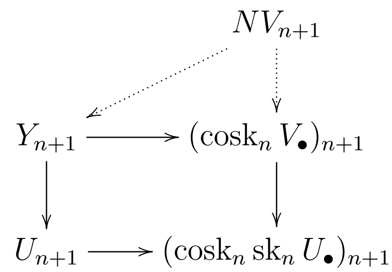
$$\begin{array}{ccc} & & \operatorname{sk}_n U_\bullet \\ & \nearrow & \downarrow \\ V_\bullet & \longrightarrow & X \end{array}$$

such that the bottom morphism is a hypercover.

Remark that even if the left adjoint $i_{n!}$ of the skeleton functor sk_n does not exist in general, we do have $i_{n!}V_\bullet$ thanks to the splitting assumption. So the last condition is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} & & U_\bullet \\ & \nearrow & \downarrow \\ i_{n!}V_\bullet & \longrightarrow & X \end{array}$$

Consider a fibered diagram



Since the bottom morphism is a smooth covering, we can pick an étale surjection $NV_{n+1} \to (\operatorname{cosk}_n V_\bullet)_{n+1}$ which lifts $Y_{n+1} \to (\operatorname{cosk}_n V_\bullet)_{n+1}$. Since each V_i is étale over X, so is their limit $(\operatorname{cosk}_n V_\bullet)_{n+1}$. In particular, NV_{n+1} is étale over X via the coskeleton. Now the morphism

$$NV_{n+1} \to (\operatorname{cosk}_n V_\bullet)_{n+1}$$

extends the n-truncated simplicial scheme $V_\bullet$ to a $(n+1)$-truncated simplicial scheme $W_\bullet$ in a way that $W_i = V_i$ for $0 \le i \le n$ and

$$W_{n+1} = (i_{n!}V_\bullet)_{n+1} \coprod NV_{n+1}$$

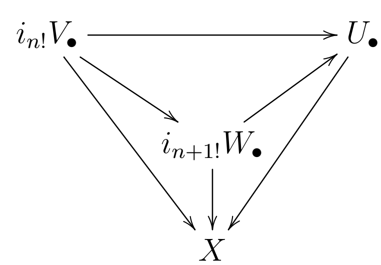
From the splitting condition, each $NV_i \to V_i$ is an open immersion and so NV_i is étale over X. Hence,

$$(i_{n!}V_\bullet)_{n+1} = \coprod_{[n+1]\to[i], i\le n} NV_i$$

is étale over X, which implies W_{n+1} is étale over X. It follows then that $W_\bullet$ is a hypercover. Indeed, we only need to check at degree $n+1$ because $W_\bullet$ extends $V_\bullet$, which is a hypercover over X. For the degree $n+1$, the étale morphism

$$(i_{n!}V_\bullet)_{n+1} \coprod NV_{n+1} \to (\operatorname{cosk}_n V_\bullet)_{n+1}$$

is a surjection because $NV_{n+1} \to (\operatorname{cosk}_n V_\bullet)_{n+1}$ is so. Finally, we have a commutative diagram



where the morphisms $i_{n+1!}W_\bullet$ to $U_\bullet$ (resp. to X) is induced by $NV_{n+1} \to Y_{n+1} \to U_{n+1}$ (resp. $NV_{n+1} \to X$). By taking inductive limits, we are done. □

Corollary 3.2.9. *Let X be a scheme. Then the functor*

$$\mathrm{HR}(\acute{E}t(X)) \to \mathrm{HR}(LFS(X))$$

between two cofiltered categories is cofinal.

3.3. **Topological types of Schemes.**

3.3.1. Given a S-scheme X, the representable functor $h_{X\to S} : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf on the big fppf site on S. The representable functor is restricted to a sheaf on LFÉ(S), although it is not representable unless the structure morphism is locally of finite type. We abusively denote by X the restricted sheaf. Moreover, we view X as a constant simplicial sheaf on LFÉ(S), in turn, as a pro-simplicial sheaf on the site.

Definition 3.3.2. A *topological type of a scheme* X over S is the pro-simplicial set

$$h(X/S) := \mathbf{L}\Pi_S(X)$$

where $\Pi_S : \mathrm{LF\acute{E}}(S)^{\sim} \to \mathbf{Set}$ is the connected component functor. A *topological type of a simplicial scheme* $X_\bullet$ over S is the pro-simplicial set

$$h(X_\bullet/S) := \mathbf{L}\Pi_S(X_\bullet)$$

Remark 3.3.3.

(i) One can compute the topological types over any base scheme due to the globalization lemma 2.3.30. Indeed, if $X_\bullet$ is a scheme over S and $S \to T$ is a locally finite type morphism of schemes with T locally noetherian, then there is a strict weak equivalence $h(X_\bullet/S) \to h(X_\bullet/T)$. So theoretically speaking, one can compute every topological type over $\operatorname{Spec}\mathbb{Z}$. In particular, assuming the structure morphism $X \to S$ is locally of finite type, there is a strict weak equivalence

$$\mathbf{L}\Pi_X(*_X) \to h(X/S)$$

of pro-simplicial sets. So we could have defined the topological type of X to be the topological type of the topos associated to the site LFÉ(X).

(ii) Thanks to 3.2.3, we could have used the topoi associated to the sites LFS(S) or Ét(S) to define the topological type $h(X/S)$.

Example 3.3.4. As an immediate application of 2.3.19, we calculate the topological type of $\operatorname{Spec} k$ with k a field. Fix a separable closure k^{sep} of k. We work with the small étale site on $\operatorname{Spec} k$ whose associated topos is equivalent to the classifying topos BG of the absolute Galois group $G = Gal(k^{\mathrm{sep}}/k)$. Therefore, $h(\operatorname{Spec} k)$ is $K(G, 1)$.

3.3.5. Let X be a locally noetherian scheme. Consider the topological type $\mathbf{L}\Pi_X(*_X)$ of the small étale topos on X. Up to strict weak equivalence, this is nothing but the topological type $h(X) := h(X/\operatorname{Spec}\mathbb{Z})$ of X over $\operatorname{Spec}\mathbb{Z}$. Barnea-Schlank have proved that [2, 8.3] if one applies the natural functor

$$\mathrm{pro}-\mathbf{SSet} \to \mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$$

to the topological type, one obtains an isomorphism

$$\mathbf{L}\Pi_X(*_X) \to h_{\mathrm{AM}}(X)$$

in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$ where $h_{\mathrm{AM}}(X)$ is the étale homotopy type of X in the sense of Artin-Mazur [1, §9]. Since a strict weak equivalence of pro-simplicial sets induces an isomorphism in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$, it also follows that the topological type $h(X)$ is isomorphic to Artin-Mazur's homotopy type as pro-objects in the homotopy category of simplicial sets.

3.3.6. The upshot of the globalization lemma 2.3.30 lies in the definition of topological types of simplicial schemes. Indeed, we regard a simplicial schemes as a single object in the pro-category of simplicial sheaves on LFÉ(S) and derive the object to obtain its topological type. On the other hand, Friedlander defined the étale topological types of simplicial schemes by introducing rigid hypercovers. These two approaches are compatible:

Proposition 3.3.7. *Let $X_\bullet$ be a locally noetherian simplicial scheme. Then the étale topological type $h_{\mathrm{F}}(X_\bullet)$ defined by Friedlander [8, 4.4] is isomorphic to the topological type $h(X_\bullet) := h(X_\bullet/\operatorname{Spec}\mathbb{Z})$ as pro-objects in the homotopy category of simplicial sets.*

Proof. Recall that the natural functor

$$\mathrm{pro}-\mathbf{SSet} \to \mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$$

factors through the homotopy category of pro-simplicial sets:

$$\begin{array}{ccc} \mathrm{pro}-\mathbf{SSet} & \longrightarrow & \mathbf{Ho}(\mathrm{pro}-\mathbf{SSet}) \\ & \searrow & \downarrow \\ & & \mathrm{pro}-\mathbf{Ho}(\mathbf{SSet}) \end{array}$$

On one hand, 2.3.41 gives an isomorphism

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} h(X_n) \to h(X_\bullet)$$

in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$. On the other hand, there is an isomorphism

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} h_{\mathrm{F}}(X_n) \to h_{\mathrm{F}}(X_\bullet)$$

in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$ due to Isaksen [14, 3.3]. For schemes, both Friedlander'sétale topological types and our topological types agree with Artin-Mazur's étale homotopy type. So the result follows from the lemma below. □

Lemma 3.3.8. *Let $X_\bullet$ and $Y_\bullet$ be simplicial pro-simplicial sets. Assume that for each n there is an isomorphism $X_n \to Y_n$ in $\mathrm{pro}-\boldsymbol{Ho}(\boldsymbol{SSet})$. Then there is a canonical isomorphism*

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} X_n \xrightarrow{\sim} \operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} Y_n$$

in $\mathrm{pro}-\boldsymbol{Ho}(\boldsymbol{SSet})$.

Proof. Note that $\operatorname{hocolim}_{[n]\in\Delta^{\mathrm{op}}} X_n$ is isomorphic to the realization $|[n] \mapsto X_n|$ in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$. Recall that the realization is the coequalizer of the diagram

$$\coprod_{([n]\to[m])\in\Delta} X_m \otimes \Delta[n] \rightrightarrows \coprod_{[n]\in\Delta} X_n \otimes \Delta[n]$$

The isomorphism $X_n \to Y_n$ in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$ induces an isomorphism

$$X_n \otimes h_{[m]} \to Y_n \otimes h_{[m]}$$

in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})$ because every pro-simplicial set is cofibrant and so the derived functor of $(-) \otimes h_{[m]}$ extends to the homotopy category. Following Isaksen [13, 9.1], we understand the coequalizer as a colimit indexed by a cofinite directed set. Then it follows from [13, 9.6] that the natural functor pro $-$ **SSet** $\to$ pro $-$ **Ho**(**SSet**) preserves the colimit, which completes the proof. □

3.4. **Topological types of Algebraic spaces.**

3.4.1. Recall that an algebraic space X over S is a sheaf on the big étale site on S. This is restricted to the sheaf on LFÉ(S). As in the case of schemes, we view X as a constant simplicial sheaf on LFÉ(X), in turn, as a pro-simplicial sheaf on the site. We extend the definition of topological types of (simplicial) schemes to (simplicial) algebraic spaces.

Definition 3.4.2. A *topological type of an algebraic space* X over S is the pro-simplicial set

$$h(X/S) := \mathbf{L}\Pi_S(X)$$

where $\Pi_S : \text{LFÉ}(S)^{\sim} \to \mathbf{Set}$ is the connected component functor. A *topological type of a simplicial algebraic space* $X_\bullet$ over S is the pro-simplicial set

$$h(X_\bullet/S) := \mathbf{L}\Pi_S(X_\bullet)$$

Theorem 3.4.3. (Simplicial descent) *Let $X_\bullet$ be a simplicial algebraic spaces over S. There is an isomorphism*

$$\operatorname*{hocolim}_{[n]\in\Delta^{\mathrm{op}}} h(X_n/S) \xrightarrow{\sim} h(X_\bullet/S)$$

of pro-simplicial sets in the homotopy category of pro-simplicial sets. Furthermore, these two pro-simplicial sets are strictly weakly equivalent.

Proof. Follows from the definition of topological types and the simplicial descent 2.3.41. □

3.4.4. Since an algebraic space admits an étale surjection from a scheme, one can try to understand the topological type of algebraic space via the topological type of the scheme. The notion of hypercovers connects these two topological types. Remark that the category of algebraic spaces has all finite limits and thus coskeleton functor cosk_n is representable for every $n \geq 0$.

3.4.5. Throughout the rest of the subsection we will apply the general theory of $\mathcal{P}$-hypercovers 2.3.45 to the topos associated to the site LFÉ(S) with $\mathcal{P}$ smooth surjections of algebraic spaces.

3.4.6. Recall that smooth surjections of algebraic spaces are stable under base change, composition, and contains all isomorphisms. Furthermore, the following lemma shows that they are epimorphisms:

Lemma 3.4.7. *Let $X \to Y$ be a smooth surjection of algebraic spaces over S. Then it is an epimorphism of sheaves on LFÉ(S).*

Proof. This is an immediate consequence of the definition of smooth surjections of algebraic spaces. □

Definition 3.4.8. A *smooth hypercover* is a $\mathcal{P}$-hypercover in the setup 3.4.5.

Theorem 3.4.9. (Hypercover descent) *Let $U_{\bullet\bullet} \to X_\bullet$ be a smooth hypercover of a simplicial algebraic spaces $X_\bullet$ over S. Then it induces a strict weak equivalence*

$$h(\Delta U_{\bullet\bullet}/S) \to h(X_\bullet/S)$$

of topological types.

Proof. An immediate consequence of 2.3.51. □

3.4.10. As a consequence of the hypercover descent 3.4.9, we can compute the topological type of an algebraic space X via the topological type of any simplicial algebraic space $U_\bullet$ with $U_\bullet \to X$ a smooth hypercover. A theoretical aspect of this consequence gives an extrinsic definition of topological types of algebraic spaces. That is, by the definition of algebraic spaces, one can choose an étale surjection $U \to \mathcal{X}$ with X a scheme. Then its 0-coskeleton $\mathrm{cosk}_0(U/\mathcal{X})$ gives a smooth hypercover

$$\mathrm{cosk}_0(U/X) \to X,$$

which shows the existence of a smooth hypercover by a simplicial scheme. Therefore, one could have defined the topological type of an algebraic space by choosing any smooth hypercover that is a simplicial scheme and define the topological type of the algebraic space to be the topological type of the simplicial scheme. That the independence of choice of smooth hypercovers by simplicial algebraic schemes follows from the intrinsic definition of topological types of algebraic spaces. Indeed, no matter how one chooses a smooth hypercover by a simplicial algebraic scheme, there is a strict weak equivalence between the topological type of the simplicial algebraic scheme and the topological type of the algebraic space.

However, the extrinsic definition itself is good enough to define the topological types of algebraic spaces because one can prove the independence of choice of hypercovers without using the intrinsic definition for algebraic spaces:

Lemma 3.4.11. *Let $U_\bullet \to X$ (resp. $V_\bullet \to X$) be a smooth hypercover of an algebraic space X over S by a simplicial algebraic space $U_\bullet$ (resp. $V_\bullet$). Then two topological types $h(U_\bullet/S)$ and $h(V_\bullet/S)$ are strictly weakly equivalent.*

Proof. An immediate consequence of 2.3.52. □

Theorem 3.4.12. (Simplicial hypercover descent) *Let $U_\bullet \to X$ be a smooth hypercover of an algebraic space X over S by a simplicial algebraic space $U_\bullet$. There is an isomorphism*

$$\underset{[n]\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}\, h(U_n/S) \xrightarrow{\sim} h(X/S)$$

of pro-simplicial sets in the homotopy category of pro-simplicial sets. Furthermore, these two pro-simplicial sets are strictly weakly equivalent.

Proof. An immediate consequence of 2.3.53. □

3.5. **Topological types of Algebraic stacks.**

3.5.1. The theory of topological types of algebraic spaces does not work verbatim for algebraic stacks because algebraic stacks cannot be regarded as sheaves. Nevertheless, we can still define topological type of algebraic stacks by using hypercovers.

Definition 3.5.2. Let $\mathcal{X}/S$ be an algebraic stack. A site *LFÉ($\mathcal{X}$)* is defined as following. An object is a pair (Y, y), where $y : Y \to \mathcal{X}$ is a locally of finite type morphism over S with Y an algebraic space. A morphism

$$(Y, y) \to (Z, z)$$

is a pair (h, h^b) where $h : Y \to Z$ is a morphism of algebraic spaces and $h^b : y \to z \circ h$ is a 2-morphism of functors. A collection of maps

$$\{(h_i, h_i^b) : (Y_i, y_i) \to (Y, y)\}$$

is a covering if the underlying collection of morphisms of algebraic spaces $\{y_i : Y_i \to Y\}$ is an étale covering. That is, each y_i is étale and $\coprod Y_i \to Y$ is surjective.

Lemma 3.5.3. *The topos LFÉ($\mathcal{X}$) is locally connected.*

Proof. The forgetful functor

$$\mathrm{LF\acute{E}}(\mathcal{X}) \to \mathrm{LF\acute{E}}(S) : (Y \to \mathcal{X}) \mapsto Y$$

is continuous and cocontinuous. So the pull-back functor of the morphism of topoi

$$\mathrm{LF\acute{E}}(\mathcal{X})^{\sim} \to \mathrm{LF\acute{E}}(S)^{\sim}$$

admits a left adjoint. Then the statement follows from 3.2.5 that $\mathrm{LF\acute{E}}(S)^{\sim}$ is locally connected. □

Remark 3.5.4. One may define LFS($\mathcal{X}$) for an algebraic stack $\mathcal{X}$ whose objects are smooth $\mathcal{X}$-morphism of algebraic stacks. This could be used to develop the theory of topological types for algebraic stacks. Also, the small site Ét($\mathcal{X}$) can be used for Deligne-Mumford stacks.

Definition 3.5.5. A *topological type of an algebraic stack* $\mathcal{X}$ over S is the pro-simplicial set

$$h(\mathcal{X}/S) := \mathbf{L}\Pi_{\mathcal{X}}(*_{\mathrm{LF\acute{E}}(\mathcal{X})^{\sim}})$$

where $\Pi_{\mathcal{X}} : \mathrm{LF\acute{E}}(\mathcal{X})^{\sim} \to \mathbf{Set}$ is the connected component functor.

Remark 3.5.6. Thanks to the definition, we can consider topological types of classifying stacks $\mathcal{B}G$ with G a smooth, not necessarily étale, group scheme. For example, the multiplicative group scheme $\mathbb{G}_m$.

3.5.7. As promised at the beginning, we can think of topological types of algebraic stacks via hypercovers. We need a lemma:

Lemma 3.5.8. *Let $X \to \mathcal{X}$ be a smooth surjection from a scheme X to an algebraic stack $\mathcal{X}$. Then the morphism*

$$h_{X \to \mathcal{X}} \to *_{LF\acute{E}(\mathcal{X})^{\sim}}$$

of sheaves on LFÉ($\mathcal{X}$) is an epimorphism.

Proof. This is an immediate consequence of the definition of the smooth surjection $X \to \mathcal{X}$. □

Theorem 3.5.9. *Let $\mathcal{X}/S$ be an algebraic stack. For any smooth surjection $X \to \mathcal{X}$ with X a scheme, there is an isomorphism*

$$h(\mathrm{cosk}_0(X/\mathcal{X})) \xrightarrow{\sim} h(\mathcal{X})$$

of pro-simplicial sets in the homotopy category of pro-simplicial sets. Furthermore, these pro-simplicial sets are strictly weakly equivalent.

Proof. An immediate consequence of 2.3.57 from 3.5.8. □

Corollary 3.5.10. *Let $\mathcal{X}$ be an algebraic stack with a smooth surjection $U \to \mathcal{X}$ with U an algebraic space. Then the topological type $h(\mathcal{X})$ is strictly weakly equivalent to the topological type $h(\mathrm{cosk}_0(U/\mathcal{X}))$ of the simplicial algebraic space $\mathrm{cosk}_0(U/\mathcal{X})$. In particular, if X is an algebraic space over S and G/S is a smooth group scheme which acts on X over S, then the topological type $h([X/G])$ of the quotient stack $[X/G]$ is strictly weakly equivalent to the topological type $h(B(G,X,S))$ of the simplicial algebraic space $B(G,X,S) := \mathrm{cosk}_0(X/[X/G])$.*

3.5.11. One can use any smooth hypercover of algebraic stacks to compute topological types:

Theorem 3.5.12. *Let $U_\bullet \to \mathcal{X}$ be a smooth hypercover of an algebraic stack $\mathcal{X}/S$. Then the canonical map of topological types*

$$h(U_\bullet) \to h(\mathcal{X})$$

is a strict weak equivalence.

Proof. An immediate consequence of 2.3.51. □

3.5.13. Let us consider the algebraic stacks counterpart of 3.4.10. We only have the half of the result due to the way we defined the topological types of algebraic stacks.

We have defined the topological types of algebraic stacks in a way that not depending on any hypercovers. This can be linked to the hypercovers by 3.5.12. Since every algebraic stack admits a smooth surjection from a scheme, one could have defined the topological type of an algebraic stack $\mathcal{X}$ by choosing any smooth hypercover that is a simplicial algebraic spaces and define the topological type of the algebraic stack $\mathcal{X}$ to be the topological type of the simplicial algebraic space. Again, the intrinsic definition shows the the independence of choice of smooth hypercovers by simplicial algebraic spaces.

However, if we define topological types of algebraic stacks by choosing hypercovers, then it is hard to prove the independence of choice of hypercovers without using the intrinsic definition we have. This is due to the lack of the stacky counterpart for 2.3.51 and 2.3.52.

3.6. **Cohomology of topological types.**

3.6.1. For a pointed connected algebraic stack $\mathcal{X}$, Behrang Noohi [25, §4] associated the Galois category of locally constant sheaves to define the fundamental group π_1^{N} of the algebraic stack $\mathcal{X}$. We compare it to the fundamental group of the topological type $h(\mathcal{X})$:

Proposition 3.6.2. *For a pointed connected algebraic stack $\mathcal{X}$, the profinite completion $h(\mathcal{X})\widehat{\ }$ of the fundamental group of the topological type $h(\mathcal{X})$ is isomorphic to Noohi's fundamental group $\pi_1^{\mathrm{N}}(\mathcal{X})$.*

Proof. This follows from 2.4.8 as both classify finite torsors. □

3.6.3. Fix an abelian group Λ throughout this subsection.

3.6.4. For a scheme (resp. an algebraic space) X locally of finite type over S, we have from 3.2.3 combined with 2.4.14 that the following cohomology

$$H^*(X_{\text{ét}}, \underline{\Lambda}), H^*(\text{LFÉ}(X)^{\sim}, \underline{\Lambda}), H^*(\text{LFS}(X)^{\sim}, \underline{\Lambda})$$

are all isomorphic to the cohomology

$$H^*(h(X/S), \Lambda)$$

of the topological type $h(X/S)$.

3.6.5. Similarly, for a simplicial scheme (resp. a simplicial algebraic space) $X_\bullet$ that is locally of finite type over S, we have from 3.2.3 combined with 2.4.16 that the following cohomology

$$H^*(S_{\text{ét}}/X_\bullet, \underline{\Lambda}), H^*(\text{LFÉ}(S)^{\sim}/X_\bullet, \underline{\Lambda}), H^*(\text{LFS}(S)^{\sim}/X_\bullet, \underline{\Lambda})$$

are all isomorphic to the cohomology

$$H^*(h(X_\bullet/S), \Lambda)$$

of the topological type $h(X_\bullet/S)$.

3.6.6. A similar result holds for Deligne-Mumford stacks. However, we should be careful for algebraic stacks. The small étale topology for algebraic stacks are not the right topology to work with (see 3.5.4). Other than that, we still have that for an algebraic stack $\mathcal{X}$ locally of finite type over S, the following cohomology

$$H^*(\text{LFÉ}(\mathcal{X})^{\sim}, \underline{\Lambda}), H^*(\text{LFS}(\mathcal{X})^{\sim}, \underline{\Lambda})$$

are all isomorphic to the cohomology

$$H^*(h(\mathcal{X}/S), \Lambda)$$

of the topological type $h(\mathcal{X}/S)$.

3.6.7. For an algebraic stack $\mathcal{X}/S$, consider the full subcategories

$$\text{LFS}^{\text{sp}}(\mathcal{X}) \text{ (resp. } \text{LFS}^{\text{sch}}/(\mathcal{X})) \subset \text{LFS}(\mathcal{X})$$

consisting of paris (X, x) where X is an algebraic space (resp. a scheme). With the induced topologies, they all induce equivalent topoi. So we can use these topoi to compute cohomology.

4. Topological types with group actions

Throughout this section X is a scheme over a field k with the structure morphism $f : X \to \operatorname{Spec} k$ unless stated otherwise. Such a scheme X has an action of the Galois group $G := \text{Gal}(k^{\text{sep}}/k)$. We study the topological type $h(X)$ of X with respect to the Galois action.

4.1. **Relative topological types.**

4.1.1. In order to study the Galois action, we use the relative homotopy introduced by Barnea-Schlank [2, §8.1]. Note that the structure morphism $f : X \to \operatorname{Spec} k$ induces a morphism of topoi

$$f = (f^*, f_*) : X_{\text{ét}} \to (\operatorname{Spec} k)_{\text{ét}}$$

With respect to Barnea-Schlank's model category structures 2.2.17, the functor

$$L_{f^*} : \text{pro} - X_{\text{ét}}^{\Delta^{\text{op}}} \to \text{pro} - (\operatorname{Spec} k)_{\text{ét}}^{\Delta^{\text{op}}},$$

a left adjoint to the pull-back functor f^* on the pro-categories, is left Quillen.

Definition 4.1.2. The *relative topological type* $h_k(X)$ of a scheme X over k is the pro-simplicial set

$$h_k(X) := \mathbf{L}L_{f^*}(*)$$

where $*$ is a final object in $X_{\text{ét}}^{\Delta^{\text{op}}}$.

4.1.3. The small étale topos $(\operatorname{Spec} k)_{\text{ét}}$ is equivalent to the category $G - \mathbf{Set}$ of discrete G-sets. So the relative topological type $h_k(X)$ naturally encodes the Galois action as it is a pro-object in the category of simplicial discrete G-sets.

4.1.4. Recall from 2.3.19 that there is an adjoint triple

$$(\Pi_G, \Gamma^*, \Gamma_*) : \text{pro} - (G - \mathbf{SSet}) \to \text{pro} - \mathbf{SSet}$$

which is induced by the morphism of topoi

$$G - \mathbf{Set} \to \mathbf{Set}$$

whose pull-back Γ^* admits a left adjoint Π_G that sends a discrete G-set into its quotient.

4.1.5. The relationship between the usual topological type $h(X/k)$ of X over k and the relative topological type is simple:

Proposition 4.1.6. *The pro-simplicial set $h_k(X)/G$ which is the relative topological type of X over k taken quotient by G, and the topological type $h(X/k)$ of X are strictly weakly equivalent.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X_{\text{ét}} & \longrightarrow & (\operatorname{Spec} k)_{\text{ét}} \\ & \searrow & \downarrow \\ & & \mathbf{Set} \end{array}$$

of topoi, which induces a zig-zag strict weak equivalence between derived objects $\mathbf{L}\Pi(*)$ and $(\mathbf{L}\Pi_G \circ \mathbf{L}L_{f^*})(*) = \mathrm{L}\Pi_G(h_k(X)) = h_k(X)/G$. As $\mathbf{L}\Pi(*)$ can be identified with $h(X/k)$ by 2.3.30, the statement follows. □

4.1.7. Given a variety X over $\mathbb{R}$, David Cox showed [5, 1.1] that the étale homotopy type of X is the homotopy orbit space of the étale homotopy type of $\bar{X} := X \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$ with respect to the Galois action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. This result was generalized by Gereon Quick [28, 5.3] to an arbitrary base field. We will see our proposition above 4.1.6 is a generalization of Quick's result (see 5.1.26).

4.1.8. Denote by X^{sep} the base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} k^{\mathrm{sep}}$ of X to k^{sep}. The following lemma is used to study the relationship between the relative topological type $h_k(X)$ and the topological type of $h(X^{\mathrm{sep}}/k^{\mathrm{sep}})$.

Lemma 4.1.9. *For the commutative diagram of schemes*

$$\begin{array}{ccc} X^{\mathrm{sep}} & \xrightarrow{p_X} & X \\ {\scriptstyle f^{\mathrm{sep}}}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} k^{\mathrm{sep}} & \xrightarrow[p]{} & \operatorname{Spec} k \end{array}$$

there is a canonical isomorphism of functors

$$f^* \circ p_* \xrightarrow{\sim} (p_X)_* \circ (f^{\mathrm{sep}})^*$$

on the small étale sheaves.

Proof. This is a consequence of the analysis of sheaves on a projective limit of schemes, and cohomology and base change of proper morphisms. Indeed, the scheme $\operatorname{Spec} k^{\mathrm{sep}}$ is the limit of the projective system $\{\operatorname{Spec} L : k \subset L \subset k^{\mathrm{sep}}$ is a finite separable extension $\}$. Then one can reduce the statement to the case for the finite morphism $\operatorname{Spec} L \to \operatorname{Spec} k$ where the result is well-known. □

Proposition 4.1.10. *There is a strict weak equivalence*

$$h(X^{\mathrm{sep}}/k^{\mathrm{sep}}) \to p^*(h_k(X))$$

of pro-simplicial sets.

Proof. Consider a cofibrant replacement $H \to *$ of $*$ in $\mathrm{pro}-X_{\text{ét}}^{\Delta^{\mathrm{op}}}$. It pulls back to a trivial fibration $(p_X)^*(H) \to *$. To begin with, we claim that $(p_X)^*(H)$ is cofibrant. So consider a lifting problem

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow \\ (p^{\mathrm{sep}})^*(H) & \longrightarrow & B \end{array}$$

in $\mathrm{pro}-(X^{\mathrm{sep}})_{\text{ét}}^{\Delta^{\mathrm{op}}}$. Since trivial local fibrations of simplicial sheaves on X^{sep} form generating trivial fibrations for $\mathrm{pro}-(X^{\mathrm{sep}})_{\text{ét}}^{\Delta^{\mathrm{op}}}$ (see [2, 4.1]), we may assume $A \to B$ is both a local weak equivalence and a local fibration of simplicial sheaves. Then by adjunction, it suffices to show that $(p_X)_*$ preserves a morphism that is both a local weak equivalence and a local fibration. Recall from 2.1.10 that those morphisms are described in terms of finite limits and epimorphisms. Since $(p_X)_*$ is a right adjoint, it only remains to prove that it preserves epimorphisms. This follows from [22, Tag 04C2] because $p : \operatorname{Spec} k^{\mathrm{sep}} \to \operatorname{Spec} k$ is integral.

Now consider a cofibrant replacement $H^{\mathrm{sep}} \to *$ of $*$ in $\mathrm{pro}-(X^{\mathrm{sep}})_{\text{ét}}^{\Delta^{\mathrm{op}}}$. Then we can choose a lift $d : H^{\mathrm{sep}} \to (p_X)^*(H)$. Since $L_{(f^{\mathrm{sep}})^*}$ preserves a weak equivalence between cofibrant objects, we obtain a strict weak equivalence

$$L_{(f^{\mathrm{sep}})^*}(H^{\mathrm{sep}}) \to L_{(f^{\mathrm{sep}})^*}((p_X)^*(H))$$

Note that upon the identification of étale topos of k^{sep} with the category of sets, by 2.3.30, we can identify the object $L_{(f^{\text{sep}})^*}(H^{\text{sep}})$ on the left with the topological type $h(X^{\text{sep}}/k^{\text{sep}})$ of X^{sep}. Furthermore, by 4.1.9, there is an isomorphism

$$(L_{(f^{\text{sep}})^*} \circ (p_X)^*)(H) \to (p^* \circ L_{f^*})(H) = p^* h_k(X)$$

Therefore, we obtain the strict weak equivalence in the statement. □

Remark 4.1.11.

(i) The pull-back $p^* : G - \mathbf{Set} \to \mathbf{Set}$ sends a discrete G-set into its underlying set. So the proposition above says that the topological type $h(X^{\text{sep}}/k^{\text{sep}})$ of X^{sep} is strictly weakly equivalent to the underlying pro-simplicial set of the relative topological type $h_k(X)$ of X over k. This result seems already known to Barnea-Schlank.

(ii) Although the Galois group G acts on $h(X^{\text{sep}})$, it is not clear whether the action is continuous or not. However, the Galois action on the relative type $h_k(X)$ is continuous by definition. Therefore, the relative typological type $h_k(X)$ can be thought of the replacement of the topological type $h(X^{\text{sep}})$.

Remark 4.1.12. One can generalize 4.1.9 and 4.1.10 as following. Let $p : S' \to S$ be an integral morphism of schemes. For a S-scheme X, consider a fibered diagram

$$\begin{array}{ccc} X' & \xrightarrow{p'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{p} & S \end{array}$$

Then the canonical morphism of functors

$$f^* \circ p_* \to (p')_* \circ (f')^*$$

is still an isomorphism ([9, 5.9.7]). The argument in 4.1.10 works verbatim to establish a strict weak equivalence

$$h_{S'}(X') \to p^*(h_S(X))$$

4.1.13. As an immediate consequence of the remark above, one obtains the invariance of topological types for a separably closed field with respect to its algebraic closure:

Proposition 4.1.14. *Let X be a scheme over a separably closed field k. For an algebraic closure $\bar{k}$ of k and $\bar{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$, there is a strict weak equivalence*

$$h(\bar{X}) \to h(X)$$

of the topological types.

Proof. Note that the small étale topoi for $\operatorname{Spec} k$ and $\operatorname{Spec} \bar{k}$ are both identified with the category of sets. So in this case, the relative topological types $h_{\bar{k}}(\bar{X})$ and $h_k(X)$ are the usual topological types $h(\bar{X})$ and $h(X)$ respectively. So the result follows from 4.1.12. □

4.1.15. For an algebraic space X/S, there is an equivalence of categories between the small étale topos on X and the category of data $(\{F_U\}, \rho_f)$ where F_U is a small étale sheaf on a scheme U for each étale morphism $U \to X$ and for each morphism of schemes $f : V \to U$ over X, ρ_f is an isomorphism $f^{-1}F_U \to F_V$. This date is subject to the condition $g^{-1}\rho_f \circ \rho_g = \rho_{g \circ f}$ for any morphism $g : W \to V$ and $f : V \to U$ of schemes over X. Consequently, the statements 4.1.9, 4.1.10, and 4.1.12 for algebraic spaces can be reduced to the case of schemes, and hence are still valid. We obtain the following generalization of 4.1.14:

Corollary 4.1.16. *Let X be an algebraic space over a separably closed field k. For an algebraic closure $\bar{k}$ of k and $\bar{X} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$, there is a strict weak equivalence*

$$h(\bar{X}) \to h(X)$$

of the topological types.

4.2. **A scheme with an abstract group action.** In this subsection we prove a similar result to 4.1.6 when X admits an abstract group action.

4.2.1. Let X be a scheme over a base scheme S. If an abstract group G acts on the scheme, then there is an induced quotient stack $[X/G]$. We then study the G-action on the topological type $h([X/G])$ of the quotient stack in the relative setting. There is a morphism of topoi

$$f : [X/G]_{\text{ét}} \to G - \mathbf{Set}$$

where the pull-back maps a G-set S into the quotient stack $[X \times S/G]$ over $[X/G]$, and the push-forward sends an étale sheaf F on $[X/G]$ into the G-set $F(X)$.

4.2.2. Just like the scheme case 4.1.6, we take the group action into account:

Definition 4.2.3. *The topological type with G-action $h^G([X/G])$* is the pro-object $\mathbf{L}L_{f^*}(*)$ in the category of simplicial G-sets where $*$ is a final object of the small étale topos $[X/G]_{\text{ét}}$.

Remark 4.2.4. The small étale topos computes the usual topological type of the quotient stack $[X/G]$ as it is Deligne-Mumford.

Proposition 4.2.5. *Two pro-simplicial sets $h^G([X/G])/G$ and $h([X/G])$ are strictly weakly equivalent.*

Proof. The same argument as in 4.1.6 applied to the following commutative diagram of topoi works:

$$\begin{array}{ccc} [X/G]_{\text{ét}} & \longrightarrow & G - \mathbf{Set} \\ & \searrow & \downarrow \\ & & \mathbf{Set} \end{array}$$

□

5. Completion of topological types

In this section we study profinite completion of topological types. On the first half, we show that 4.1.6 recovers Quick's result [28, 5.3] after profinitely completion. For the second half, we generalize Artin-Mazur's comparison theorem [1, 12.9] to simplicial schemes and to algebraic stacks.

5.1. **Completions.** We follow Quick for the profinite completion of (pro-)simplicial sets [27] and for the equivariant completion of simplicial sets with group actions [29].

5.1.1. Let $\widehat{\mathcal{E}}$ be the category of compact, Hausdorff, and totally disconnected topological spaces. The category is equivalent to the pro-category of finite sets. The forgetful functor

$$\widehat{\mathcal{E}} \to \mathbf{Set}$$

admits a left adjoint which is denote by $\widehat{(\cdot)}$ and called the *profinite completion* of sets.

5.1.2. The category of simplicial objects in $\widehat{\mathcal{E}}$ is denoted by $\widehat{\mathbf{SSet}}$ and we call its objects *profinite spaces.* The forgetful functor

$$\widehat{\mathbf{SSet}} \to \mathbf{SSet}$$

admits a left adjoint

$$\widehat{(\cdot)} : \mathbf{SSet} \to \widehat{\mathbf{SSet}},$$

which is called *profinite completion* of simplicial sets.

Remark 5.1.3. The completion of (pro)-simplicial sets was first considered by Artin-Mazur in their work of étale homotopy types. The comparison with their work is given in 5.2.13.

Definition 5.1.4. ([27, 2.6]) A morphism $f : X \to Y$ of profinite spaces is a *weak equivalence* if the following holds:

(i) $\pi_0(X) \to \pi_0(Y)$ is an isomorphisms of profinite sets,
(ii) $\pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism of profinite groups for each $x \in X_0$, and
(iii) For each $n \geq 0$, $H^n(Y; \mathcal{M}) \to H^n(X; f^*\mathcal{M})$ is an isomorphism for every local coefficient system $\mathcal{M}$ of finite abelian groups on Y (see [27, §2.2] for more details).

5.1.5. (*Quick's model category structure on the category of profinite spaces* [27, 2.12]) There is a model category structure on the category $\widehat{\mathbf{SSet}}$ of profinite spaces; if $f : X \to Y$ is a morphism of profinite spaces,

(i) f is a *weak equivalence* of profinite spaces,
(ii) f is a *cofibration* if it is a monomorphism, and
(iii) f is a *fibration* if it has the right lifting property with respect to all trivial cofibrations.

Lemma 5.1.6. ([27, 2.28]) *The adjunction*

$$(\widehat{(\cdot)}, |-|) : \boldsymbol{SSet} \to \widehat{\boldsymbol{SSet}}$$

is a Quillen adjunction.

Remark 5.1.7. A weak equivalence of profinite spaces is completely characterized by homotopy groups in the following sense:

Lemma 5.1.8. *Let $f : X \to Y$ be a morphism of profinite spaces. Then it is a weak equivalence if and only if it induces an isomorphism*

$$\pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x))$$

of profinite homotopy groups (profinite sets for $n = 0$) for every $n \geq 0$ and every $x \in X_0$.

Proof. Considering the functorial fibrant replacements of X and Y, we may assume that X and Y are fibrant by the 2-out-of-3 property of weak equivalences. Then f is a weak equivalence if and only if the map $|f| : |X| \to |Y|$ of underlying simplicial sets is a weak equivalence if and only if $\pi_n(|X|) \to \pi_n(|Y|)$ is an isomorphism for $n \geq 0$. Recall from [30, 2.9] that the profinite homotopy groups of a pointed fibrant profinite space is isomorphic to the usual homotopy group of its underlying simplicial set. Therefore, all the equivalent conditions are also equivalent to that $\pi_n(X) \to \pi_n(Y)$ is an isomorphism for $n \geq 0$. □

5.1.9. We study profinite completion of topological types which are pro-simplicial sets:

Definition 5.1.10. ([27, §2.7]) Let $X : I \to \mathbf{SSet} : i \mapsto X_i$ be a pro-simplicial set. The *profinite completion* $\widehat{X}$ of X is the profinite space

$$\widehat{X} = \varprojlim_{i \in I} \widehat{X_i}$$

That is, take a level-wise completion of simplicial sets, and then pass to the limit in $\widehat{\mathbf{SSet}}$.

5.1.11. As we study profinitely completed relative topological types (4.1.2) with respect to the Galois action, it is necessarily to build a model category on the category of profinite spaces with group action. We follow Quick [28, 2.17].

5.1.12. Fix a profinite group G. A profinite G-space is a profinite space equipped with a level-wise compatible continuous G-action. That is, a profinite set X with continuous G-action on each X_n such that the map $X_n \to X_m$ is G-equivariant for each $d : [m] \to [n]$. Denote by $G - \widehat{\mathbf{SSet}}$ the category of profinite G-spaces with G-equivariant morphisms.

One can consider profinite completion with respect to G-action. Indeed, the forgetful functor

$$| - | : G - \widehat{\mathbf{SSet}} \to |G| - \mathbf{SSet}$$

that maps a profinite G-space into its underlying simplicial $|G|$-set admits a left adjoint

$$\widehat{(\cdot)}_G : |G| - \mathbf{SSet} \to G - \widehat{\mathbf{SSet}}$$

This functor is called *G-equivariant profinite completion* ([29, §4.1]). Here $|G|$ is the underlying group of the profinite group G.

Lemma 5.1.13. *The adjunction*

$$(\widehat{(\cdot)}_G, | - |) : |G| - \boldsymbol{SSet} \to G - \widehat{\boldsymbol{SSet}}$$

is a Quillen adjunction.

Proof. The forgetful functor preserves fibrations and trivial fibrations. Indeed, a morphism $f : X \to Y$ of profinite G-spaces is a fibration (resp. a trivial fibration) if and only if its underlying morphism of profinite spaces is a fibration (resp. a trivial fibration). Then from 5.1.6 the underlying morphism of simplicial sets is a fibration (resp. a trivial fibration), which is equivalent to that the underlying morphism of simplicial $|G|$-sets of f is a fibration (resp. a trivial fibration). □

Remark 5.1.14. $|G| - \mathbf{SSet}$ is the category of simplicial $|G|$-sets whereas $G - \mathbf{SSet}$ is the category of simplicial discrete G-sets.

Definition 5.1.15. Let $X : I \to G - \mathbf{SSet} : i \mapsto X_i$ be a pro-object in the category of simplicial discrete G-sets. The *G-equivariant profinite completion* $\widehat{X}_G$ of X is the profinite space

$$\widehat{X}_G := \varprojlim_{i \in I} \widehat{X_{iG}}$$

That is, apply the forgetful functor $G - \mathbf{SSet} \to |G| - \mathbf{SSet}$, take a level-wise G-equivariant completion of simplicial $|G|$-sets, and then pass to the limit in $G - \widehat{\mathbf{SSet}}$.

5.1.16. (*Quick's model category structure on the category of profinite G-spaces* [28, 2.17]) There is a model category structure on the category $G - \widehat{\mathbf{SSet}}$ of profinite spaces with continuous G-action; if $f : X \to Y$ is a morphism of profinite G-spaces,

(i) f is a *weak equivalence* if its underlying morphism of profinite spaces is a weak equivalence,
(ii) f is a *fibration* if its underlying morphism of profinite spaces is a fibration, and
(iii) f is a *cofibration* if it has the right lifting property with respect to all trivial fibrations.

5.1.17. As pointed out earlier in 4.1.7, Quick generalized Cox's result. Concretely, for a geometrically connected variety over a field k and the Galois group G, he proved [28, 3.5] that the canonical map

$$c\widehat{\acute{\mathrm{E}}\mathrm{t}}X^{\mathrm{sep}} \times_G EG \to \widehat{\acute{\mathrm{E}}\mathrm{t}}X$$

is a weak equivalence of profinite spaces where $\widehat{\acute{\mathrm{E}}\mathrm{t}}X$ is the profinite completion of the étale topological type of X defined in the sense of Friedlander, and the left-most object $c\hat{\acute{\mathrm{E}}}\mathrm{t}X^{\mathrm{sep}}$ is defined to be

$$\varprojlim_{L} \widehat{\acute{\mathrm{E}}\mathrm{t}}X_L$$

where L runs over all finite Galois extension of k in k^{sep}. Remark from [28, 3.3] that the canonical map

$$\widehat{\acute{\mathrm{E}}\mathrm{t}}X^{\mathrm{sep}} \to \varprojlim_{L} \widehat{\acute{\mathrm{E}}\mathrm{t}}X_L$$

is a weak equivalence of profinite spaces. The reason why he had to replace $\widehat{\acute{\mathrm{E}}\mathrm{t}}X^{\mathrm{sep}}$ by $c\widehat{\acute{\mathrm{E}}\mathrm{t}}X^{\mathrm{sep}}$ is that in general one does not know whether the canonical G-action on $\widehat{\acute{\mathrm{E}}\mathrm{t}}X^{\mathrm{sep}}$ is continuous or not. Namely, we do not know in general whether $\hat{\acute{\mathrm{E}}}\mathrm{t}X^{\mathrm{sep}}$ is a profinite G-space or not. Whereas the G-action on $\varprojlim_L \widehat{\acute{\mathrm{E}}\mathrm{t}}X_L$ is continuous because the action on $\hat{\acute{\mathrm{E}}}\mathrm{t}X_L$ factors through the action by $\mathrm{Gal}(L/k)$.

Remark 5.1.18. We will prove that our theorem 4.1.6 is, at the level of pro-simplicial sets, a generalization of Quick's result. In particular, we recover his result after profinite completion. Also, observe that the relative topological type carries continuous Galois action and so one does not need to replace it by the limit over finite Galois extensions.

5.1.19. Throughout the rest of the subsection we fix a scheme X a field k. The Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is denoted by G.

5.1.20. Recall from 4.1.4 that there is the adjoint triple

$$(\Pi, \Gamma^*, \Gamma_*) : \mathrm{pro} - (G - \mathbf{SSet}) \to \mathrm{pro} - \mathbf{SSet}$$

The profinite version is the adjoint triple

$$(\Pi, \Gamma^*, \Gamma_*) : G - \widehat{\mathbf{SSet}} \to \widehat{\mathbf{SSet}}$$

Just like the case when G is a discrete group 2.3.18, for a profinite G-space X, there is a weak equivalence of profinite spaces

$$X \times_G EG := (X \times EG)/G = \Pi(X \times EG) \to \mathbf{L}\Pi(X)$$

because $EG \to *$ is a trivial fibration ([28, 2.17]) and $X \times EG$ is cofibrant ([28, 2.18]) in $G - \widehat{\mathbf{SSet}}$. Therefore,

$$\mathbf{L}\Pi : G - \widehat{\mathbf{SSet}} \to \widehat{\mathbf{SSet}}$$

can be understood as the profinite version of the Borel construction.

5.1.21. To prove the compatibility with Quick's result, we state and prove two lemmas:

Lemma 5.1.22. *There is a commutative diagram*

$$\begin{array}{ccc} G - \boldsymbol{SSet} & \xrightarrow{\Pi} & \boldsymbol{SSet} \\ \downarrow & & \\ |G| - \boldsymbol{SSet} & & \downarrow \\ \downarrow & & \\ G - \widehat{\boldsymbol{SSet}} & \xrightarrow{\Pi} & \widehat{\boldsymbol{SSet}} \end{array}$$

of categories where the left vertical arrow is a forgetful functor followed by the G-equivariant completion, the right vertical arrow is the completion, and two horizontal arrows are quotients by G.

Proof. Let X be a simplicial discrete G-set. Along the bottom-left corner (resp. top-right corner) of the diagram, one gets $\widehat{X}_G/G$ (resp. $\widehat{X/G}$). For a profinite space Y,

$$\begin{aligned} \mathrm{Mor}_{\widehat{\mathbf{SSet}}}(\widehat{X}_G/G, Y) &= \mathrm{Mor}_{G-\widehat{\mathbf{SSet}}}(\widehat{X}_G, Y) \\ &= \mathrm{Mor}_{|G|-\mathbf{SSet}}(X, |Y|) \\ &= \mathrm{Mor}_{\mathbf{SSet}}(X/G, |Y|) \\ &= \mathrm{Mor}_{\widehat{\mathbf{SSet}}}(\widehat{X/G}, Y) \end{aligned}$$

Whenever necessary, the underlying simplicial set $|Y|$ of Y (resp. Y itself) is endowed with the trivial $|G|$-action (resp. the trivial G-action). The statement follows from the Yoneda lemma. ☐

Lemma 5.1.23. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{pro} - (G - \widehat{\boldsymbol{SSet}}) & \longrightarrow & \mathrm{pro} - \widehat{\boldsymbol{SSet}} \\ \downarrow & & \downarrow \\ G - \widehat{\boldsymbol{SSet}} & \longrightarrow & \widehat{\boldsymbol{SSet}} \end{array}$$

of categories where the vertical arrows are limit functors and the horizontal arrows are quotient by G.

Proof. Let $X = (X_i)$ be a pro-object in $G - \widehat{\mathbf{SSet}}$. The assertion is an isomorphism

$$\varprojlim(X_i/G) = (\varprojlim X_i)/G$$

where the limit on the left is of profinite spaces and on the right is of profinite G-spaces. If we endow X_i/G the trivial G-action, then the limit of profinite G-spaces X_i/G in $G - \widehat{\mathbf{SSet}}$ is isomorphic to $\varprojlim(X_i/G)$ which is the limit of profinite spaces in $\widehat{\mathbf{SSet}}$. Therefore, it suffices to prove the assertion in the category of profinite G-spaces. This follows from that a cofiltered limit commutes with finite colimits in the category of profinite G-spaces, which can be checked level-wise. Hence it is enough to prove that a cofiltered limit commutes with finite colimits in the category of profinite sets with continuous G-action. However, the category is isomorphic to the pro-category of finite sets with G-action where the result follows from [13, 6.1]. □

Remark 5.1.24. Isaksen [13, 6.1] showed that a cofiltered limit commutes with finite colimits for a pro-category associated to a category $\mathcal{C}$. The statement is true under the assumption that the category $\mathcal{C}$ is complete and cocomplete. Actually, his proof shows that the theorem is still true for the category $\mathcal{C}$ that has finite limits and finite colimits, which we used in the previous lemma.

Proposition 5.1.25. *There is a commutative diagram*

$$\begin{array}{ccc}
\mathrm{pro} - (G - \boldsymbol{SSet}) & \longrightarrow & \mathrm{pro} - \boldsymbol{SSet} \\
\downarrow & & \downarrow \\
G - \widehat{\boldsymbol{SSet}} & \longrightarrow & \widehat{\boldsymbol{SSet}}
\end{array}$$

of categories where the left vertical arrow is the G-equivariant completion, the right vertical arrow is the completion, and the horizontal arrows are quotient by G.

Proof. The diagram in the assertion is a composition of two diagrams:

$$\begin{array}{ccc}
\mathrm{pro} - (G - \mathbf{SSet}) & \longrightarrow & \mathrm{pro} - \mathbf{SSet} \\
\downarrow & & \downarrow \\
\mathrm{pro} - (G - \widehat{\mathbf{SSet}}) & \longrightarrow & \mathrm{pro} - \widehat{\mathbf{SSet}} \\
\downarrow & & \downarrow \\
G - \widehat{\mathbf{SSet}} & \longrightarrow & \widehat{\mathbf{SSet}}
\end{array}$$

The top diagram commutes by 5.1.22 and the bottom one by 5.1.23. □

Theorem 5.1.26. *Let X be a scheme over a field k. Then the completion $\widehat{h(X)}$ of the topological type $h(X)$ of X is weakly equivalent to the Borel construction*

$$\widehat{h_k(X)}_G \times_G EG$$

of the G-equivariant completion (5.1.12) of the relative topological type $h_k(X)$ with respect to the Galois group $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

Proof. We apply left derived functor to the top-right and the left-bottom arrows of the diagram in 5.1.25. Furthermore, we decompose each one as a composition of two left derived functors. For this, we prove that the left derived functors are well-defined for each side of the diagram. The well-definedness of the top and bottom arrows follows from 5.1.20. For the right arrow, note that the profinite completion of simplicial sets 5.1.6 and the cofiltered limit functor of profinite spaces preserve weak equivalences [27, 2.14]. Therefore, the profinite completion of pro-simplicial sets preserves weak equivalences and so the left derived functor is well-defined. Lastly, for the left arrow, we prove that it sends a trivial cofibration between cofibrant objects in $\mathrm{pro}-(G-\mathbf{SSet})$ into a weak equivalence in $(G-\widehat{\mathbf{SSet}})$. Recall that the left arrow is the compositions of three arrows

$$\mathrm{pro}-(G-\mathbf{SSet}) \to \mathrm{pro}-(|G|-\mathbf{SSet}) \to \mathrm{pro}-(G-\widehat{\mathbf{SSet}}) \to (G-\widehat{\mathbf{SSet}})$$

A trivial cofibration between cofibrant objects in $\mathrm{pro}-(G-\mathbf{SSet})$ maps into a weak equivalence between cofibrant objects in $\mathrm{pro}-(|G|-\mathbf{SSet})$ because a weak equivalence of simplicial discrete G-sets induces a weak equivalence of simplicial $|G|$-sets and every object in $\mathrm{pro}-(|G|-\mathbf{SSet})$ is cofibrant. Then from the Quillen adjunction 5.1.13

$$\widehat{(\cdot)}_G : |G|-\mathbf{SSet} \to G-\widehat{\mathbf{SSet}}$$

it maps into a weak equivalence. Therefore, it suffices to show that a cofiltered limit functor of profinite G-spaces preserves a weak equivalence. However, this follows from [27, 2.14] because the underlying profinite space of the limit of profinite G-spaces is the limit of underlying profinite spaces.

So far, we have shown that each side of the square diagram defines the left derived functor. Consider the relative topological type $h_k(X)$. Along the derived top arrow, one gets, up to strict weak equivalences, $h_k(X)/G$ because $h_k(X)$ is cofibrant. From 4.1.6, $h_k(X)/G$ and $h(X/k)$ are strictly weakly equivalent. Then along the derived right vertical arrow, we get $\widehat{h(X)}$, up to weakly equivalent objects. On the other hand, the derived left vertical arrow gives the G-equivariant completion $\widehat{h_k(X)}_G$ of $h_k(X)$ 5.1.15, up to weak equivalence. Then the derived bottom arrow sends $\widehat{h_k(X)}_G$ into its Borel construction $\widehat{h_k(X)}_G \times_G EG$ (5.1.20), up to weakly equivalent objects. □

Remark 5.1.27.

(i) This theorem is in the same spirit as Quick's result [28, 3.1]. He applied the Borel construction to $h(X^{\mathrm{sep}})$, up to the continuity issue. This is similar to what we did because 4.1.9 says that the underlying pro-simplicial set of the relative topological type $h_k(X)$ is the topological type $h(X^{\mathrm{sep}})$.

(ii) In the statement of the theorem, the continuity issue that Quick had disappeared due to the use of the relative topological type which is an object in $\mathrm{pro}-(G-\mathbf{SSet})$ where the continuity issue is already taken care of.

(iii) In another paper by Quick [31], he avoided the continuity issue in a different way. For a geometrically connected variety X over a field k, assume further that it is quasi-projective. He used Friedlander's rigid Čech étale topological type $(X/k)_{\mathrm{rét}}$ (see [7, 3.1] for more details) to consider $\acute{\mathrm{E}}\mathrm{t}(X)$ as a pro-simplicial set over BG. Then the profinite model of $(X/k)_{\mathrm{rét}}$ over $(\operatorname{Spec} k/k)_{\mathrm{rét}} = BG$ has the homotopy type of the G-homotopy orbits of $(X^{\mathrm{sep}}/k)_{\mathrm{rét}}$ (see [31, p.13] for more details). In particular, the continuity issue is resolved. Also, his approach is at the level of pro-simplicial sets, and hence it sits between 5.1.26 and 4.1.6. Note also that 5.1.26 works for every scheme over k.

5.2. **Comparison theorems.** In this subsection we prove that for a complex variety, one can compute its topological type via the underlying topological space of its analytification, after profinite completion. We then extend this classical result of Artin-Mazur into the case of simplicial schemes and algebraic stacks.

Definition 5.2.1. (cf. 3.2.1) The *big étale site* **An** is the category of complex analytic spaces. A collection of morphisms $\{Y_i \to Y\}$ is a covering of Y if each morphism $Y_i \to Y$ is étale and the map

$$\coprod_{i \in I} Y_i \to Y$$

is surjective.

For an analytic space X, the *small étale site* $\mathbf{An}(X)$ is the category of analytic spaces étale over X. A collection of morphisms $\{Y_i \to Y\}$ is a covering of Y if the map

$$\coprod_{i \in I} Y_i \to Y$$

is surjective. Denote by $X_{\text{ét}}$ the associated topos.

5.2.2. Let X be a locally of finite type scheme over $\mathbb{C}$. There is an associated complex analytic space X^{an}. This construction is functorial and in fact the functor

$$\text{LFÉ}/\mathbb{C} \to \mathbf{An} : X \mapsto X^{\text{an}}$$

is continuous and commutes with finite limits. Therefore it induces a morphism of topoi

$$\mathbf{An}^{\sim} \to (\text{LFÉ}/\mathbb{C})^{\sim}$$

By 2.3.32, this morphism in turn induces a map of topological types

$$h(X^{\text{an}}) \to h(X)$$

Definition 5.2.3. Let X be a complex analytic space. The site $\text{Ét}(|X|)$ is defined as following. An object is a local homeomorphism from a topological space Y to the underlying topological space $|X|$ of X, and morphisms are continuous maps over $|X|$. A collection of maps $\{Y_i \to Y\}$ is a covering of Y if the map

$$\coprod_{i \in I} Y_i \to Y$$

is surjective.

Remark 5.2.4. The small étale site $\text{Ét}(X)$ is isomorphic to the site $\text{Ét}(|X|)$. Moreover, the topos associated to the site $\text{Ét}(|X|)$ is equivalent to the topos associated to the usual topology on $|X|$. So, one concludes that the small étale topos $X_{\text{ét}}$ is equivalent to the usual topos associated to the topological space $|X|$.

5.2.5. For a locally of finite type scheme X over $\mathbb{C}$, we abusively denote by $X(\mathbb{C})$ both for the underlying topological space of its associated analytic space X^{an} and the usual topos associated to it.

5.2.6. As an immediate consequence of the previous remark, for a locally of finite type scheme X over $\mathbb{C}$, there is an equivalence of topoi

$$(X^{\text{an}})_{\text{ét}} \simeq X(\mathbb{C})$$

between the small étale topos (5.2.1) of the analytic space X^{an} and the usual topos associated to the underlying topological space $X(\mathbb{C})$ of X^{an}.

5.2.7. Like the case of schemes, replacing the small site by the big site for analytic spaces is a key toward topological types of simplicial analytic spaces. By the same argument as in the étale topology of schemes, the following lemma follows immediately:

Lemma 5.2.8. (cf. 3.2.3) *Let X be an analytic space X. Then the inclusion functor*

$$j : \boldsymbol{An}(X) \to \boldsymbol{An}/X$$

from the small étale site of X to the big étale site localized by X is cocontinuous, continuous and commutes with finite limits.

5.2.9. Let X be a locally of finite type scheme over $\mathbb{C}$. Denote by $X(\mathbb{C})$ the topological type $h(X^{\text{an}})$ of X^{an} as an object in the topos $\mathbf{An}^{\sim}$.

Remark 5.2.10. For a locally paracompact topological space, the topological type of the usual topos of the topological space is just the space itself ([1, 12.1]). So we use the same notation for a topological space, its associated topos, and its topological type. From this point of view, the notation $X(\mathbb{C})$ means any of the underlying topological space of X^{an}, its associated topos, and its topological type. Recall from 5.2.6 that the last one is isomorphic to the topological type of the topos $(X^{\text{an}})_{\text{ét}}$. However, the topological type of the small étale topos $(X^{\text{an}})_{\text{ét}}$ is the topological type $h(X^{\text{an}})$ of X^{an}, up to strict weak equivalence by 5.2.8

5.2.11. Let X be a locally of finite type scheme over $\mathbb{C}$. Denote by $h(X)\widehat{}$ (resp. $\widehat{h(X)}$) the profinite completion (5.1.10) of the pro-simplicial set $h(X)$ which is the topological type of X (resp. of the topological space $X(\mathbb{C})$).

5.2.12. Recall from [1, 11.1] that $\mathbf{Ho}(\mathbf{SSet})_{\text{fin}}$ is the full subcategory of $\mathbf{Ho}(\mathbf{SSet})$ consisting of simplicial sets whose homotopy groups are all finite. The Artin-Mazur completion theorem [1, 3.4] says that the inclusion functor

$$\text{pro} - \mathbf{Ho}(\mathbf{SSet})_{\text{fin}} \to \text{pro} - \mathbf{Ho}(\mathbf{SSet})$$

admits a left adjoint. Denote by $\hat{X}^{\text{AM}}$ the Artin-Mazur completion of a pro-simplicial set X. On the other hand, for a profinite space X, one can associate a pro-object in $\mathbf{Ho}(\mathbf{SSet})_{\text{fin}}$ (see [27, p.604] for details), which is denoted by X^{AM}.

So for a pro-simplicial set X, one can consider two different pro-objects in $\mathbf{Ho}(\mathbf{SSet})_{\text{fin}}$. One is the Artin-Mazur completion $\widehat{X}^{\text{AM}}$ and the other is $(\widehat{X})^{\text{AM}}$, resulting from the profinite completion $\widehat{X}$ of X. The following proposition by Quick shows that these two pro-objets have the isomorphic homotopy groups after passing to the limit. Here we give a more detailed proof due to the missing technical details.

Proposition 5.2.13. ([27, 2.33]) *Let X be a pointed connected pro-simplicial set. Then after passing to the limits, the pro-homotopy groups of $(\hat{X})^{\text{AM}}$ and $\hat{X}^{\text{AM}}$ are isomorphic. That is, there is an isomorphism*

$$\pi_n(\hat{X}) \xrightarrow{\sim} \pi_n(\hat{X}^{\text{AM}})$$

for $n \geq 1$ where the right one taken the limit.

Proof. The pro-homotopy group of $(\hat{X})^{\mathrm{AM}}$, after passing to the limit, is isomorphic to the profinite homotopy group of the profinite completion $\hat{X} = \varprojlim_{i\in I} \hat{X}_i$ where $X = (X_i)$ (see [29, p.436]). Then

$$\pi_n(\widehat{X}) \simeq \pi_n(\varprojlim_{i\in I} \widehat{X_i}) \simeq \varprojlim_{i\in I} \pi_n(\widehat{X_i})$$

The second isomorphism follows from the lemma below 5.2.14 that the profinite homotopy group functor commutes with a cofiltered limit of profinite spaces. By the repeated application of [29, p.436], $\pi_n(\widehat{X_i})$ is isomorphic to $\pi_n((\widehat{X_i})^{\mathrm{AM}})$ taken the limit.

On the other hand, for each i, $(\widehat{X_i})^{\mathrm{AM}}$ is isomorphic to $\widehat{X_i}^{\mathrm{AM}}$ by the discussion in [27, p.604]. So by passing to the limits,

$$\pi_n((\widehat{X_i})^{\mathrm{AM}}) \simeq \pi_n(\widehat{X_i}^{\mathrm{AM}})$$

Now recall from [1, 3.9] that the Artin-Mazur completion of X is isomorphic to the cofiltered limit in $\mathrm{pro}-\mathbf{Ho}(\mathbf{SSet})_{\mathrm{fin}}$ of the Artin-Mazur completion of each X_i. So after passing to the limit,

$$\pi_n(\widehat{X}^{\mathrm{AM}}) \simeq \varprojlim_{i\in I} \pi_n(\widehat{X_i}^{\mathrm{AM}})$$

where $\pi_n(\widehat{X_i}^{\mathrm{AM}})$ is taken by the limit. □

Lemma 5.2.14. *Let $X : I \to \widehat{\mathbf{SSet}}$ be a cofiltered diagram of profinite spaces. For each $n \geq 0$, the map between profinite groups (profinite sets for $n = 0$)*

$$\pi_n(\varprojlim_{i\in I} X_i) \to \varprojlim_{i\in I} \pi_n(X_i)$$

is an isomorphism.

Proof. Apply the fibrant replacement functor F in $\widehat{\mathbf{SSet}}$ to X. We then have an induced map of limits

$$\varprojlim_{i\in I} X_i \to \varprojlim_{i\in I} FX_i$$

This map is a weak equivalence of profinite spaces because it comes from a level-wise weak equivalence of profinite spaces ([27, 2.14]). In particular, we have an induced isomorphism of homotopy groups by 5.1.8. We also have the isomorphisms

$$\pi_n(\varprojlim_{i\in I} FX_i) \simeq \varprojlim_{i\in I} \pi_n(FX_i) \simeq \varprojlim_{i\in I} \pi_n(X_i)$$

The first follows as the homotopy group functor commutes with a cofiltered limit of fibrant profinite spaces, and the second is an application of 5.1.8. So we obtain the desired isomorphism

$$\pi_n(\varprojlim_{i\in I} X_i) \xrightarrow{\sim} \varprojlim_{i\in I} \pi_n(X_i)$$

□

5.2.15. The following comparison theorem is a restatement of Artin-Mazur's comparison theorem ([1, 12.9]) in the language of topological types.

Theorem 5.2.16. (Comparison) *Let X be a pointed finite type scheme over $\mathbb{C}$. Then the map*

$$\widehat{X(\mathbb{C})} \to \widehat{h(X)}$$

of profinite completions of topological types is a weak equivalence of profinite spaces.

Proof. We may assume X is connected. Indeed, the topological type functor preserves coproducts 2.3.24, the profinite completion functor commutes with finite colimits, and a coproduct of weak equivalences between cofibrant objects is a weak equivalence [4, 1.2.5].

When taking Artin-Mazur's completion, the statement follows from their comparison theorem [1, 12.9]. In particular, they induce isomorphic pro-homotopy groups and hence isomorphisms of those pro-groups after passing to the limit. By 5.2.13, the map in the statement induces an isomorphism of profinite homotopy groups. The proof is completed by characterization of weak equivalence 5.1.8. □

Theorem 5.2.17. (Simplicial comparison) *Let $X_\bullet$ be a pointed finite type simplicial scheme over $\mathbb{C}$. Then the map*

$$\widehat{X_\bullet(\mathbb{C})} \to \widehat{h(X_\bullet)}$$

of the profinite completions of topological types is a weak equivalence of profinite spaces.

Proof. By the same reason in the proof of 5.2.16, we may assume $X_\bullet$ cannot be written as a disjoint union of non-empty simplicial finite type schemes over $\mathbb{C}$. Then the map induces isomorphic fundamental groups by the corresponding result from Friedlander [8, 8.4]. Note that we use the compatibility between our topological type and Friedlander 3.3.7, and the result on homotopy groups with respect to completions 5.2.13. It is a classical result that there is an isomorphism of cohomology groups, which completes the proof (cf. 5.1.4). □

Example 5.2.18.

(i) Consider the classifying stack $\mathcal{B}\mathbb{G}_m$ of the multiplicative group scheme $\mathbb{G}_m$ over $\mathbb{C}$. By the hypercover descent 3.5.9, the topological type $h(\mathcal{B}\mathbb{G}_m)$ of the classifying stack is strictly weakly equivalent to the topological type $h(B\mathbb{G}_m)$ of the simplicial scheme $B\mathbb{G}_m$. After profinite completion, by 5.2.17, $h(\mathcal{B}\mathbb{G}_m)\hat{}$ is weakly equivalent to $(BS^1)\hat{}$ where BS^1 is the classifying space of the unit circle. It is well-known that BS^1 is $\mathbb{C}P^\infty$ which is $K(\mathbb{Z},2)$. Therefore, $h(\mathcal{B}\mathbb{G}_m)\hat{}$ is weakly equivalent to $K(\mathbb{Z},2)\hat{}$.

(ii) More generally, for the classifying stack $\mathcal{B}GL_n$ of the general linear group scheme GL_n over $\mathbb{C}$ for $n \geq 1$, $h(\mathcal{B}GL_n)\hat{}$ is weakly equivalent to $BGL_n(\mathbb{C})\hat{}$. Since $BGL_n(\mathbb{C})$ is the Grassimannian $G(n,\mathbb{C}^\infty)$ of n-dimensional subspaces in $\mathbb{C}^\infty$, $h(\mathcal{B}GL_n)\hat{}$ is weakly equivalent to $G(n,\mathbb{C}^\infty)\hat{}$. Remark that for a finite coefficient group, the cohomology groups of a profinite completion of a pro-simplicial set coincide with the cohomology groups of the pro-simplicial set. Therefore, our result recovers that

$$H^*(BGL_n, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[c_1, c_2, \cdots, c_n] = H^*(G(n,\mathbb{C}^\infty), \mathbb{Q}_\ell)$$

where c_i's are the universal Chern classes of degree $2i$. Each side is well-known to algebraic geometers and topologists respectively.

5.2.19. The simplicial comparison theorem 5.2.17 leads to the comparison theorem for algebraic stacks. We make it precise for the rest of the subsection.

5.2.20. Let X be a topological space. Denote by $\mathrm{op}(X)$ the site induced by the usual topology on X. We call it the small topological site. Also consider the big topological site $\mathrm{Top}(X)$ whose objects are continuous maps over X and coverings are the usual open coverings. It follows that the functor

$$\mathrm{Op}(X) \to \mathrm{Top}(X)$$

satisfies the assumptions in 2.3.27. So we can use either the small or the big site to compute topological types of the associated topoi.

Now consider the site Top defined by $\mathrm{Top}(*)$ where $*$ is a final object in the category of topological spaces. Then by the 2.3.30, one can compute the topological types above in this site as well.

Remark 5.2.21. For the topos associated to Top, we take epimorphisms as the class of $\mathcal{P}$ for the theory of $\mathcal{P}$-hypercovers (see 2.3.45).

5.2.22. Recall from [26, §20] that there is a functor from the category of locally of finite type algebraic stacks over $\mathbb{C}$ to the category of stacks over Top. Denote by $\mathcal{X}^{\mathrm{top}}$ the image of $\mathcal{X}$ under the functor.

Theorem 5.2.23. (Stacky Comparison) *Let $\mathcal{X}$ be a finite type algebraic stack over $\mathbb{C}$. Then the map*

$$\widehat{h(\mathcal{X}^{\mathrm{top}})} \to \widehat{h(\mathcal{X})}$$

of the profinite completions of topological types is a weak equivalence of profinite spaces.

Proof. There is an induced epimorphism $X^{\mathrm{top}} \to \mathcal{X}^{\mathrm{top}}$. Since the analytification commutes with finite limits, the analytification of the simplicial scheme $\mathrm{cosk}_0(X/\mathcal{X})$ is simply $\mathrm{cosk}_0(X^{\mathrm{top}}/\mathcal{X}^{\mathrm{top}}))$ which is a $\mathcal{P}$-hypercover. By the hypercover descent 2.3.51, there are strict weak equivalences

$$h(\mathrm{cosk}_0(X/\mathcal{X})) \to h(\mathcal{X})$$

and

$$h(\mathrm{cosk}_0(X^{\mathrm{top}}/\mathcal{X}^{\mathrm{top}})) \to h(\mathcal{X}^{\mathrm{top}})$$

The result follows from 5.2.17. □

5.2.24. Concretely, for a group scheme G over $\mathbb{C}$ acting on a scheme $X/\mathbb{C}$, there is a weak equivalence

$$h([X(\mathbb{C})/G(\mathbb{C})])^{\widehat{}} \to h([X/G])^{\widehat{}}$$

of profinite completions of topological types of quotient stacks. In particular, we obtain a weak equivalence

$$h(\mathcal{B}G(\mathbb{C}))^{\widehat{}} \to h(\mathcal{B}G)^{\widehat{}}$$

for classifying stacks.

5.3. Profiniteness of topological types.

Definition 5.3.1. Let $X : I \to \mathbf{Set}$ be a pro-set. Its *profinite completion* $\widehat{X}$ is the profinite set

$$\varprojlim_{i \in I} \widehat{X_i}$$

where $\widehat{X_i}$ is the profinite completion of the set X_i as in 5.1.1 and the limit is taken in the category of profinite sets.

Remark 5.3.2. This notion of profinite completion of pro-sets is compatible with the profinite completion of sets 5.1.1, the profinite completion of simplicial sets 5.1.2, and the profinite completion of pro-simplicial sets 5.1.10.

Lemma 5.3.3. *Let X be a simplicial set. Then there is a canonical isomorphism of profinite sets*

$$\pi_0(X)\widehat{}\,\mathrm{Im} \xrightarrow{s} \pi_0(\widehat{X})$$

Proof. The statement is immediate from that for any finite set S,

$$\mathrm{Mor}_{\mathbf{Set}}(\pi_0(X), S) = \mathrm{Mor}_{\mathbf{SSet}}(X, S) = \mathrm{Mor}_{\widehat{\mathbf{SSet}}}(\widehat{X}, S) = \mathrm{Mor}_{\widehat{\mathcal{E}}}(\pi_0(\widehat{X}), S)$$

□

Corollary 5.3.4. *Let X be a pro-simplicial set. Then there is a canonical isomorphism of profinite sets*

$$\pi_0(X)\widehat{} \xrightarrow{\sim} \pi_0(\widehat{X})$$

Proof. Say $X : I \to \mathbf{SSet} : i \mapsto X_i$. Then

$$\pi_0(X)\widehat{} = \varprojlim_{i \in I} (\pi_0(X_i))\widehat{} \simeq \varprojlim_{i \in I} \pi_0(\widehat{X_i}) \simeq \pi_0(\varprojlim_{i \in I} \widehat{X_i}) = \pi_0(\widehat{X})$$

where the first isomorphism is by 5.3.3 and the second isomorphism from the property that π_0 commutes with cofiltered limits. □

Proposition 5.3.5. *Let $F_\bullet$ be a simplicial object in a topos T. There is a canonical isomorphism of profinite sets*

$$\pi_0(h(F_\bullet)\widehat{}) \xrightarrow{\sim} \pi_0(L_{\Gamma^*}(F_\bullet)\widehat{})$$

Proof. Consider the commutative diagram of profinite sets:

$$\begin{array}{ccc} \pi_0(h(F_\bullet))\widehat{} & \longrightarrow & \pi_0(h(F_\bullet)\widehat{}) \\ \downarrow & & \downarrow \\ \pi_0(L_{\Gamma^*}(F_\bullet))\widehat{} & \longrightarrow & \pi_0(L_{\Gamma^*}(F_\bullet)\widehat{}) \end{array}$$

The top (resp. bottom) horizontal map is an isomorphism by 5.3.4 (resp. 5.3.3). On the other hand, the left vertical map is an isomorphism by 2.4.2. Therefore, the right vertical map is also an isomorphism. □

5.3.6. For a pointed simplicial set X, the canonical map

$$(\pi_1(X))\widehat{\ } \to \pi_1(\widehat{X})$$

is an isomorphism of profinite groups ([27, 2.1]). We extend this result to pointed pro-simplicial sets:

Proposition 5.3.7. *Let X be a pointed pro-simplicial set X. Then there is a canonical isomorphism of profinite groups*

$$\pi_1(X)\widehat{\ } \xrightarrow{\sim} \pi_1(\widehat{X})$$

Proof. Say $X : I \to \mathbf{SSet} : i \mapsto X_i$. It is enough to show that for any finite group G, the canonical map

$$\mathrm{Mor}_{\mathrm{Profinite}}(\pi_1(\widehat{X}), G) \to \mathrm{Mor}_{\mathrm{Profinite}}(\pi_1(X)\widehat{\ }, G)$$

is an isomorphism where Profinite denote the category of profinite groups.

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{Profinite}}(\pi_1(\widehat{X}), G) &= H^1(\widehat{X}; G) && (5.3.7.1)\\
&= \varinjlim_{i \in I^{\mathrm{op}}} H^1(\widehat{X_i}; G) && (5.3.7.2)\\
&= \varinjlim_{i \in I^{\mathrm{op}}} H^1(X_i; G) && (5.3.7.3)\\
&= \varinjlim_{i \in I^{\mathrm{op}}} \mathbf{Ho}_{\mathbf{SSet}}(X_i, K(G,1)) && (5.3.7.4)\\
&= \mathbf{Ho}_{\mathrm{pro-}\mathbf{SSet}}(X, K(G,1)) && (5.3.7.5)\\
&= H^1(X, G) && (5.3.7.6)\\
&= \mathrm{Mor}_{\text{pro-Gps}}(\pi_1(X), G) && (5.3.7.7)\\
&= \mathrm{Mor}_{\mathrm{Profinite}}(\pi_1(X)\widehat{\ }, G) && (5.3.7.8)
\end{aligned}$$

where (5.3.7.5) follows from [27, 2.9]. □

5.3.8. Recall from [34, **0**.23.2.1] that a local ring A is *unibranch* if A_{red} is a domain and if the integral closure of A_{red} is local. We say that A is *geometrically unibranch* if it is unibranch and the residue field field of the integral closure of A_{red} is purely inseparable over the residue field of A. A scheme X is *geometrically unibranch* if for every point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is geometrically unibranch.

5.3.9. The implication of the geometrically unibranch condition is the profinite theorem of Artin-Mazur [1, 11.1]: For a pointed, connected, geometrically unibranch, and noetherian scheme X, the étale homotopy type is profinite. i.e., all homotopy pro-groups are pro-finite. This profinite theorem was generalized to simplicial schemes by Friedlander [8, 7.3]: For a pointed simplicial scheme $X_\bullet$ such that each X_n is noetherian, connected, and geometrically unibranch, the étale topological type is profinite. i.e., all homotopy pro-groups are pro-finite.

5.3.10. The property that a scheme is geometrically unibranch is local in the étale topology. So we say that an algebraic space is *geometrically unibranch* if there is an étale surjection $U \to X$ with U a geometrically unibranch scheme.

Proposition 5.3.11. *Let X/S be a quasi-compact, quasi-separated, and geometrically unibranch algebraic space. Then its topological type $h(X)$ is profinite. i.e., $\pi_n(X)$ is profinite for each $n \geq 0$.*

Proof. Choose an étale cover $U \to X$ with U a scheme. By the hypercover descent 3.4.9 there is a strict weak equivalence

$$h(\mathrm{cosk}_0(U/X)) \to h(X)$$

Therefore, the result follows from the case of simplicial schemes [8, 7.3]. □

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