Random Exponential Attractor and Equilibrium for a Stochastic Reaction-diffusion Equation with Multiplicative Noise

Gang Wang[*]

School of Science, Hubei University of Technology, Wuhan, China
*Corresponding author: wgfeiyu@sina.com

Abstract In this paper, we present a result on existence of exponential attractors for abstract random dynamical systems, and then give a criterion for exponentially attractive property of random attractors. As an application, we first prove that the random dynamical system generated by a stochastic reaction-diffusion equation possesses a random exponential attractor. Then we show that the unique random equilibrium when the nonlinearity satisfies some restrictive condition is exactly an exponential attractor.

Keywords: random dynamical system, random exponential attractor, random equilibrium, stochastic reaction-diffusion equation

1. Introduction

In this paper, we consider the asymptotic behavior of solutions to the following stochastic reaction-diffusion equation (SRDE) with multiplicative noise:

$$du - \left(\Delta u - f\left(u\right)\right)dt = bu \circ dW\left(t\right), x \in D, \qquad (1.1)$$

with the initial-boundary value conditions

$$\begin{cases} u\left(x,\tau\right) = u_0\left(x\right), x \in D, \\ u = 0, \qquad on\ \partial D. \end{cases} \qquad (1.2)$$

where $\left[\tau,t\right] \subset \mathbb{R}$ and $D \subset \mathbb{R}^n$ is a bounded open set with regular boundary ∂D and $W\left(t\right)$ is a two-sided real-valued Wiener process on a probability space which will be specified later.

The nonlinearity $f \in C^1\left(\mathbb{R}, \mathbb{R}\right)$ satisfies the following conditions:

$$c_1 \left|u\right|^p - c_2 \leq f\left(u\right)u \leq c_3 \left|u\right|^p + c_4, \qquad (1.3)$$

$$f'\left(u\right) \geq \gamma \qquad (1.4)$$

for some $p \geq 2$, $\gamma \in \mathbb{R}$, $c_i > 0$ $\left(i = 1,...,4\right)$ and for all $u \in \mathbb{R}$.

The asymptotic behavior of a random dynamical system (RDS) is captured by random attractors, which were first introduced in [5,11]. They are compact invariant random sets attracting all the orbits, but the attraction to it may be arbitrary. This drawback can be overcome by creating the notion of exponential attractor, which is a compact, positively invariant set of finite dimension and exponentially attract each orbit at an exponential rate. The existence of exponential attractors for deterministic case has been extensively studied since 1994, [7] ([3,6,8,9,10]). The concept of random exponential attractors was first introduced by A. Shirikyan and S. Zelik in [12]. They construct a random exponential attractor for abstract RDS and study its dependence on a parameter. In this paper, we devote to construct an exponential attractor for RDS and discuss the exponential attractive property of a random attractor. Firstly, we extend the deterministic result in [9] to stochastic case. Since we mainly concentrate on the exponential attractive property, we don't intend to discuss the time regularity of exponential attractors and its dependence on a parameter as in [12]. We then prove that a random attractor is actually an exponential attractor when the RDS satisfies Lipschitz continuity with small coefficient. Finally, we apply the abstract results to Eq.(1.1) to show that the corresponding RDS possesses an exponential attractors. When the derivative of the nonlinearity satisfies some restrictive condition, the random attractor become a point, i.e., the random equilibrium, and it attracts every orbit exponentially.

We organize this paper as follows. In section 2, we recall some basic notions of random attractors for RDS. In section 3, we present our main results and give the proofs. In section 4, we show our application to Eq.(1.1).

Throughout this paper, we denote by $\left\|\cdot\right\|_X$ the norm of Banach space X. The norm of $L^2\left(D\right)$ is written as $\left\|\cdot\right\|$. $\mathcal{A}\left(\omega\right)$ denotes the random attractor for RDS $\left\{\phi\left(t,\omega\right)\right\}_{t \geq 0}$ in a Banach space X.

2. Preliminaries and Main Results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{B}(X)$ be the Borel σ-algebra of X. In this paper, the term $\mathbb{P}$-a.s. (the abbreviation for $\mathbb{P}$ almost surely) denote that an event happens with probability one. In other words, the set of possible exceptions may be non-empty, but it has probability zero. Moreover, we need the following definitions, see [2,4,5,13] for more details.

Definition 2.1. $\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T}\right)$ $(T = \mathbb{Z} \text{ or } \mathbb{R})$ is called a (discrete or continuous) metric dynamical system (MDS) if $\theta : T \times \Omega \to \Omega$ is $(\mathcal{B}(T) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_s \circ \theta_t$ or all $s, t \in T$ and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in T$

Definition 2.2. *The RDS on X over an MDS $\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in T}\right)$ is a mapping $\phi : T^+ \times \Omega \times X \to X$, $(t, \omega, x) \mapsto \phi(t, \omega, x)$ which is $\left(\mathcal{B}(T^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X)\right)$-measurable and satisfies for $\mathbb{P}$-a.s.$\omega \in \Omega$,*

(i) $\phi(0, \omega, \cdot) = id$ on X;

(ii) $\phi(t + s, \omega, \cdot) = \phi(t, \theta_s \omega, \cdot) \circ \phi(s, \omega, \cdot)$ (cocycle property) on X for all $s, t \in T^+$.

An RDS is said to be continuous on X if $\phi(t, \omega) : X \to X$ is continuous for all $t \in T^+$ and $\mathbb{P}$-a.s.$\omega \in \Omega$.

Definition 2.3. A random bounded set $\{B(\omega)\}_{\omega \in \Omega}$ of X is called tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for $\mathbb{P}$-a.s.$\omega \in \Omega$,

$$\lim_{t \to \infty} e^{-\beta t} d\left(B(\theta_{-t}\omega)\right) = 0 \text{ for all } \beta > 0,$$

where $d(B) = \sup_{x \in B} \|x\|_X$.

Definition 2.4. *(1) A random set $\{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ is said to be a random absorbing set for ϕ if for every $\{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$, there exists $T_0(\omega) > 0$ such that for $\mathbb{P}$-a.s.$\omega \in \Omega$,*

$$\phi(t, \omega, D(\omega)) \subset B(\theta_t \omega), \text{ for all } t \geq T_0(\omega).$$

(2) A random set $\{C(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ is said to be $\mathcal{D}$-pullback attracting if for any $\{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$, we have for $\mathbb{P}$-a.s.$\omega \in \Omega$,

$$\lim_{t \to \infty} d\left(\phi(t, \omega, D(\omega)), C(\theta_t \omega)\right) = 0,$$

where $d(C_1, C_2)$ denotes the Hausdorff semi-distance between C_1 and C_2 in Y, given by

$$d(C_1, C_2) = \sup_{x \in C_1} \inf_{y \in C_2} \|x - y\|_X \text{ for } C_1, C_2 \subset X.$$

(3) A random set $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ is said to be a random attractor if the following conditions are satisfied for $\mathbb{P}$-a.s.$\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;

(ii) $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ is invariant, that is $\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega)$ for all $t \geq 0$;

(iii) $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ attracts every random set in $\mathcal{D}$.

(4) A random set $\{\mathcal{M}(\omega)\}_{\omega \in \Omega}$ is said to be a random exponential attractor if the following conditions are satisfied for $\mathbb{P}$-a.s.$\omega \in \Omega$,

(i) $\mathcal{M}(\omega)$ is compact;

(ii) $\{\mathcal{M}(\omega)\}_{\omega \in \Omega}$ is positively-invariant, that is, $\phi(t, \omega, \mathcal{M}(\omega)) \subset \mathcal{M}(\theta_t \omega)$ for all $t \geq 0$;

(iii) $\{\mathcal{M}(\omega)\}_{\omega \in \Omega}$ attracts every random set in $\mathcal{D}$ exponentially, that is, there is $\beta = \beta(\omega) > 0$ such that for $\{\mathcal{M}(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$

$$d\left(\phi(t, \omega, D(\omega)), M(\theta_t \omega)\right) \leq c(\omega) e^{-\beta t}, t \geq 0;$$

(iv) $\{\mathcal{M}(\omega)\}_{\omega \in \Omega}$ has finite fractal dimension, that is, there exists a number $d > 0$ such that

$$\dim_f (\mathcal{M}(\omega)) \leq d.$$

Our main results read as:

Theorem 2.1. Let X and X_1 be Banach spaces such that X_1 is compactly embedded in X. Assume that $\mathcal{B}(\omega)$ is positively invariant under a nonlinear map $S(\omega)$, and, for $\mathbb{P}-a.s.\omega \in \Omega$, $S(\omega)$ can be decomposed into a sum of two maps

$$S(\omega) = S_0(\omega) + K(\omega), S_0(\omega) : \mathcal{B}(\omega) \to X, \\ K(\omega) : \mathcal{B}(\omega) \to X, \quad (2.1)$$

and, for $\mathbb{P}-a.s.$ $\omega \in \Omega$, and any $x_1, x_2 \in \mathcal{B}(\omega)$, there exist $0 < \alpha(\omega) < 1$ and $\kappa(\omega)$ such that

$$\|S_0(\omega) x_1 - S_0(\omega) x_2\|_X \leq \alpha(\omega) \|x_1 - x_2\|_X, \quad (2.2)$$

and

$$\|K(\omega) x_1 - K(\omega) x_2\|_{X_1} \leq \kappa(\omega) \|x_1 - x_2\|_X. \quad (2.3)$$

Then, the discrete RDS $\left(S^n(\omega), \mathcal{B}(\omega)\right)$ possesses a random exponential attractor.

In particular, when $S_0(\omega) = 0$, we have

Theorem 2.2. Let X and X_1 be Banach spaces such that X_1 is compactly embedded in X. Let $\mathcal{B}(\omega)$ be a bounded random set positively invariant under $S(\omega)$. Assume that, for $\mathbb{P}-a.s.$ $\omega \in \Omega$,

$$\|S(\omega) x_1 - S(\omega) x_2\|_{X_1} \leq \kappa(\omega) \|x_1 - x_2\|_X, \\ x_1, x_2 \in \mathcal{B}(\omega), \quad (2.4)$$

Then the discrete $RDS\left(S^n(\omega), \mathcal{B}(\omega)\right)$ possesses a random exponential attractor.

Theorem 2.3. *Let X and X_1 be Banach spaces such that X_1 is compactly embedded in X. Let $\mathcal{B}(\omega)$ be a bounded random set positively invariant under $S(\omega)$. Assume that, for $\mathbb{P}-a.s.\omega \in \Omega$,*

$$\left\|S(\omega)x_1 - S(\omega)x_2\right\|_{X_1} \le \kappa(\omega)\|x_1 - x_2\|_X, \quad (2.5)$$
$$x_1, x_2 \in \mathcal{B}(\omega),$$

and

$$\left\|S(\omega)x_1 - S(\omega)x_2\right\|_X \le \nu(\omega)\|x_1 - x_2\|_X, \quad (2.6)$$
$$x_1, x_2 \in \mathcal{B}(\omega),$$

where $0 < \nu(\omega) < 1$. Then the random exponential attractor is identical with the random attractor $\mathcal{A}(\omega)$, i.e., $\mathcal{A}(\omega)$ attracts every obit exponentially.

Assume that $\phi(t,\omega)$, $t \in \mathbb{R}^+$ is an RDS on X over an $\mathrm{MDS}\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\right)$, we define

$$S^n(\omega) = \phi\left(nT^*, \omega\right) \text{ for some } T^* > 0.$$

Then, using the cocycle property of $\phi(t,\omega)$, we have

$$S^n(\omega)$$
$$= \phi\left(T^*, \theta_{(n-1)T^*}\omega\right) \circ \phi\left(T^*, \theta_{(n-2)T^*}\omega\right) \circ ... \circ \phi\left(T^*, \omega\right)$$
$$= S\left(\theta_{(n-1)T^*}\omega\right) \circ S\left(\theta_{(n-2)T^*}\omega\right) ... \circ S(\omega)$$
$$= S\left(\Theta_{n-1}\omega\right) \circ S\left(\Theta_{n-2}\omega\right) \circ ... S(\omega).$$

This implies that $\left\{S^n(\omega)\right\}_{n\in\mathbb{N}}$ is a discrete RDS over the $\mathrm{MDS}\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{n\in\mathbb{Z}}\right)$ on X, where

$$\Theta_n(\omega) = \theta_{nT^*}(\omega).$$

In the following, we still use $\left(\Omega, \mathcal{F}, \mathbb{P}, (\theta_n)_{n\in\mathbb{Z}}\right)$ instead of $\left(\Omega, \mathcal{F}, \mathbb{P}, (\Theta_n)_{n\in\mathbb{Z}}\right)$.

Once the existence of exponential attractors for discrete case is proved the result for the continuous case follows in a standard manner (e.g., see [7]).

Theorem 2.4. *Suppose that there is a $T^* > 0$ such that $S(\omega) = \phi\left(T^*, \omega\right)$ satisfies of theorem 2.1, and the map $F(t,\omega,x) = \phi(t,\omega)x$ is Hölder continuous from $[0,T] \times X$ into X for any $T > 0$. Then $\phi(t,\omega)$ has a random exponential attractor.*

Next, we construct $\mathcal{B}(\omega)$ based on the random attractor $\mathcal{A}(\omega)$.

Lemma 2.5. *For any fixed $\epsilon_0 > 0$, there exists an integer m0 such that for any $m \ge m_0$*

$$S^m(\omega)\left(B_{\epsilon_0}\mathcal{A}(\omega)\right) \subset B_{\epsilon_0}\mathcal{A}(\theta_m\omega).$$

Furthermore,

$$S(\theta_m\omega): \overline{S^m(\omega)\left(B_{\epsilon_0}\mathcal{A}(\omega)\right)}^X \to \overline{S^m(\theta\omega)\left(B_{\epsilon_0}\mathcal{A}(\theta\omega)\right)}^X.$$

Proof. Since $\mathcal{A}(\omega)$ is the random attractor, $B_{\epsilon_0}\mathcal{A}(\omega)$ is a random absorbing set for any $\epsilon_0 > 0$. Thus, the first assertion follows from the definition of random absorbing set.

By the continuity of $S(\omega)$ on $B_{\epsilon_0}\mathcal{A}(\omega)$ and the cocycle property, we get

$$S(\theta_m\omega)\overline{S^m(\omega)\left(B_{\epsilon_0}\mathcal{A}(\omega)\right)}^X$$
$$= \overline{S(\theta\omega)S^m(\omega)\left(B_{\epsilon_0}\mathcal{A}(\omega)\right)}^X$$
$$= \overline{S(\theta_m\omega) \circ S(\theta_{m-1}\omega) \circ ... \circ S(\theta\omega) \circ S(\omega)\left(B_{\epsilon_0}\mathcal{A}(\omega)\right)}^X$$
$$\subset \overline{S(\theta_m\omega) \circ S(\theta_{m-1}\omega) \circ ... \circ S(\theta\omega) \circ S(\omega)\left(B_{\epsilon_0}\mathcal{A}(\theta\omega)\right)}^X$$
$$= \overline{S^m(\theta\omega)\left(B_{\epsilon_0}\mathcal{A}(\theta\omega)\right)}^X.$$

The proof is complete.

Set $\mathcal{B}'(\omega) = \overline{S^m(\omega)\left(B_{\epsilon_0}\mathcal{A}(\theta\omega)\right)}^X$ and $\mathcal{B}(\omega) = \mathcal{B}'(\theta_{-m}\omega)$ for any fixed $m \ge m_0$, then we have

Lemma 2.6. $\mathcal{A}(\theta_k\omega) \subset S^k(\omega)\mathcal{B}(\omega) \subset \mathcal{B}(\theta_k\omega)$ *for any $k \in \mathbb{N}$. Furthermore, $\mathcal{B}(\omega)$ is a random absorbing set for $\left\{S^n(\omega)\right\}_{n\in\mathbb{N}}$.*

Proof. On one hand, from lemma 2.5, we have

$$S^k(\theta_m\omega)\mathcal{B}'(\omega) = S(\theta_{m+k-1}\omega) \circ ... \circ S(\theta_m\omega)\mathcal{B}'(\omega)$$
$$\subset S(\theta_{m+k-1}\omega) \circ ... \circ S(\theta_{m+1}\omega)\mathcal{B}'(\theta\omega)$$
$$\subset S(\theta_{m+k-1}\omega) \circ S(\theta_{m+2}\omega)\mathcal{B}'(\theta_2\omega) \subset \mathcal{B}'(\theta_k\omega).$$

By replacing $\theta_m\omega$ by ω, we get

$$S^k(\omega)\mathcal{B}'(\theta_{-m}\omega) \subset \mathcal{B}'(\theta_{k-m}\omega),$$

this implies

$$S^k(\omega)\mathcal{B}(\omega) \subset \mathcal{B}'(\theta_k\omega).$$

On the other hand, since $\mathcal{A}(\omega) \subset \mathcal{B}(\omega)$, we get

$$\mathcal{A}(\theta_k\omega) = S^k(\omega)\mathcal{A}(\omega) \subset S^k(\omega)\mathcal{B}(\omega).$$

Thus, the first assertion hold. For any $\left\{C(\omega)\right\}_{\omega\in\Omega} \in \mathcal{D}$, since $B_{\epsilon_0}\left(\mathcal{A}(\omega)\right)$ is absorbing for $\left\{S^n(\omega)\right\}_{n\in\mathbb{N}}$, there exists $k_0 > 0$ such that for all $k > k_0$,

$$S^k(\omega)C(\omega) \subset B_{\epsilon_0}\left(\mathcal{A}(\theta_k\omega)\right).$$

Therefore

$$
\begin{aligned}
S^{k+m}(\omega)C(\omega) &= S(\theta_{k+m-1}\omega)\circ...\circ S^k(\omega)C(\omega)\\
&\subset S^m(\theta_k\omega)B_{\epsilon_0}\left(\mathcal{A}(\theta_k\omega)\right)\\
&\subset \overline{S^m(\theta_k\omega)B_{\epsilon_0}\left(\mathcal{A}(\theta_k\omega)\right)}^X\\
&= \mathcal{B}'(\theta_k\omega) = \mathcal{B}(\theta_{k+m}\omega) \ \forall k \geq k_0.
\end{aligned}
$$

The proof is complete.

Proof of Theorem 2.1. We choose $\theta = \theta(\omega) > 0$ such that $2(\theta+\alpha) < 1$. Since $\mathcal{B}(\omega)$ is bounded for $\mathbb{P}-a.s.\omega \in \Omega$, there exists a ball $B_X(R,x_0)$ of radius $R = R(\omega)$ centered in $\mathcal{B}(\omega)$ which contains $\mathcal{B}(\omega)$. Setting $E_0 = V_0 = \{x_0\}$. It follows from (2.3) that the X_1-ball $B_{X_1}(\kappa R, Kx_0)$ covers $K(B(\omega))$. Since the embedding $X_1 \subset X$ is compact, we can cover the X_1-ball $B_{X_1}(\kappa R, Kx_0)$ by a finite number of θR balls in X with centers y_i. Moreover, the finite number of ball in this covering has the following estimate

$$
\begin{aligned}
N_{\theta R}&\left(B_{X_1}\left(\kappa R, K(x_0), X\right)\right)\\
&= N_{\theta R}\left(B_{X_1}(\kappa R, 0), X\right)\\
&= N_{\theta/\kappa}\left(B_{X_1}(1,0), X\right) =: N(\theta,\omega).
\end{aligned}
$$

This implies that

$$K\left(\mathcal{B}(\omega)\right) \subset \cup_{i=1}^{n(\theta,\omega)} B_X(\theta R, y_i). \tag{2.7}$$

It follows from (2.2) we get

$$S_0\left(\mathcal{B}(\omega)\right) \subset B_X\left(\alpha R, S_0(x_0)\right). \tag{2.8}$$

Combining (2.7) and (2.8), we conclude that

$$S(\omega)\mathcal{B}(\omega) \subset \cup_{i=1}^{n(\theta,\omega)} B_X\left((\theta+\alpha)R, \overline{y}_i\right), \tag{2.9}$$

where $\overline{y}_i = y_i + S_0(x_0)$.

Now, we enlarge the radius twice so that

$$S(\omega)\mathcal{B}(\omega) \subset \cup_{i=1}^{n(\theta,\omega)} B_X\left(2(\theta+\alpha)R, x_i\right), \tag{2.10}$$

and $x_i \in S(\omega)\mathcal{B}(\omega)$.

We set

$$V_{1,i_1} := \left\{x_i : i \in \{1,...,N(\theta,\omega)\}\right\}, \tag{2.11}$$

$$E^{(1)} := S(\omega)E^{(0)} \cup V_{1,i_1} \subset S(\omega)\mathcal{B}(\omega). \tag{2.12}$$

Applying the above covering process to every ball in the right-hand side in (2.10), we can generate the kth generation of centers in $S^k(\omega)\mathcal{B}(\omega)$ such that

$$V_{k,i_1,...,i_{k-1}} := \left\{x_{i_1,...,i_{k-1},j} : j \in \{1,...,N(\theta,\omega)\}\right\}, \tag{2.13}$$

$$E^{(k)} := S(\omega)E^{(k-1)} \cup V_{k,i_1,...,i_{k-1}} \subset S^k(\omega)\mathcal{B}(\omega). \tag{2.14}$$

Therefore, for any $k \in \mathbb{N}$ we find sets $E^{(k)}$, enjoy the following properties:

$$E^{(k)} \subset S^k(\omega)\mathcal{B}(\omega); \tag{2.15}$$

$$S(\omega)E^{(k)} \subset E^{(k+1)}; \tag{2.16}$$

$$\#E^{(k)} \leq N^{k+1}(\theta,\omega); \tag{2.17}$$

$$dist_X\left(S^k(\omega)\mathcal{B}(\omega), E^{(k)}\right) \leq \left(2(\alpha+\theta)\right)^k R. \tag{2.18}$$

Now, we can construct the random exponential attractor for $S(\omega)$ as follows:

$$\mathcal{M}'(\omega) = \bigcup_{k}^{\infty} E^{(k)}, \ \mathcal{M}(\omega) = \overline{\mathcal{M}'(\omega)}^X. \tag{2.19}$$

Considering $\mathcal{M}(\omega)$ as deterministic sets with parameter ω, then we can show that $\mathcal{M}(\omega)$ satisfies the conditions in definition 2.4 (4) (see [8] for deterministic case). Thus, $\mathcal{M}(\omega)$ is a random exponential attractor for S^k. The proof is completed.

Proof of Theorem 2.3. For any $x \in \mathcal{B}(\omega)$, $y \in \mathcal{A}(\theta_k\omega)$, from (2.6) and the invariant of $\mathcal{A}(\omega)$, we get

$$
\begin{aligned}
\left\|S^k(\omega)x - y\right\| &= \left\|S^k(\omega)x - S^k(\omega)x'\right\|\\
&= \left\|S(\theta_{k-1}\omega)\circ...\circ S(\omega)x - S(\theta_{k-1}\omega)\circ...\circ S(\omega)x'\right\|\\
&\leq \nu^k(\omega)\|x - x'\|,
\end{aligned}
$$

where $x' \in \mathcal{A}(\omega)$, $y = S^k(\omega)x'$. Therefore

$$
\begin{aligned}
d&\left(S^k(\omega)\mathcal{B}(\omega), \mathcal{A}(\theta_k\omega)\right)\\
&\leq \nu^k(\omega)d\left(\mathcal{B}(\omega), \mathcal{A}(\omega)\right) =: c_0(\omega)\nu^k(\omega).
\end{aligned} \tag{2.20}
$$

Combine (2.15) and (2.20), we can choose sets $F^{(k)} \subset \mathcal{A}(\theta_k\omega)$ satisfying (2.15)-(2.18) with (2.18) replaced by

$$dist_X\left(S^k(\omega)\mathcal{B}(\omega), F^{(k)}\right) \leq c(\omega)\eta^k,$$

for some $0 < \eta = \eta(\omega) < 1$. Therefore, the exponential attractor constructed in the proof of Theorem 2.1 is identical with the random attractor. The proof is complete.

3. Applications

In this section, we apply the above results to show that the RDS generated by Eq. (1.1) possess a random exponential attractor. To this end, we need to convert the stochastic equation into a deterministic equation with a

random parameter. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ is the corresponding Wienner measure on $(\Omega, \mathcal{F})$. Then we identify $\omega(t)$ with $W(t)$, i.e.,

$$W(t) = W(t, \omega) = \omega(t), t \in \mathbb{R}.$$

Define the time shift by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic MDS. We introduce an Otnstein-Uhlenbeck process

$$z(\theta_t \omega) := \int_{-\infty}^{0} e^{\tau}(\theta_t \omega)(\tau) d\tau, t \in \mathbb{R},$$

and it solves the Itô equation

$$dz + z dt = dW(t).$$

From [14], it is known that the random variable $z(\omega)$ is tempered, and there is a θ_t-invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathbb{P}$ measure such that for every $\omega \in \tilde{\Omega}$, $t \to z(\theta_t \omega)$ is continuous in t and

$$\lim_{t \to \pm\infty} \frac{|z(\theta_t \omega)|}{|t|} = 0, \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t z(\theta_s \omega) ds = 0. \quad (3.1)$$

Moreover, there exists a tempered random variable $r_1(\omega) > 0$ such that

$$|z(\theta_t \omega)| \le e^{|t|} r_1(\omega), t \in \mathbb{R}. \quad (3.2)$$

Setting $\alpha(\omega) = e^{-bz(\omega)}$, from (3.1) one can easily show that $\alpha(\omega)$ and $\alpha^{-1}(\omega)$ are temperate, and $\alpha(\theta_t(\omega))$ is continuous in t for $\mathbb{P} - a.s. \omega \in \Omega$. Therefore, by using Proposition 4.3.3 in [1], for any $\epsilon > 0$ there exists ϵ-slowly varying random variable $r_2(\omega) > 0$ such that

$$\frac{1}{r_2(\omega)} \le \alpha(\omega) \le r_2(\omega),$$

and $r_2(\omega)$ satisfies, for $\mathbb{P} - a.s. \omega \in \Omega$,

$$e^{-\epsilon|t|} r_2(\omega) \le r_2(\theta_t \omega) \le e^{\epsilon|t|} r_2(\omega), t \in \mathbb{R}.$$

Therefore, for $\mathbb{P} - a.s. \omega \in \Omega$,

$$e^{-\epsilon|t|} r_2^{-1}(\omega) \le \alpha(\theta_t \omega) \le e^{\epsilon|t|} r_2(\omega), t \in \mathbb{R}. \quad (3.3)$$

If we set $r(\omega) = \max(r_1(\omega), r_2(\omega))$, we can get from (3.2) and (3.3) that

$$|z(\theta_t \omega)| \le e^{|t|} r(\omega); \quad (3.4)$$

$$e^{-\epsilon|t|} r^{-1}(\omega) \le \alpha(\theta_t \omega) \le e^{\epsilon|t|} r(\omega), \quad (3.5)$$

for all $t \in \mathbb{R}$ and $\mathbb{P} - a.s. \omega \in \Omega$, and $r(\omega)$ is also tempered.

Let $v(t) = \alpha(\theta_t \omega) u(t)$, and we can consider the following evolution equation with random coefficients but without white noise:

$$\frac{\partial v}{\partial t} - \Delta v + \alpha(\theta_t \omega) f(\alpha^{-1}(\theta_t \omega) v) = bz(\theta_t \omega) v, \quad (3.6)$$

with Dirichlet boundary condition

$$v|_{\partial D} = 0, \quad (3.7)$$

and initial condition

$$v(\tau, x) = v(\tau) = \alpha(\theta_\tau \omega) u_0. \quad (3.8)$$

From [15] we see that for $\mathbb{P} - a.s. \omega \in \Omega$ and for all $v_0 \in L^2(D)$, the parameterized evolution equation (3.6)-(3.8) with conditions (1.3)-(1.4) has a unique solution

$$v(\cdot, \omega; \tau, v_0) \in C([\tau, \infty); L^2(D))$$
$$\cap L_{loc}^p([\tau, \infty); L^p(D)) \quad (3.9)$$
$$\cap L_{loc}^2([\tau, \infty); H_0^1(D)).$$

Furthermore, $v(t, \omega; \tau, v_0)$ is continuous with respect to v_0 in $L^2(D)$, for all $t \ge \tau$ and $\mathbb{P} - a.s. \omega \in \Omega$.

Then $v(t, \omega; \tau, u(\tau)) = \alpha^{-1}(\theta_t \omega) v(t)$ is a solution of (1.1)-(1.2) with $u_0 = \alpha^{-1}(\theta_\tau \omega) v_0$. We now define a mapping $\phi : \mathbb{R}^+ \times \Omega \times L^2(D) \to L^2(D)$ by

$$\phi(t - \tau, \theta_\tau \omega, u(\tau)) = \alpha^{-1}(\theta_t \omega) v(t, \omega; \tau, \alpha(\theta_\tau \omega) u(\tau)). (3.10)$$

Then ϕ is a continuous RDS on $L^2(D)$ and an RDS on $H_0^1(D)$ respectively associated with the SRDE (1.1)-(1.2) on D.

Theorem 3.1. *([15]) Assume that (1.3)-(1.4) hold. Then the RDS ϕ generated by (1.1)-(1.2) has a unique random attractor $\mathcal{A}(\omega)$ in $L^2(D)$.*

Theorem 3.2. *Assume that (1.3)-(1.4) hold. Let $u_{01}, u_{02} \in \mathcal{B}(\omega)$. Then for $\mathbb{P} - a.s. \omega \in \Omega$, there exists $T(\omega, \epsilon) < 0$, such that the solution $u(t, \omega; \tau, u_0)$ of (1.1)-(1.2) satisfies that for all $t \le T(\omega, \epsilon)$,*

$$\|u(0, \omega; t, u_{01}) - u(0, \omega; t, u_{02})\|_{H_0^1}$$
$$\le c(\omega) e^{2(\lambda_1 + \gamma)t} e^{-2(b+1)\epsilon t} \|u_{02} - u_{01}\|. \quad (3.11)$$

where $c(\omega)$ is a random variable and λ_1 is the first eigenvalue of $-\Delta$.

Proof. Let $v_i(t) = v(t, \omega; \tau, u_{0i})$, $v_{0i} = \alpha(\theta_t \omega) u_{0i}$, $i = 1, 2$, and $w(t) = v_2(t) - v_1(t)$. Then $w(t)$ satisfies the following equation

$$w_t - \Delta w + l(t)w = bz(\theta_t \omega)w. \qquad (3.12)$$

where $l(t) = \int_0^1 f'\left(s\alpha^{-1}(\theta_t\omega)v_1(t) + (1-s)\alpha^{-1}(\theta_t\omega)v_2(t)\right)ds$, and $l(t) \geq \gamma$ by (1.4).

Multiply the above equation with w to get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \|\nabla w(t)\|^2 + \int_D l(t)|w(t)|^2$$
$$= bz(\theta_t\omega)\|w(t)\|^2. \qquad (3.13)$$

Thus,

$$\frac{d}{dt}\|w(t)\|^2 + 2\lambda_1\|w(t)\|^2 + 2\gamma\|w(t)\|^2$$
$$\leq 2bz(\theta_t\omega)\|w(t)\|^2. \qquad (3.14)$$

We multiply both side with $e^{2\int_0^t (\lambda_1 + \gamma - bz(\theta_r\omega))dr}$ to get

$$\frac{d}{dt}e^{2\int_0^t (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\|w(t)\|^2 \leq 0. \qquad (3.15)$$

Integrating the above inequality in $[\tau, t]$ we obtian

$$\|w(t)\|^2 \leq e^{2\int_t^\tau (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\|w(\tau)\|^2$$
$$= e^{2\int_t^\tau (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\alpha^2(\theta_\tau\omega)\|u_{02} - u_{01}\|^2. \qquad (3.16)$$

Next, we take inner product of (3.12) with $-\Delta w$ in $L^2(D)$, and use $l(t) \geq \gamma$ to get

$$\frac{d}{dt}\|\nabla w\|^2 + 2\|\Delta w\|^2 + 2\gamma\|\nabla w\|^2 \leq 2bz(\theta_t\omega)\|\nabla w\|^2. \ (3.17)$$

That is

$$\frac{d}{dt}\|\nabla w\|^2 \leq 2(bz(\theta_t\omega) - \gamma)\|\nabla w\|^2. \qquad (3.18)$$

Integrating (3.13) from t to $t+1$ and using (3.16), it yields

$$\int_t^{t+1}\|\nabla w(s)\|^2 ds$$
$$\leq \frac{1}{2}\|w(t)\|^2 + \int_t^{t+1}(bz(\theta_s\omega) - \gamma)\|w(s)\|^2 ds$$
$$\leq \frac{1}{2}e^{2\int_t^\tau (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\alpha^2(\theta_\tau\omega)\|u_{02} - u_{01}\|^2$$
$$+ \int_t^{t+1}(bz(\theta_s\omega) - \gamma)e^{2\int_s^\tau (\lambda_1 + \gamma - bz(\theta_r\omega))dr}ds\|u_{02} - u_{01}\|^2. \qquad (3.19)$$

Combining (3.18) and (3.19) using Uniform Gronwall's Lemma (note that the Uniform Gronwall's inequality also hold when the right-hand side of (3.19) dependent on t), it yields

$$\|\nabla w(t+1)\|^2$$
$$\leq e^{2\int_t^{t+1}(bz(\theta_s\omega)-\gamma)ds}[\frac{1}{2}e^{2\int_t^\tau (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\alpha^2(\theta_\tau\omega)$$
$$+ \int_t^{t+1}(bz(\theta_s\omega) - \gamma)e^{2\int_s^\tau (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\alpha^2(\theta_\tau\omega)ds]$$
$$\times \|u_{02} - u_{01}\|^2. \qquad (3.20)$$

Let $t = -1$, then replace τ by t to get

$$\|\nabla w(0)\|^2$$
$$\leq e^{2\int_{-1}^0 (bz(\theta_s\omega)-\gamma)ds}[\frac{1}{2}e^{2\int_{-1}^t (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\alpha^2(\theta_t\omega)$$
$$+ \int_{-1}^0 (bz(\theta_s\omega) - \gamma)e^{2\int_s^t (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\alpha^2(\theta_t\omega)ds]$$
$$\times \|u_{02} - u_{01}\|^2$$
$$\leq c'(\omega)\alpha^2(\theta_t\omega)e^{2\int_0^t (\lambda_1 + \gamma - bz(\theta_r\omega))dr}$$
$$[e^{2\int_{-1}^0 (\lambda_1 + \gamma - bz(\theta_r\omega))dr}$$
$$+ \int_{-1}^0 e^{2\int_s^0 (\lambda_1 + \gamma - bz(\theta_r\omega))dr}ds]\|u_{02} - u_{01}\|^2$$
$$\leq c(\omega)e^{-2\epsilon t}e^{2\int_0^t (\lambda_1 + \gamma - bz(\theta_r\omega))dr}\|u_{02} - u_{01}\|^2$$
$$\leq c(\omega)e^{-2\epsilon t}e^{2(\lambda_1 + \gamma)t}e^{-2b\int_0^t (z(\theta_r\omega))dr}\|u_{02} - u_{01}\|^2$$
$$= c(\omega)e^{-2\epsilon t}e^{2(\lambda_1 + \gamma)t}e^{-2bt\left(\frac{1}{t}\int_0^t (z(\theta_r\omega))dr\right)}\|u_{02} - u_{01}\|^2. \qquad (3.21)$$

We have used (3.5) in (3.21). By (3.1), there is $T(\epsilon, \omega) < -1$, for $t \leq T(\epsilon, \omega)$

$$\|\nabla w(0)\|^2 \leq c(\omega)e^{-2\epsilon t}e^{2(\lambda_1 + \gamma)t}e^{-2b\epsilon t}\|u_{02} - u_{01}\|^2$$
$$= c(\omega)e^{2(\lambda_1 + \gamma)t}e^{-2(b+1)\epsilon t}\|u_{02} - u_{01}\|^2. \qquad (3.22)$$

Finally, by the relationship between u and v we obtain

$$\|u(0, \omega; t, u_{01}) - u(0, \omega; t, u_{02})\|_{H_0^1}$$
$$= \alpha^{-1}(\omega)\|v(0, \omega; t, u_{02}) - v(0, \omega; t, u_{01})\| \qquad (3.23)$$
$$\leq c(\omega)e^{2(\lambda_1 + \gamma)t}e^{-2(b+1)\epsilon t}\|u_{02} - u_{01}\|^2.$$

The proof is complete.

Theorem 3.3. *Assume that (1.3)-(1.4) hold. Then the RDS generated by Eq (1.1) has a random exponential attractor in $L^2(D)$.*

Proof. From theorem 2.2 and theorem 3.2, we see that, for some fixed $t = T^* \leq T(\epsilon, \omega)$, the discrete RDS$\left(S^n(\omega), \mathcal{B}(\omega)\right)$ possesses a random exponential attractor in $L^2(D)$, where $S^n(\omega) = \phi\left(nT^*, \omega\right)$. Moreover, by an elementary process, one can easily show that $\phi(t, \omega)$ is Hölder continuous from $[0,T] \times L^2(D)$ into $L^2(D)$, then by theorem 2.4 we obtain that the RDS$\left(\phi(t, \omega), \mathcal{B}(\omega)\right)$ has a random exponential attractor in $L^2(D)$. The proof is complete.

When the constant γ in (1.4) satisfies $\gamma + \lambda_1 > 0$, the random attractor reduces to a single point, i.e., a random equilibrium (see [15]). Moreover, we have

Theorem 3.4. *Assume that (1.3)-(1.4) hold and $\gamma + \lambda_1 > 0$. Then the unique random equilibrium attracts every obit exponentially.*

Proof. From (3.11) we get that for all $t \leq T(\epsilon, \omega)$

$$\left\| u(0, \omega; t, u_{01}) - u(0, \omega; t, u_{02}) \right\|_{H_0^1} \qquad (3.24)$$
$$\leq c(\omega) e^{2(\lambda_1 + \gamma)t} e^{-2(b+1)\epsilon t} \left\| u_{02} - u_{01} \right\|.$$

Using poincaré inequality, we obtain

$$\left\| u(0, \omega; t, u_{01}) - u(0, \omega; t, u_{02}) \right\| \qquad (3.25)$$
$$\leq \frac{1}{\lambda_1} c(\omega) e^{2(\lambda_1 + \gamma)t} e^{-2(b+1)\epsilon t} \left\| u_{02} - u_{01} \right\|.$$

Since $\gamma + \lambda_1 > 0$, we choose ϵ small enough and $t \leq T(\epsilon, \omega)$ such that

$$\frac{1}{\lambda_1} c(\omega) e^{2(\lambda_1 + \gamma)t} e^{-2(b+1)\epsilon t} < 1 \qquad (3.26)$$

for $\mathbb{P} - a.s. \omega \in \Omega$. Then from (3.24) and (3.25), the conditions in theorem 2.3 are satisfied. From theorem 2.3 we arrive at our conclusion. The proof is complete.

4. Conclusion

In this paper, we have constructed exponential attractors for abstract RDS and discussed the exponential attractive property of a random attractor. Moreover, we have applied our abstract results to a stochastic reaction-diffusion equation. The abstract results presented in this paper have widely applications in RDS generated by many other stochastic partial differential equations, and these results will be applied in our future study.

Acknowledgments

References

[1] Arnold L., *Random Dynamical Systems,* Springer-Verlag, Berlin, 1998.

[2] Bates P. W., Lu K., Wang B., "Random attractors for stochastic reaction-diffusion equations on unbounded domains," *J. Dynam. Differential Equations* 13 (2001) 791-806.

[3] Babin A., Nicolaenko B., "Exponential attractors of reaction-diffusion systems on unbounded domain," *J. Differential Equations* 7 (1995) 567-590.

[4] Crauel H., Debussche A., Flandoli F., "Random attractors," *J. Dynam. Differential Equations* 9 (1997) 307-341.

[5] Crauel H., Flandoli F., "Attractors for random dynamical systems," *Probab. Theory Related Fields* 100 (1994) 365-393.

[6] Dung L., Nicolaenko B., "Exponential Attractors in Banach Spaces," *J. Dynam. Differential Equations* 13 (2001) 791-806.

[7] Eden A., Foias C., Nicolaenko B., Temam R., *Exponential Attractors for Dissipative Evolution Equations*, in: Research in Applied Mathematics, vol. 37, John Wiley, New York, 1994.

[8] Efendiev M., Miranville A., Zelik S., "Exponential attractors for a singularly perturbed Cahn-Hilliard system," *Math. Nachr.* 272 (2004) 11-31.

[9] Efendiev M., Miranville A., Zelik S., "Exponential attractors for a nonlinear reactiondi ffusion system in $\mathbb{R}^3$," *C. R. Acad. Sci. Paris, Sér.* I 330 (2000) 713-718.

[10] Málek J., Pražák D., "Large time behavior via the method of l-trajectories," *J. Differential Equations* 181 (2002) 243-279.

[11] Schmalfuss B., *Backword cocycle and attractors of stochastic differential equations* in: V. Reitmann, T. Riedrich, N. Koksch(Eds.), International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior, Technische Universität, Dresden, 1992, pp. 185-192.

[12] Shirikyan A., Zelik S., "Exponential attractors for random dynamical systems and applications," *Stoch. PDE: Anal. Comp.* 1 (2013) 241-281.

[13] Wang G, Tang Y. B. "(L^2, H^1)-random attractors for stochastic reaction-diffusion equation on unbounded domains," *Abstr. Appl. Anal.* 2013(2013), Article ID 279509, 23 pages.

[14] Wang Z. J., Zhou S. F., "Random attractor for stochastic reaction-diffusion equation with multiplicative noise on unbounded domains," *J. Math. Anal. Appl.* 384 (1) (2011) 160-172.

[15] Zhao W. Q., "H^1-random attractors and random equilibria for stochastic reaction-diffusion equations with multiplicative noises," *Commun. Nonlinear Sci. Numer. Simulat.* (2013) 2707-2721.