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Composition Operators on Lorentz-Karamata-Bochner Spaces

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Abstract In this paper, study of the composition operators on Lorentz-Karamata-Bochner spaces and characterization of the properties like boundedness, closedness and essential range of these operators on the space has been made.

Keywords: *slowly varying function, Lorentz-Karamata-Bochner spaces, composition operators, closed range, essential range*

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1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. A measurable transformation T is said to be non-singular if $\mu\left(T^{-1}(A)\right) = 0$ whenever $\mu(A) = 0$ for every $A \in \mathcal{A}$.

If T is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ. Hence, by Radon-Nikodym theorem there exists a unique non-negative essentially bounded function f_T such that $\mu\left(T^{-1}\right)(A) = \int_A f_T \, d\mu$ for $A \in \mathcal{A}$.

Let f be any complex-valued measurable function. For $s \geq 0$, the *distribution function* μ_f of f is defined as

$$\mu_f(s) = \mu\{\omega \in \Omega : |f(\omega)| > s\}.$$

The *non-increasing rearrangement* f^* of f is defined as

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}, \text{ for all } t \geq 0.$$

The *maximal (average) operator* is given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

One can refer to [4] for the properties of these functions.

Definition 1. *A positive and Lebesgue measurable function b is said to be slowly varying (s.v.) on (0, ∞) if, for each $\epsilon > 0$, $t^{\epsilon}b(t)$ is equivalent to a non-decreasing function and $t^{-\epsilon}b(t)$ is equivalent to a non-increasing function on (0, ∞).*

Given a s.v. function b on (0, ∞), we denote by γ_b the positive function defined by

$$\gamma_b(t) = b\left(\max\left\{t, \frac{1}{t}\right\}\right)$$

For various properties of slowly varying function we can refer to [4,10].

For $1 < p < \infty$, $1 \leq q < \infty$ and for a measurable function f on Ω, define

$$\| f \|_{p,q;\alpha} = \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^{**}(t) \right\|_{q;(0,\infty)}$$

$$= \left(\int_0^{\infty} \left(t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^{**}(t) \right)^q dt \right)^{\frac{1}{q}}$$

The Lorentz-Karamata space $L_{p,q;b}$ introduced in [4] is the set of all measurable functions f on Ω such that

$$\|f\|_{p,q;b} < \infty.$$

Let $f : \Omega \to X$ be a strongly measurable function on a Banach space X. Define a function $\|f\|$ as

$$\|f\|(w) = \|f(w)\|$$

for all $\omega \in \Omega$. Then the Lorentz-Karamata-Bochner space $L_{p,q;b}(\Omega, X)$ is a rearrangement invariant-Bochner space for $p, q \in (0, \infty)$ where the norm is given as

$$\|f\|_{p,q;b} = \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \left\| f^{**} \right\|(t) \right\|_{q;(0,\infty)}$$

The Lorentz-Karamata space $L_{p,q;b}$ is a Banach space and we still have the density of simple functions in it and its dual is

$$L^*_{p,q;b}(\Omega, X) = L_{p',q';b^{-1}}(\omega, X^*)$$

where X^* has the Radon-Nikodym property. For every $g \in L_{p',q';b^{-1}}$, we can find a bounded linear functional $F_g \in (L_{p,q;b})^* = L_{p',q';b^{-1}}$ defined as

$$F_g(f) = \int f\, g d\mu$$

for all $f \in L_{p,q;b}(\Omega, X)$. For each $g \in L_{p',q';b^{-1}}(\omega, X^*)$, there exists a unique $T^{-1}(\mathcal{A})$ measurable function $E(g)$ such that

$$\int f\, g d\mu = \int f\, E(g) d\mu,$$

for each $T^{-1}(\mathcal{A})$ measurable function f for which the left integral exists. $E(g)$ is called the conditional expectation [11] of g with respect to $T^{-1}(\mathcal{A})$. The operator P_T defined as

$$P_T g = f_T . E(g) \circ T^{-1}$$

is called Frobenius Perron and f_T is the Radon-Nikodym derivative of μT^{-1} with respect to μ. It satisfies the property

$$E(g) \circ T^{-1} = f \text{ if and only if } E(g) = f \circ T$$

Let T be a non-singular measurable transformation on Ω then the composition operator C_T from $L_{p,q;b}(\Omega, X)$ into the space of strongly measurable functions $L(\Omega, X)$ is given by

$$(C_T f)(\omega) = f(T(\omega))$$

for all $\omega \in \Omega$. An operator T is called Fredholm if $R(T)$ is closed, dim $N(T) < \infty$ and *dim* $N(T^*) < \infty$ where $R(T)$, $N(T)$ and $N(T^*)$ denote the range, kernel and cokernel of T. B(X) denotes the space of all bounded linear operators on X. Multiplication operators on this space are already studied in [5] and on different spaces in [1,2,3,6,7,8,9,12]. In this paper, we discuss about the composition operators on the Lorentz-Karamata-Bochner space and study its various properties like boundedness, closedness and compactness.

2. Composition Operators

Theorem 2.1. *A non-singular transformation* $T : \Omega \to \Omega$ *induces the composition operator* C_T *if and only if for some k > 0,*

$$\mu T^{-1}(A) \le k\mu(A)$$

for $A \in \mathcal{A}$.

Proof. Suppose that the composition operator is bounded on $L_{p,q;b}(\Omega, X)$. Then there exists $K > 0$ such that

$$\|C_T f\|_{p,q;b} \le K \|f\|_{p,q;b}$$

Let x_0 be the fixed element of X with $\|x_0\| = 1$. Define the characteristic function χ_A for each measurable subset A of $\mathcal{A}$ by,

$$\chi_A(\omega) = \begin{cases} x_0, & \text{if } \omega \in A \\ 0, & \text{otherwise} \end{cases}.$$

Then we find that

$$\|\chi_A\|^*(t) = \chi_{[0,\mu(A)]}(t)$$

and

$$\|\chi_A\|^{**}(t) = \begin{cases} 1, & \text{if } \mu(A) \le t \\ \dfrac{\mu(A)}{t}, & \text{if } \mu(A) \le t \end{cases}.$$

This gives

$$\|\chi_A\|_{p,q;b} \approx (\mu(A))^{\frac{1}{p}} \left(\gamma_b^m(\mu(A))\right).$$

and

$$\|C_T \chi_A\|_{p,q;b} \approx \left(\mu\left(T^{-1}(A)\right)\right)^{\frac{1}{p}} \left(\gamma_b^m\left(\mu\left(T^{-1}(A)\right)\right)\right).$$

Thus, we get

$$\|C_T \chi_A\|_{p,q;b} \le K \|\chi_A\|_{p,q;b}$$
$$\left(\mu\left(T^{-1}(A)\right)\right)^{\frac{1}{p}} \left(\gamma_b^m\left(\mu\left(T^{-1}(A)\right)\right)\right)$$
$$\le K(\mu(A))^{\frac{1}{p}} \left(\gamma_b^m(\mu(A))\right)$$
$$\mu\left(T^{-1}(A)\right) \le k\mu(A), where$$
$$k = \left(\frac{\left(\gamma_b^m(\mu(A))\right)}{\left(\gamma_b^m\left(\mu\left(T^{-1}(A)\right)\right)\right)} \right)^p.$$

Conversely, suppose the given condition holds. Then

$$\mu_{C_T f}(s) = \mu\{\omega \in \Omega : \|C_T f\|(\omega) > s\}$$
$$= \mu\{\omega \in \Omega : \|f \circ T\|(\omega) > s\}$$
$$= \mu T^{-1}\{\omega \in \Omega : \|f\|(\omega) > s\}$$
$$\le k\mu\{\omega \in \Omega : \|f\|(\omega) > s\}$$
$$= \mu_f(s)$$

and

$$\|C_T f\|^*(t) = \inf\{s > 0 : \mu_{C_T f}(s) \le t\}$$
$$\le \inf\{s > 0 : k\mu_f(s) \le t\} = k\|f\|^*(t)$$

Thus

$$\|C_T f\|^*(t) \le k\|f\|^*(t).$$

Also

$$\|C_T f\|^{**} \le k\|f\|^{**}(t).$$

Therefore

$$\|C_T f\|_{p,q;b} = \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t)\|C_T f\|^{**}(t) L_q(0,\infty)\right\|$$

$$\le \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) k\|f\|^{**}(t) L_q(0,\infty)\right\| = k\|f\|_{p,q;b}.$$

Thus, C_T is a bounded operator on $L_{p,q;b}(\Omega, X)$.

Corollary 2.2. *A measurable transformation T induces the composition operator C_T on $L_{p,q;b}(\Omega, X)$ if and only if μT^{-1} is absolutely continuous with respect to μ and f_T belongs to $L^\infty(\Omega)$.*

Theorem 2.3. *If C_T is the composition operator on $L_{p,q;b}(\Omega, X)$. Then C_T is measure preserving if and only if C_T is an isometry.*

Proof. Suppose that T is measure preserving them

$$\mu T^{-1}(A) = \mu(A)$$

for all $A \in \mathcal{A}$.

The distribution function of C_T becomes

$$\mu_{C_T f}(s) = \mu\{\omega \in \Omega : \|C_T f\|(\omega) > s\}$$
$$= \mu\{\omega \in \Omega : \|f \circ T\|(\omega) > s\}$$
$$= \mu\{\omega \in \Omega : \|f\|(\omega) > s\}$$
$$= \mu_f(s)$$

and

$$\|C_T f\|^*(t) = \|f\|^*(t).$$

Also

$$\|C_T f\|^{**} = \|f\|^{**}(t).$$

This gives

$$\|C_T f\|_{p,q;b} = \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t)\|C_T f\|^{**}(t) L_q(0,\infty)\right\|$$

$$= \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t)\|f\|^{**}(t) L_q(0,\infty)\right\|$$

$$= \|f\|_{p,q;b}.$$

Converse of the theorem is obvious.

Example 1. *Let $\Omega = \mathbb{R}$ with Lebesgue measure and X be any Banach space. Define*

$$T(\omega) = a\omega + b \quad a \ne 0,1$$

Then T is a non-singular transformation on Ω which is not measure preserving. Hence C_T is not an isometry on $L_{p,q;b}(\Omega, X)$.

Theorem 2.4. *If C_T is a composition operator on $L_{p,q;b}(\Omega, X)$. Then C_T has closed range if and only if there exists ϵ > 0 such that $f_T(\omega) \ge \epsilon$ for almost all $\omega \in S$, the support of f_T.*

Proof. Suppose f_T is bounded away from zero then there exists a positive real number ϵ, such that

$$f_T(\omega) \ge \epsilon$$

for almost all $\omega \in S$.

$$\mu \circ T^{-1}(A) = \int_A f_T(\omega)\, d\mu \ge \epsilon\mu(A)$$

where

$$A = \{\omega \in S : |f(\omega)| > s\}$$
$$\mu_{C_T f}(s) = \mu\{\omega \in \Omega : |f \circ T(\omega)| > s\}$$
$$= \mu T^{-1}\{\omega \in S : |f(\omega)| > s\}$$
$$\ge \epsilon\mu\{\omega \in S : |f(\omega)| > s\}$$
$$\mu_{C_T f}(s) \ge \epsilon\, \mu_f(s)$$

and

$$\|f \circ T\|^*(\epsilon t) \ge \|f\|^*(t), \text{ for all } t \ge 0$$

and

$$\|f \circ T\|^{**}(\epsilon t) \ge \|f\|^{**}(t), \text{ for all } t \ge 0$$

This gives

$$\|C_T f\|_{p,q;b} \ge \epsilon^{\frac{1}{p}} \|f\|_{p,q;b}.$$

Thus, C_T has closed range.

Conversely, C_T has closed range then there exists $\epsilon > 0$ such that

$$\|C_T f\|_{p,q;b} \ge \epsilon\, \|f\|_{p,q;b}$$

for all $f \in L_{p,q;b}(S)$.

Choose a natural number *n* such that $\frac{1}{n} < \epsilon$. Let if possible, $\mu(B) > 0$ where $B = \left\{x \in S : f_T(x) < \frac{1}{n^p}\right\}$.

Then $\mu(B) < \frac{1}{n^p}\mu(B)$. Then

$$\|\chi_B \circ T\|^*(t) \le \frac{1}{n^p}\|\chi_B\|^*\left(n^p T\right), \text{ for all } t > 0$$

and

$$\|\chi_B \circ T\|^{**}(t) \le \frac{1}{n^p}\|\chi_B\|^{**}\left(n^p T\right), \text{ for all } t > 0$$

This gives

$$\|C_T\chi_B\|_{p,q;b}^q \le \frac{1}{n^p}\|\chi_B\|_{p,q;b}^q$$

which is a contradiction. Hence f_T is bounded away from zero.

Theorem 2.5. *If C_T is a composition operator on $L_{p,q;b}(\Omega, X)$. Then C_T has dense range in $L_{p,q;b}\left(\Omega, T^{-1}(\mathcal{A}), \mu\right)$.*

Proof. We will consider two cases:

Case 1. When $\mu(\Omega) < \infty$. Then $\chi_A \in L_{p,q;b}\left(\Omega, T^{-1}(\mathcal{A}), \mu\right)$ so we can obtain $B \in \mathcal{A}$ such that

$$\chi_A = \chi_{T^{-1}(B)} = C_T\chi_B .$$

Thus C_T belong to the range of C_T and hence all simple functions of $L_{p,q;b}\left(\Omega, T^{-1}(\mathcal{A}), \mu\right)$ belong to $R(C_T)$ where $R(C_T)$ denotes the range of C_T. Hence, range of C_T is dense in $L_{p,q;b}\left(\Omega, T^{-1}\mathcal{A}, \mu\right)$.

Case 2. When $\mu(\Omega) = \infty$. Let $g \in R(C_T)$ then there is a sequence of functions $\langle g_n \rangle$ in $R(C_T)$ converging to g in $L_{p,q;b}(\Omega, \mathcal{A}, \mu)$. Since $g_n \in R(C_T)$,

$$g_n = C_T f_n .$$

Clearly, each g_n is $T^{-1}(\mathcal{A})$ measurable and hence g is also $T^{-1}(\mathcal{A})$ measurable. Now suppose $g = \chi_A$. By adjusting f on a set of measure zero, suppose $A = T^{-1}(B)$ for some $B \in \Omega$. Since $(\Omega, \mathcal{A}, \mu)$ is a σ-finite space

$$B = \bigcup_{n=1}^{\infty} B_n$$

where $\mu(B_n) < \infty$ for each n and $\langle B_n \rangle$ is an increasing sequence of measurable sets.

This gives

$$\left\|C_T\chi_{B_n} - f\right\| = \left\|C_T\chi_{B_n} - C_T\chi_B\right\| = \left\|\left(\chi_{B_n} - \chi_B\right)\circ T\right\|$$

which converges to zero. Thus $R(C_T)$ is dense in $L_{p,q;b}\left(\Omega, T^{-1}(\mathcal{A}), \mu\right)$.

Theorem 2.6. *T inducing the composition operator C_T on $L_{p,q;b}(\Omega, \mathcal{A}, \mu)$ is a surjection if and only if f_T is bounded away from zero on its support and $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

Proof. Suppose C_T is a surjection. Then from the last theorem C_T has closed range if and only if f_T is bounded away from zero on its support. Let $A \in \mathcal{A}$ be of finite measure. Since C_T is a surjection there exist $f \in L_{p,q;b}(\Omega, \mathcal{A}, \mu)$ such that $C_T f = \chi_A$. Let

$$B = \{\omega : \omega \in \Omega \ \text{ and } \ f(\omega) = 1\} .$$

Then

$$C_T\chi_B = \chi_A .$$

Hence, $T^{-1}(B) = A$. Thus $\mathcal{A} \subseteq T^{-1}(\mathcal{A})$. Thus $\mathcal{A} = T^{-1}(\mathcal{A})$. Converse is obvious.

Corollary 2.7. *A composition operator C_T on $\mathbb{B}(\Omega, X)$, has dense range if and only if $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

Theorem 2.8. *If T induces a composition operator on $\doteq(\Omega, X)$, then C_T^*, the adjoint of C_T is P_T.*

Proof. Let $A \in \mathcal{A}$ be such that $\mu(A) < \infty$. Then for $g \in L_{p',q';b^{-1}}$

$$\left(C_T^* Fg\right)(\chi_A) = F_g(C_T\chi_A) = \int (\chi_A \circ T).g\,d\mu$$

$$= \int E(g).\chi_A \circ T d\mu = \int E(g)\circ T^{-1}.\chi_A d\mu T^{-1}$$

$$= \int E(g)\circ T^{-1}.\chi_A f_T d\mu = \int F_{E(g)\circ T^{-1} f_T(\chi_A)}$$

By identifying $g \in L_{p',q';b^{-1}}$ with the functional $F_g \in \left(L_{p,q;b}\right)^* = L_{p',q';b^{-1}}$, we get

$$C_T^* g = \left(E(g)\circ T^{-1}\right).f_T = P_T g .$$

Theorem 2.9. *If T induces a composition operator on $\mathcal{B}(\Omega, X)$, then $N\left(C_T^*\right)$ is either zero dimensional or infinite dimensional.*

Proof. Suppose $g \in N\left(C_T^*\right)$ and $g \ne 0$. Let

$$A = \{\omega \in \Omega : g(\omega) \ne 0\}$$

then $\mu(A) \ne 0$. Let $\langle A_n \rangle$ be a sequence of disjoint measurable subsets of A such that

$$A = \bigcup_{n=1}^{\infty} A_n$$

where $\mu(A_n) < \infty$. For each $n \in \mathbb{N}$, let $g_n = g.\chi_A \circ T$. For each n,

$$C_T^*(g_n) f = \int (g.\chi_A \circ T)(f \circ T)d\mu$$

$$= \int g.(\chi_A f \circ T)d\mu = C_T^*(g)(\chi_A f) = 0.$$

Therefore $\{g_n : n \ge 1\}$ is a linearly independent subset of $N\left(C_T^*\right)$. Hence, if $N\left(C_T^*\right)$ is not zero dimensional, it is infinite dimensional.

Theorem 2.10. *If T induces a composition operator on $\mathcal{B}(\Omega, X)$. Then C_T is invertible if and only if C_T is Fredholm.*

Proof. If C_T is invertible then C_T is Fredholm. Conversely, let C_T be Fredholm then $N(C_T)$ and $N\left(C_T^*\right)$ are both finite dimensional and are of zero dimension. Therefore C_T is injective and has dense range. Since $R(C_T)$ is closed, therefore C_T is surjective. Thus C_T is invertible.

Definition 2. *For a strongly measurable function* $f:\Omega \to \mathcal{B}(X)$*, the set*

$$ess_f = \left\{\lambda \in \Omega : \mu\left(\left\| f(\omega) - \lambda \right\| < \epsilon\right) \neq 0 \ \forall \ \epsilon > 0\right\}$$

is called the essential range of f.

Theorem 2.11. **[11]** *If* C_T *is a composition operator on* $L_{p,q;b}(\Omega, X)$*, then the following are equivalent:*

(i) C_T *is injective.*

(ii) f *and* $f \circ T$ *have the same essential ranges for every* $f \in L_{p,q;b}(\Omega, X)$.

(iii) μ *is absolutely continuous with respect to* $\mu \circ T^{-1}$.

(iv) f_T *is different from zero almost everywhere.*

References

[1] S.C. Arora, G. Datt and S. Verma, Multiplication and composition operators on Lorentz-Bochner spaces, Osaka J. Math. 45 (2008), 629-641.

[2] S.C Arora, G. Datt and S. Verma, Multiplication operators on Lorentz spaces, Indian J. Math. 48 (2006), 317-329.

[3] P. Bala, A. Gupta, N. Bhatia, Multiplication operators on Orlicz-Lorentz sequence spaces, Int. Journal of Math. Analysis 7 (2013), no. 30, 1461-1469.

[4] D.E. Edmunds and W.D. Evans, Hardy operators, function spaces and embeddings, Springer Monographs in Mathematics Springer-Verlag, Belin, 2004.

[5] I. Eryilmaz, Multiplication operators on Lorentz-Karamata-Bochner spaces, Math. Slovaca 62 (2012), no. 2, 293-300.

[6] B.S. Komal and S. Gupta, Multiplication operators between Orlicz spaces, Integral Equations Operator Theory 41 (2001), 324-330.

[7] V.N. Mishra, Some problems on Approximations of Functions in Banach spaces, PhD Thesis (2007) Indian Instiitute of Technology, Roorkee-247 667, Uttarakhand, India.

[8] V. N. Mishra, K. Khatri, L. N. Mishra, and Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, Journal of Inequalities and Applications 2013, 2013: 586.

[9] V. N. Mishra and L. N. Mishra, “Trigonometric Approximation of Signals (Functions) in Lp (p≥1)-Norm,” International Journal of Contemporary Mathematical Sciences, Vol. 7, No. 19, 2012, pp. 909-918.

[10] E.Seneta, Regularly Varying Functions, Lecture Notes in Math. 508, Springer-Verlag, Berlin, 1976.

[11] R.K. Singh and J.S. Manhas. Composition operators on function spaces, North Holland Math. Studies 179, North Holland, Amsterdam, 1993.

[12] H. Takagi and K. Yokouchi, Multiplication and composition operators between two Lp-spaces, In: Function Spaces (Edwardsville, IL, 1998). Contemp. Math. 232, Amer. Math. Soc., Providence, RI, 1999, pp. 321-338.