MUMFORD'S CONJECTURE - A TOPOLOGICAL OUTLOOK

Ulrike Tillmann

Abstract. We give an expository account of the proof of Mumford's conjecture on the stable, rational cohomology of moduli spaces of algebraic curves in its generalized form emphasizing some of the ideas that led up to it.

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1. A historical introduction.

This is in essence an expanded version of my talk at a conference in honor of David Mumford on the occasion of his 70th birthday in 2007. Coincidentally it was also the 150th anniversary of Riemann's influential paper *Theorie der Abel'schen Funktionen* [R], the starting point for the study of moduli spaces.

In his paper Riemann considers how the complex structure of the surfaces associated to a multi-valued complex function changes when one continuously varies the parameters of the function. He concludes that when the genus of the surface is $g \geq 2$ the isomorphism class depends on $3g-3$ complex variables, which he calls "Moduln" of the classes. Thus Riemann understands the complex dimension of his space to be $3g-3$. He also introduces its name, moduli space, into the mathematical literature.

The moduli space $\mathcal{M}_g$ of Riemann surfaces (or complex curves) of genus g has several constructions. One of these comes from complex analysis via Teichmüller spaces. We will discuss this in more detail below as it relates easily to the topological point of view. But it was Mumford who introduced $\mathcal{M}_g$ into algebraic geometry. A precise definition and construction of the coarse moduli space for smooth complex curves was given in his far-reaching 1965 book *Geometric Invariant Theory*, [M1]. A few years later he constructed a natural compactification $\overline{\mathcal{M}}_g$ in his paper with Deligne [DM]. In this construction, the added points in the compactification again correspond to natural geometric objects, so called stable curves.

The moduli space and its compactification were much studied by algebraic geometers from this time on, but for progress on the (co)homology of $\mathcal{M}_g$ we have to wait until the early 1980s when we suddenly see several developments in topology and geometry at the same time.

On the algebraic geometric side, Mumford initiates the systematic study of the Chow ring of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ in his 1983 paper *Towards an Enumerative Geometry of the Moduli Space of Curves*, [M2]. The idea here is, we quote, "to take as a model for this the enumerative geometry of the Grassmannians", and in analogy to the Chern classes Mumford defines and studies certain tautological classes κ_i.

The rational cohomology of $\mathcal{M}_g$ is the same as that of the mapping class group Γ_g, the group of isotopy classes of diffeomorphisms of a surface of genus g, as we will explain below. This group in turn has been studied extensively in algebra and topology. At around the same time as Mumford studied the Chow ring of $\mathcal{M}_g$, Hatcher and Thurston [HT] found a presentation for Γ_g. This work then lead to ground-breaking work by Harer. He computed the second cohomology (thus confirming a conjecture by Mumford on the Picard group of the moduli spaces) [H1], and, what is more important for this paper, proved in [H2] that their cohomology is independent of the genus g in degrees small relative to g. Miller [Mi] in turn, and independently Morita [Mo], used this stability result to show that in the stable range the rational cohomology contains a polynomial algebra on Mumford's κ_i classes. Mumford suggested in [M2] that

> "... it seems reasonable to guess, in view of the results of Harer and Miller, that in low dimensions $H^i(\mathcal{M}_g) \otimes \mathbb{Q}$ is a polynomial algebra in the κ_i."

This has since been known as the Mumford conjecture and is now a theorem due to Madsen and Weiss [MW].

We will explain some of the ideas that led to the proof of Mumford's conjecture. This will include some discussion of topological and conformal field theory. The first proof is a tour de force. Since then the arguments have been simplified significantly in [GMTW]. Our discussion will be based on that.

Several expository accounts of the proof of Mumford's conjecture and related results have been written. Among these we recommend [K] and [Mad1]. We also highly recommend Hatcher's account of the proof in [Ha] which is based on further simplifications and generalizations due to Galatius [G2] and Galatius and Randal-Williams [GRW].

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2. Topological models for $\mathcal{M}_g$ and $\mathcal{S}$.

The purpose of this section is to construct a topological moduli space $\mathcal{M}_g^{top}$ of surfaces and compare it with Riemann's moduli space. It will have the property that any smooth map $f : X \to \mathcal{M}_g^{top}$ defines a smooth bundle of genus g surfaces over a manifold X. Indeed, $\mathcal{M}_g^{top}$ is the space of un-parameterized genus g surfaces smoothly embedded in $\mathbb{R}^\infty$.

2.1. Construction.

The construction of the topological moduli space can be made quite generally for any closed orientable manifold W. We let $\mathrm{Emb}(W;\mathbb{R}^n)$ be the space of smooth embeddings of W in n-dimensional real Euclidean space with the C^∞-topology, and let

$$\mathrm{Emb}(W;\mathbb{R}^\infty) := \lim_{n\to\infty} \mathrm{Emb}(W;\mathbb{R}^n)$$

be the union of these. It is the space of all smooth *parameterized* sub-manifolds in $\mathbb{R}^\infty$ diffeomorphic to W. By the Whitney embedding theorem [W], $\mathrm{Emb}(W;\mathbb{R}^\infty)$ is contractible. The group of orientation preserving diffeomorphisms $\mathrm{Diff}(W)$ of W acts freely on it by precomposition: for a diffeomorphism ϕ and an embedding h we define $h.\phi := h \circ \phi$. The topological moduli space is the associated orbit space,

$$\mathcal{M}^{top}(W) := \mathrm{Emb}(W;\mathbb{R}^\infty)/\mathrm{Diff}(W),$$

the space of all smooth *un-parameterized* sub-manifolds of $\mathbb{R}^\infty$ diffeomorphic to W.

Its associated universal W-bundle is given by the Borel construction

$$\mathrm{Emb}(W;\mathbb{R}^\infty) \times_{\mathrm{Diff}(W)} W \longrightarrow \mathcal{M}^{top}(W)$$

which restricted to $\mathrm{Emb}(W;\mathbb{R}^n)$ is a smooth fiber bundle in the infinite dimensional smooth category (see [KM, Section 44]). In particular, our topological moduli space is a classifying space for the group of diffeomorphisms,

$$\mathcal{M}^{top}(W) = B\mathrm{Diff}(W).$$

We need a variant of the above construction when W has non-empty boundary. More precisely, W will be a cobordism between an incoming manifold M_0 and an outgoing manifold M_1. In other words, the boundary of W is subdivided into two parts such that $\partial W = \overline{M}_0 \sqcup M_1$ where $\overline{M}_0$ denotes the manifold M_0 with the opposite orientation. Consider the space of smooth embeddings $\mathrm{Emb}(W, \overline{M}_0 \sqcup M_1; [a_0, a_1] \times \mathbb{R}^{\infty-1})$ that map a collar of M_0 to a standard collar of $\{a_0\} \times \mathbb{R}^{\infty-1}$, and similarly M_1 to a standard collar of $\{a_1\} \times \mathbb{R}^{\infty-1}$. The group of diffeomorphisms $\mathrm{Diff}^{\Omega}(W)$ of W that map $\overline{M}_0$ to $\overline{M}_0$, M_1 to M_1 and preserve a collar acts freely on this space of embeddings. Define

$$\mathcal{M}^{top,\Omega}(W) := \mathrm{Emb}(W, \overline{M}_0 \sqcup M_1; [a_0, a_1] \times \mathbb{R}^{\infty-1}, \partial)/\mathrm{Diff}^{\Omega}(W). \tag{2.1}$$

Again, the embedding space is contractible and $\mathcal{M}^{top,\Omega}(W)$ has the homotopy type of $B\mathrm{Diff}^{\Omega}(W)$. On the other hand, if we restrict the embedding space to those that have a fixed image of M_0 and M_1, the resulting orbit space has the homotopy type of $B\mathrm{Diff}(W, \partial)$, the classifying space of those diffeomorphisms that fix a collar of the boundary pointwise, see [MT, Section 2]. We will in particular be interested in the case when W is an oriented surface $F_{g,n}$ of genus g with n boundary components.

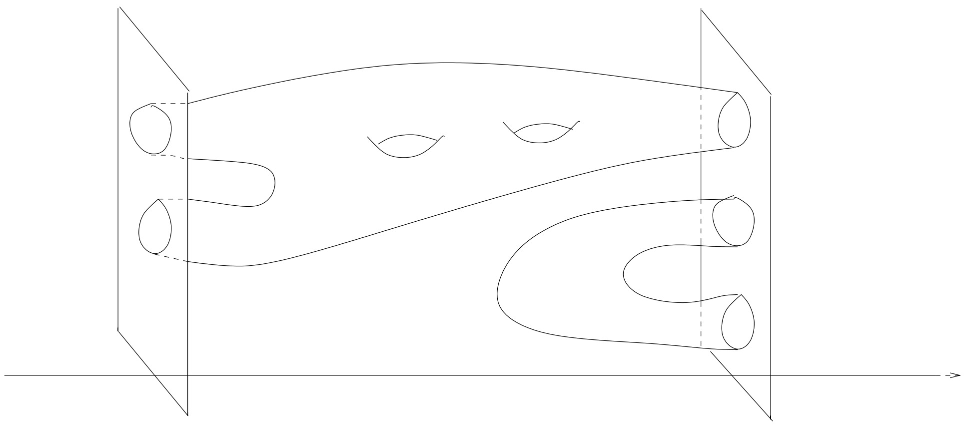

Figure 1: *Embedded cobordism W with $\partial W = \overline{M}_0 \sqcup M_1$.*

2.2. The relation to Riemann's moduli space $\mathcal{M}_g$.

There are several constructions of the moduli space of Riemann surfaces $\mathcal{M}_g$. We have already mentioned the construction by Mumford via geometric invariant theory [M1]. A somewhat different approach is via Teichmüller theory.

Let $g > 1$, and consider the space $\mathcal{S}(F_g)$ of almost complex structures on the oriented surface F_g which are compatible with the orientation. This is the space of sections of the $\mathrm{GL}_2^+(\mathbb{R})/\mathrm{GL}_1(\mathbb{C})$ bundle associated to the tangent bundle of F_g. On surfaces, every almost complex structure integrates to a complex structure.

(This follows immediately, for example, from the Newlander-Nirenberg theorem [NN] because the Nijenhuis tensor vanishes identically for surfaces.) One identifies two complex structures on F_g when one is the pull-back via a diffeomorphism of the other. Thus the moduli space of complex surfaces is the orbit space

$$\mathcal{M}_g := \mathcal{S}(F_g)/\mathrm{Diff}(F_g).$$

Let $\mathrm{Diff}_1(F_g)$ denote the connected component of the identity in $\mathrm{Diff}(F_g)$. This is a normal subgroup and its quotient is the mapping class group $\Gamma_g := \pi_0\mathrm{Diff}(F_g)$. If we define Teichmüller space as

$$\mathcal{T}_g := \mathcal{S}(F_g)/\mathrm{Diff}_1(F_g)$$

then

$$\mathcal{M}_g = \mathcal{T}_g/\Gamma_g.$$

It is well-known that $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$ and hence contractible. Furthermore, Earle and Eells [EE] show that the natural projection from $\mathcal{S}(F_g)$ to $\mathcal{T}_g$ has local sections so that

$$\mathrm{Diff}_1(F_g) \longrightarrow \mathcal{S}(F_g) \longrightarrow \mathcal{T}_g$$

is a fiber bundle. It follows that the fiber $\mathrm{Diff}_1(F_g)$ is contractible as both $\mathcal{S}(F_g)$ and $\mathcal{T}_g$ are. On the other hand, the group of holomorphic self-maps of a surface is finite (of order less than $84(g-1)$ by Hurwitz' formula). Thus all the stabilizer groups of the action of Γ_g on $\mathcal{T}_g$ are finite. Furthermore, the action is proper. One can use this to prove that $\mathcal{M}_g$ has the same rational cohomology as the classifying space $B\Gamma_g$. Hence,

$$\boxed{\mathcal{M}_g^{top} \simeq B\mathrm{Diff}(F_g) \simeq B\Gamma_g \sim_{\mathbb{Q}} \mathcal{M}_g.} \tag{2.1}$$

Here $\sim_{\mathbb{Q}}$ indicates that the two spaces have isomorphic rational cohomology.

The constructions above can be extended to define a moduli space $\mathcal{M}_{g,n}^k$ of surfaces with k punctures and n marked points with a unit tangent vector. For this we need to consider the group of diffeomorphisms that fix the marked points as well as the unit vectors. Note that this group of diffeomorphisms is homotopy equivalent to its subgroup of diffeomorphisms that fix small disks around the n marked points with unit tangent vector. The associated mapping class group is denoted by $\Gamma_{g,n}^k$. Earle and Eells' result also applies to this case whenever $2-2g-n-k<0$, see [ES]. Holomorphic self maps of degree one that fix a point and a tangent vector at that point have to be the identity (by the Identity Theorem). Thus for $n>0$, the stabilizer subgroups for the action of this mapping class group on the corresponding Teichmüller space are all trivial. Similarly, the maximal number of fixed points of a holomorphic map of a surface of genus g is $2g+2$, and thus again for $k>2g+2$ the stabilizer subgroups are trivial. We thus have the following homotopy equivalences

$$\boxed{\mathcal{M}_{g,n}^{top,k} \simeq B\mathrm{Diff}(F_{g,n}^k) \simeq B\Gamma_{g,n}^k \simeq \mathcal{M}_{g,n}^k} \tag{2.2}$$

when $n>0$ or $k>2g+2$; otherwise the equivalence $\simeq$ on the right has to be replaced again by a rational equivalence $\sim_{\mathbb{Q}}$.

2.3. Cobordism categories.

As we will see in section 4.4, a key new idea in studying the topology of moduli spaces was to study them all together as part of a category. The motivation for this came from conformal and topological field theory.

Segal's category: Segal's axiomatic approach [S4] to conformal field theory is based on a symmetric monoidal category $\mathcal{S}$ of Riemann surfaces. It has one object for each natural number n representing n disjoint copies of the unit circle. Its space of morphisms from n to m is the union of the moduli spaces of complex surfaces (possibly not connected) of genus g with n source and m target boundary components, and in particular contains all moduli spaces $\mathcal{M}_{g,n+m}$ for $g \geq 0$. The composition is given by gluing, and the monoidal structure is given by disjoint union, which is clearly symmetric. In Segal's setting a conformal field theory is a symmetric monoidal functor from $\mathcal{S}$ to an appropriate category of topological vector spaces in which the monoidal structure is defined by the tensor product.

Cobordism categories: In a similar way, the topological moduli spaces also form a category, which we denote by Cob_d. In Cob_d an embedded d-dimensional cobordism W as in figure 1 is a morphism between its boundary components M_0 and M_1. More precisely, the space of objects in Cob_d is the union over all $a \in \mathbb{R}$ of the space of embedded $(d-1)$-dimensional closed, oriented manifolds in $\{a\} \times \mathbb{R}^{\infty-1}$. Similarly, the space of morphisms is the union for all intervals $[a_0, a_1]$ of the spaces of embedded oriented cobordisms as defined in (2.1). Two morphisms $W \subset [a_0, a_1] \times \mathbb{R}^{\infty-1}$ and $W' \subset [a_0', a_1'] \times \mathbb{R}^{\infty-1}$ are composable if the target boundary M_1 of W is the same as the source boundary M_0' of W'. In particular, we must have $a_1 = a_0'$. The composition is then given by gluing the two cobordism along their common boundary:

$$W \cup W' \quad \subset \quad [a_0, a_1'] \times \mathbb{R}^{\infty-1}.$$

Given two objects M_0 and M_1 (with $a_0 < a_1$) the space of morphisms between them is

$$\mathrm{Cob}_d(M_0, M_1) \simeq \coprod_W B\mathrm{Diff}(W, \partial),$$

where the disjoint union is taken over all diffeomorphism classes of cobordisms W with boundary $\overline{M}_0 \sqcup M_1$. In particular, when $d = 2$ and M_0 and M_1 consist of respectively n and m circles, this will contain spaces homotopic to $B\Gamma_{g,n+m}$ for all $g \geq 0$.

The category Cob_d is a model for the source category of d-dimensional topological field theories. Such theories are in an appropriate sense symmetric monoidal functors from Cob_d to a category of vector spaces or some generalization thereof. We refer to [L] were these theories are studied in detail. Indeed, Theorem 7.1 below formed the inspiration for the main theorem in [L] that classifies so called extended topological field theories.

3. Classifying spaces and group completion - a tutorial

We are taking a detour here to explain some of the topological machinery that will be used later. Indeed, essential step in the proof of the Mumford conjecture is to identify the classifying space of the cobordism category Cob_2 with help of the group completion theorem. Nevertheless, the reader might find it more convenient to skip this section and come back to it when the need arises.

3.1. The nerve of a category.

Like moduli spaces, classifying spaces of groups are representing spaces: The set of homotopy classes of maps from a space X to the classifying space BG of a group G is in one to one correspondence with the set of isomorphism classes of principal G-bundles on X. In particular, BG has a universal G-bundle (corresponding to the identity map).

Given a group G, the classifying space is (by definition) only determined up to homotopy, and there are many ways of constructing BG. In Section 2, in our identification of the topological moduli space $\mathcal{M}^{top}(W)$ with $B\mathrm{Diff}(W)$ we relied on the fact that BG is the quotient space of a good, free G action on a contractible space. Many classifying spaces can be constructed ad hoc like this.

We will now present a functorial construction that can be generalized to categories. (A group G is identified with the category containing one object $*$ and morphism set G.)

The nerve $N_\bullet\mathcal{C}$ of a category $\mathcal{C}$ is a simplicial space with 0-simplices the space of objects and n-simplices the space of n composable morphisms. Boundary maps are given by composition of morphisms:

$$\partial_i(f_1,\ldots,f_n) = (f_1,\ldots,f_i \circ f_{i+1},\ldots f_n)$$

when $i \neq 0, n$, and ∂_0 and ∂_n drop the first respectively last coordinate. The i-th degeneracy map is defined by inserting an identity morphism at the i-th place. Recall, every simplicial space $X_\bullet$ has a realization

$$|X_\bullet| := (\bigsqcup_{n\geq 0} \triangle^n \times X_n)/\sim$$

where the identifications are generated by the boundary and degeneracy maps. $|X_\bullet|$ is given the compactly generated topology induced by the topology of the standard n-simplex $\triangle^n$ and the topology on X_n. The classifying space of a category is the realization of its nerve,

$$B\mathcal{C} := |N_\bullet\mathcal{C}|.$$

It is not difficult to see that this construction is functorial. It takes functors between categories to continuous maps between their classifying spaces. Furthermore, natural transformations between two functors induce homotopies between the induced maps. These construction go back to Grothendieck and in our topological setting to Segal [S1].

Proposition 3.1. *If $\mathcal{C}$ has a terminal object, then its classifying space $B\mathcal{C}$ is contractible.*

Proof: Let x_0 be a terminal object in $\mathcal{C}$. Consider the functor $F : \mathcal{C} \to \mathcal{C}$ that sends every object to x_0 and every morphism to the identity morphism of x_0. We define a natural transformation τ from the identity functor of $\mathcal{C}$ to F by setting the value at x to be the unique morphism from x to x_0. Then the diagram below commutes for all morphisms $f : x \to y$:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\tau(x)"'] & y \arrow[d, "\tau(y)"'] \\ x_0 \arrow[r, "="] & x_0. \end{tikzcd}$$

Thus τ gives rise to a retract of $B\mathcal{C}$ to a point. □

3.1.1. Example: For an object x_0 in the category $\mathcal{C}$, the over category $\mathcal{C} \downarrow x_0$ is the category with objects given by all morphisms $x \to x_0$ and morphisms between $x \to x_0$ and $y \to x_0$ given by maps $f : x \to y$ that make the obvious triangle commute. The over category has terminal object $id : x_0 \to x_0$.

The n simplices of the nerve of $\mathcal{C} \downarrow x_0$ can be identified with $n+1$-tuples of composable morphisms $(f_0, f_1, \dots, f_n)$ where the target of f_0 is x_0. When $\mathcal{C}$ is a group G then G acts freely on the nerve (and the classifying space) of $G \downarrow *$ by left multiplication on f_0, and the quotient is just the nerve of G. Thus we see that the classifying space of the category G is the quotient of a free G action on the contractible space $EG := B(G \downarrow *) \simeq *$, and is indeed a model for BG.

3.2. A functorial map and group completion.

Unlike in the case of a group G, it is not so easy to see what $B\mathcal{C}$ classifies for an arbitrary category, and we will not pursue this question here. Nevertheless we want to study the closely related question of how much information the classifying space retains about the category.

Let $\mathcal{C}(a, b)$ be the space of morphisms between two objects a and b. From the definition of the classifying space in section 3.1, we see there is a natural map from $\triangle^1 \times \mathcal{C}(a, b)$ to $B\mathcal{C}$. The adjoint map defines the important map

$$\sigma : \mathcal{C}(a, b) \longrightarrow \Omega_{a,b} B\mathcal{C}.$$

which associates to every morphism in $\mathcal{C}(a, b)$ a path in $B\mathcal{C}$ from the point corresponding to the source a to the one corresponding to the target b.

When $\mathcal{C}$ is the group G, σ gives the well-known homotopy equivalence

$$G \simeq \Omega BG.$$

For example, we have $B\mathbb{Z} \simeq \mathbb{R}/\mathbb{Z} \simeq S^1$ and hence $\Omega B\mathbb{Z} \simeq \Omega S^1 \simeq \mathbb{Z}$. The above also holds for groups up to homotopy, i.e. when $\mathcal{C}$ is a monoid M whose connected components $\pi_0 M$ form a group. Thus in these cases the classifying space contains up to homotopy the same information as the category.

More generally, when $\mathcal{C}$ is a monoid, σ is the group completion map. Recall, every (discrete) monoid M has a group completion, its Grothendieck group $\mathcal{G}(M)$. For the free monoid $M = \mathbb{N}$, one adds the inverses to obtain $\mathcal{G}(\mathbb{N}) = \mathbb{Z}$. More generally, one can construct

$$\mathcal{G}(M) = < M \,|\, m_1.m_2 = m_1 m_2 \,,\, m_1, m_2 \in M >$$

as the free group on the elements in M with relations $m_1.m_2 = m_1m_2$ for all elements $m_1, m_2 \in M$. When M is a topological monoid this purely algebraic definition is replaced by the homotopy theoretic construction ΩBM. This generalizes the discrete construction in that $\pi_0(\Omega BM) = \mathcal{G}(\pi_0(M))$.

The important but somewhat mysterious group completion theorem asserts that the homology of ΩBM is related to the homology of M. There are several versions of this theorem. The following is due to McDuff and Segal [McDS].

For simplicity assume that $M = \coprod_{n\geq 0} M_n$ is the disjoint union of connected components, one for each integer n. Right multiplication by an element in M_1 takes M_n to M_{n+1}. Let $M_\infty := \operatorname{hocolim}_{n\to\infty} M_n$ be the homotopy limit.

Group Completion Theorem 3.2. *Assume that left multiplication by any element in M defines an isomorphism on $H_*(M_\infty)$. Then*

$$H_*(\mathbb{Z} \times M_\infty) \simeq H_*(\Omega BM).$$

Note that the two spaces are generally not homotopy equivalent. While a loop space has always abelian fundamental group, M_∞ has generally a very complicated and non-abelian fundamental group. But in many cases of interest the spaces $\mathbb{Z} \times M_\infty$ and ΩBM are related by the plus construction. Recall, when the fundamental group π_1 of a space X has perfect commutator subgroup $[\pi_1, \pi_1]$ one may apply Quillen's plus construction which attaches to each generator of the perfect commutator subgroup a disk and furthermore attaches certain 3-cells so that the homology remains unchanged. The resulting space X^+ has now fundamental group $H_1(X)$ and the same homology as X.

3.2.1. Example: One of the most basic examples is the monoid $M = \coprod_{n\geq 0} B\Sigma_n$, the disjoint union of classifying spaces of symmetric groups. Then $M_\infty = B\Sigma_\infty$ where Σ_∞ is the union of all the Σ_n. In this case the Barratt-Priddy-Quillen theorem [BP][Q] identifies ΩBM with $\Omega^\infty S^\infty = \lim_{n\to\infty} \Omega^n S^n$, the space of based maps from the n-sphere to itself as $n \to \infty$. Furthermore, the commutator subgroup of Σ_∞ is the infinite alternating group which is perfect. Hence we have a homotopy equivalence

$$\mathbb{Z} \times B\Sigma_\infty^+ \simeq \Omega^\infty S^\infty.$$

Under certain circumstances, the group completion theorem for monoids can be generalized to categories in a suitable way. Indeed, if x_0 is an object in $\mathcal{C}$, consider the morphism sets $\mathcal{C}(x, x_0)$. The monoid $\mathcal{C}(x_0, x_0)$ acts on the right and we can form a homotopy limit

$$\mathcal{C}_\infty(x) = \operatorname{hocolim}_{\mathcal{C}(x_0,x_0)} \mathcal{C}(x, x_0).$$

In [T1] we noted the following generalization of the group completion theorem.

Theorem 3.3. *Assume that $\mathcal{C}$ is connected and for any object y left multiplication by any morphism in $\mathcal{C}(y, x)$ induces an isomorphism on $H_*(\mathcal{C}_\infty(x))$. Then*

$$H_*(\mathcal{C}_\infty(x)) = H_*(\Omega B\mathcal{C}).$$

3.3. Additional categorical structure and infinite loop spaces.

The nerve construction and the realization functor commute with Cartesian products (when using the compactly generated topology on product spaces). Thus a monoidal structure on $\mathcal{C}$ makes $B\mathcal{C}$ into a monoid.

But more is true. Higher categorical structure translates into higher commutativity of the multiplication on the classifying space. The following table summarizes the situation. Recall that if $B\mathcal{C}$ is connected or its connected components form a group than we do not need to take its group completion.

(3.4)

$\mathcal{C}$	$\Omega B(B\mathcal{C})$
monoidal	$\Omega -$ space
braided monoidal	$\Omega^2 -$ space
symmetric monoidal	$\Omega^\infty -$ space

We call Y an Ω^n-space if there is a space Y_n such that $Y \simeq \Omega^n Y_n := \mathrm{map}_*(S^n, Y_n)$, the space of maps from an n-sphere to Y_n that take the basepoint to the basepoint. A space Y is an infinite-loop space if it is an Ω^n-space for every n and $\Omega Y_{n+1} = Y_n$. The fact that symmetric monoidal categories give rise to infinite loop spaces is a theorem due to Segal [S3], see also [May]. The analogue for braided monoidal categories was considered later by Fiedorowicz, see [SW].

Infinite loop spaces are relatively well-behaved. In Section 6.3 we will make use of the Dyer-Lashof operations that act on the homology of infinite loop spaces. Here we just recall that every infinite loop space Y, together with a choice of deloopings, gives rise to a generalized cohomology theory by setting $h^n(X) = [X, Y_n]$, the set of homotopy classes of maps from X to Y_n. By using the adjunction between based loops and suspension we see immediately that

$$h^{n+1}(X) := [X, Y_{n+1}] = [X, \Omega Y_n] = [\Sigma X, Y_n] = h^n(\Sigma X)$$

as to be expected.

3.2.2. Example: The category of finite sets and their isomorphisms is symmetric monoidal under the disjoint union operation. Its classifying space is homotopic to M from Example 3.2.1 and its group completion $\Omega^\infty S^\infty$ is clearly an infinite loop space. The associated generalized (co)homology theory is stable (co)homotopy theory.

4. Stable (co)homology and product structures.

We recall Harer's homology stability theorem for the mapping class group and Mumford's conjecture on the rational stable (co)homology. At first it might seem surprising that the (co)homology $B\Gamma_\infty$ should be easier to understand than that of $B\Gamma_g$. The main reason for this is the existence of products.

4.1. Cohomology classes.

Consider a smooth bundle of orientable surfaces $\pi : E \to M$ over a manifold M with fiber of type F_g. Let $T^\pi E$ be the subbundle of the tangent bundle TE of E which contains all vertical tangent vectors, i.e. those in the kernel of the differential $d\pi$. This is an oriented 2-dimensional vector bundle and hence has an Euler class $e \in H^2(E;\mathbb{Z})$. Define

$$\kappa_i := (-1)^{i+1}\pi_*(e^{i+1}) \in H^{2i}(M;\mathbb{Z})$$

where the wrong way map π_* is the Gysin or 'integration over the fiber' map. These classes were first constructed in [Mu] in the algebraic-geometric setting. The topological analogues were studied in [Mi] and [Mo].

Another family of classes can be defined as follows. Let $\pi : E \to M$ be as above and consider the associated Hodge bundle $\mathcal{H}(E)$. In topological terms its fibers can be identified with $H^1(E_b;\mathbb{R}) \simeq \mathbb{R}^{2g} \simeq \mathbb{C}^g$. Define

$$s_i := i!\, ch_i(\mathcal{H}(E))$$

where $i!\, ch_i$ denotes the ith Chern character class.

The classifying map $M \to BU$ for the bundle $\mathcal{H}(E)$ can be shown to factor through

$$B\mathrm{Diff}(F_g) \simeq B\Gamma_g \to B\mathrm{Sp}(\mathbb{Z}) \to B\mathrm{Sp}(\mathbb{R}) \simeq BU$$

where the first map is given by the action of the mapping class group on the first cohomology group $H^1(F_g;\mathbb{Z})$. It is well-known by Borel [B] that for even i the classes $i!ch_i$ must vanish when pulled back to $H^*(B\mathrm{Sp}(\mathbb{Z});\mathbb{Q})$. For odd i, using the Grothendieck-Riemann-Roch theorem and the Atiyah-Singer index theorem for families respectively, Mumford [Mu] and Morita [Mo] proved the following identity in rational cohomology

$$s_{2i-1} = (-1)^i (\frac{B_i}{2i})\, \kappa_{2i-1}. \tag{4.1}$$

Here B_i denotes the ith Bernoulli number.

4.2. Homology stability.

Let $F^k_{g,n}$ be an oriented surface of genus g with n boundary components and k punctures, and let $\mathrm{Diff}(F^k_{g,n};\partial)$ its group of orientation preserving diffeomorphisms that restrict to the identity near the boundary. The mapping class group is its group of connected components

$$\Gamma^k_{g,n} := \pi_0(\mathrm{Diff}(F^k_{g,n};\partial)).$$

Let $F^k_{g,n} \hookrightarrow F^{k'}_{g',n'}$ be an inclusion such that each of the n boundary components of the subsurface either coincide with one of the boundary components of the bigger surfaces or lie entirely in its interior. By extending diffeomorphisms via the identity this inclusion induces homomorphisms of diffeomorphism groups and mapping class groups

$$\Gamma^k_{g,n} \longrightarrow \Gamma^{k'}_{g',n'}.$$

Theorem 4.2. *For $k = k'$ the induced maps on homology*

$$H_*(B\Gamma^k_{g,n}) \to H_*(B\Gamma^k_{g',n'})$$

are isomorphisms in degrees $* \leq 2g/3 - 2/3$.

The part of the homology that does not change when increasing the genus or changing the number of boundary components is called *stable.* The homology stability theorem for mapping class groups was first proved by John Harer [H2]. Subsequently, Ivanov [I] and most recently Boldsen [B] and Randal-Williams [RW1] improved the stable range. The quoted range is the best known and (essentially) best possible. For a more precise statement see also Wahl's survey in this volume [Wa2].

We are led to study the limit group

$$\Gamma^k_{\infty,n} := \lim_{g\to\infty} \Gamma^k_{g,n+1}.$$

The homology of $B\Gamma^k_{\infty,n}$ is thus the stable homology of the mapping class group. In particular it is independent of n. The dependence on k is also completely understood and follows from the independence on n. Restricting the diffeomorphisms to a neighborhood of the punctures defines a map to $\Sigma_k \wr GL^+(2,\mathbb{R}) \simeq \Sigma_k \wr SO(2)$ which records the permutation of the punctures and the twisting around them. In stable homology this is a split surjection, see [BT]. Thus by the Künneth theorem, the following can be deduced.

Theorem 4.3. *With field coefficients*

$$H_*(B\Gamma^k_{g,n}; \mathbb{F}) \simeq H_*(B\Gamma_{g,n}; \mathbb{F}) \oplus H_*(B\Sigma_k \wr SO(2); \mathbb{F})$$

in degrees $* \leq 2g/3 - 2/3$.

We may therefore now concentrate on the stable cohomology of mapping class groups of surfaces without punctures and one boundary component. (We note here that if $* \leq k/2$ the homology groups are also independent of k. This is a consequence of Theorem 4.3 and the homology stability for configuration spaces.)

4.3. Pair of pants product.

Two surfaces $F_{g,1}$ and $F_{h,1}$ can be glued to a pair of pants surface to form a surface $F_{g+h,1}$ as illustrated in Figure 2. Again by extending diffeomorphisms via the identity, this construction can be used to define a map of mapping class groups

$$\Gamma_{g,1} \times \Gamma_{h,1} \longrightarrow \Gamma_{g+h,1}$$

which in turn defines a product on the disjoint union $M = \coprod_{g\geq 0} B\Gamma_{g,1}$. With this product M is a topological monoid. The following was first noted by Ed Miller [Mi].

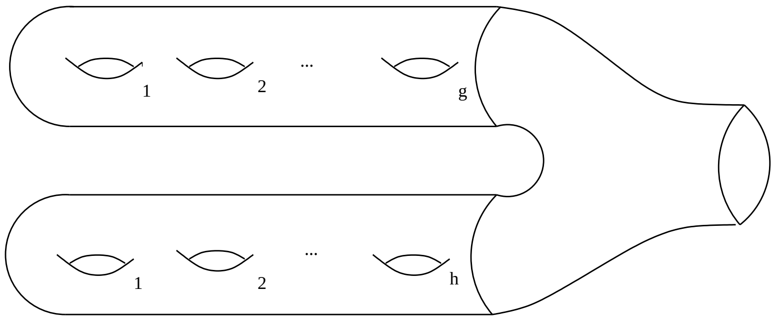

Figure 2: *Pair of pants product for surfaces.*

Proposition 4.4. *The group completion of* $M = \bigsqcup_{g\geq 0} B\Gamma_{g,1}$ *is a double loop space. In particular,*

$$H_*(\mathbb{Z} \times B\Gamma_\infty) = H_*(\Omega BM)$$

is a commutative and co-commutative Hopf algebra.

Proof: The product is commutative up to conjugation by an element $\beta_{g,h}$ in $\Gamma_{g+h,1}$ that interchanges the two 'legs' in Figure 2 by a half twist of the 'waist'. It is a standard fact from group cohomology, see [Br], that conjugation induces the identity map in homology. Thus the product is commutative in homology and the Group Completion Theorem 3.2 can be applied. This gives the claimed identity of homology groups.

But more is true. The element $\beta_{g,h}$ can be interpreted as a braid element in the mapping class group of the pair of pants surface (in this context, we allow the interchanging of boundary components). With this it is now formal to prove that the monoid M is a braided category and its group completion is a double loop space according to the table (3.4). See also [FS]. □

The homology of each $B\Gamma_{g,1}$ is finitely generated (indeed one can construct models for $B\Gamma_{g,1}$ that are finite cell complexes, see for example [Bö]). Harer's homology stability theorem implies now that also the homology of $B\Gamma_\infty$ is finitely generated in every degree, i.e. is of finite type. Rationally, finite type commutative and co-commutative Hopf algebras are polynomial algebras on even degree generators tensored exterior algebras on odd degree generators by the structure theorem of Milnor-Moore [MM]. To prove the theorem below, it is thus enough to show that the κ_i classes do not vanish and are indecomposable. This is done by Miller [Mi]. He shows that κ_i vanishes on decomposable elements in the homology and extends a method of Atiyah [A] to construct inductively surfaces bundles over $2i$-dimensional manifolds (indeed, smooth projective algebraic varieties) with non-zero κ_i number. Independently, in [Mo] Morita proves the same theorem by a more direct method.

Theorem 4.5. $H^*(B\Gamma_\infty;\mathbb{Q}) \supset \mathbb{Q}[\kappa_1, \kappa_2, \dots]$.

This led Mumford to conjecture that the above inclusion is indeed an equality as we quoted in the introduction.

4.4. Infinite loop space structure.

We saw in Proposition 4.4 that by considering the union of all $B\Gamma_{g,1}$ one could show that $B\Gamma_\infty^+$ has the homotopy type of a double loop spaces. This was thought to be best possible, see for example [FS]. The study of conformal (and topological) field theories on the other hand led one to consider surfaces of any genus with an arbitrary number of boundary components and several connected components at the same time. This we will see below allows one to show that $B\Gamma_\infty^+$ is an infinite loop space [T1]. This discovery in turn led Madsen to guess the homotopy type of $B\Gamma_\infty^+$ and to the generalized Mumford conjecture in [MT], which now is the Madsen-Weiss theorem, see Theorem 5.1.

As we explained in section 2.3, Segal's category $\mathcal{S}$ of Riemann surfaces plays a central role in conformal field theory, and the category Cob_d in topological field theory. One way to study these categories is to study their associated classifying spaces. The question then is (1) whether we can understand the homotopy type of $B\mathcal{S}$ and $B\mathrm{Cob}_d$ and (2) how it relates to that of their morphisms spaces. The first question is answered for Cob_d in Theorem 7.1 below. The latter can be answered for $d = 2$ and when we restrict the category $\mathcal{S}$ and Cob_2 to subcategories $\mathcal{S}^\partial$ and Cob_2^∂ in which every connected component of every morphisms has at least one target boundary. This means in particular that no closed surface will be part of any morphisms, nor will the disk when considered as a cobordism from a circle to the empty manifold. The conformal and topological theories defined on these subcategories will therefore not necessarily contain a trace, and define - in the terminology of [S4] - non-compact theories. A version of the following theorem was first proved in [T1].

Theorem 4.6. $\Omega B\mathit{Cob}_2^\partial \simeq \Omega B\mathcal{S}^\partial \simeq \mathbb{Z} \times B\Gamma_\infty^+$.

Proof: We want to apply Harer's homology stability theorem and the group completion theorem for categories. Consider $\mathcal{S}^\partial$. The surfaces in $\mathcal{S}^\partial(n,1)$ have to be all connected because there is only one target boundary circle. Indeed, using the homotopy equivalences (2.2),

$$\mathcal{S}^\partial(n,1) \simeq \coprod_{g\geq 0} B\Gamma_{g,n+1} \quad \text{and} \quad \mathcal{S}^\partial_\infty(n) \simeq \mathbb{Z} \times B\Gamma_{\infty,n}.$$

So by Harer's homology stability, Theorem 4.2, the group completion theorem for categories, Theorem 3.3, can be applied (with $x_0 = 1$) to give the statement of the theorem. The same argument applies to Cob_2^∂. See [T1] or [GMWT; section 7] for details. □

The reason we needed to restrict to these subcategories is that Harer's theorem applies only to connected surfaces, and yields homology isomorphisms in all degrees only for surfaces of infinite genus. Thus in order to apply Theorem 3.3, (the connected components of) the colimit of morphism spaces should have the homology type of $B\Gamma_{\infty,n}$ and cannot have any other components such as those coming from closed surfaces.

These categories have another important feature. Segal's category $\mathcal{S}$ is symmetric monoidal under disjoint union. (Cob_2 is also a symmetric monoidal category but only up to homotopy in a sense we will not make precise here.) This property is inherited by the subcategory $\mathcal{S}^\partial$. Thus, by the results tabled in (3.4), we have the following immediate consequence, see [T1].

Corollary 4.7. *$\mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space.*

5. Generalized Mumford conjecture.

The Mumford conjecture postulates that the stable rational cohomology of Riemann's moduli space is a polynomial algebra on the κ_i classes. The generalized Mumford conjecture lifts this identity of rational cohomology to a homotopy equivalence of spaces with the advantage that also integral and torsion information can be deduced.

We will first define certain infinite dimensional spaces in order to state the generalized Mumford conjecture and derive consequences for the rational cohomology. The motivation why these spaces are the natural ones to look at can be found in section 7.1, which readers may prefer to read first.

5.1. The space $\Omega^\infty\mathbf{MTSO}(d)$

Let $\mathrm{Gr}(d,n)$ be the Grassmannian manifold of oriented d-planes in $\mathbb{R}^{d+n}$. It has two canonical bundles: the universal d-dimensional bundle

$$U_{d,n} := \{(P,v) \in \mathrm{Gr}(d,n) \times \mathbb{R}^{d+n} \,|\, v \in P\}$$

and its orthogonal complement, the n-dimensional bundle $U_{d,n}^\perp$. We will be using the latter. The natural inclusion $\mathbb{R}^{d+n} \to \mathbb{R}^{d+n+1}$ induces an inclusion

$$\mathrm{Gr}(d,n) \longrightarrow \mathrm{Gr}(d,n+1).$$

The pull-back of $U_{d,n+1}^\perp$ under this map is $U_{d,n}^\perp \oplus \mathbb{R}$ and on the level of Thom spaces (or one-point compactifications) this gives a map

$$\Sigma(\mathrm{Th}(U_{d,n}^\perp)) = \mathrm{Th}(U_{d,n}^\perp \oplus \mathbb{R}) \longrightarrow \mathrm{Th}(U_{d,n+1}^\perp);$$

here ΣX denotes the suspension of X. We use this to define

$$\Omega^\infty\mathbf{MTSO}(d) := \lim_{n\to\infty} \Omega^{d+n}\mathrm{Th}(U_{d,n}^\perp)$$

where the limit is taken by sending a map $S^{d+n} \to \mathrm{Th}(U_{d,n}^\perp)$ via its suspension to $S^{d+n+1} \to \Sigma(\mathrm{Th}(U_{d,n}^\perp)) \to \mathrm{Th}(U_{d,n+1}^\perp)$.

Note that the Thom class of the vector bundle $U_{d,n}^\perp$ has degree n. In the limit spaces, after taking the $(d+n)$th loop space, this class is pushed down into (virtual) dimension $-d$.

We can now formulate the generalized Mumford conjecture which was first stated in [MT] and is now known as the Madsen-Weiss Theorem [MW].

Theorem 5.1. *There exists a weak homotopy equivalence*[1]

$$\alpha : \mathbb{Z} \times B\Gamma_\infty^+ \xrightarrow{\simeq} \Omega^\infty \mathbf{MTSO}(2).$$

We will describe the map α in detail in section 7.1.

Though the spaces $\Omega^\infty \mathbf{MTSO}(d)$ are very large, they are nevertheless relatively well understood. They are built from classical geometric objects and are closely related to the space $\Omega^\infty \mathbf{MSO}$ representing oriented cobordism theory, whose ith homotopy group is the group of oriented i-dimensional closed smooth manifolds up to cobordisms, see [St]. To make the relation more precise, consider the d-fold deloopings $\Omega^{\infty-d}\mathbf{MTSO}(d) := \lim_{n\to\infty} \Omega^n \mathrm{Th}(U_{d,n}^\perp)$ so that the Thom class is now in dimension zero. The natural inclusions of Grassmannians

$$\mathrm{Gr}(d,n) \longrightarrow \mathrm{Gr}(d+1,n)$$

induce inclusions $\Omega^{\infty-d}\mathbf{MTSO}(d) \to \Omega^{\infty-(d+1)}\mathbf{MTSO}(d+1)$ which define a natural filtration of

$$\Omega^\infty \mathbf{MSO} \simeq \lim_{d\to\infty} \Omega^{\infty-d}\mathbf{MTSO}(d).$$

Moreover, the filtration quotients have a nice description. For a topological space X with basepoint, let $\Omega^\infty\Sigma^\infty(X) := \lim_{n\to\infty} \Omega^n\Sigma^n(X)$; this is the free infinite loop spaces on X satisfying the appropriate universal property. Adding the universal bundle $U_{d,n}$ to the orthogonal complement $U_{d,n}^\perp$ defines a map of Thom spaces

$$\mathrm{Th}(U_{d,n}^\perp) \longrightarrow \mathrm{Th}(U_{d,n} \oplus U_{d,n}^\perp) = \mathrm{Th}(\mathbb{R}^{d+n}) = \Sigma^{d+n}(\mathrm{Gr}(d,n)),$$

and in the limit as $n \to \infty$ a map

$$\omega : \Omega^\infty \mathbf{MTSO}(d) \longrightarrow \Omega^\infty\Sigma^\infty(BSO(d)_+).$$

Here X_+ denotes the space X with a disjoint basepoint. The proof of the following result may be found in [GMTW, Section 3].

Proposition 5.2. *There is a fibration up to homotopy*

$$\Omega^\infty \mathbf{MTSO}(d) \xrightarrow{\omega} \Omega^\infty\Sigma^\infty(BSO(d)_+) \longrightarrow \Omega^\infty \mathbf{MTSO}(d-1).$$

5.2. Rational cohomology and Mumford's conjecture.
In contrast to the homotopy groups, the rational (co)homology of $\Omega^\infty \mathbf{MTSO}(d)$ is well-understood and can be computed by standard methods in algebraic topology. For a connected component (all are homotopic), we have

[1] A weak homotopy equivalence induces a bijection between path components, and for each path component an isomorphism on all homotopy groups and hence, by Whitehead's theorem, on all homology groups.

Proposition 5.3. $H^*(\Omega_0^\infty \mathbf{MTSO}(d)) \otimes \mathbb{Q} = \bigwedge(H^{>d}(BSO(d))[-d] \otimes \mathbb{Q})$.

$\bigwedge(V^*)$ denotes the free graded commutative algebra on a graded vector space V^*, i.e. the tensor product of the polynomial algebra on the even degree generators and the exterior algebra on the odd degree generators. The V^* in question here is given by $V^n = H^{d+n}(BSO(d)) \otimes \mathbb{Q}$. In particular, for $d = 2$,

$$H^*(BSO(2)) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[e] \qquad \text{with } \deg(e) = 2.$$

Thus there is one polynomial generator $\kappa_1, \kappa_2, \ldots$ in $H^*(\Omega_0^\infty \mathbf{MTSO}(2)) \otimes \mathbb{Q}$ for each $e^2, e^3, \ldots$. The Mumford conjecture thus follows. (We will explain in more detail how κ_i is related to e^{i+1} in section 7.1, once we have defined the map α.)

Corollary 5.4. $H^*(B\Gamma_\infty) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$.

Proof of Proposition 5.3: A result of Serre [S] says that for an $(n-1)$-connected, compact space X the Hurewicz map $\pi_k(X) \to H_k(X)$ is a rational isomorphism in degrees $k < 2n - 1$. Note that the Thom space of an n-dimensional vector bundle is $(n-1)$-connected. We apply first Serre's result and then the Thom isomorphism theorem to get: for a fixed k and large enough n,

$$\begin{aligned}
\pi_k(\Omega_0^\infty \mathbf{MTSO}(d)) \otimes \mathbb{Q} &= \pi_k(\Omega^{d+n} \mathrm{Th}(U_{d,n}^\perp)) \otimes \mathbb{Q} \\
&= H_{k+d+n}(\mathrm{Th}(U_{d,n}^\perp)) \otimes \mathbb{Q} \\
&= H_{k+d}(\mathrm{Gr}(d,n)) \otimes \mathbb{Q} \\
&= H_{k+d}(BSO(d)) \otimes \mathbb{Q}.
\end{aligned}$$

By a theorem of Milnor and Moore [MM], the rational homology of any connected double loop space X is the free graded commutative algebra on its rational homotopy groups

$$\bigwedge(\pi_*(X) \otimes \mathbb{Q}) \simeq H_*(X, \mathbb{Q}).$$

By taking duals we get the result of the proposition. □

6. Divisibility and torsion in the stable (co)homology.

The generalized Mumford conjecture gives much more information than just the rational information. Indeed, it gives us exactly as much stable information as we can understand about the space $\Omega^\infty \mathbf{MTSO}(2)$. The limits to this are essentially the same as the limits of our understanding of stable homotopy theory itself.

6.1. Divisibility of the κ_i classes.

Harer in [H1] proved that $\kappa_1 \in H^2(B\Gamma_\infty) \simeq \mathbb{Z}$ is divisible by 12. To generalize this result to higher κ_i classes we will work modulo torsion and write

$$H^*_{free}(B\Gamma_\infty) := H^*(B\Gamma_\infty; \mathbb{Z})/\mathrm{Torsion}$$

for the integral lattice in $H^*(B\Gamma_\infty; \mathbb{Q})$. The following theorem was proved in [GMT].

Theorem 6.1. *Let D_i be the maximal divisor of κ_i in $H^*_{free}(B\Gamma_\infty)$. Then for all $i \geq 1$*

$$D_{2i} = 2 \qquad \text{and} \qquad D_{2i-1} = Den(\frac{B_i}{2i}).$$

As before, B_i denotes the i-th Bernoulli number and Den is the function that takes a rational number when expressed as a fraction in its lowest terms to its denominator. By a theorem of von Staudt it is well-known that Den(B_i) is the product of all primes p such that $p-1$ divides $2i$, and that a prime divides Den$(B_i/2i)$ if and only if it divides Den(B_i). So in terms of their p-adic valuation the D_i are determined by the formula

$$\nu_p(D_i) = \begin{cases} 1 + \nu_p(i+1) & \text{if } i+1 = 0 \mod (p-1) \\ 0 & \text{if } i+1 \neq 0 \mod (p-1), \end{cases}$$

and $D_1 = 2^2.3$, $D_3 = 2^3.3.5$, $D_5 = 2^2.3^2.7 \ldots$.

A closely related result, also proved in [GMT], is the following theorem which was first conjectured by Akita [Ak] and which motivated Theorem 6.1.

Theorem 6.2. *The mod p reduction κ_i in $H^{2i}(B\Gamma_\infty; \mathbb{F}_p)$ vanishes if and only if $i+1 \equiv 0 \, mod \, (p-1)$.*

We offer some remarks on these theorems and their proofs but need to refer to [GMT] for the details.

The "only if " part of Theorem 6.2 is an immediate consequence of Theorem 6.1. For if $\kappa_i = 0$ in $H^{2i}(B\Gamma_\infty; \mathbb{F}_p)$ then p divides κ_i and in particular its reduction to the free part. So we must have $\nu_p(D_i) \neq 0$. The "if" part of Theorem 6.2 follows by a computation in the Stiefel-Whitney classes and their odd prime analogue, the Wu classes.

Vice versa, Theorem 6.2 immediately implies that $D_{2i} \geq 2$ in Theorem 6.1. The lower bound in the odd case follows from (4.1). To prove that these are also the correct upper bounds certain surface bundles with structure groups $\mathbb{Z}/p^n$ for primes p and any integer n are considered.

Clearing denominators in equation (4.1) one is naturally led to ask whether the relation between s_{2i-1} and κ_{2i-1} holds in integral cohomology, see [Ak]. However, Madsen [Mad2] has recently shown that this is not the case. He proves nevertheless that the κ_i classes can be replaced by some other, rationally equivalent classes $\bar{\kappa}_i$ such that the integral equation holds for these.

6.2. Comparison with $H^*(BU)$.

Mumford's conjecture states in particular that the stable rational cohomology of the mapping class group is formally isomorphic to the rational cohomology of the infinite complex Grassmannian manifold, BU. We can make this relation more precise. Indeed, we can describe a map that induces this rational isomorphism as follows.

Let $L : \mathbb{C}P^\infty \to BU$ be the map that classifies the canonical (complex) line bundle on $\mathbb{C}P^\infty$. By Bott periodicity, BU is an infinite loop spaces. Hence the map L can be extended to a map from the free infinite loop space on $\mathbb{C}P^\infty = BSO(2)$ using the universal property. On composition with α and ω one gets a map

$$\mathbb{Z} \times B\Gamma_\infty^+ \overset{\alpha}{\simeq} \Omega^\infty \mathbf{MTSO}(2) \xrightarrow{\omega} \Omega^\infty \Sigma^\infty (BSO(2)_+) \xrightarrow{L} \mathbb{Z} \times BU.$$

Each of these maps induces an isomorphism in rational cohomology: α is a homotopy equivalence; ω is by Proposition 5.2 part of a fiber sequence where one of the terms $\Omega^\infty \mathbf{MTSO}(1) \simeq \Omega^{\infty+1} S^\infty$ has only trivial rational cohomology; and it is well-known that L is split surjective with a fiber that has only torsion cohomology, see [S2]. In [MT] we showed that through this map the kappa classes correspond to the integral Chern character classes

$$\kappa_i = \alpha^*(\omega^*(L^*(i!\, ch_i))).$$

This rational isomorphism between cohomology groups can be strengthened. Working p-locally the following result is obtained in [GMT].

Theorem 6.4. *For odd primes p there is an isomorphism of Hopf algebras over the p-local integers*

$$H^*_{free}(B\Gamma_\infty; \mathbb{Z}_{(p)}) \simeq H^*(BU; \mathbb{Z}_{(p)}).$$

This fails for $p = 2$ where the algebra $H^*_{free}(B\Gamma_\infty; \mathbb{Z}_{(2)})$ is not polynomial.

6.3. Torsion classes and $\mathbb{F}_p$-homology.

When working with $\mathbb{F}_p$-coefficients we are able to find infinitely many families of torsion classes in the stable homology. Each family is essentially a copy of $H_*(B\Sigma_\infty; \mathbb{F}_p) = H_*(\Omega_0^\infty S^\infty; \mathbb{F}_p)$ from Example 3.2.1. We will describe these classes with reference to the fibration sequence in Proposition 5.2 and the map ω.

Every infinite loop space has a product, and hence its homology has an induced product. Just as Steenrod operations measure for topological spaces the failure of the cup product in cohomology to be commutative at the chain level, so there are Dyer-Lashof operations that measure for infinite loop spaces the failure of the product in homology to be commutative at the chain level.

To be more precise, consider sequences $I = (\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k)$ of non-negative integers with

$$\varepsilon_j \in \{0, 1\}, \quad s_j \geq \varepsilon_j, \quad ps_j - \varepsilon_j \geq s_{j-1},$$

and define

$$e(I) = 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} (2s_j(p-1) - \varepsilon_j),$$

$$b(I) = \varepsilon_1, \qquad d(I) = \sum_{j=1}^{k} (2s_j(p-1) - \varepsilon_j).$$

For any infinite loop space Z and each I there is a homology (Dyer-Lashof) operation,

$$Q^I : H_q(Z; \mathbb{F}_p) \longrightarrow H_{q+d(I)}(Z; \mathbb{F}_p)$$

which can be non-zero only if $e(I) + b(I) \geq q$.

We can now describe $H_*(\Omega^\infty\Sigma^\infty(BSO(2)_+); \mathbb{F}_p)$ as the free Dyer-Lashof algebra on generators $a_i \in H_{2i}(BSO(2); \mathbb{F}_p)$. Explicitly, if

$$T = \{Q^I a_q \mid q \geq 0,\ e(I) + b(I) > 2q\},$$

then

$$H_*(\Omega^\infty\Sigma^\infty(BSO(2)_+); \mathbb{F}_p) = \bigwedge(T) \otimes \mathbb{F}_p[\mathbb{Z}],$$

where $\bigwedge(T)$ now denotes the free graded commutative $\mathbb{F}_p$-algebra generated by T, see [CLM; p. 42].

By constructing certain surface bundles with finite structure groups $\mathbb{Z}/p^n$ a partial splitting (after p-adic completion) of the map

$$\omega \circ \alpha : \mathbb{Z} \times B\Gamma^\infty \longrightarrow \Omega^\infty\Sigma^\infty(BSO(2)_+)$$

was constructed in [MT], and the following could be deduced.

Proposition 6.6. *The mod p homology $H_*(B\Gamma_\infty; \mathbb{F}_p)$ contains the free commutative algebra $\bigwedge(T')$ where*

$$T' = \{Q^I a_q \in T \mid q \neq -1\,(mod\ p-1)\}.$$

A subset of the family associated to a_0 had previously been found by Charney and Lee [CL]. Using the infinite loop space structure on $B\Gamma_\infty^+$ the whole family could be detected, see [T2]. A complete description of the $\mathbb{F}_p$-homology of $\Omega^\infty\mathbf{MTSO}(2)$ has been given in [G1]. Galatius' computation is a careful analysis of the fibration in Proposition 5.2 and related spaces using amongst other tools the Eilenberg-Moore spectral sequence.

6.4. Odd dimensional unstable classes.

Let Γ be a finitely generated, virtually torsion free group, and let Γ' be a torsion free subgroup of finite index. Recall, the orbifold Euler characteristic of Γ is defined by

$$\chi(\Gamma) := \frac{e(\Gamma')}{[\Gamma : \Gamma']}$$

where $e(\Gamma') = \Sigma_{n\geq 0}(-1)^n \dim H_i(\Gamma'; \mathbb{Q})$ denotes the ordinary Euler characteristic of Γ'. In the 1980s Harer and Zagier [HZ] computed the orbifold Euler characteristic of the mapping class group.

Theorem 6.7. *As before, let B_g denote the gth Bernoulli number. Then*

$$\chi(\Gamma_g^s) = (-1)^s \frac{(2g+s-3)!}{2g(2g-2)!} B_g.$$

From this Harer and Zagier deduced a formula for the ordinary Euler characteristic for Γ_g and concluded that the Betti numbers grow exponentially. Furthermore, the Euler characteristic is often negative and hence there must be a large number of odd dimensional classes. By the confirmation of the Mumford conjecture, we now know that all these odd-dimensional classes must be unstable and that the stable classes form only a small proportion of the cohomology classes.

7. Towards a proof.

We will restrict ourselves to defining the map α and outlining the proof as presented in [GMTW] which is a simplification of the original one in [MW].

7.1. The map α

To motivate the generalized Mumford conjecture and the definition of the map α it is helpful to recall the construction of the κ_i classes from Section 4.1. For a smooth surface bundle $\pi : E \to B$ over a manifold B with fiber F_g let $T^\pi E$ denote the vertical tangent bundle with Euler class $e \in H^2(E;\mathbb{Z})$. Then,

$$\kappa_i := (-1)^{i+1}\pi_!(e^{i+1}) \in H^{2i}(B;\mathbb{Z}).$$

We are used to characteristic classes as cohomology classes of some universal spaces that get pulled back under a classifying map. For example, given a complex vector bundle $V \to B$ that is classified by a map $f_V : B \to BU$ its i-th Chern class $c_i(V) := f^*(c_i)$ is the pull-back of the universal Chern class $c_i \in H^{2i}(BU;\mathbb{Z})$. Similarly, one would like to interpret the κ_i classes as pull-backs from some universal space.

The definition of κ_i above leads us to consider a wrong way map $B - -- > E$ followed by the map $f_{T^\pi E} : E \to \mathrm{Gr}(2,\infty)$ that classifies the vertical tangent bundle. Such a wrong way map can be constructed via the Thom collapse map as follows. Embed the bundle E in $\mathbb{R}^N \times B$ such that the projection onto B corresponds to $\pi : E \to B$. Choose a suitable neighborhood of E in $\mathbb{R}^N \times B$ such that it can be identified with the fiberwise normal bundle $N^\pi E$ of E in $\mathbb{R}^N \times B$, which is complementary to the vertical tangent bundle $T^\pi E$. Taking Thom spaces (or one-point compactifications) and collapsing the outside of the neighborhood to a point gives a map

$$Th(\mathbb{R}^N \times B) \simeq \Sigma^N(B_+) \xrightarrow{c} Th(N^\pi E)$$

where B_+ as before denotes B with a disjoint basepoint. We compose this with the map

$$f_{T^\pi E} : Th(N^\pi E) \longrightarrow Th(U^\perp_{2,N-2})$$

induced by the map that classifies the vertical tangent bundle on E. Taking adjoints and letting $N \to \infty$ we get the desired map

$$\alpha : B_+ \longrightarrow \Omega^\infty \mathbf{MTSO}(2) = \lim_{N\to\infty} \Omega^N Th(U^\perp_{2,N-2}). \tag{7.1}$$

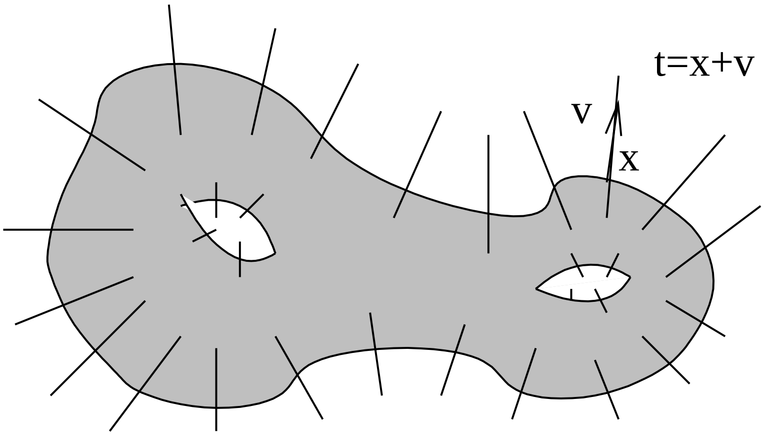

Figure 3: *The map* $\alpha : \mathcal{M}_g^{top} \to \Omega^\infty \mathbf{MTSO}(2)$.

For the universal surface bundle over topological moduli spaces, $B = \mathcal{M}_g^{top}$, the map α has a particularly nice and easy description. Recall, a point in $\mathcal{M}_g^{top}$ is represented by an embedded surface $F \subset \mathbb{R}^N$ of genus g for some N. We choose a suitable neighborhood $O_F \supset F$ and define $\alpha(F) \in \Omega^\infty \mathbf{MTSO}(2)$ as

$$\begin{aligned} \alpha(F) : S^N &\longrightarrow Th(U^\perp_{2,N-2}) \\ t &\mapsto \begin{cases} \infty & \text{if } t \notin O_F, \\ (T_xF, v) & \text{if } t \in O_F \text{ and } t = x + v. \end{cases} \end{aligned}$$

Thus when $t \in O_F \subset \mathbb{R}^N \cup \{\infty\} = S^N$ and is written as $t = x + v$ where x is the closest point on the surface and v is a normal vector to the tangent spaces at x, t is mapped to $(T_xF, v) \in U^\perp_{2,N-2}$. When $t \notin O_F$, t is mapped to the point at infinity. See Figure 3.

7.2. Cohomological interpretation.

We will now explain how α relates to the definition of κ_i in cohomological terms. Recall, by definition the Gysin or 'integration over the fiber' map $\pi_!$ is the Thom isomorphism for $N^\pi E$ followed by the Thom collapse map:

$$H^{*+2}(E) \simeq H^{*+2+(N-2)}(Th(N^\pi(E)) \xrightarrow{c^*} H^{*+N}(\Sigma(B_+)) \simeq H^*(B).$$

If we denote by $\sigma^N(x)$ the N-fold suspension of a class $x \in H^k(E)$ and use the symbol λ_- for Thom classes, then we can write

$$\pi_!(x) = (\sigma^N)^{-1}(c^*(\lambda_{N^\pi E} \, . \, x)).$$

Next consider the map

$$s : Th(N^\pi E) \to Th(T^\pi E \oplus N^\pi E) = Th(\mathbb{R}^N \times E) = \Sigma^N(E_+)$$

induced by the inclusion $N^\pi E \to T^\pi E \oplus N^\pi E = \mathbb{R}^N \times E$ of the fiberwise normal bundle into its sum with the fiberwise tangent bundle. (s may also be thought of as arising from the zero section of a 2-dimensional bundle over $Th(N^\pi E)$.) We have the identities

$$(7.2) \qquad s^*(\sigma^N(x)) = s^*(\lambda_{\mathbb{R}^N \times E} \,.\, x) = s^*(\lambda_{T^\pi E} \,.\, \lambda_{N^\pi E} \,.\, x) = e(T^\pi E) \,.\, \lambda_{N^\pi E} \,.\, x.$$

Here the first equality is the identification of the suspension isomorphism with the Thom isomorphism for the trivial bundle. The second equality follows because the Thom class of a direct sum of vector bundles is the product of their Thom classes. Finally for the equality on the right, first note that the Thom isomorphism is an isomorphism of modules over the cohomology of the base space and that a map of bundles (such as s) induces a map of modules. This gives $s^*(\lambda_{N^\pi E}) = \lambda_{N^\pi E}$ and $s^*(\lambda_{T^\pi E}) = 1 \,.\, e(T^\pi E)$ where $e(T^\pi E) \in H^2(E)$ is the Euler class of $T^\pi E \to E$.

Now compose s with the Thom collapse map c to give a map

$$\Sigma^N(B_+) \xrightarrow{c} Th(N^\pi E) \xrightarrow{s} \Sigma^N(E_+),$$

and an induced map, the transfer map trf, in cohomology

$$H^*(E) = H^{*+N}(\Sigma^N(E_+)) \xrightarrow{(s \circ c)^*} H^{*+N}(\Sigma^N(B_+)) = H^*(B).$$

It maps an element $x \in H^*(E)$ to

$$\mathrm{trf}(x) = (\sigma^N)^{-1}(c^*(s^*(\sigma^N(x)))).$$

It follows now immediately from (7.2) that

$$\mathrm{trf}(x) = \pi_!\,(e(T^\pi E) \,.\, x).$$

In the universal case, after looping, s gives the map

$$\omega : \Omega^\infty \mathbf{MTSO}(2) \longrightarrow \Omega^\infty \Sigma^\infty (BSO(2)_+)$$

from section 5.1. A calculation just as in the proof of Proposition 5.3 (only easier) shows that for any connected space X of finite type

$$\pi_k(\Omega^\infty \Sigma^\infty (X_+)) \otimes \mathbb{Q} = H_k(X) \otimes \mathbb{Q}$$

and

$$H^*(\Omega^\infty \Sigma^\infty (X_+)) \otimes \mathbb{Q} = \bigwedge (H^*(X) \otimes \mathbb{Q}).$$

In particular

$$H^*(\Omega^\infty \Sigma^\infty (BSO(2)_+)) \otimes \mathbb{Q} = \mathbb{Q}[s_0, s_1, s_2, \dots]$$

where each s_i has degree $2i$ and corresponds under this isomorphism to $e^i \in H^{2i}(BSO(2))$. We summarize our discussion with the identities

$$\boxed{\kappa_i := \pi_!(e^{i+1}) = \mathrm{trf}(e^i) = \omega^* \alpha^*(s_i).}$$

The above discussion gives the last identity only rationally. But it holds also integrally; see [MT] or [GMT].

7.3. Cobordism categories and their classifying spaces.

The constructions and computations above make equally good sense for a closed oriented manifold W of any dimension d. In particular we have a map

$$\alpha : \mathcal{M}^{top}(W) \longrightarrow \Omega^\infty \mathbf{MTSO}(d).$$

In case of a manifold (with or without boundary) which is embedded in $[a_0, a_1] \times \mathbb{R}^{\infty-1}$ we can modified the construction so that $\alpha(W)$ is now a map from $[a_0, a_1] \wedge S_+^{N-1}$. After taking adjoints this yields a map

$$\mathcal{M}^{top,\Omega}(W) \longrightarrow \mathrm{map}([a_0, a_1], \Omega^{\infty-1}\mathbf{MTSO}(d)).$$

These maps fit together to define a functor

$$\mathrm{Cob}_d \longrightarrow \mathcal{P}(\Omega^{\infty-1}\mathbf{MTSO}(d))$$

from the d-dimensional cobordism category (as defined in section 2.3) to the path category of $\Omega^{\infty-1}\mathbf{MTSO}(d)$. Recall, the objects in the path category of a space X are the points in X. The morphism space $\mathcal{P}X(x_0, x_1)$ is the space of continuous paths in X from x_0 to x_1. Furthermore, the classifying space of $\mathcal{P}X$ is homotopic to X. (This can be proved by an application of Theorem 3.3.) Let α denote again the map induced on classifying spaces:

$$\alpha : B\mathrm{Cob}_d \longrightarrow B(\mathcal{P}(\Omega^{\infty-1}\mathbf{MTSO}(d))) \simeq \Omega^{\infty-1}\mathbf{MTSO}(d).$$

The main theorem of [GMTW] states that this map is a weak homotopy equivalence.

Theorem 7.1. $\alpha : B\mathit{Cob}_d \simeq \Omega^{\infty-1}\mathbf{MTSO}(d)$.

Idea of proof: There are essentially two steps. First, it is not too difficult to show that $B\mathrm{Cob}_d$ is weakly homotopic to the space $\mathcal{T}_d$ of embedded d-dimensional manifolds without boundary that are closed subsets of the infinite tube $\mathbb{R}\times[0,1]^{\infty-1}$. The topology on $\mathcal{T}_d$ is here such that manifolds can be pushed away to infinity (unlike in the topology used when defining our topological moduli spaces in section 2.1). One can construct a map as follows. $\mathcal{T}_d$ is the realization of a constant simplicial space and $B\mathrm{Cob}_d$ is the realization of the nerve of Cob_d. An n-simplex of the latter defines an element in $\mathcal{T}$ after extending the composed cobordism to both $\pm\infty$ by gluing infinite cylinders to its boundaries. This is indeed a weak homotopy equivalence on n-simplices and hence on the realization.

Secondly, by an adaption of the classical arguments in cobordism theory (see [St]), one can now use transversality and Phillip's submersion theorem [P] to show that an element of $\pi_n(\Omega^{\infty-1}\mathbf{MTSO}(d))$ gives rise to (a cobordism class of) a triple (E, π, f) where $\pi : E^{d+n} \to S^n$ is a submersion and $f : E^{n+d} \to \mathbb{R}$ is proper, and hence an element in $\pi_n(\mathcal{T}_d)$. This defines an isomorphism between the homotopy groups and hence the result follows. $\square$

As in section 4.4, let Cob_d^∂ denote the subcategory of Cob_d in which every connected component of a cobordism has non-trivial target boundary. We need the following weak homotopy equivalence, also proved in [GMTW] for $d > 1$ and in [Ra] for $d = 1$.

Theorem 7.2. $BCob_d^\partial \simeq BCob_d$.

Idea of proof: The proof is quite technical but essentially consists of a surgery argument the basic idea of which is quite easy to explain. Work with the space $\mathcal{T}_d$ as in the proof of Theorem 7.1. Essentially one wants to show that the space of all manifolds of dimension d has the same homotopy type as the space of manifolds without certain local maxima (relative to the projection onto the first coordinate) as they would give rise to cobordisms that are not in Cob_d^∂. (The inverse of the weak equivalence $B\mathrm{Cob}_d \to \mathcal{T}_d$ given in the proof of Theorem 7.1 takes a manifold W in $\mathbb{R} \times [0,1]^{\infty-1}$ and restricts it to $[a_0, a_n] \times [0,1]^{\infty-1}$ for some transversal walls $\{a_i\} \times \mathbb{R}^{\infty-1}$, $i = 0, \dots, n$.) Grab these forbidden local maxima on W and pull to the right along the first coordinate axis so that in a neighborhood of each local maxima the manifold grows a very long nose reaching to $+\infty$. Again it is important here that the topology on $\mathcal{T}_d$ is such that manifolds 'disappear' at infinity. $\square$

This result provides the link between Theorem 7.1 and Theorem 4.6. Together they prove the generalized Mumford conjecture, Theorem 5.1:

$$\boxed{\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega B\mathrm{Cob}_2^\partial \simeq \Omega B\mathrm{Cob}_2 \simeq \Omega^\infty \mathbf{MTSO}(2).}$$

We emphasize that the first homotopy equivalence depends crucially on homology stability which allowed us to apply the group completion theorem for categories. The homology stability theorem also allows us to state the following immediate consequence. Recall from section 3.2 that for every category $\mathcal{C}$ there is map $\sigma : \mathcal{C}(a,b) \to \Omega_{a,b}B\mathcal{C}$ from the morphism spaces between two objects a and b to the space of paths from a to b in $B\mathcal{C}$. Taking $a = b = \emptyset$ this gives a map from $\mathcal{M}_g^{top}$ to $\Omega B\mathrm{Cob}_2 \simeq \Omega^\infty \mathbf{MTSO}(2)$ which is of course the map α defined above. Furthermore, the computations in section 7.2 show that the component of the image is determined by (half) the Euler characteristic, κ_0. We thus have the following result.

Corollary 7.3. *The map $\alpha : \mathcal{M}_g^{top} \to \Omega_{1-g}^\infty \mathbf{MTSO}(2)$ induces an isomorphism in homology for degrees $* \leq 2g/3 - 2/3$.*

7.3.1 Relation to Segal's category $\mathcal{S}$: An embedded, oriented surface $F \subset \mathbb{R}^\infty$ inherits a metric and hence an induced almost complex structure. There is a unique complex structure that is compatible with this almost complex structure. By assigning to F this complex curve we can define a functor

$$\mathcal{F} : \mathrm{Cob}_2 \longrightarrow \mathcal{S}.$$

For the subcategories Cob_2^∂ and $\mathcal{S}^\partial$, $\mathcal{F}$ induces a homotopy equivalence between all morphisms spaces $\mathrm{Cob}_2^\partial(M_0, M_1)$ and $\mathcal{S}^\partial(F(M_0), F(M_1))$ because for surfaces with boundary the topological and Riemann's moduli spaces have the same homotopy type, see (2.2). Indeed, it is also not hard to see that $\mathcal{F}$ induces the homotopy equivalence of classifying spaces in Theorem 4.6. However, this is not the case for

Cob_2 and $\mathcal{S}$. In particular, the topological moduli space for the oriented sphere is homotopic to $BSO(3)$ while Riemann's moduli space for the sphere is a point. Furthermore, the map $BSO(3) \to \Omega B\mathrm{Cob}_2 \simeq \Omega^\infty \mathbf{MTSO}(2)$ is non-trivial in rational cohomology. Thus $\mathcal{F}$ does not induce a homotopy equivalence between $B\mathrm{Cob}_2$ and $B\mathcal{S}$, not even rationally (!). The homotopy type of $B\mathcal{S}$ remains unknown.

8. Epilogue.

We have concentrated on a treatment of the Mumford conjecture in its topological form. There have been several expansions and other developments. We conclude by briefly mentioning some of these.

8.1. One extension concerns the question of background spaces. In physics strings are considered that move in some background space X. To study these, the category of Cob_2 is enriched to $\mathrm{Cob}_2(X)$ where all surfaces are equipped with a continuous map to X. Similarly for higher dimensions. Theorem 7.1 and 7.2 generalize to this situation and we have (see [GMTW])

$$B\mathrm{Cob}_d^\partial(X) \simeq B\mathrm{Cob}_d(X) \simeq \Omega^{\infty-1}(\mathbf{MTSO}(d) \wedge X_+).$$

This we can reinterpret to say that $h_*(X) = \pi_*(\Omega^\infty B\mathrm{Cob}_d(X))$ is the generalized homology theory associated to $\Omega^\infty \mathbf{MTSO}(d)$. In particular it can be computed for different backgrounds using a Mayer-Vietoris sequence. Furthermore, for $d = 2$ and simply connected X, Cohen and Madsen [CM] prove the analogue of Harer's stability theorem in this context. This computes the stable homology of the topological moduli space $\mathcal{M}_g^{top}(X)$ of surfaces of genus g with maps to X.

8.2. Algebraic geometers are in particular interested in the compactified moduli space $\overline{\mathcal{M}}_g$. The methods used to prove the Mumford conjecture have so far been of limited success in understanding the topology of $\overline{\mathcal{M}}_g$. Galatius and Eliashberg [EG] have however been able to prove a version of the Madsen-Weiss theorem for partially compactified moduli spaces which contain only surfaces with no separating nodal curves. See also [EbGi].

8.3. For simplicity we have restricted our attention here to surfaces and manifolds more generally that are oriented. Analogues of both Theorem 7.1 and 7.2 hold much more generally for manifolds with arbitrary tangential structure, see [GMTW]. Such tangential structure can be defined for any Serre fibration $\theta : B(d) \to BO(d)$. A θ-structure on a manifold M is then a lift through θ of the classifying map of the tangent bundle $f_{TM} : M \to BO(d)$. Wahl [Wa1] proves the analogue of Harer's homology stability for non-orientable surfaces and thus, by taking θ to be the identity map, is able to deduce the analogue of the Mumford conjecture in this context. If $\mathcal{N}_\infty$ denotes the limit of the mapping class groups of non-orientable surfaces then for classes ξ_i of dimension $4i$

$$H^*(B\mathcal{N}_\infty) \otimes \mathbb{Q} = \mathbb{Q}[\xi_1, \xi_2, ...].$$

Similar results for spin and more exotic structures on surfaces have been proved by Randal-Williams [RW2], and earlier by Tilman Bauer [Ba]. The situation for higher dimensional manifolds is more complicated though some progress has been made recently by Hatcher for certain 3-dimensional manifolds and Galatius and Randal-Williams for certain even dimensional manifolds.

8.4. A group closely related to the mapping class group is the automorphism group of a free group, $\mathrm{Aut}F_n$. By considering a moduli space of graphs embedded in $\mathbb{R}^\infty$ Galatius [G2] was able to show the analogue of Mumford's conjecture for these groups:

$$H^*(B\mathrm{Aut}F_\infty) \otimes \mathbb{Q} = \mathbb{Q}.$$

As mentioned already in the introduction, the proof of this in [G2] introduces simplifications and generalizations to the main results of [GMTW], Theorems 7.1 and 7.2.

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Mathematical Institute
24-29 St. Giles Street
Oxford OX1 3LB
UK
e-mail: tillmann@maths.ox.ac.uk