# Math Handbook

# of Formulas, Processes and Tricks

## Geometry


**Prepared by: Earl L. Whitney, FSA, MAAA**

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**Page Description**

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## Useful Websites

**Wolfram Math World** – Perhaps the premier site for mathematics on the Web. This site contains definitions, explanations and examples for elementary and advanced math topics.

mathworld.wolfram.com/

**Mathguy.us** – Developed specifically for math students from Middle School to College, based on the author's extensive experience in professional mathematics in a business setting and in math tutoring. Contains free downloadable handbooks, PC Apps, sample tests, and more.

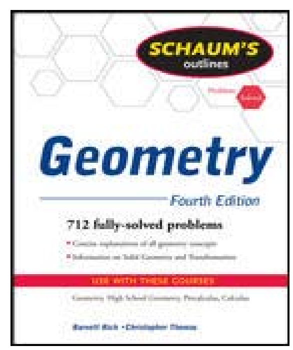
www.mathguy.us

**California Standard Geometry Test** – A standardized Geometry test released by the state of California. A good way to test your knowledge.

www.cde.ca.gov/ta/tg/sr/documents/rtqgeom.pdf

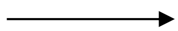
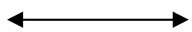
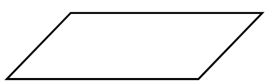
## Schaum's Outlines

An important student resource for any high school math student is a Schaum's Outline. Each book in this series provides explanations of the various topics in the course and a substantial number of problems for the student to try. Many of the problems are worked out in the book, so the student can see examples of how they should be solved.



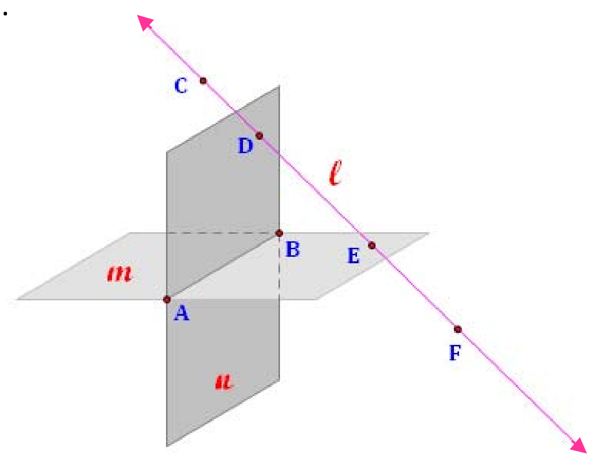

Schaum's Outlines are available at Amazon.com, Barnes & Noble and other booksellers.

# Geometry
## Points, Lines & Planes

| Item | Illustration | Notation | Definition |
|---|---|---|---|
| **Point** | • | $A$ | A location in space. |
| **Segment** | ___ | $\overline{AB}$ | A straight path that has two endpoints. |
| **Ray** |  | $\overrightarrow{AB}$ | A straight path that has one endpoint and extends infinitely in one direction. |
| **Line** |  | $\ell$ or $\overleftrightarrow{AB}$ | A straight path that extends infinitely in both directions. |
| **Plane** |  | $m$ or $\overleftrightarrow{AB}$ | A flat surface that extends infinitely in two dimensions. |

**Collinear points** are points that lie on the same line.

**Coplanar points** are points that lie on the same plane.




In the figure at right:

- $A, B, C, D, E$ and $F$ are points.
- $\ell$ is a line
- $m$ and $n$ are planes.

In addition, note that:

- $C, D, E$ and $F$ are **collinear points.**
- $A, B$ and $E$ are **coplanar points.**
- $A, B$ and $D$ are **coplanar points.**
- Ray $\overrightarrow{EF}$ goes off in a southeast direction.
- Ray $\overrightarrow{EC}$ goes off in a northwest direction.
- Together, rays $\overrightarrow{EF}$ and $\overrightarrow{EC}$ make up line $\ell$.
- Line $\ell$ intersects both planes $m$ and $n$.

**Note:** In geometric figures such as the one above, it is important to remember that, even though planes are drawn with edges, they extend infinitely in the 2 dimensions shown.

An **intersection** of geometric shapes is the set of points they share in common.

$\ell$ and $m$ intersect at point $E$.

$\ell$ and $n$ intersect at point $D$.

$m$ and $n$ intersect in line $\overleftrightarrow{AB}$.

# Geometry
# Segments, Rays & Lines

## Some Thoughts About …

### Line Segments

- Line segments are generally named by their endpoints, so the segment at right could be named either $\overline{AB}$ or $\overline{BA}$.


- Segment $\overline{AB}$ contains the two endpoints (**A** and **B**) and all points on line $\overleftrightarrow{AB}$ that are between them.

### Rays

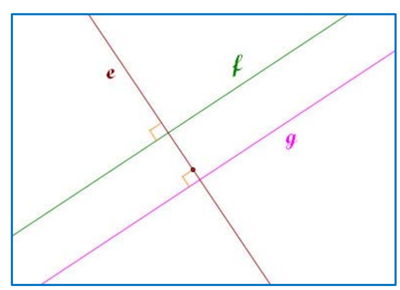
- Rays are generally named by their single endpoint, called an **initial point**, and another point on the ray.


- Ray $\overrightarrow{AB}$ contains its initial point **A** and all points on line $\overleftrightarrow{AB}$ in the direction of the arrow.
- Rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are not the same ray.
- If point **O** is on line $\overleftrightarrow{AB}$ and is between points **A** and **B**, then rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are called **opposite rays**. They have only point O in common, and together they make up line $\overleftrightarrow{AB}$.

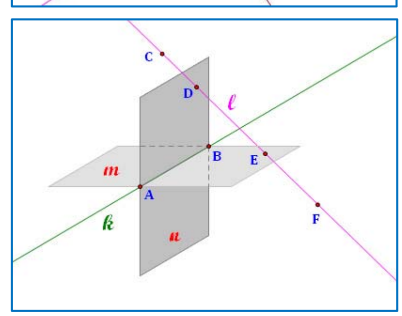

### Lines

- Lines are generally named by either a single script letter (e.g., $\ell$) or by two points on the line (e.g.,. $\overleftrightarrow{AB}$).


- A line extends infinitely in the  directions shown by its arrows.
- Lines are **parallel** if they are in the same plane and they never intersect. Lines $f$ and $g$, at right, are parallel.




- Lines are **perpendicular** if they intersect at a 90° angle. A pair of perpendicular lines is always in the same plane. Lines $f$ and $e$, at right, are perpendicular. Lines $g$ and $e$ are also perpendicular.
- Lines are **skew** if they are not in the same plane and they never intersect. Lines $k$ and $\ell$, at right, are skew. (Remember this figure is 3-dimensional.)



# Geometry

## Distance Between Points

Distance measures how far apart two things are. The distance between two points can be measured in any number of dimensions, and is defined as the length of the line connecting the two points. Distance is always a positive number.

### 1-Dimensional Distance

In one dimension the distance between two points is determined simply by subtracting the coordinates of the points.

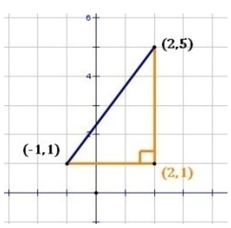
Example: In this segment, the distance between -2 and 5 is calculated as: $5 - (-2) = 7.$


### 2-Dimensional Distance

In two dimensions, the distance between two points can be calculated by considering the line between them to be the hypotenuse of a right triangle. To determine the length of this line:

- Calculate the difference in the x-coordinates of the points
- Calculate the difference in the y-coordinates of the points
- Use the Pythagorean Theorem.

This process is illustrated below, using the variable ***"d"*** for distance.




Example: Find the distance between (-1,1) and (2,5). Based on the illustration to the left:

*x-coordinate difference:* $2 - (-1) = 3.$

*y-coordinate difference:* $5 - 1 = 4.$

Then, the distance is calculated using the formula: $d^2 = (3^2 + 4^2) = (9 + 16) = 25$

So, $\boldsymbol{d = 5}$

If we define two points generally as ***($x_1, y_1$)*** and ***($x_2, y_2$)***, then a 2-dimensional distance formula would be:

$$\boldsymbol{distance = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

# Geometry
## Distance Formula in "*n*" Dimensions

**ADVANCED**

The distance between two points can be generalized to ***"n"*** dimensions by successive use of the Pythagorean Theorem in multiple dimensions. To move from two dimensions to three dimensions, we start with the two-dimensional formula and apply the Pythagorean Theorem to add the third dimension.

### 3 Dimensions

Consider two 3-dimensional points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$. Consider first the situation where the two z-coordinates are the same. Then, the distance between the points is 2-dimensional, i.e., $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

We then add a third dimension using the Pythagorean Theorem:

$$distance^2 = d^2 + (z_2 - z_1)^2$$

$$distance^2 = \left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)^2 + (z_2 - z_1)^2$$

$$distance^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

And, finally the **3-dimensional difference formula**:

$$distance = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

### n Dimensions

Using the same methodology in "*n*" dimensions, we get the generalized **n-dimensional difference formula** (where there are *n* terms beneath the radical, one for each dimension):

$$distance = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + \cdots + (w_2 - w_1)^2}$$

Or, in higher level mathematical notation:
The distance between 2 points $A=(a_1, a_2, \ldots, a_n)$ *and* $B=(b_1, b_2, \ldots, b_n)$ is

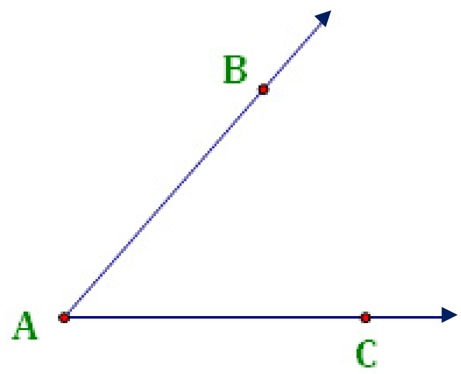
$$d(A,B) = |A - B| = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

# Geometry
# Angles

## Parts of an Angle

An **angle** consists of two rays with a common endpoint (or, initial point).

- Each ray is a **side** of the angle.
- The common endpoint is called the **vertex** of the angle.




## Naming Angles

Angles can be named in one of two ways:

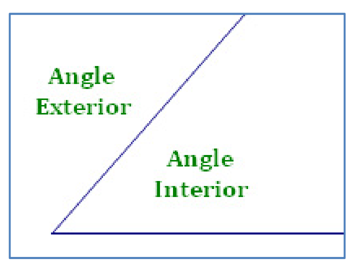
- **Point-vertex-point method.** In this method, the angle is named from a point on one ray, the vertex, and a point on the other ray. This is the most unambiguous method of naming an angle, and is useful in diagrams with multiple angles sharing the same vertex. In the above figure, the angle shown could be named $\angle BAC$ or $\angle CAB$.
- **Vertex method.** In cases where it is not ambiguous, an angle can be named based solely on its vertex. In the above figure, the angle could be named $\angle A$.

## Measure of an Angle

There are two conventions for measuring the size of an angle:

- **In degrees**. The symbol for degrees is $^{o}$. There are $360^{o}$ in a full circle. The angle above measures approximately $\mathbf{45^{o}}$ (one-eighth of a circle).
- **In radians**. There are $2\pi$ radians in a complete circle. The angle above measures approximately $\frac{1}{4}\pi$ **radians**.

## Some Terms Relating to Angles



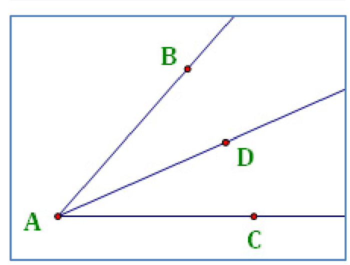

**Angle interior** is the area between the rays.

**Angle exterior** is the area not between the rays.

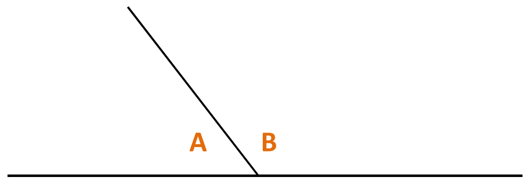
**Adjacent angles** are angles that share a ray for a side. $\angle BAD$ and $\angle DAC$ in the figure at right are adjacent angles.

**Congruent angles** area angles with the same measure.

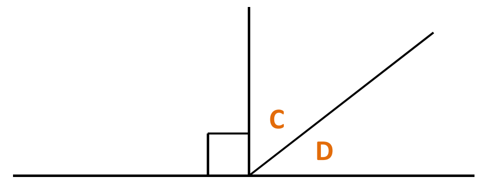
**Angle bisector** is a ray that divides the angle into two congruent angles. Ray $\overrightarrow{AD}$ bisects $\angle BAC$ in the figure at right.



# Geometry
## Types of Angles




Supplementary Angles




Complementary Angles

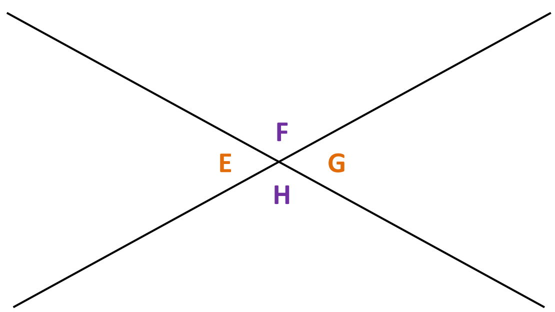
Angles **A** and **B** are supplementary.

Angles **A** and **B** form a linear pair.

$m\angle A + m\angle B = 180^0$

Angles **C** and **D** are complementary.

$m\angle C + m\angle D = 90^0$




Vertical Angles

Angles which are opposite each other when two lines cross are vertical angles.

Angles **E** and **G** are vertical angles.
Angles **F** and **H** are vertical angles.

$m\angle E = m\angle G \quad and \quad m\angle F = m\angle H$

In addition, each angle is supplementary to the two angles adjacent to it. For example:

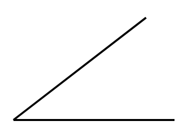
Angle **E** is supplementary to Angles **F** and **H**.

An acute angle is one that is less than $90^0$. In the illustration above, angles **E** and **G** are acute angles.

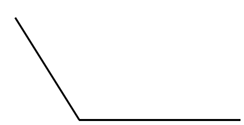
A right angle is one that is exactly $90^0$.

An obtuse angle is one that is greater than $90^0$. In the illustration above, angles **F** and **H** are obtuse angles.

A straight angle is one that is exactly $180^0$.



Acute



Obtuse

Right



Straight

# Geometry
## Conditional Statements

A conditional statement contains both a hypothesis and a conclusion in the following form:

If **hypothesis**, then **conclusion**.

For any conditional statement, it is possible to create three related conditional statements, as shown below. In the table, $p$ is the hypothesis of the original statement and $q$ is the conclusion of the original statement.

Statements linked below by red arrows must be either both true or both false.

| Type of Conditional Statement | Example Statement is: |
|---|---|
| **Original Statement:** If $p$, then $q$. $(p \rightarrow q)$<br>• Example: If a number is divisible by 6, then it is divisible by 3.<br>• The original statement may be either true or false. | TRUE |
| **Converse Statement:** If $q$, then $p$. $(q \rightarrow p)$<br>• Example: If a number is divisible by 3, then it is divisible by 6.<br>• The converse statement may be either true or false, and this does not depend on whether the original statement is true or false. | FALSE |
| **Inverse Statement:** If not $p$, then not $q$. $(\sim p \rightarrow \sim q)$<br>• Example: If a number is not divisible by 6, then it is not divisible by 3.<br>• The inverse statement is always true when the converse is true and false when the converse is false. | FALSE |
| **Contrapositive Statement:** If not $q$, then not $p$. $(\sim q \rightarrow \sim p)$<br>• Example: If a number is not divisible by 3, then it is not divisible by 6.<br>• The Contrapositive statement is always true when the original statement is true and false when the original statement is false. | TRUE |

**Note also that:**

- When two statements must be either both true or both false, they are called **equivalent statements**.
  - The original statement and the contrapositive are equivalent statements.
  - The converse and the inverse are equivalent statements.
- If both the original statement and the converse are true, the phrase **"if and only if"** (abbreviated **"iff"**) may be used. For example, **"A number is divisible by 3 iff the sum of its digits is divisible by 3."**

# Geometry
# Basic Properties of Algebra

## Properties of Equality and Congruence.

| Property | Definition for Equality | Definition for Congruence |
|---|---|---|
| | For any real numbers **a**, **b**, and **c**: | For any geometric elements **a**, **b** and **c**. (e.g., segment, angle, triangle) |
| Reflexive Property | $a = a$ | $a \cong a$ |
| Symmetric Property | $If\ a = b, then\ b = a$ | $If\ a \cong b, then\ b \cong a$ |
| Transitive Property | $If\ a = b\ and\ b = c, then\ a = c$ | $If\ a \cong b\ and\ b \cong c, then\ a \cong c$ |
| Substitution Property | If $a = b$, then either can be substituted for the other in any equation (or inequality). | If $a \cong b$, then either can be substituted for the other in any congruence expression. |

## More Properties of Equality. For any real numbers **a**, **b**, and **c**:

| Property | Definition for Equality |
|---|---|
| Addition Property | $If\ a = b, then\ a + c = b + c$ |
| Subtraction Property | $If\ a = b, then\ a - c = b - c$ |
| Multiplication Property | $If\ a = b, then\ a \cdot c = b \cdot c$ |
| Division Property | $If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$ |

## Properties of Addition and Multiplication. For any real numbers **a**, **b**, and **c**:

| Property | Definition for Addition | Definition for Multiplication |
|---|---|---|
| Commutative Property | $a + b = b + a$ | $a \cdot b = b \cdot a$ |
| Associative Property | $(a + b) + c = a + (b + c)$ | $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ |
| Distributive Property | $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ | |

# Geometry
# Inductive vs. Deductive Reasoning

## Inductive Reasoning

**Inductive reasoning** uses observation to form a hypothesis or conjecture. The hypothesis can then be tested to see if it is true. The test must be performed in order to confirm the hypothesis.

**Example:** Observe that the sum of the numbers 1 to 4 is $(4 \cdot 5/2)$ and that the sum of the numbers 1 to 5 is $(5 \cdot 6/2)$. Hypothesis: the sum of the first n numbers is $(n * (n+1)/2)$. Testing this hypothesis confirms that it is true.

## Deductive Reasoning

**Deductive reasoning** argues that if something is true about a broad category of things, it is true of an item in the category.

> **Example:** All birds have beaks. A pigeon is a bird; therefore, it has a beak.

There are two key types of deductive reasoning of which the student should be aware:

- **Law of Detachment.** **Given that $p \rightarrow q$, if $p$ is true then $q$ is true.** In words, if one thing implies another, then whenever the first thing is true, the second must also be true.

  **Example:** Start with the statement: "If a living creature is human, then it has a brain." Then because you are human, we can conclude that you have a brain.

- **Syllogism.** **Given that $p \rightarrow q$ and $q \rightarrow r$, we can conclude that $p \rightarrow r$.** This is a kind of transitive property of logic. In words, if one thing implies a second and that second thing implies a third, then the first thing implies the third.

  **Example:** Start with the statements: "If my pencil breaks, I will not be able to write," and "if I am not able to write, I will not pass my test." Then I can conclude that "If my pencil breaks, I will not pass my test."

# Geometry
# An Approach to Proofs

Learning to develop a successful proof is one of the key skills students develop in geometry. The process is different from anything students have encountered in previous math classes, and may seem difficult at first. Diligence and practice in solving proofs will help students develop reasoning skills that will serve them well for the rest of their lives.

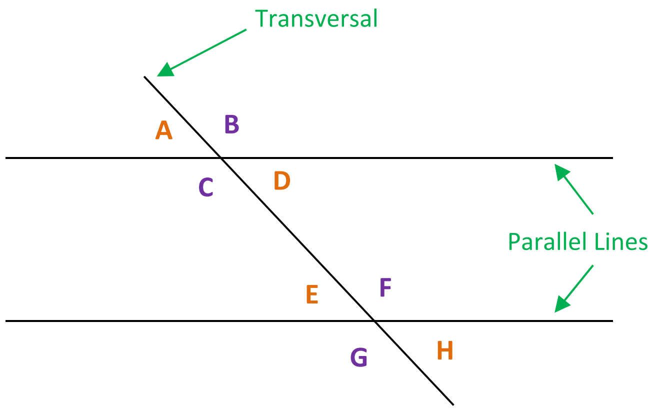
## Requirements in Performing Proofs

- Each proof starts with a set of "givens," statements that you are supplied and from which you must derive a "conclusion." Your mission is to start with the givens and to proceed logically to the conclusion, providing reasons for each step along the way.
- Each step in a proof builds on what has been developed before. Initially, you look at what you can conclude from the" givens." Then as you proceed through the steps in the proof, you are able to use additional things you have concluded based on earlier steps.
- Each step in a proof must have a valid reason associated with it. So, each statement in the proof must be furnished with an answer to the question: "Why is this step valid?"

## Tips for Successful Proof Development

- At each step, think about what you know and what you can conclude from that information. Do this initially without regard to what you are being asked to prove. Then look at each thing you can conclude and see which ones move you closer to what you are trying to prove.
- Go as far as you can into the proof from the beginning. If you get stuck, …
- Work backwards from the end of the proof. Ask yourself what the last step in the proof is likely to be. For example, if you are asked to prove that two triangles are congruent, try to see which of the several theorems about this is most likely to be useful based on what you were given and what you have been able to prove so far.
- Continue working backwards until you see steps that can be added to the front end of the proof. You may find yourself alternating between the front end and the back end until you finally bridge the gap between the two sections of the proof.
- Don't skip any steps. Some things appear obvious, but actually have a mathematical reason for being true. For example, $a = a$ might seem obvious, but "obvious" is not a valid reason in a geometry proof. The reason for $a = a$ is a property of algebra called the "reflexive property of equality." Use mathematical reasons for all your steps.

# Geometry
## Parallel Lines and Transversals




**Alternate**: refers to angles that are on opposite sides of the transversal.

**Consecutive:** refers to angles that are on the same side of the transversal.

**Interior**: refers to angles that are between the parallel lines.

**Exterior**: refers to angles that are outside the parallel lines.

### Corresponding Angles

**Corresponding Angles** are angles in the same location relative to the parallel lines and the transversal. For example, the angles on top of the parallel lines and left of the transversal (i.e., top left) are corresponding angles.

Angles **A** and **E (top left)** are **Corresponding Angles**. So are angle pairs **B** and **F (top right)**, **C** and **G (bottom left)**, and **D** and **H (bottom right)**. Corresponding angles are congruent.

### Alternate Interior Angles

Angles **D** and **E** are **Alternate Interior Angles**. Angles **C** and **F** are also alternate interior angles. Alternate interior angles are congruent.

### Alternate Exterior Angles

Angles **A** and **H** are **Alternate Exterior Angles**. Angles **B** and **G** are also alternate exterior angles. Alternate exterior angles are congruent.

### Consecutive Interior Angles

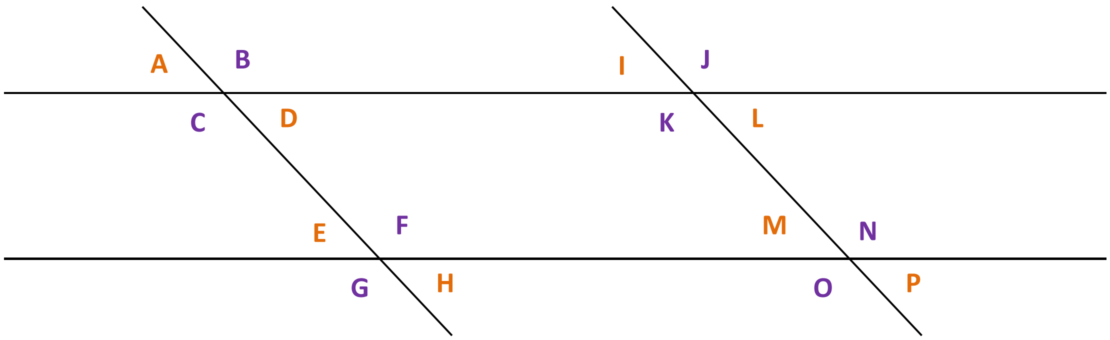
Angles **C** and **E** are **Consecutive Interior Angles**. Angles **D** and **F** are also consecutive interior angles. Consecutive interior angles are supplementary.

*Note that angles* ***A****,* ***D****,* ***E****, and* ***H*** *are congruent, and angles* ***B****,* ***C****,* ***F****, and* ***G*** *are congruent. In addition, each of the angles in the first group are supplementary to each of the angles in the second group.*

# Geometry
# Multiple Sets of Parallel Lines

## Two Transversals

Sometimes, the student is presented two sets of intersecting parallel lines, as shown above. Note that each pair of parallel lines is a set of transversals to the other set of parallel lines.



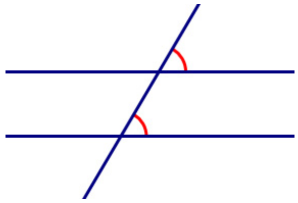

In this case, the following groups of angles are congruent:

- Group 1: Angles **A**, **D**, **E**, **H**, **I**, **L**, **M** and **P** are all congruent.
- Group 2: Angles **B**, **C**, **F**, **G**, **J**, **K**, **N**, and **O** are all congruent.
- Each angle in the Group 1 is supplementary to each angle in Group 2.

# Geometry
# Proving Lines are Parallel

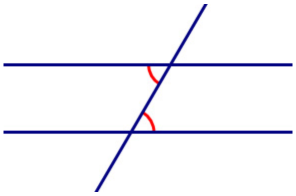
The properties of parallel lines cut by a transversal can be used to prove two lines are parallel.

## Corresponding Angles



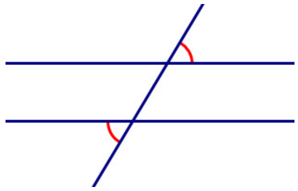
If two lines cut by a transversal have congruent corresponding angles, then the lines are parallel. Note that there are 4 sets of corresponding angles.

## Alternate Interior Angles



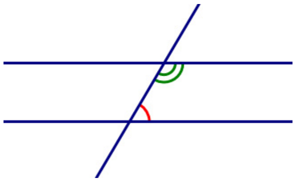
If two lines cut by a transversal have congruent alternate interior angles congruent, then the lines are parallel. Note that there are 2 sets of alternate interior angles.

## Alternate Exterior Angles



If two lines cut by a transversal have congruent alternate exterior angles, then the lines are parallel. Note that there are 2 sets of alternate exterior angles.

## Consecutive Interior Angles

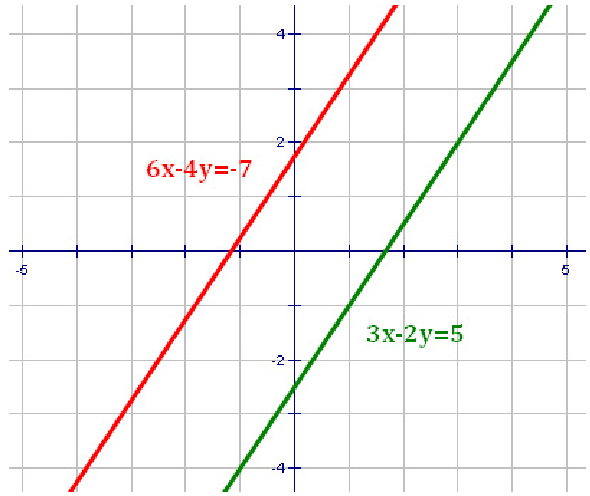


If two lines cut by a transversal have supplementary consecutive interior angles, then the lines are parallel. Note that there are 2 sets of consecutive interior angles.

# Geometry
# Parallel and Perpendicular Lines in the Coordinate Plane

## Parallel Lines

**Two lines are parallel if their slopes are equal.**




- In $y = mx + b$ form, if the values of $m$ are the same.

  **Example:** $y = 2x - 3$ and
  $y = 2x + 1$

- In Standard Form, if the coefficients of $x$ and $y$ are proportional between the equations.

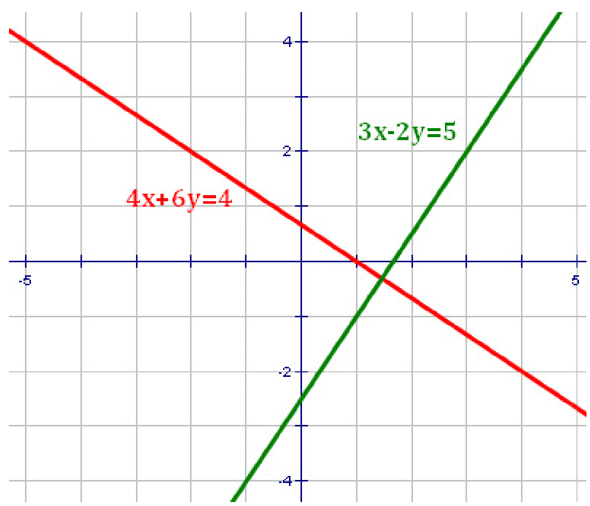
  **Example:** $3x - 2y = 5$ and
  $6x - 4y = -7$

- Also, if the lines are both vertical (i.e., their slopes are undefined).

  **Example:** $x = -3$ and
  $x = 2$

## Perpendicular Lines

**Two lines are perpendicular if the product of their slopes is $-1$.** That is, if the slopes have different signs and are multiplicative inverses.




- In $y = mx + b$ form, the values of $m$ multiply to get $-1$..

  **Example:** $y = 6x + 5$ and
  $y = -\frac{1}{6}x - 3$

- In Standard Form, if you add the product of the $x$-coefficients to the product of the $y$-coefficients and get zero.

  **Example:** $4x + 6y = 4$ and
  $3x - 2y = 5$ because $(4 \cdot 3) + \big(6 \cdot (-2)\big) = 0$

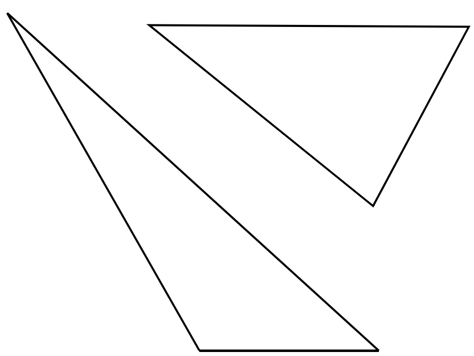
- Also, if one line is vertical (i.e., $m$ is undefined) and one line is horizontal (i.e., $m = 0$).

  **Example:** $x = 6$ and
  $y = 3$

# Geometry
## Types of Triangles

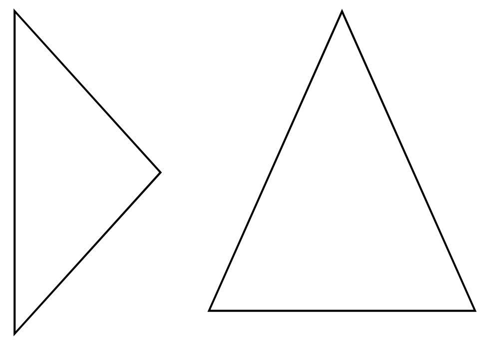
### Scalene

A Scalene Triangle has 3 sides of different lengths. Because the sides are of different lengths, the angles must also be of different measures.



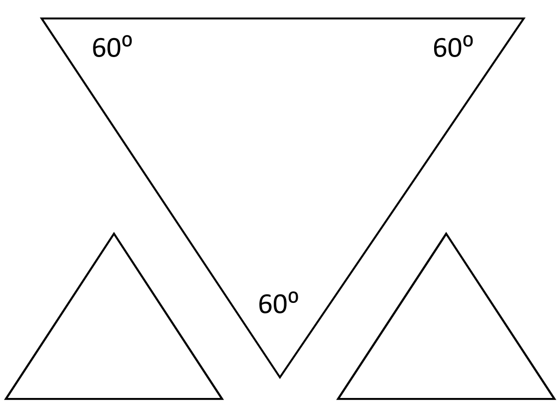
### Isosceles

An Isosceles Triangle has 2 sides the same length (i.e., congruent). Because two sides are congruent, two angles must also be congruent.



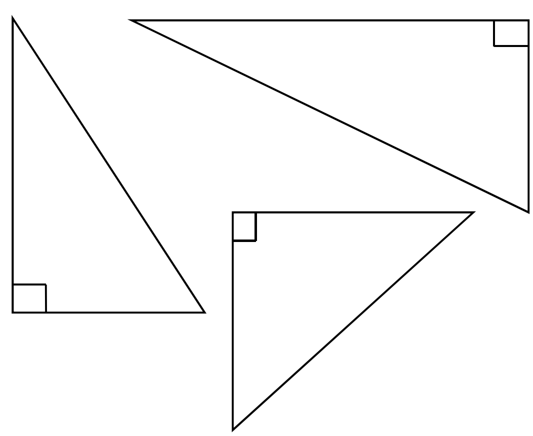
### Equilateral

An Equilateral Triangle has all 3 sides the same length (i.e., congruent). Because all 3 sides are congruent, all 3 angles must also be congruent. This requires each angle to be $60^o$.




### Right

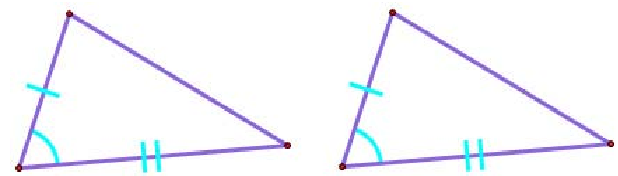
A Right Triangle is one that contains a $90^o$ angle. It may be scalene or isosceles, but cannot be equilateral. Right triangles have sides that meet the requirements of the Pythagorean Theorem.



# Geometry
# Congruent Triangles

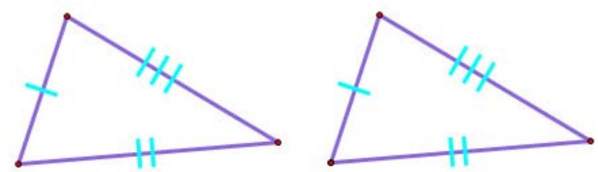
The following theorems present conditions under which triangles are congruent.

## Side-Angle-Side (SAS) Congruence



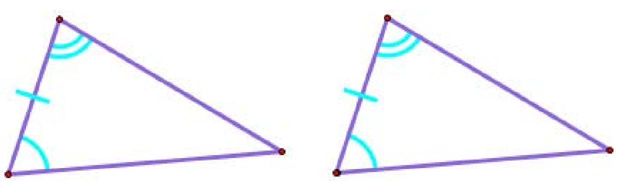
**SAS congruence** requires the congruence of two sides and the angle between those sides. Note that there is no such thing as SSA congruence; the congruent angle must be between the two congruent sides.

## Side-Side-Side (SSS) Congruence



**SSS congruence** requires the congruence of all three sides. If all of the sides are congruent then all of the angles must be congruent. The converse is not true; there is no such thing as AAA congruence.

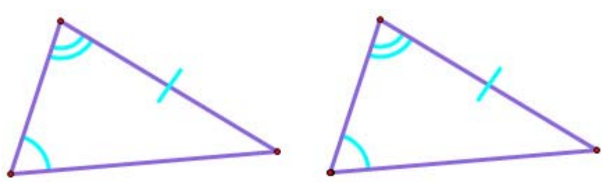
## Angle-Side-Angle (ASA) Congruence



**ASA congruence** requires the congruence of two angles and the side between those angles.

*Note: ASA and AAS combine to provide congruence of two triangles whenever any two angles and any one side of the triangles are congruent.*

## Angle-Angle-Side (AAS) Congruence



**AAS congruence** requires the congruence of two angles and a side which is not between those angles.

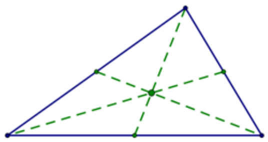
## CPCTC

**CPCTC** means **"corresponding parts of congruent triangles are congruent."** It is a very powerful tool in geometry proofs and is often used shortly after a step in the proof where a pair of triangles is proved to be congruent.

# Geometry
# Centers of Triangles

The following are all points which can be considered the center of a triangle.

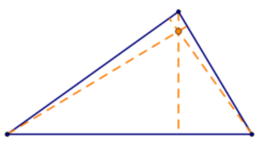
### Centroid (Medians)



The centroid is the intersection of the three medians of a triangle. A median is a line segment drawn from a vertex to the midpoint of the line opposite the vertex.

- The centroid is located 2/3 of the way from a vertex to the opposite side. That is, the distance from a vertex to the centroid is double the length from the centroid to the midpoint of the opposite line.
- The medians of a triangle create 6 inner triangles of equal area.

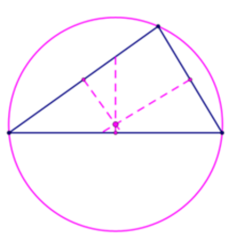
### Orthocenter (Altitudes)



The orthocenter is the intersection of the three altitudes of a triangle. An altitude is a line segment drawn from a vertex to a point on the opposite side (extended, if necessary) that is perpendicular to that side.

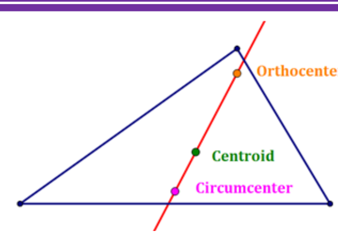
- In an acute triangle, the orthocenter is inside the triangle.
- In a right triangle, the orthocenter is the right angle vertex.
- In an obtuse triangle, the orthocenter is outside the triangle.

### Circumcenter (Perpendicular Bisectors)



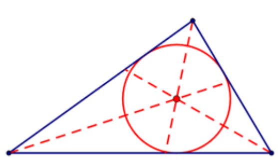
The circumcenter is the intersection of the perpendicular bisectors of the three sides of the triangle. A perpendicular bisector is a line which both bisects the side and is perpendicular to the side. The circumcenter is also the center of the circle circumscribed about the triangle.

- In an acute triangle, the circumcenter is inside the triangle.
- In a right triangle, the circumcenter is the midpoint of the hypotenuse.
- In an obtuse triangle, the circumcenter is outside the triangle.




**Euler Line**: Interestingly, the centroid, orthocenter and circumcenter of a triangle are collinear (i.e., lie on the same line, which is called the Euler Line).

### Incenter (Angle Bisectors)

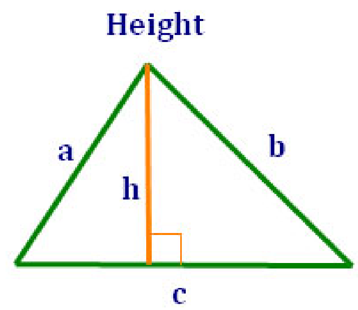


The incenter is the intersection of the angle bisectors of the three angles of the triangle. An angle bisector cuts an angle into two congruent angles, each of which is half the measure of the original angle. The incenter is also the center of the circle inscribed in the triangle.

# Geometry
## Length of Height, Median and Angle Bisector

### Height

The formula for the length of a height of a triangle is derived from Heron's formula for the area of a triangle:



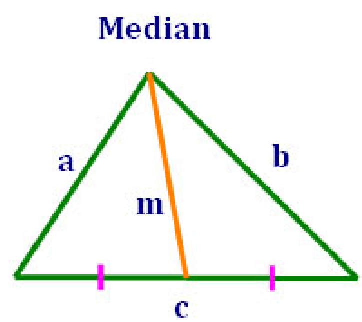

$$h = \frac{2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{c}$$

where, $s = \frac{1}{2}(a+b+c)$, and

$a, b, c$ are the lengths of the sides of the triangle.

### Median

The formula for the length of a median of a triangle is:

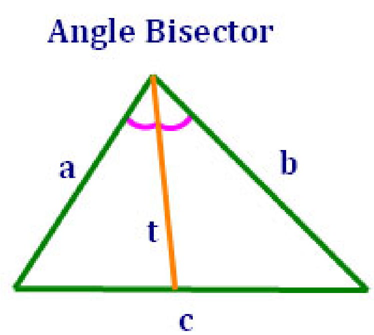



$$m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c}$$

where, $a, b, c$ are the lengths of the sides of the triangle.

### Angle Bisector

The formula for the length of an angle bisector of a triangle is:

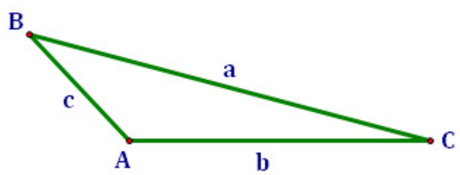



$$t = \sqrt{ab\left(1 - \frac{c^2}{(a+b)^2}\right)}$$

where, $a, b, c$ are the lengths of the sides of the triangle.

# Geometry
# Inequalities in Triangles

Angles and their opposite sides in triangles are related. In fact, this is often reflected in the labeling of angles and sides in triangle illustrations.




Angles and their opposite sides are often labeled with the same letter. An upper case letter is used for the angle and a lower case letter is used for the side.

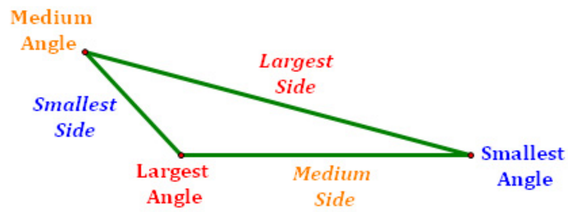
The relationship between angles and their opposite sides translates into the following triangle inequalities:

$$\text{If}\quad m\angle C < m\angle B < m\angle A,\quad \text{then}\quad c < b < a$$

$$\text{If}\quad m\angle C \le m\angle B \le m\angle A,\quad \text{then}\quad c \le b \le a$$

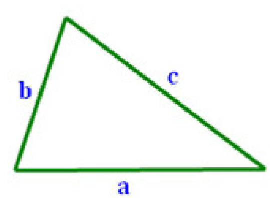
That is, in any triangle,

- The largest side is opposite the largest angle.
- The medium side is opposite the medium angle.
- The smallest side is opposite the smallest angle.




## Other Inequalities in Triangles

**Triangle Inequality:** The sum of the lengths of any two sides of a triangle is greater than the length of the third side. This is a crucial element in deciding whether segments of any 3 lengths can form a triangle.




$$a + b > c \quad \text{and} \quad b + c > a \quad \text{and} \quad c + a > b$$

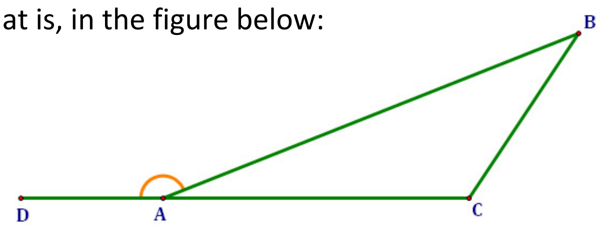
**Exterior Angle Inequality:** The measure of an external angle is greater than the measure of either of the two non-adjacent interior angles. That is, in the figure below:

$$m\angle DAB > m\angle B \quad \text{and} \quad m\angle DAB > m\angle C$$

**Note:** the Exterior Angle Inequality is much less relevant than the Exterior Angle Equality.

**Exterior Angle Equality:** The measure of an external angle is equal to the sum of the measures of the two non-adjacent interior angles. That is, in the figure below:

$$m\angle DAB = m\angle B + m\angle C$$



# Geometry
# Polygons - Basics

## Basic Definitions

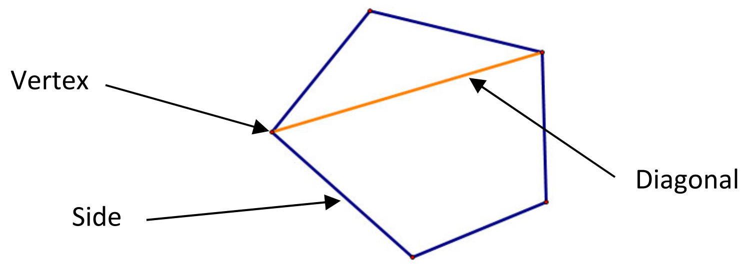
**Polygon:** a closed path of three or more line segments, where:

- no two sides with a common endpoint are collinear, and
- each segment is connected at its endpoints to exactly two other segments.

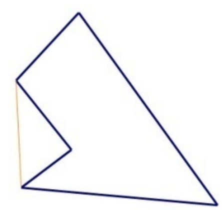
**Side:** a segment that is connected to other segments (which are also sides) to form a polygon.

**Vertex:** a point at the intersection of two sides of the polygon. (plural form: **vertices**)

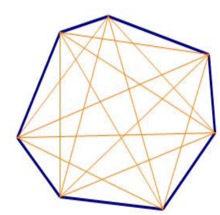
**Diagonal:** a segment, from one vertex to another, which is not a side.




**Concave:** A polygon in which it is possible to draw a diagonal "outside" the polygon. (Notice the orange diagonal drawn outside the polygon at right.) Concave polygons actually look like they have a "cave" in them.



**Convex:** A polygon in which it is not possible to draw a diagonal "outside" the polygon. (Notice that all of the orange diagonals are inside the polygon at right.) Convex polygons appear more "rounded" and do not contain "caves."



## Names of Some Common Polygons

| Number of Sides | Name of Polygon | Number of Sides | Name of Polygon |
|---|---|---|---|
| 3 | Triangle | 9 | Nonagon |
| 4 | Quadrilateral | 10 | Decagon |
| 5 | Pentagon | 11 | Undecagon |
| 6 | Hexagon | 12 | Dodecagon |
| 7 | Heptagon | 20 | Icosagon |
| 8 | Octagon | $n$ | $n$-gon |

Names of polygons are generally formed from the Greek language; however, some hybrid forms of Latin and Greek (e.g., undecagon) have crept into common usage.

# Geometry
# Polygons – More Definitions

## Definitions

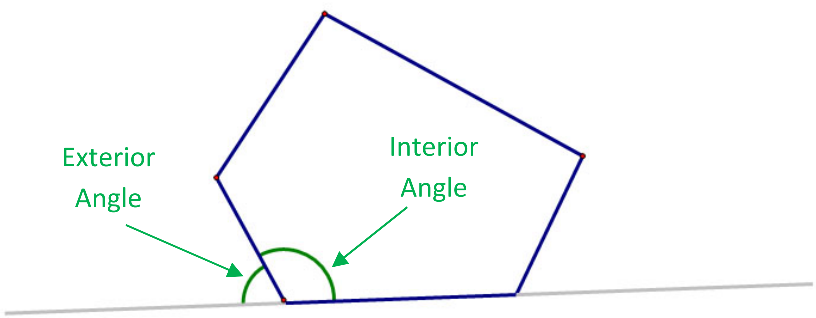
**Equilateral:** a polygon in which all of the sides are equal in length.

**Equiangular:** a polygon in which all of the angles have the same measure.

**Regular:** a polygon which is both equilateral and equiangular. That is, a ***regular polygon*** is one in which all of the sides have the same length and all of the angles have the same measure.

**Interior Angle:** An angle formed by two sides of a polygon. The angle is inside the polygon.

**Exterior Angle:** An angle formed by one side of a polygon and the line containing an adjacent side of the polygon. The angle is outside the polygon.




**"Advanced" Definitions:**

**Simple Polygon:** *a polygon whose sides do not intersect at any location other than its endpoints.* Simple polygons always divide a plane into two regions – one inside the polygon and one outside the polygon.

**Complex Polygon:** *a polygon with sides that intersect someplace other than their endpoints* (i.e., not a simple polygon). Complex polygons do not always have well-defined insides and outsides.

**Skew Polygon:** *a polygon for which not all of its vertices lie on the same plane.*

## How Many Diagonals Does a Convex Polygon Have?

Believe it or not, this is a common question with a simple solution. Consider a polygon with $n$ sides and, therefore, $n$ vertices.

- Each of the $n$ vertices of the polygon can be connected to $(n-3)$ other vertices with diagonals. That is, it can be connected to all other vertices except itself and the two to which it is connected by sides. So, there are $[\,n \cdot (n-3)]$ lines to be drawn as diagonals.
- However, when we do this, we draw each diagonal twice because we draw it once from each of its two endpoints. So, the number of diagonals is actually half of the number we calculated above.
- Therefore, **the number of diagonals in an $n$-sided polygon is:**

$$\frac{n \cdot (n-3)}{2}$$

# Geometry
# Interior and Exterior Angles of a Polygon

## Interior Angles

The sum of the interior angles in an $n$-sided polygon is:

$$\sum = (n-2) \cdot 180°$$

If the polygon is regular, you can calculate the measure of each interior angle as:

$$\frac{(n-2) \cdot 180°}{n}$$

**Notation:** The Greek letter **"Σ"** is equivalent to the English letter **"S"** and is math short-hand for a summation (i.e., addition) of things.

| Interior Angles | | |
|---|---|---|
| **Sides** | **Sum of Interior Angles** | **Each Interior Angle** |
| 3 | $180^o$ | $60^o$ |
| 4 | $360^o$ | $90^o$ |
| 5 | $540^o$ | $108^o$ |
| 6 | $720^o$ | $120^o$ |
| 7 | $900^o$ | $129^o$ |
| 8 | $1,080^o$ | $135^o$ |
| 9 | $1,260^o$ | $140^o$ |
| 10 | $1,440^o$ | $144^o$ |

## Exterior Angles

No matter how many sides there are in a polygon, the sum of the exterior angles is:

$$\sum = 360^0$$

If the polygon is regular, you can calculate the measure of each exterior angle as:

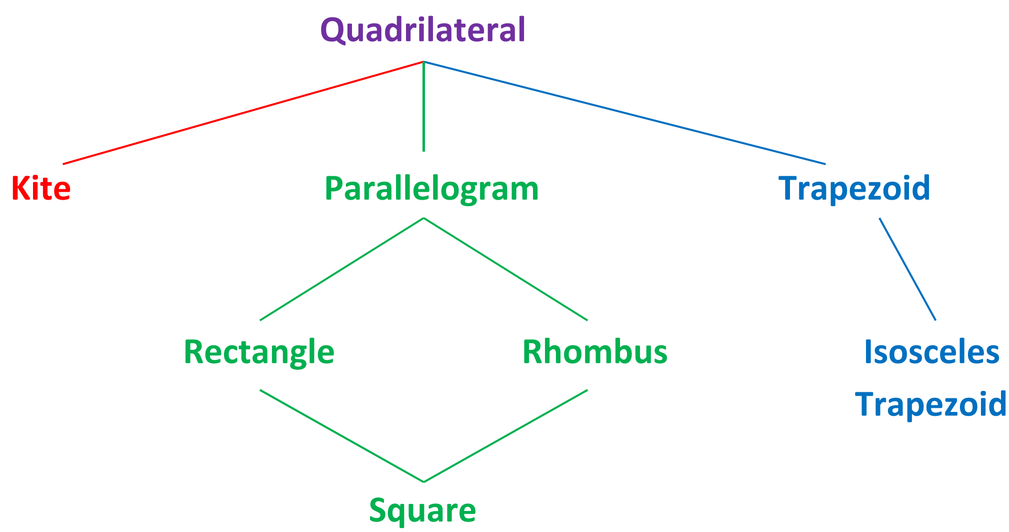
$$\frac{360^0}{n}$$

| Exterior Angles | | |
|---|---|---|
| **Sides** | **Sum of Exterior Angles** | **Each Exterior Angle** |
| 3 | $360^o$ | $120^o$ |
| 4 | $360^o$ | $90^o$ |
| 5 | $360^o$ | $72^o$ |
| 6 | $360^o$ | $60^o$ |
| 7 | $360^o$ | $51^o$ |
| 8 | $360^o$ | $45^o$ |
| 9 | $360^o$ | $40^o$ |
| 10 | $360^o$ | $36^o$ |

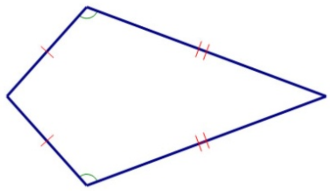
# Geometry
## Definitions of Quadrilaterals

| **Name** | **Definition** |
|---|---|
| **Quadrilateral** | A polygon with 4 sides. |
| **Kite** | A quadrilateral with two consecutive pairs of congruent sides, but with opposite sides not congruent. |
| **Trapezoid** | A quadrilateral with exactly one pair of parallel sides. |
| **Isosceles Trapezoid** | A trapezoid with congruent legs. |
| **Parallelogram** | A quadrilateral with both pairs of opposite sides parallel. |
| **Rectangle** | A parallelogram with all angles congruent (i.e., right angles). |
| **Rhombus** | A parallelogram with all sides congruent. |
| **Square** | A quadrilateral with all sides congruent and all angles congruent. |

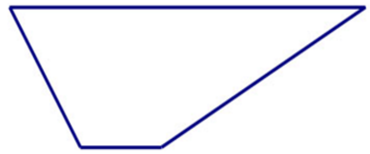
### Quadrilateral Tree:



# Geometry
# Figures of Quadrilaterals

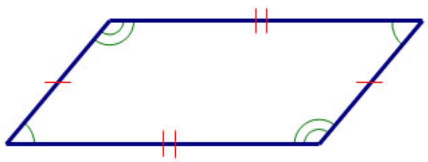


## Kite
- 2 consecutive pairs of congruent sides
- 1 pair of congruent opposite angles
- Diagonals perpendicular

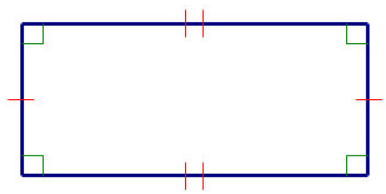


## Trapezoid
- 1 pair of parallel sides (called "bases")
- Angles on the same "side" of the bases are supplementary

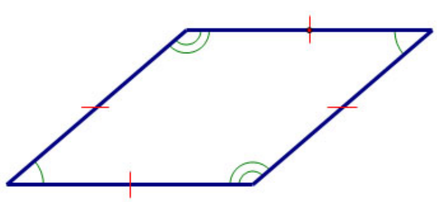
## Isosceles Trapezoid
- 1 pair of parallel sides
- Congruent legs
- 2 pair of congruent base angles
- Diagonals congruent



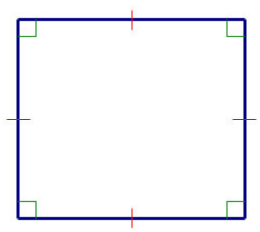
## Parallelogram
- Both pairs of opposite sides parallel
- Both pairs of opposite sides congruent
- Both pairs of opposite angles congruent
- Consecutive angles supplementary
- Diagonals bisect each other



## Rectangle
- Parallelogram with all angles congruent (i.e., right angles)
- Diagonals congruent



## Rhombus
- Parallelogram with all sides congruent
- Diagonals perpendicular
- Each diagonal bisects a pair of opposite angles



## Square
- Both a Rhombus and a Rectangle
- All angles congruent (i.e., right angles)
- All sides congruent

# Geometry
## Characteristics of Parallelograms

| Characteristic | Square | Rhombus | Rectangle | Parallelogram |
|---|---|---|---|---|
| 2 pair of parallel sides | ✓ | ✓ | ✓ | ✓ |
| Opposite sides are congruent | ✓ | ✓ | ✓ | ✓ |
| Opposite angles are congruent | ✓ | ✓ | ✓ | ✓ |
| Consecutive angles are supplementary | ✓ | ✓ | ✓ | ✓ |
| Diagonals bisect each other | ✓ | ✓ | ✓ | ✓ |
| All 4 angles are congruent (i.e., right angles) | ✓ | | ✓ | |
| Diagonals are congruent | ✓ | | ✓ | |
| All 4 sides are congruent | ✓ | ✓ | | |
| Diagonals are perpendicular | ✓ | ✓ | | |
| Each diagonal bisects a pair of opposite angles | ✓ | ✓ | | |

Notes: Red ✓-marks are conditions sufficient to prove the quadrilateral is of the type specified.
Green ✓-marks are conditions sufficient to prove the quadrilateral is of the type specified if the quadrilateral is a parallelogram.

# Geometry
# Parallelogram Proofs

## Proving a Quadrilateral is a Parallelogram

To prove a quadrilateral is a parallelogram, prove any of the following conditions:

1. Both pairs of opposite sides are parallel. (note: this is the definition of a parallelogram)
2. Both pairs of opposite sides are congruent.
3. Both pairs of opposite angles are congruent.
4. An interior angle is supplementary to both of its consecutive angles.
5. Its diagonals bisect each other.
6. A pair of opposite sides is both parallel and congruent.

## Proving a Quadrilateral is a Rectangle

To prove a quadrilateral is a rectangle, prove any of the following conditions:

1. All 4 angles are congruent.
2. It is a parallelogram and its diagonals are congruent.

## Proving a Quadrilateral is a Rhombus

To prove a quadrilateral is a rhombus, prove any of the following conditions:

1. All 4 sides are congruent.
2. It is a parallelogram and Its diagonals are perpendicular.
3. It is a parallelogram and each diagonal bisects a pair of opposite angles.

## Proving a Quadrilateral is a Square

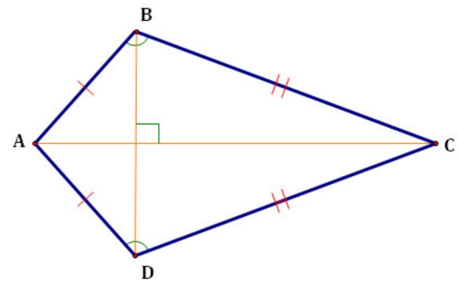
To prove a quadrilateral is a square, prove:

1. It is both a Rhombus and a Rectangle.

# Geometry
# Kites and Trapezoids

## Facts about a Kite

To prove a quadrilateral is a kite, prove:



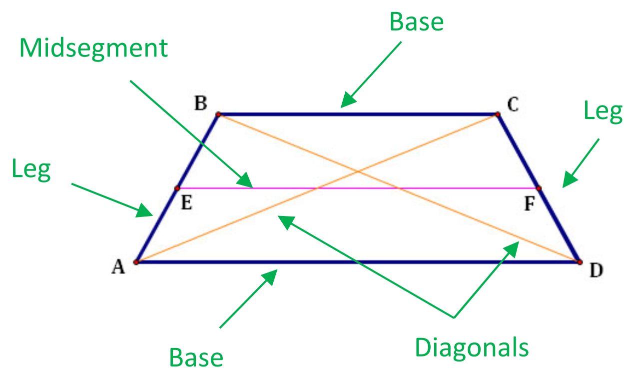

- It has two pair of congruent sides.
- Opposite sides are not congruent.

Also, if a quadrilateral is a kite, then:

- Its diagonals are perpendicular
- It has exactly one pair of congruent opposite angles.

## Parts of a Trapezoid

Trapezoid ABCD has the following parts:




- $\overline{AD}$ and $\overline{BC}$ are **bases**.
- $\overline{AB}$ and $\overline{CD}$ are **legs**.
- $\overline{EF}$ is the **midsegment**.
- $\overline{AC}$ and $\overline{BD}$ are **diagonals**.
- Angles $A$ and $D$ form a pair of **base angles**.
- Angles $B$ and $C$ form a pair of **base angles**.

## Trapezoid Midsegment Theorem

The **midsegment** of a trapezoid is parallel to each of its bases and: $EF = \frac{1}{2}\,(AD + BC)$.

## Proving a Quadrilateral is an Isosceles Trapezoid

To prove a quadrilateral is an isosceles trapezoid, prove any of the following conditions:

1. It is a trapezoid and has a pair of congruent legs. (definition of isosceles trapezoid)
2. It is a trapezoid and has a pair of congruent base angles.
3. It is a trapezoid and its diagonals are congruent.

# Geometry
# Introduction to Transformation

A **Transformation** is a mapping of the pre-image of a geometric figure onto an image that retains key characteristics of the pre-image.

## Definitions

The **Pre-Image** is the geometric figure before it has been transformed.

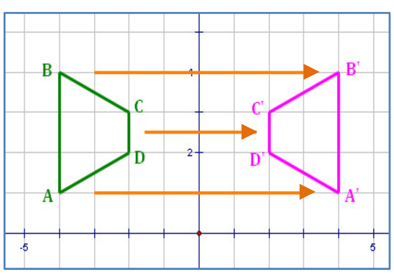
The **Image** is the geometric figure after it has been transformed.

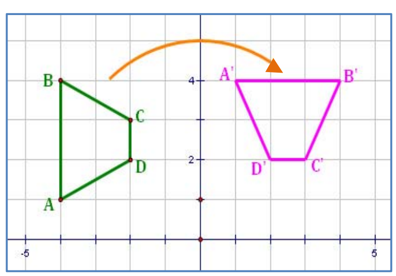
A **mapping** is an association between objects. Transformations are types of mappings. In the figures below, we say *$ABCD$ is mapped onto $A'B'C'D'$*, or $ABCD \longrightarrow A'B'C'D'$. The order of the vertices is critical to a properly named mapping.

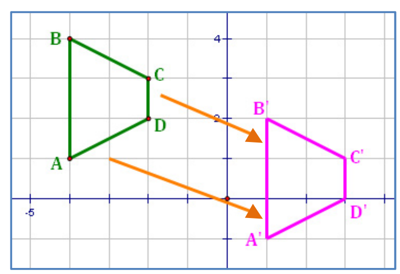
An **Isometry** is a one-to-one mapping that preserves lengths. Transformations that are isometries (i.e., preserve length) are called **rigid transformations**.

## Isometric Transformations

| Reflection | Rotation | Translation |
|---|---|---|
| **Reflection** is flipping a figure across a line called a "mirror." The figure retains its size and shape, but appears "backwards" after the reflection. | **Rotation** is turning a figure around a point. Rotated figures retain their size and shape, but not their orientation. | **Translation** is sliding a figure in the plane so that it changes location but retains its shape, size and orientation. |




## Table of Characteristics of Isometric Transformations

| Transformation | Reflection | Rotation | Translation |
|---|---|---|---|
| **Isometry (Retains Lengths)?** | Yes | Yes | Yes |
| **Retains Angles?** | Yes | Yes | Yes |
| **Retains Orientation to Axes?** | No | No | Yes |

# Geometry
# Introduction to Transformation (cont'd)

## Transformation of a Point

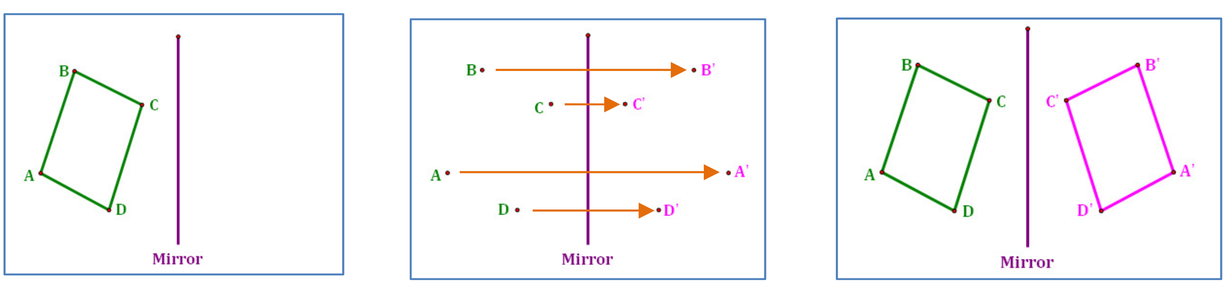
A point is the easiest object to transform. Simply reflect, rotate or translate it following the rules for the transformation selected. By transforming key points first, any transformation becomes much easier.

## Transformation of a Geometric Figure

To transform any geometric figure, it is only necessary to transform the items that define the figure, and then re-form it. For example:

- To transform a **line segment**, transform its two endpoints, and then connect the resulting images with a line segment.
- To transform a **ray**, transform the initial point and any other point on the ray, and then construct a ray using the resulting images.
- To transform a **line**, transform any two points on the line, and then fit a line through the resulting images.
- To transform a **polygon**, transform each of its vertices, and then connect the resulting images with line segments.
- To transform a **circle**, transform its center and, if necessary, its radius. From the resulting images, construct the image circle.
- To transform **other conic sections (parabolas, ellipses and hyperbolas)**, transform the foci, vertices and/or directrix. From the resulting images, construct the image conic section.

**Example:** Reflect Quadrilateral ABCD



# Geometry
# Reflection

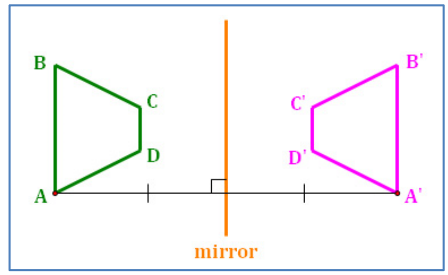
## Definitions

**Reflection** is flipping a figure across a mirror.

The **Line of Reflection** is the mirror through which the reflection takes place.

Note that:

- The line segment connecting corresponding points in the image and pre-image is bisected by the mirror.
- The line segment connecting corresponding points in the image and pre-image is perpendicular to the mirror.




## Reflection through an Axis or the Line $y = x$

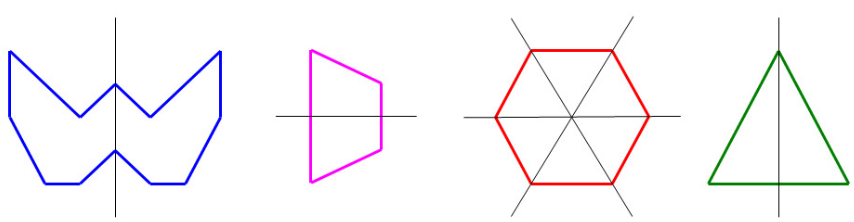
Reflection of the point (a, b) through the $x$- or $y$-*axis* or the line $y = x$ gives the following results:

| Pre-Image Point | Mirror Line | Image Point |
|---|---|---|
| (a, b) | x-axis | (a, -b) |
| (a, b) | y-axis | (-a, b) |
| (a, b) | the line: $y = x$ | (b, a) |

**If you forget the above table**, start with the point $(3, 2)$ on a set of coordinate axes. Reflect the point through the selected line and see which set of "a, b" coordinates works.

## Line of Symmetry

A **Line of Symmetry** is any line through which a figure can be mapped onto itself. The thin black lines in the following figures show their axes of symmetry:

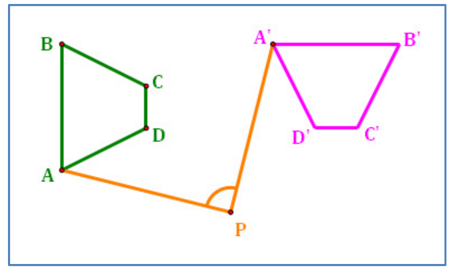


# Geometry
# Rotation

## Definitions

**Rotation** is turning a figure by an angle about a fixed point.

The **Center of Rotation** is the point about which the figure is rotated. Point ***P***, at right, is the center of rotation.




The **Angle of Rotation** determines the extent of the rotation. The angle is formed by the rays that connect the center of rotation to the pre-image and the image of the rotation. Angle ***P***, at right, is the angle of rotation. Though shown only for Point $A$, the angle is the same for any of the figure's 4 vertices.

**Note:** In performing rotations, it is important to indicate the direction of the rotation – clockwise or counterclockwise.

## Rotation about the Origin

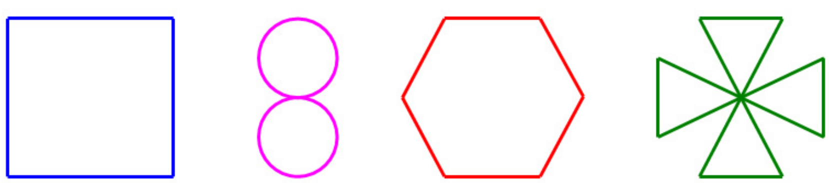
Rotation of the point (a, b) about the origin (0, 0) gives the following results:

| Pre-Image Point | Clockwise Rotation | Counterclockwise Rotation | Image Point |
|---|---|---|---|
| (a, b) | $90^0$ | $270^0$ | (b, -a) |
| (a, b) | $180^0$ | $180^0$ | (-a, -b) |
| (a, b) | $270^0$ | $90^0$ | (-b, a) |
| (a, b) | $360^0$ | $360^0$ | (a, b) |

**If you forget the above table**, start with the point (3, 2) on a set of coordinate axes. Rotate the point by the selected angle and see which set of "a, b" coordinates works.

## Rotational Symmetry

A figure in a plane has **Rotational Symmetry** if it can be mapped onto itself by a rotation of $180^0$ or less. Any regular polygon has rotational symmetry, as does a circle. Here are some examples of figures with rotational symmetry:



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# Geometry
# Rotation by $90^o$ about a Point $(x_0, y_0)$

Rotating an object by $90^o$ about a point involves rotating each point of the object by $90^o$ about that point. For a polygon, this is accomplished by rotating each vertex and then connecting them to each other, so you mainly have to worry about the vertices, which are points. The mathematics behind the process of rotating a point by $90^o$ is described below:

Let's define the following points:

- The point about which the rotation will take place: $(x_0, y_0)$
- The initial point (before rotation): $(x_1, y_1)$
- The final point (after rotation): $(x_2, y_2)$

The problem is to determine $(x_2, y_2)$ if we are given $(x_0, y_0)$ and $(x_1, y_1)$. It involves 3 steps:

1. Convert the problem to one of rotating a point about the origin (a much easier problem).
2. Perform the rotation.
3. Convert the result back to the original set of axes.

We'll consider each step separately and provide an example:

**Problem: Rotate a point by $90^o$ about another point.**

**Step 1: Convert the problem to one of rotating a point about the origin:**

First, we ask how the point $(x_1, y_1)$ relates to the point about which it will be rotated $(x_0, y_0)$ and create a new ("translated") point. This is essentially an "axis-translation," which we will reverse in Step 3.

| General Situation | Example |
|---|---|
| Points in the Problem<br>• Rotation Center: $(x_0, y_0)$<br>• Initial point: $(x_1, y_1)$<br>• Final point: $(x_2, y_2)$ | Points in the Problem<br>• Rotation Center: (2, 3)<br>• Initial point: (-2, 1)<br>• Final point: to be determined |
| Calculate a new point that represents how $(x_1, y_1)$ relates to $(x_0, y_0)$. That point is:<br>$(x_1\text{-}x_0, y_1\text{-}y_0)$ | Calculate a new point that represents how (-2, 1) relates to (2, 3). That point is:<br>(-4, -2) |

The next steps depend on whether we are making a clockwise or counter clockwise rotation.

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# Geometry
# Rotation by 90° about a Point (cont'd)

## *Clockwise Rotation:*

### Step 2: Perform the rotation about the origin:

Rotating by 90° clockwise about the origin (0, 0) is simply a process of switching the x- and y-values of a point and negating *the new y-term*. That is (x, y) becomes (y, -x) after rotation by 90°.

| General Situation | Example |
| --- | --- |
| Pre-rotated point (from Step 1):<br>$(x_1-x_0,\ y_1-y_0)$<br>Point after rotation:<br>$(y_1-y_0,\ -x_1+x_0)$ | Pre-rotated point (from Step 1):<br>(-4, -2)<br>Point after rotation:<br>(-2, 4) |

### Step 3: Convert the result back to the original set of axes.

To do this, simply add back the point of rotation (which was subtracted out in Step 1.

| General Situation | Example |
| --- | --- |
| Point after rotation:<br>$(y_1-y_0,\ -x_1+x_0)$<br>Add back the point of rotation $(x_0,\ y_0)$:<br>$(y_1-y_0+x_0\ ,\ -x_1+x_0+y_0)$<br>which gives us the values of $(x_2,\ y_2)$ | Point after rotation:<br>(-2, 4)<br>Add back the point of rotation (2, 3):<br>(0, 7) |

Finally, look at the formulas for $x_2$ and $y_2$:

**Clockwise Rotation**

$$x_2 = y_1 - y_0 + x_0$$

$$y_2 = -x_1 + x_0 + y_0$$

*Notice that the formulas for clockwise and counter-clockwise rotation by 90° are the same except the terms in blue are negated between the formulas.*

***Interesting note:*** *If you are asked to find the point about which the rotation occurred, you simply substitute in the values for the starting point* $(x_1, y_1)$ *and the ending point* $(x_2, y_2)$ *and solve the resulting pair of simultaneous equations for* $x_0$ *and* $y_0$*.*

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# Geometry
# Rotation by $90^o$ about a Point (cont'd)

## *Counter-Clockwise Rotation:*

### Step 2: Perform the rotation about the origin:

Rotating by $90^o$ counter-clockwise about the origin (0, 0) is simply a process of switching the x- and y-values of a point and negating *the new x-term*. That is (x, y) becomes (-y, x) after rotation by $90^o$.

| General Situation | Example |
|---|---|
| Pre-rotated point (from Step 1): $(x_1-x_0,\ y_1-y_0)$<br>Point after rotation: $(-y_1+y_0,\ x_1-x_0)$ | Pre-rotated point (from Step 1): (-4, -2)<br>Point after rotation: (2, -4) |

### Step 3: Convert the result back to the original set of axes.

To do this, simply add back the point of rotation (which was subtracted out in Step 1.

| General Situation | Example |
|---|---|
| Point after rotation: $(-y_1+y_0,\ x_1-x_0)$<br>Add back the point of rotation $(x_0,\ y_0)$: $(-y_1+y_0+x_0\ ,\ x_1-x_0+y_0)$<br>which gives us the values of $(x_2,\ y_2)$ | Point after rotation: (2,-4)<br>Add back the point of rotation (2, 3): (4, -1) |

Finally, look at the formulas for $x_2$ and $y_{2:}$

**Counter-Clockwise Rotation**

$$x_2 = -y_1 + y_0 + x_0$$

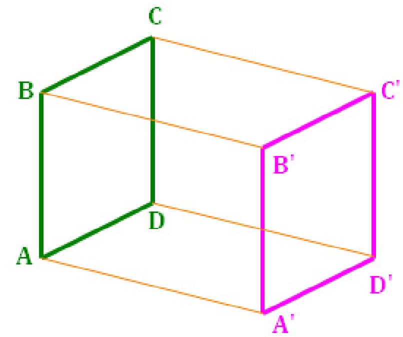
$$y_2 = x_1 - x_0 + y_0$$

*Notice that the formulas for clockwise and counter-clockwise rotation by $90^o$ are the same except the terms in blue are negated between the formulas.*

***Interesting note:*** *The point half-way between the clockwise and counter-clockwise rotations of $90^o$ is the center of rotation itself,* ***$(x_0, y_0)$****. In the example,* ***(2, 3)*** *is halfway between* ***(0, 7)*** *and* ***(4, -1)****.*

# Geometry
# Translation

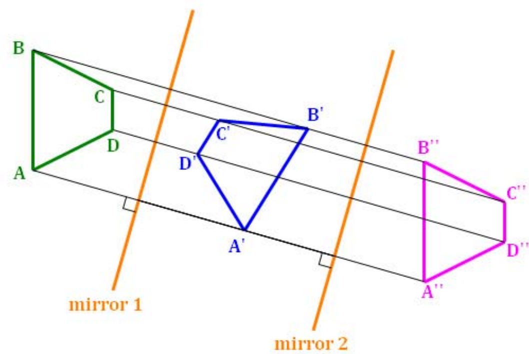
## Definitions




**Translation** is sliding a figure in the plane. Each point in the figure is moved the same distance in the same direction. The result is an image that looks the same as the pre-image in every way, except it has been moved to a different location in the plane.

Each of the four orange line segments in the figure at right has the same length and direction.

## When Two Reflections = One Translation




If two mirrors are parallel, then reflection through one of them, followed by a reflection through the second is a translation.

In the figure at right, the black lines show the paths of the two reflections; this is also the path of the resulting translation. Note the following:

- The distance of the resulting translation (e.g., from $\boldsymbol{A}$ to $\boldsymbol{A''}$) is double the distance between the mirrors.
- The black lines of movement are perpendicular to both mirrors.

## Defining Translations in the Coordinate Plane (Using Vectors)

A translation moves each point by the same distance in the same direction. In the coordinate plane, this is equivalent to moving each point the same amount in the $x$*-direction* and the same amount in the $y$*-direction*. This combination of $x$- and $y$*-direction* movement is described by a mathematical concept called a **vector**.

In the above figure, translation from $\boldsymbol{A}$ to $\boldsymbol{A''}$ moves *10* in the $x$*-direction* and the *-3* in the $y$*-direction*. In vector notation, this is: $\overrightarrow{\boldsymbol{AA''}} = \langle \mathbf{10}, -\mathbf{3} \rangle$. Notice the "half-ray" symbol over the two points and the funny-looking brackets around the movement values.

So, the translation resulting from the two reflections in the above figure moves each point of the pre-image by the vector $\overrightarrow{\boldsymbol{AA''}}$. Every translation can be defined by the vector representing its movement in the coordinate plane.

# Geometry
# Compositions

When multiple transformations are combined, the result is called a **Composition of the Transformations**. Two examples of this are:

- Combining two reflections through parallel mirrors to generate a translation (see the previous page).
- Combining a translation and a reflection to generate what is called a **glide reflection**. The glide part of the name refers to translation, which is a kind of gliding of a figure on the plane.

**Note:** In a **glide reflection**, if the line of reflection is parallel to the direction of the translation, it does not matter whether the reflection or the translation is performed first.

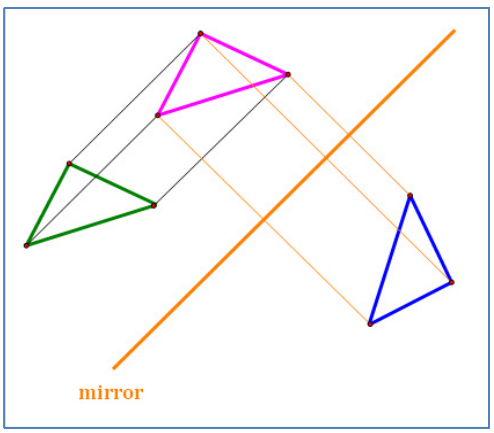



**Figure 1: Translation followed by Reflection.**

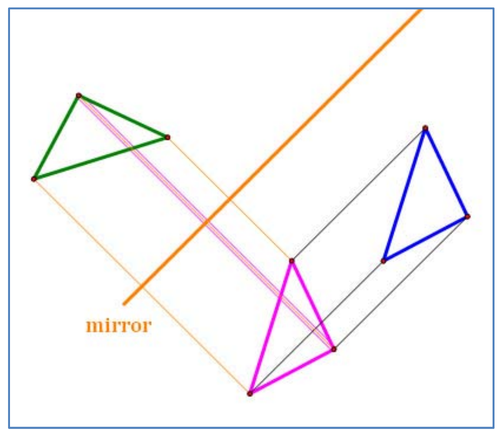



**Figure 2: Reflection followed by Translation.**

## Composition Theorem

The composition of multiple isometries is as Isometry. Put more simply, if transformations that preserve length are combined, the composition will preserve length. This is also true of compositions of transformations that preserve angle measure.

## Order of Composition

Order matters in most compositions that involve more than one class of transformation. If you apply multiple transformations of the same kind (e.g., reflection, rotation, or translation), order generally does not matter; however, applying transformations in more than one class may produce different final images if the order is switched.

# Geometry
# Ratios Involving Units

## Ratios Involving Units

When simplifying ratios containing the same units:

- Simplify the fraction.
- Notice that the units disappear. They behave just like factors; if the units exist in the numerator and denominator, the cancel and are not in the answer.

**Example:**

$$\frac{3\ inches}{12\ inches} = \frac{1}{4}$$

**Note:** the unit "inches cancel out, so the answer is $\frac{1}{4}$, not $\frac{1}{4}\ inch$.

When simplifying ratios containing different units:

- Adjust the ratio so that the numerator and denominator have the same units.
- Simplify the fraction.
- Notice that the units disappear.

**Example:**

$$\frac{3\ inches}{2\ feet} = \frac{3\ inches}{(2\ feet)\cdot(12\ inches/foot)} = \frac{3\ inches}{24\ inches} = \frac{1}{8}$$

## Dealing with Units

Notice in the above example that units can be treated the same as factors; they can be used in fractions and they cancel when they divide. This fact can be used to figure out whether multiplication or division is needed in a problem. Consider the following:

**Example:** How long did it take for a car traveling at 48 miles per hour to go 32 miles?

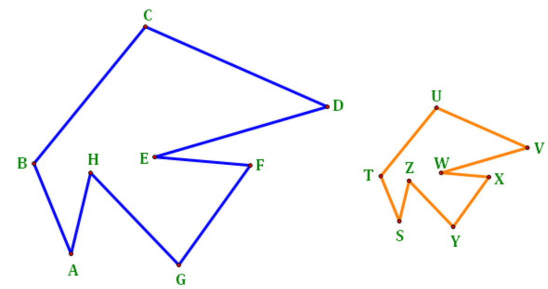
Consider the units of each item: $32\ miles$ $48\ \frac{miles}{hour}$

- If you multiply, you get: $(32\ miles)\cdot\left(48\frac{miles}{hour}\right) = 1{,}536\frac{miles^2}{hour}$. This is clearly wrong!
- If you divide, you get: $(32\ miles) \div \left(48\frac{miles}{hour}\right) = \frac{32}{48}\ miles\cdot\left(\frac{hour}{miles}\right) = \frac{2}{3}\,hour$. Now, this looks reasonable. Notice how the "$miles$" unit cancel out in the final answer.

Now you could have solved this problem by remembering that $distance\ = rate\cdot time$, or $d = rt$. However, paying close attention to the units also generates the correct answer. In addition, **the "units" technique always works, no matter what the problem!**

# Geometry
# Similar Polygons

## In similar polygons,




- Corresponding angles are congruent, and
- Corresponding sides are proportional.

Both of these conditions are necessary for two polygons to be similar. Conversely, when two polygons are similar, all of the corresponding angles are congruent and all of the sides are proportional.

## Naming Similar Polygons

Similar polygons should be named such that corresponding angles are in the same location in the name, and the order of the points in the name should "follow the polygon around."

**Example:** The polygons above could be shown similar with the following names:

$$ABCDEFGHI \sim STUVWXYZ$$

It would also be acceptable to show the similarity as:

$$DEFGHIABC \sim VWXYZSTU$$

Any names that preserve the order of the points and keeps corresponding angles in corresponding locations in the names would be acceptable.

## Proportions

One common problem relating to similar polygons is to present three side lengths, where two of the sides correspond, and to ask for the length of the side corresponding to the third length.

**Example:** In the above similar polygons, if $BC = 20, EF = 12, and\ WX = 6, what\ is\ TU?$

This problem is solvable with proportions. To do so properly, it is important to relate corresponding items in the proportion:

$$\frac{BC}{TU} = \frac{EF}{WX} \quad \longrightarrow \quad \frac{20}{TU} = \frac{12}{6} \quad \longrightarrow \quad TU = 10$$

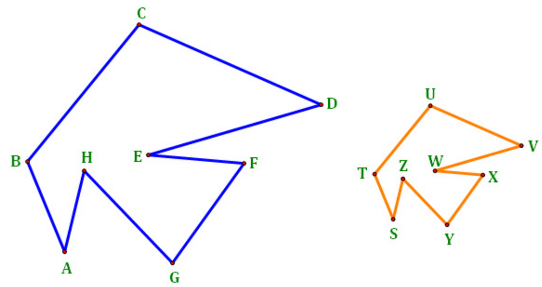
Notice that the left polygon is represented on the top of both proportions and that the left-most segments of the two polygons are in the left fraction.

# Geometry
# Scale Factors of Similar Polygons

From the similar polygons below, the following is known about the lengths of the sides:

$$\frac{AB}{ST} = \frac{BC}{TU} = \frac{CD}{UV} = \frac{DE}{VW} = \frac{EF}{WX} = \frac{FG}{XY} = \frac{GH}{YZ} = \frac{HA}{ZA} = k$$

That is, the ratios of corresponding sides in the two polygons are the same and they equal some constant $k$, called the **scale factor** of the two polygons. The value of $k$, then, is all you need to know to relate corresponding sides in the two polygons.




## Finding the Missing Length

Any time the student is asked to find the missing length in similar polygons:

- Look for two corresponding sides for which the values are known.
- Calculate the value of $k$.
- Use the value of $k$ to solve for the missing length.

$k$ is a measure of the relative size of the two polygons. Using this knowledge, it is possible to put into words an easily understandable relationship between the polygons.

- Let Polygon 1 be the one whose sides are in the numerators of the fractions.
- Let Polygon 2 be the one whose sides are in the denominators of the fractions.
- Then, it can be said that **Polygon 1 is $k$ times the size of the Polygon 2**.

**Example:** In the above similar polygons, if $BC = 20, EF = 12, and\ WX = 6, what\ is\ TU?$

Seeing that $EF$ and $WX$ relate, calculate:

$$\frac{EF}{WX} = \frac{12}{6} = 2 = k$$

Then solve for $TU$ based on the value of $k$:

$$\frac{BC}{TU} = k \quad \rightarrow \quad \frac{20}{TU} = 2 \quad \rightarrow \quad TU = 10$$

Also, since $k = 2$, the length of every side in the blue polygon is double the length of its corresponding side in the orange polygon.

# Geometry
# Dilation of Polygons

A dilation is a special case of transformation involving similar polygons. It can be thought of as a transformation that creates a polygon of the same shape but a different size from the original. Key elements of a dilation are:

- Scale Factor – The scale factor of similar polygons is the constant $k$ which represents the relative sizes of the polygons.
- Center – The center is the point from which the dilation takes place.

Note that $k > 0$ and $k \neq 1$ in order to generate a second polygon. Then,

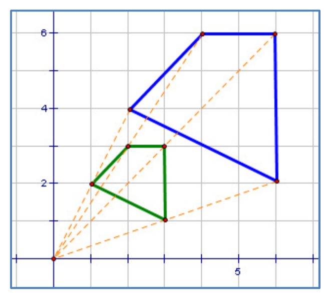
- If $k > 1$, the dilation is called an "enlargement."
- If $k < 1$, the dilation is called a "reduction."

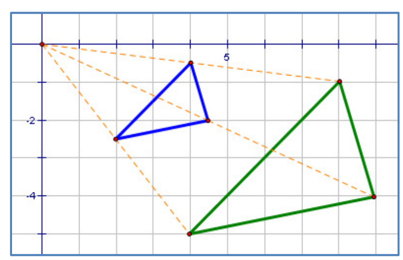
## Dilations with Center (0, 0)

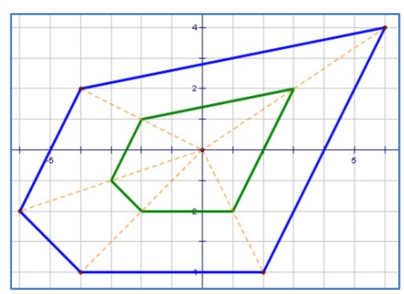
In coordinate geometry, dilations are often performed with the center being the origin $(0, 0)$. In that case, to obtain the dilation of a polygon:

- Multiply the coordinates of each vertex by the scale factor $k$, and
- Connect the vertices of the dilation with line segments (i.e., connect the dots).

**Examples:**



In the following examples:

- The green polygon is the original.
- The blue polygon is the dilation.
- The dashed orange lines show the movement away from (enlargement) or toward (reduction) the center, which is the origin in all 3 examples.

Notice that, in each example:

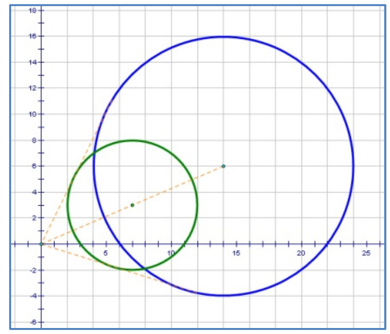
$$\begin{pmatrix} \textit{distance from center} \\ \textit{to a vertex of the} \\ \textit{dilated polygon} \end{pmatrix} = k \cdot \begin{pmatrix} \textit{distance from center} \\ \textit{to a vertex of the} \\ \textit{original polygon} \end{pmatrix}$$

This fact can be used to construct dilations when coordinate axes are not available. Alternatively, the student could draw a set of coordinate axes as an aid to performing the dilation.

**ADVANCED**

# Geometry
# More on Dilation

## Dilations of Non-Polygons



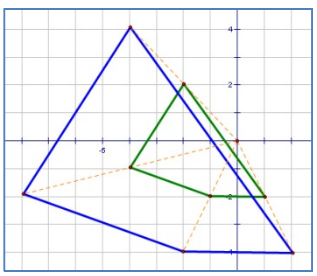
Any geometric figure can be dilated. In the dilation of the green circle at right, notice that:

- The dilation factor is 2.
- The original circle has center $(7, 3)$ and radius $= 5$.
- The dilated circle has center $(14, 6)$ and radius $= 10$.

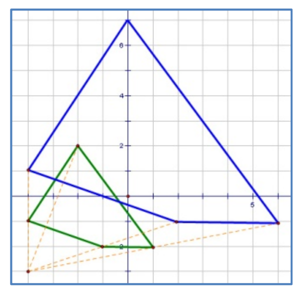
So, the center and radius are both increased by a factor of $k = 2$. This is true of any figure in a dilation with the center at the origin. All of the key elements that define the figure are increased by the scale factor $k$.

## Dilations with Center $(a, b)$

In the figures below, the green quadrilaterals are dilated to the blue ones with a scale factor of $k = 2$. Notice the following:

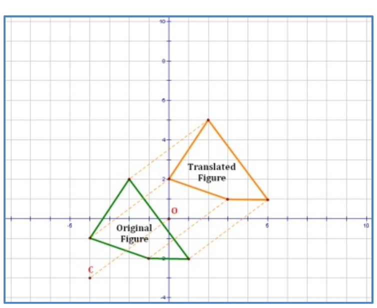


In the figure to the left, the dilation has center $(0, 0)$, whereas in the figure to the right, the dilation has center $(-4, -3)$. The size of the resulting figure is the same in both cases (because $k = 2$ in both figures), but the location is different.

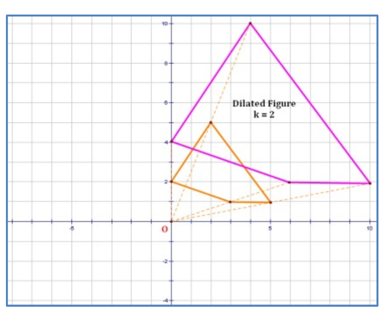


Graphically, the series of transformations that is equivalent to a dilation from a point $(a, b)$ other than the origin is shown below. Compare the final result to the figure above (right).

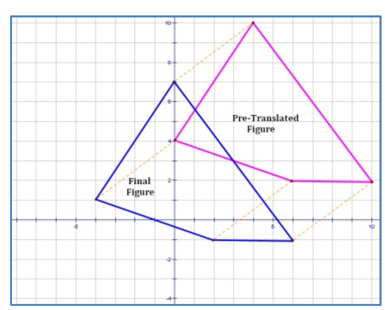
- Step 1: Translate the original figure by $(-a, -b)$ to reset the center at the origin.
- Step 2: Perform the dilation.
- Step 3: Translate the dilated figure by $(a, b)$. These steps are illustrated below.




Step 1




Step 2

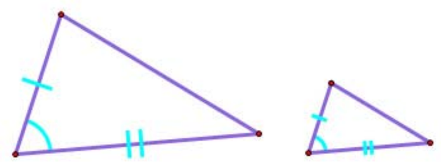



Step 3

# Geometry
# Similar Triangles

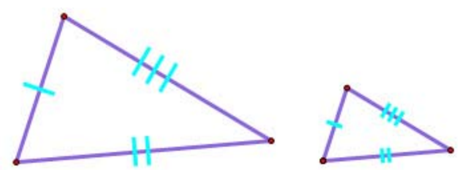
The following theorems present conditions under which triangles are similar.

## Side-Angle-Side (SAS) Similarity



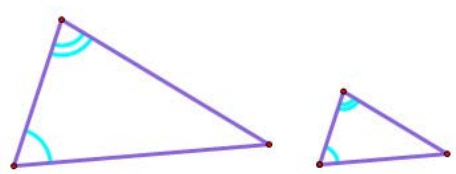
**SAS similarity** requires the proportionality of two sides and the congruence of the angle between those sides. Note that there is no such thing as SSA similarity; the congruent angle must be between the two proportional sides.

## Side-Side-Side (SSS) Similarity



**SSS similarity** requires the proportionality of all three sides. If all of the sides are proportional, then all of the angles must be congruent.

## Angle--Angle (AA) Similarity

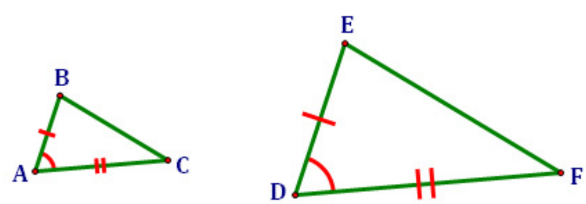


**AA similarity** requires the congruence of two angles and the side between those angles.

## Similar Triangle Parts

In similar triangles,

- Corresponding sides are proportional.
- Corresponding angles are congruent.




Establishing the proper names for similar triangles is crucial to line up corresponding vertices. In the picture above, we can say:

$$\Delta ABC \sim \Delta DEF \quad \text{or} \quad \Delta BCA \sim \Delta EFD \quad \text{or} \quad \Delta CAB \sim \Delta FDE \quad \text{or}$$
$$\Delta ACB \sim \Delta DFE \quad \text{or} \quad \Delta BAC \sim \Delta EDF \quad \text{or} \quad \Delta CBA \sim \Delta FED$$

All of these are correct because they match corresponding parts in the naming. Each of these similarities implies the following relationships between parts of the two triangles:

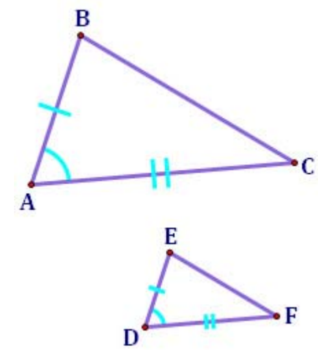
$$\angle A \cong \angle D \quad \text{and} \quad \angle B \cong \angle E \quad \text{and} \quad \angle C \cong \angle F$$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

# Geometry
# Proportion Tables for Similar Triangles

## Setting Up a Table of Proportions

It is often useful to set up a table to identify the proper proportions in a similarity. Consider the figure to the right. The table might look something like this:




| Triangle | Left Side | Right Side | Bottom Side |
|---|---|---|---|
| **Top Δ** | $AB$ | $BC$ | $CA$ |
| **Bottom Δ** | $DE$ | $EF$ | $FD$ |

The purpose of a table like this is to organize the information you have about the similar triangles so that you can readily develop the proportions you need.

## Developing the Proportions

To develop proportions from the table:

- Extract the columns needed from the table:

| | |
|---|---|
| $AB$ | $BC$ |
| $DE$ | $EF$ |

- Eliminate the table lines.
- Replace the horizontal lines with "division lines."
- Put an equal sign between the two resulting fractions:

$$\frac{AB}{DE} = \frac{BC}{EF}$$

**Also from the above table,**

$$\frac{AB}{DE} = \frac{CA}{FD}$$

$$\frac{BC}{EF} = \frac{CA}{FD}$$

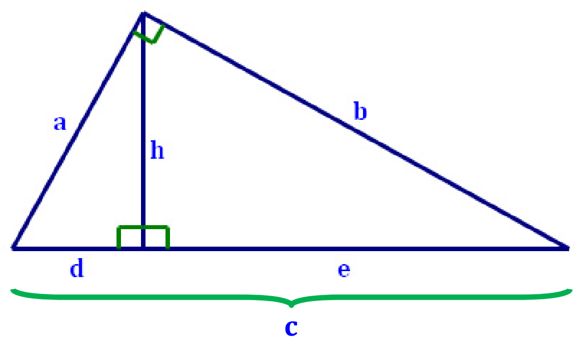
### Solving for the unknown length of a side:

You can extract any two columns you like from the table. Usually, you will have information on lengths of three of the sides and will be asked to calculate a fourth.

Look in the table for the columns that contain the 4 sides in question, and then set up your proportion. Substitute known values into the proportion, and solve for the remaining variable.

# Geometry
## Three Similar Triangles

A common problem in geometry is to find the missing value in proportions based on a set of three similar triangles, two of which are inside the third. The diagram often looks like this:




**Pythagorean Relationships**

| | |
|---|---|
| Inside triangle on the left: | $d^2 + h^2 = a^2$ |
| Inside triangle on the right: | $h^2 + e^2 = b^2$ |
| Outside (large) triangle: | $a^2 + b^2 = c^2$ |

## Similar Triangle Relationships

Because all three triangles are similar, we have the relationships in the table below. These relationships are not obvious from the picture, but are very useful in solving problems based on the above diagram. Using similarities between the triangles, 2 at a time, we get:

| From the two inside triangles | From the inside triangle on the left and the outside triangle | From the inside triangle on the right and the outside triangle |
|---|---|---|
| $\frac{h}{d} = \frac{e}{h}$ or $h^2 = d \cdot e$ | $\frac{a}{d} = \frac{c}{a}$ or $a^2 = d \cdot c$ | $\frac{b}{e} = \frac{c}{b}$ or $b^2 = e \cdot c$ |
| **The height squared = the product of: the two parts of the base** | **The left side squared = the product of: the part of the base below it and the entire base** | **The right side squared = the product of: the part of the base below it and the entire base** |

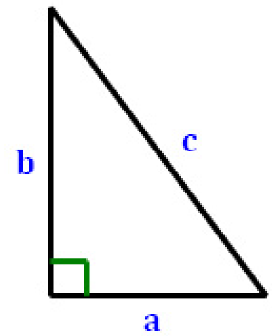
# Geometry
# Pythagorean Theorem

In a right triangle, the Pythagorean Theorem says:

$$a^2 + b^2 = c^2$$

where,

- $a$ and $b$ are the lengths of the legs of a right triangle, and
- $c$ is the length of the hypotenuse.

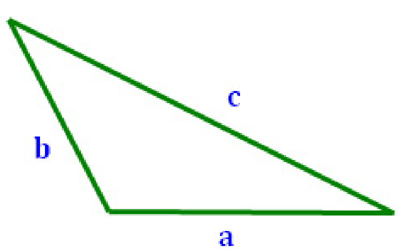



## Right, Acute, or Obtuse Triangle?

In addition to allowing the solution of right triangles, the Pythagorean Formula can be used to determine whether a triangle is a right triangle, an acute triangle, or an obtuse triangle.

To determine whether a triangle is obtuse, right, or acute:

- Arrange the lengths of the sides from low to high; call them $a$, $b$, and $c$, in increasing order
- Calculate: $a^2, b^2$, and $c^2$.
- Compare: $a^2 + b^2$ vs. $c^2$
- Use the illustrations below to determine which type of triangle you have.




**Obtuse Triangle**

$a^2 + b^2 < c^2$

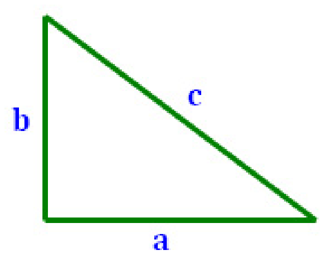
**Example:**

Triangle with sides: 7, 9, 12

$7^2 + 9^2 \ \ vs. \ \ 12^2$

$49 + 81 < 144$

$\rightarrow$ ***Obtuse Triangle***




**Right Triangle**

$a^2 + b^2 = c^2$

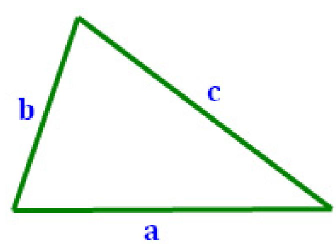
**Example:**

Triangle with sides: 6, 8, 10

$6^2 + 8^2 \ \ vs. \ \ 10^2$

$36 + 64 = 100$

$\rightarrow$ ***Right Triangle***




**Acute Triangle**

$a^2 + b^2 > c^2$

**Example:**

Triangle with sides: 5, 8, 9

$5^2 + 8^2 \ \ vs. \ \ 9^2$

$25 + 64 > 81$

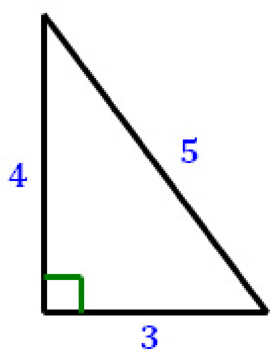
$\rightarrow$ ***Acute Triangle***

# Geometry
# Pythagorean Triples

**Pythagorean Theorem:** $a^2 + b^2 = c^2$

Pythagorean triples are sets of 3 positive integers that meet the requirements of the Pythagorean Theorem. Because these sets of integers provide “pretty” solutions to geometry problems, they are a favorite of geometry books and teachers. Knowing what triples exist can help the student quickly identify solutions to problems that might otherwise take considerable time to solve.

## 3-4-5 Triangle Family



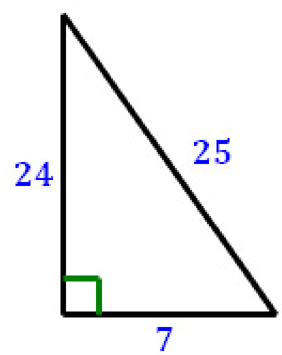

**Sample Triples**

3-4-5
6-8-10
9-12-15
12-16-20
30-40-50

$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

## 7-24-25 Triangle Family



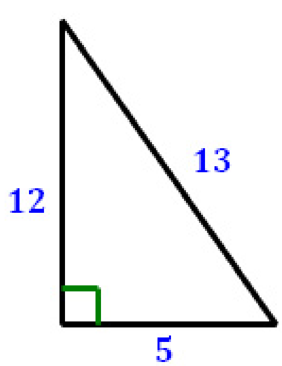

**Sample Triples**

7-24-25
14-48-50
21-72-75
. . .
70-240-250

$7^2 + 24^2 = 25^2$

$49 + 576 = 625$

## 5-12-13 Triangle Family



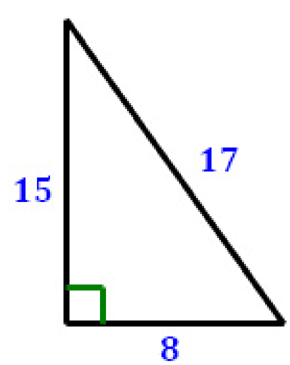

**Sample Triples**

5-12-13
10-24-26
15-36-39
. . .
50-120-130

$5^2 + 12^2 = 13^2$

$25 + 144 = 169$

## 8-15-17 Triangle Family




**Sample Triples**

8-15-17
16-30-34
24-45-51
. . .
80-150-170

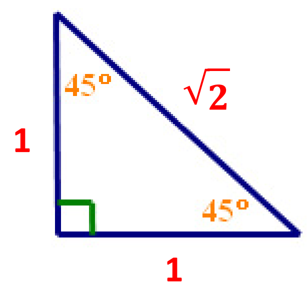
$8^2 + 15^2 = 17^2$

$64 + 225 = 289$

# Geometry
## Special Triangles

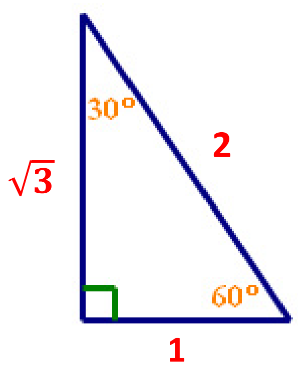
The relationship among the lengths of the sides of a triangle is dependent on the measures of the angles in the triangle. For a right triangle (i.e., one that contains a 90$^o$ angle), two special cases are of particular interest. These are shown below:

### 45$^o$-45$^o$-90$^o$ Triangle




In a **45$^o$-45$^o$-90$^o$ triangle**, the congruence of two angles guarantees the congruence of the two legs of the triangle. The proportions of the three sides are: $\mathbf{1 : 1 : \sqrt{2}}$. That is, the two legs have the same length and the hypotenuse is $\sqrt{2}$ times as long as either leg.

### 30$^o$-60$^o$-90$^o$ Triangle




In a **30$^o$-60$^o$-90$^o$ triangle**, the proportions of the three sides are: $\mathbf{1 : \sqrt{3} : 2}$. That is, the long leg is $\sqrt{3}$ times as long as the short leg, and the hypotenuse is $\mathbf{2}$ times as long as the short leg.

In a right triangle, we need to know the lengths of two sides to determine the length of the third. The power of the relationships in the special triangles lies in the fact that we need only know the length of one side of the triangle to determine the lengths of the other two sides.

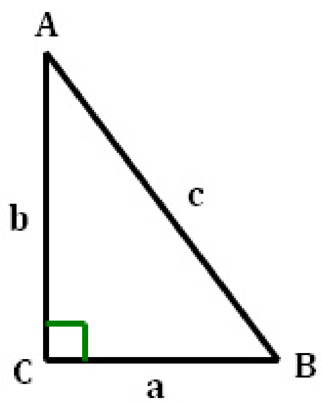
**Example Side Lengths**

| 45$^o$-45$^o$-90$^o$ Triangle | |
|---|---|
| $1 : 1 : \sqrt{2}$ | $2 : 2 : 2\sqrt{2}$ |
| $\sqrt{2} : \sqrt{2} : 2$ | $\sqrt{3} : \sqrt{3} : \sqrt{6}$ |
| $3\sqrt{2} : 3\sqrt{2} : 6$ | $25 : 25 : 25\sqrt{2}$ |

| 30$^o$-60$^o$-90$^o$ Triangle | |
|---|---|
| $1 : \sqrt{3} : 2$ | $2 : 2\sqrt{3} : 4$ |
| $\sqrt{2} : \sqrt{6} : 2\sqrt{2}$ | $\sqrt{3} : 3 : 2\sqrt{3}$ |
| $3\sqrt{2} : 3\sqrt{6} : 6\sqrt{2}$ | $25 : 25\sqrt{3} : 50$ |

# Geometry
# Trig Functions and Special Angles

## Trigonometric Functions




**SOH-CAH-TOA**

$\sin = \dfrac{opposite}{hypoteneuse}$ $\quad \sin A = \dfrac{a}{c}$ $\quad \sin B = \dfrac{b}{c}$

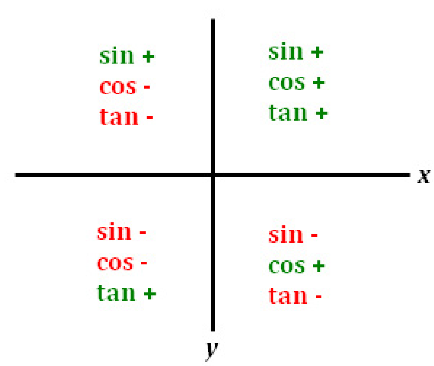
$\cos = \dfrac{adjacent}{hypoteneuse}$ $\quad \cos A = \dfrac{b}{c}$ $\quad \cos B = \dfrac{a}{c}$

$\tan = \dfrac{opposite}{adjacent}$ $\quad \tan A = \dfrac{a}{b}$ $\quad \tan B = \dfrac{b}{a}$

## Special Angles

| Trig Functions of Special Angles | | | | |
|---|---|---|---|---|
| **Radians** | **Degrees** | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
| 0 | $0^o$ | $\dfrac{\sqrt{0}}{2} = 0$ | $\dfrac{\sqrt{4}}{2} = 1$ | $\dfrac{\sqrt{0}}{\sqrt{4}} = 0$ |
| $\pi/6$ | $30^o$ | $\dfrac{\sqrt{1}}{2} = \dfrac{1}{2}$ | $\dfrac{\sqrt{3}}{2}$ | $\dfrac{\sqrt{1}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ |
| $\pi/4$ | $45^o$ | $\dfrac{\sqrt{2}}{2}$ | $\dfrac{\sqrt{2}}{2}$ | $\dfrac{\sqrt{2}}{\sqrt{2}} = 1$ |
| $\pi/3$ | $60^o$ | $\dfrac{\sqrt{3}}{2}$ | $\dfrac{\sqrt{1}}{2} = \dfrac{1}{2}$ | $\dfrac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$ |
| $\pi/2$ | $90^o$ | $\dfrac{\sqrt{4}}{2} = 1$ | $\dfrac{\sqrt{0}}{2} = 0$ | *undefined* |

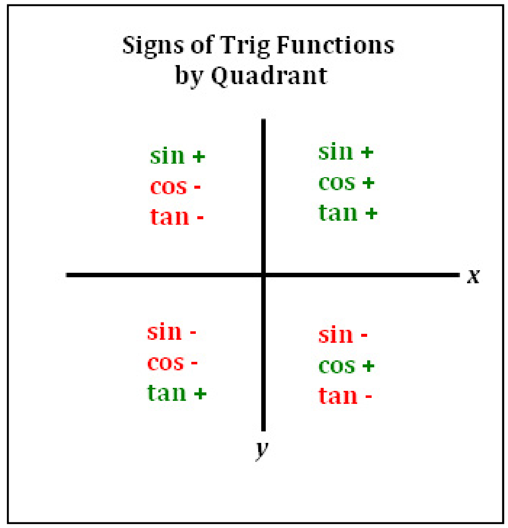
Signs of Trig Functions
by Quadrant



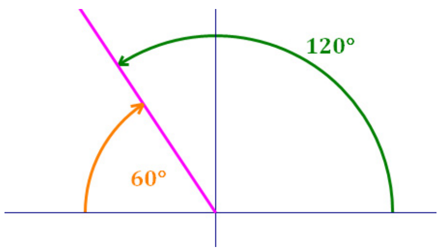
# Geometry
## Trigonometric Function Values in Quadrants II, III, and IV

In quadrants other than Quadrant I, trigonometric values for angles are calculated in the following manner:

- Draw the angle $\boldsymbol{\theta}$ on the Cartesian Plane.
- Calculate the measure of the angle from the x-axis to $\boldsymbol{\theta}$.
- Find the value of the trigonometric function of the angle in the previous step.
- Assign a "+" or "–" sign to the trigonometric value based on the function used and the quadrant $\boldsymbol{\theta}$ is in.




**Examples:**




**$\theta$ in Quadrant II** – Calculate: $(180^0 - m\angle\theta)$

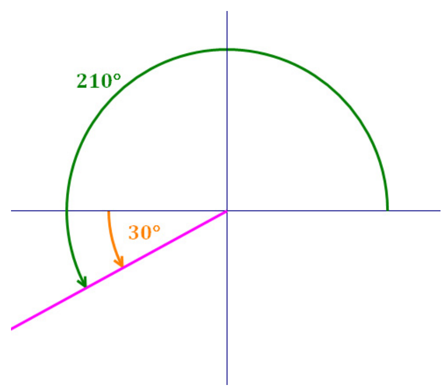
For $\theta = 120^0$, base your work on $180° - 120° = 60°$

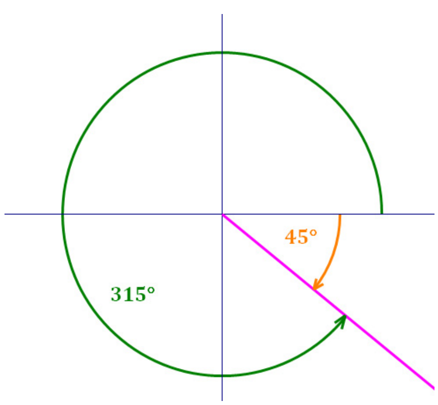
$\sin 60° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\sin 120° = \frac{\sqrt{3}}{2}}$

**$\theta$ in Quadrant III** – Calculate: $(m\angle\theta - 180^0)$

For $\theta = 210^0$, base your work on $210° - 180° = 30°$

$\cos 30° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\cos 210° = -\frac{\sqrt{3}}{2}}$



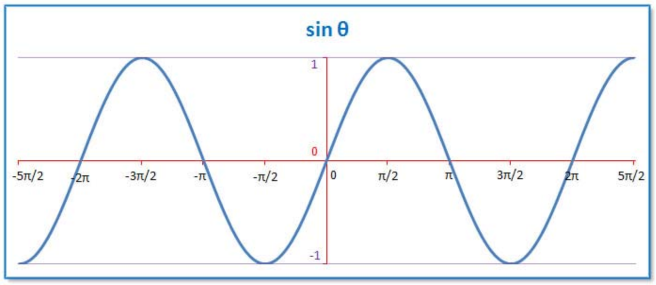



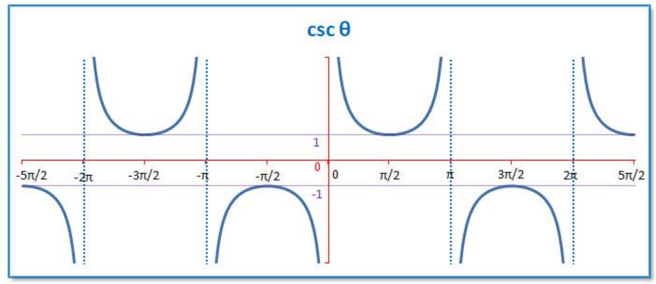

**$\theta$ in Quadrant IV** – Calculate: $(360^0 - m\angle\theta)$

For $\theta = 315^0$, base your work on $360° - 315° = 45°$

$\tan 45° = 1$, so: $\mathbf{\tan 315° = -1}$

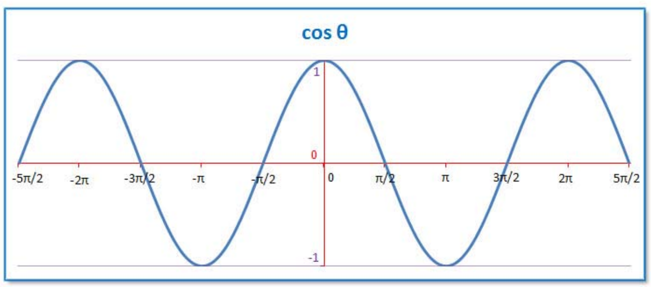
# Geometry
## Graphs of Trigonometric Functions

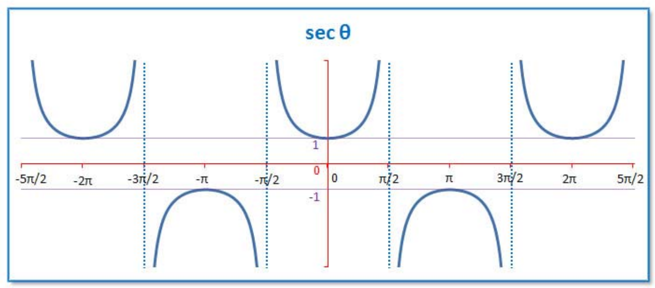



The sine and cosecant functions are inverses. So:

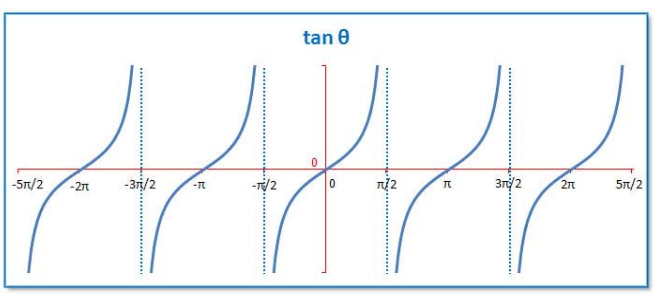
$$\sin\theta = \frac{1}{\csc\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}$$

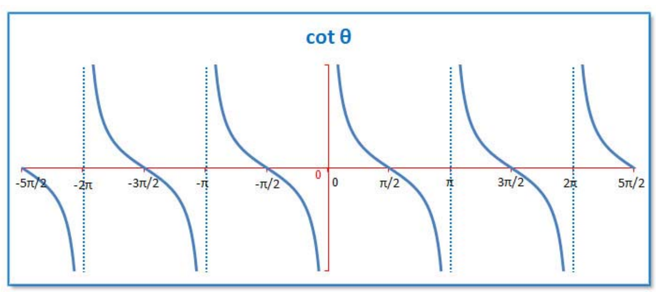



The cosine and secant functions are inverses. So:

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta}$$



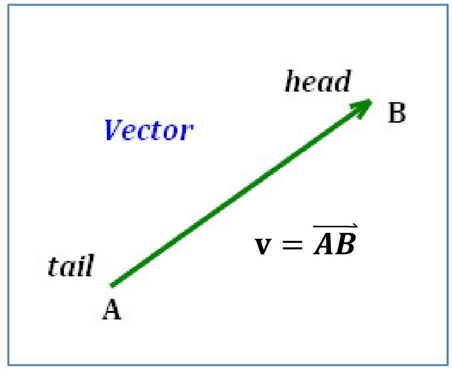




The tangent and cotangent functions are inverses. So:

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta}$$

# Geometry
# Vectors

## Definitions




- A **vector** is a geometric object that has both magnitude (length) and direction.
- The **Tail** of the vector is the end opposite the arrow. It represents where the vector is moving from.
- The **Head** of the vector is the end with the arrow. It represents where the vector is moving to.
- The **Zero Vector** is denoted $\mathbf{0}$. It has zero length and all the properties of zero.
- Two vectors are **equal** is they have both the same magnitude and the same direction.
- Two vectors are **parallel** if they have the same or opposite directions. That is, if the angles of the vectors are the same or $180^o$ different.
- Two vectors are **perpendicular** if the difference of the angles of the vectors is $90^o$ or $270^o$.

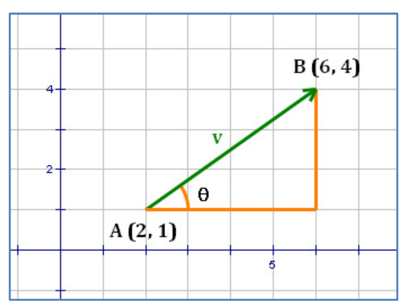
## Magnitude of a Vector

The distance formula gives the magnitude of a vector. If the head and tail of vector $\mathbf{v}$ are the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the magnitude of $\mathbf{v}$ is:

$$|\mathbf{v}| = \left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BA}\right|$. The directions of the two vectors are opposite, but their magnitudes are the same.

## Direction of a Vector




The direction of a vector is determined by the angle it makes with a horizontal line. In the figure at right, the direction is the angle $\boldsymbol{\theta}$. The value of $\boldsymbol{\theta}$ can be calculated based on the lengths of the sides of the triangle the vector forms.

$$\tan\theta = \frac{3}{4} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

where the function $\mathbf{tan^{-1}}$ is the inverse tangent function. The second equation in the line above reads "$\theta$ is the angle whose tangent is $\frac{3}{4}$."

# Geometry
# Operations with Vectors

It is possible to operate with vectors in some of the same ways we operate with numbers. In particular:

## Adding Vectors

Vectors can be added in rectangular form by separately adding their *x*- and *y-components*. In general,

$$\mathbf{u} = \langle u_1, u_2 \rangle$$

$$\mathbf{v} = \langle v_1, v_2 \rangle$$

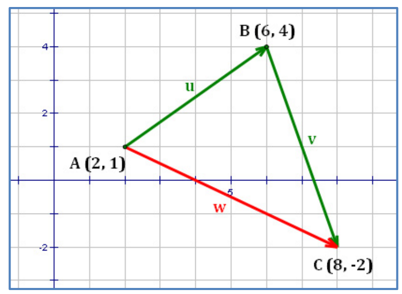
$$\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$$

**Example:** In the figure at right,

$$\mathbf{u} = \langle 4, 3 \rangle$$

$$\mathbf{v} = \langle 2, -6 \rangle$$

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = \langle 4, 3 \rangle + \langle 2, -6 \rangle = \langle 6, -3 \rangle$$




## Vector Algebra

| | | |
|---|---|---|
| $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | $\mathrm{a} \cdot (\mathbf{u} + \mathbf{v}) = (\mathrm{a} \cdot \mathbf{u}) + (\mathrm{a} \cdot \mathbf{v})$ |
| $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{w} + \mathbf{v})$ | $\mathbf{0} \cdot \mathbf{u} = \mathbf{0}$ | $(\mathrm{a} + \mathrm{b}) \cdot \mathbf{u} = (\mathrm{a} \cdot \mathbf{u}) + (\mathrm{b} \cdot \mathbf{u})$ |
| $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | $1 \cdot \mathbf{u} = \mathbf{u}$ | $(\mathrm{ab}) \cdot \mathbf{u} = \mathrm{a} \cdot (\mathrm{b} \cdot \mathbf{u}) = \mathrm{b} \cdot (\mathrm{a} \cdot \mathbf{u})$ |

## Scalar Multiplication

Scalar multiplication changes the magnitude of a vector, but not the direction. In general,

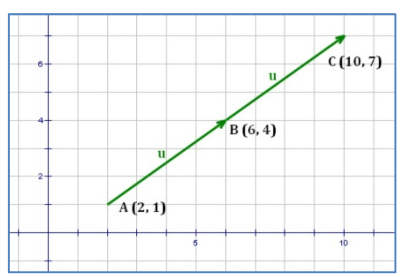
$$\mathbf{u} = \langle u_1, u_2 \rangle$$

$$k \cdot \mathbf{u} = \langle k \cdot u_1, k \cdot u_2 \rangle$$

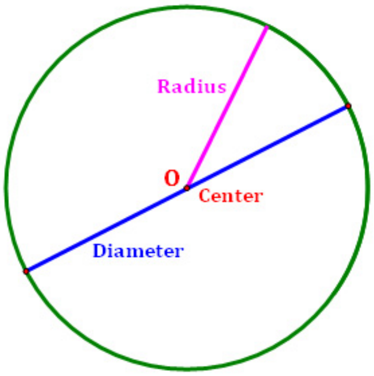
In the figure at right,

$$\mathbf{u} = \langle 4, 3 \rangle$$

$$2 \cdot \mathbf{u} = 2 \cdot \langle 4, 3 \rangle = \langle 8, 6 \rangle$$



# Geometry
## Parts of Circles




**Center** – the middle of the circle. All points on the circle are the same distance from the center.

**Radius** – a line segment with one endpoint at the center and the other endpoint on the circle. The term "radius" is also used to refer to the distance from the center to the points on the circle.

**Diameter** – a line segment with endpoints on the circle that passes through the center.

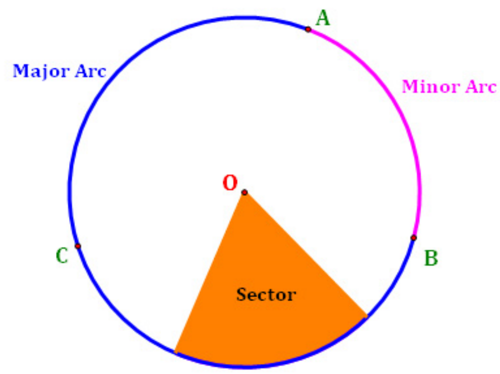
**Arc –** a path along a circle.

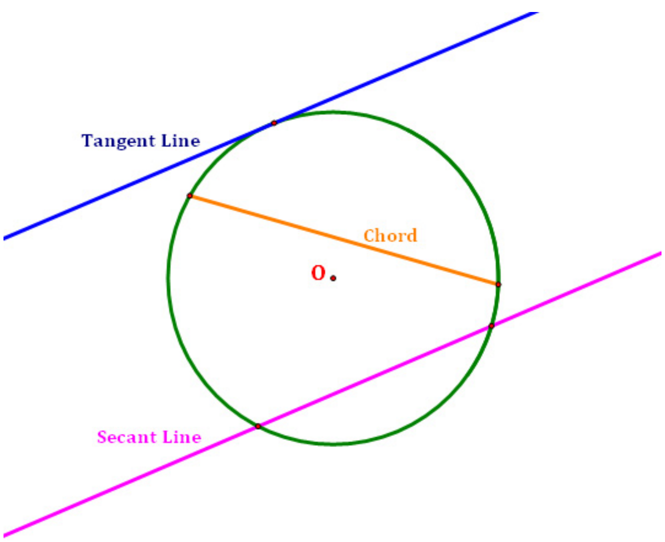
**Minor Arc** – a path along the circle that is less than 180°.

**Major Arc** – a path along the circle that is greater than 180°.

**Semicircle** – a path along a circle that equals 180°.

**Sector** – a region inside a circle that is bounded by two radii and an arc.



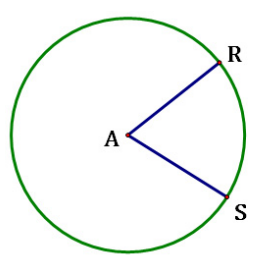

**Secant Line** – a line that intersects the circle in exactly two points.

**Tangent Line**– a line that intersects the circle in exactly one point.

**Chord** – a line segment with endpoints on the circle that does not pass through the center.

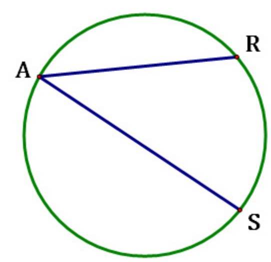
# Geometry
## Angles and Circles

### Central Angle



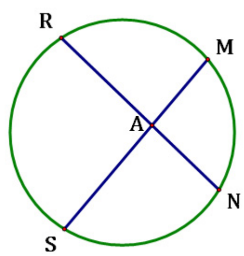

$m\angle A = m\,\widehat{RS}$

### Inscribed Angle




$m\angle A = \frac{1}{2}\,m\,\widehat{RS}$

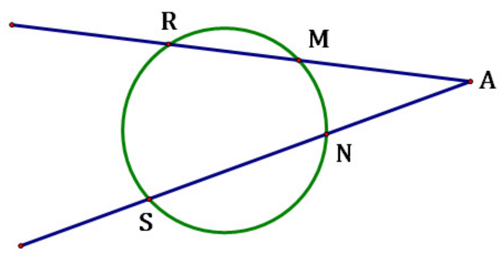
### Vertex inside the circle




$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} + m\,\widehat{MN}\right)$

$RA \cdot AN = SA \cdot AM$

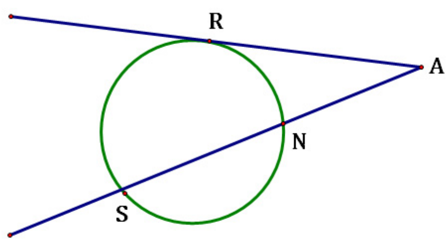
### Vertex outside the circle




$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} - m\,\widehat{MN}\right)$

$AM \cdot AR = AN \cdot AS$

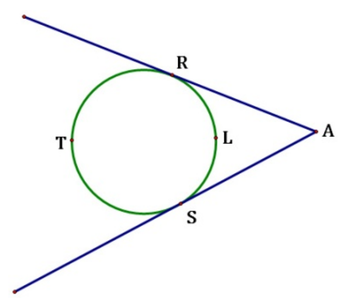
### Tangent on one side




$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} - m\,\widehat{RN}\right)$

$AR^2 = AN \cdot AS$

### Tangents on two sides




$m\angle A = \frac{1}{2}\left(m\,\widehat{RTS} - m\,\widehat{RLS}\right)$

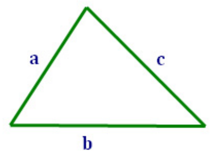
$AR = AS$

# Geometry
# Perimeter and Area of a Triangle

## Perimeter of a Triangle

The perimeter of a triangle is simply the sum of the measures of the three sides of the triangle.

$$P = a + b + c$$

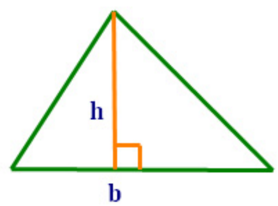



## Area of a Triangle

There are two formulas for the area of a triangle, depending on what information about the triangle is available.

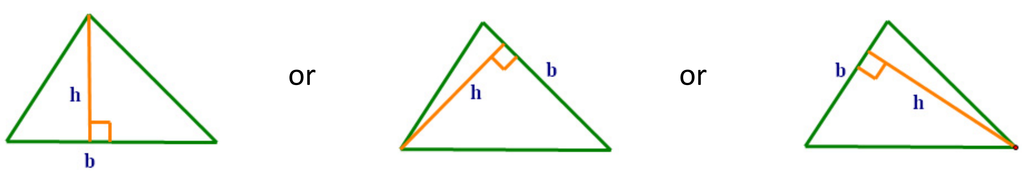
**Formula 1:** The formula most familiar to the student can be used when the base and height of the triangle are either known or can be determined.

$$A = \frac{1}{2}bh$$

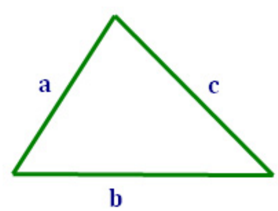



where, $b$ is the length of the base of the triangle.
$h$ is the height of the triangle.

**Note:** The base can be any side of the triangle. The height is the measure of the altitude of whichever side is selected as the base. So, you can use:




**Formula 2:** Heron's formula for the area of a triangle can be used when the lengths of all of the sides are known. Sometimes this formula, though less appealing, can be very useful.




$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where, $s = \frac{1}{2}P = \frac{1}{2}(a+b+c)$. **Note:** $s$ is sometimes called the semi-perimeter of the triangle.

$a, b, c$ are the lengths of the sides of the triangle.

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# Geometry
# More on the Area of a Triangle

## Trigonometric Formulas

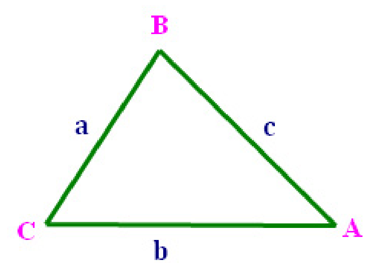
The following formulas for the area of a triangle come from trigonometry. Which one is used depends on the information available:

**Two angles and a side:**

$$A = \frac{1}{2} \cdot \frac{a^2 \cdot \sin B \cdot \sin C}{\sin A} = \frac{1}{2} \cdot \frac{b^2 \cdot \sin A \cdot \sin C}{\sin B} = \frac{1}{2} \cdot \frac{c^2 \cdot \sin A \cdot \sin B}{\sin C}$$

**Two sides and an angle:**

$$A = \frac{1}{2}\, ab\, \sin C = \frac{1}{2}\, ac\, \sin B = \frac{1}{2}\, bc\, \sin A$$

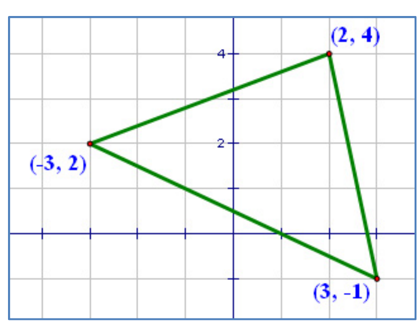



## Coordinate Geometry

If the three vertices of a triangle are displayed in a coordinate plane, the formula below, using a determinant, will give the area of a triangle.

Let the three points in the coordinate plane be: $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$. Then, the area of the triangle is one half of the absolute value of the determinant below:

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$



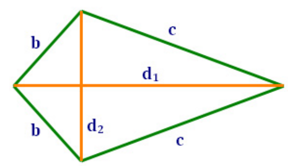
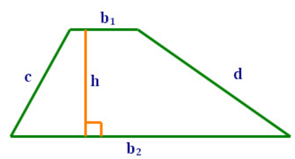
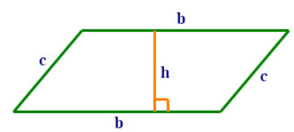
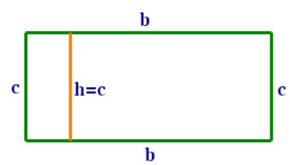
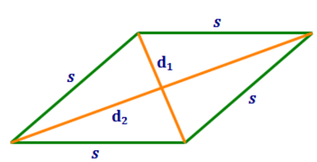
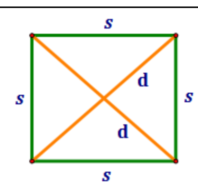

**Example:** For the triangle in the figure at right, the area is:

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} 2 & 4 & 1 \\ -3 & 2 & 1 \\ 3 & -1 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \cdot \left| \left( 2 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 4 \begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix} \right) \right| = \frac{1}{2} \cdot 27 = \frac{27}{2}$$

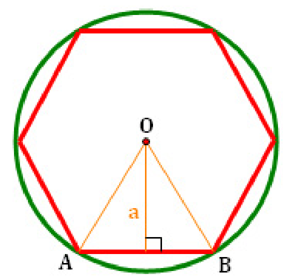
## Geometry

## Perimeter and Area of Quadrilaterals

| Name | Illustration | Perimeter | Area |
|---|---|---|---|
| **Kite** |  | $P = 2b + 2c$ | $A = \frac{1}{2}(d_1 d_2)$ |
| **Trapezoid** |  | $P = b_1 + b_2 + c + d$ | $A = \frac{1}{2}(b_1 + b_2)h$ |
| **Parallelogram** |  | $P = 2b + 2c$ | $A = bh$ |
| **Rectangle** |  | $P = 2b + 2c$ | $A = bh$ |
| **Rhombus** |  | $P = 4s$ | $A = bh = \frac{1}{2}(d_1 d_2)$ |
| **Square** |  | $P = 4s$ | $A = s^2 = \frac{1}{2}(d^2)$ |

# Geometry
## Perimeter and Area of Regular Polygons

### Definitions – Regular Polygons




- The **center** of a polygon is the center of its circumscribed circle. Point $\boldsymbol{O}$ is the center of the hexagon at right.
- The **radius** of the polygon is the radius of its circumscribed circle. $\overline{\boldsymbol{OA}}$ and $\overline{\boldsymbol{OB}}$ are both radii of the hexagon at right.
- The **apothem** of a polygon is the distance from the center to the midpoint of any of its sides. **a** is the apothem of the hexagon at right.
- The **central angle** of a polygon is an angle whose vertex is the center of the circle and whose sides pass through consecutive vertices of the polygon. In the figure above, $\angle \boldsymbol{AOB}$ is a central angle of the hexagon.

### Area of a Regular Polygon

$$\boldsymbol{A = \frac{1}{2} aP}$$

where, $a$ is the apothem of the polygon

$P$ is the perimeter of the polygon

### Perimeter and Area of Similar Figures

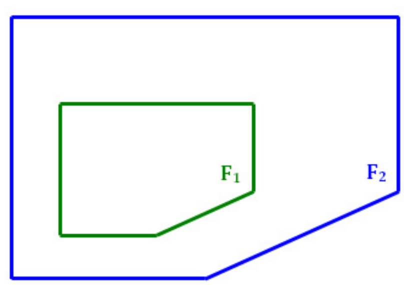
Let $\mathbf{k}$ be the scale factor relating two similar geometric figures $\mathbf{F_1}$ and $\mathbf{F_2}$ such that $\mathbf{F_2 = k \cdot F_1}$.

Then,

$$\frac{\textbf{Perimeter of } \mathbf{F_2}}{\textbf{Perimeter of } \mathbf{F_1}} = \mathbf{k}$$

and

$$\frac{\textbf{Area of } \mathbf{F_2}}{\textbf{Area of } \mathbf{F_1}} = \mathbf{k^2}$$



# Geometry
## Circle Lengths and Areas

### Circumference and Area

$C = 2\pi \cdot r$ is the **circumference** (i.e., the perimeter) of the circle.

$A = \pi r^2$ is the **area** of the circle.

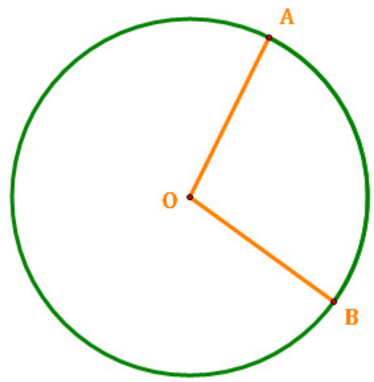
where: $r$ is the radius of the circle.

### Length of an Arc on a Circle

A common problem in the geometry of circles is to measure the length of an arc on a circle.

**Definition:** An arc is a segment along the circumference of a circle.

$$arc\ length = \frac{m\widehat{AB}}{360} \cdot C$$




where: $m\angle\widehat{AB}$ is the measure (in degrees) of the arc. Note that this is also the measure of the central angle $\angle AOB$.

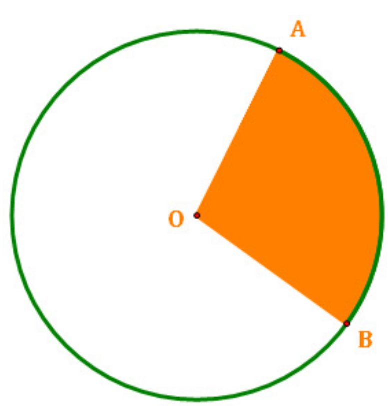
$C$ is the circumference of the circle.

### Area of a Sector of a Circle

Another common problem in the geometry of circles is to measure the area of a sector a circle.

**Definition:** A sector is a region in a circle that is bounded by two radii and an arc of the circle.

$$sector\ area = \frac{m\widehat{AB}}{360} \cdot A$$




where: $m\angle\widehat{AB}$ is the measure (in degrees) of the arc. Note that this is also the measure of the central angle $\angle AOB$.

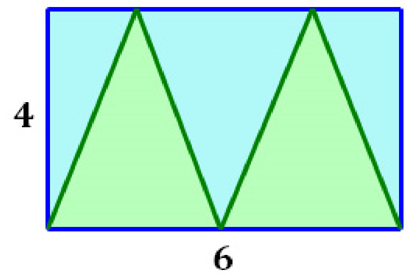
$A$ is the area of the circle.

# Geometry
## Area of Composite Figures

To calculate the area of a figure that is a composite of shapes, consider each shape separately.

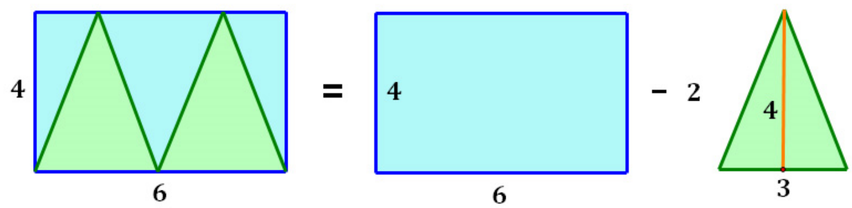
**Example 1:**

Calculate the area of the blue region in the figure to the right.




To solve this:

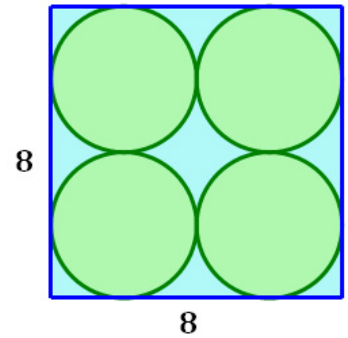
- Recognize that the figure is the composite of a rectangle and two triangles.
- Disassemble the composite figure into its components.
- Calculate the area of the components.
- Subtract to get the area of the composite figure.




$$\textit{Area of Region} = (4\cdot 6) - 2\left(\frac{1}{2}\cdot 4\cdot 3\right) = 24 - 12 = 12$$

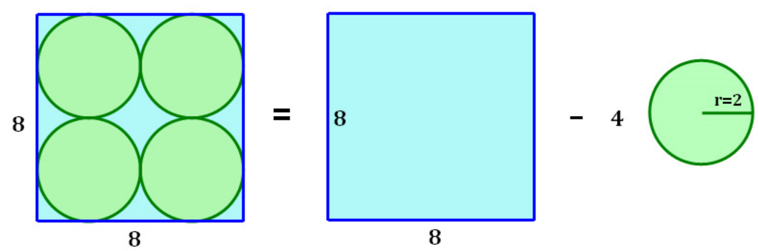
**Example 2:**

Calculate the area of the blue region in the figure to the right.




To solve this:

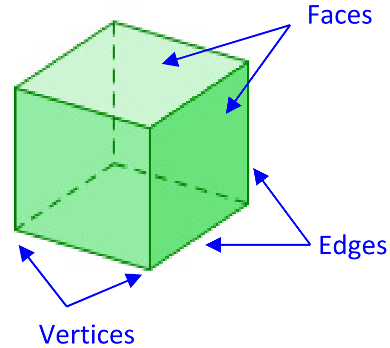
- Recognize that the figure is the composite of a square and a circle.
- Disassemble the composite figure into its components.
- Calculate the area of the components.
- Subtract to get the area of the composite figure.




$$\textit{Area of Region} = 8^2 - 4(\pi\cdot 2^2) = 64 - 16\pi \sim 13.73$$

# Geometry
# Polyhedra

## Definitions




- A **Polyhedron** is a 3-dimensional solid bounded by a series of polygons.
- **Faces** are the polygons that bound the polyhedron.
- An **Edge** is the line segment at the intersection of two faces.
- A **Vertex** is a point at the intersection of two edges.
- A **Regular** polyhedron is one in which all of the faces are the same regular polygon.
- A **Convex** Polyhedron is one in which all diagonals are contained within the interior of the polyhedron. A **Concave** polyhedron is one that is not convex.
- A **Cross Section** is the intersection of a plane with the polyhedron.

## Euler's Theorem

Let: $F$ = the number of faces of a polyhedron.
$V$ = the number of vertices of a polyhedron.
$E$ = the number of edges of a polyhedron.

Then, for any polyhedron that does not intersect itself,

$$\boldsymbol{F + V = E + 2}$$

**Euler's Theorem Example:**

The cube above has …

- 6 faces
- 8 vertices
- 12 edges

$$6 + 8 = 12 + 2 \quad ✓$$

## Calculating the Number of Edges

The number of edges of a polyhedron is one-half the number of sides in the polygons it comprises. Each side that is counted in this way is shared by two polygons; simply adding all the sides of the polygons, therefore, double counts the number of edges on the polyhedron.

**Example:** Consider a soccer ball. It is polyhedron made up of 20 hexagons and 12 pentagons. Then the number of edges is:

$$E = \frac{1}{2} \cdot [(20 \cdot 6) + (12 \cdot 5)] = 90$$

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# Geometry
# A Hole in Euler's Theorem

**Topology** is a branch of mathematics that studies the properties of objects that are preserved through manipulation that does not include tearing. An object may be stretched, twisted and otherwise deformed, but not torn. In this branch of mathematics, a donut is equivalent to a coffee cup because both have one hole; you can deform either the cup or the donut and create the other, like you are playing with clay.

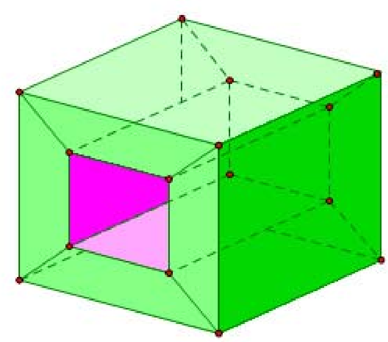
All of the usual polyhedra have no holes in them, so Euler's Equation holds. What happens if we allow the polyhedra to have holes in them? That is, what if we consider topological shapes different from the ones we normally consider?

## Euler's Characteristic

When Euler's Equation is rewritten as $\boldsymbol{F - E + V = 2}$, the left hand side of the equation is called the **Euler Characteristic**.

*The Euler Characteristic of a shape is:* $\boldsymbol{F - E + V}$

## Generalized Euler's Theorem



Let: $F$ = the number of faces of a polyhedron.

$V$ = the number of vertices of a polyhedron.

$E$ = the number of edges of a polyhedron.

$g$ = the number of holes in the polyhedron. $g$ is called the **genus** of the shape.

Then, for any polyhedron that does not intersect itself,

$$\boldsymbol{F - E + V = 2 - 2g}$$

Note that the value of Euler's Characteristic can be negative if the shape has more than one hole in it (i.e., if $g \geq 2$)!

**Example:**

The cube with a tunnel in it has …

$V = 16$

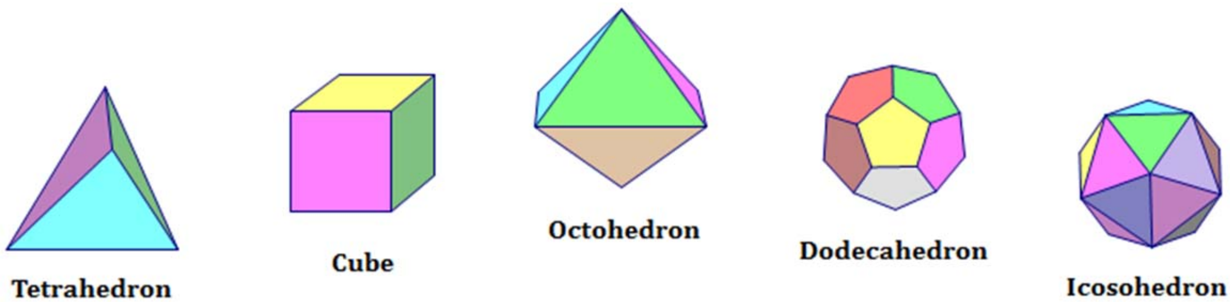
$E = 32$

$F = 16$

so, $\boldsymbol{V - E + F = 0}$

# Geometry
# Platonic Solids

A **Platonic Solid** is a convex regular polyhedron with faces composed of congruent convex regular polygons. There five of them:




## Key Properties of Platonic Solids

It is interesting to look at the key properties of these regular polyhedra.

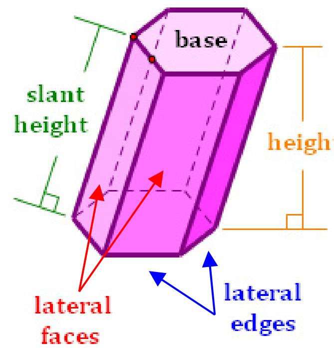
| Name | Faces | Vertices | Edges | Type of Face |
|---|---|---|---|---|
| **Tetrahedron** | 4 | 4 | 6 | Triangle |
| **Cube** | 6 | 8 | 12 | Square |
| **Octahedron** | 8 | 6 | 12 | Triangle |
| **Dodecahedron** | 12 | 20 | 30 | Pentagon |
| **Icosahedron** | 20 | 12 | 30 | Triangle |

Notice the following patterns in the table:

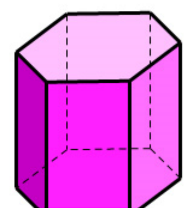
- All of the numbers of faces are even. Only the cube has a number of faces that is not a multiple of 4.
- All of the numbers of vertices are even. Only the octahedron has a number of faces that is not a multiple of 4.
- The number of faces and vertices seem to alternate (e.g., cube 6-8 vs. octahedron 8-6).
- All of the numbers of edges are multiples of 6.
- There are only three possibilities for the numbers of edges – 6, 12 and 30.
- The faces are one of: regular triangles, squares or regular pentagons.

# Geometry
# Prisms

## Definitions




- A **Prism** is a polyhedron with two congruent polygonal faces that lie in parallel planes.
- The **Bases** are the parallel polygonal faces.
- The **Lateral Faces** are the faces that are not bases.
- The **Lateral Edges** are the edges between the lateral faces.
- The **Slant Height** is the length of a lateral edge. Note that all lateral edges are the same length.
- The **Height** is the perpendicular length between the bases.



Right Hexagonal Prism

- A **Right Prism** is one in which the angles between the bases and the lateral edges are right angles. Note that in a right prism, the height and the slant height are the same.
- An **Oblique Prism** is one that is not a right prism.
- The **Surface Area** of a prism is the sum of the areas of all its faces.
- The **Lateral Area** of a prism is the sum of the areas of its lateral faces.

## Surface Area and Volume of a Right Prism

**Surface Area:** $\boldsymbol{SA = Ph + 2B}$

**Lateral SA:** $\boldsymbol{SA = Ph}$

**Volume:** $\boldsymbol{V = Bh}$

where, $P = the\ perimeter\ of\ the\ base$
$h = the\ height\ of\ the\ prism$
$B = the\ area\ of\ the\ base$

## Cavalieri's Principle

If two solids have the same height and the same cross-sectional area at every level, then they have the same volume. This principle allows us to derive a formula for the volume of an oblique prism from the formula for the volume of a right prism.

## Surface Area and Volume of an Oblique Prism

**Surface Area:** $\boldsymbol{SA = LSA + 2B}$

**Volume:** $\boldsymbol{V = Bh}$

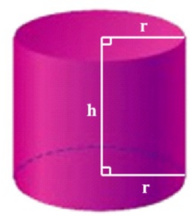
where, $LSA = the\ lateral\ surface\ area$
$h = the\ height\ of\ the\ prism$
$B = the\ area\ of\ the\ base$

The lateral surface area of an oblique prism is the sum of the areas of the faces, which must be calculated individually.

# Geometry
# Cylinders

## Definitions

- A **Cylinder** is a figure with two congruent circular bases in parallel planes.
- The **Axis** of a cylinder is the line connecting the centers of the circular bases.
- A cylinder has only one **Lateral Surface**. When deconstructed, the lateral surface of a cylinder is a rectangle with length equal to the circumference of the base.
- There are no **Lateral Edges** in a cylinder.
- The **Slant Height** is the length of the lateral side between the bases. Note that all lateral distances are the same length. The slant height has applicability only if the cylinder is oblique.
- The **Height** is the perpendicular length between the bases.




- A **Right Cylinder** is one in which the angles between the bases and the lateral side are right angles. Note that in a right cylinder, the height and the slant height are the same.
- An **Oblique Cylinder** is one that is not a right cylinder.

- The **Surface Area** of a cylinder is the sum of the areas of its bases and its lateral surface.
- The **Lateral Area** of a cylinder is the areas of its lateral surface.

## Surface Area and Volume of a Right Cylinder

**Surface Area:** $SA = Ch + 2B$

$= 2\pi rh + 2\pi r^2$

**Lateral SA:** $SA = Ch = 2\pi rh$

**Volume:** $V = Bh = \pi r^2 h$

where, $C = \textit{the circumference of the base}$

$h = \textit{the height of the cylinder}$

$B = \textit{the area of the base}$

$r = \textit{the radius of the base}$

## Surface Area and Volume of an Oblique Cylinder

**Surface Area:** $SA = Pl + 2B$

**Volume:** $V = Bh = \pi r^2 h$

* A **right section** of an oblique cylinder is a cross section perpendicular to the axis of the cylinder.

where, $P = \textit{the perimeter of a}$ right section* $\textit{of the cylinder}$

$l = \textit{the slant height of the cylinder}$

$h = \textit{the height of the cylinder}$

$B = \textit{the area of the base}$

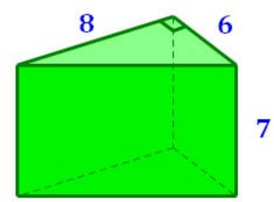
$r = \textit{the radius of the base}$

# Geometry
## Surface Area by Decomposition

Sometimes the student is asked to calculate the surface are of a prism that does not quite fit into one of the categories for which an easy formula exists. In this case, the answer may be to decompose the prism into its component shapes, and then calculate the areas of the components. Note: this process also works with cylinders and pyramids.

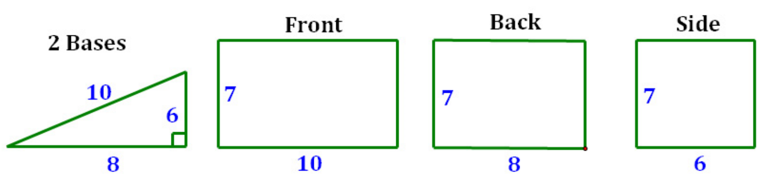
## Decomposition of a Prism

To calculate the surface area of a prism, decompose it and look at each of the prism's faces individually.




**Example:** Calculate the surface area of the triangular prism at right.

To do this, first notice that we need the value of the hypotenuse of the base. Use the Pythagorean Theorem or Pythagorean Triples to determine the missing value is $\boldsymbol{10}$. Then, decompose the figure into its various faces:



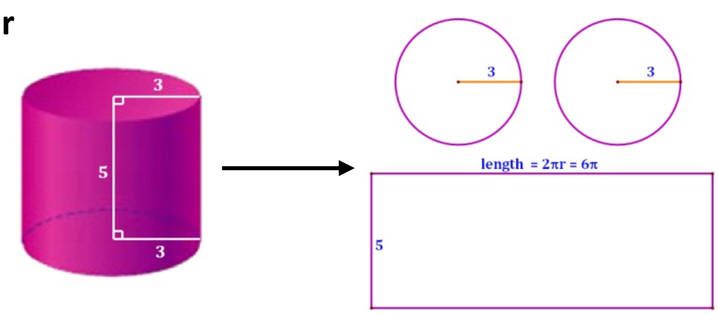

The surface area, then, is calculated as:

$$SA = (2\ Bases) + (Front) + (Back) + (Side)$$

$$SA = 2 \cdot \left(\frac{1}{2} \cdot 6 \cdot 8\right) + (10 \cdot 7) + (8 \cdot 7) + (6 \cdot 7) = 216$$

## Decomposition of a Right Cylinder

The cylinder at right is decomposed into two circles (the bases) and a rectangle (the lateral face).



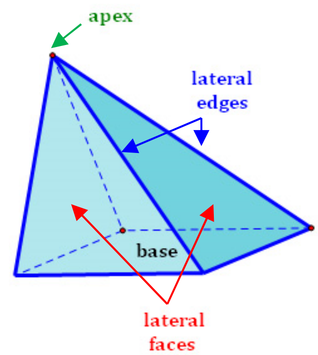

The surface area, then, is calculated as:

$$SA \;=\; (2\ tops) + (lateral\ face)$$

$$SA \;=\; 2 \cdot (\pi \cdot 3^2) + (6\pi \cdot 5) \;=\; 48\pi \;\sim\; 150.80$$

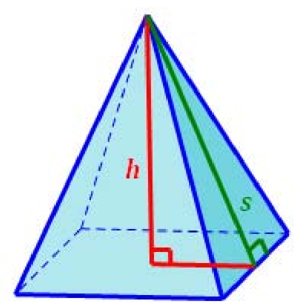
# Geometry
# Pyramids

## Pyramids




- A **Pyramid** is a polyhedron in which the base is a polygon and the lateral sides are triangles with a common vertex.
- The **Base** is a polygon of any size or shape.
- The **Lateral Faces** are the faces that are not the base.
- The **Lateral Edges** are the edges between the lateral faces.
- The **Apex** of the pyramid is the intersection of the lateral edges. It is the point at the top of the pyramid.
- The **Slant Height** of a regular pyramid is the altitude of one of the lateral faces.
- The **Height** is the perpendicular length between the base and the apex.

- A **Regular Pyramid** is one in which the lateral faces are congruent triangles. The height of a regular pyramid intersects the base at its center.
- An **Oblique Pyramid** is one that is not a right pyramid. That is, the apex is not aligned directly above the center of the base.




- The **Surface Area** of a pyramid is the sum of the areas of all its faces.
- The **Lateral Area** of a pyramid is the sum of the areas of its lateral faces.

## Surface Area and Volume of a Regular Pyramid

**Surface Area:** $SA = \frac{1}{2}Ps + B$

**Lateral SA:** $SA = \frac{1}{2}Ps$

**Volume:** $V = \frac{1}{3}Bh$

where, $P = the\ perimeter\ of\ the\ base$
$s = the\ slant\ height\ of\ the\ pyramid$
$h = the\ height\ of\ the\ pyramid$
$B = the\ area\ of\ the\ base$

## Surface Area and Volume of an Oblique Pyramid

**Surface Area:** $SA = LSA + B$

**Volume:** $V = \frac{1}{3}Bh$

where, $LSA = the\ lateral\ surface\ area$
$h = the\ height\ of\ the\ pyramid$
$B = the\ area\ of\ the\ base$

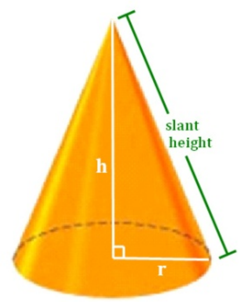
> The lateral surface area of an oblique pyramid is the sum of the areas of the faces, which must be calculated individually.

# Geometry
# Cones

## Definitions

- A **Circular Cone** is a 3-dimensional geometric figure with a circular base which tapers smoothly to a vertex (or apex). The apex and base are in different planes. Note: there is also an elliptical cone that has an ellipse as a base, but that will not be considered here.
- The **Base** is a circle.
- The **Lateral Surface** is area of the figure between the base and the apex.
- There are no **Lateral Edges** in a cone.
- The **Apex** of the cone is the point at the top of the cone.
- The **Slant Height** of a cone is the length along the lateral surface from the apex to the base.
- The **Height** is the perpendicular length between the base and the apex.

- A **Right Cone** is one in which the height of the cone intersects the base at its center.
- An **Oblique Cone** is one that is not a right cone. That is, the apex is not aligned directly above the center of the base.




- The **Surface Area** of a cone is the sum of the area of its lateral surface and its base.
- The **Lateral Area** of a cone is the area of its lateral surface.

## Surface Area and Volume of a Right Cone

**Surface Area:** $SA = \pi rs + \pi r^2$

**Lateral SA:** $SA = \pi rs$

**Volume:** $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$

where, $r = the\ radius\ of\ the\ base$
$s = the\ slant\ height\ of\ the\ cone$
$h = the\ height\ of\ the\ cone$
$B = the\ area\ of\ the\ base$

## Surface Area and Volume of an Oblique Cone

**Surface Area:** $SA = LSA + \pi r^2$

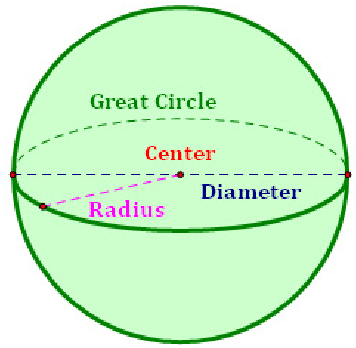
**Volume:** $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$

where, $LSA = the\ lateral\ surface\ area$
$r = the\ radius\ of\ the\ base$
$h = the\ height\ of\ the\ cone$

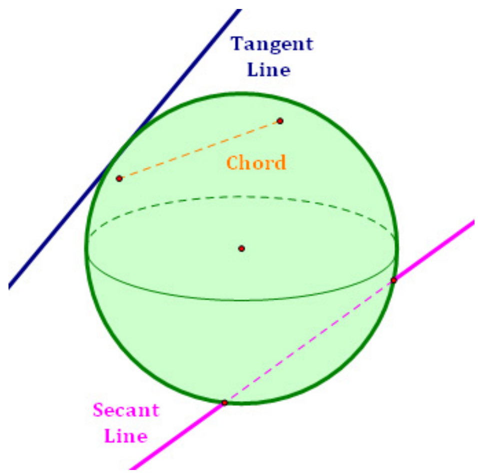
There is no easy formula for the lateral surface area of an oblique cone.

# Geometry
# Spheres

## Definitions




- A **Sphere** is a 3-dimensional geometric figure in which all points are a fixed distance from a point. A good example of a sphere is a ball.
- **Center** – the middle of the sphere. All points on the sphere are the same distance from the center.
- **Radius** – a line segment with one endpoint at the center and the other endpoint on the sphere. The term "radius" is also used to refer to the distance from the center to the points on the sphere.




- **Diameter** – a line segment with endpoints on the sphere that passes through the center.
- **Great Circle** – the intersection of a plane and a sphere that passes through the center.
- **Hemisphere** – half of a sphere. A great circle separates a plane into two hemispheres.
- **Secant Line** – a line that intersects the sphere in exactly two points.
- **Tangent Line**– a line that intersects the sphere in exactly one point.
- **Chord** – a line segment with endpoints on the sphere that does not pass through the center.

## Surface Area and Volume of a Sphere

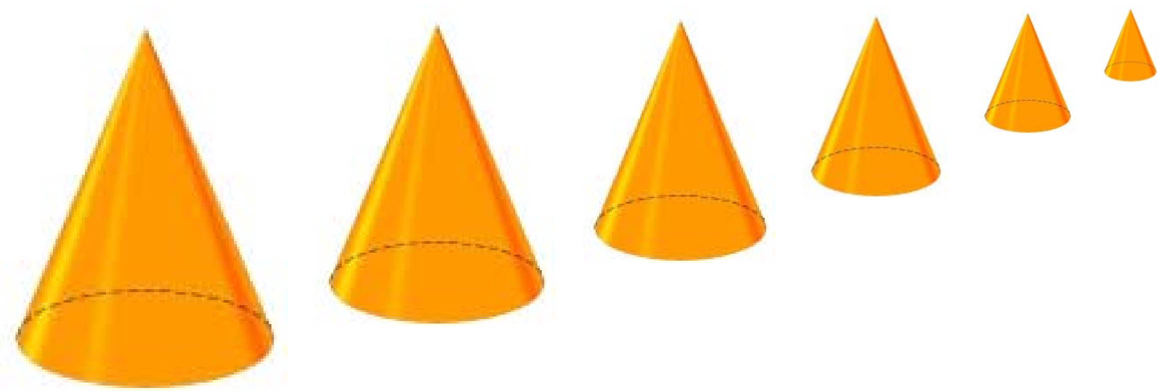
**Surface Area:** $SA = 4\pi r^2$

**Volume:** $V = \frac{4}{3}\pi r^3$

where, $r = the\ radius\ of\ the\ sphere$

# Geometry
# Similar Solids

**Similar Solids** have equal ratios of corresponding linear measurements (e.g., edges, radii). So, all of their key dimensions are proportional.



## Edges, Surface Area and Volume of Similar Figures

Let $\mathbf{k}$ be the scale factor relating two similar geometric solids $\mathbf{F_1}$ and $\mathbf{F_2}$ such that $\mathbf{F_2 = k \cdot F_1}$. Then, for corresponding parts of $\mathbf{F_1}$ and $\mathbf{F_2}$,

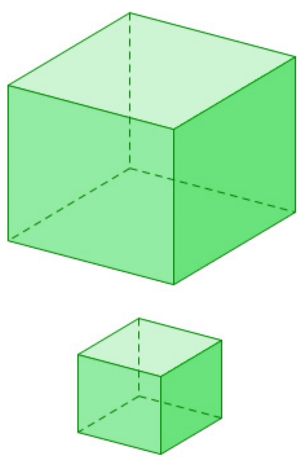
$$\frac{\textbf{Edge of } \mathbf{F_2}}{\textbf{Edge of } \mathbf{F_1}} = \mathbf{k}$$

and

$$\frac{\textbf{Surface Area of } \mathbf{F_2}}{\textbf{Surface Area of } \mathbf{F_1}} = \mathbf{k^2}$$

And

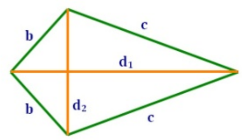
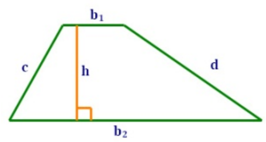
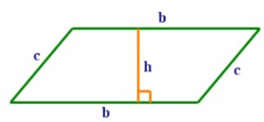
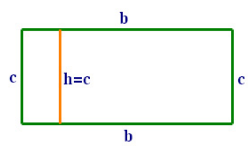
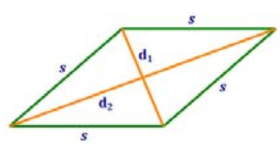
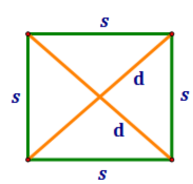
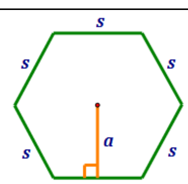
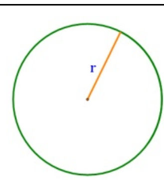
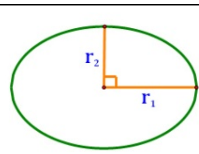
$$\frac{\textbf{Volume of } \mathbf{F_2}}{\textbf{Volume of } \mathbf{F_1}} = \mathbf{k^3}$$



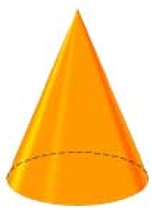
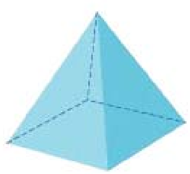
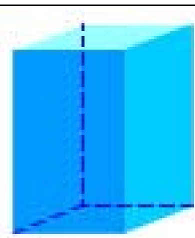
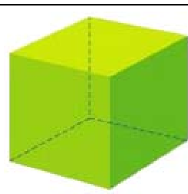
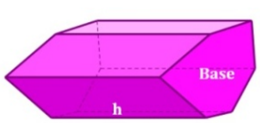
These formulas hold true for any corresponding portion of the figures. So, for example:

$$\frac{\text{Total Edge Length of } F_2}{\text{Total Edge Length of } F_1} = k \qquad \frac{\text{Area of a Face of } F_2}{\text{Area ofa Face of } F_1} = k^2$$

## Geometry
## Summary of Perimeter and Area Formulas – 2D Shapes

| Shape | Figure | Perimeter | Area |
|---|---|---|---|
| **Kite** |  | $P = 2b + 2c$<br>$b, c = sides$ | $A = \frac{1}{2}(d_1 d_2)$<br>$d_1, d_2 = diagonals$ |
| **Trapezoid** |  | $P = b_1 + b_2 + c + d$<br>$b_1, b_2 = bases$<br>$c, d = sides$ | $A = \frac{1}{2}(b_1 + b_2)h$<br>$b_1, b_2 = bases$<br>$h = height$ |
| **Parallelogram** |  | $P = 2b + 2c$<br>$b, c = sides$ | $A = bh$<br>$b = base$<br>$h = height$ |
| **Rectangle** |  | $P = 2b + 2c$<br>$b, c = sides$ | $A = bh$<br>$b = base$<br>$h = height$ |
| **Rhombus** |  | $P = 4s$<br>$s = side$ | $A = bh = \frac{1}{2}(d_1 d_2)$<br>$d_1, d_2 = diagonals$ |
| **Square** |  | $P = 4s$<br>$s = side$ | $A = s^2 = \frac{1}{2}(d_1 d_2)$<br>$d_1, d_2 = diagonals$ |
| **Regular Polygon** |  | $P = ns$<br>$n = number\ of\ sides$<br>$s = side$ | $A = \frac{1}{2}\ a \cdot P$<br>$a = apothem$<br>$P = perimeter$ |
| **Circle** |  | $C = 2\pi r = \pi d$<br>$r = radius$<br>$d = diameter$ | $A = \pi r^2$<br>$r = radius$ |
| **Ellipse** |  | $P \approx 2\pi\sqrt{\frac{1}{2}({r_1}^2 + {r_2}^2)}$<br>$r_1 = major\ axis\ radius$<br>$r_2 = minor\ axis\ radius$ | $A = \pi r_1 r_2$<br>$r_1 = major\ axis\ radius$<br>$r_2 = minor\ axis\ radius$ |

# Geometry
## Summary of Surface Area and Volume Formulas – 3D Shapes

| Shape | Figure | Surface Area | Volume |
|---|---|---|---|
| Sphere |  | $SA = 4\pi r^2$<br>$r = radius$ | $V = \frac{4}{3}\pi r^3$<br>$r = radius$ |
| Right Cylinder |  | $SA = 2\pi rh + 2\pi r^2$<br>$h = height$<br>$r = radius\ of\ base$ | $V = \pi r^2 h$<br>$h = height$<br>$r = radius\ of\ base$ |
| Cone |  | $SA = \pi rl + \pi r^2$<br>$l = slant\ height$<br>$r = radius\ of\ base$ | $V = \frac{1}{3}\pi r^2 h$<br>$h = height$<br>$r = radius\ of\ base$ |
| Square Pyramid |  | $SA = 2sl + s^2$<br>$s = base\ side\ length$<br>$l = slant\ height$ | $V = \frac{1}{3}s^2 h$<br>$s = base\ side\ length$<br>$h = height$ |
| Rectangular Prism |  | $SA = 2 \cdot (lw + lh + wh)$<br>$l = length$<br>$w = width$<br>$h = height$ | $V = lwh$<br>$l = length$<br>$w = width$<br>$h = height$ |
| Cube |  | $SA = 6s^2$<br>$s = side\ length\ (all\ sides)$ | $V = s^3$<br>$s = side\ length\ (all\ sides)$ |
| General Right Prism |   | $SA = Ph + 2B$<br>$P = Perimeter\ of\ Base$<br>$h = height\ (or\ length)$<br>$B = area\ of\ Base$ | $V = Bh$<br>$B = area\ of\ Base$<br>$h = height$ |

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