

Provident sets and rudimentary set forcing

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Abstract. Using the theory of rudimentary recursion and provident sets developed in a previous paper, we give a treatment of set forcing appropriate for working over models of a theory **PROVI** which may plausibly claim to be the weakest set theory supporting a smooth theory of set forcing, and of which the minimal model is Jensen's J_ω . Much of the development is rudimentary or at worst given by rudimentary recursions with parameter the notion of forcing under consideration. Our development eschews the power set axiom. We show that the forcing relation for $\dot{\Delta}_0$ wffs is propagated through our hierarchies by a rudimentary function, and we show that the construction of names for the values of rudimentary and rudimentarily recursive functions is similarly propagated. Our main result is that a set-generic extension of a provident set is provident.

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0: Introduction

There is a certain finitely axiomatisable theory which we call **PROV**, which is weaker than Kripke-Platek set theory **KP**, but stronger than Gandy-Jensen set theory, **GJ**. All three theories are true in $\mathbf{HF} = V_\omega = J_1 = L_\omega$; if an axiom of infinity be added to each theory, giving the theories **KPI**, **PROVI** and **GJI**, the minimal transitive models are then respectively the Jensen fragments $J_{\omega_1^{CK}}$, J_ω and J_2 .

The *provident* sets are **HF** and the transitive models of **PROVI**. We show that every provident set A supports the definition of the forcing relation $\Vdash^{\mathbb{P}}$ when $\mathbb{P} \in A$; our main result is that a set-generic extension of a provident set is provident.

For most of this paper we avoid use of the power set axiom; the paper [M4] discusses the problems and possibilities of set forcing over models of Mac Lane or of Zermelo set theory, two theories which include the power set axiom.

We draw on the results of *Rudimentary recursion, gentle functions and provident sets* [MB], and in particular we make heavy use of the rudimentary function \mathbb{T} which was introduced in *Weak systems of Gandy, Jensen and Devlin* [M3]: its properties are that if u is transitive, then $\mathbb{T}(u)$ is transitive, with u both a member and a subset of it; every member of $\mathbb{T}(u)$ is a subset of u ; further, the union over all n of $\mathbb{T}^n(u)$ is the rudimentary closure of $u \cup \{u\}$.

Provident sets

Let p be a set. Call a function $x \mapsto F(x)$ *p-rud-rec* (short for *p-rudimentarily recursive*) if there is a rud function H such that for every set x

$$F(x) = H(p, F \upharpoonright x).$$

Examples: the rank function, ϱ , and transitive closure, tcl , are \emptyset -rud-rec; the evaluation $\text{val}_{\mathcal{G}}(\cdot)$ of the names of a forcing language using a generic \mathcal{G} is \mathcal{G} -rud-rec. The axioms of **PROV** are such that its transitive models are those transitive sets A such that for each p , each p -rud-rec F and each $x \in A$, $F(x)$ is in A .

Let c be a transitive set; using \mathbb{T} we define in §2 a hierarchy giving an initial segment of $L(c)$ by a recursion on the ordinals. The novelty of the definition is that the whole of c is not included at the start, but its members are fed in according to their rank: if we put $c_\nu = \{x \in c \mid \varrho(x) < \nu\}$, then the following c -rud recursion on the ordinals holds:

$$c_0 = \emptyset; \quad c_{\nu+1} = c \cap \{x \mid x \subseteq c_\nu\}; \quad c_\lambda = \bigcup \{c_\nu \mid \nu < \lambda\} \text{ at limit } \lambda.$$

The *canonical progress towards c* is the hierarchy P_ν^c defined by setting

$$P_0^c = \emptyset; \quad P_{\nu+1}^c = \mathbb{T}(P_\nu^c) \cup \{c_\nu\} \cup c_{\nu+1}; \quad P_\lambda^c = \bigcup \{P_\nu^c \mid \nu < \lambda\} \text{ at limit } \lambda.$$

0.0 REMARK As $c_\nu = c \cap P_\nu^c$, we might have defined P_ν^c by a single c -rudimentary recursion on ordinals:

$$P_0^c = \emptyset; \quad P_{\nu+1}^c = \mathbb{T}(P_\nu^c) \cup \{c \cap P_\nu^c\} \cup (c \cap \{x \mid x \subseteq P_\nu^c\}); \quad P_\lambda^c = \bigcup_{\nu < \lambda} P_\nu^c.$$

The axiomatization of **PROV** may then be summarised as

extensionality
 + the empty set exists
 + all rudimentary functions are defined everywhere
 + every set has a rank
 + every set has a transitive closure
 + for every transitive c and ordinal ν the set P_ν^c exists

Set-forcing over provident sets

Let A be a transitive model of **PROVI**; let \mathbb{P} be a separative partial ordering which is a member of A . Many functions and relations involved in the development of forcing, though not known explicitly to be \mathbb{P} -rud-rec, may be called *essentially* \mathbb{P} -rud-rec in that their restriction, or the restriction of their characteristic functions, to the stages P_ν^c of a progress exhibits the same pattern of uniform affine delay proved in [MB, section 5] to hold for truly \mathbb{P} -rud-rec functions.

The first goal of the paper is to prove that for each ℓ the forcing relation $p \Vdash^{\mathbb{P}} \varphi$, restricted to those sentences of the language of forcing that are Δ_0 and of length at most ℓ , is, in that sense, essentially \mathbb{P} -rud-rec.

The second goal is to analyse the construction of names for the values of functions applied to objects in a generic extension. We speak of this task as the construction of *nominators* for the functions concerned.

The first stage of that is to show that to each rudimentary function R , say of two variables, there is an essentially \mathbb{P} -rud-rec function $R^{\mathbb{P}}$ of two variables such that for (A, \mathbb{P}) -generic \mathcal{G} and all x, y in A

$$\text{val}_{\mathcal{G}}(R^{\mathbb{P}}(x, y)) = R(\text{val}_{\mathcal{G}}(x), \text{val}_{\mathcal{G}}(y)).$$

We shall then find \mathbb{P} -rud rec functions $\varrho^{\mathbb{P}}$ and $\text{tcl}^{\mathbb{P}}$ that similarly build names for the rank and the transitive closure of a given object from its forcing name.

Finally we must build names for the stages of a progress $\nu \mapsto P_{\nu}^d$ for d a transitive set in the generic extension. It is here that functions of affine delay prove to be insufficient; when $\varrho(\mathbb{P})$ is small, we may use certain functions of larger delay under which provident sets are nevertheless closed; in the general case we must re-interpret our universe in terms of rudimentary recursion from the ternary relation $p \Vdash^{\mathbb{P}} \underline{a} = \underline{b}$.

The main theorem will then follow easily for provident sets of the form P_{θ}^e , and will immediately extend to all provident sets containing \mathbb{P} , using the result from [MB] that every provident set is the union of a directed family of sets of the form P_{θ}^c .

1: Heuristic

We begin with some reminders of the general character of forcing: the present discussion is heuristic, to give the reader a feel for the way the forcing relation will operate. In particular, the methods used in this section for naming old and new objects are, dangerously, simpler than the methods of the formal development to be given in subsequent sections.

Suppose we face the following challenge:

given a transitive M , to find a transitive $N \supseteq M$ with $On \cap N = On \cap M$ but where N contains a subset of ω not in M .

If M is admissible, such an N will necessarily violate the axiom of constructibility, for $(L)^N = (L)^M$. Thus we are aiming to add a set $a \subseteq \omega$ to M .

1.0 We begin by asking questions about a : suppose we have some information p about a : what statements about a will be forced to be true? For example if p is the statement that $5 \in a$, then **Not all members of a are even** is forced by p .

Our beginning intuition for forcing is the idea that we have pieces of partial information about the new object we are adding, and that we build up a picture of the new model from this partial information.

Our pieces of information are called *conditions*, and to start with we suppose that the collection of conditions is a set, \mathbf{P} . Experience shows that we should make the following assumptions about \mathbf{P} :

(1.0.0) \mathbf{P} is partially ordered by a relation \leq ; if $p \leq q$ we think that p contains **more** information than q .

(1.0.1) To get something interesting we allow the possibility of two conditions being *incompatible*: we say that p is compatible with q if there is some r stronger than both: $r \leq p$ & $r \leq q$; and we say that p is incompatible with q , in symbols $p \perp q$, if no such r exists.

(1.0.2) We assume that any condition can be strengthened in two incompatible ways:

$$\forall p \exists q: \leq p \exists r: \leq p \quad q \perp r$$

(1.0.3) We suppose that \mathbf{P} has a greatest element $\mathbf{1} = \mathbf{1}^{\mathbf{P}}$, where this condition is the one that gives us no information at all. Thus $\mathbf{1}^{\mathbf{P}}$ is compatible with every condition.

(1.0.4) Finally we suppose that \mathbf{P} is *separative*: that is,

$$\forall p \forall q (p \not\leq q \implies \exists r: \leq p \quad r \perp q).$$

Such a \mathbf{P} is called a *notion of forcing*. Let us look at two examples.

Cohen's original forcing:

We take

$$\mathbf{P} = \bigcup \{ {}^n 2 \mid n \in \omega \} = {}^{<\omega} 2$$

the set of finite maps from ω to $2 = \{0, 1\}$.

We define the ordering by reverse inclusion:

$$p \leq^{\mathbf{P}} q \iff q \subseteq p.$$

This forcing is intended to add a “generic” $a : \omega \rightarrow 2$. We take a symbol \dot{a} and use it as a name for the new a that we are trying to add.

The intended meaning of a condition $p : n \rightarrow \omega$ is that $\dot{a} \restriction n = p$. So if $n = 6$ and $p(3) = 1$, then p will force the statement that $\dot{a}(\hat{3}) = \hat{1}$.

We suppose that we are in one universe, which we call the ground model, describing a larger universe, which we call the generic extension; the new objects are only partially known to us, so we use dotted letters as names for them, as \dot{a} . The objects in the ground model are fully known to us, and we name them with hatted letters: thus $\hat{3}$ is our name for 3 in the language of forcing.

Continuing our discussion of Cohen's original, we show now that the new real \dot{a} is not the same as any old real \hat{b} : precisely, we prove the following:

1.1 PROPOSITION *Let $b : \omega \rightarrow 2$. Then $\forall p \exists q : \leq p \ q \Vdash^{\mathbf{P}} \hat{b} \neq \dot{a}$.*

Notice the topological flavour to this proposition: it is saying that the set of conditions forcing a certain statement is *dense*. Indeed we may topologise \mathbf{P} so that is exactly what is happening.

Proof of 1.1: given p , let $n = \text{dom}(p)$; n is of course the least natural number not in the domain of p . Look at $b(n)$, and let $q = p \cup \{(1 - b(n), n)\}$. ¬ (1.1)

So in any model in which everything that is forced on a dense set is true \dot{a} will be a new subset of ω .

This notion of density is central to the concept of forcing. One of the properties of the forcing relation, which we shall refer to as the *density property*, is that

$$p \Vdash^{\mathbf{P}} \varphi \iff \forall q : \leq p \ \exists r : \leq q \ r \Vdash^{\mathbf{P}} \phi.$$

As we progressively extend the definition of the forcing relation to ever wider classes of formulæ, we shall check at each stage that the density property and other characteristic properties of forcing are preserved.

Another example

Let η be an infinite ordinal. This time take

$$\mathbf{P} = \{ p \mid \exists n : \in \omega \ p : n \xrightarrow{1-1} \eta \}.$$

As before order by reverse inclusion:

$$p \leq^{\mathbf{P}} q \iff q \subseteq p.$$

This forcing adds a generic $\dot{f} : \hat{\omega} \xrightarrow{1-1} \hat{\eta}$: a condition p with domain n is a description of $\dot{f} \restriction \hat{n}$.
So

$$\mathbb{1}^{\mathbf{P}} \Vdash^{\mathbf{P}} \dot{f} \restriction \ell(\hat{p}) = \hat{p}.$$

1.2 EXERCISE $\mathbb{1}^{\mathbf{P}} \Vdash^{\mathbf{P}} \dot{f}$ is 1-1.

1.3 PROPOSITION $\mathbb{1}^{\mathbf{P}} \Vdash^{\mathbf{P}} \hat{\eta}$ is countable.

Proof: by a density argument. Given p , n not in $\text{dom}(p)$ and an ordinal $\xi < \eta$, suppose that ξ is not in the image of p . we find $q \leq^{\mathbf{P}} p$ such that $n \in \text{dom}(q)$ and $q(n) = \xi$.

Thus we have shown that $\forall \xi : \leq \eta \ \forall p \exists q : \leq p \ q \Vdash^{\mathbf{P}} \hat{\xi} \in \text{the image of } \dot{f}$.

2: Forcing in provident sets

Let M be a provident set, and $\mathbb{P} = \langle \mathbf{P}, \mathbf{1}^{\mathbf{P}}, \leq \rangle$ a separative partial order in M , with a top point $\mathbf{1} = \mathbf{1}^{\mathbf{P}}$. We suppose that $\omega \in M$.

We aim to define within M a relation \Vdash , more exactly $\Vdash^{\mathbf{P}}$, describing an extension $M[\mathcal{G}]$ of M , where \mathcal{G} is an (M, \mathbb{P}) -generic filter. Each object in M potentially names an element of $M[\mathcal{G}]$. \Vdash is a relation between elements of \mathbf{P} and sentences in a language of set theory that we shall gradually build up. In fact the full relation can only be defined schematically within M .

This language will start from two two-place relations $=$ and ϵ and will broadly resemble the formal languages introduced in the paper *The Strength of Mac Lane Set Theory* [M2]. We shall use our devices of dots and type-writer face as before; but the constants will play a different role, and hence we shall use a different mark. To each set x in the universe corresponds a name \underline{x} for an object in the generic extension. Thus the statement $p \Vdash^{\mathbf{P}} \underline{x} \in \underline{y}$ expresses information about the evaluation of the objects x and y functioning as names of sets to be created in the forcing extension given by the notion of forcing \mathbf{P} .

2.0 DEFINITION $p \Vdash_0 \underline{a} \in \underline{b} \iff_{\text{df}} (p, a) \in b$.

\Vdash_0 is our first approximation to the relation \Vdash .

2.1 LEMMA If $p \Vdash_0 \underline{a} \in \underline{b}$ then $a \in \bigcup \bigcup b$.

2.2 DEFINITION In future we shall write $\bigcup^2 x$ for $\bigcup \bigcup x$.

2.3 LEMMA \Vdash_0 is Δ_0 , indeed rudimentary.

2.4 REMARK For relations, Δ_0 and rud are the same: cf Devlin [De, VI.1.5].

2.5 DEFINITION $p \Vdash_1 \underline{a} \in \underline{b} \iff_{\text{df}} \exists q: \in \bigcup^2 b [q \geq p \ \& \ (q, a) \in b]$.

2.6 LEMMA For all $p \in \mathbb{P}$, a and b :

$$(2.6.0) \quad p \Vdash_0 \underline{a} \in \underline{b} \implies p \Vdash_1 \underline{a} \in \underline{b};$$

$$(2.6.1) \quad \text{if } p \Vdash_1 \underline{a} \in \underline{b} \text{ then } a \in \bigcup^2 b;$$

$$(2.6.2) \quad \Vdash_1 \text{ is rudimentary in } \mathbb{P}.$$

2.7 LEMMA If $p \Vdash_1 \underline{a} \in \underline{b}$ and $r \leq p$ then $r \Vdash_1 \underline{a} \in \underline{b}$.

This last statement shows that \Vdash_1 improves \Vdash_0 and starts to resemble a forcing relation.

We define the relation $p \Vdash \underline{b} = \underline{c}$ by recursion:

2.8 DEFINITION $p \Vdash \underline{b} = \underline{c} \iff_{\text{df}}$
 $\forall \beta: \in \bigcup^2 b \ \forall r: \leq p \ [r \Vdash_1 \underline{\beta} \in \underline{b} \implies \exists t: \leq r \ \exists \gamma: \in \bigcup^2 c (t \Vdash \underline{\beta} = \underline{\gamma} \ \& \ t \Vdash_1 \underline{\gamma} \in \underline{c})] \ \&$
 $\forall \gamma: \in \bigcup^2 c \ \forall r: \leq p \ [r \Vdash_1 \underline{\gamma} \in \underline{c} \implies \exists t: \leq r \ \exists \beta: \in \bigcup^2 b (t \Vdash \underline{\gamma} = \underline{\beta} \ \& \ t \Vdash_1 \underline{\beta} \in \underline{b})]$

The above definition is \mathbb{P} -rud recursive in a suitable sense, which we must now articulate, and therefore will succeed in provident sets of which \mathbb{P} is a member, or, more generally, in \mathbb{P} -provident sets.

2.9 DEFINITION Let $\chi_{\mathbb{P}}(p, b, c)$ be the characteristic function of the relation $p \Vdash^{\mathbf{P}} \underline{b} = \underline{c}$, so that it takes the value 1 if $p \Vdash^{\mathbf{P}} \underline{b} = \underline{c}$ and 0 otherwise.

Our plan is to show that the graph of $\chi_{\mathbb{P}}$ on transitive sets is definable by a \mathbb{P} -rudimentary recursion.

2.10 THE DEFINABILITY LEMMA “ f is a $\chi_{\mathbb{P}}$ attempt” is $\Delta_0(\mathbb{P}, f)$.

Proof: We must first say that everything in the domain of f is an ordered triple, of which the first component is a member of \mathbf{P} ; and whenever $(p, b, c) \in \text{Dom}(f)$ and β and γ are in b and c respectively, and $q \in \mathbf{P}$ then (q, c, b) and (q, β, γ) are in the domain too. But all that is $\Delta_0(\mathbb{P}, f)$.

Then we must say that f respects the recursive definition: but all that is also a Δ_0 statement about \mathbb{P} and f . (2.10)

2.11 THE PROPAGATION LEMMA Let $F(u) = \chi_{\mathbb{P}} \upharpoonright (\mathbb{P} \times u \times u)$. There is a rudimentary function $H_{\mathbb{P}}$ such that for any transitive P , if $P \subseteq P^+ \subseteq \mathcal{P}(P)$,

$$F(P^+) = H_{\mathbb{P}}(\mathbb{P}, F(P), P^+).$$

In the following argument, and elsewhere, $(\cdot)_i^3$ are basic “un-tripling” functions such that for a poset $\mathbb{P} = (\mathbf{P}, \mathbf{1}^{\mathbf{P}}, \leq)_3$, $(\mathbb{P})_0^3 = \mathbf{P}$, $(\mathbb{P})_1^3 = \mathbf{1}^{\mathbf{P}}$, and $(\mathbb{P})_2^3 = \leq$. On this occasion, but not in future, the restricted nature of a quantifier such as $\forall r:\leq p$ has been made manifest by re-writing it as $\forall r:\in(\leq\text{“}\{p\}\text{”})$.

Proof of the Propagation Lemma: Let $\Psi(x, f, p, b, c)$ be the Δ_0 formula

$$\begin{aligned} & \forall \beta:\in \bigcup^2 b \ \forall r:\in ((x)_2^3\text{“}\{p\}\text{”}) \ [r \Vdash_1 \beta \in \underline{b} \implies \exists t:\in ((x)_2^3\text{“}\{r\}\text{”}) \ \exists \gamma:\in \bigcup^2 c \ (f(t, \beta, \gamma) = 1 \ \& \ t \Vdash_1 \gamma \in \underline{c})] \\ & \& \ \forall \gamma:\in \bigcup^2 c \ \forall r:\in ((x)_2^3\text{“}\{p\}\text{”}) \ [r \Vdash_1 \gamma \in \underline{c} \implies \exists t:\in ((x)_2^3\text{“}\{r\}\text{”}) \ \exists \beta:\in \bigcup^2 b \ (f(t, \gamma, \beta) = 1 \ \& \ t \Vdash_1 \beta \in \underline{b})] \end{aligned}$$

Define $H_=(x, f, v)$ to be

$$\left(\{0, 1\} \times ((x)_0^3 \times (v \times v)) \right) \cap \left(\{ (1, p, b, c)_4 \mid_{p, b, c} \Psi(x, f, p, b, c) \} \cup \{ (0, p, b, c)_4 \mid_{p, b, c} \neg \Psi(x, f, p, b, c) \} \right). \quad \dashv (2.11)$$

2.12 DEFINITION $\chi_=_\vdash P =_{\text{df}} \{ \chi_=_\vdash (\mathbf{P} \times u \times u) \mid u \text{ transitive } \& u \in P \}$.

Propagation of $\chi_=_$

The progress P_ν^e was defined in §5 of *Rudimentary Recursion* for c a transitive set. We could continue to work with progresses of the above kind, but a problem would then arise at the end of the paper, in the proof that a set-generic extension of a provident set is provident. It is better to change tack now and work with other progresses, which might be called construction from e as a set and $\chi_=_$ as a predicate, with the definition of $\chi_=_$ evolving during the construction.

2.13 DEFINITION Let e be a transitive set of which \mathbb{P} is a member; let $\eta = \varrho(\mathbb{P})$. We define by a p -rudimentary recursion a sequence $((e_\nu, P_\nu^{e;=}, \chi_\nu^e)_\nu$ of triples, thus obtaining a new progress $(P_\nu^{e;=})_\nu$. For every ν , e_ν will be defined as before; for $\nu \leq \eta$ we set $P_\nu^{e;=} = P_\nu^e$; for $\nu < \eta$, we set $\chi_\nu^e = \emptyset$ but at η , we set $\chi_\eta^e = \chi_=_\vdash P_\eta^e$, which will be a set by the last Corollary. Thereafter we set

$$\begin{aligned} e_{\nu+1} &= e \cap \{x \mid x \subseteq e_\nu\} & e_\lambda &= \bigcup_{\nu < \lambda} e_\nu \\ P_{\nu+1}^{e;=} &= \mathbb{T}(P_\nu^{e;=}) \cup \{e_\nu\} \cup e_{\nu+1} \cup \{\chi_\nu^e \cap P_\nu^{e;=}\} & P_\lambda^{e;=} &= \bigcup_{\nu < \lambda} P_\nu^{e;=} \\ \chi_{\nu+1}^e &= H_=(\mathbb{P}, \chi_\nu^e, P_{\nu+1}^{e;=}) & \chi_\lambda^e &= \bigcup_{\nu < \lambda} \chi_\nu^e \end{aligned}$$

2.14 PROPOSITION Let e be transitive, with $\mathbb{P} \in e$, and let θ be indecomposable and strictly greater than $\varrho(\mathbb{P})$. Then $P_\theta^{e;=} = P_\theta^e$.

Proof : First consider the *special case* that $\theta > \varrho(e)$. by [MB, **where ?**], P_θ^e is provident and therefore supports all p -rud recursions with $p \in P_\theta^e$; the sequence of triples $((e_\nu, P_\nu^{e;=}, \chi_\nu^e)_\nu$ is defined by such a recursion, with parameter the triple $(e, \mathbb{P}, \chi_{\varrho(\mathbb{P})}^e)_3$. So the left side is included in the right. On the other hand, $(P_\nu^{e;=})_{\nu \leq \theta}$ is a θ -progress, continous at θ ; $e \in P_{\varrho(e)+1}^{e;=}$; and so by [MB, Proposition 5.36], the right side is included in the left.

Now for the *general case*: the special case tells us that for each ζ with $\varrho(\mathbb{P}) < \zeta < \theta$, $P_\theta^{e;=} = P_\theta^{e_\zeta}$. Taking the union over all such ζ gives $P_\theta^{e_\theta;=} = P_\theta^{e;=}$; the equalities $P_\theta^{e;=} = P_\theta^{e_\theta;=}$ and $P_\theta^{e_\theta} = P_\theta^e$, proved (as) in [MB, Proposition 5.61], complete the proof. \dashv (2.14)

This reconstruction of P_θ^e shortens the delay for most χ_ν^e .

2.15 PROPOSITION For any ordinal $\nu \geq \eta$, any limit ordinal $\lambda > \eta$ and $k \in \omega$,

$$\begin{aligned} (2.15.0) \quad & \chi_\nu^e = \chi_=_\vdash P_\nu^{e;=}; \\ (2.15.1) \quad & \chi_\nu^e \subseteq P_{\nu+6}^{e;=}; \\ (2.15.2) \quad & \chi_\lambda^e \subseteq P_\lambda^{e;=}; \\ (2.15.3) \quad & \chi_=_\vdash P_\nu^{e;=} \in P_{\nu+12}^{e;=}. \end{aligned}$$

Proof : (2.15.0) is true by definition for $\nu = \eta$; thereafter, the function H_- preserves its truth at successor stages, and at limit stages, we simply take unions on both sides.

(2.15.1): the ‘6’ reflects the delay in the creation of Kuratowski ordered pairs.

(2.15.2): at limit stages, that delay no longer exists.

(2.15.3): Fix $\nu \geq \eta$. Let $\chi^+ = \chi_{\nu+6}^e \cap P_{\nu+6}^{e;=}$; then $\chi^+ \in P_{\nu+7}^{e;=}$ by definition of the progress. Since $\chi_\nu^e \subseteq P_{\nu+6}^{e;=}$, $\chi_\nu^e = \chi^+ \cap (2 \times (\mathbf{P} \times (P_\nu^{e;=} \times P_\nu^{e;=})))$.

By [MB, Lemmata 5.11 and 5.12], $x, y \in P_\zeta^{e;=} \implies x \cap y = x \setminus (x \setminus y) \in P_{\zeta+2}^{e;=}$ & $x \times y \in P_{\zeta+3}^{e;=}$.

$P_\nu^{e;=} \in P_{\nu+1}^{e;=}$, so $P_\nu^{e;=} \times P_\nu^{e;=} \in P_{\nu+4}^{e;=}$, $\mathbf{P} \times (P_\nu^{e;=} \times P_\nu^{e;=}) \in P_{\nu+7}^{e;=}$ and $\{0, 1\} \times (\mathbf{P} \times (P_\nu^{e;=} \times P_\nu^{e;=})) \in P_{\nu+10}^{e;=}$.

We conclude that $\chi_\nu^e \in P_{\nu+12}^{e;=}$. + (2.15)

Propagation of χ_ϵ

We may now define $p \Vdash a \in b$:

2.16 DEFINITION $p \Vdash a \in b \iff_{\text{df}} \forall s: \leq p \exists t: \leq s \exists \beta: \in \bigcup^2 b [t \Vdash \beta = a \text{ \& } t \Vdash_1 \beta \in b]$.

2.17 REMARK This is not a definition by recursion: indeed it is visibly rudimentary in $p \Vdash b = c$.

2.18 DEFINITION Let $\chi_\epsilon(p, a, b)$ be the characteristic function of the relation $p \Vdash^{\mathbf{P}} a \in b$.

2.19 PROPOSITION There is a natural number s_ϵ such that for each ordinal $\nu \geq \eta$, $\chi_\epsilon \upharpoonright P_\nu^{e;=} \in P_{\nu+s_\epsilon}^{e;=}$.

Proof : There are rudimentary functions R and S such that

$$\chi_\epsilon \upharpoonright P_\nu^{e;=} = 2 \times \text{Dom}(\chi_\nu^e) \cap (\{(1, p, a, b)_4 \mid R(p, a, b, \chi_\nu^e) = 1\} \cup \{(0, p, a, b)_4 \mid R(p, a, b, \chi_\nu^e) = 0\}) = S(\chi_\nu^e)$$

We may take $s_\epsilon = 12 + c_S$. + (2.19)

Familiar properties of forcing

We check as our definition of forcing develops that it has the expected density properties, and we establish familiar properties of equality and the substitution properties of $=$ for ϵ :

2.20 PROPOSITION If $p \Vdash b = c$ and $q \leq p$ then $q \Vdash b = c$.

$$p \Vdash a = b \iff \forall q: \leq p \exists r: \leq q r \Vdash a = b$$

2.21 PROPOSITION For all $p \in \mathbb{P}$, a, b and c :

(2.21.0) $\forall p \forall b p \Vdash b = b$;

(2.21.1) if $p \Vdash b = a$ then $p \Vdash a = b$;

(2.21.2) if $p \Vdash a = b$ and $p \Vdash b = c$ then $p \Vdash a = c$.

Proof : 1) let b be a counter-example of minimal rank. The definition of $p \Vdash b = b$ involves various $r, \beta \in \bigcup^2 b$, for which $r \Vdash \beta = \beta$ by the minimality condition on b .

2) from the symmetry of the definition.

3) If $q \Vdash_1 \alpha \in a$ then $\exists r \leq q \exists \beta (r \Vdash_1 \beta \in b \text{ \& } r \Vdash \alpha = \beta)$, so $\exists s \leq r \exists \gamma (s \Vdash_1 \gamma \in c \text{ \& } s \Vdash \beta = \gamma)$; the t we seek is s ; $s \Vdash \alpha = \beta \wedge \beta = \gamma$; so, assuming we have minimised the rank of a possible failure \bar{b} , $s \Vdash \alpha = \gamma$ and $s \Vdash_1 \gamma \in c$, as required. + (2.21)

2.22 LEMMA $q \leq p \text{ \& } p \Vdash a \in b \implies q \Vdash a \in b$.

2.23 LEMMA $p \Vdash a \in b \iff \forall q: \leq p \exists r: \leq q r \Vdash a \in b$

2.24 PROPOSITION If $p \Vdash_1 a \in b$ then $p \Vdash a \in b$.

Proof : let $r \leq p$: takes $s = r$ and $\beta = a$; then $s \Vdash \beta = a$ and $s \Vdash_1 \beta \in b$. + (2.24)

2.25 PROPOSITION If $p \Vdash a \in b$ and $p \Vdash a = c$ then $p \Vdash c \in b$.

Proof : Let $s \leq p$. We seek $t \leq s$ and $\beta \in \bigcup^2 b$ such that $t \Vdash \beta = c$ and $t \Vdash_1 \beta \in b$. We know that there are $t \leq s$ and $\beta \in \bigcup^2 b$ such that $t \Vdash \beta = a$ and $t \Vdash_1 \beta \in b$; since $p \Vdash a = c$ and $t \leq p$, $t \Vdash \beta = c$. + (2.25)

2.26 PROPOSITION If $p \Vdash a \in b$ and $p \Vdash b = d$, then $p \Vdash a \in d$.

Proof : Let $s \leq p$. We seek $t \leq s$ and $\delta \in \bigcup^2 d$ such that $t \Vdash \underline{\delta} = \underline{a}$ and $t \Vdash_1 \underline{\delta} \in \underline{d}$. Since $p \Vdash \underline{a} \in \underline{b}$, there are $r \leq s$ and $\beta \in \bigcup^2 b$ such that $r \Vdash \underline{\beta} = \underline{a}$ & $r \Vdash_1 \underline{\beta} \in \underline{b}$. Since $p \Vdash \underline{b} = \underline{d}$ and $r \leq p$, there are $t \leq r$ and $\delta \in \bigcup^2 d$ such that $t \Vdash \underline{\beta} = \underline{\delta}$ and $t \Vdash_1 \underline{\delta} \in \underline{d}$; as $t \Vdash \underline{\beta} = \underline{\delta}$ and $t \Vdash \underline{\beta} = \underline{a}$, $t \Vdash \underline{\delta} = \underline{a}$. ¬ (2.26)

Forcing the negation of a statement

We have defined $p \Vdash \dot{\Phi}$ for $\dot{\Phi}$ of the form $\underline{x} \in \underline{y}$ or $\underline{x} = \underline{y}$, and now wish to define $p \Vdash \neg \dot{\Phi}$ in these cases. The definition is characteristic of forcing; and we will maintain it as we extend our definition of forcing to ever larger classes of formulæ.

2.27 DEFINITION $p \Vdash \neg \dot{\Phi} \iff \forall q: \leq p \ q \nVdash \dot{\Phi}$.

We shall use this definition in our next proposition: notice that it has the immediate consequence that

$$\forall p \exists q: \leq p \ [q \Vdash \dot{\Phi} \vee q \Vdash \neg \dot{\Phi}].$$

Further it renders *Modus Ponens* effective:

2.28 PROPOSITION If $p \Vdash \Phi \longrightarrow \Psi$ and $p \Vdash \Phi$ then $p \Vdash \Psi$.

Proof : if $p \Vdash \neg \Phi \vee \Psi$ then $\forall q: \leq p \ \exists r: \leq q \ (r \Vdash \neg \Phi \vee r \Vdash \Psi)$; but the first alternative is impossible for $r \leq p$ if $p \Vdash \Phi$, and so $r \Vdash \Psi$; by density $p \Vdash \Psi$. ¬ (2.28)

2.29 REMARK We have given this proof now; it only applies, of course, to those formulæ for which a forcing definition has been given.

Installing the ground model

Before we turn to the definition of the generic extension of a model, we identify objects that will serve as names for the elements of the ground model, and thus ensure that our generic structure is indeed an extension of our ground model. It is convenient to assume that $\mathbf{1}^P$ is actually the ordinal 1.

2.30 DEFINITION Set $\hat{y} =_{\text{df}} \{(\mathbf{1}^P, \hat{x}) \mid x \in y\}$.

This is a rudimentary recursion in the parameter $\mathbf{1}^P$, being of the form

$$F(a) = G(\mathbf{1}^P, F \upharpoonright a)$$

where G is the rudimentary function $(i, f) \mapsto \{i\} \times \text{Im}(f)$; thus with our convention that $\mathbf{1}^P = 1$, the recursion may be regarded as pure.

2.31 LEMMA If $q \Vdash_1 \underline{\xi} \in \hat{x}$ then $\exists a: \in x \ \xi = \hat{a}$.

Proof : if $q \leq p$ and $(p, \xi) \in \hat{x}$ then $p = \mathbf{1}^P$ and $\xi = \hat{a}$ for some $a \in x$. ¬ (2.31)

2.32 PROPOSITION For all x and y , the following hold:

$$\begin{aligned} x \in y &\implies \mathbf{1} \Vdash \underline{\hat{x}} \in \hat{y} \\ x = y &\implies \mathbf{1} \Vdash \underline{\hat{x}} = \hat{y} \\ \exists p \ p \Vdash \underline{\hat{x}} = \hat{y} &\implies x = y \\ x \neq y &\implies \mathbf{1} \Vdash \neg(\underline{\hat{x}} = \hat{y}) \\ \exists p \ p \Vdash \underline{\hat{x}} \in \hat{y} &\implies x \in y \\ x \notin y &\implies \mathbf{1} \Vdash \neg(\underline{\hat{x}} \in \hat{y}) \end{aligned}$$

Proof : If $x \in y$, then $(\mathbf{1}, \hat{x}) \in \hat{y}$, so $\mathbf{1} \Vdash_0 \underline{\hat{x}} \in \hat{y}$ and so $\mathbf{1} \Vdash \underline{\hat{x}} \in \hat{y}$. If $x = y$, $\hat{x} = \hat{y}$ and so $\mathbf{1} \Vdash \underline{\hat{x}} = \hat{y}$, by 2.13.

We prove the third line inductively: suppose $p \Vdash \underline{\hat{x}} = \hat{y}$ for $a \in x$ and $r \leq p$, $r \Vdash_1 \underline{\hat{a}} \in \hat{x}$, there will be a b and $s \leq r$ with $s \Vdash \underline{\hat{a}} = \underline{\hat{b}}$ and $s \Vdash_1 \underline{\hat{b}} \in \hat{y}$; but then $b = \hat{c}$ for some $c \in y$; so $s \Vdash \underline{\hat{a}} = \underline{\hat{c}}$ by 2.13 so by induction $a = c$; and thus $x \subseteq y$; similarly $y \subseteq x$ and so $x = y$.

The next line is the contrapositive, by definition of forcing for negation.

If $p \Vdash \underline{\hat{x}} \in \hat{y}$ then for some $r \leq p$ and some b , $r \Vdash \underline{\hat{x}} = \underline{\hat{b}}$ and $r \Vdash_1 \underline{\hat{b}} \in \hat{y}$ so for some $c \in y$, b is \hat{c} ; so $r \Vdash \underline{\hat{x}} = \underline{\hat{c}}$; by line 3, $x = c$, and thus $x \in y$.

Thus the fifth line is proved; and the sixth is its contrapositive. ¬ (2.32)

3: Extension of the definition of forcing to all $\dot{\Delta}_0$ sentences

So far we have set up the beginnings of a definition of forcing, for atomic sentences and their negations. We wish to extend the definition of \Vdash to all $\dot{\Delta}_0$ sentences of the forcing language, on these lines:

$$\begin{aligned} 3.0 \text{ PROPOSED DEFINITION} \quad & p \Vdash \varphi \wedge \vartheta \iff p \Vdash \varphi \ \& \ p \Vdash \vartheta \\ & p \Vdash \neg \varphi \iff \forall q: \leq p \ q \nVdash \varphi \\ & p \Vdash \bigwedge \mathfrak{x}: \epsilon \underline{y} \varphi(\mathfrak{x}) \iff \forall q: \leq p \ \forall (s, \beta): \epsilon \underline{y} \ (q \leq s \implies \exists r: \leq q \ r \Vdash \varphi[\underline{\beta}]) \end{aligned}$$

In fact the forcing relation will prove to be a union of an ω -sequence of relations, each of uniform affine delay.

The annotated language \mathcal{F}

We must describe our language of forcing in greater detail. The first step is to define a language $\mathcal{F} \subseteq \mathbf{HF}$, which is a first-order language with no constants, with the two binary predicate symbols $=$ and ϵ , connectives \neg and \wedge , and the restricted quantifier $\bigwedge \mathfrak{x}: \epsilon \mathfrak{y}$, where in the rules of formation \mathfrak{y} is required to be a distinct variable from \mathfrak{x} . There are no unrestricted quantifiers. Other propositional connectives and the existential restricted quantifier $\bigvee \mathfrak{x}: \epsilon \mathfrak{y}$ may be introduced by definition.

We shall need the customary notions of free and bound occurrence of a variable in a formula, and we imagine that each formula of \mathcal{F} is accompanied by an annotation saying which occurrences of variables are bound by which occurrences of quantifiers. As we build up formulæ, we have to update the annotations, and we imagine all that going on inside \mathbf{HF} .

We then define \mathcal{L}^u as the language resulting from \mathcal{F} by permitting constants \underline{a} for $a \in u$, and \mathcal{E}^u to be the set of sentences of \mathcal{L}^u , meaning those wffs with no free variables. If u is rud closed and non-empty, and the map $a \mapsto \underline{a}$ is basic, then $\mathcal{F} \subseteq \mathcal{L}^u \subseteq \mathcal{E}^u$.

3.1 DEFINITION We define the tree-rank τ of a formula, the substitution of a constant for a free occurrence of a variable in a formula, and, when the formula is a sentence, the set $\text{Rub}(\varphi)$ of sentences to which reference will be made when deciding whether $p \Vdash \varphi$. In the following, \mathfrak{x} and \mathfrak{y} are distinct formal variables.

ψ atomic:

$$\begin{aligned} \tau(\varphi) &= 0; & \text{Rub}(\psi) &= \emptyset. \\ \text{Subst}(\mathfrak{x} = \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) &= \underline{\alpha} = \underline{\alpha}, & \text{Subst}(\mathfrak{x} \epsilon \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) &= \underline{\alpha} \epsilon \underline{\alpha}, \\ \text{Subst}(\mathfrak{x} = \mathfrak{y}, \mathfrak{x}/\underline{\alpha}) &= \underline{\alpha} = \mathfrak{y}, & \text{Subst}(\mathfrak{y} = \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) &= \mathfrak{y} = \underline{\alpha}, \\ \text{Subst}(\mathfrak{x} \epsilon \mathfrak{y}, \mathfrak{x}/\underline{\alpha}) &= \underline{\alpha} \epsilon \mathfrak{y}, & \text{Subst}(\mathfrak{y} \epsilon \mathfrak{x}, \mathfrak{x}/\underline{\alpha}) &= \mathfrak{y} \epsilon \underline{\alpha}. \end{aligned}$$

$\psi = \vartheta \wedge \varphi$:

$$\begin{aligned} \tau(\psi) &= \max(\tau(\vartheta), \tau(\varphi)) + 1 & \text{Rub}(\psi) &= \{\vartheta, \varphi\} \\ \text{Subst}(\psi, \mathfrak{x}/\underline{\alpha}) &= \text{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}) \wedge \text{Subst}(\varphi, \mathfrak{x}/\underline{\alpha}). \end{aligned}$$

$\psi = \neg \vartheta$

$$\begin{aligned} \tau(\psi) &= \tau(\vartheta) + 1; & \text{Rub}(\psi) &= \{\vartheta\} \\ \text{Subst}(\psi, \mathfrak{x}/\underline{\alpha}) &= \neg \text{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}). \end{aligned}$$

$\psi = \bigwedge \mathfrak{y}: \epsilon \mathfrak{x} \vartheta$

$$\begin{aligned} \tau(\psi) &= \tau(\vartheta) + 1; \\ \text{Subst}(\psi, \mathfrak{x}/\underline{\alpha}) &= \bigwedge \mathfrak{y}: \epsilon \underline{\alpha} \text{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}). \end{aligned}$$

$\psi = \bigwedge \mathfrak{y}: \epsilon \underline{a} \vartheta$

$$\begin{aligned} \tau(\psi) &= \tau(\vartheta) + 1; & \text{Rub}(\psi) &= \{\text{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}) \mid \exists p: \in \mathbf{P} \ (p, \alpha) \in a\} \\ \text{Subst}(\psi, \mathfrak{x}/\underline{\alpha}) &= \bigwedge \mathfrak{y}: \epsilon \underline{a} \text{Subst}(\vartheta, \mathfrak{x}/\underline{\alpha}). \end{aligned}$$

3.2 REMARK As we have defined it above, viewing formulæ as trees, substitution is \emptyset -rud rec. But we can improve that to saying that substitution is rudimentary, if we view formulæ as annotated sequences.

3.3 REMARK To go from $\bigwedge \mathfrak{x}: \epsilon \underline{a} \varphi(\mathfrak{x})$ to $\{\varphi[\underline{\alpha}] \mid \exists s: \in \mathbf{P} \ (s, \alpha) \in a\}$ is to form the image of the substitution function, and is thus rudimentary. The annotations will tell us where are the free occurrences of \mathfrak{x} in φ .

3.4 DEFINITION Let $\chi_{\#}^{\ell} \upharpoonright u$, for u a transitive set and $k \in \omega$, be the characteristic function of the forcing relation restricted to those Δ_0 sentences φ of the forcing language with $\tau(\phi) \leq \ell$ and all a with \underline{a} occurring in φ being in u .

3.5 PROPOSITION (3.5.0) $\chi_{\#}^0 \upharpoonright u$ is rudimentary in $\chi_{=} \upharpoonright u$;

(3.5.1) for each ℓ , $\chi_{\#}^{\ell+1} \upharpoonright u$ is rudimentary in $\chi_{\#}^{\ell} \upharpoonright u$, and thus rudimentary in $\chi_{=} \upharpoonright u$;

(3.5.2) For each ℓ there is a natural number s_{ℓ} such that for each ordinal $\nu \geq \eta$, $\chi_{\#}^{\ell} \upharpoonright P_{\nu}^{e;=} \in P_{\nu+s_{\ell}}^{e;=}$.

Proof : For (3.5.0), note that $\chi_{\#}^0 \upharpoonright u$ is rudimentary in $\chi_{=} \upharpoonright u$ and $\chi_{\epsilon} \upharpoonright u$, which is rudimentary in $\chi_{=} \upharpoonright u$.

(3.5.1): the passage from $\chi_{\#}^{\ell} \upharpoonright u$ to $\chi_{\#}^{\ell+1} \upharpoonright u$ is rudimentary in \mathbb{P} and Subst, being given by these clauses:

$$\begin{aligned} \chi_{\#}^{\ell+1}(p, \varphi \wedge \vartheta, \vec{b}, \vec{c}) &= \inf\{\chi_{\#}^{\ell}(p, \varphi, \vec{b}), \chi_{\#}^{\ell}(p, \vartheta, \vec{c})\} \\ \chi_{\#}^{\ell+1}(p, \neg\varphi, \vec{a}) &= \inf\{\chi_{\#}^{\ell}(q, \varphi, \vec{a}) \mid q \leq p\} \\ \chi_{\#}^{\ell+1}(p, \bigwedge \mathfrak{x} : \epsilon \eta \varphi, \eta/\underline{a}, \vec{b}) &= \begin{cases} 1 & \text{if } \forall q : \leq p \forall (s, \alpha) : \epsilon a (q \leq s \implies \exists r : \leq q \chi_{\#}^{\ell}(r, \varphi, \mathfrak{x}/\underline{\alpha}, \eta/\underline{a}, \vec{b}) = 1) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The notation η/\underline{a} indicates that the free occurrences of the variable η have been replaced by occurrences of the constant \underline{a} . Only certain substitutions have been indicated explicitly: we think of wffs as accompanied by annotations, as described above; so, for example, in the longest line of the above equations, where visually φ should be $\text{Subst}(\varphi, \mathfrak{x}/\underline{\alpha})$, the details of the substitutions would be in the annotations.

(3.5.2): argue as in the proof of Proposition 2.19. ¬ (3.5)

Rudimentary generation of the sentences of the forcing language

The following outlines an alternative argument.

Suppose that u is transitive, that $u \subseteq u^+ \subseteq \mathcal{P}(u)$, and that S is a subset of \mathcal{E}^u , so that S is a set of sentences, all of whose constants are in u . We define a ternary function $H_{\mathcal{E}}$ which will yield a larger set of sentences, all of whose constants are in u^+ , thus:

$$\begin{aligned} H_{\mathcal{E}}(\mathbb{P}, S, u^+) &=_{\text{df}} S \cup \{\underline{a} = \underline{b} \mid a, b \in u^+\} \\ &\cup \{\underline{a} \in \underline{b} \mid a, b \in u^+\} \\ &\cup \{\varphi \wedge \vartheta \mid \varphi, \vartheta \in S\} \\ &\cup \{\neg\varphi \mid \varphi \in S\} \\ &\cup \{\bigwedge \mathfrak{x} : \epsilon \underline{a} \varphi \mid a \in u^+ \ \& \ \forall (p, \alpha) : \epsilon a (p \in \mathbb{P} \implies \text{Subst}(\varphi, \mathfrak{x}/\underline{\alpha}) \in S)\} \end{aligned}$$

3.6 LEMMA $H_{\mathcal{E}}$ is rudimentary in the parameter **HF**; more exactly, it is rudimentary in the subset of **HF** that codes the annotation, described above, of formulae of the constant-free language \mathcal{F} .

3.7 LEMMA $S \subseteq \mathcal{E}^u \implies H_{\mathcal{E}}(\mathbb{P}, S, u^+) \subseteq \mathcal{E}^{u^+}$

3.8 LEMMA For each $\varphi \in H_{\mathcal{E}}(\mathbb{P}, S, u^+)$, $\text{Rub}(\varphi) \subseteq S$.

3.9 LEMMA There is a rudimentary function $H_{\#}$ such that for every \mathbb{P} , u and S as above,

$$\chi_{\#} \upharpoonright H_{\mathcal{E}}(\mathbb{P}, S, u^+) = H_{\#}(\mathbb{P}, \chi_{\#} \upharpoonright S, u^+).$$

3.10 LEMMA Suppose that $(u_n)_{n \leq \omega}$ is a strict continuous progress, and that u_0 is a rud-closed transitive set. Suppose that $\mathcal{E}_0 = \mathcal{E}^{u_0}$, that for each n , $\mathcal{E}_{n+1} = H_{\mathcal{E}}(\mathbb{P}, \mathcal{E}_n, u_{n+1})$, and put $\mathcal{E}_{\omega} = \bigcup_{n < \omega} \mathcal{E}_n$. Then $\mathcal{E}_{\omega} = \mathcal{E}^{u_{\omega}}$.

Proof : Note that if $\vartheta \in \text{Rub}(\varphi)$ then $\tau(\vartheta) < \tau(\varphi)$. Now prove by induction on $k \in \omega$ that for each $n \in \omega$, if $\tau(\varphi) = k$ and all a with \underline{a} occurring in φ are in u_{n+1} then $\varphi \in \mathcal{E}_{n+k+1}$. ¬ (3.10)

Lemma 3.9 plays the role of the Propagation Lemma for $\chi_{\#}$, and at a limit stage $u = P_{\lambda}^c$, Lemma 3.10 will prove that $\chi_{\#}$ is total on \mathcal{E}^u ; these two plus a definability argument will yield

3.11 THEOREM Let $\eta_0 = \varrho(\mathbb{P}) + 1$ and let c be a transitive set of which \mathbb{P} is a member. Then for each limit ordinal λ , $\chi_{\#}$ is total on $\mathbb{P} \times \mathcal{F} \times P_{\lambda}^c$, and $\chi_{\#} \upharpoonright P_{\lambda}^c$ is a subset of $P_{\eta_0+\lambda}^c$ and is uniformly definable over it.

Proof : If $\lambda = \eta + \omega$ where η is a non-successor, we suppose the Theorem already established for P_η^c ; we consider the progression with $u_n = P_{\eta+n}^c$, and must define $p \Vdash \varphi$ for $\varphi \in \mathcal{E}^{u_\omega}$; by Lemma 3.10, φ is in some \mathcal{E}_{n+1} , which by Lemma 3.8 is closed under Rub, in the sense that $\vartheta \in \mathcal{E}_{n+1} \implies \text{Rub}(\vartheta) \subseteq \mathcal{E}_{n+1}$. By Lemma 3.9, we can build $\chi_\# \upharpoonright \mathcal{E}_{n+1}$ inside $P_{\eta_0+\lambda}^c$. So the definition of $p \Vdash \varphi$ over P_λ^c will take the expected form “there is a set \mathcal{E} closed under Rub with $\phi \in \mathcal{E}$ and a function χ defined on \mathcal{E} that satisfies the recursive definition of $\chi_\#$; moreover, $\chi(p, \varphi, \vec{a}) = 1$.” + (3.11)

Forcing $\dot{\Delta}_0$ statements

3.12 PROPOSITION For every $\dot{\Delta}_0$ wff φ ,

$$\begin{aligned} q \leq p \Vdash \varphi &\implies q \Vdash \varphi; \\ p \Vdash \varphi &\iff \forall q: \leq p \exists r: \leq q \ q \Vdash \varphi, \\ p \Vdash \bigvee \mathbf{x}: \epsilon y \ \varphi &\iff \forall q: \leq p \exists r: \leq q \exists (t, \beta): \epsilon y \ (r \leq t \ \& \ r \Vdash \varphi[\underline{\beta}]); \\ p \Vdash \varphi[\underline{\alpha}] \ \& \ \underline{\alpha} = \underline{\beta} &\implies p \Vdash \varphi[\underline{\beta}]. \end{aligned}$$

We shall often use the following general principle in our development.

3.13 PROPOSITION Suppose we have a name z such that $\forall p \forall \alpha \ (p \Vdash_0 \underline{\alpha} \in z \implies p \Vdash \varphi[\underline{\alpha}])$ for some formal wff φ . Then $\forall p \forall \alpha \ (p \Vdash \underline{\alpha} \in z \implies p \Vdash \varphi[\underline{\alpha}])$.

Proof : We gradually weaken the hypothesis. Suppose that $p \Vdash_1 \underline{\alpha} \in z$. Then for some $q \geq p$, $(q, \alpha) \in z$, so $q \Vdash_0 \underline{\alpha} \in z$; so q and therefore also p forces $\varphi[\underline{\alpha}]$.

Now suppose that $p \Vdash \underline{\alpha} \in z$. This tells us that

$$\forall q: \leq p \exists r: \leq q \exists \beta \ r \Vdash_1 \underline{\beta} \in z \ \& \ r \Vdash \underline{\alpha} = \underline{\beta}.$$

So for such r , $r \Vdash \varphi[\underline{\beta}]$; and so $r \Vdash \varphi[\underline{\alpha}]$. The class of such r being dense below p , $p \Vdash \varphi[\underline{\alpha}]$. + (3.13)

Axioms of Extensionality and Foundation

We may now prove that the Axiom of Extensionality is forced: that reduces to proving the following

3.14 PROPOSITION If $p \Vdash \bigwedge \mathbf{x}: \epsilon \underline{a} \ \mathbf{x} \in \underline{b}$ and $p \Vdash \bigwedge \mathbf{x}: \epsilon \underline{b} \ \mathbf{x} \in \underline{a}$, then $p \Vdash \underline{a} = \underline{b}$,

the proof of which presents no difficulty.

3.15 PROPOSITION \Vdash Foundation

Proof : Given x , consider $A = \{a \mid \exists p: \in \mathbf{P} \ (p, a) \in x\}$. A is a Δ_0 sub-class of $\bigcup^2 x$ and therefore a set; assuming it to be non-empty, let c be an element of A of minimal rank. Then $c \in A$ but $\bigcup^2 c \cap A$ is empty; so if $(p, c) \in x$, $p \Vdash \underline{c} \in \underline{x} \ \& \ \underline{c} \dot{\cap} \underline{x} = \dot{\emptyset}$. Thus

$$\Vdash \underline{x} \neq \dot{\emptyset} \longrightarrow \bigvee \eta: \epsilon \underline{x} \ \eta \dot{\cap} \underline{x} = \dot{\emptyset}. \quad + (3.15)$$

Preservation of $\dot{\Delta}_0$ statements about the ground model

Now that we have defined forcing for Δ_0 statements and have seen how elements of the ground model are named in the language of forcing, we may verify that Δ_0 statements about them, if true, are forced.

3.16 LEMMA $\forall p: \in \mathbf{P} \ \forall y \ (p \Vdash \bigwedge \mathbf{x}: \epsilon \hat{y} \ \varphi \iff \forall x: \epsilon y \ p \Vdash \varphi(\hat{x}))$.

Proof : $p \Vdash \bigwedge \mathbf{x}: \epsilon \hat{y} \ \varphi \iff \forall q: \leq p \ \forall (s, \beta): \epsilon \hat{y} \ (q \leq s \implies \exists r: \leq q \ r \Vdash \varphi[\underline{\beta}])$

$$\begin{aligned} &\iff \forall q: \leq p \ \forall x: \epsilon y \ (\exists r: \leq q \ r \Vdash \varphi(\hat{x})) \\ &\iff \forall x: \epsilon y \ \forall q: \leq p \ (\exists r: \leq q \ r \Vdash \varphi(\hat{x})) \\ &\iff \forall x: \epsilon y \ p \Vdash \varphi(\hat{x}). \end{aligned} \quad + (3.16)$$

3.17 PROPOSITION Let $\Phi(x_1, \dots, x_n)$ be a Δ_0 statement true of a_1, \dots, a_n . Then $\mathbf{1}^{\mathbf{P}} \Vdash \dot{\Phi}[\hat{a}_1 \dots \hat{a}_n]$.

Proof : for Φ either atomic or the negation of atomic, this was proved in Proposition 2.28. We now proceed by induction on the length of Φ ; propositional connectives are easily handled, as a 0-1 law applies in this context; and the last lemma covers restricted quantifiers. + (3.17)

The above is a schema expressed in the metalanguage: the version when we quantify over $\dot{\Delta}_0$ wffs in the language of discourse would read

3.18 PROPOSITION Let $\varphi(x_1, \dots, x_n)$ be a $\dot{\Delta}_0$ statement such that $\models^0 \varphi[a_1, \dots, a_n]$. Then $\mathbf{1}^{\mathbf{P}} \Vdash \varphi[\hat{a}_1 \dots \hat{a}_n]$.

4: Generic extensions of a transitive set

Let M be a transitive set and \mathbb{P} a notion of forcing in M . The aim in life of an (M, \mathbb{P}) -generic filter \mathcal{G} is to create a transitive set $M[\mathcal{G}]$ out of the Lindenbaum algebra of the language of forcing, with the property that what is true in the model is what is forced by some $p \in \mathcal{G}$. That principle is known as the Forcing Theorem.

4.0 DEFINITION Δ is *dense open* in \mathbb{P} if $\forall p: \in \mathbb{P} \exists q: \in \Delta \ q \leq p$ and $\forall p: \in \Delta \ \forall q: \leq p \ q \in \Delta$.

4.1 DEFINITION $\mathcal{G} \subseteq \mathbb{P}$ is (M, \mathbb{P}) -generic if $\forall p: \in \mathcal{G} \ \forall q: \in \mathcal{G} \ \exists r: \in \mathcal{G} \ r \leq p \ \& \ r \leq q$, $\forall p: \in \mathcal{G} \ \forall q \ p \leq q \implies q \in \mathcal{G}$, and $\mathcal{G} \cap \Delta \neq \emptyset$ for each $\Delta \in M$ that is dense open in \mathbb{P} .

For forcing over models of ZF that would suffice to prove the Forcing Theorem; but for models of certain weaker theories such as KP, it is inadequate: we shall find in a later section that if M is admissible, we must require \mathcal{G} to meet all dense open subsets of \mathbb{P} that are unions of a Σ_1 and a Π_1 class over M if we are to show that $M[\mathcal{G}]$ will be admissible. Ironically, for models of the still weaker theory PROVI, all is well again: we shall show in this section that if a filter \mathcal{G} is generic as defined above, then the Forcing Theorem will hold for $\dot{\Delta}_0$ formulæ.

We know from its being a filter that \mathcal{G} will be *consistent* in the sense that for no sentence φ of the language $\mathcal{L}^{\mathbb{P}}$ can there be a $p \in \mathcal{G}$ with $p \Vdash^{\mathbb{P}} \varphi$ and a $q \in \mathcal{G}$ with $q \Vdash^{\mathbb{P}} \neg \varphi$. Let $\Delta(\varphi) = \{p \in \mathbb{P} \mid p \Vdash^{\mathbb{P}} \varphi \vee p \Vdash^{\mathbb{P}} \neg \varphi\}$: this is a dense open subclass of \mathbb{P} . If φ is Δ_0 , $\Delta(\varphi)$ will be Δ_1^{KP} , and thus a set of M . If \mathcal{G} meets $\Delta(\varphi)$, then some $p \in \mathcal{G}$ decides φ in the sense that either it forces φ or forces $\neg \varphi$. We shall refer to this property as the *completeness* of \mathcal{G} .

4.2 DEFINITION Suppose now that \mathcal{G} is (M, \mathbb{P}) -generic. Define (externally to M) $\text{val}_{\mathcal{G}} : M \rightarrow V$ by

$$\text{val}_{\mathcal{G}}(b) = \{\text{val}_{\mathcal{G}}(a) \mid \exists p: \in \mathcal{G} \ (p, a) \in b\}.$$

4.3 REMARK This is a rudimentary recursion with parameter \mathcal{G} : $\phi(b) = H(\mathcal{G}, \phi \upharpoonright b)$ where $H(g, x) =_{\text{df}} x \text{“}(\text{Dom}(x)) \text{“}g$. In [M4] it is shown that certain transitive models of Zermelo set theory fail to support such recursions: thus it is necessary to assume that PROVI is true in the “background” set theory.

4.4 REMARK An immediate consequence of the definition is that if $p \in \mathcal{G}$ and either $p \Vdash_0 \underline{a} \in \underline{a}$ or $p \Vdash_1 \underline{a} \in \underline{a}$, then $\text{val}_{\mathcal{G}}(a) \in \text{val}_{\mathcal{G}}(a)$.

4.5 PROPOSITION For all a and b the following hold:

$$(4.6) \quad \text{val}_{\mathcal{G}}(a) = \text{val}_{\mathcal{G}}(b) \iff \exists p: \in \mathcal{G} \ p \Vdash \underline{a} = \underline{b}$$

$$(4.7) \quad \text{val}_{\mathcal{G}}(a) \in \text{val}_{\mathcal{G}}(b) \iff \exists p: \in \mathcal{G} \ p \Vdash \underline{a} \in \underline{b}$$

We divide the proof into four lemmata.

4.8 LEMMA If $p \in \mathcal{G}$ and $p \Vdash \underline{a} = \underline{b}$ then $\text{val}_{\mathcal{G}}(a) = \text{val}_{\mathcal{G}}(b)$.

Proof by induction on rank: Let $x \in \text{val}_{\mathcal{G}}(a)$. Let $(q_0, \alpha) \in a$, with $q_0 \in \mathcal{G}$ and $\text{val}_{\mathcal{G}}(\alpha) = x$. Let $q \in \mathcal{G}$ be below both p and q_0 . Consider the class

$$\mathbb{P} \cap \{r \mid r \leq q \ \& \ \exists \beta: \in \bigcup^2 b \ r \Vdash_1 \underline{\beta} \in \underline{b} \ \& \ r \Vdash \underline{\alpha} = \underline{\beta}\}.$$

That is dense below q , and is a set by $\dot{\Delta}_0$ separation, once one has replaced the predicate $r \Vdash \underline{\alpha} = \underline{\beta}$ by an evaluation by an appropriate fragment of χ_{\pm} . It is therefore met by \mathcal{G} ; so let $r \in \mathcal{G}$ be below q and $\beta \in \bigcup^2 b$ with $r \Vdash \underline{\alpha} = \underline{\beta}$ and $r \Vdash_1 \underline{\beta} \in \underline{b}$.

From the second property of r , $\text{val}_{\mathcal{G}}(\beta) \in \text{val}_{\mathcal{G}}(b)$, and from the first, applying the induction hypothesis, $\text{val}_{\mathcal{G}}(\alpha) = \text{val}_{\mathcal{G}}(\beta)$; thus $x \in \text{val}_{\mathcal{G}}(b)$; as x was arbitrary, $\text{val}_{\mathcal{G}}(a) \subseteq \text{val}_{\mathcal{G}}(b)$.

A similar argument shows that $\text{val}_{\mathcal{G}}(b) \subseteq \text{val}_{\mathcal{G}}(a)$. (4.8)

4.9 LEMMA If $p \in \mathcal{G}$ and $p \Vdash \underline{a} \in \underline{b}$ then $\text{val}_{\mathcal{G}}(a) \in \text{val}_{\mathcal{G}}(b)$.

Proof: $\{r \in \mathbb{P} \mid \exists \beta: \in \bigcup^2 b \ r \Vdash \underline{\beta} = \underline{a} \ \& \ r \Vdash_1 \underline{\beta} \in \underline{b}\}$ is dense below p and is a set, and so there is an $r \in \mathcal{G}$ and a $\beta \in \bigcup^2 b$ such that $r \Vdash \underline{\beta} = \underline{a}$, which by the previous lemma implies that $\text{val}_{\mathcal{G}}(\beta) = \text{val}_{\mathcal{G}}(a)$, and, by Remark 4.4, such that $\text{val}_{\mathcal{G}}(\beta) \in \text{val}_{\mathcal{G}}(b)$; so $\text{val}_{\mathcal{G}}(a) \in \text{val}_{\mathcal{G}}(b)$. (4.9)

4.10 LEMMA If $\text{val}_{\mathcal{G}}(a) = \text{val}_{\mathcal{G}}(b)$, then for some $p \in \mathcal{G}$, $p \Vdash \underline{a} = \underline{b}$.

Proof by induction: We show first that $\exists p : \in \mathcal{G} \ p \Vdash \bigwedge \mathfrak{x} : \epsilon \underline{a} \ \mathfrak{x} \in \underline{b}$. If not, then by density, $\exists p : \in \mathcal{G} \ p \Vdash \bigvee \mathfrak{x} : \epsilon \underline{a} \ \neg(\mathfrak{x} \in \underline{b})$; indeed there will then exist $p \in \mathcal{G}$ and $\alpha \in \bigcup^2 a$ such that $p \Vdash_1 \underline{\alpha} \in \underline{a}$ and $p \Vdash \neg(\underline{\alpha} \in \underline{b})$. Given such p and α , $\text{val}_{\mathcal{G}}(\alpha) \in \text{val}_{\mathcal{G}}(a)$; so there is a $\beta \in \bigcup^2 b$ and a $q \in \mathcal{G}$ with $(q, \beta) \in b$, and $\text{val}_{\mathcal{G}}(\alpha) = \text{val}_{\mathcal{G}}(\beta)$.

By the induction hypothesis, there will be an $r \in \mathcal{G}$, which we may suppose to be below both q and p , such that $r \Vdash \underline{\alpha} = \underline{\beta}$ and $r \Vdash_1 \underline{\beta} \in \underline{b}$; so $r \Vdash \underline{\alpha} \in \underline{b}$, contrary to our hypothesis on p .

A similar argument will show that $\exists p : \in \mathcal{G} \ p \Vdash \bigwedge \mathfrak{x} : \epsilon \underline{b} \ \mathfrak{x} \in \underline{a}$; and we may now invoke the fact that Extensionality is forced, to conclude that there is a $p \in \mathcal{G}$ such that $p \Vdash \underline{a} = \underline{b}$. \dashv (4.10)

4.11 LEMMA If $\text{val}_{\mathcal{G}}(a) \in \text{val}_{\mathcal{G}}(b)$, then for some $p \in \mathcal{G}$, $p \Vdash \underline{a} \in \underline{b}$.

Proof : The hypothesis implies that there are $q \in \mathcal{G}$ and c such that $(q, c) \in b$ and $\text{val}_{\mathcal{G}}(a) = \text{val}_{\mathcal{G}}(c)$. By the previous lemma, there is a p_0 in \mathcal{G} such that $p \Vdash \underline{a} = \underline{c}$; then if $p \in \mathcal{G}$ is below both p_0 and q , $p \Vdash_1 \underline{c} \in \underline{b}$ and so $p \Vdash \underline{a} \in \underline{b}$. \dashv (4.11)

The proof of Proposition 4.5 is now complete. \dashv (4.5)

4.12 DEFINITION $M^{\mathbf{P}}[\mathcal{G}] =_{\text{df}} \{\text{val}_{\mathcal{G}}(a) \mid a \in M\}$.

We check that $M \cup \{\mathcal{G}\} \subseteq M^{\mathbf{P}}[\mathcal{G}]$. For showing that $M \subseteq M^{\mathbf{P}}[\mathcal{G}]$, we use our names \hat{x} :

4.13 PROPOSITION For all $x \in M$, $\text{val}_{\mathcal{G}}(\hat{x}) = x$.

Proof : an easy application of Proposition 2.28. \dashv (4.13)

We have a canonical name for \mathcal{G} :

4.14 DEFINITION Let $\dot{\mathcal{G}} =_{\text{df}} \{(p, \hat{p}) \mid p \in \mathbf{P}\}$.

$\dot{\mathcal{G}} \in M$ as M is provident and $a \mapsto \hat{a}$ is rud rec.

4.15 PROPOSITION $\text{val}_{\mathcal{G}}(\dot{\mathcal{G}}) = \mathcal{G}$.

Proof : Both sides equal $\{\text{val}_{\mathcal{G}}(\hat{p}) \mid p \in \mathcal{G}\}$. \dashv (4.15)

4.16 COROLLARY $\mathcal{G} \in M[\mathcal{G}]$.

4.17 THE FORCING THEOREM Given A , \mathbb{P} and \mathcal{G} ; for each $\dot{\Delta}_0$ formula φ and a_1, \dots, a_n in A :

$$A^{\mathbf{P}}[\mathcal{G}] \models \varphi[\text{val}_{\mathcal{G}}(a_1), \dots, \text{val}_{\mathcal{G}}(a_n)] \iff \exists p : \in \mathcal{G} \ (p \Vdash^{\mathbf{P}} \varphi[\underline{a}_1, \dots, \underline{a}_n])^A.$$

Proof : First, the case of atomic φ is covered by Proposition 4.5.

For Boolean conjunctions: $\exists s : \in \mathcal{G} \ (s \leq p \ \& \ s \leq q)$ iff p and q are both in \mathcal{G} .

For negations: there is a dense class to be met by \mathcal{G} , and we must show that the class in question is a set. Note that $\mathbf{P} \cap \{t \mid \neg \exists r \leq t \ (r \Vdash \neg \varphi[\underline{\beta}])\}$ is a member of the provident set that is the ground model: take an attempt at $\chi_{=}$ that covers all; then this set is obtainable as a separator that is Δ_0 in that attempt.

Now consider the problem of a restricted quantifier. Suppose $p \Vdash \bigwedge \mathfrak{x} : \epsilon \underline{b} \ \varphi(\mathfrak{x})$, and let $y = \text{val}_{\mathcal{G}}(b)$. Suppose $A[\mathcal{G}] \models x \in y$: let $x = \text{val}_{\mathcal{G}}(a)$. Then there is a q in \mathcal{G} and an η such that $q \Vdash_1 \underline{\eta} \in \underline{y}$ and $q \Vdash \underline{a} = \underline{\eta}$. Then densely below q , there are r such that $r \Vdash \neg \varphi[\underline{\eta}]$. So some such r is in \mathcal{G} ; so $A[\mathcal{G}] \models \varphi[\text{val}_{\mathcal{G}}(\eta)]$; but $\text{val}_{\mathcal{G}}(\eta) = \text{val}_{\mathcal{G}}(a) = x$. Thus $A[\mathcal{G}] \models \bigwedge \mathfrak{x} : \epsilon y \ \varphi(\mathfrak{x})$.

Conversely, suppose that $A[\mathcal{G}] \models \bigwedge \mathfrak{x} : \epsilon y \ \varphi(\mathfrak{x})$, and suppose that $b \in A$ and that $y = \text{val}_{\mathcal{G}}(b)$. Let

$$X = \mathbb{P} \cap \{t \mid \exists \beta : \in \bigcup^2 b \ t \Vdash_1 \underline{\beta} \in \underline{b} \ \& \ \neg \exists r \leq t \ (r \Vdash \neg \varphi[\underline{\beta}])\}.$$

X is a set and is downwards closed, i.e. open in the usual topology on \mathbb{P} . Let $\Delta = X \cup \{p \mid \mathcal{O}_p \cap X = \emptyset\}$, where $\mathcal{O}_p = \{q \mid q \leq p\}$.

Δ is a dense open set, and so meets \mathcal{G} . Let $p \in \mathcal{G} \cap \Delta$. If $p \in X$, then for some $\beta \in \bigcup^2 b$, $p \Vdash_1 \underline{\beta} \in \underline{b}$, but for no $r \leq p$ does $r \Vdash \neg \varphi[\underline{\beta}]$; so $p \Vdash \neg \neg \varphi[\underline{\beta}]$; so $A[\mathcal{G}] \models \neg \varphi[\text{val}_{\mathcal{G}}(\beta)]$; but $\text{val}_{\mathcal{G}}(\beta) \in \text{val}_{\mathcal{G}}(b)$.

Thus $p \notin X$; and so there is no q below p with $q \in X$. So

$$\forall q \leq p \ \forall \beta : \in \bigcup^2 b \ (q \Vdash_1 \underline{\beta} \in \underline{b} \implies \exists r \leq q \ r \Vdash \neg \varphi[\underline{\beta}]) :$$

which says precisely that $p \Vdash \bigwedge \mathfrak{x} : \epsilon \underline{b} \ \varphi(\mathfrak{x})$. \dashv (4.17)

5: Construction of nominators for rudimentary functions

Our aim in this section is to prove a theorem about the construction of names in a provident set for the values of a rudimentary function in a set-generic extension of that set. We state it now, and shall restate it later in a more precise form.

5.0 THEOREM *Let R be a rudimentary function of some number of arguments. Then there is a function $R^{\mathbb{P}}$, of the same number of arguments, with the property that if A is a provident set and $\mathbb{P} \in A$ a notion of forcing, then A is closed under $R^{\mathbb{P}}$ and, further, if \mathcal{G} is an (A, \mathbb{P}) -generic, then (to take the case of a function of two variables) for all x and y in A , $\text{val}_{\mathcal{G}}(R^{\mathbb{P}}(x, y)) = R(\text{val}_{\mathcal{G}}(x), \text{val}_{\mathcal{G}}(y))$.*

DEFINITION We shall call the function $R^{\mathbb{P}}$ the *nominator* of the function R . Usually its definition is uniform in \mathbb{P} and A , and we shall see that when R is gentle its nominator will be multi-gentle.

I henceforth use the phrase “Cohen term” to speak of the value of a nominator given some arguments.

5.1 COROLLARY *Let A be provident, $\mathbb{P} \in A$ and \mathcal{G} (A, \mathbb{P}) -generic. Then $A[\mathcal{G}]$ is rud closed and so a model of GJ_0 .*

Proof of the Corollary: suppose (to take a function of two variables) that $R(x, y)$ is a rudimentary function and that x and y are in $A[\mathcal{G}]$. Then there are a and b in A so that $x = \text{val}_{\mathcal{G}}(a)$ and $y = \text{val}_{\mathcal{G}}(b)$. Applying the nominator of R , the corresponding Cohen term $R^{\mathbb{P}}(a, b)$ exists in A : let $z = \text{val}_{\mathcal{G}}(R^{\mathbb{P}}(a, b))$. Then $z \in A[\mathcal{G}]$, and by the theorem $R(x, y) = z$. Since $R(x, y) = z$ is a Δ_0 statement, and therefore absolute for transitive sets containing x, y and z , we know that it is true in $A[\mathcal{G}]$ that $R(x, y) = z$. \dashv (5.1)

Some general lemmata about forcing

5.2 LEMMA *If $a \subseteq b$, $\text{val}_{\mathcal{G}}(a) \subseteq \text{val}_{\mathcal{G}}(b)$.*

Proof: $\text{val}_{\mathcal{G}}(a) = \{\text{val}_{\mathcal{G}}(\alpha) \mid \exists p: \mathcal{G}(p, \alpha) \in a\} \subseteq \{\text{val}_{\mathcal{G}}(\beta) \mid \exists p: \mathcal{G}(p, \beta) \in b\} = \text{val}_{\mathcal{G}}(b)$. \dashv (5.2)

5.3 LEMMA *Let u be transitive. Then $\text{val}_{\mathcal{G}}(u)$ is transitive.*

Proof: If $x \in \text{val}_{\mathcal{G}}(u)$, $\exists \alpha \exists p: \mathcal{G}(p, \alpha) \in u$ & $\text{val}_{\mathcal{G}}(\alpha) = x$. But $\alpha \in \bigcup^2 u \subseteq u$ so $\alpha \subseteq u$; so $\text{val}_{\mathcal{G}}(\alpha) \subseteq \text{val}_{\mathcal{G}}(u)$. \dashv (5.3)

5.4 We note alternative ways of naming an object. For given y , put

$$\mathcal{A}_0(y) = \{(p, x) \mid p \Vdash_0 x \in y\}$$

$$\mathcal{A}_1(y) = \{(p, x) \mid p \Vdash_1 x \in y\}$$

$$\mathcal{A}(y) = \{(p, x) \mid p \Vdash x \in y\}$$

5.5 REMARK $\mathcal{A}_0(y) \subseteq y$; $\mathcal{A}_0(y) \subseteq \mathcal{A}_1(y) \subseteq \mathbf{P} \times \bigcup^2 y$, so $\mathcal{A}_0(y)$ and $\mathcal{A}_1(y)$ are sets, whereas $\mathcal{A}(y)$ is a proper class whose intersection with a set will be a set provided one has rud rec separation.

5.6 REMARK $\mathcal{A}_0(\mathcal{A}_1(y)) = \mathcal{A}_1(y) = \mathcal{A}_1(\mathcal{A}_0(y)) = \mathcal{A}_1(\mathcal{A}_1(y))$; $\mathcal{A}_0(\mathcal{A}_0(y)) = \mathcal{A}_0(y)$.

LEMMA *If $q \Vdash_1 \underline{w} \in \mathcal{A}_1(y)$, then $q \Vdash_1 \underline{w} \in y$; if $q \Vdash_1 \underline{w} \in y$ then $q \Vdash_0 \underline{w} \in \mathcal{A}_1(y)$*

5.7 LEMMA *Let $\mathbb{P} \in \mathbf{M}$ and let \mathcal{G} be (\mathbf{M}, \mathbb{P}) -generic. Then if $y \in \mathbf{M}$, $\text{val}_{\mathcal{G}}(\mathcal{A}_0(y)) = \text{val}_{\mathcal{G}}(\mathcal{A}_1(y)) = \text{val}_{\mathcal{G}}(y)$.*

Proof: That $\text{val}_{\mathcal{G}}(y) = \text{val}_{\mathcal{G}}(\mathcal{A}_0(y)) \subseteq \text{val}_{\mathcal{G}}(\mathcal{A}_1(y))$ is immediate from the definition of $\text{val}_{\mathcal{G}}(\cdot)$ and Remark 5.3. It remains to show that $\text{val}_{\mathcal{G}}(\mathcal{A}_1(y)) \subseteq \text{val}_{\mathcal{G}}(y)$.

Suppose that $p \in \mathcal{G}$ and $z \in M$ are such that $p \Vdash z \in \mathcal{A}_1(y)$ and $p \Vdash z \notin y$. Then there are $w \in M$ and $q \in \mathcal{G}$ with $q \leq p$, $q \Vdash_1 \underline{w} \in \mathcal{A}_1(y)$, (so by the previous Lemma, $q \Vdash \underline{w} \in y$), but also $q \Vdash \underline{w} = z$ and therefore $q \Vdash \underline{w} \notin y$, a contradiction. \dashv (5.7)

The proof

To prove the Theorem we begin by working through the list of nine rudimentary functions given in *Weak Systems*, and show how, for such a function F , given names (in the ground model) for its arguments in the generic extension we may build names for its values. Certain of the nominators are rudimentary, even

basic, functions, of their arguments, others will be of affine delay in \mathbb{P} . We shall do the rudimentary ones first. We assume that $\mathbb{P} = (\mathbf{P}, \mathbf{1}^{\mathbf{P}}, \leq^{\mathbf{P}})_3$.

Although we shall not always obtain a nominator for a rudimentary function as a rudimentary function of the names of its arguments, as we shall see with $x \smallfrown y$, we may check as we go that we always find a nominator that is rudimentary in the relation $\chi_{\#}^{\ell}$ for some ℓ .

5.8 REMARK Composition of rudimentary nominators will of course be rudimentary.

A basic nominator for $\{x, y\}$

5.9 DEFINITION $\{a, b\}^{\mathbb{P}} =_{\text{df}} \{(\mathbf{1}^{\mathbf{P}}, a)_2, (\mathbf{1}^{\mathbf{P}}, b)_2\}$

5.10 LEMMA $\{a, b\}^{\mathbb{P}}$ is a basic function of the variables shown.

5.11 PROPOSITION $\text{val}_{\mathcal{G}}(\{a, b\}^{\mathbb{P}}) = \{\text{val}_{\mathcal{G}}(a), \text{val}_{\mathcal{G}}(b)\}$.

5.12 COROLLARY $\mathbf{1} \Vdash^{\mathbb{P}} \underline{\{a, b\}^{\mathbb{P}}} = \dot{\underline{\{a, b\}}}$.

Basic nominators for ordered pairs and triples

These can be obtained by composition.

5.13 DEFINITION $\{x\}^{\mathbb{P}} =_{\text{df}} \{(\mathbf{1}^{\mathbf{P}}, x)_2\}$.

5.14 DEFINITION $(x, y)_2^{\mathbb{P}} =_{\text{df}} \{\{x\}^{\mathbb{P}}, \{x, y\}^{\mathbb{P}}\}^{\mathbb{P}}$.

5.15 DEFINITION $A_2^{\mathbb{P}}(x, y, z) =_{\text{df}} \{x, (y, z)_2^{\mathbb{P}}\}^{\mathbb{P}}$.

5.16 DEFINITION $(x, y, z)_3^{\mathbb{P}} =_{\text{df}} (x, (y, z)_2^{\mathbb{P}})_2^{\mathbb{P}}$.

5.17 LEMMA The four functions just introduced are basic functions of the variables shown.

A basic nominator for $x \cup y$

5.18 DEFINITION $x \cup^{\mathbb{P}} y =_{\text{df}} x \cup y$.

A basic nominator for $\bigcup x$

5.19 DEFINITION $\bigcup^{\mathbb{P}} x =_{\text{df}} (\mathbf{P} \times \bigcup^5 x) \cap \{(p, \alpha) \mid \exists (q, \beta) : \in x (p \leq q \ \& \ p \Vdash_1 \underline{\alpha} \in \underline{\beta})\}$.

5.20 REMARK $\bigcup^{\mathbb{P}}$ is a basic function, being the application of a Δ_0 separator.

5.21 LEMMA If $p \Vdash_0 \underline{\gamma} \in \underline{\bigcup^{\mathbb{P}} x}$, then $(p, \gamma)_2 \in \mathbf{P} \times \bigcup^5 x$.

5.22 LEMMA If $p \Vdash_1 \underline{\gamma} \in \underline{\bigcup^{\mathbb{P}} x}$, then $(p, \gamma)_2 \in \mathbf{P} \times \bigcup^5 x$.

5.23 REMARK Hence if $p \Vdash^{\mathbf{P}} \underline{\gamma} \in \underline{\bigcup^{\mathbb{P}} x}$, then there are many β and t (dense below p) for which $t \Vdash \underline{\beta} = \underline{\gamma}$ and $t \Vdash_1 \underline{\beta} \in \underline{\bigcup^{\mathbb{P}} x}$.

5.24 PROPOSITION $\text{val}_{\mathcal{G}}(\bigcup^{\mathbb{P}} x) = \bigcup \text{val}_{\mathcal{G}}(x)$.

5.25 COROLLARY $\mathbf{1} \Vdash^{\mathbb{P}} \underline{\bigcup^{\mathbb{P}} x} = \dot{\bigcup} x$.

A basic nominator for $x \times y$

5.25 DEFINITION $x \times^{\mathbb{P}} y =_{\text{df}} \{(p, (\alpha, \beta)_2^{\mathbb{P}}) \mid p \Vdash_1 \underline{\alpha} \in \underline{x} \ \& \ p \Vdash_1 \underline{\beta} \in \underline{y}\}$

5.26 REMARK $\cdot \times^{\mathbb{P}} \cdot$ is evidently rudimentary, but it is actually basic in \mathbb{P} , being the result of applying applying a Δ_0 separator to the set $\mathbf{P} \times [\{\mathbf{1}^{\mathbf{P}}\} \times [\{\mathbf{1}^{\mathbf{P}}\} \times (x \cup y)]^{\leq 2}]^{\leq 2}$.

5.27 PROPOSITION $\text{val}_{\mathcal{G}}(x \times^{\mathbb{P}} y) = \text{val}_{\mathcal{G}}(x) \times \text{val}_{\mathcal{G}}(y)$.

5.28 COROLLARY $\mathbf{1}^{\mathbf{P}} \Vdash^{\mathbb{P}} \underline{x \times^{\mathbb{P}} y} = \underline{x} \dot{\times} \underline{y}$.

A basic nominator for $[x]^1$

5.29 DEFINITION $F_1^{\mathbb{P}}(x) =_{\text{df}} \{(p, \{\alpha\}^{\mathbb{P}}) \mid (p, \alpha) \in x\}$.

Again, this rudimentary term can be shown to be basic, using the fact that $F_1^{\mathbb{P}}(x) \subseteq \mathbb{P} \times (\{1^{\mathbb{P}}\} \times \bigcup^2 x)$.

5.30 PROPOSITION $\Vdash [\underline{x}]^1 = \underline{F_1^{\mathbb{P}}(x)}$

Proof: If $q \Vdash_1 \underline{z} \in \underline{F_1^{\mathbb{P}}(x)}$, then there is a $p \geq q$ with $(p, z)_2 \in F_1^{\mathbb{P}}(x)$, so that there is an α with $(p, \alpha)_2 \in x$, $1 \Vdash \underline{z} = \dot{\{\alpha\}}$ and $q \Vdash_1 \alpha \in \underline{x}$.

Conversely, if $q \Vdash \bigvee \eta : \epsilon \underline{x} \underline{z} = \dot{\{\eta\}}$, then there are y and $r \leq q$ with $r \Vdash_1 \underline{y} \in \underline{x}$ and $r \Vdash^{\mathbb{P}} \underline{z} = \dot{\{y\}}$. So $(p, y)_2 \in x$ for some $p \geq r$, so that $(p, \{y\}^{\mathbb{P}})_2 \in F_1^{\mathbb{P}}(x)$, $r \Vdash_1 \{y\}^{\mathbb{P}} \in \underline{F_1^{\mathbb{P}}(x)}$ and so $r \Vdash^{\mathbb{P}} \underline{z} \in \underline{F_1^{\mathbb{P}}(x)}$. \dashv (5.30)

A basic nominator for $[x]^{\leq 2}$

5.31 REMARK $[x]^{\leq 2}$ is easier to get than $[x]^2$, because the latter will require us to be certain that two names denote different things; we could obtain such a term by using the identity $[x]^{\leq 2} = \bigcup(x \times x)$; the following is slightly simpler.

5.32 DEFINITION $F_{\leq 2}^{\mathbb{P}}(x) =_{\text{df}} \{(r, \{\alpha, \beta\}^{\mathbb{P}}) \mid r \Vdash_1 \alpha \in \underline{x} \ \& \ r \Vdash_1 \beta \in \underline{x}\}$.

5.33 REMARK That function is basic since its value is a Δ_0 subset of $\mathbb{P} \times \bigcup((\{1^{\mathbb{P}}\} \times \bigcup^2 x) \times (\{1^{\mathbb{P}}\} \times \bigcup^2 x))$.

5.34 PROPOSITION $\Vdash [\underline{x}]^{\leq 2} = \underline{F_{\leq 2}^{\mathbb{P}}(x)}$

Proof: If $t \Vdash^{\mathbb{P}} \underline{a} \in \underline{F_{\leq 2}^{\mathbb{P}}(x)}$, there is an $s \leq t$ and a b such that $s \Vdash \underline{a} = \underline{b}$ and $s \Vdash_1 \underline{b} \in \underline{F_{\leq 2}^{\mathbb{P}}(x)}$. So there are an $r \geq s$, α and β with $b = \{\alpha, \beta\}^{\mathbb{P}}$, and conditions p and q with

$$(r \leq p \ \& \ r \leq q \ \& \ (p, \alpha) \in x \ \& \ (q, \beta) \in x) :$$

so that $s \Vdash^{\mathbb{P}} \underline{a} = \underline{b} = \dot{\{\alpha, \beta\}} \in [\underline{x}]^{\leq 2}$.

If $s \Vdash^{\mathbb{P}} \underline{a} \in [\underline{x}]^{\leq 2}$ then there are $t \leq s$, α, β , such that $t \Vdash_1 \alpha \in \underline{x} \ \& \ t \Vdash_1 \beta \in \underline{x} \ \& \ t \Vdash^{\mathbb{P}} \underline{a} = \dot{\{\alpha, \beta\}}$, so that there are p and q with $t \leq p, t \leq q, (p, \alpha) \in x$ and $(q, \beta) \in x$; so $(t, \{\alpha, \beta\}^{\mathbb{P}})_2 \in F_{\leq 2}^{\mathbb{P}}(x)$, so $t \Vdash^{\mathbb{P}} \underline{a} \in \underline{F_{\leq 2}^{\mathbb{P}}(x)}$. \dashv (5.34)

A basic nominator for u^*

We recall the definition:

$$u^* =_{\text{df}} u \cup [u]^{\leq 2} \cup (u \times u)$$

and that for u transitive, u^* is transitive.

Then a basic nominator for it can be found by composition using the preceding ones.

Nominators of affine delay for the other rudimentary generators

We show that for the remaining rud generators we get terms of the form $G(\mathbb{P}, x, y) \cap A$ where G is a rudimentary function and A is a separator that is rud in an appropriate segment of χ_{\neq} . It is the definition of A that will give us the desired uniformity.

REMARK That that should be so is suggested by our theory of companions, at least for DB functions. Each of them has a 2-companion W that is generated by \bigcup and \times and is therefore such that $W^{\mathbb{P}}$ is basic; so if $R(x, y) \subseteq W(\{x, y\})$, then we may expect $R^{\mathbb{P}}(x, y)$ to be of the form $W^{\mathbb{P}}(\{x, y\}^{\mathbb{P}} \cap \{(p, \alpha) \mid p \Vdash \underline{\alpha} \in \dot{R}(\underline{x}, \underline{y})\})$; and as $z \in R(x, y)$ is Δ_0 , $\underline{\alpha} \in \dot{R}(\underline{x}, \underline{y})$ will be $\dot{\Delta}_0$.

$x \searrow^{\mathbb{P}} y$: Set

$$x \searrow^{\mathbb{P}} y =_{\text{df}} \mathcal{A}_1(x) \cap \{(p, \alpha) \mid p \Vdash^{\mathbb{P}} \underline{\alpha} \notin \underline{y}\}$$

Then $x \searrow^{\mathbb{P}} y$, being a subclass of $\mathbb{P} \times \bigcup^2 x$ will be a set if \mathbb{P} is, being the application of a separator that is Δ_0 in some appropriate attempt at χ_{\neq} .

Let $z = x \setminus^{\mathbb{P}} y$. For each p and α with $p \Vdash_0 \underline{\alpha} \in \underline{z}$, $p \Vdash \alpha \notin \underline{y} \wedge \underline{\alpha} \in \underline{x}$ so by our general principle, the same is true for each p and α with $p \Vdash \alpha \in z$. Hence $\Vdash \forall t: \epsilon \underline{z} [t \in x \wedge t \notin y]$.

Conversely, suppose that $q \Vdash \underline{\beta} \in \underline{x} \wedge \underline{\beta} \notin \underline{y}$. We seek $\bar{s} \leq r$ and $(\bar{t}, \alpha) \in z$ with $\bar{s} \leq \bar{t}$ and $\bar{s} \Vdash \underline{\alpha} = \underline{\beta}$. We know that

$$\exists s: \leq r \exists (t, \alpha): \epsilon x \ s \leq t \ \& \ s \Vdash \underline{\alpha} = \underline{\beta}$$

so that for such an s , $s \Vdash_1 \underline{\alpha} \in \underline{x}$ and $s \Vdash \underline{\alpha} \notin \underline{y}$, so $(s, \alpha) \in z$. Hence we may take $\bar{s} = \bar{t} = s$.

R_3 : domain:

5.35 DEFINITION $\text{Dom}^{\mathbb{P}}(x) =_{\text{df}} (\mathbb{P} \times \bigcup^{10} x) \cap \{(p, \alpha)_2 \mid \forall q: \leq p \ \exists r: \leq q \ \exists \beta: \epsilon \bigcup^{10} x \ r \Vdash^{\mathbb{P}} (\beta, \alpha)_2^{\mathbb{P}} \in \underline{x}\}$

5.36 PROPOSITION $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_3^{\mathbb{P}}(x)} = \dot{R}_3(\underline{x})$.

R_5 : $x \cap \{(a, b)_2 \mid a \in b\}$:

5.37 DEFINITION $R_5^{\mathbb{P}}(x) =_{\text{df}} \{(p, \gamma) \mid \exists \alpha: \epsilon \bigcup^{10} x \ \exists \beta: \epsilon \bigcup^{10} x \ \gamma = (\alpha, \beta)_2^{\mathbb{P}} \ \& \ p \Vdash^{\mathbb{P}} \underline{\alpha} \in \underline{\beta}\}$.

That is a set since $(\alpha, \beta)_2^{\mathbb{P}}$ is basic, so we can easily find a companion (*i.e.* a bounding set), and then apply the separator induced by the relation $p \Vdash \underline{\alpha} \in \underline{\beta}$.

5.38 PROPOSITION $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_5^{\mathbb{P}}(x)} = \dot{R}_5(\underline{x})$.

R_6 : first twirl: $\{\langle b, a, c \rangle \mid \langle a, b, c \rangle \in x\}$:

5.38 DEFINITION $R_6^{\mathbb{P}}(x) =_{\text{df}}$

$$\{(p, \delta) \mid \exists (q, \tau): \epsilon x \ \exists \alpha: \epsilon \bigcup^l x \ \exists \beta: \epsilon \bigcup^m x \ \exists \gamma: \epsilon \bigcup^n x \ [q \geq p \ \& \ \delta = (\beta, \alpha, \gamma)_3^{\mathbb{P}} \ \& \ p \Vdash^{\mathbb{P}} \underline{\tau} = (\alpha, \beta, \gamma)_3^{\mathbb{P}}]\}.$$

We have defined $(\cdot, \cdot, \cdot)_3^{\mathbb{P}}$ above; it is basic; so we can use it to predict the whereabouts of δ . l, m, n must then be given appropriate values.

5.39 PROPOSITION $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_6^{\mathbb{P}}(x)} = \dot{R}_6(\underline{x})$.

R_7 : second twirl: $\{\langle b, c, a \rangle \mid \langle a, b, c \rangle \in x\}$:

5.39 DEFINITION $R_7^{\mathbb{P}}(x) =_{\text{df}}$

$$\{(p, \delta) \mid \exists (q, \tau): \epsilon x \ \exists \alpha: \epsilon \bigcup^l x \ \exists \beta: \epsilon \bigcup^m x \ \exists \gamma: \epsilon \bigcup^n x \ [q \geq p \ \& \ \delta = (\beta, \gamma, \alpha)_3^{\mathbb{P}} \ \& \ p \Vdash^{\mathbb{P}} \underline{\tau} = (\alpha, \beta, \gamma)_3^{\mathbb{P}}]\}.$$

5.40 PROPOSITION $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_7^{\mathbb{P}}(x)} = \dot{R}_7(\underline{x})$.

A_{14} : $x^{\{\{w\}\}}$:

5.41 DEFINITION $A_{14}^{\mathbb{P}}(x, w) =_{\text{df}} (\mathbb{P} \times \bigcup^{10} x) \cap \{(p, \alpha) \mid p \Vdash^{\mathbb{P}} (\alpha, w)_2^{\mathbb{P}} \in \underline{x}\}$

5.42 PROPOSITION $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{A_{14}^{\mathbb{P}}(x, w)} = \dot{A}_{14}(\underline{x}, \underline{w})$.

R_8 : $\{x^{\{\{w\}\}} \mid w \in y\}$

5.43 DEFINITION $R_8^{\mathbb{P}}(x, y) =_{\text{df}} \{(p, \gamma) \mid \exists (q, \beta): \epsilon y \ p \leq q \ \& \ \gamma = A_{14}^{\mathbb{P}}(x, \beta)\}$.

5.44 LEMMA $R_8^{\mathbb{P}}(x, y)$ is a set.

Proof: Define $F(x, y) =_{\text{df}} ((\mathbb{P} \times \bigcup^{10} x) \times \bigcup^2 y) \cap \{((p, \alpha)_2, \beta)_2 \mid p \in \mathbb{P} \ \& \ p \Vdash^{\mathbb{P}} \{\alpha, \beta\}_2^{\mathbb{P}} \in \underline{x}\}$. Note that if u is any transitive set containing \mathbb{P} , x and y , $F \upharpoonright (u \times u)$ is rudimentary in $\chi_{=} \upharpoonright (\mathbb{P} \times u \times u)$.

Set $G(x, y, \beta) = A_{14}(F(x, y), \beta)$. Then G is rudimentary in F and $\beta \in \bigcup^2 y \implies A_{14}^{\mathbb{P}}(x, \beta) = G(x, y, \beta)$.

Now set $H(x, y) = \{G(x, y, \beta) \mid \beta \in \bigcup^2(y)\}$. Then H is rudimentary in G , and $R_8^{\mathbb{P}}(x, y)$ is the result of applying a Δ_0 separator to $\mathbb{P} \times H(x, y)$.

Thus there is a rudimentary function E such that for all such u , $R_8^{\mathbb{P}} \upharpoonright (u \times u) = E(\chi_{=} \upharpoonright (\mathbb{P} \times u \times u))$.

→ (5.44)

5.45 PROPOSITION $\mathbf{1}^{\mathbb{P}} \Vdash^{\mathbb{P}} \underline{R_8^{\mathbb{P}}(x, y)} = \dot{R}_8(\underline{x}, \underline{y})$.

5.46 REMARK Suppose that $Q(\vec{x}) = R(S(\vec{x}), T(\vec{x}))$, where Q, R, S, T are rudimentary, and that we have already found functions $R^{\mathbb{P}}, S^{\mathbb{P}}, T^{\mathbb{P}}$ as in the statement of the theorem. We may obtain $Q^{\mathbb{P}}$ by composition: define $Q^{\mathbb{P}}(\vec{x}) = R^{\mathbb{P}}(S^{\mathbb{P}}(\vec{x}), T^{\mathbb{P}}(\vec{x}))$.

The proof of Theorem 5.0 as stated is now complete.

→ (5.0)

We must now prove that each of these functions is of uniform finite delay.

Propagation of nominators for rudimentary functions

5.47 PROPOSITION *Let R be a rudimentary function of some number of arguments, and let $R^{\mathbb{P}}$ be the corresponding function of names that we have defined. There is a natural number s_R such that whenever e is a transitive set with $\mathbb{P} \in e$, and ν is an ordinal not less than $\varrho(\mathbb{P})$,*

$$R^{\mathbb{P}} \upharpoonright P_{\nu}^{e;=} \in P_{\nu+s_R}^{e;=}.$$

Proof : We have seen that the nominators for R_0 , R_2 and R_4 proved to be themselves rudimentary, and hence s_R can be taken in these cases to be 1 plus the rudimentary constant for $u \mapsto R \upharpoonright u$. The nominators corresponding to the other functions in the standard generating set are all rudimentary in appropriate fragments of $\chi_{=}$, and so the proof in those cases follows from the corresponding result for $\chi_{=}$. We give the argument for R_8 .

Let $\nu \geq \eta$. We know that $\chi_{=} \upharpoonright P_{\nu}^{e;=} \in P_{\nu+12}^{e;=}$, and that for some rudimentary function E , $R_8^{\mathbb{P}} \upharpoonright P_{\nu}^{e;=} = E(\chi_{=} \upharpoonright P_{\nu}^{e;=})$, so we may take $s_{R_8} = 12 + c_E$.

Once the theorem has been established for the nine generators, it remains only to observe that the property in question is preserved under composition. If, for example, $Q(\vec{x}) = R(S(\vec{x}), T(\vec{x}))$, then s_Q can be taken to be $c_R + \max\{c_S, c_T\} + c_o$, where c_o is the constant of the rudimentary function that composes fragments of $R^{\mathbb{P}}$, $S^{\mathbb{P}}$ and $T^{\mathbb{P}}$ to a fragment of $Q^{\mathbb{P}}$. ⊢ (5.47)

5.48 REMARK At this point we know that all the axioms of \mathbf{GJ}_0 are forced by the trivial condition.

No new ordinals !

There is a long-established principle that a generic extension will contain no ordinals not in the ground model. In [M4] an admittedly pathological example of forcing over an improvident but transitive model of Zermelo set theory is presented where this principle breaks down. So our task here is to present a proof, working in the theory \mathbf{PROVI} , of the following:

5.49 PROPOSITION *If $p \Vdash \underline{x} \in \dot{O}_n$ then $\exists q: \leq p \exists \zeta: \leq \varrho(x) \ q \Vdash \hat{\zeta} = \underline{x}$.*

Plainly the statement of the Proposition requires every ordinal to have a hat; but hatting is $1^{\mathbb{P}}$ -rud rec, so available in \mathbf{PROVI} . Proposition 3.16 then yields

5.50 PROPOSITION *For each ordinal η , $1 \Vdash \hat{\eta}$ is an ordinal.*

The second requirement is that there should be enough set theory to prove that the principle of trichotomy for two ordinals is forced. So let ζ and η be ordinals.

5.51 LEMMA *Either $\zeta \cap \eta = \eta$ or $\zeta \cap \eta \in \eta$.*

Proof : $\eta \setminus \zeta$ if non-empty has, by foundation, a least element, ξ say; then show that $\xi = \eta \cap \zeta$. ⊢ (5.51)

5.52 TRICHOTOMY FOR ORDINALS $\zeta \in \eta$, $\zeta = \eta$ or $\eta \in \zeta$.

Proof : Consider the four statements

- | | |
|----|-----------------------------|
| 1a | $\zeta \cap \eta = \eta$ |
| 1b | $\zeta \cap \eta \in \eta$ |
| 2a | $\zeta \cap \eta = \zeta$ |
| 2b | $\zeta \cap \eta \in \zeta$ |

We know that [(1a or 1b) and (2a or 2b)] holds. Of the four possibilities, (1b and 2b) is impossible, as it would imply $\zeta \cap \eta \in \zeta \cap \eta$, contradicting foundation; the three disjuncts of the proposition correspond to (1b and 2a), (1a and 2a), (1a and 2b). ⊢ (5.52)

The final requirement is that rank should be definable in the ground model; but ϱ is \emptyset -rud rec.

5.53 LEMMA $\Vdash \dot{O}n$ is transitive.

5.54 LEMMA There are no $p \in \mathbf{P}$ and x such that $p \Vdash \underline{x} \in \dot{O}n \wedge \widehat{\underline{\rho(x)}} \in \underline{x}$.

Proof : suppose such an x exists; let it be a member of the transitive set u . By rewriting its definition in terms of the attempts $\chi_{\#} \upharpoonright u$, $\rho \upharpoonright u$ and $\hat{\cdot} \upharpoonright \rho(u)$, we see that the class

$$u \cap \{x \mid \exists p \in \mathbf{P} \ p \Vdash \underline{x} \in \dot{O}n \ \& \ p \Vdash \widehat{\underline{\rho(x)}} \in \underline{x}\}$$

is a set by Δ_0 separation, and non-empty by the initial supposition. Call that set A .

Let x be a member of A with $\rho(x)$ minimal. Then $x \in A$ and $\bigcup^2 x \cap A = \emptyset$. Let $\eta = \rho(x)$, and let $p \Vdash \underline{x} \in \dot{O}n \wedge \hat{\eta} \in \underline{x}$. So $\exists q \leq p \ \exists r \leq q \ (r, y) \in x \ \& \ q \Vdash \hat{\eta} = \underline{y}$.

Let $\zeta = \rho(y)$. Since $y \in \bigcup^2 x$, $\zeta \in \eta$, so by Proposition 3.16, $\mathbb{1} \Vdash \hat{\zeta} \in \hat{\eta}$; $q \Vdash \hat{\eta} = \underline{y}$, so $q \Vdash \widehat{\underline{\rho(y)}} \in \underline{y}$; so $y \in A$, in contradiction to the choice of x . (5.54)

We complete the proof of Proposition 5.43 by noting that the law of trichotomy for ordinals is forced:

5.55 LEMMA \Vdash Trichotomy for ordinals

Proof : We have just seen that Trichotomy for ordinals is provable in \mathbf{GJ}_0 , and we know that all axioms of \mathbf{GJ}_0 are forced. (5.55)

Now Lemma 5.48 implies that if $p \Vdash \underline{x} \in \dot{O}n$, and $\eta = \rho(x)$ then $p \Vdash \hat{\eta} \notin \underline{x}$. By trichotomy, $p \Vdash \underline{x} \in \hat{\eta} \vee \underline{x} = \hat{\eta}$; which implies that there are $q \leq p$ and $\zeta \leq \eta$ with $q \Vdash \underline{x} = \hat{\zeta}$ as required. (5.43)

5.56 REMARK In section 6 of [M2], a forcing construction is done over a non-standard model \mathfrak{N} , and it was there blithely stated without proof that the generic extension would bring no new “ordinals” Fortunately the model \mathfrak{N} was power-admissible, and therefore certainly a model of \mathbf{PROVI} , which is a sub-theory of \mathbf{KP} , so that the present remarks justify that blithe statement; that is reassuring in view of the somewhat pathological models presented in [M4].

We record two related arguments.

5.57 LEMMA $\rho(\text{val}_G(x)) \leq \rho(x)$.

Proof :

$$\begin{aligned} \rho(\text{val}_G(x)) &= \sup\{\rho(\text{val}_G(y)) + 1 \mid (1, y) \in x\} \\ &\leq \sup\{\rho(y) + 1 \mid (1, y) \in x\} \\ &\leq \rho(x) \end{aligned}$$

(5.57)

5.58 LEMMA If $p \Vdash \hat{\zeta} \in \underline{x}$ then $\zeta < \rho(x)$.

Proof by eps-recursion on x : if $p \Vdash \hat{\zeta} \in \underline{x}$ then there are q and r with $q \leq p$, $q \leq r$, $(r, y) \in x$ and $q \Vdash \hat{\zeta} = \underline{y}$. Hence for all $\eta < \zeta$, $q \Vdash \hat{\eta} \in \underline{y}$ and so by induction, $\eta < \rho(y)$, so $\zeta \leq \rho(y) < \rho(x)$. (5.58)

6: Construction of rudimentarily recursive nominators for rank and transitive closure

Rank and transitive closure are pure rud rec; we show here that \mathbb{P} -rud rec nominators exist for them.

6.0 LEMMA *Let A be provident and closed under F and F^{\bullet} : for example if F is rud of rud rec. Then*

$$\text{val}_G(\{(p, F(y)) \mid_{p,y} (p, y) \in x\}) = \{\text{val}_G(F(y)) \mid_y \exists p: \in G (p, y) \in x\}.$$

Proof : Let $Z = \{(p, F(y)) \mid_{p,y} (p, y) \in x\}$. Z is in A by the hypotheses concerning the closure of A under F , F^{\bullet} and related functions. Then $\text{val}_G(Z) = \{\text{val}_G(z) \mid_z \exists p: \in G (p, z) \in Z\}$.

So if $w \in \text{val}_G(Z)$, $\exists z \exists p: \in G [w = \text{val}_G(z) \ \& \ (p, z) \in Z]$. So $\exists y[(p, y) \in x \ \& \ z = F(y)]$. So $w = \text{val}_G(F(y))$ where for some $p \in G (p, y) \in x$. So the LHS is contained in the RHS.

Conversely, if $(p, y) \in x$ and $p \in G$, then $(p, F(y)) \in Z$ and $p \in G$; so $\text{val}_G(F(y)) \in \text{val}_G(Z)$. \dashv (6.0)

Let $S(\cdot)$ be the basic function $z \mapsto z \cup \{z\}$.

6.1 LEMMA *There is a rud function $S^{\mathbb{P}}(\cdot)$ such that $\text{val}_G(S^{\mathbb{P}}(x)) = S(\text{val}_G(x))$.*

Proof : by composition. \dashv (6.1)

6.2 DEFINITION $\varrho^{\mathbb{P}}(x) =_{\text{df}} \bigcup^{\mathbb{P}} \{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(y)) \mid (p, y) \in x \ \& \ p \in \mathbb{P}\}$

6.3 REMARK $\varrho^{\mathbb{P}}$ is rud rec in the parameter \mathbb{P} .

6.4 LEMMA *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$, $\text{val}_G(\varrho^{\mathbb{P}}(x)) = \varrho(\text{val}_G(x))$.*

REMARK That all makes sense: if x is in A , the name $\varrho^{\mathbb{P}}(x)$ is in A . Note that $\varrho(\text{val}_G(x))$ is evaluated in the universe. At present we do not know that the evaluation can be carried out in $A^{\mathbb{P}}[G]$.

Proof :

$$\begin{aligned} \varrho(\text{val}_G(x)) &= \bigcup \{\varrho(y) + 1 \mid_y y \in \text{val}_G(x)\} && \text{definition of } \varrho \\ &= \bigcup \{\varrho(\text{val}_G(w)) + 1 \mid_w \exists p: \in G (p, w) \in x\} && \text{definition of } \text{val}_G(x) \\ &= \bigcup \{\text{val}_G(\varrho^{\mathbb{P}}(w)) + 1 \mid_w \exists p: \in G (p, w) \in x\} && \text{induction hypothesis} \\ &= \bigcup \{\text{val}_G(S^{\mathbb{P}}(\varrho^{\mathbb{P}}(w))) \mid_w \exists p: \in G (p, w) \in x\} && \text{property of } S^{\mathbb{P}} \\ &= \bigcup \text{val}_G(\{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(w))) \mid_{p,w} (p, w) \in x\}) && \text{by Lemma 6.0} \\ &= \text{val}_G(\bigcup^{\mathbb{P}} \{(p, S^{\mathbb{P}}(\varrho^{\mathbb{P}}(w))) \mid_{p,w} (p, w) \in x\}) && \text{property of } \bigcup^{\mathbb{P}} \\ &= \text{val}_G(\varrho^{\mathbb{P}}(x)), && \text{by the definition of } \varrho^{\mathbb{P}}. \end{aligned} \quad \dashv (6.4)$$

6.5 DEFINITION $\text{tcl}^{\mathbb{P}}(x) =_{\text{df}} x \cup^{\mathbb{P}} \bigcup^{\mathbb{P}} \{(p, \text{tcl}^{\mathbb{P}}(z)) \mid (p, z) \in x\}$.

6.6 REMARK $\text{tcl}^{\mathbb{P}}$ is rud rec in the parameter \mathbb{P} .

6.7 LEMMA *Let A be provident, and $\mathbb{P} \in A$. For all $x \in A$, $\text{val}_G(\text{tcl}^{\mathbb{P}}(x)) = \text{tcl}(\text{val}_G(x))$.*

Proof : by similar reasoning.

$$\begin{aligned} \text{tcl}(\text{val}_G(x)) &= \text{val}_G(x) \cup \bigcup \{\text{tcl}(y) \mid_y y \in \text{val}_G(x)\} && \text{definition of tcl} \\ &= \text{val}_G(x) \cup \bigcup \{\text{tcl}(y) \mid_y \exists p: \in G \exists z (p, z) \in x \ \& \ y = \text{val}_G(z)\} && \text{definition of val}(x) \\ &= \text{val}_G(x) \cup \bigcup \{\text{tcl}(\text{val}_G(z)) \mid_z \exists p: \in G (p, z) \in x\} \\ &= \text{val}_G(x) \cup \bigcup \{\text{val}_G(\text{tcl}^{\mathbb{P}}(z)) \mid_z \exists p: \in G (p, z) \in x\} && \text{induction hypothesis} \\ &= \text{val}_G(x) \cup \bigcup \text{val}_G(\{(p, \text{tcl}^{\mathbb{P}}(z)) \mid_{p,z} (p, z) \in x\}) && \text{Lemma 6.0} \\ &= \text{val}_G(x \cup^{\mathbb{P}} \bigcup^{\mathbb{P}} \{(p, \text{tcl}^{\mathbb{P}}(z)) \mid_{p,z} (p, z) \in x\}) && \text{properties of } \cup^{\mathbb{P}} \text{ and } \bigcup^{\mathbb{P}} \\ &= \text{val}_G(\text{tcl}^{\mathbb{P}}(x)), && \text{by the definition of tcl}^{\mathbb{P}}. \end{aligned} \quad \dashv (6.7)$$

6.8 PROPOSITION *Let A be provident, and $\mathbb{P} \in A$, and let G be $(A, \mathbb{P}, \dot{\Delta}_0)$ -generic. Then $A^{\mathbb{P}}[G]$ is closed under rank and transitive closure.*

6.9 REMARK A similar result will hold whenever F is rud rec, given by G , where $G^{\mathbb{P}}$ is rudimentary; $G^{\mathbb{P}}$ may be permitted to have as a parameter a name \underline{a} for a parameter $\text{val}_G(a)$ in the extension.

We pause for breath. The next stage will be to show that the generic extension is closed under the formation of certain canonical progresses; but we digress to discuss the case of primitive recursively closed sets, which is now easy.

7: Construction of primitive recursive nominators for primitive recursive functions

Jensen and Karp give, following Gandy, this definition: there are some initial functions, which are all rudimentary; two versions of substitution: $F(\vec{x}, \vec{y}) = G(\vec{x}, H(\vec{x}), \vec{y})$ and $F(\vec{x}, \vec{y}) = G(H(\vec{x}), \vec{y})$; and this recursion schema:

$$F(z, \vec{x}) = G(\bigcup \{F(u, \vec{x}) \mid u \in z\}, z, \vec{x}).$$

7.0 LEMMA *Let A be transitive and primitive recursively closed, and let F be primitive recursive. Then*

$$\text{val}_G(\{(p, F(y)) \mid_{p,y} (p, y) \in x\}) = \{\text{val}_G(F(y)) \mid_y \exists p: \in G (p, y) \in x\}.$$

Proof : as before. + (7.0)

For notational simplicity there is only one x in the following, but it could easily be replaced by a finite sequence.

7.1 PROPOSITION *Let A be transitive and primitive recursively closed. Let $\mathbb{P} \in A$, and let \mathcal{G} be (A, \mathbb{P}) -generic. Suppose that $G(f, z, x)$ is primitive recursive in the parameter \mathbb{P} , and that it has a primitive recursive nominator $G^\mathbb{P}$, so that for all f, z, x in A ,*

$$\text{val}_G(G^\mathbb{P}(f, z, x)) = G(\text{val}_G(f), \text{val}_G(z), \text{val}_G(x)).$$

Suppose that $F(z, x) = G(\bigcup \{F(u, x) \mid u \in z\}, z, x)$. Define $F^\mathbb{P}$ by

$$F^\mathbb{P}(z, x) = G^\mathbb{P}(\bigcup^\mathbb{P}(\{(p, F^\mathbb{P}(u, x)) \mid_{p,u} (p, u) \in z\}), z, x).$$

Then $F^\mathbb{P}$ is primitive recursive in the parameter \mathbb{P} , and for all z, x in A ,

$$\text{val}_G(F^\mathbb{P}(z, x)) = F(\text{val}_G(z), \text{val}_G(x));$$

so that $F^\mathbb{P}$ is a primitive recursive nominator for F .

Proof : for fixed x by recursion on z :

$$\begin{aligned} F(\text{val}_G(z), \text{val}_G(x)) &= G(\bigcup \{F(w, \text{val}_G(x)) \mid_w w \in \text{val}_G(z)\}, \text{val}_G(z), \text{val}_G(x)) \\ &= G(\bigcup \{F(\text{val}_G(u), \text{val}_G(x)) \mid_u \exists p: \in \mathcal{G} (p, u) \in z\}, \text{val}_G(z), \text{val}_G(x)) \\ &= G(\bigcup \{\text{val}_G(F^\mathbb{P}(u, x)) \mid_u \exists p: \in \mathcal{G} (p, u) \in z\}, \text{val}_G(z), \text{val}_G(x)) \\ &= G(\bigcup \text{val}_G(\{(p, F^\mathbb{P}(u, x)) \mid_{p,u} (p, u) \in z\}), \text{val}_G(z), \text{val}_G(x)) \\ &= G(\text{val}_G(\bigcup^\mathbb{P}(\{(p, F^\mathbb{P}(u, x)) \mid_{p,u} (p, u) \in z\})), \text{val}_G(z), \text{val}_G(x)) \\ &= G^\mathbb{P}(\bigcup^\mathbb{P}(\{(p, F^\mathbb{P}(u, x)) \mid_{p,u} (p, u) \in z\}), z, x) \\ &= \text{val}_G(F^\mathbb{P}(z, x)) \end{aligned} \quad + (7.1)$$

The above confirms an observation made some years ago by Jensen:

7.2 COROLLARY *A set-generic extension of a primitive recursively closed set is primitive recursively closed.*

8: Construction of nominators for the stages of a progress.

Let e be a transitive set in the ground model of which \mathbb{P} is a member, and let θ be indecomposable, exceeding the rank of e . P_θ^e is provident. Let \dot{d} be the Cohen term $\hat{e} \cup \{\dot{\mathcal{G}}\}^\mathbb{P}$, so that $\text{val}_{\mathcal{G}}(\dot{d})$ will be the transitive set $d = e \cup \{\mathcal{G}\}$.

8.0 REMARK \dot{d} will be a member of $P_{\varrho(\mathbb{P})+k}^e$ for some (small) k , given the definition of $\dot{\mathcal{G}}$, our convention that $\mathbb{1} = 1$ and the fact that $\hat{\cdot}$ is $\mathbb{1}$ -rud rec.

Our task is to build for each $\nu < \theta$ a name $N(\nu)$ for the stage P_ν^d of the progress towards d .

A simplified progress

Now $\varrho(\mathcal{G}) \leq \varrho(\mathbf{P}) < \varrho(\mathbb{P})$, so that for $\nu \geq \eta$, $d_\nu = e_\nu \cup \{\mathcal{G}\}$. It might be that $\varrho(\mathcal{G}) < \varrho(\mathbf{P})$; to avoid building names which make allowance for that uncertainty, we shall build names for the terms of a slightly different progress $(Q_\nu^d)_\nu$.

8.1 DEFINITION

$$\begin{aligned} \text{for } \nu < \eta, \quad Q_\nu^d &= P_\nu^e; & Q_\eta^d &= P_\eta^e \cup \{\mathcal{G}\}; \\ \text{for } \nu \geq \eta, \quad Q_{\nu+1}^d &= \mathbb{T}(Q_\nu^e) \cup \{d_\nu\} \cup d_{\nu+1}; & Q_\lambda^d &= \bigcup_{\nu < \lambda} Q_\nu^d \quad \text{for } \lambda = \bigcup \lambda > \eta. \end{aligned}$$

8.2 PROPOSITION *If θ is indecomposable, then Q_θ^d is provident and equals P_θ^d .*

Proof : by [MB, Proposition 55.54].

⊢ (8.2)

Names using dynamic predicates

With that in mind, we now define names $N(\nu)$ such that $\text{val}_{\mathcal{G}}(N(\nu)) = Q_\nu^d$.

8.3 DEFINITION $\dot{d}_\nu =_{\text{df}} \hat{e}_\nu \cup \{\dot{\mathcal{G}}\}^\mathbb{P}$ for $\nu \geq \eta$

$$\begin{aligned} \text{for } \nu < \eta, \quad N(\nu) &= \widehat{P}_\nu^e; & N(\eta) &= \widehat{P}_\eta^e \cup \{\dot{\mathcal{G}}\}^\mathbb{P}; \\ \text{for } \nu \geq \eta, \quad N(\nu+1) &= \mathbb{T}^\mathbb{P}(N(\nu)) \cup \{\dot{d}_\nu\} \cup \dot{d}_{\nu+1}; & N(\lambda) &= \bigcup^\mathbb{P} \{(\mathbf{1}^\mathbf{P}, N(\nu)) \mid \nu < \lambda\} \quad \text{for } \lambda = \bigcup \lambda > \eta \end{aligned}$$

8.4 LEMMA *For $\nu \geq \eta$, $N(\nu) \in P_{\nu+\omega}^{e; \mathbb{P}}$.*

Proof by cases: for $\nu = \eta$, by inspection; for successor ordinals, by knowledge of the birthday of $\mathbb{T}^\mathbb{P}$; for limit λ by knowledge of the delay of $\bigcup^\mathbb{P}$. ⊢ (8.4)

8.5 PROPOSITION *Each $N(\nu)$ for $\nu < \theta$ is in P_θ^e .*

Proof : All those names are in $P_\theta^{e; \mathbb{P}}$, which was shown in Proposition 2.18 to equal P_θ^e . ⊢ (8.5)

8.6 PROPOSITION *Let \mathcal{G} be (P_θ^e, \mathbb{P}) generic and let $\nu < \theta$. Then $\text{val}_{\mathcal{G}}(N(\nu)) = Q_\nu^d$.*

Proof : by induction on ν . ⊢ (8.6)

9: Generic extensions of provident sets and of Jensen fragments

We are now in a position to prove the following theorem:

9.0 THEOREM *Let θ be an indecomposable ordinal strictly greater than the rank of a transitive set e which contains the notion of forcing, \mathbb{P} . Let \mathcal{G} be (P_θ^e, \mathbb{P}) -generic. Then $(P_\theta^e)^\mathbb{P}[\mathcal{G}] = P_\theta^{e \cup \{\mathcal{G}\}}$ and hence is provident.*

Proof : $(P_\theta^e)^\mathbb{P}[\mathcal{G}]$ contains $P_\theta^{e \cup \{\mathcal{G}\}}$, as we have for each $\nu < \theta$ built a name in P_θ^e that evaluates under \mathcal{G} to $Q_\nu^{e \cup \{\mathcal{G}\}}$, and we know by Proposition 8.4 that $Q_\nu^{e \cup \{\mathcal{G}\}}$ equals $P_\theta^{e \cup \{\mathcal{G}\}}$.

For the converse direction, we know that $P_\theta^{e \cup \{\mathcal{G}\}}$ is provident, and has \mathcal{G} as a member and hence can support the \mathcal{G} -rudimentary recursion defining $\text{val}_\mathcal{G}(\cdot)$. Further $P_\theta^{e \cup \{\mathcal{G}\}}$ includes $(P_\nu^e)_\nu$, which is defined by an e -rudimentary recursion, and so includes $(P_\theta^e)^\mathbb{P}[\mathcal{G}]$. ⊢ (9.0)

REMARK Thus, in this special case, a generic extension of a model of **PROVI** is a model of **PROVI**. We shall use this result to establish it more generally.

REMARK Theorem 9.0 remains true if the hypothesis on θ is weakened to requiring that $\theta > \varrho(\mathbb{P})$.

Proof that a generic extension of a provident set is provident.

9.1 THEOREM *Let A be provident, $\mathbb{P} \in A$ and \mathcal{G} (A, \mathbb{P}) -generic. Then $A^\mathbb{P}[\mathcal{G}]$ is provident.*

Proof : Let $\theta =_{\text{df}} \text{On} \cap A$ and let $T = \{c \mid c \in A \ \& \ c \text{ is transitive} \ \& \ \mathbb{P} \in c\}$. Then

$$A = \bigcup \{P_\theta^c \mid c \in T\},$$

since the union on the right contains each element of A and is contained in A . It follows that

$$A^\mathbb{P}[\mathcal{G}] = \bigcup_{c \in T} (P_\theta^c)^\mathbb{P}[\mathcal{G}]$$

By Theorem 9.0, as each P_θ^c is provident and contains \mathbb{P} ,

$$A^\mathbb{P}[\mathcal{G}] = \bigcup_{c \in T} P_\theta^{c \cup \{\mathcal{G}\}}$$

and each $P_\theta^{c \cup \{\mathcal{G}\}}$ is provident. Now in [MB, Proposition 5.52] we proved the

LEMMA *If θ is indecomposable and D is a collection of transitive sets each of rank less than θ and such that the pair of any two is a member of a third, then $\bigcup_{d \in D} P_\theta^d$ is provident.*

Take $D = \{c \cup \{\mathcal{G}\} \mid c \in T\}$ to complete the proof. ⊢ (9.1)

Genericity at every limit level

9.2 PROPOSITION *Let e be a transitive set with $\mathbb{P} \in e$. Let θ be indecomposable, greater than $\varrho(e)$. Let λ be a limit ordinal not less than θ . Let $\kappa \geq \lambda + \omega$, and let \mathcal{G} be (P_κ^e, \mathbb{P}) generic. Put $d = e \cup \{\mathcal{G}\}$; then d is also transitive of rank $< \theta$. Suppose that $P_\lambda^e[\mathcal{G}] = P_\lambda^d$. Then $P_{\lambda+\omega}^e[\mathcal{G}] = P_{\lambda+\omega}^d$.*

Proof: At this level, where we are above the rank of both e and d , $P_{\nu+1}^e = \mathbb{T}(P_\nu^e)$ and $P_{\nu+1}^d = \mathbb{T}(P_\nu^d)$.

(i) $P_\lambda^d = \text{val}_{\mathcal{G}}(P_\lambda^e)$: for as λ is a limit ordinal, P_λ^e is rud closed.

Hence

$$(ii) \quad \begin{aligned} \mathbb{T}(P_\lambda^d) &= \text{val}_{\mathcal{G}}(\mathbb{T}^{\mathbb{P}}(P_\lambda^e)) \\ &\in P_{\lambda+\omega}^e[\mathcal{G}] \quad \text{as } \mathbb{T}^{\mathbb{P}}(P_\lambda^e) \in P_{\lambda+\omega}^e. \end{aligned}$$

Iterating \mathbb{T} , we see that

$$P_{\lambda+\omega}^d \subseteq P_{\lambda+\omega}^e[\mathcal{G}].$$

(iii) In the other direction, both e and \mathcal{G} are in P_θ^d ; as $\theta \leq \lambda$, $P_{\lambda+\omega}^d$ will be both e - and \mathcal{G} -provident; so $P_{\lambda+\omega}^e \subseteq P_{\lambda+\omega}^d$, and therefore as $\text{val}_{\mathcal{G}}(\cdot)$ is \mathcal{G} -rud rec, $P_{\lambda+\omega}^e[\mathcal{G}] \subseteq P_{\lambda+\omega}^d$. \dashv (9.2)

9.3 COROLLARY *If $J_\nu^{\mathbb{P}}[\mathcal{G}] = J_\nu(\mathcal{G})$, and $\nu \geq \varrho(\mathbb{P}).\omega$, then $J_{\nu+1}^{\mathbb{P}}[\mathcal{G}] = J_{\nu+1}(\mathcal{G})$.*

9.4 THEOREM *Let $\mathbb{P} \in J_\xi$, where ξ is indecomposable. Let \mathcal{G} be \mathbb{P} -generic over L . Then for each ordinal $\zeta \geq \xi$, $J_\zeta(\mathcal{G}) = J_\zeta^{\mathbb{P}}[\mathcal{G}]$: in particular each set in $J_\zeta(\mathcal{G})$ is $\text{val}_{\mathcal{G}}(a)$ for some $a \in J_\zeta$.*

Here is an application, which fleshes out an argument outlined in a letter from Sy Friedman.

9.5 PROPOSITION *Let $\theta < \eta \leq \zeta < \xi$ be ordinals, with η indecomposable. Suppose that $\mathbb{P} \in J_\eta$ and that \mathcal{G} is (J_ξ, \mathbb{P}) generic. Let $x \subseteq \theta$, with $x \in J_\xi$ and $x \in J_\zeta(\mathcal{G})$. Then $x \in J_\zeta$.*

Proof: $\hat{x} \in J_\xi$. Since $J_\zeta^{\mathbb{P}}[\mathcal{G}] = J_\zeta(\mathcal{G})$, $x = \text{val}_{\mathcal{G}}(y)$ for some $y \in J_\zeta$. Therefore some condition p in \mathcal{G} forces $\underline{y} = \underline{\hat{x}}$; so $x = \{\nu < \theta \mid p \Vdash \underline{\hat{\nu}} \in \underline{y}\}$. The map $\nu \mapsto \hat{\nu}$, restricted to the ν less than θ , is in $J_{\theta+1}$, and the relation $p \Vdash \underline{\hat{\nu}} \in \underline{z}$ is rudimentary in $\chi_{=}$; an appropriate segment of that characteristic function is in J_ζ , by propagation starting from η ; and therefore $x \in J_\zeta$. \dashv (9.5)

10: Persistence of certain systems

Extension of the definition of forcing to all wffs

We may extend the definition of forcing, schematically, to all wffs, thus:

10-0 DEFINITION $p \Vdash \bigwedge x \dot{\Phi} \iff \forall x p \Vdash \dot{\Phi}(x)$

10-1 PROPOSITION $p \Vdash \bigvee x \dot{\Phi} \iff \forall q: \leq p \exists r: \leq q \exists x r \Vdash \dot{\Phi}(x)$
 $p \Vdash \bigvee x: \epsilon y \dot{\Phi} \iff \forall q: \leq p \exists r: \leq q \exists (t, \beta): \epsilon y (r \leq t \ \& \ r \Vdash \dot{\Phi}(\beta))$

Thus (in KP) if \mathbb{P} is a set, \Vdash restricted to Σ_1 wffs will be Σ_1 ; and in ZF, forcing for Σ_n wffs is Σ_n .

10-2 PROPOSITION $p \Vdash \dot{\Phi} \iff \forall q \leq p \exists r \leq q r \Vdash \dot{\Phi}$
 $\Vdash \underline{x} = \underline{\alpha} \wedge \dot{\Phi}(\underline{\alpha}) \longrightarrow \dot{\Phi}(\underline{x}).$

Proof: already proved for atomic wffs, an easy induction thereafter. \dashv

10-3 EXERCISE Show that if $p \Vdash \bigwedge x (\dot{\Phi} \longrightarrow \Psi(x))$ and $\dot{\Phi}$ is a sentence of $\mathcal{L}^{\mathbb{P}}$, (so that, intuitively, x has no free occurrence in $\dot{\Phi}$), then $p \Vdash \dot{\Phi} \longrightarrow \bigwedge x \Psi(x)$.

This exercise, coupled with our remarks about *Modus Ponens* above, ensure that we may apply mathematical reasoning to statements in our forcing language.

10-4 REMARK There is a point to be made here, similar to the problem of defining truth for all formulae. We have defined the forcing relation χ_{\Vdash} for all wffs φ of our formal language that are $\dot{\Delta}_0$ by a single definition, similar to our defining truth for $\dot{\Delta}_0$ wffs by recursion over the transitive closure of the parameters occurring; in the case of forcing we need to consider the transitive closure of $\{\mathbb{P}\}$ and the names occurring in the wff.

But we are not able to make a single definition for all formulae with arbitrarily many unrestricted quantifiers, but must introduce them schematically. This would become very apparent in the Boolean-valued presentation of forcing, where truth values are assigned in a complete Boolean algebra, and we must invoke the axiom of replacement for each quantifier to see that the supremum over a class is actually the supremum over a set.

For the moment, of course, we only need this definition for the very small number of unrestricted quantifiers, two or three, required for reasoning in KP.

The persistence of KPI

The system KPI, as presented in [M2], may be obtained by adding to the axioms of PROVI the schemes of Π_1 foundation and Δ_0 collection. As we have proved the persistence of PROVI, it only remains to discuss those two schemes.

Π_1 foundation

We exploit the fact that in KP, $\Delta_0(\Pi_1)$ predicates (that is, Π_1 preceded by a Δ_0 string of restricted quantifiers) are equivalent to Π_1 ones. First a general discussion:

Let A be a $\Delta_0(\Pi_1)$ class. Put

$$\begin{aligned} B &=_{\text{df}} \{ \xi \mid \xi \in A \vee \exists \zeta: \epsilon \xi \zeta \in A \} \\ C &=_{\text{df}} \{ \xi \mid \xi \in A \vee \exists \zeta: \epsilon \xi \zeta \in A \vee \exists \zeta: \epsilon \xi \exists \eta: \epsilon \zeta \eta \in A \} \\ D &=_{\text{df}} \{ \xi \mid \text{tcl}(\xi) \cap A \neq \emptyset \} \end{aligned}$$

10-5 LEMMA (KP) Each of B , C , and D is Π_1^{KP} .

Proof: For D , express $\text{tcl}(\xi) \cap A \neq \emptyset$ as

$$\forall f (\text{ if } f \text{ is an attempt at the function } \text{tcl} \text{ and } \xi \in \text{Dom } f \text{ then } \exists a: \epsilon f(\xi) a \in A)$$

which is $\Pi_1(\Delta_0(\Pi_1))$ and thus Π_1^{KP} .

\dashv (10-5)

10-6 LEMMA (KP)

- (10-6-0) If ξ is B -minimal, then $\xi \in A$ but $(\xi \cup \bigcup \xi) \cap A = \emptyset$;
 (10-6-1) If ξ is C -minimal, then $\xi \in A$ but $(\xi \cup \bigcup \xi \cup \bigcup^2 \xi) \cap A = \emptyset$;
 (10-6-2) If ξ is D -minimal, then $\xi \in A$ but $\text{tel}(\xi) \cap A = \emptyset$.

Now let Φ be $\dot{\Delta}_0(\Pi_1)$; put $A = \{\xi \mid \exists p : \in \mathbb{P} \ p \Vdash \Phi[\xi]\}$. Then A is $\dot{\Delta}_0(\Pi_1)$. Form c , and, assuming $A \neq \emptyset$, let ξ be C -minimal. Then for some p , $p \Vdash \Phi[\xi]$, but p will also force that ξ is Φ -minimal.

Restricted collection

10-7 PROPOSITION $\Vdash^{\mathbb{P}} \Delta_0$ -collection.

Suppose that $p \Vdash \bigwedge \mathfrak{x} : \epsilon a \bigvee \eta \dot{\Phi}$, where $\dot{\Phi}$ is Δ_0 . This will expand to a statement of the form

$$\forall x : \in \text{Dom}(a) \ \forall q : \leq p \ \exists r : \leq q \ \exists \text{ a name } t \text{ such that } q \Vdash_1 \underline{x} \in \underline{a} \implies r \Vdash \dot{\Phi}[t, \underline{x}].$$

But we know that the forcing relation for Δ_0 wffs is Δ_1 , and so in KP we may find a set v containing at least one such name for each $x \in \text{Dom}(a)$. Form the set $w = \{(\mathbb{1}^{\mathbb{P}}, a) \mid a \in v\}$: then we shall find that \underline{w} will do. ¬ (10-7)

10-8 REMARK Collection is more natural than replacement in the context of forcing.

Proof that a generic extension of an admissible set is admissible

NOTE that this discussion seems to be aimed at the ill-founded case.

10-9 LEMMA For each Δ_0 sentence φ of $\mathcal{L}^{\mathbb{P}}$, say with constants $\underline{a}_1 \dots \underline{a}_n$,

$$M^{\mathbb{P}}[G] \models \varphi[\text{val}_G(a_1), \dots, \text{val}_G(a_n)] \iff \exists p : \in G \ p \Vdash^{\mathbb{P}} \varphi[\underline{a}_1, \dots, \underline{a}_n].$$

We shall prove this by induction on the length of φ . For the atomic cases of the definition of forcing, this is the content of the previous proposition in the light of the putative lemma, now a definition.

The propositional connectives are covered by G 's being a consistent and complete filter.

The problem of handling restricted quantifiers reduces to showing that if $\exists p : \in G \ p \Vdash^{\mathbb{P}} \bigvee \mathfrak{x} : \epsilon \underline{a} \ \varphi(\mathfrak{x})$ then $\exists q : \in G \ \exists (r, \xi) : \in a \ q \leq p \ \& \ q \leq r \ \& \ q \Vdash^{\mathbb{P}} \varphi[\xi]$. But the class

$$\{q \in \mathbb{P} \mid q \text{ incompatible with } p \vee \exists (r, \xi) : \in a \ q \leq p \ \& \ q \leq r \ \& \ q \Vdash^{\mathbb{P}} \varphi[\xi]\}$$

is dense open and Δ_1^{KP} ; hence in M and so met by G . ¬ (10-9)

10-10 LEMMA If G meets all dense open $\Sigma_1(M)$ sub-classes of \mathbb{P} then for each Σ_1 sentence $\varphi[\underline{a}_1, \dots, \underline{a}_n]$ as above,

$$M^{\mathbb{P}}[G] \models \varphi[\text{val}_G(a_1), \dots, \text{val}_G(a_n)] \iff \exists p : \in G \ p \Vdash^{\mathbb{P}} \varphi[\underline{a}_1, \dots, \underline{a}_n].$$

Further, if φ is Π_2 , then

$$\exists p : \in G \ p \Vdash^{\mathbb{P}} \varphi[\underline{a}_1, \dots, \underline{a}_n] \implies M^{\mathbb{P}}[G] \models \varphi[\text{val}_G(a_1), \dots, \text{val}_G(a_n)]$$

Proof : The second clause follows easily from the first. The proof of the first reduces to showing that for Δ_0 Φ ,

$$\exists p : \in G \ p \Vdash \bigvee x \ \Phi(x) \iff \exists q : \in G \ \exists a \ q \Vdash \Phi(\underline{a}).$$

To do the less trivial direction, suppose that $p \Vdash \bigvee \mathfrak{x} \ \Phi$. The class

$$\{q \in \mathbb{P} \mid q \text{ incompatible with } p \vee \exists a \ q \Vdash \Phi(\underline{a})\}$$

is dense open and $\Sigma_1(M)$; so G meets it. ¬ (10-10)

10-11 THEOREM If M is admissible, $\mathbb{P} \in M$ and G is an (M, \mathbb{P}) generic filter meeting each dense open subclass of M that is the union of a $\Sigma_1(M)$ and a $\Pi_1(M)$ class, then $M^{\mathbb{P}}[G]$ is admissible.

Proof : Remark first that much of KP is a Π_2 theory: the elementary axioms and indeed the Δ_0 separation scheme amount to saying that the universe is closed under the eight basic rudimentary functions: so for each R_i (say of two variables) we assert that $\forall x \forall y \exists z z = R_i(x, y)$. This is Π_2 ; we know that all these assertions are forced by $\mathbb{1}^{\mathbb{P}}$, and hence by the last lemma are true in $M^{\mathbb{P}}[G]$. Transitive containment and, with a little manipulation, Π_1 foundation can also be obtained this way.

Our only problem therefore is with Δ_0 collection. Suppose therefore that Ψ is Δ_0 , that $\mathbf{a} = \text{val}_{\mathcal{G}}(a) \in M[G]^{N1}$ and that

$$M[G] \models \bigwedge \mathbf{x} : \epsilon \tilde{\mathbf{a}} \vee \eta \Psi(\mathbf{x}, \eta).$$

Let

$$\Delta =_{\text{df}} \{p \mid p \Vdash^{\mathbb{P}} \bigwedge \mathbf{x} : \epsilon \underline{a} \vee \eta \Psi \vee \exists (q, \alpha) : \epsilon a \ p \leq q \ \& \ p \Vdash^{\mathbb{P}} \bigwedge \eta \neg \Psi(\underline{\alpha}, \eta)\}.$$

Then Δ is dense open, and is the union of a Σ_1^{KP} class and a Π_1^{KP} class. By hypothesis, $G \cap \Delta \neq \emptyset$.

Hence there must be a $p \in G$ such that $p \Vdash^{\mathbb{P}} \bigwedge \mathbf{x} : \epsilon \underline{a} \vee \eta \Psi$, the other half of the dense set being excluded by our assumption on $M[G]$. But we know from Proposition 10.7 that then $p \Vdash^{\mathbb{P}} \bigvee \mathbf{v} \bigwedge \mathbf{x} : \epsilon \underline{a} \vee \eta : \epsilon \mathbf{v} \Psi(\mathbf{x}, \eta)$; as $p \in G$, and this statement is Σ_1 , $M^{\mathbb{P}}[G] \models \bigvee \mathbf{v} \bigwedge \mathbf{x} : \epsilon \tilde{\mathbf{a}} \vee \eta : \epsilon \mathbf{v} \Psi(\mathbf{x}, \eta)$, as required. \dashv

Persistence of Σ_1 separation

To see that the Forcing Theorem, the principle that “what is true is what is forced”, holds for Σ_1 wffs, we shall need Σ_1 separation in the ground model.

10.12 PROPOSITION Suppose that M is a provident set modelling Σ_1 separation, $\mathbb{P} \in M$ and $\mathcal{G} (M, \mathcal{P}, \dot{\Delta}_0)$ generic. Let $\Psi(\cdot, \cdot)$ be $\dot{\Delta}_0$. Then the following are equivalent for each $x \in M$:

- (i) $M[\mathcal{G}] \models \bigvee \eta \Psi(\eta; \text{val}_{\mathcal{G}}(x))$;
- (ii) $\exists p : \epsilon \mathcal{G} \ M \models p \Vdash \bigvee \eta \Psi(\eta; \underline{x})$.

Proof : If (i) then $\exists y : \epsilon M \ M[\mathcal{G}] \models \Psi[\text{val}_{\mathcal{G}}(y), \text{val}_{\mathcal{G}}(x)]$; so by the Forcing theorem for restricted wffs, some p in \mathcal{G} forces $\Psi[y, \underline{x}]$; and then this p trivially forces $\bigvee \eta \Psi(\eta; \underline{x})$.

Conversely, if $p_0 \in \mathcal{G}$ forces $\bigvee \eta \Psi(\eta; \underline{x})$ then the class $E =_{\text{df}} \{p \mid \exists y \ p \Vdash \Psi(y; \underline{x})\}$, is dense below p_0 ; but E is defined by a Σ_1 prefix to a rud rec property, and so it will follow from Σ_1 separation in M that E is a set in M and therefore meets \mathcal{G} . Let $p \in E \cap \mathcal{G}$. Then for some $y \in M$, $p \Vdash \Psi[y, \underline{x}]$, so $M[\mathcal{G}] \models \Psi[\text{val}_{\mathcal{G}}(y), \text{val}_{\mathcal{G}}(x)]$, whence $M[\mathcal{G}] \models \bigvee \eta \Psi(\eta; \text{val}_{\mathcal{G}}(x))$. \dashv (10.12)

10.13 PROPOSITION Suppose that M is a provident set modelling Σ_1 separation, $\mathbb{P} \in M$ and $\mathcal{G} (M, \mathcal{P}, \dot{\Delta}_0)$ generic. Then Σ_1 separation is true in $M[\mathcal{G}]$.

Proof : Let $\text{val}_{\mathcal{G}}(a) \in M[\mathcal{G}]$. In M , let $A = \{(p, x) \mid p \Vdash_1 \underline{x} \in \underline{a}\}$. Then $\text{val}_{\mathcal{G}}(a) = \text{val}_{\mathcal{G}}(A)$. $A \in M$, as it is recoverable from $\mathbb{P} \times \bigcup^2 a$ by using a basic separator.

Now, in M , set $B = A \cap \{(p, x) \mid \exists y \ p \Vdash \Psi[y, \underline{x}]\}$. $B \in M$, as it is recoverable from A by using a rud rec separator.

LEMMA In $M[\mathcal{G}]$, $\text{val}_{\mathcal{G}}(B) = \text{val}_{\mathcal{G}}(A) \cap \{x \mid \exists y \Psi(y, x)\}$.

Proof : $\text{val}_{\mathcal{G}}(B) = \{\text{val}_{\mathcal{G}}(x) \mid \exists p : \epsilon \mathcal{G} \ p \Vdash_1 \underline{x} \in \underline{a} \ \& \ \exists y : \epsilon M \ p \Vdash \Psi[y, \underline{x}]\}$, so certainly in $M[\mathcal{G}]$, $\text{val}_{\mathcal{G}}(B) \subseteq \text{val}_{\mathcal{G}}(A) \cap \{x \mid \exists y \Psi(y, x)\}$.

Suppose that $\text{val}_{\mathcal{G}}(x) \in \text{val}_{\mathcal{G}}(a)$ and that $M[\mathcal{G}] \models \bigvee \eta \Psi(\eta, \text{val}_{\mathcal{G}}(x))$; then for some $y \in M$, $M[\mathcal{G}] \models \Psi[\text{val}_{\mathcal{G}}(y), \text{val}_{\mathcal{G}}(x)]$; so for some $q \in \mathcal{G}$, $q \Vdash \underline{x} \in \underline{a} \wedge \Psi[y, \underline{x}]$. So there is a $p \in \mathcal{G}$ with $p \leq q$ and $\exists x_1 : \epsilon M$ such that

$$p \Vdash \underline{x} = \underline{x}_1 \ \& \ p \Vdash_1 \underline{x}_1 \in \underline{a} \ \& \ p \Vdash \Psi[y, \underline{x}_1].$$

But then $(p, x_1) \in B$, and $\text{val}_{\mathcal{G}}(x) = \text{val}_{\mathcal{G}}(x_1) \in \text{val}_{\mathcal{G}}(B)$. \dashv (10.13)

^{N1} a comment on the notation: a is in M and is a name in the forcing for \mathbf{a} ; \underline{a} is used in the forcing language to remind us that a is not being spoken of as itself but as a name for an as yet uncreated object; on the other hand, once the model $M[G]$ exists we may discuss what sentences are true in it, in terms of the usual truth predicate \models and the associated language; $\tilde{\mathbf{a}}$ is a name for \mathbf{a} in that language.

Persistence of full separation

For a class Γ of wffs, such as $\Sigma_\mathfrak{k}$ or $\Pi_\mathfrak{k}$, let $FT(\Gamma)$ be the principle that for set forcing over a provident model M , if p forces some Φ in Γ , then Φ will be true in $M[\mathcal{G}]$ whenever $p \in \mathcal{G}$; let $\Sigma_1\Gamma$ be the class of wffs of the form $\exists x\Psi$ where $\Psi \in \Gamma$; and let $\Vdash\Gamma$ be the class of wffs of the form $p \Vdash \Phi$ where Φ is in Γ .

Then the arguments given for Σ_1 show more generally that

10·14 THEOREM

- (i) if $FT(\Sigma_\mathfrak{k})$ then $FT(\Pi_{\mathfrak{k}+1})$;
- (ii) if $FT(\Pi_\mathfrak{k})$ and $\Sigma_1(\Vdash\Pi_\mathfrak{k})$ separation holds in M then $FT(\Sigma_{\mathfrak{k}+1})$;
- (iii) if $FT(\Sigma_\mathfrak{k})$ and $(\Vdash\Sigma_\mathfrak{k})$ separation holds in M , then $\Sigma_\mathfrak{k}$ separation holds in all set-generic extensions of M .

10·15 COROLLARY $FT(\Delta_0)$ holds for all provident sets; so any set-generic extension of a provident set which models full separation will also be provident and model full separation; moreover the forcing theorem will hold for all wffs.

10·16 REMARK We would have to take $\Sigma_\mathfrak{k}$ in a strict sense, as we have no mechanism for removing Δ_0 prefixes to a $\Sigma_\mathfrak{k}$ formula. On the other hand, it seems that \mathcal{G} meeting each dense subset of \mathbb{P} will suffice.

10·17 REMARK The class $\Vdash\Gamma$ will generally be larger than Γ , as Δ_0 quantifiers such as $\forall q:\leq r \exists r:\leq q$ will usually be interpolated between successive unrestricted quantifiers of a formula in Γ . Of course if appropriate forms of Collection hold in M , these surplus restricted quantifiers can be absorbed; and they can also be absorbed in contexts such as $V = L$ when the Lemma of Sy Friedman discussed in §5 of [M2] holds.

10·18 REMARK We can reduce the amount of separation required to hold in M in proving the forcing theorem if, instead, we require the generic \mathcal{G} to meet certain dense definable classes. On the other hand, that device apparently cannot be used to show that separation holds in the extension where it did not hold in the ground model; which raises the following question.

10·19 PROBLEM Is it possible for a set-generic extension to satisfy more separation than held in the ground model ?

11: Definition of generic filter and extension in the ill-founded case

There are some points to be clarified about the construction of a generic extension of an ill-founded model \mathfrak{N} . There are three problems: one is to show that the generic filter can be built, then that the model can be defined, and finally that truth respects forcing.

We suppose that we are treating forcing in the manner of Shoenfield, in which every element of the ground model is interpreted as a name for a member of the extension.

If one were dealing with a well-founded model N , one would proceed by first choosing a filter \mathcal{G} that meets every subclass of \mathbf{P} that is definable over N , so that we may call \mathcal{G} (N, \mathbf{P}) -generic, and then making the following recursive definition:

11.0 PUTATIVE DEFINITION Define (externally to N) $\text{val}_{\mathcal{G}} : N \rightarrow V$ by

$$\text{val}_{\mathcal{G}}(b) = \{\text{val}_{\mathcal{G}}(a) \mid \exists p \in \mathcal{G} \ (p, a) \in b\}.$$

Then we would prove the following

11.1 PUTATIVE LEMMA For all a and b the following hold:

$$(11.2) \quad \text{val}_{\mathcal{G}}(a) \in \text{val}_{\mathcal{G}}(b) \iff \exists p \in \mathcal{G} \ p \Vdash \underline{a} \in \underline{b}$$

$$(11.3) \quad \text{val}_{\mathcal{G}}(a) \subseteq \text{val}_{\mathcal{G}}(b) \iff \exists p \in \mathcal{G} \ p \Vdash \underline{a} \subseteq \underline{b}$$

$$(11.4) \quad \text{val}_{\mathcal{G}}(a) = \text{val}_{\mathcal{G}}(b) \iff \exists p \in \mathcal{G} \ p \Vdash \underline{a} = \underline{b}$$

In our present context, the model \mathfrak{N} is ill-founded, and so *prima facie* we cannot carry out that recursive definition. However we may choose \mathcal{G} as before, meeting every \mathfrak{N} -definable subclass of \mathbf{P} . Then we treat the above Lemma as a definition:

11.5 DEFINITION Define for all a and b in \mathfrak{N} the following equivalence relation:

$$a \equiv_{\mathcal{G}} b \iff \exists p \in \mathcal{G} \ p \Vdash \underline{a} = \underline{b}$$

Let $\Omega = \Omega_{\mathcal{G}}$ be the set of equivalence classes. Write $[a]_{\mathcal{G}}$ for the $\equiv_{\mathcal{G}}$ -equivalence class of $a \in \mathfrak{N}$.

Define a relation $\in_{\mathcal{G}}$ on Ω by

$$[a]_{\mathcal{G}} \in_{\mathcal{G}} [b]_{\mathcal{G}} \iff \exists p \in \mathcal{G} \ p \Vdash \underline{a} \in \underline{b}$$

That that relation is independent of the chosen representatives a, b , of their equivalence classes follows from general facts about forcing established within \mathfrak{N} .

Then $(\Omega, \in_{\mathcal{G}})$ is a perfectly reasonable **countable** set with a two-place relation on it, and we can ask which of the sentences of the language of set theory are true in that model when we interpret $=$ by equality and \in by $\in_{\mathcal{G}}$.

We establish the familiar principle that what is true in this model is what is forced by some member of \mathcal{G} : but the proof of that relies entirely on the fact that \mathcal{G} meets all the necessary dense classes, and makes no use of the well-foundedness of the model under consideration.

The Forcing Theorem in the general case

We wish to prove that

$$(N, S) \models \Phi[(a)_{\mathcal{F}}, (b)_{\mathcal{F}}] \iff \exists p \in \mathcal{F} \ (M, R) \models p \Vdash \Phi[\underline{a}, \underline{b}].$$

11.6 REMARK That notation hints at a conflict of language level. We have $\dot{\Delta}_0$ wffs which are sets, and over the set of which we can quantify; we are using these wffs, when their formal free variables are interpreted by constants, in two contexts; in our current universe, for which we have a truth definition \models^0 and in the generic extension via the definition of forcing $p \Vdash \varphi$.

So really the above principle, for $\dot{\Delta}_0$ wffs, should be written

$$(N, S) \models \models^0 \varphi[(a)_{\mathcal{F}}, (b)_{\mathcal{F}}] \iff \exists p \in \mathcal{F} \ (M, R) \models p \Vdash \varphi[\underline{a}, \underline{b}],$$

and then the apparent conflict of language level will have been resolved.

When unrestricted quantifiers are then “added by hand” there is no further problem.

The principle holds for atomic wffs by definition of the relation S .

For propositional connectives, the induction will advance by general properties of the forcing relation: we are repeatedly saying that there is some dense subset of the forcing in the model, which the generic has to meet.

For quantifiers, the discussion will focus on the dense class of those p for which there is an object in M which names a witness to the existential statement under consideration. Why should the generic meet that class ?

We have two answers: either a sufficient amount of separation is true in the ground model to conclude that the class is actually a set; or we require the generic to meet all dense classes of that definability level.

For forcing over models of **PROV**, we can certainly establish the principle for $\dot{\Delta}_0$ wffs. That might suffice for our purposes; if the model is one of full Zermelo, then we shall get the principle for all wffs.

Once that has been done, we may strengthen the ties between \mathfrak{Q} and \mathfrak{N} , by showing that we may treat \mathfrak{Q} as an extension of \mathfrak{N} by considering the map $x \mapsto [\hat{x}]$; we may also show that \mathcal{G} is in \mathfrak{Q} , being $[\hat{\mathcal{G}}]$. Here \hat{x} is the canonical forcing name for the member x of the ground model, defined recursively inside \mathfrak{N} , (using which we may define a predicate \hat{V} of the forcing language for membership of the ground model) and $\hat{\mathcal{G}}$ is the canonical forcing name for the generic being added.

We show that every name has a unique rank of \mathfrak{N} attached, chosen from all possible ones by the completeness of \mathcal{G} . Those ranks are simply the ordinals of \mathfrak{N} .

So loosely we may say that the extension \mathfrak{Q} is no more ill-founded than is the starting model \mathfrak{N} . Further, \mathfrak{Q} considers itself to be a generic extension of \mathfrak{N} via \mathbf{P} and \mathcal{G} , the corresponding statement about $\hat{\mathbf{P}}$ and $\hat{\mathcal{G}}$ being forced. Hence inside \mathfrak{Q} the recursive definition of $\text{val}_{\mathcal{G}} : \mathfrak{N} \rightarrow \mathfrak{Q}$ by

$$\text{val}_{\mathcal{G}}(b) = \{\text{val}_{\mathcal{G}}(a) \mid \exists p : \in \mathcal{G} \langle p, a \rangle \in b\}.$$

succeeds, using the predicate \hat{V} identifying the members of \mathfrak{N} .

11.7 PROPOSITION $[a]_{\mathcal{G}} = \text{val}_{\mathcal{G}}(a)$.

Proof : we do this by recursion inside the ill-founded model.

⊢ (11.7)

11.8 HISTORICAL NOTE Since Cohen’s creation of forcing as a construction of extensions of models of full ZF, many people have examined the possibility of forcing over models of weaker systems of set theory, to say nothing of those who have transplanted Cohen’s ideas to other areas of enquiry outside set theory. Forcing over admissible sets was studied briefly by Barwise in his 1967 Stanford thesis, at greater length by Jensen in an originally unpublished treatise [J3] on admissibility that contained a proof of his celebrated “sequence-of-admissibles” theorem, in Steel’s 1978 paper [St1], in Sacks’ study [Sa], and in numerous writings of Sy Friedman such as his papers [F1] and [F2], which latter expounds *inter alia* that result of Jensen. In this connection, the referee draws my attention to the expositions of Ershov, (the paper [E1], with a correction following a critical review by Blass; and the book [E2] which discusses forcing over models of KPU) and of Zarach [Z2].

The paper of Hauser [Ha] and the as yet unpublished notes of Steel [St2] contain explorations of forcing over transitive sets which, whilst not required to be admissible, are nevertheless assumed to possess certain fine-structural properties.

A paper of Feferman gave an application of forcing in the context of second order arithmetic; this theme was developed in an expository article of Scott, and in lectures by Jensen at the 1967 UCLA meeting. The referee suggests that a bridge between the work of Feferman and the ideas of this paper might result if it were to be shown, as is indeed the case, that all axioms of PROVI are theorems of the set-theoretic variant $\text{ATR}_0^{\text{set}}$ described in [Si2, §VII.3], of the well-known system ATR_0 . With his permission we report that François Dorais is investigating this question and writes that PROVI (with set-foundation rather than Π_1 foundation) is interpretable in a subsystem of second order arithmetic, between ACA_0 and ATR_0 , which allows arithmetic transfinite recursion along every proper initial segment of ω^2 . The subsystem concerned is slightly stronger than ACA_0^+ , but much weaker than ATR_0 .

Something of the interplay between analysis and set theory is to be seen in a paper of Zarach and an unpublished manuscript of Gandy.

The referee also draws attention to the use of class forcing over admissible sets, which, it is hoped, might form the subject of a further paper. An early paper on class forcing in the context of Morse–Kelley theory is [Ch]. Of the papers of Zarach, [Z1] cites a preprint form of [Ch]. It discusses forcing with classes in the context of ZF-. [Z2] cites [Z1] and [Ch]: it discusses set forcing over admissible sets and certain cases of class forcing. [Z4] cites [Z1], [Z2] and [Z3]; it does both set and class forcing over models of ZF-. [Z3] cites none of the above, but it might be re-read in the light of the theory of rudimentary recursion.

11.9 REMARK A possible line of attack is this: suppose that M is a transitive model of some class theory, so that M has members of all ranks $\leq \lambda$, where λ is a limit ordinal. For example, in the case of Morse–Kelley, M might be $V_{\kappa+1}$ where κ is a strongly inaccessible cardinal. Pass to the provident closure, N of M , as defined in [M4] and [MB]. N will be of height $\lambda\omega$. Now the class forcing one had in mind for M will be a member of N , and therefore we can treat the problem as one of set forcing over the provident set N . The attraction of this approach is that names for members of N can be explicitly laid out, since the ordinals in N are all of the form $\lambda n + \zeta$ where $\zeta < \lambda$.

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