FORMAL SCHEMES AND GROTHENDIECK EXISTENCE

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ABSTRACT. I wrote some notes for myself while preparing for Student Arithmetic Geometry Seminar talk on Fri, Oct 3, 2014. The abstract was "I will give an introduction to formal schemes, state a version of Grothendieck existence theorem, and discuss applications to deformation theory."

CONTENTS

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1. INTRODUCTION

1.1. **References.** [9, Section II.9], [6, Chapter 8], [2, Chapter 7], http://mathoverflow.net/q/47993/15505

1.2. **Conventions.** All rings are commutative rings with unit.

1.3. **Outline.** (INCOMPLETE :)

(1)
(2)
(3)
(4)
(5)

Date: Fri, Oct 3, 2014.

2. Completions

Given a ring A, an ideal $\mathfrak{A}$, and an A-module M, we denote the $\mathfrak{a}$-adic completion of M by $\widehat{M} := \varprojlim M/\mathfrak{a}^n M$.

Theorem 1. [9, II.9.3A] Let A be a Noetherian ring, and $\mathfrak{a}$ an ideal of A.

(a) $\widehat{\mathfrak{a}}$ is an ideal of $\widehat{A}$, and for any n, we have $\widehat{\mathfrak{a}}^n = \widehat{\mathfrak{a}^n} = \mathfrak{a}^n \widehat{A}$ and $\widehat{A}/\widehat{\mathfrak{a}}^n \simeq A/\mathfrak{a}^n$.
(b) If M is a finitely generated A-module, then $\widehat{M} \simeq M \otimes_A \widehat{A}$.
(c) The functor $M \mapsto \widehat{M}$ is an exact functor

$$\{\text{fin. gen. } A\text{-mod}\} \to \{\text{fin. gen. } \widehat{A}\text{-mod}\} .$$

(d) $\widehat{A}$ is a Noetherian ring.
(e) If $(\{M_n\}_{n\geq 1}, \{\varphi_{n,m} : M_m \to M_n\}_{m\geq n})$ is an inverse system, where each M_n is a finitely generated $A/\mathfrak{a}^n$-module, each $\varphi_{n,m}$ is surjective with $\ker \varphi_{n,m} = I^n M_m$, then $M := \varprojlim M_n$ is a finitely generated $\widehat{A}$-module, and for each n, we have $M_n \simeq M/\mathfrak{a}^n M$.

Proof. (a) [1, Proposition 10.15]
(b) [1, Proposition 10.13]
(c) [1, Proposition 10.12] If M is a finitely generated A-module, then $\widehat{M}$ is a finitely generated $\widehat{A}$-module since $\widehat{M} \simeq M \otimes_A \widehat{A}$ by (b).
(d) [1, Theorem 10.26]
(e) (See also Theorem 38.) Note that the conditions imply in particular that, for $m \geq n$, we have $M_n = M_m \otimes_{A/\mathfrak{a}^m} A/\mathfrak{a}^n$.

□

Lemma 2. [1, Proposition 10.14] Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, and $\widehat{A}$ the $\mathfrak{a}$-adic completion of A. Then $A \to \widehat{A}$ is a flat ring homomorphism.

Proof. (Finitely generated) Ideal Criterion for Flatness [11, Lemma (9.26)] □

Proposition 3. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, and $\widehat{A}$ the $\mathfrak{a}$-adic completion of A. Then $\dim A \leq \dim \widehat{A}$.

Proof. Going-Down for flat algebras [11, Theorem (14.11)] and Lemma 2. □

Theorem 4. [1, Corollary 11.19] Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. Then $\dim \widehat{A} = \dim A$.

Question 5. What can we say, other than Proposition 3, about the dimension of the $\mathfrak{p}$-adic completion of A if A not necessarily local or $\mathfrak{p}$ is not necessarily prime? (INCOMPLETE :)

3. Adic rings

Definition 6. [2, Section 7.1] A topological ring A is a ring equipped with a topology such that the addition and multiplication maps $A \times A \to A$ are continuous, when $A \times A$ is given the product topology.

Definition 7. [2, Section 7.1] Let A be a ring, and $\mathfrak{a}$ an ideal. There is a unique topology on A making it a topological ring such that the collection of ideals $\{\mathfrak{a}^n\}_{n\geq 1}$ form a basis of neighborhoods of 0 in A. Namely, a subset $U \subset A$ is open if for each $x \in U$ there exists $n \geq 1$ such that $x + \mathfrak{a}^n \subset U$. The resulting topology is called the $\mathfrak{a}$-adic topology on A. A topological ring A is called an adic ring if its topology is the $\mathfrak{a}$-adic topology for some ideal $\mathfrak{a}$. Any such $\mathfrak{a}$ is called an ideal of definition.

Example 8. An example of a topological ring which is not adic is $A = \mathbb{C}$ with the analytic topology.

Given a topological ring, $\widehat{A}$ is the set of Cauchy sequences in A; it is a ring equipped with a canonical ring homomorphism $A \to \widehat{A}$. We say that A is separated if $A \to \widehat{A}$ is injective,[1] and complete if $A \to \widehat{A}$ is surjective.

If A has the $\mathfrak{a}$-adic topology, then there is a canonical homomorphism of topological rings $\widehat{A} \simeq \varprojlim A/\mathfrak{a}^n$, where each $A/\mathfrak{a}^n$ is given the discrete topology and $\varprojlim A/\mathfrak{a}^n$ has the coarsest topology such that each projection $\varprojlim A/\mathfrak{a}^n \to A/\mathfrak{a}^n$ is continuous.

4. Admissible rings

Definition 9. [2, page 160] We say that a topological ring A is admissible if the following conditions are satisfied:

(1) A is linearly topologized, i.e. there is a basis of neighborhoods $(I_\lambda)_{\lambda\in\Lambda}$ of 0 consisting of ideals of A.
(2) A has an ideal of definition, i.e. there is an open ideal $\mathfrak{a}$ such that for any neighborhood U of 0, there exists $n \geq 0$ such that $\mathfrak{a}^n \subset U$.
(3) A is separated and complete.

5. Definition of Formal schemes

Definition 10. [9, pg. 194], [8, I, Definition (10.8.4)] Let X be a Noetherian scheme, and let $i : Y \to X$ be a closed subscheme with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. For any $n \geq 0$, the ringed space $X_n = (Y, i^{-1}(\mathcal{O}_X/\mathcal{I}^{n+1}))$ is a Noetherian scheme; there are canonical morphisms $X_n \to X$ and $X_n \to X_m$ for $m \geq n$ which makes $\{X_n\}_{n\geq 0}$ a directed system of schemes. We define the formal completion of X along Y, denoted $(\widehat{X}, \mathcal{O}_{\widehat{X}})$, to be the following ringed space. We take $|\widehat{X}| = |Y|$, and on it the sheaf of rings

$$\mathcal{O}_{\widehat{X}} := \varprojlim i^{-1}(\mathcal{O}_X/\mathcal{I}^n) \ . \tag{1}$$

Given a coherent[2] sheaf $\mathscr{F}$ on X, define the completion of $\mathscr{F}$ along Y to be the sheaf

$$\widehat{\mathscr{F}} := \varprojlim i^{-1}(\mathscr{F}/\mathcal{I}^n\mathscr{F}) \ . \tag{2}$$

[1]If A is Noetherian local, then Krull intersection theorem [11, Theorem (18.29)] implies that A is automatically separated.

[2]Defined as in [9, pg. 111]; essentially the same as "of finite type" defined in [8, 0, (5.2.1)]; my impression is that it doesn't matter for Hartshorne because coherent sheaves are discussed usually on Noetherian schemes, where the two notions coincide.

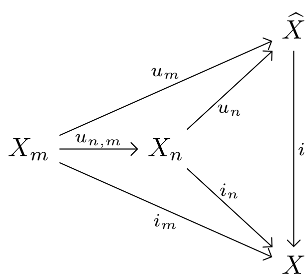
Then $\widehat{\mathscr{F}}$ has a natural structure of an $\mathcal{O}_{\widehat{X}}$-module. If X is an affine scheme, say $X = \operatorname{Spec} A$ with $Y = \operatorname{Spec} A/I$, then we say that $\widehat{X}$ is an affine Noetherian formal scheme, and denote $\widehat{X} = \operatorname{Spf} A$.[3]

Definition 11 ($\widehat{X}$ is topologically ringed space). For any open subset U of $\widehat{X}$, give $(i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(U)$ the discrete topology and $\mathcal{O}_{\widehat{X}}(U)$ the coarsest topology such that each projection $\mathcal{O}_{\widehat{X}}(U) \to (i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(U)$ is continuous. Suppose $V \subset U$ is an inclusion of open subsets of X. The topology on $\mathcal{O}_{\widehat{X}}(V)$ is generated by fibers of the map $\mathcal{O}_{\widehat{X}}(V) \to (i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(V)$ for $n \geq 1$. Thus, to check that the restriction maps $\mathcal{O}_{\widehat{X}}(U) \to \mathcal{O}_{\widehat{X}}(V)$ are continuous ring homomorphisms, it suffices to check that the inverse image under $\mathcal{O}_{\widehat{X}}(U) \to \mathcal{O}_{\widehat{X}}(V)$ of a fiber of $\mathcal{O}_{\widehat{X}}(V) \to (i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(V)$ is open in $\mathcal{O}_{\widehat{X}}(U)$. This is the same as the inverse image of an element of $(i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(V)$ under the map $\mathcal{O}_{\widehat{X}}(U) \to (i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(U) \to (i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(V)$. Thus $\mathcal{O}_{\widehat{X}}$ is a sheaf of topological rings.

Remark 12. Let $X, Y, \mathcal{I}, X_n$ be as in Definition 10. The completion $(\widehat{X}, \mathcal{O}_{\widehat{X}})$ comes equipped with a canonical morphism of locally ringed spaces

$$(i, i^\sharp) : (\widehat{X}, \mathcal{O}_{\widehat{X}}) \to (X, \mathcal{O}_X) \tag{3}$$

where $i : |\widehat{X}| \to |X|$ is the closed immersion and $i^\sharp : i^{-1}\mathcal{O}_X \to \mathcal{O}_{\widehat{X}} = \varprojlim i^{-1}(\mathcal{O}_X/\mathcal{I}^n)$ is the canonical morphism induced by $i^{-1}\mathcal{O}_X \to i^{-1}(\mathcal{O}_X/\mathcal{I}^n)$ for all n.



There are also canonical morphisms of locally ringed spaces $u_n : X_n \to \widehat{X}$ corresponding to the canonical projections $\varprojlim i^{-1}(\mathcal{O}_X/\mathcal{I}^n) \to i^{-1}(\mathcal{O}_X/\mathcal{I}^n)$, and the composition $X_n \to \widehat{X} \to X$ is the canonical morphism $i_n : X_n \to X$.

Remark 13.
(1) If V is an open subset of $\widehat{X}$, then $\mathcal{O}_{\widehat{X}}(V) = \varprojlim((\mathcal{O}_X(U)/\mathcal{I}^n)(U))$ for any open subset U of X such that $U \cap \widehat{X} = V$, by Lemma 47.
(2) If we "complete X along X", i.e. if one takes $i : Y \to X$ to be the closed immersion $\mathrm{id} : X \to X$ in Definition 10 (so that $\mathcal{I} = 0$), then $\widehat{X}$ and X are isomorphic as locally ringed spaces (since i is a homeomorphism and $\varprojlim i^{-1}(\mathcal{O}_X/\mathcal{I}^n) \simeq \varprojlim \mathcal{O}_X \simeq \mathcal{O}_X$).
(3) With $X, \widehat{X}$ as in Definition 10, we have that $|\widehat{X}|$ is a Noetherian topological space, since $|Y|$ is.
(4) Taking the coherent sheaf $\mathscr{F} = \mathcal{O}_X$ in Equation (2), we have $\widehat{\mathcal{O}_X} = \mathcal{O}_{\widehat{X}}$.

Definition 14. [9, pg. 194] A Noetherian formal scheme is a locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ which has a finite open cover $\{\mathfrak{U}_i\}_{i \in I}$ such that, for each i, the pair

[3]Note that, by abuse of notation, there is no mention of the ideal I.

$(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}_i})$ is isomorphic as a locally ringed space to the completion of some Noetherian scheme X_i along a closed subscheme Y_i. A morphism of Noetherian formal schemes is a morphism as locally ringed spaces. A sheaf $\mathfrak{F}$ of $\mathcal{O}_{\mathfrak{X}}$-modules is said to be coherent[4] if there exists a finite open cover $\mathfrak{U}_i \simeq \widehat{X_i}$ as above, and for each i there is a coherent sheaf $\mathscr{F}_i$ on X_i such that $\mathfrak{F}|_{\mathfrak{U}_i} \simeq \widehat{\mathscr{F}_i}$ as $\mathcal{O}_{\widehat{X_i}}$-modules via the given isomorphism $\mathfrak{U}_i \simeq \widehat{X_i}$.

Definition 15. [8, I, Definition (10.4.5)] Let $\mathfrak{X}, \mathfrak{Y}$ be Noetherian formal schemes. A morphism of ringed spaces $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be a morphism of formal schemes if it is a morphism of topologically ringed spaces, and, for each $x \in \mathfrak{X}$, the induced map $\mathcal{O}_{\mathfrak{Y},f(x)} \to \mathcal{O}_{\mathfrak{X},x}$ is a local homomorphism.

Remark 16.
1. Every Noetherian scheme is a Noetherian formal scheme; see Remark 13(2).
2. Let $X, Y, \widehat{X}$ be as in Definition 10. Using Remark 21, we have that $\widehat{X}$ has an open cover by a finite number of affine Noetherian formal schemes. Thus it's equivalent to require in Definition 14 that $\mathfrak{X}$ have a finite open cover by "affine Noetherian formal schemes" (instead of "Noetherian formal schemes").
3. It seems that "affine Noetherian formal scheme" is to "Noetherian formal scheme" as "affine scheme" is to "scheme". (INCOMPLETE : maybe?)

Question 17. Is $\widehat{\mathscr{F}}$ a coherent $\mathcal{O}_{\widehat{X}}$-module? Yes, by [9, Corollary II.9.8]. (INCOMPLETE :)

Example 18. Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A; set $X := \operatorname{Spec} A$ and $Y := \operatorname{Spec} A/\mathfrak{a}$; let $\widehat{A}$ be the $\mathfrak{a}$-adic completion of A. Let $\widehat{X}$ be the completion of X along Y. Then $\widehat{X} = (|\operatorname{Spec} A/\mathfrak{a}|, \widehat{A})$.

1. Let $p \in \mathbb{Z}$ be a prime; set $A = \mathbb{Z}$ and $\mathfrak{a} = (p)$. Then $\widehat{X} = (\{*\}, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-adic integers.
2. With k a field, set $A = k[t]$ and $\mathfrak{a} = (t)$. Then $\widehat{X} = (\{*\}, k[[t]])$, where $k[[t]]$ is the ring of power series in t and coefficients in k.
3. Completion of singular variety at singular point (INCOMPLETE :)
4. Completion of $\mathbb{P}^n_k$ along hyperplane? (INCOMPLETE :)

Proposition 19 (Functoriality for completion along a closed subscheme). [8, I, (10.9.1), (10.9.2)] Let $f : X \to Y$ be a morphism of Noetherian schemes. Let $i : X_0 \to X$ (resp. $j : Y_0 \to Y$) be a closed subscheme of X (resp. Y) such that $f \circ i$ factors through j. Let $\widehat{X}$ (resp. $\widehat{Y}$) be the completion of X (resp. Y) along X_0 (resp. Y_0). Then there exists a canonical morphism of formal schemes $\widehat{f} : \widehat{X} \to \widehat{Y}$ such that the following diagram commutes (where the horizontal arrows are the canonical morphisms described in Remark 12).

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{i_X} & X \\ {\scriptstyle \widehat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widehat{Y} & \xrightarrow[i_Y]{} & Y \end{array}$$

[4][8, 0, (5.3.1)] defines coherent modules differently.

Proof. Use Proposition 50. By assumption, there exists $f_0 : X_0 \to Y_0$ such that $j \circ f_0 = f \circ i$. For each $n > 0$, define $X_n := (|X_0|, i^{-1}(\mathcal{O}_X/\mathcal{I}^{n+1}))$ and $Y_n := (|Y_0|, j^{-1}(\mathcal{O}_Y/\mathcal{J}^{n+1}))$. We show that there are morphisms $f_n : X_n \to Y_n$ such that the following two diagrams commute:

$$\begin{array}{ccc} X_m & \xrightarrow{u_{n,m}} & X_n \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle f_n} \\ Y_m & \xrightarrow[v_{n,m}]{} & Y_n \end{array} \qquad \begin{array}{ccc} X_n & \xrightarrow{i_n} & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f} \\ Y_n & \xrightarrow[j_n]{} & Y \end{array}$$

Fix $n > 0$. Since $f \circ i$ factors through j, by Proposition 50 we have $f^*\mathcal{J} \subset \mathcal{I}$. This implies $f^*(\mathcal{J}^n) = (f^*\mathcal{J})^{n+1} \subset \mathcal{I}^{n+1}$. Since $\mathcal{I}^{n+1}$ and $\mathcal{J}^{n+1}$ are the ideal sheaves of i_n and j_n, we have by Proposition 50 again that $f \circ i_n$ factors through j_n, say $j_n \circ f_n = f \circ i_n$. Commutativity of the left diagram follows from the fact that $f_n \circ u_{n,m}$ factors through $v_{n,m}$ and that j_n is a monomorphism (since it is a closed immersion): namely, if f'_m is the unique morphism such that $v_{n,m} \circ f'_m = f_n \circ u_{n,m}$, then f_m and f'_m are both factorizations of $f \circ i_m$ through j_m. Thus, by UMP of the colimit of $\widehat{X}$, there exists unique morphism of locally ringed spaces $\widehat{f} : \widehat{X} \to \widehat{Y}$ such that $i_Y \circ \widehat{f} = f \circ i_X$. (INCOMPLETE : Why is this a morphism of topologically ringed spaces?) □

We may ask why we do not define $\mathcal{O}_{\widehat{X}}$ to be $i^{-1}(\varprojlim \mathcal{O}_X/\mathcal{I}^n)$; this is addressed by Lemma 20, which says that it doesn't matter.

Lemma 20. The canonical morphism

$$i^{-1}(\varprojlim \mathcal{O}_X/\mathcal{I}^n) \to \varprojlim i^{-1}(\mathcal{O}_X/\mathcal{I}^n) \tag{4}$$

induced by the maps $i^{-1}(\varprojlim \mathcal{O}_X/\mathcal{I}^n) \to i^{-1}(\mathcal{O}_X/\mathcal{I}^n)$ is an isomorphism.

Proof. I don't know whether filtered colimits always commute with inverse limits (indexed by $\mathbb{N}$), but in this case i is a closed immersion of topological spaces and the supports of every $\mathcal{O}_X/\mathcal{I}^n$ and $\varprojlim \mathcal{O}_X/\mathcal{I}^n$ are contained in the image of i. Thus if $U \subset X$ is any open subset, then

$$\begin{aligned} (i^{-1}(\varprojlim \mathcal{O}_X/\mathcal{I}^n))(Y \cap U) &\overset{*}{\simeq} (\varprojlim \mathcal{O}_X/\mathcal{I}^n)(U) \\ &= \varprojlim((\mathcal{O}_X/\mathcal{I}^n)(U)) \\ &\overset{*}{\simeq} \varprojlim((i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(Y \cap U)) \\ &\simeq (\varprojlim i^{-1}(\mathcal{O}_X/\mathcal{I}^n))(Y \cap U) \end{aligned}$$

so Equation (4) is an isomorphism. (In the previous chain of isomorphisms, the isomorphisms marked with $*$ are by Lemma 47.) □

Remark 21. Completion along a closed subscheme commutes with taking open subsets: Let X and Y be as in Definition 10, and let U be a quasicompact[5] open subset of X. Let us denote by $Y \cap U$ the fiber product of $Y \times_X U$; then $\mathcal{J} := \mathcal{I}|_U$ is

[5]This is to ensure that U is also a Noetherian scheme.

the ideal sheaf of the closed immersion $j : Y \cap U \to U$. Let $\widehat{U}$ denote the completion of U along $Y \cap U$. Then

$$\begin{aligned}\mathcal{O}_{\widehat{X}}|_{Y\cap U} &= \left(\varprojlim i^{-1}(\mathcal{O}_X/\mathcal{I}^n)\right)|_{Y\cap U}\\ &= \varprojlim \left((i^{-1}(\mathcal{O}_X/\mathcal{I}^n))|_{Y\cap U}\right)\\ &= \varprojlim j^{-1}(\mathcal{O}_U/\mathcal{J}^n)\\ &= \mathcal{O}_{\widehat{U}}\ .\end{aligned}$$

Since $|Y \cap U| \to |Y|$ is an open immersion of topological spaces, we have an open immersion of locally ringed spaces $(\widehat{U}, \mathcal{O}_{\widehat{U}}) \to (\widehat{X}, \mathcal{O}_{\widehat{X}})$.

Lemma 22. [5] With the definition of Definition 10, the completion $\widehat{X}$ is a locally ringed space.

Proof. We consider the canonical morphism $X_0 \to \widehat{X}$ (which is a homeomorphism on the underlying topological spaces). Given $x \in X_0$, we have an induced map $\varphi : \mathcal{O}_{\widehat{X},x} \to \mathcal{O}_{X_0,x}$. Set $\mathfrak{n} := \varphi^{-1}(\mathfrak{m}_{X_0,x})$. Since $\mathcal{O}_{X_0,x}$ is local with maximal ideal $\mathfrak{m}_{X_0,x}$, it suffices to show that $\mathcal{O}_{\widehat{X},x} \setminus \mathfrak{n}$ consists of units. Let $f \in \mathcal{O}_{\widehat{X},x} \setminus \mathfrak{n}$, its image $\varphi(f)$ is a unit in $\mathcal{O}_{X_0,x}$, so there exists an open affine neighborhood $U = \operatorname{Spec} A \subset X_0$ of x and a regular section $f' \in \mathcal{O}_{X_0}(U) = A/I$ which maps via the canonical morphism $\mathcal{O}_{X_0}(U) \to \mathcal{O}_{X_0,x}$ to $\varphi(f)$. Restricting further, we can assume that f' is a unit in A/I; restricting further, we may assume that there exists some $f'' \in \mathcal{O}_{\widehat{X}}(U)$ which maps to f' under $\mathcal{O}_{\widehat{X}}(U) \to \mathcal{O}_{X_0}(U)$ and to f under the canonical morphism $\mathcal{O}_{\widehat{X}}(U) \to \mathcal{O}_{\widehat{X},x}$. This implies that f'' is a unit, see Proposition 48. Thus f is a unit. □

Question 23 (Conjecture). If $\mathfrak{X}$ is an affine Noetherian formal scheme that is also a scheme, then it is an affine scheme. (INCOMPLETE :) Since $i : |\widehat{X}| \to |X|$ is a closed immersion of topological spaces, it is in particular quasi-compact and quasi-separated, so $i_*\mathcal{O}_{\widehat{X}} = \varprojlim \mathcal{O}_X/\mathcal{I}^n$ is a quasi-coherent $\mathcal{O}_X$-algebra.

Question 24 (Conjecture). With $X, Y, \mathcal{I}, \widehat{X}$ as in Definition 10, suppose that $\widehat{X}$ is a scheme. Then the canonical morphism Equation (3) of locally ringed spaces $\widehat{X} \to X$ is an affine morphism of schemes. Use Question 23, if it is true. (INCOMPLETE :)

Remark 25. By definition, every affine Noetherian formal scheme comes equipped with a morphism of locally ringed spaces into an affine Noetherian scheme.

Question 26. Given an affine Noetherian formal scheme, is there a way to determine if it is a scheme? A partial result is given by Proposition 27. (INCOMPLETE :)

Proposition 27. With $A, \mathfrak{a}, \widehat{A}, X, Y, \widehat{X}$ as in Example 18, if $\dim Y < \dim X$, then $\widehat{X}$ is not an affine scheme.

Proof. We have

$$\dim \widehat{X} = \dim Y < \dim X = \dim A \overset{(1)}{\leq} \dim \widehat{A} \overset{(2)}{=} \dim \Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})$$

where (1) follows from Proposition 3 and (2) follows by definition of $\mathcal{O}_{\widehat{X}}$. □

Lemma 28. With $X, Y, \mathcal{I}, X_n$ as in Definition 10, $(\widehat{X}, \mathcal{O}_{\widehat{X}})$ is the colimit of $\{X_n\}$ in the category of locally ringed spaces.

Proof. Since the underlying topological spaces $|\widehat{X}|$ and $|Y|$ are homeomorphic, it remains to consider the structure sheaves. Then the statement follows from the fact that $\mathcal{O}_{\widehat{X}}$ is defined to be the projective limit of $\mathcal{O}_{X_n} = i^{-1}(\mathcal{O}_X/\mathcal{I}^{n+1})$ in the category of sheaves of abelian groups on Y. □

(INCOMPLETE : define Spf A; where is it defined in EGA?)

Question 29. Are the local rings of Spf A topological rings? (INCOMPLETE :)

Remark 30. If X is a Noetherian scheme and $\mathcal{I}_1$ and $\mathcal{I}_2$ are two ideal sheaves of $\mathcal{O}_X$ such that $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}_1) = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I}_2)$, then the completions defined in Definition 10 are isomorphic. This is because there exist positive integers n_1, n_2 such that $\mathcal{I}_1^{n_1} \subset \mathcal{I}_2$ and $\mathcal{I}_2^{n_2} \subset \mathcal{I}_1$ (work affine-locally and cover X with finitely many affine opens).

Question 31. In which situations is the inverse limit $\varprojlim \mathcal{O}_X/\mathcal{I}^n$ a quasicoherent sheaf on X? Let A be a ring, $\mathfrak{a}$ an ideal, and $f \in A$ any element. Localizing the projection maps $\varprojlim A/\mathfrak{a}^n \to A/\mathfrak{a}^n$ give maps $(\varprojlim A/\mathfrak{a}^n)_f \to (A/\mathfrak{a}^n)_f$, which induces a morphism

$$(\varprojlim A/\mathfrak{a}^n)_f \to \varprojlim A_f/\mathfrak{a}_f^n \tag{5}$$

which may not be an isomorphism. Consider, for example, $A = \mathbb{Z}$ and $\mathfrak{a} = (p)$; then $\varprojlim A/\mathfrak{a}^n = \mathbb{Z}_p$. Let $f \in \mathbb{Z}$ be a positive integer. If p divides f, then $V_A(\mathfrak{a}) \cap D_A(f) = \emptyset$ and $(\varprojlim A/\mathfrak{a}^n)_f = (\mathbb{Z}_p)_f = \mathbb{Q}_p$ and $\varprojlim A_f/\mathfrak{a}_f^n = \varprojlim(\mathbb{Z}/p^n\mathbb{Z})_f = 0$ (so in this case Equation (5) is not an isomorphism). If p does not divide f, then $V_A(\mathfrak{a}) \cap D_A(f) \neq \emptyset$ and $(\varprojlim A/\mathfrak{a}^n)_f = (\mathbb{Z}_p)_f = \mathbb{Z}_p$ and $\varprojlim A_f/\mathfrak{a}_f^n = \varprojlim(\mathbb{Z}/p^n\mathbb{Z})_f = \mathbb{Z}_p$. (INCOMPLETE : According to [2, page 156, Remark 7.1/9; page 160], this is an isomorphism if $\mathfrak{a}$ is finitely generated.) Perhaps we may able to use Theorem 1(b); but this requires that A_f is finitely generated over A, which is true if and only if $\frac{1}{f} \in A_f$ is integral over A.

Lemma 32. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and X a Noetherian A-scheme. There is a commutative diagram of locally ringed spaces, which is not fibered (INCOMPLETE : ?)

6. Properties of Formal Schemes

Proposition 33. [9, II.9.4] Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and $X := \operatorname{Spec} A$, $Y := V(\mathfrak{a})$, and $\mathfrak{X} := \widehat{X} = \operatorname{Spf} A$ (the completion of X along Y).

(a) $\mathfrak{I} := \mathfrak{a}^{\triangle}$ is a sheaf of ideals in $\mathcal{O}_{\mathfrak{X}}$, and for any n, $\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n \simeq (A/\mathfrak{a})^{\sim}$ as sheaves on Y.
(b) if M is a finitely generated A-module, then $M^{\triangle} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathfrak{X}}$.
(c) The functor $M \mapsto M^{\triangle}$ is an exact functor from the category of finitely generated A-modules to the category of coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Definition 34. Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a Noetherian formal scheme. A sheaf of ideals $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ is called an ideal of definition for $\mathfrak{X}$ if $\operatorname{Supp} \mathcal{O}_{\mathfrak{X}}/\mathfrak{I} = |\mathfrak{X}|$ and the locally ringed space $(|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ is a Noetherian scheme.

Remark 35. With $X, \widehat{X}, Y, \mathcal{I}, X_n$ as in Definition 10, we have an isomorphism of sheaves of rings $\mathcal{O}_{\widehat{X}}/\widehat{\mathcal{I}}^{n+1} \simeq \mathcal{O}_X/\mathcal{I}^{n+1}$ on Y; this yields an isomorphism of locally ringed spaces $(\widehat{X}, \mathcal{O}_{\widehat{X}}/\widehat{\mathcal{I}}^{n+1}) \simeq X_n$ (which is a Noetherian scheme) for all $n \geq 0$; thus every $\widehat{\mathcal{I}}^{n+1}$ is an ideal of definition for $\widehat{X}$. (INCOMPLETE : check this)

Proposition 36. [9, II.9.5] Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a Noetherian formal scheme.

(a) If $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are two ideals of definition, then there are integers $m, n > 0$ such that $\mathfrak{I}_2^m \subseteq \mathfrak{I}_1$ and $\mathfrak{I}_1^n \subseteq \mathfrak{I}_2$.
(b) There is a unique largest ideal of definition $\mathfrak{I}$, characterized by the fact that $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ is a reduced scheme. In particular, ideals of definition exist.
(c) If $\mathfrak{I}$ is an ideal of definition, so is $\mathfrak{I}^n$ for any $n > 0$.

Question 37. To what extent can we recover the original scheme X from $\widehat{X}$? [9, II.9.5] says that, given any Noetherian formal scheme $\mathfrak{X}$, we can ideals of definition (which are intrinsic) to come up with a direct system of noetherian schemes of which $\mathfrak{X}$ is the colimit. (INCOMPLETE :)

Theorem 38. [9, II.9.6] Let $\mathfrak{X}$ be a Noetherian formal scheme and let $\mathfrak{I}$ be an ideal of definition. For each $n \geq 0$, we denote $Y_n := (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^{n+1})$, which is a scheme.

(a) If $\mathfrak{F}$ is a coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules, then $\mathscr{F}_n := \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}$ is a coherent sheaf of $\mathcal{O}_{Y_n}$-modules for each $n \geq 0$, and $\mathfrak{F} \simeq \varprojlim \mathscr{F}_n$.
(b) Conversely, suppose $(\{\mathscr{F}_n\}_{n\geq 0}, \{\varphi_{n,m} : \mathscr{F}_m \to \mathscr{F}_n\})$ is an inverse system, where $\mathscr{F}_n$ is a coherent $\mathcal{O}_{Y_n}$-module and $\varphi_{n,m}$ is surjective with $\ker \varphi_{n,m} = \mathfrak{I}^n\mathscr{F}_m$. Then $\mathfrak{F} := \varprojlim \mathscr{F}_n$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module, and for each $n \geq 0$, we have $\mathscr{F}_n \simeq \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}$.

Proof. Uses Theorem 1. The condition in (b) means we have a compatible system of $\mathcal{O}_{Y_n}$-modules; having exact sequences $0 \to \mathfrak{I}^n\mathscr{F}_m \to \mathscr{F}_m \to \mathscr{F}_n \to 0$ means $\mathscr{F}_n = \mathscr{F}_m/\mathfrak{I}^n\mathscr{F}_m \simeq \mathscr{F}_m \otimes_{\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^m} (\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n)$. □

Remark 39. Using Theorem 38 and Remark 35, the pullback of coherent sheaf on $\widehat{X}$ along $X_n \to \widehat{X}$ gives a coherent sheaf on X_n, hence we obtain a functor $\mathrm{Coh}(\widehat{X}) \to \mathrm{Coh}(X_n)$ for every $n \geq 0$.

Theorem 40. [9, II.9.7] Let A be a Noetherian ring, $\mathfrak{a}$ an ideal, and assume that A is $\mathfrak{a}$-adically complete. Let $X := \operatorname{Spec} A$, $Y := V(\mathfrak{a})$, and $\mathfrak{X} = \widehat{X}$. Then the functors $M \mapsto M^{\triangle}$ and $\mathfrak{F} \mapsto \Gamma(\mathfrak{X}, \mathfrak{F})$ are exact, and inverse to each other, on the categories of finitely generated A-modules and coherent $\mathcal{O}_{\mathfrak{X}}$-modules respectively. Thus they establish an equivalence of categories. In particular, every coherent $\mathcal{O}_{\mathfrak{X}}$-module is of the form $M^{\triangle}$ for some M.

Corollary 41. [9, II.9.8] If X is any Noetherian scheme, Y a closed subscheme, and $\mathfrak{X} = \widehat{X}$ the completion along Y, then the functor $\mathscr{F} \mapsto \widehat{\mathscr{F}}$ is an exact functor from coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_{\mathfrak{X}}$-modules. Furthermore, if $\mathcal{I}$ is the sheaf of ideals of Y, and $\widehat{I}$ its completion, then we have $\widehat{\mathscr{F}}/\widehat{\mathcal{I}}^n\widehat{\mathscr{F}} \simeq \mathscr{F}/\mathcal{I}^n\mathscr{F}$ for each n, and $\widehat{\mathscr{F}} \simeq \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathfrak{X}}$.

7. Grothendieck existence theorem

See also [3, Theorem 3.4].

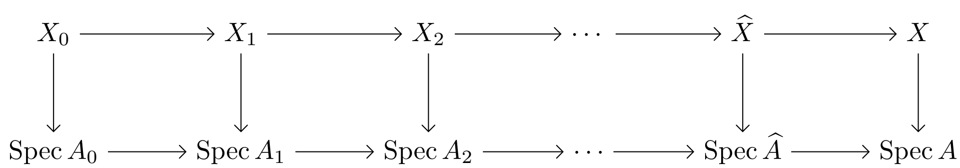
Theorem 42 (Grothendieck existence). [8, III_1, (5.1.4)] Let A be a Noetherian adic ring, $Y := \operatorname{Spec} A$, $\mathfrak{a}$ an ideal of definition of A, $Y' := \operatorname{Spec} A/\mathfrak{a}$, $f : X \to Y$ a morphism separated and of finite type, and $X' := f^{-1}(Y')$. Let $\widehat{X}$[6] (resp. $\widehat{Y} = \operatorname{Spf}(A)$) be the completion of X (resp. Y) along X' (resp. Y'), and $\widehat{f} : \widehat{X} \to \widehat{Y}$ the extension of f to the completions[7]; then, the functor $\mathscr{F} \mapsto \widehat{\mathscr{F}}$[8] is an equivalence of categories between the category of coherent $\mathcal{O}_X$-modules with support proper over Y[9] and the category of $\mathcal{O}_{\widehat{X}}$-modules with support proper over $\widehat{Y}$.

Proof. [8, III_1, Section 5.2 and 5.3; page 151–156] □

Corollary 43. [8, III_1, (5.1.6)] Assuming the conditions of Theorem 42, suppose X is proper over $Y = \operatorname{Spec} A$. Then the functor $\operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X})$ sending $\mathscr{F} \mapsto \widehat{\mathscr{F}}$ is an equivalence of categories.

Proof. Every closed subscheme of X is proper over Y, and every closed subscheme of $\widehat{X}$ is proper over $\widehat{Y}$; apply Theorem 42. (INCOMPLETE : Does it make sense to define properness for morphisms of locally ringed spaces? How to define "of finite type" for morphisms of locally ringed spaces?) (INCOMPLETE : Is it harder to show that the functor is essentially surjective; given a coherent sheaf $\mathfrak{F}$ on $\widehat{X}$, to find some coherent $\mathscr{F}$ on X whose pullback is $\mathfrak{F}$ is to "effectivize" $\mathfrak{F}$.) For "proper over $\widehat{Y}$", look at remark following statement of [6, Theorem 8.4.2]. □

7.1. **Setup.** (From [14]. See also [6, 8.1.4].) Let $(A, \mathfrak{m})$ be a complete Noetherian local ring, I an ideal, and $X \to \operatorname{Spec} A$ a proper A-scheme. For each $n \geq 0$, set $A_n := A/\mathfrak{m}^{n+1}$ and $X_n := X \times_{\operatorname{Spec} A} \operatorname{Spec} A_n$.[10] Let $\widehat{X}$ denote the completion of X along the closed subscheme X_0.



Note that, for any $m > n$, we have $X_n = X_m \times_{\operatorname{Spec} A_m} \operatorname{Spec} A_n$.

Define "$\varprojlim \operatorname{Coh}(X_n)$" to be the category whose objects are collections of data $(\mathscr{F}_n, \iota_n)_{n \geq 0}$ where $\mathscr{F}_n$ is a coherent sheaf on X_n and $\iota_n : \mathscr{F}_{n+1}|_{X_n} \to \mathscr{F}_n$ is an isomorphism.[11] Morphisms $(\mathscr{F}_n, \iota_n)_{n \geq 0} \to (\mathscr{G}_n, \eta_n)_{n \geq 0}$ are collections of morphisms $(\varphi_n)_{n \geq 0}$ where $\varphi_n : \mathscr{F}_n \to \mathscr{G}_n$ is a morphism of coherent sheaves on X_n such that the following diagram commutes:

[6]The notation "$X_{/X'}$" is also used; see [8, I, page 195].
[7]See [8, I, (10.9.1)].
[8]The notation "$\mathscr{F}_{/X'}$" is also used; see [8, I, page 194, Definition (10.8.4)].
[9](INCOMPLETE : "closed subspaces proper over")?
[10]The A_n are local Artinian rings, so the X_n are intuitively "points with fuzz on them".
[11]Here "$\mathscr{F}_{n+1}|_{X_n}$" denotes the pullback of the coherent sheaf $\mathscr{F}_{n+1}$ along the closed immersion $i_{n,n+1} : X_n \to X_{n+1}$, i.e. $i_{n,n+1}^* \mathscr{F}_{n+1}$.

$$\begin{array}{ccc} \mathscr{F}_{n+1}|_{X_n} & \xrightarrow{\iota_n} & \mathscr{F}_n \\ {\scriptstyle \varphi_{n+1}|_{X_n}}\downarrow & & \downarrow{\scriptstyle \varphi_n} \\ \mathscr{G}_{n+1}|_{X_n} & \xrightarrow[\eta_n]{} & \mathscr{G}_n \end{array}$$

(INCOMPLETE :)

There is a functor

$$F : \mathrm{Coh}(X) \to \varprojlim \mathrm{Coh}(X_n) \tag{6}$$

induced by the pullback functors $\mathrm{Coh}(X) \to \mathrm{Coh}(X_n)$, where pullback of coherent on X is coherent on X_n since $X_n \to X$ is a closed immersion of Noetherian schemes. which sends (INCOMPLETE : factors through $\mathrm{Coh}(\widehat{X})$; sends coherents on X to coherents on $\widehat{X}$ by [9, II.9.6]? just a morphism of locally ringed spaces (need noetherian?))

Lemma 44. Let X be a scheme, and $I \subset \Gamma(X, \mathcal{O}_X)$ an ideal. Let $I\mathcal{O}_X$ be the sheaf associated to the presheaf $U \mapsto I \cdot \Gamma(U, \mathcal{O}_X)$. Then $I\mathcal{O}_X$ is quasicoherent. (INCOMPLETE : why is this here?)

(INCOMPLETE : Any idea about the proof?)

8. Applications to deformation theory

Theorem 45. [6, 8.5.19] (SGA1, III 7.3) Let $(A, \mathfrak{m}, k)$ be a complete Noetherian local ring. Let $S = \operatorname{Spec} A$, $s = \operatorname{Spec} k$, and let X_0 be a smooth projective scheme over s satisfying $H^2(X_0, T_{X_0/s}) = 0$. Then there exists a proper and smooth formal scheme $\mathfrak{X}$ over $\widehat{S}$ lifting X_0. If, in addition, X_0 satisfies $H^2(X_0, \mathcal{O}_{X_0}) = 0$, then there exists a smooth projective scheme X over S such that $X_s = X_0$.

Theorem 46. [14] Let $(A, \mathfrak{m})$ be a complete Noetherian local ring, and set $A_n := A/\mathfrak{m}^{n+1}$ for $n \geq 0$. Let r be a positive integer. Suppose given for every $n \geq 0$ a closed subscheme $i_n : Z_n \to \mathbb{P}^r_{A_n}$ flat over $\operatorname{Spec} A_n$ such that $Z_{n+1} \times_{\operatorname{Spec} A_{n+1}} \operatorname{Spec} A_n \to \mathbb{P}^r_{A_n}$ is isomorphic to i_n for all $n \geq 0$. Then there exists a unique closed subscheme $Z \to \mathbb{P}^r_A$ inducing the Z_n.

$$\begin{array}{ccccccccc} Z_0 & & Z_1 & & Z_2 & & & & Z \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & & & \vdots{\scriptstyle i} \\ \mathbb{P}^r_{A_0} & \longrightarrow & \mathbb{P}^r_{A_1} & \longrightarrow & \mathbb{P}^r_{A_2} & \longrightarrow & \cdots & \longrightarrow & \mathbb{P}^r_A \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \operatorname{Spec} A_0 & \longrightarrow & \operatorname{Spec} A_1 & \longrightarrow & \operatorname{Spec} A_2 & \longrightarrow & \cdots & \longrightarrow & \operatorname{Spec} A \end{array}$$

In Theorem 46, we have

$$Z_{n+1} \times_{\operatorname{Spec} A_{n+1}} \operatorname{Spec} A_n \simeq Z_{n+1} \times_{\mathbb{P}^r_{A_{n+1}}} \mathbb{P}^r_{A_n}$$

because $\mathbb{P}^r_{A_n} \simeq \operatorname{Spec} A_n \times_{\operatorname{Spec} A_{n+1}} \mathbb{P}^r_{A_{n+1}}$. For this reason, it also follows that the projection $Z_{n+1} \times_{\operatorname{Spec} A_{n+1}} \operatorname{Spec} A_n \to \mathbb{P}^r_{A_n}$ is a closed immersion, since it is the base-change of a closed immersion (namely, i_{n+1}).

Proof. Set $X = \mathbb{P}^r_A$ and $X_n = \mathbb{P}^r_{A_n}$. Apply Corollary 43 to the ideal sheaves defining the i_n, □

(INCOMPLETE : Why do we need flatness in the above theorem?)

(INCOMPLETE : What is the deformation problem that this solves? See [12, Warning 6.1.17] and [15, Warning 2.2.3].)

(INCOMPLETE : See also Example 1.2.8(ii) in https://www.uni-due.de/~mat903/sem/ws0809/material/Minicourse_FormalGeometry.pdf for the notation $A\{T\}$.)

9. Appendix

Lemma 47. Let $i : Z \to X$ be a closed immersion of topological spaces, and let $\mathscr{F}$ be a sheaf of abelian groups on X which has support contained in $i(Z)$. If $V \subset U$ are nested open subsets of X such that $V \cap Z = U \cap Z$, then the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is an isomorphism. Thus, if $Y \subset Z$ is any open subset of Z, then $(i^{-1}\mathscr{F})(Y) = \mathscr{F}(U)$ for any open subset U of X such that $Y = U \cap Z$.

Proof. We have open cover $U = V \cup (U \setminus Z)$ of U where $V \cap (U \setminus Z) = V \setminus Z$. Then $\mathscr{F}(U \setminus Z) = 0$ and $\mathscr{F}(V \setminus Z) = 0$. Using the sheaf axioms, we have an exact sequence

$$0 \to \mathscr{F}(U) \to \mathscr{F}(V) \oplus \mathscr{F}(U \setminus Z) \to \mathscr{F}(V \setminus Z)$$

which reduces to the exact sequence $0 \to \mathscr{F}(U) \to \mathscr{F}(V) \to 0$. □

Proposition 48. Let A be a ring, $\mathfrak{a}$ an ideal, and $\widehat{A}$ the $\mathfrak{a}$-adic completion of A. Then the canonical morphism $\widehat{A} \to A/\mathfrak{a}$ sends nonunits to nonunits.[12]

Theorem 49. [7, Theorem 8] The fiber product of schemes in the category of locally ringed spaces is a scheme.

Proposition 50. [8, I, (4.4.6)] Let $f : X \to Y$ be a morphism of schemes, $i : X \to X'$ (resp. $j : Y \to Y'$) a closed subscheme of X (resp. Y) with ideal sheaf $\mathcal{I}$ (resp. $\mathcal{J}$). Then the following are equivalent:

(i) $f \circ i$ factors through j.
(ii) $(f^*\mathcal{J})\mathcal{O}_X \subset \mathcal{I}$.

[12] Inspired by Emerton's claim in http://mathoverflow.net/a/27780/15505.

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