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Supergravity

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Abstract: These are the unofficial notes for the Part III course given in Easter term 2009 in DAMTP, the University of Cambridge. The (official) books and references for this course are:

- P. van Nieuwenhuizen, Physics Report **68** (1981) 189;
- P. van Nieuwenhuizen in *Supergravity '81*, CUP, eds. Ferrara and Taylor;
- P. van Nieuwenhuizen in *Superstrings and Supergravity*, eds. Davies and Sutherland;
- D.Z. Freedman, B. de Witt in *Supersymmetry*, eds. Dietz et al.

Some examples given in the introduction section are omitted from these notes, but this should not affect understanding of subsequent material. Please direct any spotted errors, suggestions etc. to z.hu@damtp.cam.ac.uk.

Contents

1. Introduction

1.1 Conventions

Our conventions is mostly that of Hawking-Ellis and Misner-Thorne-Wheeler. For the metric tensor in general relativity, we will use the "west coast" signature $(-,+,+,+)$. The covariant derivative is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_\mu{}^\nu{}_\sigma V^\sigma.$$

Note our rather unconventional placement of indices for the connection coefficients: the leftmost index is always the differentiation index. We do not assume that the connection is symmetric, and hence the torsion may not vanish:

$$\Gamma_\mu{}^\nu{}_\sigma \neq \Gamma_\sigma{}^\nu{}_\mu, \qquad T_\mu{}^\nu{}_\sigma = \Gamma_\mu{}^\nu{}_\sigma - \Gamma_\sigma{}^\nu{}_\mu \neq 0.$$

The Riemann curvature tensor is

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_\nu{}^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\alpha \Gamma_\nu{}^\alpha{}_\sigma - \partial_\nu \Gamma_\mu{}^\rho{}_\sigma - \Gamma_\nu{}^\rho{}_\alpha \Gamma_\mu{}^\alpha{}_\sigma,$$

the Ricci tensor and Ricci scalar are

$$R_{\sigma\nu} = R^\lambda{}_{\sigma\lambda\nu}, \qquad R = R^\sigma{}_\sigma = R^{\lambda\sigma}{}_{\lambda\sigma},$$

and the Einstein field equation is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

In places we will be replacing Newton's constant by

$$\kappa^2 = \frac{8\pi G\hbar}{c^3}.$$

For spinors, our gamma matrices satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

where the Minkowski metric is $\eta = \mathrm{diag}(-1,1,1,1)$. To translate from the convention of some other books: $\gamma^\mu \to -i\gamma^\mu$.

1.2 Dimensions and Planck units

Dimensional analysis is a useful tool for working out some equations and coefficients in a physical theory, and we will use it in several places throughout the course. Here we will give a brief overview and establish notation.

The effects of quantum gravity becomes important when half of the Schwarz radius is comparable to the Compton radius

$$\frac{Gm_p}{c^2} = \frac{\hbar}{m_p c}.$$

From this we can define the Planck mass, length and time

$$
\begin{aligned}
m_p &= \sqrt{\frac{\hbar c}{G}} & &\simeq 10^{-5}\,\text{grams} \simeq 10^{19}\,\text{GeV} \\
l_p &= \sqrt{\frac{G\hbar}{c^3}} = \sqrt{\frac{\kappa^2}{8\pi}} & &\simeq 10^{-33}\,\text{cm} \\
t_p &= \sqrt{\frac{G\hbar}{c^5}} = \sqrt{\frac{\kappa^2}{8\pi c^2}} & &\simeq 10^{-44}\,\text{seconds}
\end{aligned}
$$

i.e. huge mass (compared to, say, a proton) and very small distance and intervals of time. The units of relevant quantities are

$$
\begin{aligned}
[\kappa] &= L & &\text{length} \\
[x^\mu] &= L & & \\
[R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}] &= L^{-2} & & \\
[T_{\mu\nu}] &= MT^{-2}L^{-1} & &\text{energy per unit volume} \\
[c^4/8\pi G] &= MLT^{-2} & &\text{tension.}
\end{aligned}
$$

In this course, we will set $\hbar = c = 1$.

1.3 Motivation for supergravity

Supergravity (SUGRA) is an extension of Einstein's general relativity to include supersymmetry (SUSY). General relativity demands extensions since it has shortcomings including at least the following:

- Spacetime singularities. The singularity theorems of Penrose, Hawking and Geroch shows that general relativity is incomplete: it predicts its own demise.

- Failure to unify gravity with the strong and electroweak forces. In the Einstein equation, the left hand side, i.e. spacetime geometry, is "a house of marbles", whereas the right hand side, i.e. matter fields, is "lowly hovel". Historically, Kaluza-Klein theory addressed this problem. However, it did not give realistic predictions.

- Incompatibility with quantum mechanics. Conceptually, the role of time in general relativity is very different from its role in quantum theory. If we think of the relativistic "time" as an operator, its unitarity, which is required in a consistent quantum theory, is not obvious. A Hilbert space based on curved geometry is difficult to define. Computationally, pure quantum gravity theory is not thought to be renormalizable and hence has little predictive power.

If we include supersymmetry in a theory of gravity, the situation becomes a little bit better, since the simplest example of divergences: zero point energy of the vacuum, can potentially be cancelled by super partners of ordinary particles.

1.4 Supersymmetry

We here give a very brief overview of supersymmetry, one of the ingredients of supergravity. Quantum mechanically, a supersymmetric theory is a theory in which the Hilbert space can be written as a direct sum

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$$

and there exists self-adjoint operators $Q_i = Q_i^\dagger$, $i = 1, 2, \ldots, 4\mathcal{N}$ acting on $\mathcal{H}$, which satisfy

$$\{Q_i, Q_j\} = \delta_{ij} H, \qquad [Q_i, H] = 0, \qquad Q_i \mathcal{H}_B \subseteq \mathcal{H}_F, \qquad Q_i \mathcal{H}_F \subseteq \mathcal{H}_B$$

where H is a certain hamiltonian operator. In a relativistic theory, the operators Q_i carry angular momentum $\pm\frac{1}{2}$. If a state $|\phi\rangle$ has spin s, then $Q_i|\phi\rangle$ has spin $s \pm \frac{1}{2}$. States fall into supermultiplets with respect to actions of these operators.

The energy expectation value of a state with superpartners can be calculated

$$\langle\psi|H|\psi\rangle = 2\langle\psi|Q_i^\dagger Q_i|\psi\rangle \geq 0.$$

So in a supersymmetric theory, the energy is always non-negative.

The simplest supersymmetric theory is the case $\mathcal{N} = \frac{1}{2}$. If we define

$$a = \frac{1}{\sqrt{2}}(Q_1 + iQ_2), \qquad a^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2)$$

we recover the creation and annihilation operator relations for the harmonic oscillator

$$[a, a] = 0, \qquad \{a, a^\dagger\} = H.$$

In this case, we have a single multiplet with 2 states.

In general, the multiplets are of dimension $4^{\mathcal{N}}$. These are called the long multiplets

$$\{Q_\alpha, Q_\beta\} = -C\gamma^\mu_{\alpha\beta} P_\mu, \qquad P^0 = H, \qquad C = \gamma_0, \qquad \alpha, \beta = 1, \ldots, 4.$$

For example, it is easy to construct a theory in which

$$\begin{aligned}
\{Q_1, Q_1\} &= H + P_1 \\
\{Q_2, Q_2\} &= H - P_1 \\
\{Q_3, Q_3\} &= H + P_1 \\
\{Q_4, Q_4\} &= H - P_1
\end{aligned}$$

where $H \geq |\mathbf{P}|$. For the special case where states are lightlike $H = |\mathbf{P}| = |P_1|$, half of the states will vanish, and we are left with a short multiplet.

1.5 The current status of supergravity

Currently, supergravity is generally thought of as

- a reliable approximation to M-theory at low energy;
- a valuable technical tool (e.g. Witten's proof of the positive energy theorem);
- an essential ingredient for supersymmetric phenomenology (minimal supersymmetric standard model coupled to $\mathcal{N} = 1$ supergravity);
- an essential ingredient for the AdS/CFT correspondence of Maldacena.

2. General relativity and the action principle

2.1 Moving frames

To define spin structure on spacetime, we will need to formulate general relativity in the moving frame language. Let

$$e^a = e^a{}_\mu(x) dx^\mu$$

be a basis of 1-forms. The last part of the Greek alphabet denotes world indices, i.e. of local coordinates, whereas the first part of the Latin alphabet denotes tangent space indices, i.e. of moving frames. Then

$$e_a = e^\mu{}_a(x) \tfrac{\partial}{\partial x^\mu}$$

form a basis of vector fields orthogonal to the basis of 1-forms: $e^a(e_b) = \delta^a{}_b$. In coordinates

$$e^a{}_\mu(x) e^\mu{}_b(x) = \delta^a{}_b.$$

We can think of $e^a{}_\mu$ as components of the matrix e. Then $e^\mu{}_a$ are just the components of the inverse matrix e^{-1}. To carry this analogy with matrices further, we will always write the upstairs index first, even though it really does not matter. If in an expression the contraction is not between adjacent indices, a matrix transpose is technically needed.

If we contract the tangent space index instead of the world index, we get

$$e^\mu{}_a e^a{}_\nu = \delta^\mu{}_\nu.$$

We can exchange world index for tangent space index, i.e. translating from a holonomic frame to a moving frame. For example, in the case of a vector,

$$V^\mu \to V^a = e^a{}_\mu V^\mu.$$

For our basis, the metric of the moving frame is pseudo-orthonormal

$$g_{ab} = e^\mu{}_a g_{\mu\nu} e^\nu{}_b = \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1), \qquad g_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu.$$

Tangent space index can be raised and lowered with η_{ab} and η^{ab}.

The volume form on a manifold is $\eta = \sqrt{-g}\, d^4x$ where $g = \det g_{\mu\nu}$. Since $-\det g_{\mu\nu} = -(\det e)^2$, we have

$$\begin{aligned}
\eta &= |e|\, d^4x \\
&= e^0 \wedge e^1 \wedge e^2 \wedge e^3 \\
&= e^0{}_\mu e^1{}_\nu e^2{}_\rho e^3{}_\sigma \, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma.
\end{aligned}$$

The whole expression can be checked using the definition of the determinant and the fact that $dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \epsilon^{\mu\nu\rho\sigma}$. Our convention is that $\epsilon^{0123} = 1$. Note that our wedge product includes the appropriate normalization factor.

2.2 Connection 1-forms

Let ∇ be a metric compatible connection, possibly with torsion. Acting on the basis,

$$\nabla e^a = \omega^a{}_b \otimes e^b, \qquad \omega^a{}_b = e^c \Gamma_c{}^a{}_b = \omega_\mu{}^a{}_b \, dx^\mu.$$

$\Gamma_c{}^a{}_b$ is called the Ricci rotation coefficients (old-fashioned) and $\omega^a{}_b$ the spin connection. Metric compatibility implies $\omega_{ab} = -\omega_{ba}$, and in this case,

$$\begin{aligned}
\nabla e^a &= \omega_\mu{}^a{}_b e^b{}_\nu dx^\mu \otimes dx^\nu \\
&= \tfrac{1}{2}(\omega_\mu{}^a{}_b e^b{}_\nu dx^\mu \otimes dx^\nu - e^b{}_\nu \omega_{\mu b}{}^a dx^\nu \otimes dx^\mu) \\
&= \omega^a{}_b \wedge e^b.
\end{aligned}$$

The covariant derivative acting on a vector $V = V^a e_a$ gives

$$(\nabla V)_\mu{}^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b.$$

To write this in a more compact way, we think of $\partial_\mu V^a$ as a vector-valued 1-form

$$\partial_\mu V^a \tfrac{\partial}{\partial x^\mu} \otimes e_a = dV^a \otimes e_a,$$

then

$$\nabla V^a = dV^a + \omega^a{}_b V^b.$$

Cartan's first structural equation is

$$de^a + \omega^a{}_b \wedge e^b = T^a, \qquad T^c = T_a{}^c{}_b e^a \wedge e^b$$

where the torsion form T^c is a vector-valued 2-form. This is derived as follows: expand $\nabla e^a = \omega^a{}_b \otimes e^b$ in coordinate basis

$$\partial_\mu e^a{}_\nu - \Gamma_\mu{}^\nu{}_\sigma e^a{}_\sigma - \omega_\mu{}^a{}_b e^b{}_\nu = 0$$

and antisymmetrise. We see that the our connection decomposes

$$\omega_\mu{}^a{}_b = \omega'{}_\mu{}^a{}_b(e) + K_\mu{}^a{}_b$$

where ω' denotes the Levi-Civita connection in moving frame, and K is the contorsion

$$K_\alpha{}^\mu{}_\beta = -\tfrac{1}{2}(T_\alpha{}^\mu{}_\beta + T^\mu{}_{\alpha\beta} + T^\mu{}_{\beta\alpha}).$$

In a holonomic frame, this is

$$\Gamma_\alpha{}^\mu{}_\beta = \{_\alpha{}^\mu{}_\beta\} + K_\alpha{}^\mu{}_\beta.$$

For a function f,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = -T_\mu{}^\sigma{}_v \partial_\sigma f.$$

The Riemann tensor for this connection is

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_\nu{}^\rho{}_\sigma + \Gamma_\mu{}^\rho{}_\alpha \Gamma_\nu{}^\alpha{}_\sigma - \partial_\nu \Gamma_\mu{}^\rho{}_\sigma - \Gamma_\nu{}^\rho{}_\alpha \Gamma_\mu{}^\alpha{}_\sigma.$$

We think of it as a 2-form valued matrix

$$R^{\mu}{}_{\nu} = R^{\mu}{}_{\nu\lambda\rho}\, dx^{\lambda} \wedge dx^{\rho}$$

and convert indices using frames

$$R^{a}{}_{b} = e^{a}{}_{\mu} e^{b}{}_{\nu} R^{\mu}{}_{\nu\lambda\rho}\, dx^{\lambda} \wedge dx^{\rho}$$

then, it easily follows

$$R^{a}{}_{b} = d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}.$$

The important thing to note is that $R^{a}{}_{b}$ transforms homogeneously under local Lorentz transformation (or "tetrad rotation") $\Lambda \in SO(3,1)$, while ω does not. Indeed,

$$\begin{aligned}
\tilde{e}^{a} &= \Lambda^{a}{}_{b} e^{b} \\
d\tilde{e} &= \Lambda\, de + d\Lambda \wedge e = -\Lambda\omega \wedge e + d\Lambda\, \Lambda^{-1} \wedge \tilde{e} \\
\tilde{\omega} &= \Lambda\omega\Lambda^{-1} + d\Lambda\, \Lambda^{-1} \\
\tilde{R}^{a}{}_{b} &= \Lambda^{a}{}_{c} R^{c}{}_{d} (\Lambda^{-1})^{d}{}_{b}.
\end{aligned}$$

The Ricci tensor need not be symmetric in our theory: $R_{\mu\nu} \neq R_{\nu\mu}$. We can think of it as a vector-valued 1-form $R^{a}{}_{\mu} dx^{\mu}$. The Ricci scalar is then $R = g^{\mu\nu} R_{\mu\nu} = e_{a}{}^{\mu} R^{a}{}_{\mu}(\omega)$.

2.3 Poincaré gauge theory

To make the bundle structure clear and to ease our subsequent introduction of spin structure, we think of gravity as gauge symmetries. The Poincaré group, i.e. the local symmetry group of general relativity, is the pseudo-Euclidean group $\mathbb{E}^{3,1} = O(3,1) \ltimes \mathbb{R}^{4}$, a semidirect product of rotations with translations. We can write its action on spacetime coordinates in matrix notation

$$\begin{pmatrix} \Lambda^{a}{}_{b} & a^{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{b} \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda^{a}{}_{b} x^{b} + a^{b} \\ 1 \end{pmatrix}.$$

Its Lie algebra is $\mathfrak{so}(3,1) \ltimes \mathbb{R}^{4}$. The Lie algebra of $\mathbb{R}^{4}$ is still itself, while for $O(3,1)$ it is given by

$$\Lambda^{a}{}_{b} = \delta^{a}{}_{b} + \lambda^{a}{}_{b} + \ldots, \qquad \lambda_{ab} = -\lambda_{ba}.$$

So in this gauge theory, infinitesimal gauge transformations are generated by translations and frame rotations. In any gauge field theory, the gauge field takes values in the Lie algebra of the group. So we can write a field as

$$\Phi = \begin{pmatrix} \lambda^{a}{}_{b} & a^{a} \\ 0 & 0 \end{pmatrix}$$

(c.f. the Higgs field). A covariant derivative is needed to make the gauge symmetry a local one. Following Cartan, a covariant derivative (connection) in this case is just a Lie algebra-valued 1-form,

$$\mathcal{A} = \begin{pmatrix} \omega^{a}{}_{b} & e^{a} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \omega_{\mu}{}^{a}{}_{b}\, dx^{\mu} & e^{a}{}_{\mu}\, dx^{\mu} \\ 0 & 0 \end{pmatrix}.$$

This is the simplest example of a Cartan connection. Its meaning is this: for a function f, df is "the change in f under infinitesimal displacement". But an "infinitesimal displacement" can at best be described by giving a tangent vector, therefore df is a 1-form acting on tangent vectors to give real numbers. In a similar way, this connection we have constructed measures how the Lie algebra relevant for the gauge theory changes under "infinitesimal displacement".

We can now calculate the curvature for this connection, a Lie algebra valued 2-form:

$$\begin{aligned}\mathcal{F} &= d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} \\ &= \begin{pmatrix} d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b & de^a + \omega^a{}_b \wedge e^b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R^a{}_b & T^a \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

This makes sense: T^a is a vector-valued 2-form, while $R^a{}_b$ takes value in $\mathfrak{so}(3,1)$. The torsion T^a is nothing but the "curvature for translation".

We introduce a connection symbol D for this gauge theory:

$$De^a = de^a + \omega^a{}_b \wedge e^b = T^a, \qquad D\eta_{ab} = 0, \qquad D\epsilon^{abcd} = 0.$$

Note that this does not contradict our previous result $\nabla e^a = \omega^a{}_b \wedge e^b$: ∇ is a connection on the tangent bundle whereas D is the connection for the gauge field.

To summarise, we have introduced a 4-plane bundle (fibres are isomorphic to $\mathbb{R}^4$ as vector spaces) E over our manifold M, with fibre trivialization $\{V^a\}$ and equipped with the fibre metric η_{ab}. The principal bundle P with fibres isomorphic to the Poincaré group acts on E in an affine manner. The frame fields e^a provides a local unnatural isomorphism between TM and E. The frame fields e^a are called the soldering form, and in general $de^a \neq 0$. Our point of view has changed: we started by saying that e^a are 1-forms, so at a point p they belong to T_p^*M, but now they become members of $\mathrm{Hom}(T_pM, \mathbb{R}^4)$. In a similar fashion, $e_a(p) \in \mathrm{Hom}(T_p^*M, (\mathbb{R}^4)^*)$.

2.4 Action principles

To derive the field equations in general relativity using the action principle, after we have written down the metric, we vary the metric and its first derivative

$$S = S(g_{\mu\nu}, g_{\mu\nu,\lambda}).$$

This is called the second order metric formalism. An equivalent procedure, the first order or Palatini procedure, is varying the metric and the connection independently

$$S = S(g_{\mu\nu}, \Gamma_\mu{}^\nu{}_\lambda).$$

Translating into moving frame language, we have

$$\begin{aligned} S &= S(e^a{}_\mu, e^a{}_{\mu,\lambda}) && \text{second order,} \\ S &= S(e^a{}_\mu, \omega_\mu{}^a{}_b) && \text{first order.} \end{aligned}$$

For example, the second order action for gravity with cosmological constant is

$$S = \int_M \sqrt{-g}\, d^4x \, \frac{R - 2\Lambda}{16\pi G} + \ldots$$

where the dots represent boundary terms. A first order action in terms of frame fields is

$$S = \frac{1}{2\kappa^2} \int_M |e| d^4x \, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega).$$

Let us vary this action. We need

$$\delta e^{-1} = -e^{-1} \delta e \, e^{-1}, \qquad \delta |e| = e^{\mu}{}_{a} \, \delta e^{a}{}_{\mu} = \mathrm{Tr}(e^{-1} \delta e)$$

then

$$\delta S(e, \omega) = -\frac{1}{\kappa^2} \int |e| d^4x \, G_a{}^{\mu} \delta e^{a}{}_{\mu} + \frac{1}{2\kappa^2} \int |e| d^4x \, e^{a\mu} e^{b\nu} \delta R_{ab\mu\nu}(\omega).$$

We see immediately from the first integral (variation with respect to $e^{a}{}_{\mu}$) the Einstein equation in vacuum $G_a{}^{\mu} = 0$. The second term, variation with respect to ω, should give us relations between the connection and the metric (or the soldering form).

To proceed, we adopt the Poincaré gauge theory point of view: $\omega^{a}{}_{b}$ and $R^{a}{}_{b}$ take values in $\Lambda^2(E) = \mathfrak{so}(3,1)$ and η_{ab} provides an isomorphism between E and E^*. The exterior product $e^a \wedge e^b$ is a Λ^2-valued 2-form. We need to do some algebra. First,

$$\epsilon^{abcd} A_{ab} \wedge B_{cd} = \mathrm{Tr}(A \wedge B) = \mathrm{Tr}(B \wedge A)$$

therefore

$$\begin{aligned} \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} &= \mathrm{Tr}(e \wedge e \wedge R) \\ \mathrm{Tr}(e \wedge e \wedge e \wedge e) &= -24 |e| d^4x \\ -2|e| R \, d^4x &= \mathrm{Tr}(e \wedge e \wedge R). \end{aligned}$$

To verify the second expression above, use $e^a \wedge e^b \wedge e^c \wedge e^d = |e| d^4x \, \epsilon^{abcd}$ and $\epsilon^{abcd} \epsilon_{abef} = -2(\delta^{c}{}_{e} \delta^{d}{}_{f} - \delta^{c}{}_{f} \delta^{d}{}_{e})$ and further contractions. To verify the third expression above, use $\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} = \frac{1}{2} \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}{}_{ef} e^e \wedge e^f$. Also note that, by expanding R,

$$\delta R = d\delta\omega^{a}{}_{b} + \delta\omega^{a}{}_{c} \wedge \omega^{c}{}_{b} + \omega^{a}{}_{c} \wedge \delta\omega^{c}{}_{b} = D\delta\omega.$$

Putting everything together, we have

$$\begin{aligned} \delta \int \mathrm{Tr}(e \wedge e \wedge R) &= \int \mathrm{Tr}(e \wedge e \wedge \delta R) \\ &= \int d\, \mathrm{Tr}(e \wedge e \wedge \delta\omega) - 2 \int \mathrm{Tr}(De \wedge e \wedge \delta\omega) \\ &\doteq -2 \int \mathrm{Tr}(T \wedge e \wedge \delta\omega) \end{aligned}$$

where in the last line we have thrown away a boundary term having no effect on the equations of motion (the sign "$\doteq$" will be used to denote equality up to boundary terms). Now define a tensor $\mathcal{V}$ by $\delta\omega^{cd} = e^e \mathcal{V}_e{}^{cd}$, and

$$\begin{aligned}
\mathrm{Tr}(T \wedge e \wedge \delta\omega) &= \tfrac{1}{2}\epsilon_{abcd} T_r{}^a{}_s e^r \wedge e^s \wedge e^b \wedge e^e \mathcal{V}_e{}^{cd} \\
&= \tfrac{1}{2}\epsilon_{abcd}\epsilon^{rsbe} T_r{}^a{}_s \mathcal{V}_e{}^{cd} |e| d^4x \\
&= (\delta^r{}_a \delta^s{}_c \delta^e{}_d + \text{cyclic permutations}) T_r{}^a{}_s \mathcal{V}_e{}^{cd} |e| d^4x \\
&= (T_a{}^a{}_c \delta^e{}_d + T_d{}^a{}_a \delta^e{}_c + T_c{}^e{}_d) \mathcal{V}_e{}^{cd} |e| d^4x.
\end{aligned}$$

The expression in the parentheses in the last line must vanish. After some further algebra, this is equivalent to $T^a = 0$, our equation of motion. Hence, if $T^a = 0$ by assumption, then $\delta\omega$ is a total derivative in second order formalism and we are only left with the Einstein equation. Therefore, no torsion can be present in vacuum.

Now we add to our action a matter piece $S_m(e, \omega, \psi)$ where ψ represents the matter fields. Variation gives

$$\delta S_m = \int |e| d^4x \, T_a{}^\mu \delta e^a{}_\mu + \int |e| d^4x \, S_c{}^e{}_d \mathcal{V}_e{}^{cd}$$

where $T_a{}^\mu$ is the canonical (unsymmetrized) energy momentum tensor: $T_{\mu\nu} \neq T_{\nu\mu}$ in general, and $S_c{}^e{}_d$ denotes the spin current. Besides the Einstein equation coupling the energy momentum tensor to the Einstein tensor $G_a{}^\mu = \kappa^2 T_a{}^\mu$, we also have the following equation of motion:

$$T_a{}^a{}_c \delta^e{}_d + T_d{}^a{}_a \delta^e{}_c + T_c{}^e{}_d = 2\kappa^2 S_c{}^e{}_d.$$

So if matter lagrangian contains ω explicitly, spin is a source of torsion.

Usually, the spin current vanishes for bosonic fields. For example, the lagrangian for scalar fields

$$\mathcal{L} = -\tfrac{1}{2}\sqrt{g}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

has no spin current, neither have the Maxwell lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (\text{i.e. } F = dA).$$

However, in this case a little care is needed: if we carry out the procedure of "minimal coupling" advertised in introductory general relativity courses "$\partial \to \nabla$", we must make sure that we use the Levi-Civita connection uniquely determined by the metric. Otherwise, torsion comes into play, and in general

$$F_{\mu\nu}^{\text{wrong}} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \neq \partial_\mu A_\nu - \partial_\nu A_\mu.$$

For a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\Lambda = -T_\mu{}^\sigma{}_\nu \partial_\sigma \Lambda \neq 0,$$

hence charge conservation is broken. In this case, we are just unnecessarily asking for trouble, since the exterior derivative is perfectly well-defined on curved spacetime and is coordinate-independent.

2.5 The 1.5 formalism

The 1.5 formalism is the following: for an action S, we have

$$\delta S = \frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta \omega}\delta \omega \qquad \text{first order formalism;}$$
$$\delta S = \frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta \omega}\frac{\delta \omega}{\delta e}\delta e \qquad \text{second order formalism.}$$

However, we can think of $\omega = \omega(e)$ as defined by $\delta S/\delta\omega = 0$, then the second term in the second order formalism can be ignored and we are effectively "1.5".

3. Spinors and the Dirac equation

3.1 Clifford algebra and Majorana spinors

To describe fermionic fields we must first define spinors. We will label components of the gamma matrices using the first part of the Greek alphabet, e.g. $\alpha, \beta = 1, 2, 3, 4$ (the other part of the Greek alphabet is used for holonomic coordinate indices). Our signature for the metric is $(-, +, +, +)$. Our representation of the Clifford algebra Cliff$(3, 1)$ is generated by the following four real matrices

$$\gamma^0 = \begin{pmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix},$$

which also generates $\mathbb{R}^{4\times 4}$. Note

$$\gamma^0 = -\gamma_0 = -(\gamma^0)^t, \qquad \gamma^i = \gamma_i = (\gamma^i)^t. \qquad (i = 1, 2, 3)$$

The gamma matrices relation is

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}.$$

A basis for the Clifford algebra is obtained by multiplying these matrices together. They are (the right hand side gives the number independent matrices in each category)

$$\begin{array}{rlr} \mathbb{I} & & 1 \\ \gamma^a & & 4 \\ \gamma^{ab} & = \gamma^{[a}\gamma^{b]} & 6 \\ \gamma^{abc} & = \gamma^{[a}\gamma^b\gamma^{c]} & 4 \\ \gamma^{abcd} & = \gamma^{[a}\gamma^b\gamma^c\gamma^{d]} = -\gamma_5\epsilon^{abcd} & 1 \end{array}$$

where (note everything is real)

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \qquad \gamma_5 = -(\gamma_5)^t, \qquad (\gamma_5)^2 = -\mathbb{I}.$$

The matrices $\mathbb{R}^{4\times 4}$ act on $\mathbb{R}^4$ by the usual linear action. We will write $\mathbb{M}$ for $\mathbb{R}^4$, denoting Majorana spinors. As a vector space, $\mathbb{M}$ has a symplectic form $C_{\alpha\beta} = -C_{\beta\alpha}$, and $C\gamma_a C^{-1} = -(\gamma_a)^t$. This is the representation of the charge conjugation operator exchanging particles and antiparticles. In our representation, $C = \gamma_0$. Here are some other useful identities

$$(C\gamma^a)_{\alpha\beta} = (C\gamma^a)_{\beta\alpha}, \qquad (C\gamma^{ab})_{\alpha\beta} = (C\gamma^{ab})_{\beta\alpha},$$
$$(C\gamma^{abc})_{\alpha\beta} = (C\gamma^{abc})_{\beta\alpha}, \qquad (C\gamma_5)_{\alpha\beta} = -(C\gamma_5)_{\beta\alpha}.$$

The Dirac equation is

$$(\gamma^a \partial_a + m)\psi = 0.$$

The sign of m is irrelevant: if we apply the operator in the bracket (with the opposite sign) again, we get the Klein-Gordon equation $(-\Box + m^2)\psi = 0$. Furthermore, if ψ is a solution, then $\gamma_5\psi$ is also a solution of the conjugate equation

$$(\gamma^a \partial_a - m)\gamma_5\psi = 0.$$

Lorentz transformation leaves the equation invariant by the following action on spinors: let $x^a \to \Lambda^a{}_b x^b$ be a Lorentz transformation, and infinitesimally $\Lambda^a{}_b = \exp(\lambda^a{}_b)$, then

$$\psi \to \exp(\tfrac{1}{4}\lambda_{ab}\gamma^{ab})\psi.$$

3.2 Dirac and Weyl spinors, complex structure

Once we have defined Majorana spinors as a real vector space on which the spin group acts, Dirac spinors are easy: the complexification of Majorana spinors $\mathbb{D} = \mathbb{C}^4 = \mathbb{M} \otimes_{\mathbb{R}} \mathbb{C}$. To get Weyl spinors, we claim that Dirac spinor is the direct sum of two Weyl spinors: $\mathbb{D} = \mathbb{W} \oplus \bar{\mathbb{W}}$. We make this decomposition concrete in the following way: take a Dirac spinor ψ, if $\gamma_5\psi = i\psi$, an eigenstate with eigenvalue i, then $\psi \in \mathbb{W}$, whereas if $\gamma_5\psi = -i\psi$, then $\psi \in \bar{\mathbb{W}}$. Decomposition of $\mathbb{D}$ then gives $\mathbb{W} = \mathbb{C}^2$ in some basis.

The action of the spin group $\mathrm{Spin}(3,1)$ on a Weyl spinor is via its homomorphism to $SL(2,\mathbb{C})$: if we write the spacetime coordinates in a matrix

$$X = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = X^\dagger,$$

then we see that $\det(X) = -\eta_{ab}x^a x^b$. So if $S \in SL(2,\mathbb{C})$, under the transformation

$$X \to SXS^\dagger,$$

$\det(X)$ is invariant.

Let us examine in more details the above constructions. Let $V_{\mathbb{R}}$ be an even-dimensional real vector space, and $J \in \mathrm{End}(V_{\mathbb{R}})$ a linear operator satisfying $J^2 = -\mathbb{I}$, then J acts as a complex structure on $V_{\mathbb{R}}$ and make it into a complex vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$. We can extend J to $V_{\mathbb{C}}$ by complex linearity, and a basis can be chosen such that J is diagonal on $V_{\mathbb{C}}$. This is just our construction of Dirac spinors from Majorana spinors, with $J = \gamma_5$ in our basis. The Weyl spinors are then just the eigenspaces of J.

Let us see some examples other than spinors. The electromagnetic field tensor $F_{\mu\nu}$ is completely determined by the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ in a frame, so $F_{\mu\nu} \in \mathbb{R}^6$. We have the Hodge star operator that manifests the electromagnetic duality

$$*\mathbf{E} = -\mathbf{B}, \qquad *\mathbf{B} = \mathbf{E}, \qquad (*)^2 = -1,$$

so the Hodge star can be chosen as the complex structure. A "Weyl spinor" in this case can be written as

$$\mathbf{M} = \mathbf{E} + i\mathbf{B} \in \mathbb{W}$$

and the Maxwell equations become

$$\nabla \cdot \mathbf{M} = 0, \qquad \nabla \times \mathbf{M} = i\frac{\partial \mathbf{M}}{\partial t}.$$

The symmetry group acting on the usual Maxwell theory $SO(3,1)_{\mathbb{R}}$ then becomes $SO(3)_{\mathbb{C}}$, and the group action leaves invariant

$$\mathbf{M} \cdot \mathbf{M} = \mathbf{E}^2 - \mathbf{B}^2 + 2i\mathbf{E} \cdot \mathbf{B} = F^2 + F * F.$$

Another example is an even-dimensional manifold equipped with a metric $g_{ab} = g_{(ab)}$ and a symplectic form $\omega_{ab} = \omega_{[ab]}$, both of which are covariantly constant under a connection $\nabla g_{ab} = \nabla \omega_{ab} = 0$. Then we can form a complex structure

$$J^a{}_b = g^{ac}\omega_{ab}, \qquad J^2 = -\mathbb{I}, \qquad \nabla J^a{}_b = 0.$$

The complexified tangent space decomposes into "Weyl sums" $T_{\mathbb{C}}M = \mathbb{W}M \oplus \overline{\mathbb{W}M}$, and a tangent vector decomposes into the direct sum of a holomorphic tangent vector and an antiholomorphic tangent vector. The manifold now has the structure of a Kähler manifold.

3.3 Coupling to gravity

Back to spinors, in supergravity it is convenient to work purely over the reals, and despite the above constructions, consider a Dirac spinor as the direct sum of two Majorana spinors $\mathbb{D} = \mathbb{M} \oplus \mathbb{M}$ instead of complexification. We write a Dirac spinor as ψ_i, $i = 1, 2$. Introduce a complex structure on this space

$$J = \epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon_{ij}\epsilon_{ij} = -\delta_{ik}.$$

Then J induces $\mathbb{R}^4 \oplus \mathbb{R}^4 = \mathbb{R}^8 = \mathbb{C}^4 = \mathbb{D}$. Note that ϵ_{ij} is the generator of $SO(2) = U(1)$, and hence in this notation a $U(1)$ gauge field can be written as

$$A^{ij}{}_\mu = -A^{ji}{}_\mu = \epsilon^{ij} A_\mu.$$

The complexification of a Majorana spinor can be written as $\psi_j A^{ij}$.

Now suppose a field Φ^A transforms under a representation of Spin(3,1). Let $(\Sigma_{ab})^A{}_B = -(\Sigma_{ba})^A{}_B$ be a representation of this Lie algebra

$$[\Sigma_{ab}, \Sigma_{cd}] = \eta_{ab}\Sigma_{cd} - \eta_{ac}\Sigma_{bd} - \eta_{bd}\Sigma_{ac} + \eta_{ad}\Sigma_{bc},$$

and the field transforms as

$$\Phi \to \exp(\tfrac{1}{2}\lambda_{ab}\Sigma^{ab})^A{}_B \Phi^B = S(\Lambda)\Phi.$$

For example, if Φ^a is a four-vector, then

$$(\Sigma^{ab})^e{}_f = \eta^{ae}\delta^b{}_f - \eta^{be}\delta^a{}_f$$

reproduces standard action on vectors. But for a spinor, the representation is

$$\Sigma^{ab} = \tfrac{1}{4}[\gamma^a, \gamma^b] = \tfrac{1}{2}\gamma^{[a}\gamma^{b]} = \tfrac{1}{2}\gamma^{ab}, \qquad [\gamma^{ab}, \gamma^c] = 2(\eta^{ac}\gamma^b - \eta^{bc}\gamma^a).$$

For coupling to gravity, we need a covariant derivative. We define it for the general case:

$$\nabla_\mu \Phi = \partial_\mu \Phi + \tfrac{1}{2}\omega_\mu{}^{ab}\Sigma_{ab}\Phi, \qquad \text{or} \qquad \nabla\Phi = d\Phi + \tfrac{1}{2}\omega^{ab}\Sigma_{ab}\Phi.$$

Under the action $\Phi \to \exp(\frac{1}{2}\lambda_{ab}\Sigma^{ab})\Phi$, we have $\nabla\Phi \to \exp(\frac{1}{2}\lambda_{ab}\Sigma^{ab})\nabla\Phi$. We also need

$$(\nabla_\mu\nabla_\mu - \nabla_\nu\nabla_\mu)\Phi = \tfrac{1}{2}(R_{\mu\nu}{}^{ab}\Sigma_{ab})\Phi - T_\mu{}^\sigma{}_\nu\nabla_\sigma\Phi.$$

If we specialise to a Dirac spinor ψ_a, we have

$$\nabla\psi = d\psi + \tfrac{1}{4}\omega_{ab}\gamma^{ab}\psi$$

and on background with no torsion,

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\psi = \tfrac{1}{8}R_{\mu\nu ab}\gamma^{ab}\psi.$$

We will write the gamma matrices in coordinate basis as $\gamma^\mu = e^\mu{}_a\gamma^a$, and hence

$$\nabla_\mu(\gamma^\mu)^\alpha{}_\beta = 0.$$

The Majorana and Dirac conjugate of a spinor ψ are defined as

$$\psi_M = \psi^t C = \psi^\alpha C_{\alpha\beta}, \qquad \psi_D = \psi^\dagger\beta = \psi^{*\dot{\alpha}}\beta_{\dot{\alpha}\beta} = \bar{\psi}_\beta$$

and the matrices C and β satisfies

$$C\gamma_\mu C^{-1} = -\gamma_\mu^t, \qquad \beta\gamma_\mu\beta^{-1} = -\gamma_\mu^\dagger, \qquad \beta = -\beta^\dagger, \qquad \beta\gamma_\mu = (\beta\gamma_\mu)^\dagger.$$

In our basis, we can choose $\beta = C = \gamma_0$. Majorana spinors are exactly those that satisfy $\psi_M = \psi_D$, and in our basis, a Majorana ψ is purely real.

Finally, we define a symplectic linear product

$$\psi_1^\alpha C_{\alpha\beta}\psi_2^\beta = \bar{\psi}_1\psi_2 = \begin{cases} -\bar{\psi}_2\psi_1 & \text{for commuting spinors,} \\ +\bar{\psi}_2\psi_1 & \text{for anticommuting spinors.} \end{cases}$$

Now we can write down the lagrangian in flat spacetime for a Dirac spinor

$$\mathcal{L}_{1/2} = \tfrac{1}{2}\bar{\psi}(\not{\partial} + m)\psi.$$

The second term vanishes if ψ are commuting. Also,

$$\psi^\alpha\gamma^\mu_{\alpha\beta}\partial_\mu\psi^\beta = \tfrac{1}{2}\partial_\mu(\psi^\alpha\gamma^\mu_{\alpha\beta}\psi^\beta),$$

so if the spinors are commuting, $\mathcal{L}$ is a total derivative. This motivates thinking of ψ^α as taking values in some large (strictly speaking infinite dimensional) real Grassmann algebra $\mathcal{G}$. Bosons are even elements of $\mathcal{G}$, while fermions are odd elements.

To calculate the variation, note

$$\delta(\psi^\alpha C_{\alpha\beta}\psi^\beta) = \delta\psi^\alpha C_{\alpha\beta}\psi^\beta + \psi^\alpha C_{\alpha\beta}\delta\psi^\beta = 2\delta\bar{\psi}\,\psi$$

and

$$\delta(\tfrac{1}{2}\bar{\psi}\not{\partial}\psi) = \delta\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu(\bar{\psi}\gamma^\mu_{\alpha\beta}\delta\psi)$$

so variation gives

$$\delta\mathcal{L} \doteq \delta\bar{\psi}(\not{\partial} + m)\psi = 0$$

therefore we obtain the Dirac equation

$$(\not{\partial} + m)\psi = 0.$$

The spin current for a single spinor is

$$J^\mu = \bar{\psi}\gamma^\mu\psi = \psi^\alpha\gamma^\mu_{\alpha\beta}\psi^\beta$$

This vanishes for anticommuting objects. Therefore, to have a non-vanishing spin current, we need two Majorana spinors or one Dirac spinor, in which case

$$J^\mu = \psi^\alpha_i\gamma^\mu_{\alpha\beta}\psi^\beta_j\epsilon_{ij} = \psi^\alpha_1\gamma^\mu_{\alpha\beta}\psi^\beta_2 - \psi^\alpha_1\gamma^\mu_{\alpha\beta}\psi^\beta_2 \neq 0.$$

It is easy to generalise the construction to a purely bosonic background with action

$$S_{1/2} = \frac{1}{2}\int |e| d^4x\,\bar{\psi}(\gamma^\mu\nabla_\mu + m)\psi$$

and variation with respect to ψ gives the Dirac equation in curved background

$$(\not{\nabla} + m)\psi = 0.$$

If we iterate the Dirac operator and hope to get a generalisation of the Klein-Gordon equation, we get

$$\begin{aligned}
0 &= (\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu - m^2)\psi \\
&= (\gamma^{(\mu}\gamma^{\nu)}\nabla_\mu\nabla_\nu - m^2 + \gamma^{[\mu}\gamma^{\nu]}\nabla_\mu\nabla_\nu)\psi \\
&= (\nabla^2 - m^2 + \gamma^\mu\gamma^\nu\nabla_{[\mu}\nabla_{\nu]})\psi \\
&= (\nabla^2 - m^2 + \tfrac{1}{8}\gamma^\mu\gamma^\nu R_{\mu\nu\alpha\beta}\gamma^\alpha\gamma^\beta)\psi
\end{aligned}$$

It may look like that we are getting some spin-curvature coupling in the last step, but this is false: ∇ here is the Levi-Civita connection, and we have

$$\left(\nabla^2 - m^2 - \tfrac{1}{4}R\right)\psi = 0,$$

so there is no spin current coupled to the curvature. This calculation is first done by Perez and Lichnerowicz, is mostly easily verified by substitution: one needs the following formula

$$\begin{aligned}
\gamma^a\gamma^b\gamma^c &= \gamma^{[abc]} + \eta^{ab}\gamma^c - \eta^{cd}\gamma^b + \eta^{cb}\gamma^a \\
\gamma^{[abc]} &= \gamma^a\gamma^{[bc]} - \eta^{ba}\gamma^c + \eta^{bc}\gamma^a
\end{aligned}$$

so

$$
\begin{aligned}
R_{abcd}\gamma^b\gamma^c\gamma^d &= -2R_{ab}\gamma^b \\
R_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d &= -2R \\
\gamma^{abc}R_{acef}\gamma^{ef} &= 4G^e{}_f\gamma^f
\end{aligned}
$$

and the result follows.

It should be noted that the massless Dirac equation in curved spacetime is conformally invariant: if $(g_{\mu\nu}, \psi)$ is a solution of $\nabla\!\!\!/\psi = 0$, then $(\Omega^2 g_{\mu\nu}, \psi/\Omega^{3/2})$ is a solution also. We can use this fact to deduce how R changes under Weyl rescaling.

Let us now investigate the equation under chiral rotations. A chiral rotation is

$$
\begin{aligned}
\psi &\to \exp(\theta\gamma_5)\psi = (\cos\theta + \gamma_5\sin\theta)\psi \\
\bar{\psi} &\to \bar{\psi}\exp(\theta\gamma_5) = \bar{\psi}(\cos\theta + \gamma_5\sin\theta).
\end{aligned}
$$

Since γ_5 anticommutes with γ^α, the kinetic term is invariant but the mass term is not: $\bar{\psi}m\psi \to \bar{\psi}\exp(2\theta\gamma_5)m\psi$. Massless theory has chiral invariance. But consider a theory where the mass term is replaced by

$$
m \to m_1 + \gamma_5 m_2
$$

where m_1 is called the Dirac mass, and m_2 the Majorana mass. Using chiral rotation we can eliminate the m_2 term: the quantity $\sqrt{m_1^2 + m_2^2}$ is invariant.

In general, given n Majorana spinors ψ_i

$$
\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\bar{\psi}^i\nabla\!\!\!/\psi + \frac{1}{2}\sum_{i,j=1}^{n}\bar{\psi}^i M_{ij}\psi^j,
$$

the mass matrix $M_{ij} = m_{ij}^1 + \gamma_5 m_{ij}^2$ where both m_{ij}^1 and m_{ij}^2 are symmetric, a chiral rotation of the form

$$
\exp(\alpha_{ij} + \gamma^5\beta_{ij}) \in U(n), \qquad \alpha_{ij} = -\alpha_{ji}, \qquad \beta_{ij} = \beta_{ji}
$$

can make the mass matrix M_{ij} diagonal.

3.4 Einstein-Cartan-Weyl-Sciama-Kibble theory

In this theory, one adds torsion to the connection and attempt to couple matter fields (fermions) to gravity. The lagrangian is

$$
\mathcal{L} = -\frac{R}{2}\kappa^2 + \frac{1}{2}\bar{\psi}\nabla\!\!\!/\psi.
$$

Since $\bar{\psi}\nabla\!\!\!/\psi$ contains $\omega_\mu{}^{ab}$, spin density couples to torsion. The spin density is given by

$$
S_a{}^\mu{}_b = \tfrac{1}{8}\bar{\psi}\gamma^\mu\gamma_{ab}\psi
$$

so that $S_{abc} = S_{[abc]}$, which implies $T_{abc} = T_{[abc]}$. The additional equation of motion is

$$T_{abc} = -\frac{\kappa^2}{4}\bar{\psi}\gamma_{abc}\psi.$$

In this case, we have a totally antisymmetric torsion. A theory with totally antisymmetric torsion has the following characterisation. Autoparallels are defined by solution curves to the following "geodesic equation" (the connection is not Levi-Civita, and hence the quotation marks):

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma_\mu{}^\alpha{}_\nu \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0,$$

while geodesics are defined by the extremal curves as defined by the length functional with respect to the metric, i.e. solution curves to the same equation but with the Levi-Civita connection. Now

$$\Gamma_\mu{}^\alpha{}_\nu = \{_\mu{}^\alpha{}_\nu\} + K_\mu{}^\alpha{}_\nu$$

and

$$K_{\alpha\mu\beta} = -\tfrac{1}{2}(T_{\alpha\mu\beta} + T_{\mu\alpha\beta} + T_{\mu\beta\alpha}) = -\tfrac{1}{2}T_{\alpha\mu\beta}$$

since T is totally antisymmetric. So $K_{(\alpha}{}^\mu{}_{\beta)} = 0$, and hence geodesics and autoparallels coincide.

4. Supergravity lagrangian and super invariance

4.1 Rarita-Schwinger equation in flat spacetime

We will write, as our first attempt at a supergravity lagrangian (a lagrangian for spin-$\frac{3}{2}$ fields), the following massless action for Majorana fermions:

$$S = \frac{1}{2}\int d^4x\, \bar{\psi}_a^\alpha \gamma^{abc}_{\alpha\beta}\partial_b\psi_c^\beta.$$

Note that $\gamma^{abc}_{\alpha\beta} = \gamma^{cba}_{\beta\alpha}$. Variation gives

$$\delta S \doteq \int d^4x\, \delta\bar{\psi}_a\gamma^{abc}\partial_b\psi_c$$

and the equation of motion is

$$\gamma^{abc}\partial_b\psi_c = 0.$$

This equation is invariant under chiral rotations, and also fermionic gauge transformations $\psi_a \to \psi_a + \partial_a\lambda$, analogous to the bosoinc gauge transformations for spin-1 fields.

We can simplify the equation further. Start with

$$\gamma_a\gamma^{abc} = 2\gamma^{bc}, \qquad \gamma_a\gamma^{ab} = 3\gamma^b, \qquad \gamma^{ab} = \gamma^a\gamma^b - \eta^{ab}$$

and hitting the equation with γ_a, we get

$$\partial\!\!\!/\psi_a - \partial_a\psi\!\!\!/ = 0.$$

We now attempt to answer the Cauchy question: find how many (real) functions are needed to give the Cauchy data to this equation so that it has a unique solution. We first choose a gauge, i.e. use gauge invariance to set

$$\gamma^i\psi_i = 0, \qquad \forall t, \qquad i = 1,2,3.$$

This amounts to the following. Under a gauge transformation, $\gamma^i\psi_i \to \gamma^i\psi_i + \partial\!\!\!/\lambda$, so if we "solve" the highly non-local equation $\lambda = -\frac{1}{\partial\!\!\!/_{(3)}}(\gamma^i\psi_i)$, we can set what we require to zero. Now write spatial and time part of the equation of motion separately

$$(\gamma^0\partial_0 + \underline{\gamma}\cdot\nabla)\psi_\mu - \partial_\mu(\gamma^0\psi_0 + \gamma^i\psi_i) = 0.$$

The $\mu = 0$ equation of motion then gives

$$(\underline{\gamma}\cdot\nabla)\psi_0 = 0$$

iteration gives $\nabla^2\psi_0 = 0$, so $\psi_0 = 0$. Therefore, $\gamma^i\psi_i = 0$, and we are left with the equation $\partial_i\psi^i = 0$. We conclude that, the equation of motion in this gauge is just

$$\partial\!\!\!/\psi_i = 0.$$

Now ψ_i, having a spatial vector index, includes $3 \times 4 = 12$ functions. The constraints are $\gamma^i\psi_i = 0$ and $\partial_i\psi^i = 0$, each gives four constraints on functions (these are matrix equations). So we are left with 4 free functions. Four free functions gives two degrees of freedom for Majorana fermions.

What we have done is similar to the Coulumb gauge in electromagnetism: i.e. we set

$$A_0 = 0, \qquad \nabla \cdot \mathbf{A} = 0$$

using the gauge transformation $A_i \to A_i + \partial_i \Lambda$.

We can also use a covariant gauge, the Lorenz gauge: set $\gamma^a \psi_a = 0$, then $\partial_a \psi^a = 0$ and the equation of motion is $\partial\!\!\!/ \psi_a = 0$. This is analogous to setting $\partial_\mu A^\mu = 0$ in electromagnetism.

In quantum field theory we complexify ψ^a and take it to be proportional to the plane wave solutions $e^{ik\cdot x}$. Then we have

$$k^2 = 0, \qquad k_a \psi^a = 0, \qquad \gamma^a \psi_a = 0.$$

Therefore ψ^a lies in a null plane with null normal k^a. We still have freedom to add to ψ^a any multiples of k^a. So we can set

$$k^a = (1, 1, 0, 0), \qquad \psi^a = (0, 0, \psi^2, \psi^3).$$

So we have as many degrees of freedom as two vectors and two spinors have, minus the following constraint:

$$\gamma^2 \psi^3 + \gamma^3 \psi^2 = 0.$$

Hence, the spin content of ψ_i is the following: ψ_i transforms as spin $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ under $SU(2)$, but the spin-$\frac{1}{2}$ part is killed by the constraints. So the field ψ_i is a spin-$\frac{3}{2}$ field.

To add a mass term to our gravitino, we also look for analogies with electromagnetism. The massive analogue of the Maxwell equation is the Proca equation

$$\partial_\mu F^{\mu\nu} = m^2 A^\nu$$

from the following lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2} m^2 A_\mu A^\mu.$$

This includes the Klein-Gordon equation: if we hit both sides with ∂_ν we obtain $m^2 \partial_\nu A^\nu = 0$, back substitution gives $\partial^2 A^\nu = m^2 A^\nu$.

Therefore, for gravitino, we use the following lagrangian

$$\mathcal{L} = \tfrac{1}{2} \bar{\psi}_a \gamma^{abc} \partial_b \psi_c + \tfrac{1}{2} m \bar{\psi}_a \gamma^{ab} \psi_b.$$

Note that $\gamma^{ab}_{\alpha\beta} = \gamma^{[ab]}_{\alpha\beta} = \gamma^{ab}_{(\alpha\beta)}$. The equation of motion is

$$\gamma^{abc} \partial_b \psi_c + m \gamma^{ab} \psi_b = 0.$$

Hit with ∂_a, we get

$$m \gamma^{ab} \partial_a \psi_b = 0.$$

But $\gamma^{ab} = \gamma^a \gamma^b - \eta^{ab}$, so

$$\partial \cdot \psi = \partial\!\!\!/ \psi\!\!\!/.$$

Also $\gamma_a \gamma^{ab} = 3\gamma^b$, which implies $3m\gamma^a \psi_a = 0$, or $\psi\!\!\!/ = 0$, so $\partial \cdot \psi = 0$. Now

$$\gamma^{abc} = \gamma^a \gamma^{bc} + \eta^{ac}\gamma^b - \eta^{ab}\gamma^c,$$

expanding our original equation, we get

$$\partial\!\!\!/ \psi^a - \partial^a \psi\!\!\!/ + m\gamma^a \psi\!\!\!/ - m\psi^a = 0$$

therefore our final equation of motion is

$$(\partial\!\!\!/ - m)\psi^a = 0 \qquad \text{subject to} \qquad \partial \cdot \psi = \psi\!\!\!/ = 0.$$

To analyse further, we again complexify and take $\psi^a \propto e^{ik\cdot x}$, $k^a k_a = -m^2$. Set

$$k^a = (m, 0, 0, 0), \qquad \psi_a = (0, \psi_1, \psi_2, \psi_3),$$

then

$$\gamma^1 \psi_1 + \gamma^2 \psi_2 + \gamma^3 \psi_3 = 0.$$

As a representation of $SU(2)$ we have vector together with spinor again, but this time it is $1 \oplus \frac{1}{2} \oplus \frac{3}{2} \oplus \frac{1}{2}$, with the last $\frac{1}{2}$ eliminated by the residual gauge freedom. So we have $2 \times \frac{3}{2} + 1 = 4$ degrees of freedom in the massive case.

To couple our theory to gravity, we could try the "minimal coupling" $\partial \to \nabla$, but it doesn't work. This procedure actually only works for spin 1 and below. For example, for spin 2 in flat spacetime, we have

$$\nabla^2 h_{\mu\nu} = 0, \qquad h_\mu{}^\mu = 0, \qquad \partial_\mu h^{\mu\nu} = 0,$$

but letting $\partial \to \nabla$ gives too many conditions on $h_{\mu\nu}$.

4.2 $\mathcal{N} = 1$ supergravity

We want to generalise the Rarita-Schwinger equation to curved spacetime. Again, we start with the massless case. Our action is

$$S_{3/2} = \int d^4x \sqrt{g}\, \frac{1}{2} \bar{\psi}_a (\gamma^{abc} D_b \psi_c), \qquad D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\psi_\mu$$

note that we have omitted the Levi-Civita term $\{_\mu{}^\sigma{}_\nu\}\psi_\sigma$ in $D_\mu \psi_\nu$, since it vanishes when we anti-symmetrise with respect to μ, ν. We consider $\psi = \psi_\mu dx^\mu$ as a spinor-valued 1-form. The equation of motion is

$$\gamma^{abc} D_b \psi_c = 0.$$

What about gauge invariance for this theory? Let us start with a "pure gauge" spinor

$$\psi = D\lambda = d\lambda + \tfrac{1}{2}\omega_{ab}\gamma^{ab}\lambda$$

then

$$\gamma^{abc}D_bD_c\lambda = \gamma^{abc}D_{[b}D_{c]}\lambda = 0$$

so

$$\gamma^{abc}R_{bcef}\gamma^{ef}\lambda = 0, \qquad \text{or} \qquad G^b{}_f\gamma^f\lambda = 0.$$

In general $\det(G^b{}_f\gamma^f) \neq 0$ and $\det(a^a\gamma_a) = (a^aa_a)^2$, so this term do not vanish. Therefore, spin-$\frac{3}{2}$ field in curved background with $G_{ab} \neq 0$ is inconsistent! This is a generic problem with spin greater than 1, known as the Buchdahl condition.

One method of overcoming this problem is by cancelling this term using the Einstein action. Let a symmetry transformation be

$$\delta\psi_\mu = \frac{1}{\kappa}D_\mu\epsilon$$

where ϵ is a spinor field with dimension $[\epsilon] = L^{1/2}$. Remember $[\psi] = L^{-3/2}$. Then the variation of our gravitino action gives

$$\delta S_{3/2} = \frac{1}{\kappa}\int d^4x\sqrt{-g}\,\bar{\psi}_a\gamma^{abc}D_bD_c\epsilon \doteq \frac{1}{\kappa}\int d^4x\sqrt{-g}\,\bar{\psi}G^a{}_b\gamma^b\epsilon.$$

For the Einstein action

$$S_2 = \int d^4x\sqrt{-g}\frac{R}{2\kappa^2},$$

we have

$$\delta S_2 \doteq -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,G^\mu{}_b\delta e^b{}_\mu.$$

Hence, by choosing "$\frac{1}{2} = \frac{1}{2}$", i.e. use the specific symmetry variation

$$\delta e^a{}_\mu = \bar{\psi}_\mu\gamma^a\epsilon, \qquad [e] = 0, \qquad [\kappa] = L$$

we can show that

$$\delta\left(\int d^4x\sqrt{-g}\frac{R}{2\kappa^2} + \tfrac{1}{2}\bar{\psi}_a\gamma^{abc}D_b\psi_c\right) = 0.$$

This is to the lowest order (ignoring "4-Fermi terms") a proof of the invariance of $\mathcal{N} = 1$ supergravity lagrangian under supersymmetry variations (and we see why this lagrangian is called the *super*gravity lagrangian).

For a proof of the invariance in the exact theory, in second order formalism, we write the spin connection as

$$\omega_\mu{}^{ab} = \omega'_\mu{}^{ab}(e) + \tfrac{1}{2}\kappa^2(\bar{\psi}_\mu\gamma^a\psi^b - \bar{\psi}_\mu\gamma^b\psi^a + \bar{\psi}^a\gamma_\mu\psi^b)$$

where ω' denotes the Levi-Civita connection. The expression in bracket are the 4-Fermi terms which needs to be cancelled using Fierz identities discussed later. Note that

$$T_{abc} = -\tfrac{1}{2}\kappa^2\bar{\psi}_a\gamma^b\psi_c$$

is in general not totally antisymmetric, and hence the Bianchi identities are modified

$$d\omega + \omega\wedge\omega = R, \qquad de + \omega\wedge e = T$$

which imply

$$d\omega \wedge e - \omega \wedge de = dT, \qquad (R - \omega \wedge \omega) \wedge e = \omega \wedge (T - \omega \wedge e), \qquad R \wedge e = DT.$$

An extension of this $\mathcal{N} = 1$ theory is addition of the cosmological constant, first considered by Townsend

$$D_\mu \to \hat{D}_\mu = D_\mu + \frac{1}{2} a \gamma_\mu, \qquad R \to R - 2\Lambda, \qquad \Lambda = -\frac{3}{a^2} < 0$$

where a is a constant. $\hat{D}_\mu$ is an example of a super-covariant derivative:

$$\delta \psi_\mu = \tfrac{1}{\kappa} \hat{D}_\mu \epsilon.$$

This change would appear to introduce a "mass term" into the lagrangian

$$\frac{1}{2a} \bar{\psi}_a \gamma^{abc} \gamma_b \gamma_c = -\frac{1}{2a} \bar{\psi}_a \gamma^{ac} \gamma_c$$

but this is not really a mass, since the lagrangian is still invariant. We have maintained gauge invariance at the expense of changing our ground state, i.e.

$$R_{\mu\nu} = 0 \qquad \to \qquad R_{\mu\nu} = \Lambda g_{\mu\nu}$$

and Minkowski spacetime is no longer a solution. The ground state is now $AdS_4 = SO(3,2)/SO(3,1)$. This is the homogeneous hypersurface defined in $\mathbb{E}^{3,2}$ by the equation

$$(x^1)^2 + (x^2)^2 + (x^3)^3 - (x^4)^4 - (x^5)^5 = -a^2$$

homogenous meaning that the metric tensor completely determines the Riemann tensor (no derivatives):

$$R_{\mu\nu\lambda\rho} = -\frac{1}{a^2}(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}).$$

In flat spacetime, the covariant derivative commutes: $D_{[\mu} D_{\nu]} = 0$. In anti-de Sitter space it does not, but the super covariant derivative does:

$$\hat{D}_{[\mu} \hat{D}_{\nu]} = 0.$$

This only works in anti-de Sitter spacetime, not in de Sitter spacetime, since in that case an i has to be introduced, rendering the lagrangian complex.

4.3 $\mathcal{N} = 2$ supergravity

In a $\mathcal{N} = 2$ theory, to the lowest order, our fields are the metric, the gauge field and two Majorana spinors

$$g_{\mu\nu}, \qquad A_\mu, \qquad \psi^i{}_\mu \qquad (i = 1, 2).$$

To the usual gauge lagrangian piece

$$\mathcal{L}_1 = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

we add the Pauli term

$$\mathcal{L}_{\text{Pauli}} = \frac{\kappa}{2\sqrt{2}}\bar{\psi}^i{}_\mu(F^{\mu\nu} + *F^{\mu\nu}\gamma_5)\psi^j_\nu\epsilon_{ij}, \qquad *F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}.$$

The supersymmetry variations of the various fields are

$$\delta e^a{}_\mu = \frac{1}{2}\kappa\bar{\psi}^i_\mu\gamma^a\epsilon^i, \qquad \delta A_\mu = \frac{1}{\sqrt{2}}\kappa\bar{\epsilon}^i\psi^j_\mu\epsilon_{ij}, \qquad \delta\psi^i{}_\mu = \frac{1}{\kappa}\hat{D}_\mu\epsilon^i$$

where in the last case the super-covariant derivative is

$$\hat{D}_\mu\epsilon^i = D_\mu\epsilon^i + \frac{1}{2\sqrt{2}}\kappa\epsilon^{ij}(F_{\mu\nu}\gamma^\lambda + *F_{\mu\nu}\gamma^\lambda\gamma_5)\epsilon^j.$$

A feature of this theory is that the equation of motion (but not action) is chiral-duality invariant, i.e. invariant under

$$\psi^i_\mu \to \exp(\theta\gamma_5)\psi^i, \qquad F_{\mu\nu} \to \exp(e\theta *)F_{\mu\nu}.$$

4.4 Super invariance of the exact theory

For easier manipulation in showing the super invariance of the exact theory, we write our theory using spinor-valued 1-forms $\psi = \psi_\mu dx^\mu = \psi_a e^a$. Note that the wedge product on spinors now commutes, since the wedge product and Grassmann variables both contribute a minus sign. We have

$$\psi_1 \wedge \psi_2 = \psi_2 \wedge \psi_1, \qquad \bar{\psi} \wedge \psi = 0, \qquad e^a \wedge \psi = -\psi \wedge e^a, \qquad \bar{\psi}\gamma^{abc} \wedge \psi = 0$$

while the combination $\bar{\psi}\gamma^a \wedge \psi \neq 0$. Also recall that $\gamma^{abc} = \gamma_5\gamma_d\gamma^{abcd}$ and $\epsilon^{abcd}|e|d^4x = e^1 \wedge e^2 \wedge e^3 \wedge e^4$. The gravitino part of the lagrangian is

$$\begin{aligned}
\mathcal{L}_{3/2} &= \tfrac{1}{2}\bar{\psi}_a\gamma^{abc}D_b\psi_c|e|d^4x \\
&= \tfrac{1}{2}\bar{\psi} \wedge \gamma^5\gamma \wedge D\psi \\
&= -\tfrac{1}{12}\bar{\psi} \wedge \gamma^{abc}D\psi \wedge e^d\epsilon_{abcd}
\end{aligned}$$

where $\psi = \psi_\mu dx^\mu$ is a matrix-valued 1-form and

$$D\psi = d\psi + \tfrac{1}{4}\omega_{ab}\gamma^{ab} \wedge \psi.$$

The total lagrangian is

$$\mathcal{L} = -\frac{1}{4\kappa^2}\operatorname{Tr}(e \wedge e \wedge R) + \frac{1}{2}\bar{\psi}\gamma_5\gamma \wedge D\psi.$$

If we now do the variation, we get a term proportional to $D\delta\omega^{ab}$ for variation of R^{ab} with respect to the connection, and $\bar{\epsilon}D\psi \wedge De^s$ for variation of $\delta\bar{\psi} = D\bar{\epsilon}/\kappa$ and integration by parts, $\bar{\psi}\gamma^{mnr}\gamma_{kl}\psi\epsilon_{mnrs} = -6\bar{\psi}\gamma^m\psi\epsilon_{mkls}$ for variation respect to the connection in $D\psi$ and finally $\frac{1}{2}\kappa\bar{\epsilon}\gamma^a\psi$ for variation of the "metric" e^a and using the Fierz identities. Putting everything together, we have

$$\delta\mathcal{L} \doteq \frac{1}{2\kappa^2}(\delta\omega^{mn} \wedge e^r - \frac{1}{6}\bar{\epsilon}\gamma^{mnr}D\psi) \wedge (De^s - \frac{1}{4}\kappa^2\bar{\psi}\gamma^s\psi)\epsilon_{mnrs}.$$

The interpretation of this variation depends on the formalism used: in second order formalism, $\delta\omega^{mn} = 0$ since by definition $\omega = \omega(e, \psi)$; in the first order formalism, this defines the variation of $\delta\omega^{mn}$ since we really only have δe and $\delta\psi$; in the 1.5 formalism, $\delta\omega^{mn}$ vanishes as well because of the equation of motion coming for ω^{mn}.

4.5 Fierz identities

Now we come to the Fierz identities that is referred to previously. $\text{Cliff}(3,1)$ has a basis $\mathbb{I}, \gamma_\mu, \gamma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5$ and we write them as $\{\Gamma_A\}$ for $A = 1, \ldots, 16$. We consider the matrices ${\Gamma_A}^\alpha{}_\beta$ as endomorphisms of the spinor vector space. We claim that, except for $\mathbb{I}$, $\operatorname{Tr}\Gamma_A = 0$. This can be shown by, e.g.

$$\gamma_5\gamma_\mu\gamma_5 = \gamma_\mu, \qquad \operatorname{Tr}(\gamma_5\gamma_\mu\gamma_5) = \operatorname{Tr}(\gamma_\mu) = -\operatorname{Tr}(\gamma_\mu) = 0$$

due to cyclicity of trace. For those involving γ_5, we use $\gamma_5 = -\gamma_1\gamma_5\gamma_1$. Similarly, we obtain relations for $\operatorname{Tr}\Gamma_A\Gamma_B$. Putting them together,

$$\operatorname{Tr}\Gamma_A = 0 \quad \Gamma_A \neq \mathbb{I}, \qquad \operatorname{Tr}\Gamma_A\Gamma_B = 4\eta_{AB} \quad \eta_{AB} = \operatorname{diag}(\pm 1).$$

Let us determine the signs of η_{AB}: we write (m, n) for m plus signatures and n minus signatures for generators. We get

$\mathbb{I}$	$(1,0)$
γ_μ	$(3,1)$
$\gamma_{\mu\nu}$	$(3,3)$
$\gamma_5\gamma_{\mu\nu}$	$(3,1)$
γ_5	$(0,1)$

So adding everything together, we have $\text{Cliff}(3,1) = \mathbb{R}^{10,6}$.

Note that as a vector space (ignoring the product structure), $\text{Cliff}(3,1) = \Lambda(\mathbb{R}^{3,1})$ has a natural inner product induced from η_{ab}:

$$(F_p, {F'}_p) = \frac{1}{p!}(F_{\mu_1\ldots\mu_p}{F'}_{\nu_1\ldots\nu_p}\eta^{\mu_1\nu_1}\ldots\eta^{\mu_p\nu_p}).$$

Taking into consideration of the product structure, however, the counting goes

1	$(1,0)$
e_μ	$(3,1)$
$e_\mu \wedge e_\nu$	$(3,3)$
$e_\mu \wedge e_\nu \wedge e_\lambda$	$(1,3)$
$e_\mu \wedge e_\nu \wedge e_\lambda \wedge e_\rho$	$(0,1)$

The Fierz identities is used to simplify the following expression

$$(\bar{\psi}_1 M\psi_2)(\bar{\psi}_3 N\psi_4) = \psi_{1\alpha}M^\alpha{}_\beta\psi_2^\beta\psi_{3\gamma}N^\gamma{}_\delta\psi_4^\delta.$$

Use the complete basis, we write

$$M^\alpha{}_\beta N^\gamma{}_\delta = C^{A\alpha}{}_\delta {\Gamma_A}^\gamma{}_\beta$$

multiply by $\Gamma^{B\beta}{}_\gamma$ to obtain

$$C^{A\alpha}{}_\delta = \tfrac{1}{4}M^\alpha{}_\beta N^\gamma{}_\delta \Gamma^{A\beta}{}_\gamma$$

therefore

$$M^\alpha{}_\beta = \tfrac{1}{4}(M\Gamma^a N)^\alpha{}_\delta \Gamma_A{}^\alpha{}_\beta = \tfrac{1}{4}\Gamma_A{}^\alpha{}_\delta (N\Gamma^A M)^\gamma{}_\beta.$$

Using this, for anticommuting spinors

$$(\bar\psi_1 M\psi_2)(\bar\psi_3 N\psi_4) = -\tfrac{1}{4}(\bar\psi_1 M\Gamma^A N\psi_4)(\bar\psi_3\Gamma_A\psi_2) = -\tfrac{1}{4}(\bar\psi_1\Gamma^A\psi_4)(\bar\psi_3 N\Gamma_A M\psi_2).$$

Therefore,

$$\begin{aligned}(\bar\psi_1 M\psi_2)(\bar\psi_3 N\psi_4) = &-\tfrac{1}{4}(\bar\psi_1\psi_4)(\bar\psi_3 NM\psi_2)\\ &-\tfrac{1}{4}(\bar\psi_1\gamma^\mu\psi_4)(\bar\psi_3 N\gamma_\mu M\psi_2)\\ &+\tfrac{1}{4}(\bar\psi_1\gamma_5\psi_4)(\bar\psi_3 N\gamma_5\Gamma_A M\psi_2)\\ &-\tfrac{1}{4}(\bar\psi_1\gamma_5\gamma^\mu\psi_4)(\bar\psi_3 N\gamma_5\gamma_\mu M\psi_2)\\ &+\tfrac{1}{8}(\bar\psi_1\gamma^{\mu\nu}\psi_4)(\bar\psi_3\gamma_{\mu\nu} NM\psi_2).\end{aligned}$$

The signs are opposite if the spinors are commuting. Therefore

$$\psi_1\bar\psi_2 = -\tfrac{1}{4}(\bar\psi_2\psi_1)\mathbb{I} + \tfrac{1}{4}(\bar\psi_2\gamma_5\psi_1)\gamma_5 - \tfrac{1}{4}(\bar\psi_2\gamma^\mu\psi_1)\gamma_\mu + \tfrac{1}{4}(\bar\psi_2\gamma_5\gamma^\mu\psi_1)\gamma_5\gamma_\mu + \tfrac{1}{8}(\bar\psi_2\gamma^{\mu\nu}\psi_1)\gamma_{\mu\nu}$$

so that

$$\psi_1\bar\psi_2 - \psi_2\bar\psi_1 = \tfrac{1}{2}(\bar\psi_1\gamma^\mu\psi_2)\gamma_\mu - \tfrac{1}{4}(\bar\psi_2\gamma^{\mu\nu}\psi_1)\gamma_{\mu\nu}.$$

The Fierz identities are useful in establishing the commutator of 2 supersymmetry transformations and thus checking the algebra.

The following identities

$$\gamma^\mu\gamma_\mu = 4, \quad \gamma^\lambda\gamma_\mu\gamma_\lambda = -3\gamma_\mu, \quad \gamma^\lambda\gamma_{\mu\nu}\gamma_\lambda = 0, \quad \gamma^\lambda\gamma_5\gamma_\mu\gamma_\lambda = 3\gamma_5\gamma_\mu, \quad \gamma^\lambda\gamma_5\gamma_\lambda = -4,$$

allow we to simplify expressions and we can obtain interesting consequences, including the fact that $\bar\epsilon\gamma^\mu\epsilon$ is lightlike and future-pointing for commuting Majorana spinors:

$$(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi) = 0, \qquad \psi^t\gamma_0\gamma^0\psi > 0,$$

also

$$\bar\psi\gamma_5\gamma^\mu\psi = 0,$$

by setting $M = \gamma_\lambda$, $N = \gamma^\lambda$ and $\psi_1 = \psi_2 = \psi_3 = \psi_4 = \psi$.

4.6 Supersymmetric background

In flat background, schematically a supersymmetry variation gives

$$\delta_\epsilon B \propto \bar\epsilon F, \qquad \delta_\epsilon F \propto (\gamma^\mu\partial_\mu B)\epsilon, \qquad [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \propto -\tfrac{1}{2}\bar\epsilon_1\gamma^\mu\epsilon_2\partial_\mu$$

where B is bosonic and F is fermionic. The dimensions are

$$[F] = L^{-3/2}, \qquad [B] = L^{-1}, \qquad [\epsilon] = L^{1/2}.$$

A purely bosonic background has $F = 0$, is a solution of the equation of motion, and is invariant under any supersymmetry transformations ϵ satisfying

$$(\gamma^\mu \partial_\mu B)\epsilon = 0.$$

(A background is called supersymmetric or BPS if it is invariant under action of at least one supersymmetry. This gives extra constraint on B, e.g. $B =$ constant.)

Let us give a concrete example: the Wess-Zumino model. In flat spacetime, consider free and massless fields

$$\mathcal{L} = -\tfrac{1}{2}(\partial A)^2 - \tfrac{1}{2}(\partial B)^2 + \tfrac{1}{2}\bar{\lambda}\not{\partial}\lambda.$$

Variation gives

$$\delta\mathcal{L} = \partial^2 A \delta A + \partial^2 B \delta B + \bar{\lambda}\gamma^\mu \partial_\mu \delta\lambda - \partial_\mu(\partial^\mu A \delta A + \partial^\mu B \delta B - \tfrac{1}{2}\delta\bar{\lambda}\gamma^\mu \lambda)$$

where

$$\delta A = \tfrac{1}{2}\bar{\epsilon}\lambda, \qquad \delta B = \tfrac{1}{2}\bar{\epsilon}\gamma_5\lambda, \qquad \delta\lambda = -\tfrac{1}{2}\gamma^\mu(\partial_\mu(A + \gamma_5 B))\epsilon.$$

So the equations of motion are

$$\partial^2(A + \gamma_5 B) = 0, \qquad \not{\partial}\lambda = 0.$$

And once the equation of motion is satisfied, we see that $\delta\mathcal{L} = \partial_\mu J^\mu$, where the conserved supercurrent J^μ is

$$J^\mu = -\tfrac{1}{4}(\bar{\epsilon}\partial_\mu(A + \gamma_5 B)\lambda).$$

We can calculate the commutator of two supersymmetry transformations in this theory:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \tfrac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu.$$

This is a translation, and this algebra closes "on shell", i.e. we need to impose the equations of motions to get this.

Let us try to solve the equations of motions for bosonic background. For the wave-like equation, we write

$$A + \gamma_5 B = \exp(\gamma_5 k \cdot x)(A_0 + B_0\gamma_5).$$

Substituting into the equation of motion, we see that the wave vector is lightlike

$$k_\mu k^\mu = 0.$$

In order to have bosonic solutions, the variation $\delta\lambda = 0$, so we must also have

$$(k^\mu\gamma_\mu)\epsilon = 0.$$

This is an equation of constraint on ϵ. As $\det(k^\mu\gamma_\mu) = k^2 = 0$, $k^\mu\gamma_\mu$ has a non-trivial kernel. Now let us go into the frame where

$$k^\mu = (1, 1, 0, 0)$$

we have

$$(\gamma_0 + \gamma_1)\epsilon = 0$$

which, using $(\gamma^0\gamma^1)^2 = \mathbb{I}$ and $\mathrm{Tr}(\gamma^0\gamma^1) = 0$, we get

$$\gamma^0\gamma^1\epsilon = -\epsilon.$$

Now we can introduce

$$P_+ = \tfrac{1}{2}(1 + \gamma^0\gamma^1), \qquad P_- = \tfrac{1}{2}(1 - \gamma^0\gamma^1)$$

which are projection operators

$$P_+^2 = P_+, \qquad P_-^2 = P_-, \qquad P_+P_- = P_-P_+ = 0, \qquad P_+ + P_- = \mathbb{I}.$$

These projection operators can be used to split the Majorana spinors into direct sums: $\mathbb{M} = \mathbb{M}_+ \oplus \mathbb{M}_-$ where $\psi_\pm \in \mathbb{M}_\pm$ if and only if $P_\pm\psi_\pm = \psi_\pm$ and any spinor decomposes uniquely as $\psi = \psi_+ + \psi_-$. Also note $\gamma^0\gamma^1\psi_+ = \psi_+$ and $\gamma^0\gamma^1\psi_- = \psi_-$. In terms of these operators, our Killing spinor belongs to the minus part of the space: $\epsilon \in \mathbb{M}_-$. Now write $\epsilon = a\bar{\psi}_-$ with a anticommuting and ψ_- commuting, we have

$$\begin{aligned}
\bar{\psi}_-\gamma^\mu\psi_- &= (\psi_-^t\psi_-, \psi_-^t\gamma_0\gamma^1\psi_-, \psi_-^t\gamma_0\gamma^2\psi_-, \psi_-^t\gamma_0\gamma^3\psi_-) \\
&= (\psi_-^t\psi_-, \psi_-^t\psi_-, \psi_-^t\gamma_0\gamma^2\psi_-, \psi_-^t\gamma_0\gamma^3\psi_-) \\
&= \psi^t\psi(1,1,0,0) \propto k^\mu.
\end{aligned}$$

To get the last equality, we need to do some gamma matrices manipulations:

$$\gamma_-^t\gamma_0\gamma^2\psi_- = \psi_-^t(\gamma_0\gamma^2\gamma^1\gamma^0\gamma^1)\psi_- = \psi_-^t\gamma^0\gamma^1\gamma_0\gamma^2\psi_-$$

and use

$$\gamma_-^t\gamma_0\gamma^2\psi_- = -\psi_-^t\gamma_0\gamma^2\psi_- = 0.$$

In a curved spacetime, the supersymmetry variation on the spinor field is $\delta_\epsilon\psi_\mu = \frac{1}{\kappa}\hat{D}_\mu\epsilon$ where we have used the super covariant derivative. From the previous analysis we see that there is no harm in assuming ϵ to be a commuting (and possibly Dirac) spinor. We can obtain a vector by setting $k^\mu = \bar{\psi}\gamma^\mu\psi$. Typically, k_μ is covariantly constant $\nabla_\lambda k_\mu = 0$ and Killing $\nabla_{(\nu}\kappa_{\mu)} = 0$. Moreover, $k^0 = \psi^\dagger\psi$ (remember $\bar{\psi} = \psi^\dagger\gamma^0$) in a local frame, hence $k^0 > 0$ in all Lorentz frames: it is future directed and causal. If ψ is actually Majorana, then it is lightlike. The probability current for the Dirac equation in this theory is $J^\mu = \psi^\dagger\gamma^\mu\psi$. Note that in such theories we typically get a supply of covariantly constant tensor fields like $\bar{\psi}\gamma_{\mu\nu}\psi$, etc.

In the simplest case of such a theory $\mathcal{N} = 1$, $\Lambda = 0$, we have $\hat{D} = \nabla$, the Levi-Civita connection. In this case we would have $\nabla_\mu\psi = 0$, i.e. ψ is a covariantly constant Killing spinor field. The $k^\mu = \bar{\psi}\gamma^\mu\psi$ is covariantly constant and null, also

$$\nabla_\lambda(\bar{\psi}\gamma_{\mu\nu}\psi) = 0$$

where the expression in bracket is a covariantly constant 2-form.

From the result in general relativity, if we use lightcone coordinates $u = t - x^1$ and $v = t + x^1$, then in this case we can write the metric as

$$ds^2 = -2du\, dv + H(u, x^i)du^2$$

and then

$$\frac{\partial}{\partial v} = k^\mu \frac{\partial}{\partial x^\mu}$$

is a covariantly constant null Killing vector field. If the spacetime we are considering is Ricci flat, then $\partial_i \partial_i H = 0$ for arbitrary u. An example in this case would be

$$H = \alpha(u)((x^1)^2 - (x^2)^2) + 2\beta(u)x^2x^3$$

for arbitrary $\alpha(u)$ and $\beta(u)$. Observe that the α and β represent polarisations $(+)$ and $(\times)$ of a quadruple moment, i.e. describing the behaviour of a classical graviton. Therefore, that classical gravitons are invariant under 2 of the 4 supersymmetry transformations. (For $\mathcal{N} = 1$, $\Lambda = 0$, the vacuum ground state $\mathbb{E}^{3,1}$ is invariant under all 4 supersymmetry transformations.)

4.7 Super Poincaré group and gauge theory

We now attempt to make the supersymmetry that we have used in previous discussions systematic. We will do this by introducing a superspace and a super group acting on it, and finally define a gauge theory analogous to the general relativity case. The action of super Poincaré group on a superspace is

$$\begin{pmatrix} \Lambda^a{}_b & -\frac{1}{4}\bar{\epsilon}\gamma^a & a^a \\ 0 & S(\Lambda)^\alpha{}_\beta & \epsilon^\alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^b \\ \theta^\beta \\ 1 \end{pmatrix}$$

where the action on spinors is $S(\Lambda) = \exp(\frac{1}{4}\lambda_{ab}\gamma^a\gamma^b)$ for $\Lambda^a{}_b = \exp \lambda^a{}_b$. The dimensions of various quantities are

$$[\epsilon] = L^{1/2}, \qquad [\theta] = L^{1/2}, \qquad [a] = L.$$

The "superspace" (x^b, θ^β) is identified with $\mathbb{E}^{3,1} \oplus \mathbb{M}$. We have the Grassmann variables θ and ϵ here, so we need to be careful with signs. As matrix multiplication, this is a left action, and we can interpret this group action as a semidirect product of Lorentz transformations with translations and supertranslations, where supertranslations are

$$\theta \to \theta + \epsilon$$

with the associated translation

$$x^\mu \to x^\mu - \tfrac{1}{4}\bar{\epsilon}\gamma^\mu\theta.$$

Translations and supertranslations together form an invariant subgroup of the whole super Poincaré group. The superspace is then the quotient manifold of the super Poincaré group by the Lorentz group in the usual way.

We can consider functions on this superspace $f(x,\theta)$. When we do "Taylor expansion", the series in θ terminates at finite power: to be exact, at θ^4 due to the Grassmann nature of θ. Consider

$$\begin{aligned}\delta_\epsilon f &= \delta x^\mu \frac{\partial f}{\partial x^\mu} + \delta\theta^\alpha \frac{\partial f}{\partial\theta^\alpha} \\ &= -\tfrac{1}{4}\gamma^\mu\theta\partial_\mu f + \epsilon^\alpha \frac{\partial f}{\partial\theta^\alpha} \equiv \epsilon^\alpha R_\alpha f\end{aligned}$$

where we have defined

$$R_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{4}\theta^\beta\gamma^\mu_{\beta\alpha}\frac{\partial}{\partial x^\mu}.$$

We need to be careful about position of $\delta\theta^\alpha$ in the variation since it is anticommuting. Actually, $\frac{\partial}{\partial\theta^\alpha}$ behaves like an inner multiplication. For R_α, we think of as right invariant vector fields generating left translations on superspace and thus commuting with generators of right translations

$$\{L_\alpha, R_\beta\} = 0, \qquad L_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{4}\theta^\beta\gamma^\mu_{\beta\alpha}\frac{\partial}{\partial x^\mu}$$

the analogy with right and left invariant vector fields on a Lie group is exact (note that left invariant vector fields generate right translations, etc.). To make this analogy more concrete, let us introduce left invariant 1-forms

$$\lambda^\alpha = d\theta^\alpha, \qquad \lambda^\mu = dx^\mu + \tfrac{1}{4}\theta^\alpha\gamma^\mu_{\alpha\beta}d\theta^\beta,$$

and right invariant 1-forms

$$\rho^\alpha = d\theta^\alpha, \qquad \rho^\mu = dx^\mu - \tfrac{1}{4}\theta^\alpha\gamma^\mu_{\alpha\beta}d\theta^\beta.$$

Let us recall the following Maurer-Cartan relations from the theory of Lie groups and Lie algebras, where ${C_A}^B{}_C$ are the structural constants and $A, B, \ldots = 1, 2, \ldots \dim\mathfrak{g}$:

$$\begin{aligned}d\lambda^C &= -{C_A}^C{}_B\,\lambda^A\wedge\lambda^B \\ d\rho^C &= +{C_A}^C{}_B\,\rho^A\wedge\rho^B \\ [L_A, L_B] &= +{C_A}^C{}_B\,L_C \\ [R_A, R_B] &= -{C_A}^C{}_B\,R_C \\ [L_A, R_B] &= 0 \\ {\delta^A}_B &= \rho^A(R_B) = \lambda^A(L_B).\end{aligned}$$

The Maurer-Cartan form for the Lie group can be written $g^{-1}dg = \lambda^A T_A$ where T_A are the basis for the Lie algebra. Also, $dg\,g^{-1} = \rho^A T_A$. In quantum theories, one usually defines the generators after division by i so they can act as quantum mechanical hermitian operators.

Now let us write down similar formulae for our super theory:

$$\begin{aligned}d\lambda^\mu &= +\tfrac{1}{4}d\theta^\alpha\gamma^\mu_{\alpha\beta}d\theta^\beta \\ d\rho^\mu &= -\tfrac{1}{4}d\theta^\alpha\gamma^\mu_{\alpha\beta}d\theta^\beta \\ \{R_\alpha, R_\beta\} &= +\tfrac{1}{2}\gamma^\mu_{\alpha\beta}\tfrac{\partial}{\partial x^\mu} \\ \{L_\alpha, L_\beta\} &= -\tfrac{1}{2}\gamma^\mu_{\alpha\beta}\tfrac{\partial}{\partial x^\mu},\end{aligned}$$

and now the structural constants are

$$C_\alpha{}^\mu{}_\beta = C_\beta{}^\mu{}_\alpha = -\tfrac{1}{2}\gamma^\mu_{\alpha\beta}.$$

Note that for anticommuting objects, $d\theta^\alpha \wedge d\theta^\beta = +d\theta^\beta \wedge d\theta^\alpha$. In the literature, the usual practice is to call our R_α by Q_α, the "generators" and our L_α by D_α, the "covariant derivatives". The anti-commutation relation then reads

$$\{Q_\alpha, Q_\beta\} = \tfrac{1}{2}\gamma^\mu_{\alpha\beta}\tfrac{\partial}{\partial x^\mu}.$$

Let us also use this notation from now on.

Consider $Q(\epsilon_1) = \epsilon_1^\alpha \mathbf{T}_\alpha$ where $\mathbf{T}$ is an $\mathbb{R}$-valued (anti-hermitian) matrix. To write out this matrix representation in full,

$$Q(\epsilon_1) = \begin{pmatrix} 0 & -\frac{1}{4}\bar{\epsilon}_1\gamma & 0 \\ 0 & 0 & \epsilon_1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{T}_\alpha = \begin{pmatrix} 0 & -\frac{1}{4}\gamma^\mu_{\alpha\beta} & 0 \\ 0 & 0 & \delta^\beta{}_\alpha \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{T}_\mu = \begin{pmatrix} 0 & 0 & \delta_\mu{}^\nu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with

$$\mathbf{T}_\alpha\mathbf{T}_\beta + \mathbf{T}_\beta\mathbf{T}_\alpha = -\tfrac{1}{2}\gamma^\mu_{\alpha\beta}\mathbf{T}_\mu \tag{4.1}$$

Then one can check

$$Q(\epsilon_1)Q(\epsilon_2) - Q(\epsilon_2)Q(\epsilon_1) = \epsilon_1^\alpha\mathbf{T}_\alpha\epsilon_2^\beta\mathbf{T}_\beta - \epsilon_2^\alpha\mathbf{T}_\alpha\epsilon_1^\beta\mathbf{T}_\beta = \epsilon_1^\alpha\epsilon_2^\beta\{\mathbf{T}_\alpha, \mathbf{T}_\beta\}.$$

A supersymmetry transformation acting on a superfield then can be written as

$$\delta f = \epsilon^\alpha Q_\alpha f = \epsilon^\alpha R_\alpha f.$$

Again, quantum mechanically, we would write

$$\mathbf{T}_\alpha \to \frac{\hat{Q}_\alpha}{i^{1/2}}, \qquad \mathbf{T}_\mu \to \frac{\hat{P}_\mu}{i}, \qquad \hat{P}_\mu = \frac{\partial_\mu}{i},$$

and then

$$\hat{Q}_\alpha\hat{Q}_\beta + \hat{Q}_\beta\hat{Q}_\alpha = -\tfrac{1}{2}(C\gamma^\mu)\hat{P}_\mu.$$

which is $\frac{1}{2}\hat{P}^0\delta_{\alpha\beta}$ in the rest frame. Since the left hand side is non-negative, The total expectation value is then $\langle\hat{P}_\mu\rangle \geq 0$. Taking the trace, we get $\langle\hat{P}^0\rangle > 0$, and hence $\langle\hat{P}^\mu\rangle$ is future directed timelike or null. This is the positive energy property of supersymmetric theories.

We can now formulate a super Poincaré gauge theory, similar to the Poincaré case. The connection 1-form is

$$\mathcal{A} = \begin{pmatrix} \omega^a{}_b & -\frac{1}{4}\bar{\psi}\gamma^a & e^a \\ 0 & \frac{1}{4}\omega_{ab}\gamma^{ab} & \kappa\psi \\ 0 & 0 & 0 \end{pmatrix}$$

where

$$\psi = \psi_\mu dx^\mu, \qquad [\omega] = L^{-1}.$$

The curvature is

$$\mathcal{F} = \begin{pmatrix} d\omega + \omega \wedge \omega & * & de^a + \omega^a{}_b \wedge e^b - \frac{1}{4}\kappa^2 \bar{\psi}\gamma^a\psi \\ 0 & \frac{1}{4}\gamma^{ab}(d\omega + \omega \wedge \omega) & \kappa(d\psi + \frac{1}{4}\omega_{ef}\gamma^e\gamma^f\psi) \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that now $\kappa(d\psi + \frac{1}{4}\omega_{ef}\gamma^e\gamma^f\psi)$ is the curvature associated with supertranslations, while $de^a + \omega^a{}_b \wedge e^b - \frac{1}{4}\kappa^2\bar{\psi}\gamma^a\psi$ is the curvature associated with translations. We want no curvature for translations, so setting the appropriate term to zero, we get

$$de^a + \omega^a{}_b \wedge e^b = \tfrac{1}{4}\kappa^2\bar{\psi}\gamma^a\psi = T^a$$

— fermionic contribution to torsion.

The super Poincaré group is relevant for Minkowski spacetime. For AdS_4, because $\mathrm{Spin}(3,2) = Sp(4,\mathbb{R})$, the supergroup relevant in this case is the orthosymplectic group $Osp(1|4)$, i.e. those matrices leaving invariant the quadratic form

$$\delta_{ij}x^i x^j + C_{\alpha\beta}\theta^\alpha\theta^\beta$$

where $C_{\alpha\beta} = C_{\beta\alpha}$ is the symplectic form. We can pass from $Osp(1|4)$ to the super Poincaré group by a process called Wigner-Inönü contraction, which we will outline below.

The orthosymplectic group $Osp(M|N)$ has a diagonal subgroup $SO(M) \times Sp(N,\mathbb{R})$, i.e. matrices of the form

$$\begin{pmatrix} SO(M) & \mathbf{0} \\ \mathbf{0} & Sp(N,\mathbb{R}) \end{pmatrix}$$

where elements of the symplectic group $Sp(N,\mathbb{R})$ preserves the symplectic form C_{ab}. The dimensions are

$$\dim_{\mathbb{R}} SO(M) = \tfrac{1}{2}M(M-1), \qquad \dim_{\mathbb{R}} Sp(N,\mathbb{R}) = \tfrac{1}{2}N(N+1).$$

The action of $Osp(M|N)$ on superspace is

$$\begin{pmatrix} \Lambda^i{}_j & \Lambda^i{}_b \\ \Lambda^a{}_j & \Lambda^a{}_b \end{pmatrix} \begin{pmatrix} x^j \\ \theta^b \end{pmatrix}$$

where

$$\begin{aligned} \Lambda^i{}_j &= \delta^i{}_j + \lambda^i{}_j + \ldots & \lambda_{ij} &= \delta_{ik}\lambda^k{}_j, & \lambda_{ij} &= -\lambda_{ji}, \\ \Lambda^a{}_b &= \delta^a{}_b + \lambda^a{}_b + \ldots & \lambda_{ab} &= C_{ac}\lambda^c{}_b, & \lambda_{ab} &= +\lambda_{ba}. \end{aligned}$$

For the isometry group of AdS_4, $Sp(4,\mathbb{R}) = \mathrm{Spin}(3,2)$, we have the generators

$$\Gamma_A = (\gamma_\mu, \gamma_5), \qquad A = 0,1,2,3,4$$

and

$$\Gamma_A\Gamma_B + \Gamma_B\Gamma_A = 2\eta_{AB}, \qquad \eta_{AB} = \mathrm{diag}(-1,1,1,1,-1).$$

For a real (i.e. Majorana) representation of Cliff$(4,1)$,

$$C_5 = \gamma_0\gamma_5, \qquad C_5 = -C_5^t$$

we have the basis which generate $Sp(4,\mathbb{R})$

C_5	1	skew,
$C_5\Gamma_A$	5	skew,
$C_5\Gamma_{AB}$	10	symmetric.

Pass into the supersymmetric case, $Osp(1|4)$ is the super anti-de Sitter group for $\mathcal{N}=1$, and $Osp(\mathcal{N}|4)$ is the $\mathcal{N}$-extended super anti-de Sitter group.

Now consider the generators of $SO(3,2)$, which we will call M_{AB}. In these, the generators $M_{\mu\nu}$ generate Lorentz rotations, while $M_{4,\nu}$ generators non-commuting translations:

$$\left[\frac{M_{4\mu}}{a}, \frac{M_{4\nu}}{a}\right] \propto \frac{M_{\mu\nu}}{a}$$

where a is a length scale set by the radius of the anti de Sitter space. Note the dimension

$$[M_{\mu\nu}] = 1, \qquad [P_\mu] = L$$

so the above can be written as

$$[P_\mu, P_\nu] \propto a^{-2} M_{\mu\nu}.$$

To proceed with the Wigner-Inönü contraction, we take the limit $a \to \infty$. Then in the limit $[P_\mu, P_\nu] = 0$, i.e. Minkowski spacetime. Recall that the anti de Sitter space is defined by

$$x^A x^B \eta_{AB} = -a^2,$$

so the limit $a \to \infty$ corresponds to setting the radius of curvature to infinity, and hence

$$\Lambda = -\frac{3}{a^2} \to 0.$$

If we do a similar analysis for the de Sitter spacetime, the algebra is anti-unitary instead of unitary.

5. Witten's proof of the positive energy theorem

In general relativity, the positive energy theorem states that, assuming the dominant energy condition, the mass of an asymptotically flat spacetime is non-negative; furthermore, the mass is zero only for Minkowski spacetime. This theorem is important not only in physics, but in pure mathematics as well, where it is more commonly known as the positive mass theorem. By far the simplest and most elegant proof is given by Edward Witten in 1981, which is a very good illustration of the power of using spinors and supergravity in doing essentially classical calculations. We will briefly outline Witten's proof in this section.

We begin by defining the Nestor 2-form

$$N^{\mu\nu} = \bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\rho\epsilon$$

where ϵ is a commuting Dirac spinor. To manipulate this, we will use Stokes' theorem. Let $\nabla_\nu N^{\mu\nu} = J^\mu$, and suppose Σ is a spacelike hypersurface in spacetime, e.g. a Cauchy surface. For a domain D with boundary ∂D, Stokes theorem states that

$$\frac{1}{2}\oint_{\partial D} = \nabla^{\mu\nu} d\Sigma_{\mu\nu} = \int_D J^\mu d\Sigma_\mu$$

Now, take increasing domains D such that $\lim D = \Sigma$, $\lim \partial D = S^2_\infty$. Also assume spacetime (M, g_{ab}) is asymptotically flat. Then we have

$$\begin{aligned}
\nabla_\nu N^{\mu\nu} &= \nabla_\nu\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\rho\epsilon + \bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\nu\nabla_\rho\epsilon \\
&= \nabla_\nu\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\rho\epsilon + \bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_{[\nu}\nabla_{\rho]}\epsilon \\
&= \nabla_\nu\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\rho\epsilon + \tfrac{1}{8}\bar{\epsilon}R_{\nu\rho\alpha\beta}\gamma^{\alpha\beta}\epsilon \\
&= \nabla_\nu\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\rho\epsilon + \tfrac{1}{2}\bar{\epsilon}\gamma^\lambda\epsilon G^\mu{}_\lambda \\
&= \nabla_\nu\bar{\epsilon}\gamma^{\mu\nu\rho}\nabla_\rho\epsilon + \tfrac{1}{2}\kappa^2\bar{\epsilon}\gamma^\lambda\epsilon T^\mu{}_\lambda.
\end{aligned}$$

To proceed, we assume the dominant energy condition: $T_{\alpha\beta}V^\alpha W^\beta \geq 0$. Consider this equation for all future directed timelike vectors V^α, W^β, we see that it implies $T_{00} \geq 0$. An equivalent statement is that $T_{\hat{0}\hat{0}} \geq |T_{\hat{\mu}\hat{\nu}}|$ in all Lorentz frames, where hatted indices denote local pseudo orthonormal frames. Hence, $T_{\alpha\beta}W^\beta$ is past directed and timelike for any future directed timelike W^β. Note that in our sign convention, $T^0{}_0 < 0$. Putting all these together, we see that $\frac{1}{2}\kappa^2\bar{\epsilon}\gamma^\lambda\epsilon T^\mu{}_\lambda$ is past directed and timelike.

Now the second term in the expansion of $\nabla_\nu N^{\mu\nu}$ has been taken care of, let us come to the first term. We will use a local pseudo-orthonormal frame e_i adapted to the hypersurface Σ, with e_0 orthonormal to Σ. The zeroth component of the first term in the expansion reads

$$\begin{aligned}
(\nabla_i\epsilon)^\dagger\gamma_0\gamma^{ij}\nabla_j\epsilon &= (\nabla_i\epsilon)^\dagger(\gamma^i\gamma^j - \delta^{ij})\nabla_j\epsilon \\
&= (\gamma^i\nabla_i\epsilon)^2 - (\nabla\epsilon)^2
\end{aligned}$$

where

$$\nabla_i\epsilon = (\partial_i\epsilon + \tfrac{1}{4}\omega_{irs}\gamma^r\gamma^s) + \tfrac{1}{2}\omega_{irs}\gamma^r\gamma^s\epsilon.$$

In this expression for the connection, we recognise the first term as the Levi-Civita connection, whereas the second term can be thought of the second fundamental form of Σ.

Now we impose the following conditions: first, we require $\gamma^i \nabla_i \epsilon = 0$ on Σ. This is called the Witten equation. It is different from the Dirac equation since the Dirac one uses the Levi-Civita connection induced by the metric. Second, we require the boundary condition $\epsilon = \epsilon_0$, a constant spinor at spatial infinity. These conditions together with the above calculation implies $J^\mu = \nabla_\nu N^{\mu\nu}$ is past directed and timelike.

For further manipulation of the second fundamental form introduced above, we will work in a Gaussian normal coordinate. Locally, we write the metric as

$$ds^2 = -dt^2 + g_{ij}(x,t) dx^i dx^j$$

so, to connect with our frame,

$$e^0 = dt, \qquad de^0 = 0 = -\omega_i^0 \wedge e^i, \qquad de^i = \frac{\partial e^i}{\partial t} \wedge dt + d_x e^i = -\omega^i{}_0 \wedge dt - \omega^i{}_k \wedge e^k.$$

We see the emergence of the Levi-Civita connection with respect to g_{ij} in the last expression above.

We write $-\omega^0{}_j = e^i K_{ij}$, then $-\omega_0{}^j = K_{ji} e^i$. Also note that $de^0 = 0$ is equivalent to $K_{ij} = K_{ji}$. Then,

$$\frac{\partial e_i}{\partial t} = K_{ij} e^j, \qquad \frac{\partial}{\partial t}(e^i \otimes_S e^j) = 2K_{ij} e^i \otimes_S e^j$$

so in our coordinate system,

$$K_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial t} = \frac{1}{2}\pounds_n g_{ij}.$$

The constraint equations that needs to be required in this coordinate are

$${}^{(3)}R = K_{ij}K^{ij} - K_i{}^i K_j{}^j, \qquad (K_{ij} - g_{ij} K_k{}^k)^{;l} = 0.$$

Covariantly, this is

$$K_{\alpha\beta} = (\nabla_\alpha n_\beta)_\perp = -\Gamma_\alpha{}^0{}_\beta$$

where n is the unit normal to the hypersurface, $n_\alpha n^\beta = -1$, and $\perp$ is the projection onto Σ using $h^\alpha{}_\beta = \delta^\alpha{}_\beta + n^\alpha n_\beta$.

We now take a detour and outline several applications using this second fundamental form approach. The first is that we can see for the Witten equation

$$\nabla_i \epsilon = \nabla_i^{\text{L.C.}} \epsilon + \tfrac{1}{2} K_{ij} \gamma^0 \gamma^j \epsilon,$$

multiplication by γ^i gives $\gamma^i \nabla_i \epsilon = 0$. The second is for the Einstein-Hilbert action

$$S = \frac{1}{2\kappa^2} \int_D R|e| d^4x + 2\int_{\partial D} (\operatorname{Tr} K_{ij}) \sqrt{g}\, d^3x$$

variation contains only δg terms on boundary and no terms form $\delta(\partial g_{ij}/\partial n)$. The third application is for the necessary condition for the existence of Killing spinors in anti-de Sitter spacetime. Let $A = 0, 1, 2, 3, 4$ label the coordinates in $\mathbb{E}^{3,2}$, on which the anti-de

Sitter spacetime is defined using the usual quadric. The super covariant derivatives on AdS commutes: $[\hat{D}_\mu, \hat{D}_\nu] = 0$, and the existence of Killing spinors is the existence of ϵ satisfying $\hat{D}_\mu \epsilon = 0$. This can be constructed as follows. Let ϵ now denotes a constant spinor in $\mathbb{E}^{3,2}$, so $\nabla_A \epsilon = 0$. Then it is easy to see

$$\nabla^\perp_A \epsilon = 0, \qquad \nabla^\perp_\mu \epsilon = 0$$

and in this case, $K_{\mu\nu} \propto g_{\mu\nu}$ where $g_{\mu\nu}$ is the induced metric.

Let us return to Witten's proof of the positive mass theorem. We now need to solve the equation $\gamma^i \nabla_i \epsilon = 0$ subject to $\epsilon \to \epsilon_0$ at spatial infinity. Since this equation is an elliptical equation, existence of solution follows from standard results from differential equations. For uniqueness, suppose ϵ_1 and ϵ_2 are both solutions satisfying the same boundary conditions. Then $\epsilon = \epsilon_1 - \epsilon_2$ satisfies the formula and the boundary term vanishes. Since $\nabla_\mu \bar{\epsilon} \epsilon^{\mu\nu\rho} \nabla_\rho \epsilon$ is (subject to the Witten condition) past directed and timelike, the volume term has a fixed sign. The boundary term, however, vanishes, since it only depends on ϵ_0. Hence $\epsilon_1 = \epsilon_0$.

The last step in the proof is to note that

$$\frac{2}{\kappa^2} \int_{S^2_\infty} \frac{1}{2} N_{\mu\nu} d\Sigma^{\mu\nu} = \bar{\epsilon}_0 \gamma^\mu \epsilon_0 P^{\mathrm{ADM}}_\mu \leq 0 \tag{5.1}$$

where P^{ADM}_μ is the ADM 4-momentum of spacetime. Hence P^{ADM}_μ is future directed and timelike, if $(M, g_{\mu\nu})$ is asymptotically flat.

We can identify the boundary term with the ADM mass in this theory. First we analyse in more details the ADM mass/momentum. The metric at infinity is:

$$ds^2 \to -\left(1 - \frac{2MG}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2MG}{r}} + r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

where M is the ADM mass. We can write this in isotropic coordinates

$$ds^2 = \frac{(1 - \frac{GM}{2\rho})^2}{(1 + \frac{GM}{2\rho})^2} dt^2 + \left(1 + \frac{GM}{2\rho}\right)^4 d^2\mathbf{x}.$$

This has a momentum $P^\mu = (M, 0, 0, 0)$. We can boost it such that $P^i \neq 0$. This will introduce cross terms in the metric.

We can solve the Witten equation in this metric and evaluate the boundary term. The metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1})$$

so the connection

$$\omega = O(r^{-2})$$

with a factor of M. For $\epsilon = \epsilon_0 + \ldots$, we get

$$D_\mu \epsilon = \bar{\epsilon}_0 \gamma^0 \epsilon_0 \frac{M}{r^2} + \ldots.$$

For the integrand in (5.1), it is of order $O(Mr^{-2})\epsilon$, and $d\Sigma_{\mu\nu}$ is of order $O(r^2)$. The boundary term therefore is of order $M(\bar{\epsilon}_0\gamma^0\epsilon_0)$. For its coefficient, we can either work it out (it is complicated), or we can look at the identity

$$\bar{\epsilon}_0\gamma^\mu\epsilon_0 P_\mu^{\mathrm{ADM}} = \int -T^\mu{}_\lambda\bar{\epsilon}\gamma^\lambda\epsilon dV + \frac{2}{\kappa^2}(\dots)dV.$$

Evaluating near flat spacetime, we get the integrand as

$$\int(-T^\mu{}_\lambda\bar{\epsilon}_0\gamma^\lambda\epsilon_0)d^3x + \dots$$

Now ϵ_0 is a constant spinor in Minkowski spacetime, and hence $\bar{\epsilon}_0\gamma^\mu\epsilon_0$ is a Killing vector of the background. The whole expression is like a total energy. By the linearised Einstein equation this is the total mass M of the linearised theory. However, the boundary term depends only on the asymptotic metric, not on the interior. Thus we can always identify the boundary term with the ADM mass.

We now consider the case where $P_\mu^{\mathrm{ADM}} = 0$, i.e. the case of equality in equation (5.1). Then $\nabla_i\epsilon|_\Sigma = 0$, and some components of $T^\mu{}_\nu|_\Sigma = 0$. Consider another Cauchy surface Σ', with the same boundary at infinity $\partial\Sigma' = S^2_\infty$ (i.e. a finite variation of Σ), we also have $\nabla_i\epsilon|_{\Sigma'} = 0$ and $T^\mu{}_\nu|_{\Sigma'} = 0$. Thus ϵ is actually covariantly constant, $\nabla_\mu\epsilon = 0$, and the dominant energy condition becomes $T^\mu{}_\nu = 0$. Therefore, $k^\mu = \bar{\epsilon}\gamma^\mu\epsilon$ is a covariantly constant Killing field, and $R_{\mu\nu} = 0$ (the Einstein equation). Now, if k^μ is null, and then we have gravitational waves, the spacetime is not asymptotically flat, and hence this case is not allowed. If k^μ is timelike, then it is hypersurface orthogonal by $\nabla_\lambda k_\mu = 0$ and has constant norm $g^{\mu\nu}k_\mu k_\nu$. Then the metric is ultrastatic:

$$ds^2 = -dt^2 + g_{ij}(\mathbf{x})dx^\mu dx^\nu$$

and g_{ij} must admit constant spinor $\nabla_i\epsilon = 0$. Then

$$R_{\mu\nu} = 0 \to {}^{(3)}R_{ij} = 0 \to {}^{(3)}R_{ijrs} = 0 \to g_{ij} = \delta_{ij},$$

i.e. the hypersurface is flat. This remains true in higher dimensions if we quote the theorem that asymptotically flatness and Ricci flatness together imply flatness.

There are some global issues with this approach to the proof of the positive energy theorem: the existence and uniqueness of spin structures. For the moving frames we use, global frames are not essential but convenient. If a global frame exists there is no difficulty in introducing a spin structure. However, uniqueness is more problematic. If the fundamental group is non-trivial, $H^2(M,\mathbb{Z}_2) \neq 0$, then there exists more than one spin structure. They can be odd or even as non-trivial closed curves. If no global frame exists at all, there may not be a spin structure on the manifold. The obstruction in this case is called the second Stiefel-Witney class $W_2 \in H^2(M,\mathbb{Z}_2)$, which vanishes if all 2-cycles are topologically trivial.

In four spacetime dimensions, a globally hyperbolic spacetime always admit a global framing: $M = \mathbb{R}\times\Sigma$ and Σ is spacelike. Also, every 3-manifold is parallelizable, and hence

M admits a global framing and at least one spin structure. If it is not simply connected, it may admit more than one. For our argument of the positive mass theorem to work, we need a spin structure which allows constant spinors at infinity.

If black holes are present in this spacetime, then we can work on the exterior only and use boundary condition on the horizon. Typically, $\epsilon = 0$ on horizon.

The positive mass theorem has extensions: instead of asymptotic flatness, we can require asymptotic AdS_4 behaviour, e.g. the Kottler solution

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{(\dots)} + r^2 d\Omega^2.$$

Here M is called the Abbot-Deser mass. Using $\hat{D}_\mu \epsilon$ as defined with a cosmological constant, we can prove that $M \geq 0$. The case $M = 0$ corresponds to AdS_4 spacetime. The solutions of Witten equation $\gamma^i \hat{D}_i \epsilon = 0$ tend to Killing spinors of AdS_4 at infinity.

6. Central charges and BPS states

BPS stands for Bogomol'nyi, Prasad and Sommerfield. BPS bounds refers to a series of inequalities for solutions of field equations depending only asymptotic behaviour the solutions at infinity (actually, only on the homotopy class of the solution). A BPS state is a solution which saturates this bound. In supersymmetric theories, the BPS bound usually is saturated when half of the SUSY generators are unbroken. This happens when the mass is equal to the central extension, which can be interpreted as a topological conserved charge. In this section we will investigate BPS states briefly.

For $\mathcal{N} \geq 1$, the supersymmetry algebra admits a central extension:

$$\{\hat{Q}^i_\alpha, \hat{Q}^j_\beta\} = -\tfrac{1}{2}(P_\mu C\gamma^\mu)_{\alpha\beta}\delta^{ij} + \tfrac{1}{2}C_{\alpha\beta}X^{ij} + \tfrac{1}{2}(C\gamma_5)_{\alpha\beta}Y^{ij}, \quad X^{ij} = -X^{ji}, \quad Y^{ij} = -Y^{ji},$$

where X and Y commutes with all elements of the algebra and can be thought of electric and magnetic charges respectively. In fact, we can obtain this algebra by contraction of $Osp(\mathcal{N}|4)$. Witten and Olive showed that such conserved central charges can arise as boundary terms in supersymmetric field theories.

For example, for $\mathcal{N} = 2$, $X^{ij} = X\epsilon^{ij}$, they showed that $M \geq \sqrt{X^2 + Y^2}$ and the equality case is invariant under half of the maximal supersymmetry transformations. In the rest frame, the right hand side of the algebra relation is

$$M\delta^{ij} + (X\gamma_0 + Y\gamma_0\gamma_5)\epsilon^{ij},$$

which is a 8×8 matrix. Passing into Dirac notation $\epsilon^{ij} \to \sqrt{-1} = i$, it becomes a 4×4 hermitian matrix

$$A = M + iR, \qquad R = X\gamma_0 + Y\gamma_0\gamma_5.$$

Then $R^2 = -(X^2 + Y^2)$ implies

$$A^2 = M^2 + 2iMR - R^2 = 2MA - M^2 + X^2 + Y^2$$

hence we have the characteristic equation

$$A^2 - 2MA + M^2 - (X^2 + Y^2) = 0.$$

The eigenvalues are obtained from the characteristic equation as

$$\lambda_\pm = M \pm \sqrt{X^2 + Y^2}.$$

Since $\operatorname{Tr} A = 4M$, we see that $\lambda_\pm$ are both doubly degenerate. The BPS state has $M = \sqrt{X^2 + Y^2}$ and A has a kernel the dimension of which is half of the dimension of A, i.e. half of the maximal supersymmetry. Another way to say this is that there exists 2 linear combinations of the supersymmetry generators which annihilate the BPS states.

Let us now focus on BPS states in $\mathcal{N} = 2$ supergravity. The BPS states are invariant under $4\mathcal{N}/2 = 2\mathcal{N}$ supersymmetries. After diagonalisation,

$$A = \operatorname{diag}(M + \sqrt{X^2 + Y^2}, M + \sqrt{X^2 + Y^2}, M + \sqrt{X^2 - Y^2}, M + \sqrt{X^2 - Y^2}).$$

We implement it as follows. The bosonic part of the supergravity lagrangian is the Einstein-Maxwell theory. Define

$$L_B = \frac{R}{4M^2} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu},$$

then

$$T_{\mu\nu} = (F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho})$$

satisfies the dominant energy condition. We have

$$\hat{D}_\epsilon \epsilon = \nabla_\mu \epsilon + \frac{i\kappa}{2\sqrt{2}}(F_{\rho\sigma}\gamma^\rho\gamma^\sigma)\gamma_\mu \epsilon.$$

We can now identify boundary terms. $N_{\mu\nu}$ contains $F_{\rho\sigma}$, and

$$X = \frac{Q}{\sqrt{4\pi G}}, \qquad Y = \frac{P}{\sqrt{4\pi G}}$$

where Q and P are the electric and magnetic charges. Since the electromagnetic potential is asymptotically

$$A_\mu dx^\mu \to \frac{Q}{4\pi r} + \frac{P\cos\theta}{4\pi}d\phi,$$

the first term can be thought of as the Coulomb potential and the second the Dirac monopole. Because we are ungauged since the charges are central, neither Q or P is quantised. The BPS bound on the mass is

$$M \geq \sqrt{\frac{2Q^2}{\kappa^2} + \frac{2P^2}{\kappa^2}}$$

and for BPS states, the equality holds.

We can compare this with the Reissner-Nordström black hole:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2+P^2}{4\pi r^2}\right)dt^2 + \frac{dr^2}{(\dots)} + r^2 d\Omega^2$$

in which case we have

$$M \leq \sqrt{\frac{Q^2+P^2}{4\pi G}}.$$

Then

$$\frac{r_\pm}{G} = M \pm \sqrt{M^2 - \frac{(Q^2+P^2)}{4\pi G}}.$$

The extremal case is that for $r_+ = r_-$. If we draw Penrose diagrams for the black hole case and the supergravity case, we see that they complement each other.

The Hawking temperature for this black hole can be calculated

$$T_H = \frac{1}{2\pi G}\frac{\sqrt{M^2 - \frac{1}{4\pi G}(P^2+Q^2)}}{\left(M + \sqrt{M^2 - \frac{1}{4\pi G}(P^2+Q^2)}\right)^2}$$

the temperature $T \searrow 0$ as $M \searrow \sqrt{\frac{1}{4\pi G}(P^2+Q^2)}$. For non-extremal cases, $k = \frac{\partial}{\partial t}$ becomes spacelike for $r_- < r < r_+$, so it cannot be a BPS state. Also note that the Hawking temperature is finite, so the theory can never be supersymmetric, since the Fermi-Dirac statistics is different from the Bose-Einstein statistics at all finite temperatures. The supersymmetric generators cannot even act in such a case.

The extremal black hole case is compatible with the BPS condition, for which

$$T_H = \frac{\sqrt{M^2 - \frac{1}{4\pi G}(P^2+Q^2)}}{Gr_+^2}.$$

Let us calculate the Killing spinors. $\hat{D}_\mu \epsilon = 0$ implies the Dirac equation $\gamma^\mu \nabla_\mu \epsilon = 0$, hence

$$\gamma^\mu \gamma_{\alpha\beta} \gamma_\mu = 0.$$

Let us solve the Dirac equation. If

$$ds^2 = \Omega^2 g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\mu\nu} dx^\mu dx^\nu.$$

Then $\not{D}\psi = \Omega^{-(n+1)/2} \tilde{\not{D}} \tilde{\psi}$ where $\psi = \Omega^{-(n-1)/2}\tilde{\psi}$. Introduce isotropic coordinates $A = A(\mathbf{x})$ and $B = B(\mathbf{x})$, we have

$$ds^2 = -A^2 dt^2 + B^2 d\mathbf{x}^2 = A^2 \left(-dt^2 + \frac{B^2}{A^2} d\mathbf{x}^2 \right),$$

an ultra static metric. Then

$$(\gamma^0 \partial_0 + \gamma^i \tilde{\nabla}_i)\tilde{\psi} = 0.$$

Assume $\partial_t \tilde{\psi} = 0$ on the 3-dimensional surface, we have

$$\gamma^i \tilde{\nabla}_i \tilde{\psi} = 0.$$

Then

$$\gamma^i \partial_i \tilde{\tilde{\psi}} = 0,$$

for the flat Dirac operator on $\mathbb{E}^3$. Hence

$$\psi = \frac{1}{\sqrt{AB}} \frac{1}{\sqrt{B}} \tilde{\tilde{\psi}}.$$

Let us try the solution $\tilde{\tilde{\psi}} = \epsilon_0$ a constant. Then

$$\bar{\psi}\gamma^\mu \psi \propto \bar{\epsilon}_0 \gamma^\mu \epsilon_0.$$

Now choose $\gamma_0 \epsilon_0 = i\epsilon_0$, then

$$\bar{\psi}\gamma^\mu\psi \propto k^\mu = \delta^\mu{}_0,$$

however, the factor involved here is not a constant.

In general, the solution is not regular near horizon at which $A = 0$. The exceptional case occurs when $AB = 1$. There if

$$ds^2 = -H^{-2} dt^2 + H^2 d\mathbf{x}^2, \qquad H = 1 + \sum_{a=1}^{k} \frac{GM_a}{|\mathbf{x} - \mathbf{x}_0|},$$

we have the Majumdar-Papeptrou solution. For $k = 1$, we have the extremal Reissner-Nordström solution again, and

$$\bar{\psi}\gamma^\mu\psi \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}.$$