ON THE NON-STANDARD PODLEŚ SPHERES

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ABSTRACT. It was shown in [1] that the C*-completion of Podleś' generic quantum spheres $A_{q\rho}$ [4] is independent of the parameter ρ. In the present note we provide a proof that this is not true for the $A_{q\rho}$ themselves which remained a conjecture in [1]. As a byproduct we obtain that $\mathrm{Aut}(A_{q\rho}) = \mathbb{C}^\times$

1. Introduction

The quantum spheres of Podleś [4] constitute a family of algebras $A_{q\rho}$, $q \in \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$ not a root of unity, $\rho \in \mathbb{C} \cup \{\infty\}$ that can be considered as deformations of the complex coordinate ring of the real affine variety $S^2 \subset \mathbb{R}^3$. They can be embedded as left coideal subalgebras into the standard quantized coordinate ring $\mathbb{C}_q[SL(2)]$ and become in this way the paradigmatic examples of homogeneous spaces of quantum groups. If $q \in \mathbb{R}$ and $\rho \in \mathbb{R} \cup \{\infty\}$, then $A_{q\rho}$ are $*$-subalgebras of the 'compact real form' of $\mathbb{C}_q[SL(2)]$. See e.g. [3] for details and more information.

It was shown in [1] that the C*-completion of these $*$-algebras does not depend on ρ, but it remained a conjecture that this is not the case for $A_{q\rho}$ themselves. The present contribution gives a proof of this fact, see Theorem 1 below.

It is a pleasure to thank the authors of [1] for pointing out to me this problem and for all the other discussions we had.

2. The algebras $A_{q\rho}$ and some of their properties

Let $q \in \mathbb{C}^\times$ be not a root of unity and $\rho \in \mathbb{C}$. Define $A_{q\rho}$ as the unital associative algebra with generators x_{-1}, x_0, x_1 and relations

$$x_0 x_{\pm 1} = q^{\pm 2} x_{\pm 1} x_0, \quad x_{\mp 1} x_{\pm 1} = q^{\pm 2} x_0^2 + (1 + q^{\pm 2}) \rho x_0 - 1. \tag{1}$$

Analogously one defines $A_{q\infty}$ by the relations

$$x_0 x_{\pm 1} = q^{\pm 2} x_{\pm 1} x_0, \quad x_{\mp 1} x_{\pm 1} = q^{\pm 2} x_0^2 + (1 + q^{\pm 2}) x_0. \tag{2}$$

The defining relations imply (see [3], p. 125 for the details) that the elements

$$e_{ij} := \begin{cases} x_0^i x_1^j & j \geq 0 \\ x_0^i x_{-1}^{-j} & j < 0. \end{cases}, i \in \mathbb{N}_0, j \in \mathbb{Z}$$

form a vector space basis of $A_{q\rho}$. It is immediate that $A_{q\rho}$ is $\mathbb{Z}$-graded,

$$A_{q\rho} = \bigoplus_{j \in \mathbb{Z}} A^j, \quad A^j := \mathrm{span}\{e_{ij} \mid i \in \mathbb{N}_0\} = \{f \in A_{q\rho} \mid x_0 f = q^{2j} f x_0\}.$$

We denote by I the ideal generated by x_0 and by $\pi : A_{q\rho} \to A_{q\rho}/I$ the canonical projection. Using the basis $\{e_{ij}\}$ one sees that $I = x_0 A_{q\rho} = A_{q\rho} x_0$.

Proposition 1. *$A_{q\rho}$ is an integral domain and any invertible element is a scalar.*

Supported by an EU Marie Curie postdoctoral fellowship.

Proof. $A_{q\rho}$ can be embedded into the quantized coordinate ring $\mathbb{C}_q[SL(2)]$ ([3], Proposition 4.31) which has these properties ([2], 9.1.9 and 9.1.14). □

Besides this we will need the well-known and easily verified fact that the following is a complete list of the characters of $A_{q\rho}$:

$$\begin{aligned} \rho \neq \infty, \pm i: &\quad \chi_\lambda(x_0) = 0, \chi_\lambda(x_{\pm 1}) = \lambda^{\pm 1}, \quad \lambda \in \mathbb{C}^\times, \\ \rho = \pm i: &\quad \chi_\lambda(x_0) = 0, \chi_\lambda(x_{\pm 1}) = \lambda^{\pm 1}, \quad \lambda \in \mathbb{C}^\times, \\ &\quad \chi'(x_{\pm 1}) = 0, \chi'(x_0) = \mp i, \\ \rho = \infty: &\quad \chi^\pm_\lambda(x_{\pm 1}) = \chi^\pm_\lambda(x_0) = 0, \chi^\pm_\lambda(x_{\mp 1}) = \lambda, \quad \lambda \in \mathbb{C}. \end{aligned}$$

We denote by $J \subset A_{q\rho}$ the intersection of the kernels of all characters. For $\rho \neq \infty, \pm i$ an element $x = \sum_{ij} \xi_{ij} e_{ij} \in A_{q\rho}$, $\xi_{ij} \in \mathbb{C}$, is mapped by χ_λ to $f(\lambda)$, where f is the Laurent polynomial $f(z) = \sum_{j \in \mathbb{Z}} \xi_{0j} z^j$. Thus $\chi_\lambda(x) = 0$ for all $\lambda \in \mathbb{C}^\times$ iff $f = 0$. Hence $J = I$. The same is true for $\rho = \infty$ as one checks similarly. For $\rho = \pm i$ one obtains the smaller ideal $I \cap \ker \chi'$.

3. The algebra $A_{q\rho}$ depends on ρ

The aim of this note is to prove the following fact that was conjectured in [1]:

Theorem 1. *The algebras $A_{q\rho}, A_{q\rho'}$ are isomorphic iff $\rho' = \pm\rho$ ($-\infty = \infty$).*

Proof. We first note that $A_{q\infty}$ can not be isomorphic to $A_{q\rho}$ with $\rho \neq \infty$: Otherwise $A_{q\infty}/J$ would be isomorphic to $A_{q\rho}/J$. The first algebra is isomorphic to $\mathbb{C}[z] \oplus \mathbb{C}[z]$ with $\pi(x_{\pm 1})$ as generators. This follows from adding $x_0 = 0$ to (2). For $\rho \neq \infty, \pm i$ the algebra $A_{q\rho}/J$ is instead isomorphic to $\mathbb{C}[z, z^{-1}]$ with $z^{\pm 1}$ corresponding to $\pm\pi(x_{\pm 1})$. For $\rho = \pm i$ we have $J = I \cap \ker \chi' \subset I$, and $A_{q\pm i}/I$ is as above isomorphic to $\mathbb{C}[z, z^{-1}]$. That is, this is a quotient algebra of $A_{q\pm i}/J$, hence the latter can also not be isomorphic to $A_{q\infty}/J = \mathbb{C}[z] \oplus \mathbb{C}[z]$.

Suppose now that $\psi : A_{q\rho'} \to A_{q\rho}$ is an isomorphism with $\rho, \rho' \neq \infty$. We denote by $X_i \in A_{q\rho}$ the images of the generators of $A_{q\rho'}$ under ψ.

Since X_i generate $A_{q\rho}$, $\pi(X_i)$ generate $\pi(A_{q\rho}) = \mathbb{C}[z, z^{-1}]$. This algebra is a commutative integral domain, so $\pi(X_0)\pi(X_{\pm 1}) = q^{\pm 2}\pi(X_{\pm 1})\pi(X_0)$ implies that either $\pi(X_0)$ or both $\pi(X_{\pm 1})$ vanish. But $\mathbb{C}[z, z^{-1}]$ can not be generated by a single element, so $\pi(X_0) = 0$. Hence $X_0 = \lambda_0 x_0$ for some $\lambda_0 \in A_{q\rho}$. Repeating the whole argumentation with the roles of x_i and X_i interchanged one gets $x_0 = \mu_0 X_0$, that is, $X_0 = \mu_0 \lambda_0 X_0$ for some $\mu_0 \in A_{q\rho}$. Proposition 1 now implies $\lambda_0 = \mu_0^{-1} \in \mathbb{C}^\times$.

Therefore $x_0 X_{\pm 1} = q^{\pm 2} X_{\pm 1} x_0$. Hence $X_{\pm 1} \in A^{\pm 1}$, so $X_{\pm 1} = P_\pm(x_0) x_{\pm 1}$ for some polynomials $P_\pm \in \mathbb{C}[z]$. Inserting this into (1) one sees that both $P_\pm$ must be of degree zero. So $X_i = \lambda_i x_i$ for three non-zero constants λ_i. Inserting this again into the relations (1) we get

$$q^{\pm 2}\lambda_0^2 x_0^2 + (1 + q^{\pm 2})\rho' \lambda_0 x_0 - 1 = \lambda_1 \lambda_{-1}(q^{\pm 2} x_0^2 + (1 + q^{\pm 2})\rho x_0 - 1),$$

which is equivalent to

$$\lambda_0 = \pm 1, \quad \rho' = \pm\rho, \quad \lambda_1 \lambda_{-1} = 1.$$

If conversely $\rho' = -\rho$, then it is immediate that the assignment $x_{-1}, x_0, x_1 \mapsto x_{-1}, -x_0, x_1$ extends to an isomorphism $A_{q\rho'} \to A_{q\rho}$. □

Note that we have proven en passent (for $\rho \neq \infty$, $\rho = \infty$ is treated analogously):

Corollary 1. *The map $\lambda \mapsto \sigma_\lambda$, $\sigma_\lambda(x_i) = \lambda^i x_i$ is an isomorphism $\mathbb{C}^\times \to \mathrm{Aut}(A_{q\rho})$.*

References

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