

ON THE HOCHSCHILD HOMOLOGY OF QUANTUM $SL(N)$ TOM HADFIELD¹, ULRICH KRÄHMER²

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ABSTRACT. We show that the quantized coordinate ring $\mathcal{A} := k_q[SL(N)]$ satisfies van den Bergh's analogue of Poincaré duality for Hochschild (co)homology with dualizing bimodule being \mathcal{A}_σ , the \mathcal{A} -bimodule which is \mathcal{A} as k -vector space with right multiplication twisted by the modular automorphism σ of the Haar functional. This implies that $H_{N^2-1}(\mathcal{A}, \mathcal{A}_\sigma) \cong k$, generalizing our previous result for $k_q[SL(2)]$.

1. INTRODUCTION AND STATEMENT OF THE RESULT

According to the Hochschild-Kostant-Rosenberg theorem [11], the dimension of a regular affine variety V over an algebraically closed field k of characteristic 0 can be expressed in terms of the Hochschild homology of its coordinate ring $k[V]$ as

$$(1) \quad \dim(V) = \sup\{n \geq 0 \mid \mathrm{HH}_n(k[V]) \neq 0\}.$$

However, even for well-behaved noncommutative algebras Hochschild homology is often rather degenerate. For example, the standard quantized coordinate ring $\mathcal{A} := k_q[SL(N)]$ is for generic q Auslander regular and Cohen-Macaulay with global and Gelfand-Kirillov dimension equal to the classical dimension $N^2 - 1$ of $SL(N)$ [15], but $\mathrm{HH}_n(\mathcal{A}) = 0$ for $n \geq N$ [6]. In this note we show that this “dimension drop” is overcome by passing to Hochschild homology $H_*(\mathcal{A}, \mathcal{M})$ with coefficients in a suitable bimodule \mathcal{M} .

The cosemisimple Hopf algebra structure on \mathcal{A} determines the Haar functional $h : \mathcal{A} \rightarrow k$ which is left and right invariant under the coaction of \mathcal{A} on itself, and there is a unique automorphism $\sigma \in \mathrm{Aut}(\mathcal{A})$, the so-called modular automorphism, such that $h(xy) = h(\sigma(y)x)$ for all $x, y \in \mathcal{A}$ (see [13], Section 11.3). The crucial coefficient bimodule \mathcal{M} is then \mathcal{A}_σ which is \mathcal{A} as k -vector space with bimodule structure $x \triangleright y \triangleleft z := xy\sigma(z)$. Our main result is:

Theorem 1.1. *There is an isomorphism of k -vector spaces $H_{N^2-1}(\mathcal{A}, \mathcal{A}_\sigma) \cong k$.*

For $N = 2$ this was shown by explicit calculation in [10]. The proof for arbitrary N given below relies on the following analogue of Poincaré duality for Hochschild (co)homology proven by van den Bergh:

Theorem 1.2. [19] *Let \mathcal{X} be a smooth algebra such that there exists $d_{\mathcal{X}} \in \mathbb{N}$ with $H^n(\mathcal{X}, \mathcal{X}^e) = 0$ for $n \neq d_{\mathcal{X}}$, and that $\mathcal{U}_{\mathcal{X}} := H^{d_{\mathcal{X}}}(\mathcal{X}, \mathcal{X}^e)$ is an invertible \mathcal{X} -bimodule. Then for every \mathcal{X} -bimodule \mathcal{M} we have*

$$(2) \quad H^n(\mathcal{X}, \mathcal{M}) \cong H_{d_{\mathcal{X}}-n}(\mathcal{X}, \mathcal{U}_{\mathcal{X}} \otimes_{\mathcal{X}} \mathcal{M}).$$

Here $\mathcal{X}^e := \mathcal{X} \otimes \mathcal{X}^{\text{op}}$ is the enveloping algebra of \mathcal{X} (throughout this paper an unadorned \otimes means tensor product over k), so the Hochschild homology and cohomology groups of \mathcal{X} with coefficients in \mathcal{M} are $H_n(\mathcal{X}, \mathcal{M}) = \text{Tor}_n^{\mathcal{X}^e}(\mathcal{M}, \mathcal{X})$ and $H^n(\mathcal{X}, \mathcal{M}) = \text{Ext}_{\mathcal{X}^e}^n(\mathcal{X}, \mathcal{M})$, respectively. Following [19] (erratum) an algebra \mathcal{X} is called smooth if it has finite projective dimension $\text{pd}_{\mathcal{X}^e}(\mathcal{X}) = \inf\{n \geq 0 \mid H^{n+1}(\mathcal{X}, \cdot) = 0\}$ as an \mathcal{X}^e -module. As in [3] we call $\text{pd}_{\mathcal{X}^e}(\mathcal{X})$ the dimension of \mathcal{X} and denote it by $\dim(\mathcal{X})$. As pointed out by van den Bergh, \mathcal{X} is smooth if and only if \mathcal{X}^e has finite global dimension. This follows from $\text{gldim}(\mathcal{X}) \leq \dim(\mathcal{X}) \leq \text{gldim}(\mathcal{X}^e)$ and $\dim(\mathcal{X} \otimes \mathcal{Y}) \leq \dim(\mathcal{X}) + \dim(\mathcal{Y})$ ([3], Propositions 7.4-7.6), which gives $\text{gldim}(\mathcal{X}^e) \leq \dim(\mathcal{X}^e) \leq 2 \dim(\mathcal{X}) \leq 2 \text{gldim}(\mathcal{X}^e)$. In the sequel we say that an algebra has the Poincaré duality property if it satisfies the assumptions of Theorem 1.2.

The principal technical result of this note consists of remarking successively that Theorem 1.2 applies to the quantized coordinate rings $\mathcal{B} := k_q[M(N)]$ of $N \times N$ -matrices, $\mathcal{C} := k_q[GL(N)]$ and $\mathcal{A} = k_q[SL(N)]$. Theorem 1.1 then follows from the well-known fact that the center of \mathcal{A} consists only of the scalars.

Our main motivation for studying $H_*(\mathcal{A}, \mathcal{A}_{\sigma})$ is the so-called twisted cyclic cohomology and its link to covariant differential calculi over quantum groups both due to Kustermans, Murphy and Tuset [14]. Twisted cyclic cohomology is defined by a cyclic object in the sense of Connes [4] depending on an algebra \mathcal{X} and an automorphism σ . Its underlying simplicial homology is $H_*(\mathcal{X}, \mathcal{X}_{\sigma})$ (at least when σ is diagonalizable, see Proposition 2.1 in [10]). The volume forms of covariant differential calculi over quantum groups define twisted cyclic cocycles, with the appearance of the twisting automorphism forced by the modular properties of the Haar functional that replaces the traces of Connes' original construction [4]. In view of Theorem 1.2 twisted coefficients appear very naturally also for purely homological reasons, and Theorem 1.1 and similar results for quantum hyperplanes and Podleś quantum spheres [9, 18] show that the twist determines as in the classical case a unique class of top degree in Hochschild homology.

2. PROOF OF THEOREM 1.1

We first consider the quantized coordinate ring $\mathcal{B} = k_q[M(N)]$. Recall that this has generators u_{ij} , $1 \leq i, j \leq N$, with relations

$$(3) \quad \begin{aligned} u_{ik}u_{il} &= qu_{il}u_{ik}, & u_{ik}u_{jk} &= qu_{jk}u_{ik}, & u_{ik}u_{jk} &= qu_{jk}u_{ik}, \\ u_{jk}u_{jl} &= qu_{jl}u_{jk}, & u_{il}u_{jk} &= u_{jk}u_{il}, & u_{ik}u_{jl} - u_{jl}u_{ik} &= (q - q^{-1})u_{il}u_{jk} \end{aligned}$$

for all $i < j, k < l$. Here $q \in k \setminus \{0\}$ is a fixed deformation parameter, assumed not to be a root of unity.

Proposition 2.1. \mathcal{B} has the Poincaré duality property with $d_{\mathcal{B}} = N^2$ and $\mathcal{U}_{\mathcal{B}} = \mathcal{B}_{\sigma}$, with σ defined by

$$(4) \quad \sigma(u_{ij}) := q^{2(N+1-i-j)}u_{ij}$$

We will use here and later the following Künneth-type isomorphism of Cartan and Eilenberg:

Theorem 2.2. [3], *Theorem XI.3.1.* Let k be a field, $\mathcal{A}_1, \mathcal{A}_2$ be two left Noetherian k -algebras and $\mathcal{M}_i, \mathcal{N}_i$ be finitely generated left modules over \mathcal{A}_i . Then

$$(5) \quad \bigoplus_{i+j=n} \text{Ext}_{\mathcal{A}_1}^i(\mathcal{M}_1, \mathcal{N}_1) \otimes \text{Ext}_{\mathcal{A}_2}^j(\mathcal{M}_2, \mathcal{N}_2) \cong \text{Ext}_{\mathcal{A}_1 \otimes \mathcal{A}_2}^n(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{N}_1 \otimes \mathcal{N}_2).$$

Proof of Proposition 2.1. The claim follows from Proposition 2 in [19]: As mentioned in [16] it follows from a result of Priddy ([17], Theorem 5.3) that \mathcal{B} is a graded Koszul algebra. By definition the Koszul dual $\mathcal{B}^!$ has generators \hat{u}_{ij} with relations orthogonal to (3):

$$(6) \quad \begin{aligned} \hat{u}_{ij}^2 &= 0 \quad \forall i, j, & \hat{u}_{ik}\hat{u}_{il} &= -q^{-1}\hat{u}_{il}\hat{u}_{ik}, & \hat{u}_{ik}\hat{u}_{jk} &= -q^{-1}\hat{u}_{jk}\hat{u}_{ik}, \\ \hat{u}_{ik}\hat{u}_{jk} &= -q^{-1}\hat{u}_{jk}\hat{u}_{ik}, & \hat{u}_{jk}\hat{u}_{jl} &= -q^{-1}\hat{u}_{jl}\hat{u}_{jk}, & \hat{u}_{ik}\hat{u}_{jl} &= -\hat{u}_{jl}\hat{u}_{ik}, \\ \hat{u}_{il}\hat{u}_{jk} + \hat{u}_{jk}\hat{u}_{il} &= (q^{-1} - q)\hat{u}_{ik}\hat{u}_{jl}, \end{aligned}$$

where $i < j, k < l$. These relations imply that the monomials $\hat{u}_{i_1 j_1} \cdots \hat{u}_{i_n j_n}$, $n = 1, \dots, N^2$, $i_1 j_1 \prec \dots \prec i_n j_n$ with respect to lexicographical ordering, form a k -linear basis, and that $\mathcal{B}^!$ is Frobenius with Frobenius functional $\hat{h} : \mathcal{B}^! \rightarrow k$ being projection onto the component of the longest basis element $\hat{u}_{11}\hat{u}_{12} \cdots \hat{u}_{N N-1}\hat{u}_{NN}$ (that is, for each nonzero $x \in \mathcal{B}^!$ there exists $y \in \mathcal{B}^!$ with $\hat{h}(xy) \neq 0$). The formula for σ follows by straightforward computation using the relations (6).

Smoothness of \mathcal{B} follows from some well-known facts about Koszul algebras (see e.g. the survey [7]). First, $\text{Tor}_n^{\mathcal{B}^{\sigma}}(k, k) \cong \text{Ext}_{\mathcal{B}^e}^n(k, k)$, and by Theorem 2.2 and Koszulity this can be written as $\sum_{i+j=n} \mathcal{B}_i^! \otimes \mathcal{B}_j^!$ (note that $\mathcal{B} \cong \mathcal{B}^{\text{op}}$). Thus $\text{Tor}_n^{\mathcal{B}^{\sigma}}(k, k) = 0$ for $n > 2N^2$, hence $\dim(\mathcal{B}) \leq 2N^2$ by [1], Corollary 8.7.5. \square

It was shown by Farinati that the class of algebras having the Poincaré duality property is closed under localization [5], Theorem 1.5. The quantized coordinate ring $\mathcal{C} = k_q[GL(N)]$ of the general linear group is the localization of $\mathcal{B} = k_q[M(N)]$ at the central quantum determinant [16]

$$(7) \quad \det_q = \sum_{\pi \in S_N} (-q)^{l(\pi)} u_{1\pi(1)} \cdots u_{N\pi(N)},$$

with S_N the permutation group on N elements and $l(\pi)$ the length of a permutation. This is σ -invariant, so σ passes to an automorphism of \mathcal{C} , still denoted by σ and given by (4). Proposition 2.1 now implies:

Corollary 2.3. \mathcal{C} has the Poincaré duality property with $d_{\mathcal{C}} = d_{\mathcal{B}} = N^2$ and $\mathcal{U}_{\mathcal{C}} = \mathcal{C} \otimes_{\mathcal{B}} \mathcal{U}_{\mathcal{B}} = \mathcal{C}_{\sigma}$.

The algebra $\mathcal{A} = k_q[SL(N)]$ is the quotient of \mathcal{B} by the relation $\det_q = 1$, and again by σ -invariance of \det_q , σ descends to an automorphism of \mathcal{A} . Following the strategy of Lévassieur and Stafford [15] we will use the isomorphism $\mathcal{C} \cong \mathcal{A} \otimes D$, where $D := k[t, t^{-1}]$ to deduce Poincaré duality for \mathcal{A} from \mathcal{B} via \mathcal{C} . This enables us to prove finally:

Proposition 2.4. *The algebra \mathcal{A} has the Poincaré duality property with $d_{\mathcal{A}} = N^2 - 1$ and $\mathcal{U}_{\mathcal{A}} = \mathcal{A}_{\sigma}$.*

Proof. We apply Theorem 2.2 with $\mathcal{A}_1 = \mathcal{N}_1 = \mathcal{A}^e$, $\mathcal{M}_1 = \mathcal{A}$ and $\mathcal{A}_2 = \mathcal{N}_2 = D^e$, $\mathcal{M}_2 = D$. Since $\mathcal{A} = \mathcal{A}^{\text{op}}$ (the antipode of the standard Hopf algebra structure gives an isomorphism) we have $\mathcal{A}^e \cong k_q[SL(N) \times SL(N)]$, so it is (both left and right) Noetherian by [12], Proposition 9.2.2 and further \mathcal{A} is smooth by [8]. It is elementary to show that D satisfies Poincaré duality with $d_D = 1$, $\mathcal{U}_D = D$. So $\text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, \mathcal{A}^e) \otimes D \cong \text{Ext}_{\mathcal{C}^e}^{n+1}(\mathcal{C}, \mathcal{C}^e)$ for each $n \geq 0$. So by Corollary 2.3 we have $\text{Ext}_{\mathcal{A}^e}^{N^2-1}(\mathcal{A}, \mathcal{A}^e) \otimes D = \mathcal{C}_{\sigma}$, which is $\mathcal{A}_{\sigma} \otimes D$, and all other $\text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, \mathcal{A}^e)$ vanish. The result follows. \square

Thus there is an isomorphism $H_{N^2-1}(\mathcal{A}, \mathcal{A}_{\sigma}) \cong H^0(\mathcal{A}, \mathcal{A})$. The latter is by definition the center of \mathcal{A} , and this consists only of the scalars (see e.g. [12], Theorem 9.3.20). This completes the proof of Theorem 1.1.

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