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Closed Forms of Some Definite Integrals

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Abstract This paper studies two types of definite integrals and uses Maple for verification. The closed forms of these definite integrals can be obtained using Poisson integral formula. On the other hand, some examples are used to demonstrate the calculations.

Keywords: *definite integrals, closed forms, Poisson integral formula, Maple*

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1. Introduction

In calculus and engineering mathematics, there are many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. In this paper, we study the following two types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \frac{\cos^m \theta}{r^2 - 2rs\cos(\theta-\phi) + s^2} d\theta , \qquad (1)$$

$$\int_0^{2\pi} \frac{\sin^m \theta}{r^2 - 2rs\cos(\theta-\phi) + s^2} d\theta , \qquad (2)$$

where r, s, ϕ are real numbers, $|s| < |r|$, and m is a positive integer. We can obtain the closed forms of these definite integrals using Poisson integral formula; these are the major results of this paper (i.e., Theorem A). Adams et al. [1], Nyblom [2], and Oster [3] provided some techniques to solve the integral problems. Yu [4-29], Yu and B. -H. Chen [30], Yu and Sheu [31,32,33], and T. -J. Chen and Yu [34,35,36] used some methods including complex power series method, integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, area mean value theorem, and generalized Cauchy integral formula to solve some types of integrals. In this article, some examples are used to demonstrate the proposed calculations, and the manual calculations are verified using Maple.

2. Main Results

Some notations and formulas used in this paper are introduced below.

2.1. Notations

2.1.1. Let t be a real number, the largest integer less than or equal to t is denoted as $\lfloor t \rfloor$.

2.1.2. Suppose that a is a real number, then $(a)_p = a(a-1)\cdots(a-p+1)$ for positive integers $p \le a$; $(a)_0 = 1$.

2.2. Formulas

2.2.1. Euler's formula

$e^{ix} = \cos x + i\sin x$, where $i = \sqrt{-1}$, and x is any real number.

2.2.2. DeMoivre's formula:

$(\cos x + i\sin x)^m = \cos mx + i\sin mx$, where m is any integer, and x is any real number.

2.2.3. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$**, where α, β are real numbers.**

An important formula used in this study is introduced below, which can be found in [[37], p 145].

2.2.4. Poisson integral formula:

Suppose that r, s are real numbers, and $|s| < |r|$. If f is defined and continuous on the closed disc $\{z \in C \,|\, |z| \le |r|\}$ and is analytic on the open disc $\{z \in C \,|\, |z| < |r|\}$, then:

$$f(se^{i\phi}) = \frac{r^2 - s^2}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{r^2 - 2rs\cos(\theta-\phi) + s^2} d\theta .$$

2.2.5. Binomial Theorem

$(u+v)^n = \sum_{p=0}^{n} \frac{(n)_p}{p!} u^{n-p} v^p$, where u, v are complex numbers, and n is a positive integer.

Before deriving the major results in this study, two lemmas are needed.

Lemma 1 Suppose that θ is a real number, and m is a positive integer. Then:

$$\cos^m\theta = \frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cos(m-2p)\theta + \frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!} \quad (3)$$

and

$$\sin^m\theta = \frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\left[\begin{array}{l}\cos[(m-2p)\theta]\cos\frac{(m-2p)\pi}{2}\\ +\sin[(m-2p)\theta]\sin\frac{(m-2p)\pi}{2}\end{array}\right] + \frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!} \quad (4)$$

Proof

$$\cos^m\theta = \left[\frac{1}{2}(e^{i\theta}+e^{-i\theta})\right]^m \text{ (by Euler's formula)}$$
$$= \frac{1}{2^m}\sum_{p=0}^{m}\frac{(m)_p}{p!}(e^{i\theta})^{m-p}(e^{-i\theta})^p$$
(by binomial theorem)
$$= \frac{1}{2^m}\sum_{p=0}^{m}\frac{(m)_p}{p!}e^{i(m-2p)\theta} \text{ (by DeMoivre's formula)}$$
$$= \frac{1}{2^m}\sum_{p=0}^{m}\frac{(m)_p}{p!}\cos(m-2p)\theta$$
$$= \frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cos(m-2p)\theta + \frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}$$

On the other hand,

$$\sin^m\theta = \cos^m\left(\theta-\frac{\pi}{2}\right)$$
$$= \frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cos\left[(m-2p)\left(\theta-\frac{\pi}{2}\right)\right] + \frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!} \text{ (by Eq. (3))}$$
$$= \frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\left[\begin{array}{l}\cos[(m-2p)\theta]\cos\frac{(m-2p)\pi}{2}\\ +\sin[(m-2p)\theta]\sin\frac{(m-2p)\pi}{2}\end{array}\right] + \frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!} \text{ (by Formula 2.2.3)}$$

Lemma 2 Assume that r, s, ϕ are real numbers, $|s|<|r|$, and k is a non-negative integer. Then:

$$\int_0^{2\pi}\frac{\cos k\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta = \frac{2\pi}{r^2-s^2}\left(\frac{s}{r}\right)^k\cos k\phi, \quad (5)$$

$$\int_0^{2\pi}\frac{\sin k\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta = \frac{2\pi}{r^2-s^2}\left(\frac{s}{r}\right)^k\sin k\phi \quad (6)$$

Proof Because $f(z)=z^k$ is analytic on the whole complex plane. Using Poisson integral formula yields:

$$(se^{i\phi})^k = \frac{r^2-s^2}{2\pi}\int_0^{2\pi}\frac{(re^{i\theta})^k}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta. \quad (7)$$

By Euler's formula and DeMoivre's formula, we have:

$$s^k e^{ik\phi} = \frac{r^2-s^2}{2\pi}\int_0^{2\pi}\frac{r^k e^{ik\theta}}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta. \quad (8)$$

Thus,

$$\int_0^{2\pi}\frac{e^{ik\theta}}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta = \frac{2\pi}{r^2-s^2}\left(\frac{s}{r}\right)^k e^{ik\phi} \quad (9)$$

By the equality of real parts of both sides of Eq. (9), we obtain Eq. (5). The equality of imaginary parts of both sides of Eq. (9) yields Eq. (6) holds.

In the following, we determine the closed forms of the definite integrals (1) and (2).

Theorem A If r, s, ϕ are real numbers, $|s|<|r|$, and m is a positive integer, then the definite integrals:

$$\int_0^{2\pi}\frac{\cos^m\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta = \frac{2\pi}{2^{m-1}(r^2-s^2)}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\left(\frac{s}{r}\right)^{m-2p}\cos(m-2p)\phi + \frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}\cdot\frac{2\pi}{r^2-s^2} \quad (10)$$

and

$$\int_0^{2\pi}\frac{\sin^m\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta = \frac{2\pi}{2^{m-1}(r^2-s^2)}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cdot\left(\frac{s}{r}\right)^{m-2p}\cos\left[(m-2p)\left(\phi-\frac{\pi}{2}\right)\right] + \frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}\cdot\frac{2\pi}{r^2-s^2} \quad (11)$$

Proof

$$\int_0^{2\pi}\frac{\cos^m\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta$$

$$=\int_0^{2\pi}\frac{\frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cos(m-2p)\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta$$

$$+\int_0^{2\pi}\frac{\frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta\left(\text{by Eq. }(3)\right)$$

$$=\frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\int_0^{2\pi}\frac{\cos(m-2p)\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta$$

$$+\frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}\cdot\int_0^{2\pi}\frac{1}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta$$

$$=\frac{2\pi}{2^{m-1}(r^2-s^2)}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\left(\frac{s}{r}\right)^{m-2p}\cos(m-2p)\phi$$

$$+\frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}\cdot\frac{2\pi}{r^2-s^2}$$

On the other hand,

$$\int_0^{2\pi}\frac{\sin^m\theta}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta$$

$$=\int_0^{2\pi}\frac{\frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cos[(m-2p)\theta]\cos\frac{(m-2p)\pi}{2}}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta$$

$$+\int_0^{2\pi}\frac{\frac{1}{2^{m-1}}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\sin[(m-2p)\theta]\sin\frac{(m-2p)\pi}{2}}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta$$

$$+\int_0^{2\pi}\frac{\frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}}{r^2-2rs\cos(\theta-\phi)+s^2}d\theta\left(\text{by Eq. }(4)\right)$$

$$=\frac{2\pi}{2^{m-1}(r^2-s^2)}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cos\frac{(m-2p)\pi}{2}\cdot\left(\frac{s}{r}\right)^{m-2p}\cos(m-2p)\phi$$

$$+\frac{2\pi}{2^{m-1}(r^2-s^2)}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\sin\frac{(m-2p)\pi}{2}\cdot\left(\frac{s}{r}\right)^{m-2p}\sin(m-2p)\phi$$

$$+\frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}\cdot\frac{2\pi}{r^2-s^2}\left(\text{by Eqs. }(5)\text{ and }(6)\right)$$

$$=\frac{2\pi}{2^{m-1}(r^2-s^2)}\sum_{p=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\frac{(m)_p}{p!}\cdot\left(\frac{s}{r}\right)^{m-2p}\cos\left[(m-2p)\left(\phi-\frac{\pi}{2}\right)\right]$$

$$+\frac{1+(-1)^m}{2}\cdot\frac{1}{2^m}\cdot\frac{(m)_{\lfloor m/2\rfloor}}{(\lfloor m/2\rfloor)!}\cdot\frac{2\pi}{r^2-s^2}$$

3. Examples

In the following, for the two types of definite integrals in this study, we provide some examples and use Theorem A to determine their closed forms. In addition, Maple is used to calculate the approximations of these definite integrals and their solutions for verifying our answers.

3.1. Example In Eq. (10), if $r=4, s=3, \phi=\pi/3$, and $m=5$, then the definite integral:

$$\int_0^{2\pi}\frac{\cos^5\theta}{25-24\cos(\theta-\pi/3)}d\theta=\frac{\pi}{56}\left[\left(\frac{3}{4}\right)^5\cdot\frac{1}{2}-\left(\frac{3}{4}\right)^3\cdot5+\left(\frac{3}{4}\right)\cdot5\right]=\frac{3603\pi}{114688} \qquad (12)$$

Next, we use Maple to verify the correctness of Eq. (12).

```
>evalf(int((cos(theta))^5/(25-24*cos(theta-Pi/3)),
theta=0..2*Pi),18);
0.0986952281919993812
>evalf(3603*Pi/114688,18);
0.0986952281919993812
```

On the other hand, let $r=6, s=5, \phi=\pi/4$, and $m=4$ in Eq. (10), then we obtain:

$$\int_0^{2\pi}\frac{\cos^4\theta}{61-60\cos(\theta-\pi/4)}d\theta=\frac{3263\pi}{57024}. \qquad (13)$$

We also use Maple to verify the correctness of Eq. (13).

```
>evalf(int((cos(theta))^4/(61-60*cos(theta-Pi/4)),
theta=0..2*Pi),18);
0.179766709256865449
>evalf(3263*Pi/57024,18);
0.179766709256865449
```

3.2. Example In Eq. (11), if $r=-3, s=2, \phi=-\pi/6$, and $m=3$, then the definite integral:

$$\int_0^{2\pi}\frac{\sin^3\theta}{13+12\cos(\theta+\pi/6)}d\theta=\frac{19\pi}{270}. \qquad (14)$$

Using Maple to verify the correctness of Eq. (14) as follows:

```
>evalf(int((sin(theta))^3/(13+12*cos(theta+Pi/6)),theta
=0..2*Pi),18);
0.221075038585948413
>evalf(19*Pi/270,18);
0.221075038585948413
```

In addition, let $r=4, s=-3, \phi=3\pi/4$, and $m=8$ in Eq. (11), then:

$$\int_0^{2\pi}\frac{\sin^8\theta}{25+24\cos(\theta-3\pi/4)}d\theta=\frac{1719713\pi}{29360128}. \qquad (15)$$

```
>evalf(int((sin(theta))^8/(25+24*cos(theta-3*Pi/4)),
theta=0..2*Pi),18);
0.184012744327370238
>evalf(1719713*Pi/29360128,18);
0.184012744327370238
```

4. Conclusion

In this paper, we use Poisson integral formula to solve two types of definite integrals. In fact, the applications of this formula are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers.

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