SOME ELLIPTIC PDES ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Abstract. The goal of this paper is to investigate some rigidity properties of stable solutions of elliptic equations set on manifolds with boundary.

We provide several types of results, according to the dimension of the manifold and the sign of its Ricci curvature.

Contents

1. Introduction

Let $(\mathcal{M}, \bar{g})$ be a complete, connected, smooth, $n+1$-dimensional manifold with boundary $\partial\mathcal{M}$, endowed with a smooth Riemannian metric $\bar{g} = \{\bar{g}_{ij}\}_{i,j=1,\dots,n}$.

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The volume element writes in local coordinates as

$$dV_{\bar g} = \sqrt{|\bar g|}\, dx^1 \wedge \cdots \wedge dx^n, \tag{1.1}$$

where $\{dx^1, \dots, dx^n\}$ is the basis of 1-forms dual to the vector basis $\{\partial_i, \dots, \partial_n\}$ and we use the standard notation $|\bar g| = \det(\bar g_{ij}) \geqslant 0$.

We denote by $\mathrm{div}_{\bar g} X$ the divergence of a smooth vector field X on $\mathcal{M}$, that is, in local coordinates,

$$\mathrm{div}_{\bar g} X = \frac{1}{\sqrt{|\bar g|}} \partial_i \Big(\sqrt{|\bar g|} X^i \Big),$$

with the Einstein summation convention.

We also denote by $\nabla_{\bar g}$ the Riemannian gradient and by $\Delta_{\bar g}$ the Laplace-Beltrami operator, that is, in local coordinates,

$$(\nabla_{\bar g}\phi)^i = \bar g^{ij} \partial_j \phi \tag{1.2}$$

and

$$\Delta_{\bar g}\phi = \mathrm{div}_{\bar g}(\nabla_{\bar g}\phi) = \frac{1}{\sqrt{|\bar g|}} \partial_i \Big(\sqrt{|\bar g|} \bar g^{ij} \partial_j \phi \Big),$$

for any smooth function $\phi : \mathcal{M} \to \mathbb{R}$.

We set $\langle \cdot, \cdot \rangle$ to be the scalar product induced by $\bar g$.

Given a vector field X, we also denote

$$|X| = \sqrt{\langle X, X \rangle}.$$

Also (see, for instance Definition 3.3.5 in [Jos98]), it is customary to define the Hessian of a smooth function ϕ as the symmetric 2-tensor given in a local patch by

$$(H_{\bar g}\phi)_{ij} = \partial^2_{ij}\phi - \Gamma^k_{ij} \partial_k \phi,$$

where Γ^k_{ij} are the Christoffel symbols, namely

$$\Gamma^k_{ij} = \frac{1}{2} \bar g^{hk} \left(\partial_i \bar g_{hj} + \partial_j \bar g_{ih} - \partial_h \bar g_{ij} \right).$$

Given a tensor A, we define its norm by $|A| = \sqrt{AA^*}$, where A^* is the adjoint.

The present paper is devoted to the study of special solutions of elliptic equations on manifolds with boundary and is, in some sense, a follow up to the paper by the authors and Farina (see [FSV08b]) where the case without boundary was investigated. In an Euclidean context, i.e. $\mathcal{M} = \mathbb{R}^{n+1}_+$ with the flat metric, the rigidity features of the stable solutions has been investigated in [SV09, CS09].

Boundary problems are related (via a theorem of Caffarelli and Silvestre [CS07]) to non local equations involving fractional powers of the Laplacian. An analogue of the results of [CS07] has been obtained in a geometric context, by means of scattering theory (see [FG02, GJMS92, GZ03]).

In this paper, we will focus on the following two specific models:

- **product manifolds** of the type
$$\Big(\mathcal{M} = M \times \mathbb{R}^+, \bar{g} = g + |dx|^2\Big)$$
where (M, g) is a complete, smooth Riemannian manifold without boundary, and
- **the hyperbolic halfspace**, i.e.
$$\Big(\mathcal{M} = \mathbb{H}^{n+1}, \bar{g} = \frac{|dy|^2 + |dx|^2}{x^2}\Big)$$
where $x > 0$ and $y \in \mathbb{R}^n$.

Notice that the above models comprise both the positive and the negative curvature cases.

We denote by ν the exterior derivative at points of $\partial\mathcal{M}$.

We will investigate the two following problems

$$\left\{ \begin{array}{c} \Delta_{\bar{g}} u = 0 \quad \text{in } \mathcal{M} = M \times \mathbb{R}^+, \\ \partial_\nu u = f(u) \quad \text{on } M \times \{0\} \, . \end{array} \right. \tag{1.3}$$

and

$$\left\{ \begin{array}{c} -\Delta_{\bar{g}} u - s(n-s)u = 0 \quad \text{in } \mathcal{M} = \mathbb{H}^{n+1}, \\ \partial_\nu u = f(u) \quad \text{on } \partial\mathbb{H}^{n+1}. \end{array} \right. \tag{1.4}$$

where f is a $C^1(\mathcal{M})$ nonlinearity (in fact, up to minor modifications, the proofs we present also work for locally Lipschitz nonlinearities).

The real parameter s in (1.4) is chosen to be
$$s = \frac{n}{2} + \gamma,$$
where $\gamma \in (0, 1)$.

We recall that the problem in (1.3) has been studied in the context of conformal geometry and it is related to conformally compact Einstein manifolds (see section 4.1 below for a further discussion).

We will consider weak solutions of (1.3) and (1.4). Namely, we say that u is weak solution of (1.3) if, for every $\xi \in C_0^\infty(M \times \mathbb{R})$, we have

$$\int_{\mathcal{M}} \langle \nabla_{\bar{g}} u, \nabla_{\bar{g}} \xi \rangle = \int_{\partial\mathcal{M}} f(u)\xi. \tag{1.5}$$

Analogously, we say that u is weak solution of (1.4) if, for every $\xi \in C^\infty(\mathcal{M})$, we have that

$$\int_{\mathcal{M}} \langle \nabla_g u, \nabla_g \xi \rangle - s(n-s) \int_{\mathcal{M}} u\xi = \int_{\partial\mathcal{M}} f(u)\xi. \tag{1.6}$$

We focus on an important class of solutions of (1.3) and (1.4), namely the so called stable solutions.

These solutions play an important role in the calculus of variations and are characterized by the fact that the second variation of the energy functional is non negative definite. This condition may be explicitly written in our case by saying that a solution u of either (1.3) and or (1.4) is stable if

$$\int_{\mathcal{M}} |\nabla_{\bar g}\xi|^2 dV_{\bar g} - s(n-s)\varepsilon \int_{\mathcal{M}} \xi^2 - \int_{\partial\mathcal{M}} f'(u)\xi^2\, dV_{\bar g} \geqslant 0 \tag{1.7}$$

for every $\xi \in C_0^\infty(M\times\mathbb{R})$ with $\varepsilon = 0$ in case of (1.3), and for every $\xi \in C^\infty(\mathcal{M})$ and with $\varepsilon = 1$ in case of (1.4).

1.1. **Results for product manifolds.** Now we present our results in the case of product manifolds $\mathcal{M} = M \times \mathbb{R}^+$.

Theorem 1.1. *Assume that the metric on $\mathcal{M} = M\times\mathbb{R}^+$ is given by $\bar g = g + |dx|^2$.*

Assume furthermore that M is compact and satisfies

$$Ric_g \geqslant 0$$

with Ric_g not vanishing identically.

Then every bounded stable weak solution u of (1.3) *is constant.*

We remark that the assumption on the boundedness of u is needed as the following example shows: the function $u(x,y) = x$ is a stable solution of

$$\begin{cases} \Delta_{\bar g} u = 0 & \text{in}\, M\times\mathbb{R}^+, \\ \partial_\nu u = -1 & \text{on}\, M\times\{0\}\,. \end{cases}$$

From theorem 1.1, one also obtains the following Liouville-type theorem for the half-Laplacian on compact manifolds (for the definition and basic functional properties of fractional operators see, e.g., [Kat95]):

Theorem 1.2. *Let (M,g) be a compact manifold and $u: M\to\mathbb{R}$ be a smooth bounded solution of*

$$(-\Delta_g)^{1/2} u = f(u), \tag{1.8}$$

with

$$\int_{\mathcal{M}} (|\nabla_g \xi|^2 + |\nabla_x \xi|^2) - \int_{\partial\mathcal{M}} f'(u)\xi^2 \geqslant 0, \tag{1.9}$$

for every $\xi \in C_0^\infty(\mathcal{M})$.

Assume furthermore that

$$Ric_g \geqslant 0$$

and Ric_g does not vanish identically.

Then u is constant.

Results for $(-\Delta_g)^\alpha$ with $\alpha \in (0,1)$ may be obtained with similar techniques as well.

Theorem 1.3. *Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is given by $\bar{g} = g + |dx|^2$, that M is complete, and*

$$Ric_g \geqslant 0,$$

with Ric_g not vanishing identically.

Assume also that, for any $R > 0$, the volume of the geodesic ball B_R in M (measured with respect to the volume element dV_g) is bounded by $C(R+1)$, for some $C > 0$.

Then every bounded stable weak solution u of (1.3) *is constant.*

Next theorem is a flatness result when the Ricci tensor of M vanishes identically:

Theorem 1.4. *Assume that the metric on $\mathcal{M} = M \times \mathbb{R}^+$ is given by $\bar{g} = g + |dx|^2$ and Ric_g vanishes identically.*

Assume also that, for any $R > 0$, the volume of the geodesic ball B_R in M (measured with respect to the volume element dV_g) is bounded by $C(R+1)$, for some $C > 0$.

Then for every $x > 0$ and $c \in \mathbb{R}$, every connected component of the submanifold

$$\mathcal{S}_x = \{y \in M, \;\; u(x,y) = c\}$$

is a geodesic.

1.2. **Results for the hyperbolic space.** The next theorem provides a flatness result when the manifold $\mathcal{M}$ is $\mathbb{H}^3$.

Theorem 1.5. *Let $n = 2$.*

Let u be a smooth weak solution of (1.4) and let $s = \frac{n}{2} + \gamma$ where $\gamma \in (0,1)$.

Also, suppose that either

$$\partial_{y_2} u > 0 \tag{1.10}$$

or

$$f' \leqslant 0 \;\; \text{on } \partial\mathbb{H}^{n+1}. \tag{1.11}$$

Then, for every $x > 0$ and $c \in \mathbb{R}$, each of the submanifold

$$\mathcal{S}_x = \left\{y \in \mathbb{R}^n, \;\; | \; u(x,y) = c x^{n-s}\right\}$$

is a Euclidean straight line.

As discussed in details in section 4.2, the proof of theorem 1.5 contains two main ingredients:

(1) We first notice that the metric on $\mathbb{H}^{n+1}$ is conformal to the flat metric on $\mathbb{R}^{n+1}_{+}$.
(2) We then use some results by the authors in [SV09] (see also [CS09] for related problems) to get the desired result.

The rest of this paper is structured as follows. In section 2 we prove a geometric inequality for stable solutions in product manifolds, from which we obtain the proofs of theorems 1.1–1.4, contained in section 3. Then, in section 4, we consider the hyperbolic case and we prove theorem 1.1.

2. The case of product manifolds and a weighted Poincaré inequality for stable solutions of (1.3)

Now we deal with the case of product manifolds $M \times \mathbb{R}^+$.

In order to simplify notations, we write ∇ instead of $\nabla_{\bar g}$ for the gradient on $M \times \mathbb{R}^+$ but we will keep the notation ∇_g for the Riemannian gradient on M.

Recalling (1.2), we have that

$$\nabla = (\nabla_g, \partial_x). \tag{2.12}$$

In the subsequent theorem 2.1, we obtain a formula involving the geometry, in a quite implicit way, of the level sets of stable solutions of (1.3).

Such a formula may be considered a geometric version of the Poincaré inequality, since the L^2-norm of the gradient of any test function bounds the L^2-norm of the test function itself. Remarkably, these L^2-norms are weighted and the weights have a neat geometric interpretation.

These type of geometric Poincaré inequalities were first obtained by [SZ98a, SZ98b] in the Euclidean setting, and similar estimates have been recently widely used for rigidity results in PDEs (see, for instance, [FSV08a, SV09, FV09]).

Theorem 2.1. *Let u be a stable solution of* (1.3) *such that $\nabla_g u$ is bounded.*

Then, for every $\varphi \in C_0^\infty(M \times \mathbb{R})$, the following inequality holds:

$$\begin{aligned} &\int_{M\times\mathbb{R}^+} \Big\{ Ric_g(\nabla_g u, \nabla_g u) + |H_g u|^2 - |\nabla_g |\nabla_g u||^2 \Big\} \varphi^2 \\ &\qquad \leqslant \int_{M\times\mathbb{R}^+} |\nabla_g u|^2 |\nabla \varphi|^2. \end{aligned} \tag{2.13}$$

Notice that only the geometry of M comes into play in formula (2.13).

Proof. First of all, we recall the classical Bochner-Weitzenböck formula for a smooth function $\phi : \mathcal{M} \to \mathbb{R}$ (see, for instance, [BGM71, Wan05] and references therein):

$$\frac{1}{2}\Delta_{\bar g}|\nabla_{\bar g}\phi|^2 = |H_{\bar g}\phi|^2 + \langle \nabla_{\bar g}\Delta_{\bar g}\phi, \nabla_{\bar g}\phi\rangle + Ric_{\bar g}(\nabla_{\bar g}\phi, \nabla_{\bar g}\phi). \tag{2.14}$$

The proof of theorem 2.1 consists in plugging the test function $\xi = |\nabla_g u|\varphi$ in the stability condition (1.7): after a simple computation, this gives

$$\begin{aligned} &\int_{\mathcal{M}} \varphi^2|\nabla|\nabla_g u||^2 + \frac{1}{2}\langle \nabla|\nabla_g u|^2, \nabla\varphi^2\rangle + |\nabla_g u|^2|\nabla\varphi|^2 \\ &\qquad - \int_M f'(u)|\nabla_g u|^2\varphi^2 \geqslant 0. \end{aligned} \tag{2.15}$$

Also, by recalling (2.12), we have

$$\langle \nabla|\nabla_g u|^2, \nabla\varphi^2\rangle = \langle \nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle + \partial_x|\nabla_g u|^2\partial_x\varphi^2. \tag{2.16}$$

Moreover, since M is boundaryless, we can use on M the Green formula (see, for example, page 184 of [GHL90]) and obtain that

$$\begin{aligned} \int_{\mathcal{M}} \langle \nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle &= \int_{\mathbb{R}^+}\int_M \langle \nabla_g|\nabla_g u|^2, \nabla_g\varphi^2\rangle \\ &= -\int_{\mathbb{R}^+}\int_M \Delta_g|\nabla_g u|\varphi^2 = -\int_{\mathcal{M}} \Delta_g|\nabla_g u|\varphi^2. \end{aligned} \tag{2.17}$$

Hence, using (2.14), (2.16) and (2.17), we conclude that

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{M}} \langle \nabla|\nabla_g u|^2, \nabla\varphi^2\rangle &= \frac{1}{2}\int_{\mathcal{M}} \partial_x|\nabla_g u|^2\partial_x\varphi^2 \\ &\quad - \int_{\mathcal{M}} \varphi^2\Big\{|H_g u|^2 + \langle \nabla_g\Delta_g u, \nabla_g u\rangle + Ric_g(\nabla_g u, \nabla_g u)\Big\}. \end{aligned} \tag{2.18}$$

Using the equation in (1.3), we obtain that

$$\Delta_g u = -\partial_{xx}u,$$

so (2.18) becomes

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{M}} \langle \nabla|\nabla_g u|^2, \nabla\varphi^2\rangle &= \frac{1}{2}\int_{\mathcal{M}} \partial_x|\nabla_g u|^2\partial_x\varphi^2 \\ &\quad - \int_{\mathcal{M}} \varphi^2|H_g u|^2 + \int_{\mathcal{M}} \varphi^2\langle \nabla_g\partial_{xx}u, \nabla_g u\rangle - \int_{\mathcal{M}} \varphi^2 Ric_g(\nabla_g u, \nabla_g u). \end{aligned} \tag{2.19}$$

Furthermore, Integrating by parts, we see that

$$\begin{aligned}\int_{\mathcal{M}} \partial_x |\nabla_g u| \partial_x \varphi^2 &= \int_M \int_0^{+\infty} \partial_x |\nabla_g u| \partial_x \varphi^2 \\ &= -\int_M \Big(\partial_x |\nabla_g u| \varphi^2 \Big)|_{x=0} - \int_M \int_0^{+\infty} \partial_{xx} |\nabla_g u| \varphi^2 \\ &= -\int_M \Big(\partial_x |\nabla_g u| \varphi^2 \Big)|_{x=0} - \int_{\mathcal{M}} \partial_{xx} |\nabla_g u| \varphi^2 .\end{aligned}$$

Consequently, (2.19) becomes

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{M}} \langle \nabla |\nabla_g u|^2, \nabla \varphi^2 \rangle = \\ &\quad -\int_{\mathcal{M}} \varphi^2 \Big\{ \frac{1}{2} \partial_{xx} |\nabla_g u|^2 + |H_g u|^2 + Ric_g(\nabla_g u, \nabla_g u) \Big\} \\ &\quad + \int_{\mathcal{M}} \varphi^2 \langle \nabla_g \partial_{xx} u, \nabla_g u \rangle - \frac{1}{2} \Big(\partial_x |\nabla_g u|^2 \varphi^2 \Big)|_{x=0}.\end{aligned} \tag{2.20}$$

Now, we use the boundary condition in (1.3) to obtain that, on M,

$$f'(u) \nabla_g u = \nabla_g (f(u)) = \nabla_g \partial_\nu u = -\nabla_g \partial_x u.$$

Therefore,

$$\begin{aligned}&-\frac{1}{2}\int_M \Big(\partial_x |\nabla_g u|^2 \varphi^2 \Big)|_{x=0} - \int_M \langle \nabla_g u_x, \nabla_g u \rangle \varphi^2 \\ &\qquad = \int_M f'(u) |\nabla_g u|^2 \varphi^2 .\end{aligned} \tag{2.21}$$

All in all, by collecting the results in (2.15), (2.20), and (2.21), we obtain that

$$\begin{aligned}&\int_{\mathcal{M}} \varphi^2 |\nabla |\nabla_g u||^2 - \int_{\mathcal{M}} \varphi^2 \Big\{ \frac{1}{2} \partial_{xx} |\nabla_g u|^2 + |H_g u|^2 + Ric_g(\nabla_g u, \nabla_g u) \Big\} \\ &\quad + \int_{\mathcal{M}} \varphi^2 \langle \nabla_g \partial_{xx} u, \nabla_g u \rangle + \int_{\mathcal{M}} |\nabla_g u|^2 |\nabla \varphi|^2 \geqslant 0.\end{aligned} \tag{2.22}$$

Also, we observe that

$$\begin{aligned}|\partial_x |\nabla_g u||^2 + \langle \nabla_g \partial_{xx} u, \nabla_g u \rangle - \frac{1}{2} \partial_{xx} |\nabla_g u|^2 &= \\ |\partial_x |\nabla_g u||^2 - |\partial_x \nabla_g u|^2 &\leqslant 0\end{aligned}$$

by the Cauchy-Schwarz inequality.

Accordingly,

$$|\nabla |\nabla_g u||^2 = |\nabla_g |\nabla_g u||^2 + |\partial_x |\nabla_g u||^2 \leqslant \frac{1}{2} \partial_{xx} |\nabla_g u|^2 - \langle \nabla_g \partial_{xx} u, \nabla_g u \rangle .$$

This and (2.22) give (2.13). □

3. Proof of theorems 1.1–1.4

With (2.13) at hand, one can prove theorems 1.1–1.4.

For this scope, first, we recall the following lemma, whose proof can be found in section 2 of [FSV08b].

Lemma 3.1. *For any smooth $\phi : \mathcal{M} \to \mathbb{R}$, we have that*

$$|H_{\bar g}\phi|^2 \geqslant \left|\nabla_{\bar g}|\nabla_{\bar g}\phi|\right|^2 \qquad \text{almost everywhere.} \tag{3.23}$$

Moreover, we have the following result:

Lemma 3.2. *Let u be a bounded solution of (1.3). Assume that*

$$Ric_g \geqslant 0$$

and that Ric_g does not vanish identically on M.

Suppose that

$$Ric_g(\nabla_g u, \nabla_g u) \text{ vanishes identically on } \mathcal{M}. \tag{3.24}$$

Then, u is constant on $\mathcal{M}$.

Proof. By assumption, we have that Ric_g is strictly positive definite in a suitable non empty open set $U \subseteq M$.

Then, (3.24) gives that $\nabla_g u$ vanishes identically in $U \times \mathbb{R}^+$.

This means that, for any fixed $x \in \mathbb{R}^+$, the map $U \ni y \mapsto u(x, y)$ does not depend on y. Accordingly, there exists a function $\tilde u : \mathbb{R}^+ \to \mathbb{R}$ such that $u(x, y) = \tilde u(x)$, for any $y \in U$.

Thus, from (1.3),

$$0 = \Delta_{\bar g} u = \tilde u_{xx} \qquad \text{in } U \times \mathbb{R}^+$$

and so there exist a, $b \in \mathbb{R}$ for which

$$u(x, y) = \tilde u(x) = a + bx \qquad \text{for any } x \in \mathbb{R}^+ \text{ and any } y \in U.$$

Since u is bounded, we have that $b = 0$, so u is constant in $U \times \mathbb{R}^+$.

By the unique continuation principle (see Theorem 1.8 of [Kaz88]), we have that u is constant on $M \times \mathbb{R}^+$. □

3.1. Proof of theorem 1.1.

Points in $\mathcal{M}$ will be denoted here as (x, y), with $x \in \mathbb{R}^+$ and $y \in M$.

Take φ in (2.13) to be the function

$$\varphi(x, y) = \phi(\frac{x}{R})$$

where $R > 0$ and ϕ is a smooth cut-off, that is $\phi = 0$ on $|x| \geqslant 2$ and $\phi = 1$ on $|x| \leqslant 1$.

We remark that this is an admissible test function, since M is assumed to be compact in theorem 1.1. Moreover, we remark that

$$|\nabla\varphi(x,y)| \leqslant \frac{\|\phi\|_{C^1(\mathbb{R})}\ \chi_{(0,2R)}(x)}{R}. \tag{3.25}$$

Also, since u is bounded, elliptic regularity gives that ∇u is bounded in $M \times \mathbb{R}^+$.

Therefore, using (2.13), lemma 3.1 and (3.25), we obtain

$$\int_{M\times\mathbb{R}^+} \Big\{ Ric_g(\nabla_g u, \nabla_g u) \Big\} \varphi^2 \leqslant \frac{C}{R^2} \int_{M\times(0,2R)} dV_{\bar g} \leqslant \frac{C}{R} \tag{3.26}$$

for some constant $C > 0$.

Sending $R \to +\infty$ and using the fact that $Ric_g \geqslant 0$, we conclude that $Ric_g(\nabla_g u, \nabla_g u)$ vanishes identically.

Thus, by lemma 3.2, we deduce that u is constant.

3.2. **Proof of theorem 1.2.** We put coordinates $x \in \mathbb{R}^+$ and $y \in M$ for points in $\mathcal{M} = M \times \mathbb{R}^+$.

Given a smooth and bounded $u_o : M \to \mathbb{R}$, we can define the harmonic extension $\mathcal{E}u_o : M \times \mathbb{R}^+ \to \mathbb{R}$ as the unique bounded function solving

$$\begin{cases} \Delta_{\bar g}(\mathcal{E}u_o) = 0 & \text{in } M \times \mathbb{R}^+, \\ \mathcal{E}u_o = u_o & \text{on } M \times \{0\}. \end{cases} \tag{3.27}$$

See Section 2.4 of [CSM05] for furter details.

Then, we define

$$\mathcal{L}u_o := \partial_\nu(\mathcal{E}u_o)\Big|_{x=0}. \tag{3.28}$$

We claim that, for any point in $M \to \mathbb{R}$,

$$-\partial_x(\mathcal{E}u_o) \;=\; \mathcal{E}(\mathcal{L}u_o). \tag{3.29}$$

Indeed, by differentiating the PDE in (3.27),

$$\Delta_{\bar g}\partial_x(\mathcal{E}u_o) = 0.$$

On the other hand,

$$-\partial_x(\mathcal{E}u_o)(0,y) = \partial_\nu(\mathcal{E}u_o)(0,y) = \mathcal{L}u_o,$$

thanks to (3.28).

Moreover, $\partial_x(\mathcal{E}u_o)$ is bounded by elliptic estimates, since so is u_o.

Consequently, $-\partial_x(\mathcal{E}u_o)$ is a bounded solution of (3.27) with u_o replaced by $\mathcal{L}u_o$.

Thus, by the uniqueness of bounded solutions of (3.27), we obtain (3.29).

By exploiting (3.28) and (3.29), we see that

$$\begin{aligned} \mathcal{L}^2 u_o &= \partial_\nu\big(\mathcal{E}(\mathcal{L}u_o)\big)\big|_{x=0} = -\partial_x\big(\mathcal{E}(\mathcal{L}u_o)\big)\big|_{x=0} \\ &= -\partial_x\big(-\partial_x(\mathcal{E}u_o)\big)\big|_{x=0} = \partial_{xx}(\mathcal{E}u_o)\big|_{x=0}. \end{aligned} \tag{3.30}$$

On the other hand, using the PDE in (3.27),

$$0 = \Delta_{\bar g}(\mathcal{E}u_o) = \Delta_g(\mathcal{E}u_o) + \partial_{xx}(\mathcal{E}u_o),$$

so (3.30) becomes

$$\mathcal{L}^2 u_o(y) = \partial_{xx}(\mathcal{E}u_o)(0,y) = -\Delta_g(\mathcal{E}u_o)(0,y) = -\Delta_g u_o(y),$$

for any $y \in M$, that is

$$\mathcal{L} = (-\Delta_g)^{1/2}. \tag{3.31}$$

With these observations in hand, we now take u as in the statement of theorem 1.2 and we define $v := \mathcal{E}u$.

From (3.28) and (3.31),

$$\partial_\nu v\big|_{x=0} = \partial_\nu(\mathcal{E}u)\big|_{x=0} = \mathcal{L}u = (-\Delta_g)^{1/2}u.$$

Consequently, recalling (1.8), we obtain that v is a bounded solution of (1.3).

Furthermore, the function v is stable, thanks to (1.9).

Hence v is constant by theorem 1.1, and so we obtain the desired result for $u = v|_{x=0}$.

3.3. **Proof of theorem 1.3.** Given $p = (m, x) \in M \times \mathbb{R}^+$, we define $d_g(m)$ to be the geodesic distance of m in M (with respect to a fixed point) and

$$d(p) := \sqrt{d_g(m)^2 + x^2}.$$

Let also $\hat B_R := \{p \in M \times \mathbb{R}^+ \text{ s.t. } d(p) < R\}$, for any $R > 0$. Notice that $|\nabla_g u| \in L^\infty(M \times \mathbb{R}^+)$, by elliptic estimates, and that $\hat B_R \subseteq B_R \times [0, R]$, where B_R is the corresponding geodesic ball in M.

As a consequence, by our assumption on the volume of B_R, we obtain

$$\begin{aligned} \int_{\hat B_R} |\nabla_g u|^2 \, dV_{\bar g} &\leqslant \|\nabla_g u\|^2_{L^\infty(M\times\mathbb{R}^+)} \int_{B_R \times [0,R]} dV_{\bar g} \\ &= R\, \|\nabla_g u\|^2_{L^\infty(M\times\mathbb{R}^+)} \int_{B_R} dV_g \leqslant CR(R+1)\, \|\nabla_g u\|^2_{L^\infty(M\times\mathbb{R}^+)}. \end{aligned}$$

That is, by changing name of C,

$$\int_{\hat B_R} |\nabla_g u|^2 \, dV_{\bar g} \leqslant CR^2 \qquad \text{for any } R \geqslant 1. \tag{3.32}$$

Also, since d_g is a distance function on M (see pages 34 and 123 of [Pet98]), we have that

$$|\nabla d(p)| = \frac{\big|\big(d_g(m)\nabla_g d_g(m),\, x\big)\big|}{d(p)} \leqslant 1. \tag{3.33}$$

Also, given $R \geqslant 1$, we define

$$\phi_R(p) := \begin{cases} 1 & \text{if } d(p) \leqslant \sqrt{R}, \\ (\log\sqrt{R})^{-1}\big(\log R - \log(d(p))\big) & \text{if } d(p) \in (\sqrt{R}, R), \\ 0 & \text{if } d(p) \geqslant R. \end{cases}$$

Notice that (up to a set of zero $V_{\bar{g}}$-measure)

$$|\nabla \phi_R(p)| \leqslant \frac{\chi_{\hat{B}_R \setminus \hat{B}_{\sqrt{R}}}(p)}{\log\sqrt{R}\; d(p)},$$

due to (3.33).

As a consequence,

$$\begin{aligned} (\log\sqrt{R})^2 \int_{M\times\mathbb{R}^+} |\nabla_g u|^2 |\nabla\phi_R|^2\, dV_{\bar{g}} &\leqslant \int_{\hat{B}_R\setminus\hat{B}_{\sqrt{R}}} \frac{|\nabla_g u(p)|^2}{d(p)^2}\, dV_{\bar{g}}(p) \\ &= \int_{\hat{B}_R\setminus\hat{B}_{\sqrt{R}}} |\nabla_g u(p)|^2 \Big(\frac{1}{R^2} + \int_{d(p)}^R \frac{2\,dt}{t^3}\Big)\, dV_{\bar{g}}(p) \\ &\leqslant \frac{1}{R^2}\int_{\hat{B}_R} |\nabla_g u(p)|^2\, dV_{\bar{g}}(p) + \int_{\sqrt{R}}^R \int_{\hat{B}_t} \frac{2|\nabla_g u(p)|^2}{t^3}\, dV_{\bar{g}}(p)\, dt. \end{aligned}$$

Therefore, by (3.32),

$$(\log\sqrt{R})^2 \int_{M\times\mathbb{R}^+} |\nabla_g u|^2 |\nabla\phi_R|^2\, dV_{\bar{g}} \leqslant C\left(1 + \int_{\sqrt{R}}^R \frac{2\,dt}{t}\right) \leqslant 3C\log R.$$

Consequently, from (2.13),

$$\int_{M\times\mathbb{R}^+} \Big\{Ric_g(\nabla_g u, \nabla_g u) + |H_g u|^2 - |\nabla_g|\nabla_g u||^2\Big\}\phi_R^2 \leqslant \frac{12C}{\log R}. \tag{3.34}$$

From this and (3.23), we conclude that

$$\int_{M\times\mathbb{R}^+} Ric_g(\nabla_g u, \nabla_g u)\phi_R^2 \leqslant \frac{12C}{\log R}.$$

By sending $R \to +\infty$, we obtain that $Ric_g(\nabla_g u, \nabla_g u)$ vanishes identically.

Hence, u is constant, thanks to lemma 3.2, proving theorem 1.3.

3.4. **Proof of theorem 1.4.** The proof of theorem 1.3 can be carried out in this case too, up to formula (3.34).

Then, (3.34) in this case gives that

$$\int_{M\times\mathbb{R}^+} \Big\{|H_g u|^2 - |\nabla_g|\nabla_g u||^2\Big\}\phi_R^2 \leqslant \frac{12C}{\log R}.$$

By sending $R \to +\infty$, and by recalling (3.23), we conclude that $|H_g u|$ is identically equal to $|\nabla_g|\nabla_g u||$ on $\big(M \times \{x\}\big) \cap \{\nabla_g u \neq 0\}$, for any fixed $x > 0$.

Consequently, by lemma 5 of [FSV08b], we have that for any $k = 1, \dots, n$ there exist $\kappa^k : M \to \mathbb{R}$ such that

$$\nabla_g\big(\nabla_g u\big)^k(p) = \kappa^k(p)\nabla_g u(p) \qquad \text{for any } p \in \big(M \times \{x\}\big) \cap \{\nabla_g u \neq 0\}.$$

From this and [FSV08b] (see the computation starting there on formula (23)), one concludes that every connected component of $\{y \in M,\ \ u(x,y) = c\}$ is a geodesic.

4. The case of the hyperbolic space

We now come to problem (1.4). Notice that up to now we assumed for the manifold $\mathcal{M}$ to be positively curved. We deal here with special equations on negatively curved manifolds. As a consequence, the geometric formula (2.13) is not useful since the Ricci tensor does not have the good sign and so we need a different strategy to deal with the hyperbolic case.

For this, we will make use here of the fact that the manifold $\mathbb{H}^{n+1}$ with the metric $\bar{g}\frac{|dy|^2+|dx|^2}{x^2}$ is conformal to $\mathbb{R}^{n+1}_+$ with the flat metric, and, in fact, $(\mathbb{H}^{n+1}, \bar{g})$ is the main example of conformally compact Einstein manifold, as we discuss in section 4.1 here below.

4.1. **Motivations and scattering theory.** In order to justify the study of problem (1.4), we describe the link between problem (1.4) and fractional order conformally covariant operators.

Let M be a compact manifold of dimension n. Given a metric h on M, the conformal class $[h]$ of h is defined as the set of metrics $\hat{h}$ that can be written as $\hat{h} = fh$ for a positive conformal factor f.

Let $\mathcal{M}$ be a smooth manifold of dimension $n+1$ with boundary $\partial\mathcal{M} = M$.

A function ρ is a *defining function* of $\partial\mathcal{M}$ in $\mathcal{M}$ if

$$\rho > 0 \text{ in } \mathcal{M}, \quad \rho = 0 \text{ on } \partial\mathcal{M}, \quad d\rho \neq 0 \text{ on } \partial\mathcal{M}$$

We say that g is a *conformally compact* metric on X with conformal infinity $(M, [h])$ if there exists a defining function ρ such that the manifold $(\bar{\mathcal{M}}, \bar{g})$ is compact for $\bar{g} = \rho^2 g$, and $\bar{g}|_M \in [h]$.

If, in addition $(\mathcal{M}^{n+1}, g)$ is a conformally compact manifold and $Ric_g = -ng$, then we call $(\mathcal{M}^{n+1}, g)$ a *conformally compact Einstein* manifold.

Given a conformally compact, asymptotically hyperbolic manifold $(\mathcal{M}^{n+1}, g)$ and a representative $\hat{g}$ in $[\hat{g}]$ on the conformal infinity M, there is a uniquely defining function ρ such that, on $M \times (0, \epsilon)$ in $\mathcal{M}$, g has the normal form $g = \rho^{-2}(d\rho^2 + g_\rho)$ where g_ρ is a one parameter family of metrics on M (see [GZ03] for precise statements and further details).

In this setting, the scattering matrix of M is defined as follows. Consider the following eigenvalue problem in $(\mathcal{M}, g)$, with Dirichlet boundary condition,

$$\begin{cases} -\Delta_g u_s - s(n-s)u_s &= 0 \text{ in } \mathcal{M} \\ \qquad u_s &= f \text{ on } M \end{cases} \tag{4.35}$$

for $s \in \mathbb{C}$ and f defined on M.

Problem (4.35) is solvable unless $s(n-s)$ belongs to the spectrum of $-\Delta_g$.

However,

$$\sigma(-\Delta_g) = \big[(n/2)^2, \infty\big) \cup \sigma_{pp}(\Delta_g)$$

where the pure point spectrum $\sigma_{pp}(\Delta_g)$ (i.e., the set of L^2 eigenvalues), is finite and it is contained in $(0, (n/2)^2)$.

Moreover, given any f on M, Graham-Zworski [GZ03] obtained a meromorphic family of solutions $u_s = \mathcal{P}(s)f$ such that, if $s \notin n/2 + \mathbb{N}$, then

$$\mathcal{P}(s)f = F\rho^{n-s} + H\rho^s.$$

And if $s = n/2 + \gamma$, $\gamma \in \mathbb{N}$,

$$\mathcal{P}(s)f = F\rho^{n/2-\gamma} + H\rho^{n/2+\gamma}\log\rho$$

where $F, H \in \mathcal{C}^\infty(X)$, $F|_M = f$, and F, H mod $O(\rho^n)$ are even in ρ.

It is worth mentioning that in the second case $H|_M$ is locally determined by f and $\hat{g}$. However, in the first case, $H|_M$ is globally determined by f and g. We are interested in the study of these nonlocal operators.

We define the scattering operator as $S(s)f = H|_M$, which is a meromorphic family of pseudo-differential operators in $Re(s) > n/2$ with poles at $s = n/2 + \mathbb{N}$ of finite rank residues. The relation between f

and $S(s)f$ is like that of the Dirichlet to Neumann operator in standard harmonic analysis. Note that the principal symbol is

$$\sigma\left(S(s)\right) = 2^{n-2s}\,\frac{\Gamma(n/2-s)}{\Gamma(s-n/2)}\,\sigma\left((-\Delta_g)^{s-n/2}\right)$$

The operators obtained when $s = n/2+\gamma$, $\gamma \in \mathbb{N}$ have been well studied. Indeed, at those values of s the scattering matrix $S(s)$ has a simple pole of finite rank and its residue can be computed explicitly, namely

$$Res_{s=n/2+\gamma}S(s) = c_\gamma P_\gamma, \quad c_\gamma = (-1)^\gamma[2^{2\gamma}\gamma!(\gamma-1)!]^{-1}$$

and P_γ are the conformally invariant powers of the Laplacian constructed by [FG02, GJMS92].

In particular, when $\gamma = 1$ we have the conformal Laplacian,

$$P_1 = -\Delta + \frac{n-2}{4(n-1)}R$$

and when $\gamma = 2$, the Paneitz operator

$$P_2 = \Delta^2 + \delta\left(a_n Rg + b_n Ric\right)d + \tfrac{n-4}{2}Q^n$$

We can similarly define the following fractional order operators on M of order $\gamma \in (0,1)$ as

$$P_\gamma f := d_\gamma S(n/2+\gamma)f, \quad d_\gamma = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}.$$

It is important to mention that these operators are conformally covariant. Indeed, for a change of metric $g_u = u^{\frac{4}{n-2\gamma}}g_0$, we have

$$P_\gamma^{g_u} f = u^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{g_0}\left(uf\right).$$

The following result, which can be found in [CG08], establishes a link between scattering theory on $\mathcal{M}$ and a local problem in the half-space. We provide the proof for sake of completeness.

Lemma 4.1. *Fix $0 < \gamma < 1$ and let $s = \frac{n}{2}+\gamma$. Assume that u is a smooth solution of*

$$\begin{cases} -\Delta_{\bar g}u - s(n-s)u = 0 & in \quad \mathbb{H}^{n+1}, \\ \partial_\nu u = v & on \quad \partial\mathbb{H}^{n+1}. \end{cases} \tag{4.36}$$

for some smooth function v defined on $\partial\mathbb{H}^{n+1}$.

Then the function $U = x^{s-n}u$ solves

$$\begin{cases} div(x^{1-2\gamma}\nabla U) = 0 & for\ y \in \mathbb{R}^n,\ x \in (0,+\infty) \\ U(0,.) = u|_{x=0}, & in\ \mathbb{R}^n \\ -\lim_{x\to 0} x^{1-2\gamma}\partial_x U = Cv & \end{cases} \tag{4.37}$$

for some constant C.

Proof. By the results in [GZ03], one has the following representation of u in $\mathbb{H}^{n+1}$

$$u = x^{n-s}u|_{x=0} + x^s \partial_\nu u.$$

From this we deduce that

$$U = u|_{x=0} + x^{2s-n}\partial_\nu u.$$

Since $s = \frac{n}{2} + \gamma$, we have $2s - n = 2\gamma > 0$ and then

$$U|_{x=0} = u|_{x=0}$$

and

$$-\lim_{x\to 0} x^{1-2\gamma}\partial_x U = C\partial_\nu u = Cv.$$

We now prove that U satisfies the desired equation. This only comes from the conformality of the metric on $\mathbb{H}^{n+1}$ to the flat one in the half-space. Indeed, the conformal Laplacian is given by

$$L_{\bar g} = -\Delta_{\bar g} + \frac{n-1}{4n} R_{\bar g}$$

where $R_{\bar g}$ is the scalar curvature of $\mathbb{H}^{n+1}$, which is equal to $-n(n+1)$. On the other hand, if we have $h = e^{2w}\bar g$ for some function w (i.e. the metrics h and g are conformal) then the conformal law of $L_{\bar g}$ is given by

$$L_h \psi = e^{-\frac{n+3}{2}w} L_{\bar g}(e^{\frac{n-1}{2}w}\psi)$$

for any smooth ψ.

In our case, we have $h = |dx|^2 + |dy|^2$ the flat metric on $\mathbb{R}^{n+1}_+$ and $e^w = x$. Thus, using the conformal law, we have

$$-\Delta_{\bar g}\psi = -x^2\Delta\psi + (n-1)\partial_x\psi$$

for any ψ smoothly on $\mathbb{R}^{n+1}_+$.

Plugging $\psi = u$ and using equation (4.36) leads

$$s(n-s)u = -x^2\Delta u + (n-1)x\partial_x u.$$

Finally, plugging $U = x^{s-n}u$ leads to the equation

$$\Delta U + \frac{1-2\gamma}{x}\partial_x U = 0,$$

which is equivalent to $\operatorname{div}(x^{1-2\gamma}\nabla U) = 0$. $\square$

4.2. **Proof of theorem 1.5.** Let u be a solution as requested in Theorem 1.5. By Lemma 4.1, the function U satisfies in a weak sense

$$\begin{cases} \operatorname{div}(x^{1-2\gamma}\nabla U) = 0 & \text{for } y \in \mathbb{R}^2,\ x \in (0,+\infty) \\ U(0,.) = u|_{x=0}, & \\ -\lim_{x\to 0} x^{1-2\gamma}\partial_x U = f(U). & \end{cases} \tag{4.38}$$

Notice that either $\partial_{y_2} U > 0$ or $f' \leqslant 0$, thanks to (1.10) and (1.11). Furthermore, since u is bounded, U is bounded close to $x = 0$. Additionally, we have

$$U = x^{\gamma-1} u.$$

This gives that U is bounded inside $\mathbb{H}^{n+1}_+$. So, since U agrees with u on $\partial\mathbb{H}^{n+1}_+$, we obtain that U is bounded in all of $\mathbb{H}^{n+1}_+$.

Therefore, by theorem 3 in [SV09], we have that $U(x, y) = U_o(x, \omega{\cdot}y)$, for suitable $U_o : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $\omega \in S^1$. This gives directly the desired result.

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