

Discrete Differential Calculus, Graphs, Topologies and Gauge Theory

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Abstract

Differential calculus on discrete sets is developed in the spirit of noncommutative geometry. Any differential algebra on a discrete set can be regarded as a ‘reduction’ of the ‘universal differential algebra’ and this allows a systematic exploration of differential algebras on a given set. Associated with a differential algebra is a (di)graph where two vertices are connected by at most two (antiparallel) arrows. The interpretation of such a graph as a ‘Hasse diagram’ determining a (locally finite) topology then establishes contact with recent work by other authors in which discretizations of topological spaces and corresponding field theories were considered which retain their global topological structure. It is shown that field theories, and in particular gauge theories, can be formulated on a discrete set in close analogy with the continuum case. The framework presented generalizes ordinary lattice theory which is recovered from an oriented (hypercubic) lattice graph. It also includes, e.g., the two-point space used by Connes and Lott (and others) in models of elementary particle physics. The formalism suggests that the latter be regarded as an approximation of a manifold and thus opens a way to relate models with an ‘internal’ discrete space (à la Connes et al.) to models of dimensionally reduced gauge fields. Furthermore, also a ‘symmetric lattice’ is studied which (in a certain continuum limit) turns out to be related to a ‘noncommutative differential calculus’ on manifolds.

02.40.+m, 05.50.+q, 11.15.Ha, 03.20.+i

I. INTRODUCTION

In the context of ‘noncommutative geometry’ [1,2] a generalization of the notion of differential forms (on a manifold) plays a crucial role. With any associative algebra \mathcal{A} (over \mathbb{R} or \mathbb{C}) one associates a *differential algebra* which is a \mathbb{Z} -graded associative algebra $\Omega(\mathcal{A}) = \bigoplus_{r=0}^{\infty} \Omega^r(\mathcal{A})$ (where $\Omega^r(\mathcal{A})$ are \mathcal{A} -bimodules and $\Omega^0(\mathcal{A}) = \mathcal{A}$) together with a linear operator $d : \Omega^r(\mathcal{A}) \rightarrow \Omega^{r+1}(\mathcal{A})$ satisfying $d^2 = 0$ and $d(\omega\omega') = (d\omega)\omega' + (-1)^r\omega d\omega'$ where $\omega \in \Omega^r(\mathcal{A})$. This structure has been studied for *non-commutative* algebras in many recent papers (in particular, for quantum groups, see [3] and the references given there). But even *commutative* algebras, i.e. algebras of functions on some topological space, are very much of interest in this respect for mathematics and physics. A particular example is provided [4] by models of elementary particle physics with an extended space-time of the form $M \times \mathbb{Z}_2$ where M is an ordinary four-dimensional space-time manifold. The extension of differential calculus to discrete spaces allows a corresponding extension of the Yang-Mills action to $M \times \mathbb{Z}_2$ which incorporates Higgs fields and the usual Higgs potential. Our work in [5,6] can be viewed as a lattice analogue of the \mathbb{Z}_2 calculus. In [7] we went beyond lattices to an exploration of differential calculus and gauge theory on arbitrary finite or countable sets. In particular, some ideas about ‘reductions’ of the *universal* differential algebra (the ‘universal differential envelope’ of \mathcal{A} [1,2]) arose in that work. The present paper presents a much more complete treatment of the universal differential algebra, reductions of it, and gauge theory on discrete (i.e., finite or countable) sets.

Furthermore, the formalism developed in this paper provides a bridge between noncommutative geometry and various treatments of field theories on discrete spaces (like lattice gauge theory). We may view a field theory on a discrete set as an approximation of a continuum theory, e.g., for the purpose of numerical simulations. More interesting, however, is in this context the idea that a discrete space-time could actually be more fundamental than the continuum. This idea has been discussed and pursued by numerous authors (see [8], in particular).

Discrete spaces have been used in [9] to approximate general topological spaces and manifolds, taking their global topological structure into account (see also [10]). We establish a relation with noncommutative geometry. In particular, the two-point space in Connes’ model can then be regarded as an approximation of an ‘internal’ manifold as considered in models of dimensional reduction of gauge theories (see [11] for a review). The appearance of a Higgs field and a Higgs potential in Connes’ model then comes as no surprise since this is a familiar feature in the latter context. In 1979 Manton derived the bosonic part of the Weinberg-Salam model from a 6-dimensional Yang-Mills theory on (4-dimensional) Minkowski space times a 2-dimensional sphere [12].

In section II we introduce differential calculus on a discrete set. Reductions of the ‘universal differential algebra’ are considered in section III where we discuss the relation with digraphs and Hasse diagrams (which assign a topology to the discrete set [9]). Section IV deals with gauge theory, and in particular the case of the universal differential algebra. Sections V and VI treat, respectively, the oriented and the ‘symmetric’ lattice as particular examples of graphs defining a differential algebra. Finally, section VII summarizes some of the results and contains further remarks.

II. DIFFERENTIAL CALCULUS ON A DISCRETE SET

We consider a countable set \mathcal{M} with elements i, j, \dots . Although we include the case of infinite sets (in particular when it comes to lattices) in this work, our calculations are then formal rather than rigorous (since we simply commute operators with infinite sums, for example). Either one may regard these cases as idealizations of the case of a large finite set, or one finally has to work with a representation of the corresponding differential algebra (as in [4,13], see also [7]).

Let \mathcal{A} be the algebra of \mathbb{C} -valued functions on \mathcal{M} . Multiplication is defined pointwise, i.e. $(fh)(i) = f(i)h(i)$. There is a distinguished set of functions e_i on \mathcal{M} defined by $e_i(j) = \delta_{ij}$. They satisfy the relations

$$e_i e_j = \delta_{ij} e_i \quad , \quad \sum_i e_i = \mathbf{1} \quad (2.1)$$

where $\mathbf{1}$ denotes the constant function $\mathbf{1}(i) = 1$. Each $f \in \mathcal{A}$ can then be written as

$$f = \sum_i f(i) e_i . \quad (2.2)$$

The algebra \mathcal{A} can be extended to a \mathbb{Z} -graded *differential algebra* $\Omega(\mathcal{A}) = \bigoplus_{r=0}^{\infty} \Omega^r(\mathcal{A})$ (where $\Omega^0(\mathcal{A}) = \mathcal{A}$) via the action of a linear operator $d : \Omega^r(\mathcal{A}) \rightarrow \Omega^{r+1}(\mathcal{A})$ satisfying

$$d\mathbf{1} = 0 \quad , \quad d^2 = 0 \quad , \quad d(\omega_r \omega') = (d\omega_r) \omega' + (-1)^r \omega_r d\omega' \quad (2.3)$$

where $\omega_r \in \Omega^r(\mathcal{A})$. The spaces $\Omega^r(\mathcal{A})$ of *r-forms* are \mathcal{A} -bimodules. $\mathbf{1}$ is taken to be the unit in $\Omega(\mathcal{A})$. From the above properties of the set of functions e_i we obtain

$$e_i de_j = -(de_i) e_j + \delta_{ij} de_i \quad (2.4)$$

and

$$\sum_i de_i = 0 \quad (2.5)$$

(assuming that d commutes with the sum) which shows that the de_i are linearly dependent. Let us define

$$e_{ij} := e_i de_j \quad (i \neq j) \quad , \quad e_{ii} := 0 \quad (2.6)$$

(note that $e_i de_i \neq 0$) and

$$e_{i_1 \dots i_r} := e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{r-1} i_r} \quad (2.7)$$

which for $i_k \neq i_{k+1}$ equals $e_{i_1} de_{i_2} \cdots de_{i_r}$. Then

$$e_{i_1 \dots i_r} e_{j_1 \dots j_s} = \delta_{i_r j_1} e_{i_1 \dots i_{r-1} j_1 \dots j_s} \quad (r, s \geq 1) . \quad (2.8)$$

A simple calculation shows that

$$de_i = \sum_j (e_{ji} - e_{ij}) \quad (2.9)$$

and consequently

$$df = \sum_{i,j} e_{ij} (f(j) - f(i)) \quad (\forall f \in \mathcal{A}) . \quad (2.10)$$

Furthermore,

$$\begin{aligned} de_{ij} &= de_i de_j = \sum_{k,\ell} (e_{ki} - e_{ik})(e_{\ell j} - e_{j\ell}) \\ &= \sum_{k,\ell} (\delta_{i\ell} e_{kij} - \delta_{k\ell} e_{ikj} + \delta_{kj} e_{ij\ell}) \\ &= \sum_k (e_{kij} - e_{ikj} + e_{ijk}) . \end{aligned} \quad (2.11)$$

Any 1-form ρ can be written as $\rho = \sum e_{ij} \rho_{ij}$ with $\rho_{ij} \in \mathbb{C}$ and $\rho_{ii} = 0$. One then finds

$$d\rho = \sum_{i,j,k} e_{ijk} (\rho_{jk} - \rho_{ik} + \rho_{ij}) . \quad (2.12)$$

More generally, we have

$$\begin{aligned} de_{i_1 \dots i_r} &= \sum_j \sum_{k=1}^{r+1} (-1)^{k+1} e_{i_1 \dots i_{k-1} j i_k \dots i_r} \\ &= \sum_{j_1, \dots, j_{r+1}} e_{j_1 \dots j_{r+1}} \sum_{k=1}^{r+1} (-1)^{k+1} I_{i_1 \dots \widehat{i}_k \dots i_r}^{j_1 \dots \widehat{j}_k \dots j_{r+1}} \end{aligned} \quad (2.13)$$

where a hat indicates an omission and

$$I_{i_1 \dots i_r}^{j_1 \dots j_r} := \delta_{i_1}^{j_1} \dots \delta_{i_r}^{j_r} . \quad (2.14)$$

Any $\psi \in \Omega^{r-1}(\mathcal{A})$ can be written as

$$\psi = \sum_{i_1, \dots, i_r} e_{i_1 \dots i_r} \psi_{i_1 \dots i_r} \quad (2.15)$$

with $\psi_{i_1 \dots i_r} \in \mathbb{C}$, $\psi_{i_1 \dots i_r} = 0$ if $i_s = i_{s+1}$ for some s . We thus have

$$d\psi = \sum_{i_1, \dots, i_{r+1}} e_{i_1 \dots i_{r+1}} \sum_{k=1}^{r+1} (-1)^{k+1} \psi_{i_1 \dots \widehat{i}_k \dots i_{r+1}} . \quad (2.16)$$

If no further relations are imposed on the differential algebra (see section III, however), we call it the *universal* differential algebra (it is usually called ‘universal differential envelope’ of \mathcal{A} [1,2]). This particular case will be considered in the following. The e_{ij} with $i \neq j$ are then a basis of the space of 1-forms. More generally, it can be shown that $e_{i_1 \dots i_r}$ with $i_k \neq i_{k+1}$ for $k = 1, \dots, r-1$ form a basis (over \mathbb{C}) of the space of $(r-1)$ -forms ($r > 1$). As a consequence, $df = 0$ implies $f(i) = f(j)$ for all $i, j \in \mathcal{M}$, i.e. $f = \text{const.}$. $d\rho = 0$ implies

the relation $\rho_{jk} = \rho_{ik} - \rho_{ij}$ for $i \neq j \neq k$. Hence, we have $\rho_{jk} = \rho_{0k} - \rho_{0j}$ with some fixed element $0 \in \mathcal{M}$. With $f := \sum_i \rho_{0i} e_i \in \mathcal{A}$ we find $\rho = df$. Hence every closed 1-form is exact.

Remark. The condition $d\rho = 0$ can also be written as $\rho_{ik} = \rho_{ij} + \rho_{jk}$ which has the form of the Ritz-Rydberg combination principle [14] for the frequencies of atomic spectra (see also [13]). With $\nu := \sum_{nm} \nu_{nm} e_{nm}$ the Ritz-Rydberg principle can be expressed in the simple form $d\nu = 0$ which implies $\nu = dH/h$ with the energy $H := \sum_n E_n e_n$ (and Planck's constant h). The equation of motion

$$\frac{d}{dt} \rho = \frac{i}{\hbar} [H, \rho] \quad (2.17)$$

for a 1-form ρ (with t -dependent coefficients) is equivalent to

$$\frac{d}{dt} \rho_{ij} = \frac{i}{\hbar} (E_i - E_j) \rho_{ij} \quad (2.18)$$

which appeared as an early version of the Heisenberg equation (see [15]). \square

Using (2.16), $d\psi = 0$ implies

$$\sum_{k=1}^{r+1} (-1)^k \psi_{i_1 \dots \widehat{i}_k \dots i_{r+1}} = 0 \quad (2.19)$$

which leads to the expression

$$\begin{aligned} \psi_{i_1 \dots i_r} &= \sum_{k=1}^r (-1)^{k+1} \psi_{0i_1 \dots \widehat{i}_k \dots i_r} \\ &= \psi_{0i_2 \dots i_r} - \psi_{0i_1 i_3 \dots i_r} + \dots + (-1)^{r+1} \psi_{0i_1 \dots i_{r-1}}. \end{aligned} \quad (2.20)$$

With

$$\phi := \sum_{i_1, \dots, i_{r-1}} e_{i_1 \dots i_{r-1}} \psi_{0i_1 \dots i_{r-1}} \quad (2.21)$$

we find $d\phi = \psi$. Hence, closed forms are always exact so that the cohomology of d for the universal differential algebra is trivial.

Now we introduce some general notions which are not restricted to the case of the universal differential algebra.

An *inner product* on a differential algebra $\Omega(\mathcal{A})$ should have the properties $(\psi_r, \phi_s) = 0$ for $r \neq s$ and

$$(\psi, c\phi) = c(\psi, \phi) \quad (\forall c \in \mathbb{C}) \quad , \quad \overline{(\psi, \phi)} = (\phi, \psi) \quad (2.22)$$

(a bar indicates complex conjugation). Furthermore, we should require that $(\psi, \phi) = 0 \forall \phi$ implies $\psi = 0$.

An *adjoint* $d^* : \Omega^r(\mathcal{A}) \rightarrow \Omega^{r-1}(\mathcal{A})$ of d with respect to an inner product is then defined by

$$(\psi_{r-1}, d^* \phi_r) := (d\psi_{r-1}, \phi_r) . \quad (2.23)$$

This allows us to construct a *Laplace-Beltrami operator* as follows,

$$\Delta := -d d^* - d^* d . \quad (2.24)$$

Let $\Omega(\mathcal{A})^*$ denote the \mathbb{C} -dual of $\Omega(\mathcal{A})$. The inner product determines a mapping $\psi \in \Omega(\mathcal{A}) \rightarrow \psi^\natural \in \Omega(\mathcal{A})^*$ via

$$\psi^\natural(\omega) := (\psi, \omega) \quad (\forall \omega \in \Omega(\mathcal{A})) . \quad (2.25)$$

For $c \in \mathbb{C}$ we have $(c\psi)^\natural = \bar{c}\psi^\natural$. Now we can introduce products $\bullet : \Omega^r(\mathcal{A})^* \times \Omega^{r+p}(\mathcal{A}) \rightarrow \Omega^p(\mathcal{A})$ and $\bullet : \Omega^{r+p}(\mathcal{A}) \times \Omega^p(\mathcal{A})^* \rightarrow \Omega^r(\mathcal{A})$ by

$$(\phi_p, \psi_r^\natural \bullet \omega_{r+p}) := (\psi_r \phi_p, \omega_{r+p}) \quad , \quad (\phi_p, \omega_{r+p} \bullet \psi_r^\natural) := (\phi_p \psi_r, \omega_{r+p}) \quad (2.26)$$

($\forall \phi_p, \omega_{r+p}$). They have the properties

$$(\psi\phi)^\natural \bullet \omega = \phi^\natural \bullet (\psi^\natural \bullet \omega) \quad (2.27)$$

$$\omega \bullet (\psi\phi)^\natural = (\omega \bullet \phi^\natural) \bullet \psi^\natural . \quad (2.28)$$

For an inner product such that

$$(e_{i_1 \dots i_r}, e_{j_1 \dots j_r}) := \delta_{i_1 j_1} g_{i_1 \dots i_r j_1 \dots j_r} \delta_{i_r j_r} \quad (2.29)$$

with constants $g_{i_1 \dots i_r j_1 \dots j_r}$, the \bullet -products satisfy the relations

$$f^\natural \bullet \omega = \bar{f} \omega \quad (2.30)$$

$$\omega \bullet f^\natural = \omega \bar{f} \quad (2.31)$$

$$(\psi f)^\natural \bullet \omega = \bar{f} (\psi^\natural \bullet \omega) \quad (2.32)$$

$$\omega \bullet (f \psi)^\natural = (\omega \bullet \psi^\natural) \bar{f} \quad (2.33)$$

$$\psi^\natural \bullet (f \omega) = (\bar{f} \psi)^\natural \bullet \omega \quad (2.34)$$

$$(f \omega) \bullet \psi^\natural = f (\omega \bullet \psi^\natural) \quad (2.35)$$

$$\psi^\natural \bullet (\omega f) = (\psi^\natural \bullet \omega) f \quad (2.36)$$

$$(\omega f) \bullet \psi^\natural = \omega \bullet (\psi^\natural f) . \quad (2.37)$$

Whereas our ordinary product of differential forms corresponds to the cup-product in algebraic topology, the \bullet is related to the cap-product of a cochain and a chain [16].

Let us now turn again to the particular case of the universal differential algebra on \mathcal{A} . An inner product is then determined by (2.29) with

$$g_{i_1 \dots i_r j_1 \dots j_r} := \mu_r \delta_{i_1 j_1} \cdots \delta_{i_r j_r} \sigma_{i_1 \dots i_r} \quad (2.38)$$

where $\mu_r \in \mathbb{R}^+$ and

$$\sigma_{i_1 \dots i_r} := \prod_{s=1}^{r-1} (1 - \delta_{i_s i_{s+1}}) . \quad (2.39)$$

The factor $\sigma_{i_1 \dots i_r}$ takes care of the fact that $e_{i_1 \dots i_r}$ vanishes if two neighbouring indices coincide. We then have

$$(\psi, \phi) = \mu_r \sum_{i_1, \dots, i_r} \bar{\psi}_{i_1 \dots i_r} \phi_{i_1 \dots i_r} \quad (2.40)$$

for r -forms ψ, ϕ . For the adjoint of d , we obtain the formulae

$$d^* e_{i_1 \dots i_r} = \frac{\mu_r}{\mu_{r-1}} \sum_{k=1}^r (-1)^{k+1} e_{i_1 \dots \widehat{i}_k \dots i_r} \quad (2.41)$$

and

$$d^* \psi_r = \frac{\mu_{r+1}}{\mu_r} \sum_{i_1, \dots, i_r} e_{i_1 \dots i_r} \sum_j \sum_{k=1}^{r+1} (-1)^{k+1} \psi_{i_1 \dots i_{k-1} j i_k \dots i_r} \cdot \quad (2.42)$$

Obviously, $d^* f = 0$ for $f \in \mathcal{A}$ and $(d^*)^2 = 0$. If \mathcal{M} is a *finite* set of N elements, one finds

$$\Delta f = 2N \frac{\mu_2}{\mu_1} \left(\frac{1}{N} \text{Tr} f - f \right) \quad (2.43)$$

for $f \in \mathcal{A}$ where $\text{Tr} f := \sum_i f(i)$.

For the \bullet -products we obtain

$$e_{i_1 \dots i_r}^\natural \bullet e_{j_1 \dots j_s} = \frac{\mu_s}{\mu_{s-r+1}} \delta_{i_1 j_1} \cdots \delta_{i_r j_r} e_{j_r \dots j_s} \sigma_{i_1 \dots i_r} \quad (2.44)$$

$$e_{j_1 \dots j_s} \bullet e_{i_1 \dots i_r}^\natural = \frac{\mu_s}{\mu_{s-r+1}} \delta_{i_1 j_{s-r+1}} \cdots \delta_{i_r j_s} e_{j_1 \dots j_{s-r+1}} \sigma_{i_1 \dots i_r} \cdot \quad (2.45)$$

Remark. There is a representation of the universal differential algebra such that $df = \mathbb{1} \otimes f - f \otimes \mathbb{1}$ (cf [2], for example). From this we obtain $e_{ij} = e_i \otimes e_j$ for $i \neq j$ and $e_{i_1 \dots i_r} = \sigma_{i_1 \dots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}$. Hence, $e_{i_1 \dots i_r}$ can be regarded as an r -linear mapping $\mathcal{M}^r \rightarrow \mathbb{C}$,

$$\begin{aligned} \langle e_{i_1 \dots i_r}, (j_1, \dots, j_r) \rangle &:= \sigma_{i_1 \dots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} (j_1, \dots, j_r) \\ &= \sigma_{i_1 \dots i_r} \delta_{i_1 j_1} \cdots \delta_{i_r j_r} \cdot \end{aligned} \quad (2.46)$$

Obviously, tuples (j_1, \dots, j_r) with $j_s = j_{s+1}$ for some s lie in the kernel of this mapping. We can then introduce *boundary* and *coboundary* operators, ∂ and ∂^* respectively, on ordered r -tuples of elements of \mathcal{M} via

$$\langle e_{i_1 \dots i_{r-1}}, \partial(j_1, \dots, j_r) \rangle := \langle de_{i_1 \dots i_{r-1}}, (j_1, \dots, j_r) \rangle \quad (2.47)$$

$$\langle e_{i_1 \dots i_r}, \partial^*(j_1, \dots, j_{r+1}) \rangle := \langle d^* e_{i_1 \dots i_r}, (j_1, \dots, j_{r+1}) \rangle \cdot \quad (2.48)$$

One finds

$$\partial(i_1, \dots, i_r) = \sum_{k=1}^r (-1)^{k+1} (i_1, \dots, \widehat{i}_k, \dots, i_r) \quad (2.49)$$

$$\partial^*(i_1, \dots, i_r) = \sum_i \sum_{k=1}^{r+1} (-1)^{k+1} (i_1, \dots, i_{k-1}, i, i_k, \dots, i_r) \quad (2.50)$$

and that ∂^* is the adjoint of ∂ with respect to the inner product defined by

$$((i_1, \dots, i_r), (j_1, \dots, j_s)) = \delta_{r,s} \delta_{i_1, j_1} \cdots \delta_{i_r, j_r} . \quad (2.51)$$

This construction makes contact with simplicial homology theory [16]. There, however, r -tuples are identified (up to a sign in case of oriented simplexes) if they differ only by a permutation of their vertices. In the case under consideration, we have general r -tuples, not subject to any condition at all. \square

Let $\tilde{\cdot}$ denote an involutive mapping of \mathcal{M} ($\tilde{\tilde{k}} = k$ for all $k \in \mathcal{M}$). It induces an *involution* $*$ on $\Omega(\mathcal{A})$ by requiring $(\psi\phi)^* = \phi^*\psi^*$, $(d\omega_r)^* = (-1)^r d\omega_r^*$ and $(f^*)(k) = \overline{f(\tilde{k})}$ where a bar denotes complex conjugation. Using the Leibniz rule, one finds

$$e_{k\ell}^* = -e_{\tilde{\ell}\tilde{k}} . \quad (2.52)$$

A natural involution is induced by the identity $\tilde{k} = k$. Then $e_{k\ell}^* = -e_{\ell k}$ and thus $e_{i_1 \dots i_r}^* = (-1)^{r+1} e_{i_r \dots i_1}$.

III. DIFFERENTIAL ALGEBRAS AND TOPOLOGIES

If one is interested, for example, to approximate a differentiable manifold by a discrete set, the universal differential algebra is too large to provide us with a corresponding analogue of the algebra of differential forms on the manifold. We need ‘smaller’ differential algebras. The fact that the 1-forms e_{ij} ($i \neq j$) induce via (2.7) a basis (over \mathbb{C}) for the spaces of higher forms together with the relations (2.8) offer a simple way to *reduce* the differential algebra. Setting some of the e_{ij} to zero does not generate any relations for the remaining (nonvanishing) $e_{k\ell}$ and is consistent with the rules of differential calculus. It generates relations for forms of higher grades, however. In particular, it may require that some of those $e_{i_1 \dots i_r}$ with $r > 2$ have to vanish which do not contain as a factor any of the e_{ij} which are set to zero (cf example 1 below).

The reductions of the universal differential algebra obtained in this way are conveniently represented by graphs as follows. We regard the elements of \mathcal{M} as vertices and associate with $e_{ij} \neq 0$ an arrow from i to j . The universal differential algebra then corresponds to the graph where all the vertices are connected pairwise by arrows in both directions. Deleting some of the arrows leads to a graph which represents a reduction of the universal differential algebra.

In the following we discuss some simple examples and establish a relation with topology (some related aspects are discussed in the appendix). More complicated examples are presented in sections V and VI.

Example 1. Let us consider a set of three elements with the differential algebra determined by the graph in Fig.1a.

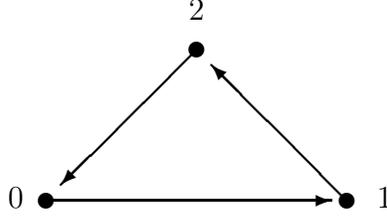


Fig.1a

The graph associated with a differential algebra on a set of three elements.

The nonvanishing basic 1-forms are then e_{01}, e_{12}, e_{20} . From these we can only build the basic 2-forms e_{012}, e_{120} and e_{201} . However, (2.11) yields

$$0 = de_{10} = \sum_{k=1}^3 (e_{k10} - e_{1k0} + e_{10k}) = -e_{120} \quad (3.1)$$

and similarly $e_{012} = 0 = e_{201}$. Hence there are no 2-forms (and thus also no higher forms). This may be interpreted in such a way that the differential algebra assigns a *one*-dimensional structure to the three-point set. Using (2.9), we have

$$de_0 = e_{20} - e_{01} \quad , \quad de_1 = e_{01} - e_{12} \quad , \quad de_2 = e_{12} - e_{20} . \quad (3.2)$$

Let us extend the graph in Fig.1a in the following way (see Fig.1b). We add new vertices corresponding to the 1-forms on the respective edges of the diagram. The arrows from the 0-form vertices to the 1-form vertices are then determined by the last equations. For example, e_{01} appears with a minus sign on the rhs of the expression for de_0 . We draw an arrow from the e_0 vertex to the new e_{01} vertex. e_{20} appears with a plus sign and we draw an arrow from the e_{20} vertex to the e_0 vertex.

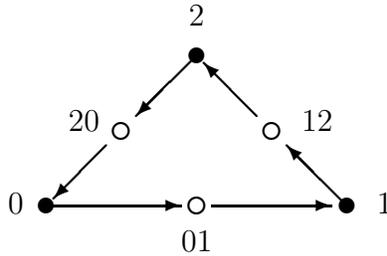


Fig.1b

The extension of the graph in Fig.1a obtained from the latter by adding new vertices corresponding to the nonvanishing basic 1-forms.

Another form of the graph is shown in Fig.1c where vertices corresponding to differential forms with the same grade are grouped together horizontally and $(r + 1)$ -forms are below r -forms.

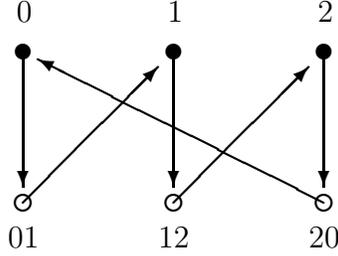


Fig.1c

The (oriented) Hasse diagram derived from the graph in Fig.1a.

The result can be interpreted as a *Hasse* diagram which determines a finite topology in the following way [9]. A vertex together with all lower lying vertices which are connected to it forms an open set. In the present case, $\{01\}, \{12\}, \{20\}, \{0, 01, 20\}, \{1, 01, 12\}, \{2, 12, 20\}$ are the open sets (besides the empty and the whole set). This is an approximation to the topology of S^1 . It consists of a chain of three open sets covering S^1 which already displays the global topology of S^1 . In particular, the fundamental group π_1 is the same as for S^1 .

In the above example we have defined the *dimension* of a differential algebra as the grade of its highest nonvanishing forms. This is probably the most fruitful such concept (others have been considered in [7,17]). Applying it to subgraphs leads to a *local* notion of dimension.

Example 2. Again, we consider a set of three elements, but this time with the differential algebra determined by the graph in Fig.2a. In this case, the nonvanishing basic 1-forms are e_{01}, e_{12}, e_{02} . From these one can only build the 2-form e_{012} . (2.11) then reads

$$de_{ij} = e_{0ij} - e_{i1j} + e_{ij2} \tag{3.3}$$

and e_{012} remains as a non-vanishing 2-form.

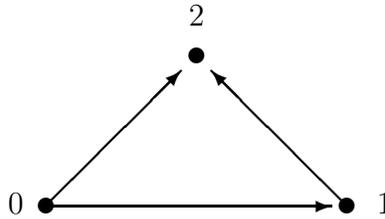


Fig.2a

The graph associated with another differential algebra on the set of three elements.

There are no higher forms so that the differential algebra assigns *two* dimensions to the three-point set. Using (2.9), we have

$$de_0 = -e_{01} - e_{02} \quad , \quad de_1 = e_{01} - e_{12} \quad , \quad de_2 = e_{02} + e_{12} . \tag{3.4}$$

The extended graph is shown in Fig.2b. Now we have an additional vertex corresponding to the two-form e_{012} with connecting arrows determined by (3.3).

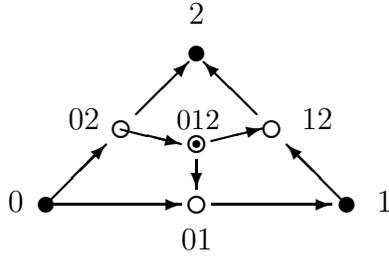


Fig.2b

The extension of the graph in Fig.2a with new vertices corresponding to nonvanishing forms of grade higher than zero.

The Hasse diagram is drawn in Fig.2c. The open sets of the corresponding topology are $\{012\}$, $\{01, 012\}$, $\{12, 012\}$, $\{02, 012\}$, $\{0, 01, 02, 012\}$, $\{1, 01, 12, 012\}$, $\{2, 12, 02, 012\}$.

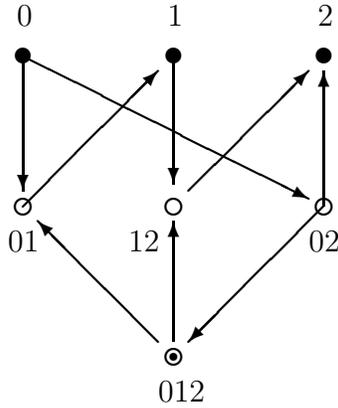


Fig.2c

The (oriented) Hasse diagram derived from the graph in Fig.2a.

The topology is shown in Fig.2d.

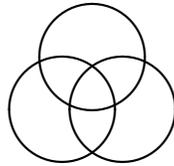


Fig.2d

The topology on the three point set determined by the Hasse diagram in Fig.2c.

Example 3. We supply a set of four elements with the differential algebra determined by the graph in Fig.3a.

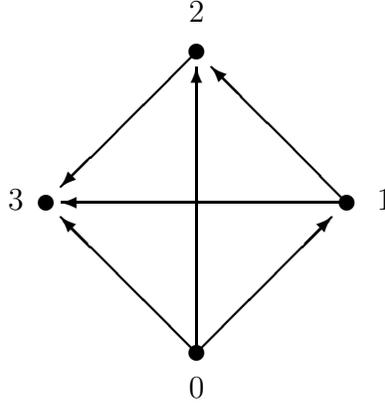


Fig.3a

A graph which determines a differential algebra on a set of four points.

The nonvanishing basic 1-forms are thus $e_{01}, e_{02}, e_{03}, e_{12}, e_{13}, e_{23}$. From these we can only build the 2-forms $e_{012}, e_{013}, e_{023}, e_{123}$ and the 3-form e_{0123} . There are no higher forms. (2.13) yields

$$\begin{aligned} de_{01} &= e_{012} + e_{013} & , & & de_{02} &= -e_{012} + e_{023} & , & & de_{12} &= e_{012} + e_{123} \\ de_{13} &= e_{013} - e_{123} & , & & de_{03} &= -e_{013} - e_{023} & , & & de_{23} &= e_{123} + e_{023} \end{aligned} \quad (3.5)$$

and

$$de_{012} = -e_{0123} \quad , \quad de_{013} = e_{0123} \quad , \quad de_{023} = -e_{0123} \quad , \quad de_{123} = e_{0123} . \quad (3.6)$$

The corresponding (oriented) Hasse diagram is drawn in Fig.3b. It determines a topology which approximates the topology of a simply connected open set in \mathbb{R}^3 .

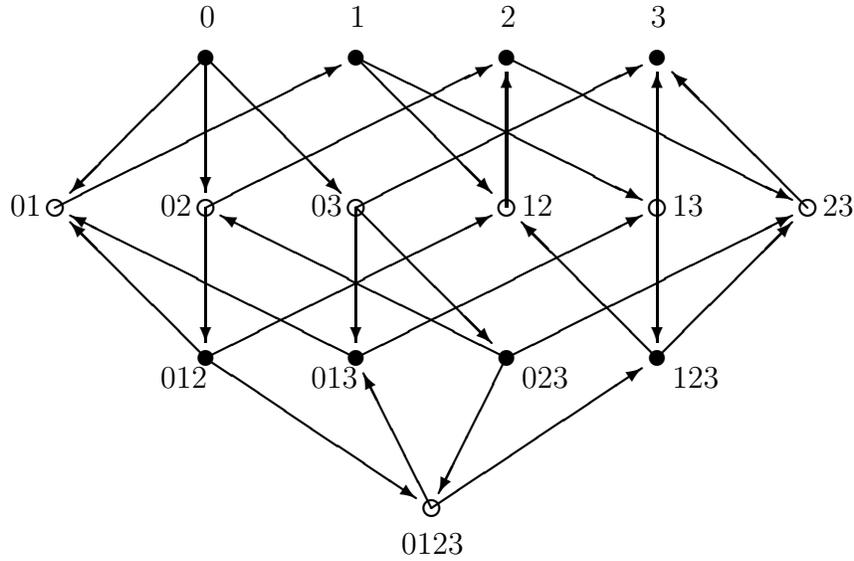


Fig.3b

The (oriented) Hasse diagram derived from the graph in Fig.3a.

More generally, we have the possibility of reductions on the level of r -forms ($r \geq 1$), i.e. we can set any of the (not already vanishing) $e_{i_1 \dots i_{r+1}}$ to zero. In example 2, we have the freedom to set the 2-form e_{012} to zero by hand. We then end up with the topology of a circle (as in example 1) instead of the topology of a disc. Such a ‘higher-order reduction’ has no effect on the remaining s -forms with $s \leq r$, but influences forms of grade higher than r . The full information about the differential algebra is then only contained in the extended graph in which *all* the nonvanishing basic forms – and not just the 0-forms – are represented as vertices. This is the (oriented) Hasse diagram.

Example 4. The graph in Fig.4a corresponds to the universal differential algebra on a set of two elements. We are allowed to omit the arrows since both directions are present. From the basic 1-forms e_{01} and e_{10} we can construct forms $e_{010101\dots}$ and $e_{101010\dots}$ of arbitrary grade.

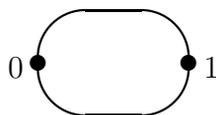


Fig.4a

The graph associated with the universal differential algebra on a two point set.

The associated (oriented) Hasse diagram is shown in Fig.4b. If we set the 2-forms e_{010} and e_{101} to zero, the Hasse diagram determines a topology which approximates the topology of the circle S^1 . If, however, we set the 3-forms e_{0101} and e_{1010} to zero, an approximation of S^2 is obtained, etc.. In this way contact is made with the work in [18].

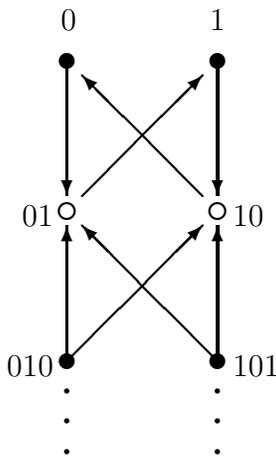


Fig.4b

The (oriented) Hasse diagram for the universal differential algebra on a set of two elements.

The two-point space can thus be regarded, in particular, as an approximation of the two-dimensional sphere. As an ‘internal space’ the latter appears, for example, in Manton’s six-

dimensional Yang-Mills model from which he obtained the bosonic sector of the Weinberg-Salam model by dimensional reduction [12]. In view of this relation, the appearance of the Higgs field in recent models of noncommutative geometry à la Connes and Lott [4] (see also section IV) may be traced back essentially to the abovementioned old result.

Let $e := e_0$. Then $e_1 = \mathbf{1} - e$, $e^2 = e$ and $(de)e + e de = de$. Comparison with Appendix B in [19] (see also [1]) shows that we are dealing with the differential envelope of the complex numbers.

Example 5. Let us consider the ‘symmetric’ graph in Fig.5. The nonvanishing basic 1-forms are $e_{01}, e_{10}, e_{12}, e_{21}$. From these one obtains the 2-forms $e_{010}, e_{012}, e_{101}, e_{210}, e_{121}, e_{212}$. As a consequence of $e_{02} = 0 = e_{20}$ we find $e_{012} = 0 = e_{210}$. The only nonvanishing basic 2-forms are thus e_{010}, e_{101} and e_{121}, e_{212} .

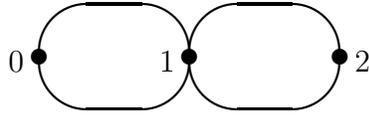


Fig.5

A ‘symmetric’ differential algebra on a three point set.

In terms of the (coordinate) function $x := 0 e_0 + 1 e_1 + 2 e_2 = e_1 + 2 e_2$ we obtain

$$e_0 = \mathbf{1} - \frac{3}{2}x + \frac{1}{2}x^2 \quad , \quad e_1 = 2x - x^2 \quad , \quad e_2 = \frac{1}{2}(x^2 - x) \quad (3.7)$$

and

$$[x, dx] = -(e_{01} + e_{10} + e_{12} + e_{21}) =: \tau \quad (3.8)$$

The 1-forms dx and τ constitute a basis of $\Omega^1(\mathcal{A})$ as a left (or right) \mathcal{A} -module. A simple calculation shows that a 1-form ω is closed iff it can be written in the form

$$\omega = c_1 (e_{01} - e_{10}) + c_2 (e_{12} - e_{21}) \quad (3.9)$$

with complex constants c_1, c_2 . Furthermore, closed 1-forms are exact. The 1-form τ introduced above is not closed.

A differential algebra with the property that $e_{ij} = 0$ for some $i, j \in \mathcal{M}$ only if also $e_{ji} = 0$ is called a *symmetric reduction* of the universal differential algebra. The associated graph will then also be called *symmetric*. The algebra considered in example 5 is of this type.

In the examples treated above, we started from a differential algebra and ended up with a topology. One can go the other way round, i.e. start with a topology, construct the corresponding Hasse diagram and add directions to its edges in accordance with the rules of differential calculus (cf [18]).

IV. GAUGE THEORY ON A DISCRETE SET

A field Ψ on \mathcal{M} is a cross section of a vector bundle over \mathcal{M} , e.g., a cross section of the trivial bundle $\mathcal{M} \times \mathbb{C}^n$. In the algebraic language the latter corresponds to the (free) \mathcal{A} -module \mathcal{A}^n (nontrivial bundles correspond to ‘finite projective modules’). We regard it as a *left* \mathcal{A} -module and consider an action $\Psi \mapsto G \Psi$ of a (local) gauge group, a subgroup of $GL(n, \mathcal{A})$ with elements $G = \sum_i G(i) e_i$, on \mathcal{A}^n . This induces on the dual (right \mathcal{A} -) module an action $\alpha \mapsto \alpha G^{-1}$.

Let us introduce *covariant exterior derivatives*

$$D\Psi = d\Psi + A \Psi \quad , \quad D\alpha = d\alpha - \alpha A \quad (4.1)$$

where A is a 1-form. These expressions are indeed covariant if A obeys the usual transformation law of a *connection 1-form*,

$$A' = G A G^{-1} - dG G^{-1} . \quad (4.2)$$

Since dG is a discrete derivative, A cannot be Lie algebra valued. It is rather an element of $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} M_n(\mathcal{A})$ where $M_n(\mathcal{A})$ is the space of $n \times n$ matrices with entries in \mathcal{A} .

As a consequence of (4.1) we have $d(\alpha \Psi) = (D\alpha) \Psi + \alpha (D\Psi)$. We could have used different connections for the module \mathcal{A}^n and its dual. The requirement of the last relation would then identify both.

We call an element $U \in \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} GL(n, \mathcal{A})$ a *transport operator* (the reason will become clear in the following) if it transforms as $U \mapsto G U G^{-1}$ under a gauge transformation. Since $U = \sum_{i,j} e_{ij} U_{ij}$ with $U_{ij} \in GL(n, \mathbb{C})$, we find

$$U'_{ij} = G(i) U_{ij} G(j)^{-1} . \quad (4.3)$$

Using (4.2), (2.8) and (2.10) it can be shown that such a transport operator is given by

$$U := \sum_{i,j} e_{ij} (\mathbf{1} + A_{ij}) \quad (4.4)$$

where $\mathbf{1}$ is the identity in the group.

The covariant derivatives introduced above can now be written as follows,

$$D\Psi = \sum_{i,j} e_{ij} \nabla_j \Psi(i) \quad , \quad \nabla_j \Psi(i) := U_{ij} \Psi(j) - \Psi(i) \quad (4.5)$$

$$D\alpha = \sum_{i,j} e_{ij} \nabla_j \alpha(i) U_{ij} \quad , \quad \nabla_j \alpha(i) := \alpha(j) U_{ij}^{-1} - \alpha(i) \quad (4.6)$$

where $\Psi = \sum_i e_i \Psi(i)$ and in the last equation we have made the additional assumption that U_{ij} is invertible.

The 1-forms e_{ij} are linearly independent, except for those which are set to zero in a reduction of the universal differential algebra. The condition of covariantly constant Ψ , i.e. $D\Psi = 0$,

thus implies $\nabla_j \Psi(i) = 0$ for those i, j for which $e_{ij} \neq 0$. This gives U_{ij} the interpretation of an operator for parallel transport from j to i . Drawing points for the elements of \mathcal{M} , we may assign to U_{ij} an arrow from j to i (see Fig.6).

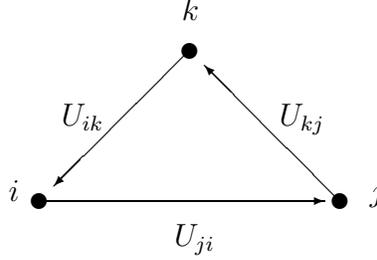


Fig.6

A visualization of U_{ij} as a transport operator from j to i .

The curvature of the connection A is given by the familiar formula

$$F = dA + A^2 \quad (4.7)$$

and transforms in the usual way, $F \mapsto G F G^{-1}$. As a 2-form, it can be written as $F = \sum_{i,j,k} e_{ijk} F_{ijk}$ and we find

$$F = \sum_{i,j,k} e_{ijk} (U_{ij} U_{jk} - U_{ik}) . \quad (4.8)$$

Remark. In general, the 2-forms e_{ijk} are not linearly independent for a given differential algebra so that $F = 0$ does not imply the vanishing of the coefficients. The latter is true, however, for the universal differential algebra. In that case vanishing F leads to $U_{ij} U_{jk} = U_{ik}$ and, in particular, $U_{ji} = U_{ij}^{-1}$. As a consequence, U is then ‘path-independent’ on \mathcal{M} . If we set $G(i) := U_{i0}$, the condition of vanishing curvature implies $U_{ij} = G(i) G(j)^{-1}$ which can also be expressed as $A = -dG G^{-1}$, i.e., the connection is ‘pure gauge’. The Bianchi identity $DF = dF + [A, F] = 0$ is a 3-form relation. But only for the universal differential algebra we can conclude that the coefficients of the basic 3-forms e_{ijkl} vanish which then leads to $F_{ijl} - F_{ikl} = F_{ijk} U_{kl} - U_{ij} F_{jkl}$. \square

In order to generalize an inner product (with the properties specified in section II) to matrix valued forms, we require that

$$(\phi, \psi) = \sum \phi_{i_1 \dots i_r}^\dagger (e_{i_1 \dots i_r}, e_{j_1 \dots j_s}) \psi_{j_1 \dots j_s} . \quad (4.9)$$

Here $\phi_{i_1 \dots i_r}$ is a matrix with entries in \mathbb{C} and $\phi_{i_1 \dots i_r}^\dagger$ denotes the hermitian conjugate matrix.

The *Yang-Mills action*

$$S_{YM} := \text{tr} (F, F) \quad (4.10)$$

is then gauge-invariant if $G^\dagger = G^{-1}$. With these tools we can now formulate gauge theory, in particular, on the differential algebras (respectively graphs) considered in the previous section. We will not elaborate these examples here but only discuss the case of the universal

differential algebra in some detail. Other examples will then be treated in sections V and VI.

The hermitian conjugation of complex matrices can be extended to matrix valued differential forms via

$$\phi^\dagger = \sum_{i_1 \dots i_r} (\phi_{i_1 \dots i_r} e_{i_1 \dots i_r})^\dagger = \sum_{i_1 \dots i_r} \phi_{i_1 \dots i_r}^\dagger e_{i_1 \dots i_r}^* \quad (4.11)$$

if an involution is given on $\Omega(\mathcal{A})$. A *conjugation* † acting on a field Ψ is a map from a left \mathcal{A} -module, \mathcal{A}^n in our case, to the dual right \mathcal{A} -module such that $(\phi \Psi)^\dagger = \Psi^\dagger \phi^\dagger$ where ϕ is an $(n \times n)$ -matrix valued differential form. Since $G^\dagger = G^{-1}$ this is in accordance with the transformation rule for Ψ^\dagger , i.e. $\Psi^\dagger \mapsto \Psi^\dagger G^{-1}$.

A. Gauge theory with the universal differential algebra

Using the inner product introduced in section II on the universal differential algebra, the *Yang-Mills action* becomes

$$S_{YM} = \sum_{i,j,k} \text{tr} (F_{ijk}^\dagger F_{ijk}) \quad (4.12)$$

(where we have set $\mu_r = 1$). Using (4.8), we get

$$S_{YM} = \text{tr} \sum_{i,j,k} (U_{jk}^\dagger U_{ij}^\dagger U_{ij} U_{jk} - U_{jk}^\dagger U_{ij}^\dagger U_{ik} - U_{ik}^\dagger U_{ij} U_{jk} + U_{ik}^\dagger U_{ik}) . \quad (4.13)$$

Variation of the Yang-Mills action with respect to the connection A , making use of (2.23) and (2.26), leads to

$$(\delta F, F) = (d\delta A + \delta A A + A \delta A, F) = (\delta A, d^* F + A^\natural \bullet F + F \bullet A^\natural) \quad (4.14)$$

from which we read off the Yang-Mills equation

$$d^* F + A^\natural \bullet F + F \bullet A^\natural = 0 . \quad (4.15)$$

In the following we will evaluate some of the formulae given above with the choice of the natural involution (cf section II) and with certain additional conditions imposed on the gauge field. The usual compatibility condition for parallel transport and conjugation is

$$(D\Psi)^\dagger = D(\Psi^\dagger) \quad (4.16)$$

which is equivalent to

$$A^\dagger = -A \quad (4.17)$$

and implies $F^\dagger = F$. Using (2.52), (4.17) becomes

$$U_{ij}^\dagger = U_{ji} \quad (4.18)$$

which implies $F_{ijk}^\dagger = F_{kji}$. Evaluating the Yang-Mills action (4.13) and the Yang-Mills equation (4.15) with (4.18), we obtain

$$S_{YM} = \text{tr} \sum_{i,j,k} (U_{kj} U_{ji} U_{ij} U_{jk} - U_{kj} U_{ji} U_{ik} - U_{ki} U_{ij} U_{jk} + U_{ki} U_{ik}) \quad (4.19)$$

and

$$\sum_k (F_{ikj} - \delta_{ij} F_{iki} - U_{ik} F_{kij} - F_{ijk} U_{kj}) = 0 \quad (4.20)$$

respectively.

Example. For $\mathcal{M} = \mathbb{Z}_2 = \{0, 1\}$ with the universal differential algebra one finds

$$\begin{aligned} F &= e_{010} (U_{01} U_{10} - \mathbf{1}) + e_{101} (U_{10} U_{01} - \mathbf{1}) \\ &= e_{010} (U_{10}^\dagger U_{10} - \mathbf{1}) + e_{101} (U_{10} U_{10}^\dagger - \mathbf{1}) \end{aligned} \quad (4.21)$$

and

$$S_{YM} = 2 \text{tr} (U_{10}^\dagger U_{10} - \mathbf{1})^2 \quad (4.22)$$

which has the form of a Higgs potential (cf [4]). \square

With U we also have $\check{U} := \sum_{i,j} e_{ij} U_{ji}^{-1}$ as a transport operator (more generally, this is the case for a differential algebra with a ‘symmetric’ graph). Hence, there is another connection,

$$\check{A} := \sum_{i,j} e_{ij} (U_{ji}^{-1} - \mathbf{1}) . \quad (4.23)$$

For the corresponding covariant exterior derivatives $\check{D}\Psi := d\Psi + \check{A}\Psi$ and $\check{D}\alpha := d\alpha - \alpha\check{A}$ one finds

$$\check{D}\Psi = \sum_{i,j} e_{ij} \check{\nabla}_j \Psi(i) \quad , \quad \check{\nabla}_j \Psi(i) := U_{ji}^{-1} \Psi(j) - \Psi(i) \quad (4.24)$$

$$\check{D}\alpha = \sum_{i,j} e_{ij} \check{\nabla}_j \alpha(i) U_{ji}^{-1} \quad , \quad \check{\nabla}_j \alpha(i) := \alpha(j) U_{ji} - \alpha(i) . \quad (4.25)$$

There are thus two different parallel transports between any two points. This suggests to look for field configurations where the two covariant derivatives associated with U and \check{U} coincide. The condition

$$(D\Psi)^\dagger = \check{D}(\Psi^\dagger) \quad (4.26)$$

is equivalent to

$$A^\dagger = -\check{A} \quad (4.27)$$

and leads to $F^\dagger = \check{F}$. Furthermore, (4.27) implies

$$U_{ij}^\dagger = U_{ij}^{-1} . \quad (4.28)$$

Using (4.18), this yields

$$U_{ij}^{-1} = U_{ji} . \quad (4.29)$$

As a consequence, we have $F_{iji} = 0$. The condition (4.27) thus eliminates the gauge field in Connes' 2-point space model [4] (see also the example given above). The Yang-Mills equation is now reduced to

$$\sum_k (U_{ij} - U_{ik} U_{kj}) = 0 . \quad (4.30)$$

If \mathcal{M} is a finite set of N elements, the last equation can be rewritten as

$$U_{ij} = \frac{1}{N} \sum_k U_{ik} U_{kj} = \frac{1}{N-2} \sum_{k \neq i, j} U_{ik} U_{kj} \quad (4.31)$$

where the last equality assumes $N \neq 2$.

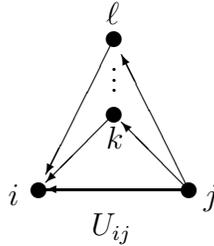


Fig.7

An illustration of equation (4.31).

(4.31) means that the parallel transport operator from j to i equals the average of all parallel transports via some other point $k \neq i, j$ (see Fig.7). Evaluation of the Yang-Mills action (4.19) with the condition (4.27) leads to the expression

$$S'_{YM} = 2 \sum_{i, j, k} \text{tr} (1 - U_{ij} U_{jk} U_{ki}) \quad (4.32)$$

which contains a sum over all parallel transport loops with three vertices (cf Fig.6). Note, however, that the above reduced Yang-Mills equations are not obtained by variation of this action with respect to U_{ij} as a consequence of the constraint (4.27).

V. LATTICE CALCULUS

In this section we choose $\mathcal{M} = \mathbb{Z}^n = \{a = (a^\mu) \mid \mu = 1, \dots, n; a^\mu \in \mathbb{Z}\}$ and consider the reduction of the universal differential algebra obtained by imposing the relations

$$e_{ab} \neq 0 \quad \Leftrightarrow \quad b = a + \hat{\mu} \quad \text{for some } \mu \quad (5.1)$$

where $\hat{\mu} := (\delta_\mu^\nu) \in \mathcal{M}$. The corresponding graph is an oriented lattice in n dimensions (a finite part of it is drawn in Fig.8).

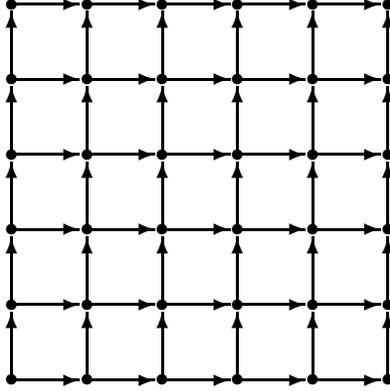


Fig.8

A finite part of the oriented lattice graph which determines the differential calculus underlying usual lattice theories.

A. Differential calculus on the oriented lattice

In the following we will use the notation $e_a^\mu := e_{a,a+\hat{\mu}}$ and

$$e_a^{\mu_1 \dots \mu_r} := e_a^{\mu_1} e_{a+\hat{\mu}_1}^{\mu_2 \dots \mu_r} . \quad (5.2)$$

In particular, $e_a^{\mu\nu} = e_{a,a+\hat{\mu},a+\hat{\mu}+\hat{\nu}} = e_{a,a+\hat{\mu}} e_{a+\hat{\mu},a+\hat{\mu}+\hat{\nu}}$. It also turns out to be convenient to introduce

$$e^\mu := \sum_a e_a^\mu \quad (5.3)$$

which satisfies $e_a^\mu = e_a e^\mu$ and, more generally,

$$e_a^{\mu_1 \dots \mu_r} = e_a e^{\mu_1} \dots e^{\mu_r} . \quad (5.4)$$

Acting with d on $e_{a,a+\hat{\mu}+\hat{\nu}} = 0$ and using (2.11), we obtain

$$e_a^{\mu\nu} + e_a^{\nu\mu} = 0 \quad (5.5)$$

and therefore

$$e^\mu e^\nu = -e^\nu e^\mu . \quad (5.6)$$

Using (2.13) we find

$$de_a = \sum_\mu [e_{a-\hat{\mu}} - e_a] e^\mu \quad , \quad de_a^\mu = \sum_\nu [e_{a-\hat{\nu}} - e_a] e^\nu e^\mu . \quad (5.7)$$

This leads to

$$de^\mu = 0 . \quad (5.8)$$

(5.7), (5.8) and the Leibniz rule allow us to calculate $d\omega$ for any form ω . Any $f \in \mathcal{A}$ can be written as a function of

$$x^\mu := \ell \sum_a a^\mu e_a \quad (5.9)$$

(where ℓ is a positive constant) since

$$f = \sum_a e_a f(\ell a) = \sum_a e_a f(x) = f(x) \quad (5.10)$$

using $e_a x^\mu = \ell e_a a^\mu$ and $\sum_a e_a = \mathbf{1}$. The differential of a function f is then given by

$$\begin{aligned} df &= \sum_{\mu,a} e_a^\mu [f(\ell(a + \hat{\mu})) - f(\ell a)] = \sum_{\mu,a} [f(x + \ell \hat{\mu}) - f(x)] e_a^\mu \\ &= \sum_\mu (\partial_{+\mu} f)(x) dx^\mu \end{aligned} \quad (5.11)$$

(where the expression $x + \ell \hat{\mu}$ should be read as $x + \ell \hat{\mu} \mathbf{1}$). We have introduced

$$(\partial_{\pm\mu} f)(x) := \pm \frac{1}{\ell} [f(x \pm \ell \hat{\mu}) - f(x)] \quad (5.12)$$

and

$$dx^\mu = \ell e^\mu . \quad (5.13)$$

Furthermore, we obtain

$$\begin{aligned} dx^\mu f(x) &= \ell \sum_a e_a^\mu f(x) = \ell \sum_a e_a^\mu f(\ell(a + \hat{\mu})) = \ell \sum_a f(\ell(a + \hat{\mu})) e_a^\mu \\ &= \ell f(x + \ell \hat{\mu}) \sum_a e_a^\mu = f(x + \ell \hat{\mu}) dx^\mu \end{aligned} \quad (5.14)$$

which shows that we are dealing with the differential calculus of [5,6] which was demonstrated to underly usual lattice theories. The 1-forms dx^μ constitute a basis of $\Omega^1(\mathcal{A})$ as a left (or right) \mathcal{A} -module. In accordance with previous results [6] we obtain

$$dx^\mu dx^\nu + dx^\nu dx^\mu = 0 \quad (5.15)$$

from (5.6). More generally, $dx^{\mu_1} \dots dx^{\mu_r} = \ell^r e^{\mu_1} \dots e^{\mu_r}$ is totally antisymmetric. For an arbitrary r -form

$$\phi = \frac{1}{r!} \sum_{a, \mu_1, \dots, \mu_r} \ell^r e_a^{\mu_1 \dots \mu_r} \phi_{\mu_1 \dots \mu_r}(\ell a) = \frac{1}{r!} \sum_{\mu_1, \dots, \mu_r} \phi_{\mu_1 \dots \mu_r}(x) dx^{\mu_1} \dots dx^{\mu_r} \quad (5.16)$$

one finds

$$d\phi = \frac{1}{r!} \sum_{\mu, \mu_1, \dots, \mu_r} (\partial_{+\mu} \phi_{\mu_1 \dots \mu_r})(x) dx^\mu dx^{\mu_1} \dots dx^{\mu_r} \quad (5.17)$$

using the rules of differentiation. With the help of (5.14), the differential of ϕ can also be expressed as

$$d\phi = [u, \phi] \quad (5.18)$$

with the graded commutator on the rhs and the 1-form

$$u := \frac{1}{\ell} \sum_{\mu} dx^{\mu} \quad (5.19)$$

which satisfies $u^2 = 0$.

Next we introduce an inner product of forms. Taking account of the identities (5.5), we set

$$(e_a^{\mu_1 \dots \mu_r}, e_b^{\nu_1 \dots \nu_r}) := \ell^{-2r} \delta_{rs} \delta_{a,b} \delta_{\mu_1 \dots \mu_r}^{\nu_1 \dots \nu_r} \quad (5.20)$$

where

$$\delta_{\mu_1 \dots \mu_r}^{\nu_1 \dots \nu_r} := \delta_{[\mu_1}^{\nu_1} \dots \delta_{\mu_r]}^{\nu_r} = \sum_{k=1}^r (-1)^{k+1} \delta_{\mu_k}^{\nu_1} \delta_{\mu_1 \dots \widehat{\mu}_k \dots \mu_r}^{\nu_2 \dots \nu_r} . \quad (5.21)$$

This is compatible with (2.29) since the rhs vanishes for $a + \hat{\mu}_1 + \dots + \hat{\mu}_r \neq b + \hat{\nu}_1 + \dots + \hat{\nu}_r$. In particular, we get $(f, h) = \sum_a \bar{f}(a) h(a)$ and

$$(\psi, \phi) = \frac{1}{r!} \sum_{a, \mu_1, \dots, \mu_r} \bar{\psi}_{\mu_1 \dots \mu_r}(a) \phi_{\mu_1 \dots \mu_r}(a) \quad (5.22)$$

for r -forms ψ, ϕ .

An adjoint d^* of the exterior derivative d can now be introduced as in section II. A simple calculation shows that

$$d^* e^{\mu_1 \dots \mu_r}(a) = \ell^{-2} \sum_{k=1}^r (-1)^{k+1} [e_{a+\hat{\mu}_k} - e_a] e^{\mu_1} \dots \widehat{e^{\mu_k}} \dots e^{\mu_r} \quad (5.23)$$

and

$$d^* \phi = -\frac{1}{(r-1)!} \sum_{\mu_1, \dots, \mu_r} \partial_{-\mu_1} \phi_{\mu_1 \dots \mu_r}(x) dx^{\mu_2} \dots dx^{\mu_r} . \quad (5.24)$$

Remark. From (5.23) and (5.20) we find

$$(e_a^{\mu_1 \dots \mu_r}, d^* e_b^{\nu_1 \dots \nu_{r+1}}) = (-1)^{r+1} \ell^{-2(r+1)} I[(a; \mu_1 \dots \mu_r), (b; \nu_1 \dots \nu_{r+1})] \quad (5.25)$$

where

$$I[(a; \mu_1 \dots \mu_r), (b; \nu_1 \dots \nu_{r+1})] = \sum_{k=1}^{r+1} (-1)^{r+1-k} (\delta_{a,b} - \delta_{a, b+\hat{\nu}_k}) \delta_{\mu_1 \dots \widehat{\mu}_k \dots \mu_r}^{\nu_1 \dots \widehat{\nu}_k \dots \nu_{r+1}} \quad (5.26)$$

is the *incidence number* of the two cells $(a; \mu_1 \dots \mu_r)$ and $(b; \nu_1 \dots \nu_{r+1})$. This relates our formalism to others used in lattice theories (cf [20], for example). \square

For the \bullet -products (cf section II) defined with respect to the inner product introduced above one can prove the relations

$$\begin{aligned} (dx^\mu)^\natural \bullet (dx^{\mu_1} \dots dx^{\mu_r}) &= \sum_{k=1}^r (-1)^{k+1} \delta_{\mu_k}^\mu dx^{\mu_1} \dots \widehat{dx^{\mu_k}} \dots dx^{\mu_r} \\ &= (-1)^{r+1} (dx^{\mu_1} \dots dx^{\mu_r}) \bullet (dx^\mu)^\natural . \end{aligned} \quad (5.27)$$

Together with the general formulae given in section II, this allows us to evaluate any expression involving a \bullet .

Let us introduce the ‘volume form’

$$\epsilon := \frac{1}{n!} \sum_{\mu_1, \dots, \mu_n} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \dots dx^{\mu_n} = \frac{\ell^n}{n!} \sum_{a, \mu_1, \dots, \mu_n} \epsilon_{\mu_1 \dots \mu_n} e_a^{\mu_1 \dots \mu_n} \quad (5.28)$$

where $\epsilon_{\mu_1 \dots \mu_n}$ is totally antisymmetric with $\epsilon_{1 \dots n} = 1$. Obviously, $d\epsilon = 0$ and $d^*\epsilon = 0$. We can now define a Hodge star-operator on differential forms as follows,

$$\star \psi := \psi^\natural \bullet \epsilon . \quad (5.29)$$

Using (2.32), we obtain

$$\star (\psi f) = \bar{f} \star \psi . \quad (5.30)$$

An application of (5.14) leads to

$$\star (f \psi) = (\star \psi) \bar{f}(x - \ell \hat{\epsilon}) \quad (5.31)$$

where $\hat{\epsilon} := \hat{1} + \dots + \hat{n}$. The usual formula

$$\star (dx^{\mu_1} \dots dx^{\mu_r}) = \frac{1}{(n-r)!} \sum_{\mu_{r+1}, \dots, \mu_n} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_{r+1}} \dots dx^{\mu_n} \quad (5.32)$$

holds as can be shown with the help of (2.27) and (5.27). It can be used to show that

$$\star \star [\psi_r(x)] = (-1)^{r(n-r)} \psi_r(x - \ell \hat{\epsilon}) . \quad (5.33)$$

A simple consequence of our definitions is

$$\epsilon \bullet \psi_r(x)^\natural = (-1)^{r(n+1)} \star \psi_r(x + \ell \hat{\epsilon}) . \quad (5.34)$$

With the help of this relation one finds

$$(\star \psi, \star \omega) = (\omega, \psi) . \quad (5.35)$$

Furthermore, we have

$$d^* \psi_r(x) = -(-1)^{n(r+1)} \star d \star \psi_r(x + \ell \hat{\epsilon}) . \quad (5.36)$$

It is natural to introduce the following notation,

$$\int \omega_n := (\epsilon, \omega_n) . \quad (5.37)$$

B. Gauge theory on a lattice

A connection 1-form can be written as $A = \sum_{\mu} A_{\mu}(x) dx^{\mu}$. Again, instead of A we consider

$$U_{\mu}(x) := \mathbf{1} + \ell A_{\mu}(x) . \quad (5.38)$$

The transformation law (4.2) for A then leads to

$$U'_{\mu}(x) = G(x) U_{\mu}(x) G(x + \ell \hat{\mu})^{-1} . \quad (5.39)$$

For the exterior covariant derivatives (4.1) we obtain

$$D\Psi = \sum_{\mu} \nabla_{\mu} \Psi(x) dx^{\mu} \quad , \quad \nabla_{\mu} \Psi(x) := \frac{1}{\ell} [U_{\mu}(x) \Psi(x + \ell \hat{\mu}) - \Psi(x)] \quad (5.40)$$

$$D\alpha = \sum_{\mu} \nabla_{\mu} \alpha(x) U_{\mu}(x) dx^{\mu} \quad , \quad \nabla_{\mu} \alpha(x) := \frac{1}{\ell} [\alpha(x + \ell \hat{\mu}) U_{\mu}(x)^{-1} - \alpha(x)] . \quad (5.41)$$

Using (5.18) and $u^2 = 0$, we find

$$F = dA + A^2 = u A + A u + A^2 = U^2 \quad (5.42)$$

for the curvature of the connection A . Here we have introduced

$$U := u + A = \frac{1}{\ell} \sum_{\mu} U_{\mu} dx^{\mu} . \quad (5.43)$$

Further evaluation of the expression for F leads to

$$F = \frac{1}{2\ell^2} \sum_{\mu, \nu} [U_{\mu}(x) U_{\nu}(x + \ell \hat{\mu}) - U_{\nu}(x) U_{\mu}(x + \ell \hat{\nu})] dx^{\mu} dx^{\nu} . \quad (5.44)$$

Imposing the compatibility condition

$$[\nabla_{\mu} \Psi(x)]^{\dagger} = \nabla_{\mu} [\Psi(x)]^{\dagger} \quad (5.45)$$

for the covariant derivative with a conjugation leads to the unitarity condition $U_{\mu}(x)^{\dagger} = U_{\mu}(x)^{-1}$. The Yang-Mills action $S_{YM} = \text{tr}(F, F)$ is now turned into the Wilson action

$$S_{YM} = \frac{1}{\ell^4} \sum_{a, \mu, \nu} \text{tr}[\mathbf{1} - U_{\mu}(\ell a) U_{\nu}(\ell(a + \hat{\mu})) U_{\mu}(\ell(a + \hat{\nu}))^{\dagger} U_{\nu}(\ell a)^{\dagger}] \quad (5.46)$$

of lattice gauge theory. We also have

$$S_{YM} = \text{tr}(F, F) = \text{tr}(\star F, \star F) = \text{tr}(\epsilon, F \star F) = \text{tr} \int F \star F . \quad (5.47)$$

The Yang-Mills equations are again obtained in the form (4.15). Evaluation using (2.44) and (2.45) leads to the lattice Yang-Mills equations

$$\begin{aligned} U_{\mu}(x) = \frac{1}{2n} \sum_{\nu} [& U_{\nu}(x) U_{\mu}(x + \ell \hat{\nu}) U_{\nu}(x + \ell \hat{\mu})^{\dagger} \\ & + U_{\nu}(x - \ell \hat{\nu})^{\dagger} U_{\mu}(x - \ell \hat{\nu}) U_{\nu}(x + \ell(\hat{\mu} - \hat{\nu}))] \end{aligned} \quad (5.48)$$

which have a simple geometric meaning on the lattice. $U_{\mu}(x)$ must be the average of the parallel transports along all neighbouring paths.

VI. THE SYMMETRIC LATTICE.

The lattice considered in the previous section had an orientation arbitrarily assigned to it (the arrows point in the direction of increasing values of the coordinates x^μ). In this section we consider a ‘symmetric lattice’, i.e. a lattice without distinguished directions. It corresponds to a ‘symmetric reduction’ (of the universal differential algebra on \mathbb{Z}^n) as defined in section III. Some of its features were anticipated in example 5 of section III.

The differential calculus associated with the symmetric lattice turns out to be a kind of discrete version of a ‘noncommutative differential calculus’ on manifolds which has been studied recently [21,22,3].

Again, we choose $\mathcal{M} = \mathbb{Z}^n$ and use the same notation as in the previous section. The reduction of the universal differential algebra associated with a ‘symmetric’ (n -dimensional) lattice is determined by

$$e_{ab} \neq 0 \quad \Leftrightarrow \quad b = a + \hat{\mu} \quad \text{or} \quad b = a - \hat{\mu} \quad \text{for some } \mu \quad (6.1)$$

where $\hat{\mu} = (\delta_\mu^\nu)$ and $\mu = 1, \dots, n$.

A. Calculus on the symmetric lattice

It is convenient to introduce a variable ϵ which takes values in $\{\pm 1\}$. Furthermore, we define $e_a^{\epsilon\mu} := e_{a, a+\epsilon\hat{\mu}}$ and, more generally,

$$e_a^{\epsilon_1\mu_1 \dots \epsilon_r\mu_r} := e_a^{\epsilon_1\mu_1} e_{a+\epsilon_1\hat{\mu}_1}^{\epsilon_2\mu_2 \dots \epsilon_r\mu_r} . \quad (6.2)$$

Acting with d on the identity $e_{a, a+\epsilon\hat{\mu}+\epsilon'\hat{\nu}} = 0$ for $\epsilon\hat{\mu} + \epsilon'\hat{\nu} \neq 0$, we obtain

$$e_a^{\epsilon\mu\epsilon'\nu} + e_a^{\epsilon'\nu\epsilon\mu} = 0 \quad (\epsilon\hat{\mu} + \epsilon'\hat{\nu} \neq 0) . \quad (6.3)$$

We supplement these relations with corresponding relations for $\epsilon\hat{\mu} + \epsilon'\hat{\nu} = 0$, namely

$$e_a^{+\mu-\mu} + e_a^{-\mu+\mu} = 0 . \quad (6.4)$$

The general case will be discussed elsewhere. In (6.4) we have simply written \pm instead of ± 1 . As a consequence of (6.4), $e_a^{\epsilon_1\mu_1 \dots \epsilon_r\mu_r}$ is totally antisymmetric in the (double-) indices $\epsilon_i\mu_i$. Introducing

$$e^{\epsilon\mu} := \sum_a e_a^{\epsilon\mu} \quad (6.5)$$

we have

$$\sum_a e_a^{\epsilon_1\mu_1 \dots \epsilon_r\mu_r} = e^{\epsilon_1\mu_1} \dots e^{\epsilon_r\mu_r} \quad (6.6)$$

and

$$e_a^{\epsilon_1\mu_1} e^{\epsilon_2\mu_2} \dots e^{\epsilon_r\mu_r} = e_a e^{\epsilon_1\mu_1} \dots e^{\epsilon_r\mu_r} . \quad (6.7)$$

As a consequence of these relations we find

$$e^{\epsilon\mu\epsilon'\nu} + e^{\epsilon'\nu\epsilon\mu} = 0 \quad (6.8)$$

and the general differentiation rule (2.13) gives

$$d[e_a^{\epsilon_1\mu_1} e^{\epsilon_2\mu_2} \dots e^{\epsilon_r\mu_r}] = \sum_{\epsilon,\mu} [e_{a-\epsilon\hat{\mu}} - e_a] e^{\epsilon\mu} e^{\epsilon_1\mu_1} \dots e^{\epsilon_r\mu_r} \quad (6.9)$$

which, in particular, implies $de^{\epsilon\mu} = 0$. As in the previous section we introduce

$$x^\mu := \ell \sum_a a^\mu e_a . \quad (6.10)$$

Every $f \in \mathcal{A}$ can be regarded as a function of x^μ (cf section V). Using (6.9) we obtain

$$dx^\mu = \ell (e^{+\mu} - e^{-\mu}) = \ell \sum_\epsilon \epsilon e^{\epsilon\mu} \quad (6.11)$$

and thus

$$e^{\epsilon\mu} = \frac{\epsilon}{2\ell} dx^\mu + \frac{1}{2\beta} \tau^\mu \quad (6.12)$$

with the 1-forms

$$\tau^\mu := \beta (e^{+\mu} + e^{-\mu}) = \beta \sum_\epsilon e^{\epsilon\mu} \quad (6.13)$$

where $\beta \neq 0$ is a real constant. The 1-forms τ^μ satisfy $d\tau^\mu = 0$. Together with dx^μ they form a basis of $\Omega^1(\mathcal{A})$ as a left (or right) \mathcal{A} -module. The differential of a function $f(x)$ can now be written as follows,

$$df = \ell \sum_{\epsilon,\mu} \epsilon \partial_{\epsilon\mu} f e^{\epsilon\mu} = \sum_\mu (\bar{\partial}_\mu f dx^\mu + \frac{\kappa}{2} \Delta_\mu f \tau^\mu) \quad (6.14)$$

where $\kappa := \ell^2/\beta$ and we have introduced the operators

$$\partial_{\epsilon\mu} f := \frac{\epsilon}{\ell} (f(x + \epsilon \ell \hat{\mu}) - f(x)) \quad (6.15)$$

$$\bar{\partial}_\mu f := \frac{1}{2} (\partial_{+\mu} f + \partial_{-\mu} f) = \frac{1}{2\ell} [f(x + \ell \hat{\mu}) - f(x - \ell \hat{\mu})] \quad (6.16)$$

$$\begin{aligned} \Delta_\mu f &:= \partial_{+\mu} \partial_{-\mu} f = \frac{1}{\ell} (\partial_{+\mu} f - \partial_{-\mu} f) \\ &= \frac{1}{\ell^2} [f(x + \ell \hat{\mu}) + f(x - \ell \hat{\mu}) - 2f(x)] . \end{aligned} \quad (6.17)$$

For the commutation relations between functions and 1-forms we find

$$e^{\epsilon\mu} f(x) = f(x + \epsilon \ell \hat{\mu}) e^{\epsilon\mu} \quad (6.18)$$

and

$$[dx^\mu, f(x)] = \frac{\kappa\beta}{2} \Delta_\mu f(x) dx^\mu + \kappa \bar{\partial}_\mu f(x) \tau^\mu \quad (6.19)$$

$$[\tau^\mu, f(x)] = \beta \bar{\partial}_\mu f(x) dx^\mu + \frac{\kappa\beta}{2} \Delta_\mu f(x) \tau^\mu . \quad (6.20)$$

Let us take a look at the continuum limit where $\ell \rightarrow 0$ and $\beta \rightarrow 0$, but $\kappa = \text{const.}$ Under the additional assumption that $\tau^\mu \rightarrow \tau$, one (formally) obtains from (6.19) and (6.20) the commutation relations

$$[dx^\mu, f(x)] = \kappa \delta^{\mu\nu} \partial_\nu f(x) \tau \quad (\text{summation over } \nu) \quad (6.21)$$

$$[\tau, f(x)] = 0 \quad (6.22)$$

with the metric tensor $\delta^{\mu\nu}$. For the differential of a (differentiable) function we get

$$df = \partial_\mu f dx^\mu + \frac{\kappa}{2} \square f \tau \quad (\text{summation over } \mu) \quad (6.23)$$

in the continuum limit. Here $\square := \sum_{\mu\nu} \delta^{\mu\nu} \partial_\mu \partial_\nu$ is the d'Alembertian of the metric $\delta^{\mu\nu}$ and ∂_μ is the ordinary partial derivative with respect to x^μ . Differential calculi of the form (6.21), (6.22) on manifolds have been investigated recently. They are related to quantum theory [21] and stochastics [22] and show up in the classical limit of (bicovariant) differential calculi on certain quantum groups [3].

Returning to the general case, we find from (6.4) the 2-form relations

$$dx^\mu dx^\nu + dx^\nu dx^\mu = 0 \quad , \quad dx^\mu \tau^\nu + \tau^\nu dx^\mu = 0 \quad , \quad \tau^\mu \tau^\nu + \tau^\nu \tau^\mu = 0 . \quad (6.24)$$

An r -form ψ can be written in the following two ways,

$$\begin{aligned} \psi &= \frac{1}{r!} \sum_{\substack{\epsilon_1, \dots, \epsilon_r \\ \mu_1, \dots, \mu_r}} \psi_{\epsilon_1 \mu_1 \dots \epsilon_r \mu_r} e^{\epsilon_1 \mu_1} \dots e^{\epsilon_r \mu_r} \\ &= \frac{1}{r!} \sum_{\mu_1, \dots, \mu_r} \sum_{k=0}^r \binom{r}{k} \psi_{\mu_1 \dots \mu_r}^{(k)} \tau^{\mu_1} \dots \tau^{\mu_k} dx^{\mu_{k+1}} \dots dx^{\mu_r} . \end{aligned} \quad (6.25)$$

Using (6.11) and (6.13) we obtain

$$\psi_{\epsilon_1 \mu_1 \dots \epsilon_r \mu_r} = \sum_{k=0}^r \frac{\ell^{r-k} \beta^k}{k! (r-k)!} \sum_{\pi \in \mathcal{S}_r} \text{sgn} \pi \epsilon_{\pi(k+1)} \dots \epsilon_{\pi(r)} \psi_{\mu_{\pi(1)} \dots \mu_{\pi(r)}}^{(k)} . \quad (6.26)$$

Furthermore, we have

$$d\psi = u \psi - (-1)^r \psi u = [u, \psi] \quad (6.27)$$

with

$$u := \sum_{\epsilon, \mu} e^{\epsilon \mu} = \frac{1}{\beta} \sum_{\mu} \tau^\mu \quad (6.28)$$

which satisfies $u^2 = 0$.

An inner product is determined by

$$(e_a^{\epsilon_1 \mu_1} \dots e^{\epsilon_r \mu_r}, e_b^{\epsilon'_1 \nu_1} \dots e^{\epsilon'_s \nu_s}) := \delta_{rs} (2\ell^2)^{-r} \delta_{a,b} \delta_{\epsilon_1 \mu_1 \dots \epsilon_r \mu_r}^{\epsilon'_1 \nu_1 \dots \epsilon'_s \nu_s} \quad (6.29)$$

and the usual rules (2.22). The adjoint d^* of d with respect to this inner product then acts as follows,

$$d^* e^{\epsilon_1 \mu_1} \dots e^{\epsilon_r \mu_r} = \frac{1}{2\ell^2} \sum_{k=1}^r (-1)^{k+1} [e_{a+\epsilon_k \hat{\mu}_k} - e_a] e^{\epsilon_1 \mu_1} \dots e^{\widehat{\epsilon_k \mu_k}} \dots e^{\epsilon_r \mu_r} . \quad (6.30)$$

More generally, for an r -form ψ we have

$$\begin{aligned} d^* \psi &= -\frac{1}{2\ell} \frac{1}{(r-1)!} \sum_{\substack{\epsilon_1, \dots, \epsilon_r \\ \mu_1, \dots, \mu_r}} \epsilon_1 \partial_{-\epsilon_1 \mu_1} \psi_{\epsilon_1 \mu_1 \dots \epsilon_r \mu_r} e^{\epsilon_2 \mu_2} \dots e^{\epsilon_r \mu_r} \\ &= \frac{1}{(r-1)!} \sum_{\mu_1, \dots, \mu_r} \sum_{k=0}^r \left[\frac{\beta}{2} \binom{r-1}{k} (\Delta_{\mu_1} \psi_{\mu_1 \dots \mu_r}^{(k)}) \tau^{\mu_2} \dots \tau^{\mu_k} dx^{\mu_{k+1}} \dots dx^{\mu_r} \right. \\ &\quad \left. + (-1)^{k+1} \binom{r-1}{k} (\bar{\partial}_{\mu_{k+1}} \psi_{\mu_1 \dots \mu_r}^{(k)}) \tau^{\mu_1} \dots \tau^{\mu_k} dx^{\mu_{k+2}} \dots dx^{\mu_r} \right] . \end{aligned} \quad (6.31)$$

For a 1-form $\rho = \sum_{\mu} (\rho_{\mu}^{(0)} dx^{\mu} + \rho_{\mu}^{(1)} \tau^{\mu})$ this reads

$$d^* \rho = - \sum_{\mu, \nu} \delta^{\mu\nu} (\bar{\partial}_{\mu} \rho_{\nu}^{(0)} - \frac{\beta}{2} \Delta_{\mu} \rho_{\nu}^{(1)}) . \quad (6.32)$$

If $\rho = df$, this implies

$$-d^* df = \sum_{\mu, \nu} \delta^{\mu\nu} (\bar{\partial}_{\mu} \bar{\partial}_{\nu} f - \frac{\ell^2}{2} \Delta_{\mu} \Delta_{\nu} f) = \sum_{\mu} \Delta_{\mu} f \quad (6.33)$$

where we made use of the identity

$$\bar{\partial}_{\mu}^2 f - \frac{\ell^2}{4} \Delta_{\mu}^2 f = \Delta_{\mu} f . \quad (6.34)$$

In the continuum limit one thus obtains $-d^* df = \delta^{\mu\nu} \partial_{\mu} \partial_{\nu} f = \square f$ (summation over μ and ν).

The involution on $\Omega(\mathcal{A})$ (induced by the identity on \mathcal{M}) introduced in section II acts on the basic 1-forms as follows,

$$(e_a^{\epsilon\mu})^* = -e_{a+\epsilon\hat{\mu}}^{-\epsilon\mu} . \quad (6.35)$$

This leads to

$$(e^{\epsilon\mu})^* = -e^{-\epsilon\mu} \quad , \quad (dx^{\mu})^* = dx^{\mu} \quad , \quad (\tau^{\mu})^* = -\tau^{\mu} . \quad (6.36)$$

The \bullet -products are again defined by (2.26), now with respect to the inner product (6.29). For their evaluation it is sufficient to know that

$$e^{\epsilon\mu} \bullet (e^{\epsilon_1\mu_1} \dots e^{\epsilon_r\mu_r}) = \frac{1}{2\ell^2} \sum_{k=1}^r (-1)^{k+1} \delta_{\epsilon\mu}^{\epsilon_k\mu_k} e^{\epsilon_1\mu_1} \dots e^{\widehat{\epsilon_k\mu_k}} \dots e^{\epsilon_r\mu_r} . \quad (6.37)$$

In particular, one obtains

$$\begin{aligned} & (\tau^\mu)^\natural \bullet (\tau^{\mu_1} \dots \tau^{\mu_s} dx^{\mu_{s+1}} \dots dx^{\mu_r}) \\ &= \frac{\beta}{\kappa} \sum_{k=1}^s (-1)^{k+1} \delta^{\mu\mu_k} \tau^{\mu_1} \dots \widehat{\tau^{\mu_k}} \dots \tau^{\mu_s} dx^{\mu_{s+1}} \dots dx^{\mu_r} \end{aligned} \quad (6.38)$$

$$\begin{aligned} & (dx^\mu)^\natural \bullet (\tau^{\mu_1} \dots \tau^{\mu_s} dx^{\mu_{s+1}} \dots dx^{\mu_r}) \\ &= \sum_{k=s+1}^r (-1)^{k+1} \delta^{\mu\mu_k} \tau^{\mu_1} \dots \tau^{\mu_s} dx^{\mu_{s+1}} \dots \widehat{dx^{\mu_k}} \dots dx^{\mu_r} \end{aligned} \quad (6.39)$$

and

$$(\tau^{\mu_1} \dots \tau^{\mu_s} dx^{\mu_{s+1}} \dots dx^{\mu_r}) \bullet (e^{\epsilon\mu})^\natural = (-1)^{r+1} (e^{\epsilon\mu})^\natural \bullet (\tau^{\mu_1} \dots \tau^{\mu_s} dx^{\mu_{s+1}} \dots dx^{\mu_r}) . \quad (6.40)$$

Using these expressions one can show that

$$(\psi, \phi) = \frac{1}{r!} \sum_{a, \mu_1, \dots, \mu_r} \sum_{k=0}^r \binom{r}{k} \left(\frac{\beta}{\kappa} \right)^k \overline{\psi}_{\mu_1 \dots \mu_r}^{(k)}(\ell a) \phi_{\mu_1 \dots \mu_r}^{(k)}(\ell a) \quad (6.41)$$

for r -forms ψ, ϕ .

B. Gauge theory on the symmetric lattice

A connection 1-form on the symmetric lattice can be expressed as

$$A = \sum_{\epsilon, \mu} A_{\epsilon\mu} e^{\epsilon\mu} = \sum_{\mu} (A_{\mu}^{(0)} dx^{\mu} + \frac{\kappa}{2} A_{\mu}^{(1)} \tau^{\mu}) \quad (6.42)$$

where $A_{\epsilon\mu} = \epsilon \ell A_{\mu}^{(0)} + (\ell^2/2) A_{\mu}^{(1)}$. The transformation rule (4.2) for a connection 1-form leads to

$$\begin{aligned} 0 = \sum_{\mu} \left\{ \left[\bar{\partial}_{\mu} G - G A_{\mu}^{(0)} + A_{\mu}^{(0)} (G + \frac{\kappa\beta}{2} \Delta_{\mu} G) + \frac{\kappa\beta}{2} A_{\mu}^{(1)} \bar{\partial}_{\mu} G \right] dx^{\mu} \right. \\ \left. + \frac{\kappa}{2} \left[\Delta_{\mu} G - G A_{\mu}^{(1)} + A_{\mu}^{(1)} (G + \frac{\kappa\beta}{2} \Delta_{\mu} G) + 2 A_{\mu}^{(0)} \bar{\partial}_{\mu} G \right] \tau^{\mu} \right\} . \end{aligned} \quad (6.43)$$

Formally, the continuum limit $\ell \rightarrow 0$ of this equation yields the familiar gauge transformation formula $\partial_{\mu} G = G A_{\mu}^{(0)} - A_{\mu}^{(0)} G$ for $A^{(0)}$ and in addition

$$\square G = -2 \sum_{\mu, \nu} \delta^{\mu\nu} A_{\nu}^{(0)} \partial_{\mu} G + G A^{(1)} - A^{(1)} G \quad (6.44)$$

where $A^{(1)} := \sum_{\mu} A_{\mu}^{(1)}$.

In terms of the basis dx^{μ}, τ^{ν} , the curvature 2-form $F = dA + A^2$ reads

$$\begin{aligned}
F &= \sum_{\mu,\nu} \left\{ \left[\bar{\partial}_\mu A_\nu^{(0)} + A_\mu^{(0)} A_\nu^{(0)} + \frac{\kappa\beta}{2} (A_\mu^{(0)} \Delta_\mu A_\nu^{(0)} + A_\mu^{(1)} \bar{\partial}_\mu A_\nu^{(0)}) \right] dx^\mu dx^\nu \right. \\
&\quad + \frac{\kappa}{2} \left[\Delta_\mu A_\nu^{(0)} - \bar{\partial}_\nu A_\nu^{(1)} - A_\nu^{(0)} A_\mu^{(1)} + A_\mu^{(1)} A_\nu^{(0)} + 2A_\mu^{(0)} \bar{\partial}_\mu A_\nu^{(0)} \right. \\
&\quad \left. \left. - \frac{\kappa\beta}{2} (A_\nu^{(0)} \Delta_\nu A_\mu^{(1)} - 2A_\mu^{(1)} \Delta_\mu A_\nu^{(0)} + A_\nu^{(1)} \bar{\partial}_\nu A_\mu^{(1)}) \right] \tau^\mu dx^\nu \right. \\
&\quad \left. + \frac{\kappa^2}{4} \left[\Delta_\mu A_\nu^{(1)} + 2A_\mu^{(0)} \bar{\partial}_\mu A_\nu^{(1)} + A_\mu^{(1)} A_\nu^{(1)} + \frac{\kappa\beta}{2} A_\mu^{(1)} \Delta_\mu A_\nu^{(1)} \right] \tau^\mu \tau^\nu \right\} \\
&=: \sum_{\mu,\nu} \left[\frac{1}{2} F_{\mu\nu}^{(0)} dx^\mu dx^\nu + F_{\mu\nu}^{(1)} \tau^\mu dx^\nu + \frac{1}{2} F_{\mu\nu}^{(2)} \tau^\mu \tau^\nu \right]. \tag{6.45}
\end{aligned}$$

Evaluation of the Yang-Mills action with the help of (6.41) leads to

$$S_{YM} = \text{tr} \sum_{a,\mu,\nu} \left[\frac{1}{2} F_{\mu\nu}^{(0)\dagger} F_{\mu\nu}^{(0)} + \frac{\beta}{\kappa} F_{\mu\nu}^{(1)\dagger} F_{\mu\nu}^{(1)} + \frac{1}{2} \left(\frac{\beta}{\kappa} \right)^2 F_{\mu\nu}^{(2)\dagger} F_{\mu\nu}^{(2)} \right] \tag{6.46}$$

where the function in square brackets has to be taken at ℓa . Obviously, because of the factors β/κ the ordinary Yang-Mills action for $A_\mu^{(0)}$ is obtained in the limit $\ell \rightarrow 0, \beta \rightarrow 0$ (with κ fixed).

Again, we introduce

$$U_{\epsilon\mu} = \mathbf{1} + A_{\epsilon\mu} = \mathbf{1} + \epsilon \ell A_\mu^{(0)} + \frac{\ell^2}{2} A_\mu^{(1)} \tag{6.47}$$

which transforms as follows,

$$U'_{\epsilon\mu}(x) = G(x) U_{\epsilon\mu}(x) G(x + \epsilon \ell \hat{\mu})^\dagger \tag{6.48}$$

(note that $G^\dagger = G^{-1}$). Using (6.18) this implies that

$$E^{\epsilon\mu} := U_{\epsilon\mu} e^{\epsilon\mu} \tag{6.49}$$

transform covariantly under a gauge transformation, i.e., $G E^{\epsilon\mu} G^\dagger = U'_{\epsilon\mu} e^{\epsilon\mu} = E'^{\epsilon\mu}$. Also covariant are the 1-forms

$$Dx^\mu := \left(\mathbf{1} + \frac{\kappa\beta}{2} A_\mu^{(1)} \right) dx^\mu + \kappa A_\mu^{(0)} \tau^\mu = U x^\mu - x^\mu U \tag{6.50}$$

where

$$U := u + A = \sum_{\epsilon,\mu} U_{\epsilon\mu} e^{\epsilon\mu} = \sum_{\epsilon,\mu} E^{\epsilon\mu} \tag{6.51}$$

with u defined in (6.28). Together with $E^{\epsilon\mu}$ the Dx^μ constitute a basis of the space of 1-forms (as a left or right \mathcal{A} -module) and allow us to read off covariant components from covariant differential forms.

For the covariant exterior derivatives (4.1) we find

$$D\Psi = U\Psi - \Psi u \quad , \quad D\alpha = u\alpha - \alpha U \quad . \quad (6.52)$$

In the following we constrain U with the conditions

$$U_{-\epsilon\mu}(x + \epsilon\ell\hat{\mu}) = U_{\epsilon\mu}(x)^\dagger = U_{\epsilon\mu}(x)^{-1} \quad (6.53)$$

(cf (4.18) and (4.29)) for a given conjugation. It may be more reasonable to dispense with the last condition in (6.53). See also the discussion in section IV.A.

For the curvature we find

$$F = U^2 = \frac{1}{2} \sum_{\epsilon,\mu,\epsilon',\nu} [U_{\epsilon\mu}(x)U_{\epsilon'\nu}(x + \epsilon\ell\hat{\mu}) - U_{\epsilon'\nu}(x)U_{\epsilon\mu}(x + \epsilon'\ell\hat{\nu})] e^{\epsilon\mu} e^{\epsilon'\nu} \quad . \quad (6.54)$$

The Yang-Mills equation

$$d^*F + A^\natural \bullet F + F \bullet A^\natural = U^\natural \bullet (U^2) + (U^2) \bullet U^\natural = 0 \quad (6.55)$$

now leads to

$$\begin{aligned} U_{\epsilon\mu}(x) &= \frac{1}{4n} \sum_{\epsilon',\nu} [U_{\epsilon'\nu}(x)U_{\epsilon\mu}(x + \ell\epsilon'\hat{\nu})U_{\epsilon'\nu}(x + \ell\epsilon\hat{\mu})^\dagger \\ &\quad + U_{\epsilon'\nu}(x - \ell\epsilon'\hat{\nu})^\dagger U_{\epsilon\mu}(x - \ell\epsilon'\hat{\nu})U_{\epsilon'\nu}(x + \ell(\epsilon\hat{\mu} - \epsilon'\hat{\nu}))] \end{aligned} \quad (6.56)$$

and the Yang-Mills action takes the form

$$\begin{aligned} S_{YM} &= \frac{1}{4\ell^4} \text{tr} \sum_{a,\epsilon,\epsilon',\mu,\nu} [1 - U_{\epsilon\mu}(\ell a)U_{\epsilon'\nu}(\ell(a + \epsilon\hat{\nu}))U_{\epsilon\mu}(\ell(a + \epsilon'\hat{\nu}))^\dagger U_{\epsilon'\nu}(\ell a)^\dagger] \\ &= \frac{1}{\ell^4} \text{tr} \sum_{a,\mu,\nu} [1 - U_{+\mu}(\ell a)U_{+\nu}(\ell(a + \hat{\mu}))U_{+\mu}(\ell(a + \hat{\nu}))^\dagger U_{+\nu}(\ell a)^\dagger] \quad . \end{aligned} \quad (6.57)$$

This is again the Wilson action (cf (5.46)). Note, however, that this result was obtained by imposing an additional constraint, the second equation in (6.53).

VII. CONCLUSIONS

We have explored differential algebras on a discrete set \mathcal{M} . In section II we introduced 1-forms e_{ij} , $i, j \in \mathcal{M}$, which generate the differential algebra over \mathbb{C} . They turned out to be particularly convenient to work with and, in particular, provided us with a simple way to ‘reduce’ the universal differential algebra to smaller differential algebras. Such ‘reductions’ of the universal differential algebra are described by certain graphs which can be related to ‘Hasse diagrams’ determining a locally finite topology. In this way, contact was made in section III with the recent work by Balachandran et al. [18] where a calculus on ‘posets’ (partially ordered sets) has been developed with the idea to discretize continuum models in such a way that important topological features of continuum physics (like winding numbers) are preserved. What we learned is that the adequate framework for doing this is (noncommutative) differential calculus on discrete sets. As a special example, the differential calculus

which corresponds to an oriented hypercubic lattice graph reproduces the familiar formalism of lattice (gauge) theories (see also [6]). This is, however, just one choice among many.

In particular, we have studied the differential calculus associated with the ‘symmetric’ hypercubic lattice graph. In a certain continuum limit, this calculus tends to a deformation of the ordinary calculus of differential forms on a manifold which is known to be related to quantum theory [21], stochastics [22] and, more exotically, differential calculus on quantum groups [3]. In the same limit, however, the Yang-Mills action on the symmetric lattice just tends to the ordinary Yang-Mills action.

In this work, we have presented a formulation of gauge theory on a discrete set or, more precisely, on graphs describing differential algebras on a discrete set. This should be viewed as a generalization of the familiar Wilson loop formulation of lattice gauge theory. A corresponding gauge theoretical approach to a discrete gravity theory will be discussed elsewhere [23]. In that case, ‘symmetric graphs’ play a distinguished role.

As already mentioned in the introduction, it seems that the relation between differential calculus on finite sets and approximations of topological spaces established in the present work allows us to understand the ‘discrete’ gauge theory models of Connes and Lott [4] (and many similar models which have been proposed after their work) as approximations of higher-dimensional gauge theory models (see [12,11], in particular). The details have still to be worked out, however.

A differential algebra provides us with a notion of locality since its graph determines a neighbourhood structure. If we supply, for example, the set \mathbb{Z} with the differential algebra such that $e_{ij} \neq 0$ iff $j = i + 1$, then some fixed i and $i + 1000$, say, are quite remote from one another in the sense that they are connected via many intermediate points. If, however, we allow also $e_{i(i+1000)} \neq 0$, then the two points become neighbours. This modification of the graph has crucial consequences since a nonvanishing e_{ij} yields the possibility of correlations between fields at the points i and j . If a set is supplied with the universal differential algebra, then correlations between any two points are allowed and will naturally be present in a field theory built on it. One could imagine that in such a field theory certain correlations are dynamically suppressed so that, e.g., a four-dimensional structure of the set is observed.

The universal differential algebra on a set \mathcal{M} corresponds to the graph where the elements of \mathcal{M} are represented by the vertices and any two points are connected by two (antiparallel) arrows. To the arrow from i to j we may assign a probability $p_{ij} \in [0, 1]$. We are then dealing with a ‘fuzzy graph’. If $p_{ij} \in \{0, 1\}$ for all $i, j \in \mathcal{M}$, we recover our concept of a reduction of the universal differential algebra. A more general formalism (allowing also values of $p_{ij} \in (0, 1)$) could describe, e.g., fluctuations in the space (-time) dimension since the latter depends on how many (direct) neighbours a given site has. This suggests a quantization of the universal differential algebra by introducing creation and annihilation operators for the 1-forms e_{ij} (and perhaps higher forms).

After completion of this work we received a preprint by Balachandran et al [24] which also relates (poset) approximations of topological spaces and noncommutative geometry although in a way quite different from ours. In particular, they associate a *non*-commutative algebra (of operators on a Hilbert space) with a poset.

APPENDIX: RELATION WITH ČECH-COHOMOLOGY

From the universal differential algebra on a discrete set \mathcal{M} one can construct a \mathbb{C} -vector space $\tilde{A}^k(\mathcal{A})$ of *antisymmetric* k -forms ($k > 0$) generated by

$$a_{i_0 \dots i_k} := e_{[i_0 \dots i_k]} \quad (\text{A.1})$$

where $e_{i_0 \dots i_k}$ is defined in (2.7) and the square brackets indicate antisymmetrization. $\tilde{A}^k(\mathcal{A})$ is *not* an \mathcal{A} -module since multiplication with e_i leaves the space. For example,

$$e_i \underbrace{(e_{ij} - e_{ji})}_{\in \tilde{A}^1(\mathcal{A})} = e_{ij} \notin \tilde{A}^1(\mathcal{A}). \quad (\text{A.2})$$

We set $\tilde{A}(\mathcal{A}) := \bigoplus_{k \geq 0} \tilde{A}^k(\mathcal{A})$ with $\tilde{A}^0(\mathcal{A}) := \mathcal{A}$. More generally, one may consider any reduction $A(\mathcal{A})$ of $\tilde{A}(\mathcal{A})$ obtained by setting some of the generators $a_{i_0 \dots i_k}$ to zero. Since

$$a_{i_0 \dots i_k} \neq 0 \Rightarrow a_{j_0 \dots j_\ell} \neq 0 \text{ if } \{j_0, \dots, j_\ell\} \subset \{i_0, \dots, i_k\} \quad (\text{A.3})$$

one finds from (2.16) that $A(\mathcal{A})$ is closed under d . Now

$$a_{i_0 \dots i_k} = e_{[i_0} \otimes \dots \otimes e_{i_k]} \quad (\text{A.4})$$

(cf the second remark in section II) yields a representation of $A(\mathcal{A})$. If $f \in A^k(\mathcal{A})$, then $f = \sum_{j_0 \dots j_k} a_{j_0 \dots j_k} f_{j_0 \dots j_k}$ with antisymmetric coefficients $f_{j_0 \dots j_k} \in \mathbb{C}$ and thus

$$f(i_0, \dots, i_k) = f_{i_0 \dots i_k}. \quad (\text{A.5})$$

Furthermore, from (2.16) we obtain

$$df(i_0, \dots, i_{k+1}) = \sum_{j=0}^{k+1} (-1)^j f(i_0, \dots, i_{j-1}, \hat{i}_j, i_{j+1}, \dots, i_k) \quad (\text{A.6})$$

where a hat indicates an omission. These formulae are reminiscent of Čech cohomology theory. The relation will be explained in the following.

Let $\mathcal{U} = \{\mathcal{U}_i \mid i \in \mathcal{M}\}$ be an open covering of a manifold M . In Čech cohomology theory a k -*simplex* is any $(k+1)$ -tuple (i_0, i_1, \dots, i_k) such that $\mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_k} \neq \emptyset$. A *Čech-cochain* is any (totally) antisymmetric mapping

$$f : (i_0, \dots, i_k) \mapsto f(i_0, \dots, i_k) \in \mathcal{K} \quad (\text{A.7})$$

where $\mathcal{K} = \mathbb{C}, \mathbb{R}, \mathbb{Z}$. The set of all k -cochains forms a \mathcal{K} -linear space $C^k(\mathcal{U}, \mathcal{K})$. The Čech-coboundary operator $d : C^k(\mathcal{U}, \mathcal{K}) \rightarrow C^{k+1}(\mathcal{U}, \mathcal{K})$ is then defined by (A.6). If \mathcal{U} is a *good* covering, then the de Rham cohomology of the manifold is isomorphic to the Čech cohomology with $\mathcal{K} = \mathbb{R}$ [25]. A covering of a manifold is ‘good’ if all finite nonempty intersections are contractible.

This suggests the following way to associate a topology with $A(\mathcal{A})$. For each element $i \in \mathcal{M}$ we have an open set \mathcal{U}_i . Intersection relations are then determined by

$$a_{i_0 \dots i_k} \neq 0 \Leftrightarrow \mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_k} \neq \emptyset. \quad (\text{A.8})$$

In algebraic topology one constructs from the intersection relations of the open sets \mathcal{U}_i a simplicial complex, the *nerve* of \mathcal{U} (see [25], for example). If $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, we connect the vertices i and j with an edge. Since the intersection relation is symmetric we may also think of drawing two antiparallel arrows between i and j (thus making contact with the procedure in section III). A triple intersection relation $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$ (where i, j, k are pairwise different) corresponds to the face of the triangle with corners i, j, k , and so forth. Instead of the simplicial complex (the nerve) obtained in this way – which need not be a simplicial approximation of the manifold (see [16]) – we can construct a Hasse diagram with the same information as follows. The first row consists of the basic vertices corresponding to the elements of \mathcal{M} respectively the open sets \mathcal{U}_i , $i \in \mathcal{M}$. The next row (below the first) consists of vertices associated with the nontrivial intersections of pairs of the open sets \mathcal{U}_i . The vertex representing $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ then gets connections with \mathcal{U}_i and \mathcal{U}_j . With each $\mathcal{U}_i \cap \dots \cap \mathcal{U}_j \neq \emptyset$ we associate a new vertex and proceed in an obvious way.

In section III we started from a differential calculus (a reduction of the universal differential calculus on \mathcal{M}) and derived a Hasse diagram from it which then determined a covering of some topological space. The covering defines a Čech complex and we have seen above that the latter is represented by some space $A(\mathcal{A})$ of antisymmetric forms.

REFERENCES

- [1] A. Connes, “Non-commutative differential geometry”, Publ. I.H.E.S. **62**, 257 (1986).
- [2] R. Coquereaux, “Noncommutative geometry and theoretical physics”, J. Geom. Phys. **6**, 425 (1989).
- [3] F. Müller-Hoissen and C. Reuten, “Bicovariant differential calculus on $GL_{p,q}(2)$ and quantum subgroups”, J. Phys. A (Math. Gen.) **26**, 2955 (1993); F. Müller-Hoissen, “Differential calculi on quantum (sub-) groups and their classical limit”, preprint, Göttingen University, hep-th/9401152, to appear in the proceedings of the workshop on *Generalized Symmetries in Physics*, ASI Clausthal, July 1993.
- [4] A. Connes and J. Lott, “Particle models and noncommutative geometry”, Nucl. Phys. B (Proc. Suppl.) **18**, 29 (1990); “The metric aspect of noncommutative geometry”, in *New Symmetry Principles in Quantum Field Theory*, edited by J. Fröhlich et al. (Plenum Press, New York, 1992) p. 53.
- [5] A. Dimakis and F. Müller-Hoissen, “Quantum mechanics on a lattice and q -deformations”, Phys. Lett. **295B**, 242 (1992).
- [6] A. Dimakis, F. Müller-Hoissen and T. Striker, “From continuum to lattice theory via deformation of the differential calculus”, Phys. Lett. **300B**, 141 (1993); “Noncommutative differential calculus and lattice gauge theory”, J. Phys. A **26**, 1927 (1993).
- [7] A. Dimakis and F. Müller-Hoissen, “Differential calculus and gauge theory on finite sets”, J. Phys. A: Math. Gen. **27**, 3159 (1994); “Differential calculus and discrete structures”, preprint, Göttingen University, hep-th/9401150, to appear in the proceedings of the workshop on *Generalized Symmetries in Physics*, ASI Clausthal, July 1993.
- [8] V. Ambarzumian and D. Iwanenko, “Zur Frage nach Vermeidung der unendlichen Selbststrückwirkung des Elektrons”, Z. Physik **64**, 563 (1930); A. Ruark, “The roles of discrete and continuous theories in physics”, Phys. Rev. **37**, 315 (1931); H.S. Snyder, “Quantized space-time”, Phys. Rev. **71**, 38 (1947); D. Finkelstein, “Space-time code”, Phys. Rev. **184**, 1261 (1969); R.P. Feynman, “Simulating physics with computers”, Int. J. Theor. Phys. **21**, 467 (1982); M. Minsky, “Cellular vacuum”, Int. J. Theor. Phys. **21**, 537 (1982); G. Feinberg, R. Friedberg, T.D. Lee and H.C. Ren, “Lattice gravity near the continuum”, Nucl. Phys. B **245**, 343 (1984); T.D. Lee, “Discrete mechanics”, in *How far are we from the gauge forces*, edited by A. Zichichi (Plenum Press, New York, 1985) p.15; H. Yamamoto, “Quantum field theory on discrete space-time”, Phys. Rev. **30**, 1727 (1984); “Quantum field theory on discrete space-time. II”, Phys. Rev. **32**, 2659 (1985); H. Yamamoto, A. Hayashi, T. Hashimoto and M. Horibe, “Towards a canonical formalism of field theory on discrete spacetime”, preprint, Fukui University, Japan (1993); L. Bombelli, J. Lee, D. Meyer and R.D. Sorkin, “Space-time as a causal set”, Phys. Rev. Lett. **59**, 521 (1987); G. ’t Hooft, “Quantization of discrete deterministic theories by Hilbert space extension”, Nucl. Phys. B **342**, 471 (1990); D. Finkelstein and J.M. Gibbs, “Quantum relativity”, Int. J. Theor. Phys. **32**, 1801 (1993).
- [9] R.D. Sorkin, “Finitary substitute for continuous topology”, Int. J. Theor. Phys. **30**, 923 (1991).
- [10] C. Aneziris, “Topology and statistics in zero dimensions”, Int. J. Theor. Phys. **33**, 535 (1994).
- [11] D. Kapetanakis and G. Zoupanos, “Coset space dimensional reduction of gauge theories”, Phys. Rep. **219**, 1 (1992).
- [12] N.S. Manton, “A new six-dimensional approach to the Weinberg-Salam model”, Nucl. Phys. B **158**, 141 (1979).

- [13] A. Connes, “Noncommutative geometry and physics”, Les Houches lecture notes, IHES/M/93/32.
- [14] W. Ritz, “Über ein neues Gesetz der Serienspektren”, *Phys. Z.* **9**, 521 (1908); J.R. Rydberg, “La distribution des raies spectrales”, in *Rapports présentés au Congrès International de Physique*, Vol. 2 (Gauthier-Villars, Paris, 1900), p. 200.
- [15] M. Born and P. Jordan, “Zur Quantentheorie”, *Zeitschr. f. Physik* **34**, 858 (1925).
- [16] J.G. Hocking and G.S. Young, *Topology* (Addison-Wesley, Reading MA, 1961).
- [17] A.V. Evako, “Dimension on discrete spaces”, preprint, Syracuse University, gr-qc/9402035.
- [18] A. P. Balachandran, G. Bimonte, E. Ercolessi and P. Teotonio-Sobrinho, “Finite approximations to quantum physics: quantum points and their bundles”, preprint, Syracuse University, SU-4240-550 (1993).
- [19] D. Kastler, *Cyclic cohomology within the differential envelope* (Hermann, Paris, 1988).
- [20] J. Rabin, “Homology theory of lattice fermion doubling”, *Nucl.Phys.* **B201**, 315 (1981); A. Guth, “Existence proof of a nonconfining phase in four-dimensional $U(1)$ lattice gauge theory”, *Phys. Rev D* **21**, 2291 (1980).
- [21] A. Dimakis and F. Müller-Hoissen, “A noncommutative differential calculus and its relation to gauge theory and gravitation”, *Int. J. Mod. Phys. A (Proc. Suppl.)* **3A**, 474 (1993); “Noncommutative differential calculus, gauge theory and gravitation”, preprint, Göttingen University, GOE-TP 33/92 (1992).
- [22] A. Dimakis and F. Müller-Hoissen, “Stochastic differential calculus, the Moyal \ast -product, and noncommutative geometry”, *Lett. Math. Phys.* **28**, 123 (1993); “Noncommutative Differential Calculus: Quantum Groups, Stochastic Processes, and the Antibracket”, preprint, Göttingen University, GOET-TP 55/93, hep-th/9401151.
- [23] A. Dimakis and F. Müller-Hoissen, “Noncommutative geometry and gravitation”, in preparation.
- [24] A. P. Balachandran, G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano and P. Teotonio-Sobrinho, “Finite quantum physics and noncommutative geometry”, preprint, Syracuse University, SU-4240-567, March 1994.
- [25] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology* (Springer, New York, 1982).