L(2,1)-Labeling in the Context of Some Graph Operations

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Abstract

Let $G = (V(G), E(G))$ be a connected graph. For integers $j \geq k$, $L(j, k)$-labeling of a graph G is an integer labeling of the vertices in V such that adjacent vertices receive integers which differ by at least j and vertices which are at distance two apart receive labels which differ by at least k. In this paper we discuss $L(2, 1)$-labeling (or distance two labeling) in the context of some graph operations.

Keywords: Graph Labeling, λ- Number, λ'- Number

1. Introduction

We begin with finite, connected, undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For standard terminology and notations we refer to (West, D., 2001). We will give brief summary of definitions and information which are prerequisites for the present work.

Definition 1.1 *Duplication* of a vertex v_k of graph G produces a new graph G' by adding a vertex v'_k with $N(v'_k) = N(v_k)$. In other words a vertex v'_k is said to be duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to v'_k also.

Definition 1.2 Let G be a graph. A graph H is called a *supersubdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} (for some m_i and $1 \leq i \leq q$) in such a way that the ends of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G.

A new family of graph introduced in (Vaidya, S., 2008, p.54-64) defined as follows.

Definition 1.3 A graph obtained by replacing each vertex of a star $K_{1,n}$ by a graph G is called *star of G* denoted as G'. The central graph in G' is the graph which replaces apex vertex of $K_{1,n}$.

Definition 1.4 If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

The unprecedented growth of wireless communication is recorded but the available radio frequencies allocated to these communication networks are not enough. Proper allocation of frequencies is demand of the time. The interference by unconstrained transmitters will interrupt the communications. This problem was taken up in (Hale, W., 1980, p.1497-1514) in terms of graph labeling. In a private communication with Griggs during 1988 Roberts proposed a variation in channel assignment problem. According to him any two close transmitters must receive different channels in order to avoid interference. Motivated by this problem the concept of L(2,1)-labeling was introduced by (Yeh, R., 1990) and

(Griggs, J., and Yeh, R., 1992, p.586-595) which is defined as follows.

Definition 1.5 For a graph G, $L(2,1)$-*labeling* (or distance two labeling) with span k is a function $f : V(G) \longrightarrow \{0, 1, \ldots, k\}$ such that the following conditions are satisfied:

(1)$|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$

(2)$|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$

In otherwords the $L(2,1)$-labeling of a graph is an abstraction of assigning integer frequencies to radio transmitters such that (1) Transmitters that are one unit of distance apart receive frequencies that differ by at least two and (2) Transmitters that are two units of distance apart receive frequencies that differ by at least one. The *span* of f is the largest number in $f(V)$. The minimum span taken over all $L(2,1)$-labeling of G, denoted as $\lambda(G)$ is called the λ-*number* of G. The minimum label in $L(2,1)$-labeling of G is assumed to be 0.

Definition 1.6 An injective $L(2,1)$-labeling is called an $L^{'}(2,1)$-labeling and the minimum span taken over all such $L^{'}(2,1)$-labeling is called $\lambda^{'}$-number of the graph.

The $L(2,1)$-labeling has been extensively studied in the recent past by many researchers (Georges, J., 1995, p.141-159), (Georges, J.P., 2001, p.28-35),(Georges, J., 1996, p.47-57),(Georges, J., 1994, p.103-111),(Liu. D., 1997, p.13-22),(Shao. Z., 2005, p.668-671). Practically it is observed that the interference might go beyond two levels. This observation motivated (Chartrand. G., 2001, p.77-85) to introduce the concept of radio labeling which is the extension of $L(2,1)$-labeling when the interference is beyond two levels to the largest possible - the diameter of G. We investigate three results corresponding to $L(2,1)$-labeling and $L^{'}(2,1)$-labeling each.

2. Main Results

Theorem 2.1 Let $C_n^{'}$ be the graph obtained by duplicating all the vertices of the cycle C_n at a time then $\lambda(C_n^{'}) = 7$. (where $n > 3$)

Proof: Let $v_1^{'}, v_2^{'}, \ldots, v_n^{'}$ be the duplicated vertices corresponding to $v_1, v_2, \ldots, v_n$ of cycle C_n.

To define $f : V(C_n^{'}) \longrightarrow N \bigcup \{0\}$, we consider following four cases.

Case 1: $n \equiv 0 (\text{mod } 3)$ (where $n > 5$)

We label the vertices as follows.

$$\begin{gathered}
f(v_i) = 0, i = 3j - 2, 1 \leq j \leq \tfrac{n}{3} \\
f(v_i) = 2, i = 3j - 1, 1 \leq j \leq \tfrac{n}{3} \\
f(v_i) = 4, i = 3j, 1 \leq j \leq \tfrac{n}{3} \\
f(v_i^{'}) = 7, i = 3j - 2, 1 \leq j \leq \tfrac{n}{3} \\
f(v_i^{'}) = 6, i = 3j - 1, 1 \leq j \leq \tfrac{n}{3} \\
f(v_i^{'}) = 5, i = 3j, 1 \leq j \leq \tfrac{n}{3}
\end{gathered}$$

Case 2: $n \equiv 1 (\text{mod } 3)$ (where $n > 5$)
We label the vertices as follows.

$$\begin{gathered}
f(v_i) = 0, i = 3j - 2, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_i) = 2, i = 3j - 1, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_i) = 4, i = 3j, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_{n-3}) = 0, f(v_{n-2}) = 3, f(v_{n-1}) = 1, f(v_n) = 4 \\
f(v_i^{'}) = 7, i = 3j - 2, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_i^{'}) = 6, i = 3j - 1, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_i^{'}) = 5, i = 3j, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_{n-3}^{'}) = 7, f(v_{n-2}^{'}) = 7, f(v_{n-1}^{'}) = 6, f(v_n^{'}) = 5
\end{gathered}$$

Case 3: $n \equiv 2 (\text{mod } 3)$ (where $n > 5$)

We label the vertices as follows.

$$\begin{gathered}
f(v_i) = 0, i = 3j - 2, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_i) = 2, i = 3j - 1, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_i) = 4, i = 3j, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\
f(v_{n-1}) = 1, f(v_n) = 3 \\
f(v_1^{'}) = 6, f(v_n^{'}) = 7
\end{gathered}$$

$$f(v_i^{'}) = 6, i = 3j - 1, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_i^{'}) = 5, i = 3j, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_i^{'}) = 7, i = 3j + 1, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor$$

Case 4: $n = 4, 5$

These cases are to be dealt separately. The $L(2, 1)$-labeling for the graphs obtained by duplicating all the vertices at a time in the cycle C_n when $n = 4, 5$ are as shown in *Fig 1*

Thus in all the possibilities $R_f = \{0, 1, 2 \ldots, 7\} \subset N \bigcup \{0\}$.

i.e. $\lambda(C_n^{'}) = 7$.

Remark The $L(2, 1)$-labeling for the graph obtained by duplicating all the vertices of the cycle C_3 is shown in *Fig 2*
Thus $R_f = \{0, 1, 2 \ldots, 6\} \subset N \bigcup \{0\}$.

i.e. $\lambda(C_3^{'}) = 6$.

Illustration 2.2 Consider the graph C_6 and duplicate all the vertices at a time. The $L(2, 1)$-labeling is as shown in *Fig 3*.

Theorem 2.3 Let $C_n^{'}$ be the graph obtained by duplicating all the vertices at a time of the cycle C_n then $\lambda^{'}(C_n^{'}) = p - 1$, where p is the total number vertices in $C_n^{'}$ (where $n > 3$).

Proof: Let $v_1^{'}, v_2^{'}, \ldots, v_n^{'}$ be the duplicated vertices corresponding to $v_1, v_2, \ldots, v_n$ of cycle C_n.

To define $f : V(C_n^{'}) \longrightarrow N \bigcup \{0\}$, we consider following two cases.

Case 1: $n > 5$

$$f(v_i) = 2i - 7, 4 \leq i \leq n$$
$$f(v_i) = f(v_n) + 2, 1 \leq i \leq 3$$
$$f(v_i^{'}) = 2i - 2, 1 \leq i \leq n$$

Now label the vertices of $C_n^{'}$ using the above defined pattern we have $R_f = \{0, 1, 2, \ldots, p - 1\} \subset N \bigcup \{0\}$

This implies that $\lambda^{'}(C_n^{'}) = p - 1$.

Case 2: $n = 4, 5$

These cases to be dealt separately. The $L^{'}(2, 1)$-labeling for the graphs obtained by duplicating all the vertices at a time in the cycle C_n when $n = 4, 5$ are as shown in the following *Fig 4*.

Remark The $L^{'}(2, 1)$-labeling for the graphs obtained by duplicating all the vertices at a time in the cycle C_3 is shown in the following *Fig 5*.

Thus $R_f = \{0, 1, 2 \ldots, 6\} \subset N \bigcup \{0\}$.

i.e. $\lambda^{'}(C_3^{'}) = 6$.

Illustration 2.4 Consider the graph C_6 and duplicating all the vertices at a time. The $L^{'}(2, 1)$-labeling is as shown in *Fig 6*.

Theorem 2.5 Let $C_n^{'}$ be the graph obtained by taking arbitrary supersubdivision of each edge of cycle C_n then

1 For n even

$$\lambda(C_n^{'}) = \Delta + 2$$

2 For n odd

$$\lambda(C_n^{'}) = \begin{cases} \Delta + 2; & if \quad s + t + r < \Delta, \\ \Delta + 3; & if \quad s + t + r = \Delta, \\ s + t + r + 2; & if \quad s + t + r > \Delta \end{cases}$$

where v_k is a vertex with label 2,
s is number of subdivision between v_{k-2} and v_{k-1},
t is number of subdivision between v_{k-1} and v_k,
r is number of subdivision between v_k and v_{k+1},
Δ is the maximum degree of $C_n^{'}$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n. Let $C_n^{'}$ be the graph obtained by arbitrary super subdivision of cycle C_n.

It is obvious that for any two vertices v_i and v_{i+2}, $N(v_i)\bigcap N(v_{i+2}) = \phi$
To define $f : V(C_n^{'}) \longrightarrow N\bigcup\{0\}$, we consider following two cases.

Case 1: C_n is even cycle

$$f(v_{2i-1}) = 0,\ 1 \leq i \leq \tfrac{n}{2}$$
$$f(v_{2i}) = 1,\ 1 \leq i \leq \tfrac{n}{2}$$

If P_{ij} is the number of supersubdivisions between v_i and v_j then for the vertex v_1, $|N(v_1)| = P_{12} + P_{n1}$. Without loss of generality we assume that v_1 is the vertex with maximum degree i.e. $d(v_1) = \Delta$. suppose $u_1, u_2.....u_\Delta$ be the members of $N(v_1)$. We label the vertices of $N(v_1)$ as follows.

$f(u_i) = 2 + i,\ 1 \leq i \leq \Delta$

As $N(v_1)\bigcap N(v_3) = \phi$ then it is possible to label the vertices of $N(v_3)$ using the vertex labels of the members of $N(v_1)$ in accordance with the requirement for $L(2, 1)$-labeling. Extending this argument recursively upto $N(v_{n-1})$ it is possible to label all the vertices of $C_n^{'}$ using the distinct numbers between 0 and $\Delta + 2$.

i.e. $R_f = \{0, 1, 2, \ldots, \Delta + 2\} \subset N\bigcup\{0\}$

Consequently $\lambda(C_n^{'}) = \Delta + 2$.

Case 2: C_n is odd cycle

Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n.

Without loss of generality we assume that v_1 is a vertex with maximum degree and v_k be the vertex with minimum degree.

Define $f(v_k) = 2$ and label the remaining vertices alternatively with labels 0 and 1 such that $f(v_1) = 0$. Then either $f(v_{k-1}) = 1$; $f(v_{k+1}) = 0$ OR $f(v_{k-1}) = 0$; $f(v_{k+1}) = 1$. We assign labeling in such a way that $f(v_{k-1}) = 1$; $f(v_{k+1}) = 0$.

Now following the procedure adapted in case (1) it is possible to label all the vertices except the vertices between v_{k-1} and v_k. Label the vertices between v_{k-1} and v_k using the vertex labels of $N(v_1)$ except the labels which are used earlier to label the vertices between v_{k-2}, v_{k-1} and between v_k, v_{k+1}.

If there are p vertices $u_1, u_2...u_p$ are left unlabeled between v_{k-1} and v_k then label them as follows,

$f(u_i)$=max{labels of the vertices between v_{k-2} and v_{k-1}, labels of the vertices between v_k and v_{k+1}} $+ i$, $1 \leq i \leq p$

Now if s is the number of subdivisions between v_{k-2} and v_{k-1}

t is the number of subdivisions between v_{k-1} and v_k

r is the number of subdivisions between v_k and v_{k+1}

then (1) $R_f = \{0, 1, 2, \ldots, \Delta + 2\} \subset N\bigcup\{0\}$, when $s + t + r < \Delta$

i.e. $\lambda(C_n^{'}) = \Delta + 2$
(2) $R_f = \{0, 1, 2, \ldots, \Delta + 3\} \subset N\bigcup\{0\}$, when $s + t + r = \Delta$
i.e. $\lambda(C_n^{'}) = \Delta + 3$
(3) $R_f = \{0, 1, 2, \ldots, s + t + r + 2\} \subset N\bigcup\{0\}$, when $s + t + r > \Delta$
i.e. $\lambda(C_n^{'}) = s + t + r + 2$

Illustration 2.6 Consider the graph C_8. The $L(2, 1)$-labeling of $C_8^{'}$ is shown in *Fig 7*.

Theorem 2.7 Let $G^{'}$ be the graph obtained by taking arbitrary supersubdivision of each edge of graph G with number of vertices $n \geq 3$ then $\lambda^{'}(G^{'}) = p - 1$, where p is the total number of vertices in $G^{'}$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of any connected graph G and let $G^{'}$ be the graph obtained by taking arbitrary supersubdivision of G. Let u_k be the vertices which are used for arbitrary supersubdivision of the edge v_iv_j where $1 \leq i \leq n$, $1 \leq j \leq n$ and $i < j$

Here k is a total number of vertices used for arbitrary supersubdivision.

We define $f : V(G^{'}) \longrightarrow N\bigcup\{0\}$ as

$f(v_i) = i - 1$, where $1 \leq i \leq n$

Now we label the vertices u_i in the following order.

First we label the vertices between v_1 and v_{1+j}, $1 \leq j \leq n$ then following the same procedure for $v_2, v_3,...v_n$

$f(u_i) = f(v_n) + i,\ 1 \leq i \leq k$

Now label the vertices of $G^{'}$ using the above defined pattern we have $R_f = \{0, 1, 2, \ldots, p - 1\} \subset N\bigcup\{0\}$

This implies that $\lambda^{'}(G^{'}) = p - 1$.

Illustration 2.8 Consider the graph P_4 and its supersubdivision. The $L^{'}(2,1)$-labeling is as shown in *Fig 8*.

Theorem 2.9 Let $C_n^{'}$ be the graph obtained by taking star of a cycle C_n then $\lambda(C_n^{'}) = 5$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and v_{ij} be the vertices of cycle C_n which are adjacent to the i^{th} vertex of cycle C_n.

To define $f : V(C_n^{'}) \longrightarrow N \bigcup \{0\}$, we consider following four cases.

<u>**Case 1:** $n \equiv 0 (\text{mod } 3)$</u>

$$\begin{gathered} f(v_i) = 0, i = 3j-2, 1 \leq j \leq \tfrac{n}{3} \\ f(v_i) = 2, i = 3j-1, 1 \leq j \leq \tfrac{n}{3} \\ f(v_i) = 4, i = 3j, 1 \leq j \leq \tfrac{n}{3} \end{gathered}$$

Now we label the vertices v_{ij} of star of a cycle according to the label of $f(v_i)$.

<u>**(1)** when $f(v_i) = 0, i = 3j-2, 1 \leq j \leq \frac{n}{3}$</u>

$$\begin{gathered} f(v_{ik}) = 3, k = 3p-2, 1 \leq p \leq \tfrac{n}{3} \\ f(v_{ik}) = 5, k = 3p-1, 1 \leq p \leq \tfrac{n}{3} \\ f(v_{ik}) = 1, k = 3p, 1 \leq p \leq \tfrac{n}{3} \end{gathered}$$

<u>**(2)** when $f(v_i) = 2, i = 3j-1, 1 \leq j \leq \frac{n}{3}$</u>

$$\begin{gathered} f(v_{ik}) = 5, k = 3p-2, 1 \leq p \leq \tfrac{n}{3} \\ f(v_{ik}) = 3, k = 3p-1, 1 \leq p \leq \tfrac{n}{3} \\ f(v_{ik}) = 1, k = 3p, 1 \leq p \leq \tfrac{n}{3} \end{gathered}$$

<u>**(3)** when $f(v_i) = 4, i = 3j, 1 \leq j \leq \frac{n}{3}$</u>

$$\begin{gathered} f(v_{ik}) = 1, k = 3p-2, 1 \leq p \leq \tfrac{n}{3} \\ f(v_{ik}) = 3, k = 3p-1, 1 \leq p \leq \tfrac{n}{3} \\ f(v_{ik}) = 5, k = 3p, 1 \leq p \leq \tfrac{n}{3} \end{gathered}$$

<u>**Case 2:** $n \equiv 1 (\text{mod } 3)$</u>

$$\begin{gathered} f(v_i) = 0, i = 3j-2, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\ f(v_i) = 2, i = 3j-1, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\ f(v_i) = 5, i = 3j, 1 \leq j \leq \lfloor \tfrac{n}{3} \rfloor \\ f(v_n) = 3 \end{gathered}$$

Now we label the vertices of star of a cycle v_{ij} according to label of $f(v_i)$.
<u>**(1)** when $f(v_i) = 0, i = 3j-2, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$</u>

$$\begin{gathered} f(v_{ik}) = 4, k = 3p-2, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor \\ f(v_{ik}) = 2, k = 3p-1, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor \\ f(v_{ik}) = 0, k = 3p, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) = 5, \\ f(v_{in}) = 1 \end{gathered}$$

<u>**(2)** when $f(v_i) = 2, i = 3j-1, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$</u>

$$\begin{gathered} f(v_{ik}) = 4, k = 3p-2, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor \\ f(v_{ik}) = 0, k = 3p-1, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor \\ f(v_{ik}) = 2, k = 3p, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor - 1 \\ f(v_{i(n-1)}) = 3, \\ f(v_{in}) = 1 \end{gathered}$$

<u>**(3)** when $f(v_i) = 5, i = 3j, 1 \leq j \leq \lfloor \frac{n}{3} \rfloor$</u>

$$f(v_{ik}) = 1,\ k = 3p-2,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 3,\ k = 3p-1,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 5,\ k = 3p,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor - 1$$
$$f(v_{i(n-1)}) = 0,$$
$$f(v_{in}) = 4$$

(4) when $f(v_i) = 3,\ i = n$

$$f(v_{ik}) = 1,\ k = 3p-2,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 5,\ k = 3p-1,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 3,\ k = 3p,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor - 1$$
$$f(v_{i(n-1)}) = 0$$
$$f(v_{in}) = 4$$

Case 3: $n \equiv 2 (\mathrm{mod}\ 3),\ n \neq 5$

$$f(v_i) = 1,\ i = 3j-2,\ 1 \le j \le \lfloor \tfrac{n}{3} \rfloor - 1$$
$$f(v_i) = 3,\ i = 3j-1,\ 1 \le j \le \lfloor \tfrac{n}{3} \rfloor - 1$$
$$f(v_i) = 5,\ i = 3j,\ 1 \le j \le \lfloor \tfrac{n}{3} \rfloor - 1$$
$$f(v_{n-4}) = 0,\ f(v_{n-3}) = 2,\ f(v_{n-2}) = 5,\ f(v_{n-1}) = 0,\ f(v_n) = 4$$

Now we label the vertices v_{ij} of star of a cycle according to the label of $f(v_i)$.

(1) when $f(v_i) = 1,\ i = 3j-2,\ 1 \le j \le \lfloor \tfrac{n}{3} \rfloor - 1$

$$f(v_{ik}) = 5,\ k = 3p-2,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 3,\ k = 3p-1,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 1,\ k = 3p,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{i(n-1)}) = 4,$$
$$f(v_{in}) = 0$$

(2) when $f(v_i) = 3,\ i = 3j-1,\ 1 \le j \le \lfloor \tfrac{n}{3} \rfloor - 1$

$$f(v_{ik}) = 0,\ k = 3p-2,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 2,\ k = 3p-1,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 4,\ k = 3p,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{i(n-1)}) = 1,$$
$$f(v_{in}) = 5$$

(3) when $f(v_i) = 5,\ i = 3j,\ 1 \le j \le \lfloor \tfrac{n}{3} \rfloor - 1$ and $i = n-2$

$$f(v_{ik}) = 1,\ k = 3p-2,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 3,\ k = 3p-1,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 5,\ k = 3p,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{i(n-1)}) = 2,$$
$$f(v_{in}) = 4$$

(4) when $f(v_i) = 0,\ i = n-4, n-1$

$$f(v_{ik}) = 3,\ k = 3p-2,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 5,\ k = 3p-1,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 1,\ k = 3p,\ 1 \le p \le \lfloor \tfrac{n}{3} \rfloor - 1$$
$$f(v_{i(n-2)}) = 2,$$
$$f(v_{i(n-1)}) = 4,$$
$$f(v_{in}) = 1$$

(5) when $f(v_i) = 2,\ i = n-3$

$$f(v_{ik}) = 4, k = 3p-2, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 0, k = 3p-1, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 2, k = 3p, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{i(n-1)}) = 5,$$
$$f(v_{in}) = 1$$

(6) when $f(v_i) = 4, i = n$

$$f(v_{ik}) = 2, k = 3p-2, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 0, k = 3p-1, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{ik}) = 4, k = 3p, 1 \leq p \leq \lfloor \tfrac{n}{3} \rfloor$$
$$f(v_{i(n-1)}) = 1,$$
$$f(v_{in}) = 5$$

Case 4: $n = 5$

This case is to be dealt separately. The $L(2, 1)$-labeling for the graph obtained by taking star of the cycle C_5 is shown in *Fig 9*. Thus in all the possibilities we have $\lambda(C_n^{'}) = 5$

Illustration 2.10 Consider the graph C_7, the $L(2, 1)$-labeling is as shown in *Fig 10*.

Theorem 2.11 Let $G^{'}$ be the graph obtained by taking star of a graph G then $\lambda^{'}(G^{'}) = p - 1$, where p be the total number of vertices of $G^{'}$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of any connected graph G. Let v_{ij} be the vertices of a graph which is adjacent to the i^{th} vertex of graph G. By the definition of a star of a graph the total number of vertices in a graph $G^{'}$ are $n(n + 1)$.

To define $f : V(G^{'}) \longrightarrow N \bigcup \{0\}$

$f(v_{i1}) = i - 1, \qquad 1 \leq i \leq n$

for $1 \leq i \leq n$ do the labeling as follows:

$$f(v_i) = f(v_{ni}) + 1$$
$$f(v_{1(i+1)}) = f(v_i) + 1$$
$$f(v_{(j+1)(i+1)}) = f(v_{j(i+1)}) + 1, 1 \leq j \leq n-1$$

Thus $\lambda^{'}(G^{'}) = p - 1 = n^2 + n - 1$

Illustration 2.12 Consider the star of a graph K_4, the $L^{'}(2, 1)$-labeling is shown in *Fig 11*.

3 Concluding Remarks

Here we investigate some new results corresponding to $L(2, 1)$-labeling and $L^{'}(2, 1)$-labeling. The λ-number is completely determined for the graphs obtained by duplicating the vertices altogether in a cycle, arbitrary supersubdivision of a cycle and star of a cycle. We also determine $\lambda^{'}$-number for some graph families. This work is an effort to relate some graph operations and $L(2, 1)$-labeling. All the results reported here are of very general nature and λ-number as well as $\lambda^{'}$-number are completely determined for the larger graphs resulted from the graph operations on standard graphs which is the salient features of this work. It is also possible to investigate some more results corresponding to other graph families.

Acknowledgement

The authors are thankful to the anonymous referee for useful suggestions and comments.

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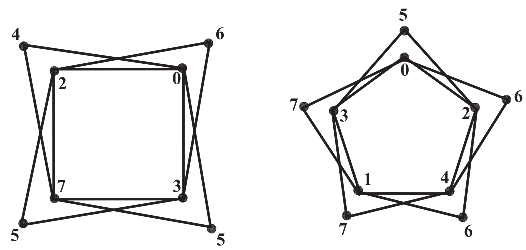

Figure 1. vertex duplication in C_4,C_5 and $L(2,1)$-labeling

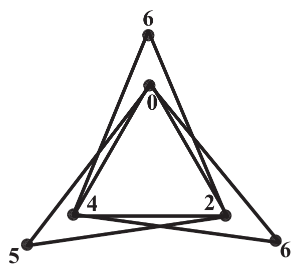

Figure 2. vertex duplication in C_3 and $L(2,1)$-labeling

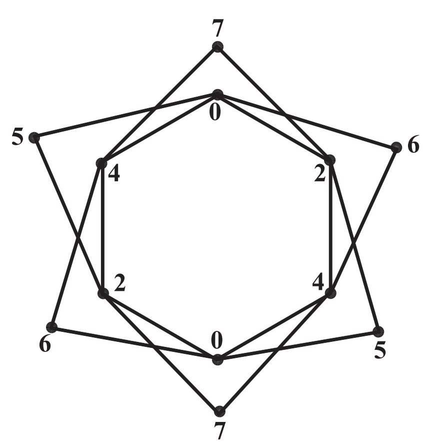

Figure 3. vertex duplication in C_6 and $L(2,1)$-labeling

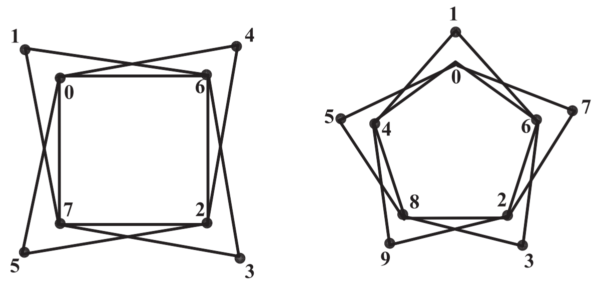

Figure 4. vertex duplication in C_4,C_5 and $L^{'}(2,1)$-labeling

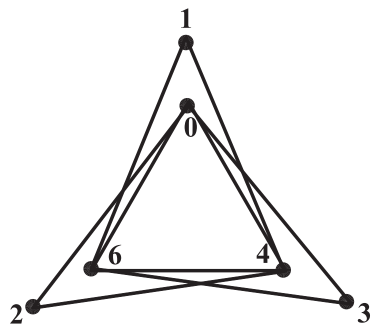

Figure 5. vertex duplication in C_3 and $L^{'}(2,1)$-labeling

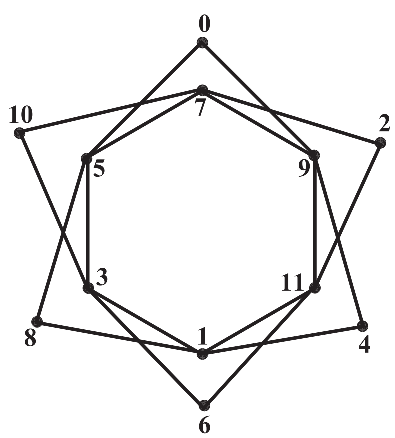

Figure 6. vertex duplication in C_6 and $L^{'}(2,1)$-labeling

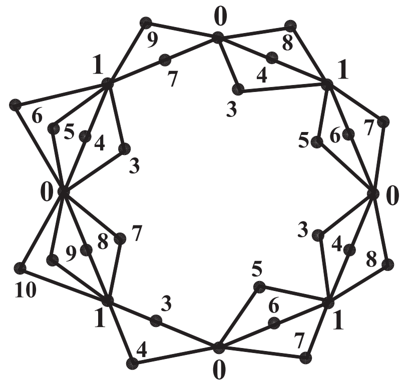

Figure 7. $L(2,1)$-labeling of C_8'

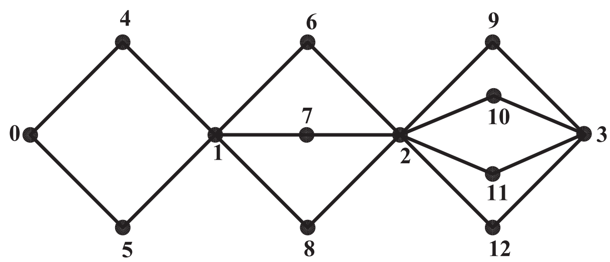

Figure 8. $L'(2,1)$-labeling of P_4'

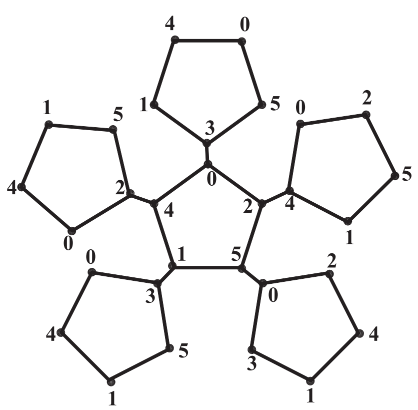

Figure 9. $L(2,1)$-labeling for star of cycle C_5

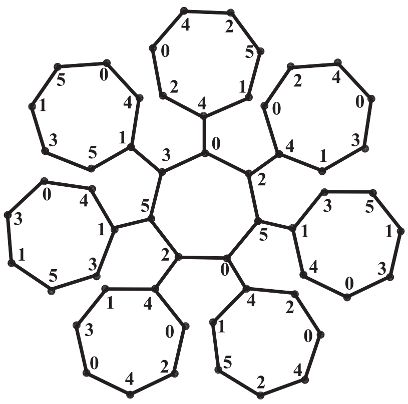

Figure 10. $L(2,1)$-labeling for star of cycle C_7

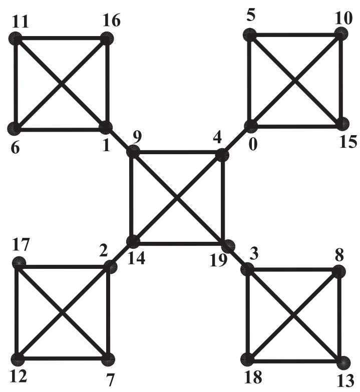

Figure 11. $L^{'}(2,1)$-labeling for star of a complete graph K_4