A Generalized Mixed Variational Inclusion Involving $(H(\cdot,\cdot),\eta)$-Monotone Operators in Banach Spaces

Zhenhui Xu

Applied Technology College, Southewst University of Scinence and Technology

Mianyang 621010, China

Tel: 86-816-633-2137 E-mail: xuzhenhui@swust.edu.cn

Zhongbao Wang

Mathematical College, Sichuan University

Chengdu 610064, China

Abstract

In this paper, we introduce a new class of monotone operators $-$ $(H(\cdot,\cdot),\eta)$- monotone operators, which generalize many existing monotone operators. The resolvent operator associated with an $(H(\cdot,\cdot),\eta)$- monotone operator is defined and its Lipschitz continuity is presented. As an application, we also consider a new generalized mixed variational inclusion involving $(H(\cdot,\cdot),\eta)$-monotone operators and construct a new algorithm for solving the generalized mixed variational inclusion. Under some suitable conditions, we prove the convergence of the iterative sequences generated by the algorithm. These results improve and generalize many corresponding results in recently literatures.

Keywords: $(H(\cdot,\cdot),\eta)$- monotone operator, Resolvent operator, Variational inclusion, Iterative algorithm, Banach spaces

1. Introduction

The resolvent operator method is an important and useful tool to study approximation solvability of nonlinear variational inequalities and variational inclusions, which are providing mathematical models to some problems arising in optimization and control, economics and engineering science. In order to study various variational inequalities and variational inclusions, Ding(2000), Huang and Fang (2003), Fang and Huang (2003), Fang et al.(2005), Verma (2006), Zhang (2007), Sun et al.(2008), Xia and Huang (2007), Feng and Ding (2009) and He et al.(2008) have introduced the concepts of η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, A-monotone operators, (A,η)-monotone operators, G-η-monotone operators, M-monotone operators in Hilbert spaces, H-monotone operators, A-monotone operators and H-η-monotone operators in Banach spaces and their resolvent operators, respectively. Further, by using the resolvent operator technique, a number of nonlinear variational inclusions and many systems of variational inequalities and variational inclusions have been studied by some authors in recent years (for example Lan (2007), Ding and Feng (2008), Peng and Zhu (2007), Zeng (2007), Ding and Wang (2009)).

Motivated and inspired by the above works, we introduce a new class of monotone operators: $(H(\cdot,\cdot),\eta)$- monotone operators, which provide a unifying framework for maximal monotone operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, A-monotone mappings, (A,η)-monotone operators, G-η-monotone operators, M-monotone operators, H-monotone operators, A-monotone operators and H-η-monotone operators. The resolvent operator associated with an $(H(\cdot,\cdot),\eta)$- monotone operator is defined and its Lipschitz continuity is presented. We also consider a new generalized mixed variational inclusion involving $(H(\cdot,\cdot),\eta)$-monotone operators and construct a new algorithm for solving the generalized mixed variational inclusion. Under some suitable conditions, we prove the convergence of iterative sequences generated by the algorithm. These results improve and generalize many known corresponding results.

2. Preliminaries

Let E be a real Banach space with dual space E^*, and the norm and the dual pair between E and E^* are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. $CB(E)$ denotes the family of all bounded closed subsets of E. The Hausdorff metric on $CB(E)$ is defined by

$$\widetilde{H}(A,B)=\max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\sup_{y\in B}\inf_{x\in A}\|x-y\|\},\ \forall\ A,B\in CB(E).$$

The normalized duality mapping $J:E\to 2^{E^*}$ on E is defined by

$$J(x)=\{f^*\in E^*\colon \langle x,f^*\rangle=\|f^*\|\|x\|,\|f^*\|=\|x\|\},\quad \forall\ \ x\in E.$$

If $E=\mathbb{H}$ is a Hilbert space, then J is the identity mapping on $\mathbb{H}$.

Lemma 2.1 (Petryshyn(1970)) Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then for all $x, y \in E$,

$$\| x + y \|^2 \leq \| x \|^2 + 2\langle y, j(x + y)\rangle, \quad \forall \ j(x + y) \in J(x + y).$$

Definition 2.1 Let $A, B : E \to E$, $T : E \to E^*$, $H : E \times E \to E^*$ and $\eta : E \times E \to E$ be five single-valued mappings.

(1) T is said to be η-monotone if $\langle T(x) - T(y), \eta(x, y)\rangle \geq 0$;

(2) T is said to be strictly η-monotone if T is η-monotone and

$$\langle T(x) - T(y), \eta(x, y))\rangle = 0$$

if and only if $x = y$;

(3) $H(A, \cdot)$ is said to be α-strongly η-monotone with respect to A if there exists a constant $\alpha > 0$ such that

$$\langle H(Ax, u) - H(Ay, u), \eta(x, y)\rangle \geq \alpha \| x - y \|^2, \qquad \forall \ x, y, u \in E;$$

(4) $H(\cdot, B)$ is said to be β-relaxed η-monotone with respect to B if there exists a constant $\beta > 0$ such that

$$\langle H(u, Bx) - H(u, By), \eta(x, y))\rangle \geq -\beta \| x - y \|^2, \qquad \forall \ x, y, u \in E;$$

(5) $H(\cdot, \cdot)$ is said to be λ-Lipschitz continuous with respect to A if there exists a constant $\lambda > 0$ such that

$$\| H(Ax, u) - H(Ay, u) \| \leq \lambda \| x - y \|, \qquad \forall \ x, y, u \in E;$$

(6) T is said to be ϵ-Lipschitz continuous if there exists a constant $\epsilon > 0$ such that

$$\| T(x) - T(y) \| \leq \epsilon \| x - y \|, \quad \forall \ x, y \in E;$$

(7) η is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\| \eta(x, y) \| \leq \tau \| x - y \|, \quad \forall \ x, y \in E.$$

Remark 2.1 If $E = \mathcal{H}$ is an Hilbert space, $\eta(x, y) = x - y, \quad \forall \ x, y \in E$, then (3) and (4) of Definition 2.1 reduce to (i) and (ii) of Definition 1.2, respectively(Sun et al.(2008)).

Definition 2.2 (Lou et al.(2008)) Let $M : E \to 2^{E^*}$ be a multi-valued mapping, $H : E \to E^*$ and $\eta : E \times E \to E$ be single-valued mappings. M is said to be

(1) monotone if $\langle x - y, u - v\rangle \geq 0, \ \forall \ u, v \in E, x \in M(u), y \in M(v)$;

(2) η-monotone if $\langle x - y, \eta(u, v)\rangle \geq 0, \ \forall \ u, v \in E, x \in M(u), y \in M(v)$;

(3) strictly η-monotone if M is η-monotone and equality holds if and only if $x = y$;

(4) r-strongly η-monotone if there exists a constant $r > 0$ such that

$$\langle x - y, \eta(u, v)\rangle \geq r \| u - v \|^2, \ \forall \ u, v \in E, \ x \in M(u) \ , y \in M(v);$$

(5) m-relaxed η-monotone if there exists a constant $m > 0$ such that

$$\langle x - y, \eta(u, v)\rangle \geq -m \| u - v \|^2, \ \forall \ u, v \in E, x \in M(u), y \in M(v);$$

(6) maximal monotone, if M is monotone and has no a proper monotone extension in E, i.e., for all $u, v_0 \in E, \ x \in M(u)$, $\langle x - y_0, u - v_0\rangle \geq 0$ implies $y_0 \in M(v_0)$;
when E is a reflexive Banach space, M is maximal monotone if and only if M is monotone and $(J + \lambda M)E = E^*$ for all $\lambda > 0$;

(7) maximal η-monotone, if M is η-monotone and has no a proper η-monotone extension in E, when E is a reflexive Banach space, M is maximal η-monotone if and only if M is η-monotone and $(J + \lambda M)E = E^*$ for all $\lambda > 0$;

(8) H-monotone, if M is monotone and $(H + \lambda M)E = E^*$ for all $\lambda > 0$;

(9) (H, η)-monotone, if M is η-monotone and $(H + \lambda M)E = E^*$ for all $\lambda > 0$;

(10) H-η-monotone, if M is m-η-relaxed monotone and $(H + \lambda M)E = E^*$ for all $\lambda > 0$.

Definition 2.3 Let $T : E \to E$ be a single-valued mapping. T is said to be δ-strongly accretive, if there exists a constant $\delta > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle T(x) - T(y), j(x - y)\rangle \geq \delta \parallel x - y \parallel^2, \quad \forall\, x, y \in E.$$

3. $(H(\cdot,\cdot),\ \eta)$**-monotone operators**

In this section, we shall introduce a new class of monotone operators $-$ $(H(\cdot,\cdot),\ \eta)$-monotone operators and discuss some properties of this class of operators

Definition 3.1 Let $H : E \times E \to E^*$, $\eta : E \times E \to E$, $A, B : E \to E$ be four single-valued mappings. Then the set-valued mapping $M : E \to 2^{E^*}$ is said to be $(H(\cdot,\cdot),\ \eta)$-monotone with respect to mappings A and B (or simply $(H(\cdot,\cdot),\ \eta)$-monotone in the sequel), if M is m-relaxed η-monotone and $(H(A, B) + \rho M)(E) = E^*$ for all $\rho > 0$.

Remark 3.1 (1) If $H(Au, Bu) = Au$, $\forall\, u \in E$, then Definition 3.1 reduces to the definition of $H - \eta$-monotone operators (Lou et al.(2008)). Hence, the class of $(H(\cdot,\cdot),\ \eta)$-monotone operators in Banach spaces provides a unifying framework for the classes of η-subdifferential operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators, (H, η)-monotone operators, G-η-monotone operators, A-monotone operators and (A, η)-monotone operators in Hilbert spaces and H-η-monotone operators, H-monotone operators, A-monotone operators in Banach spaces. We emphasize that an $(H(\cdot,\cdot), \eta)$-monotone operator in Banach spaces maps from E to E^*.

(2) If $E = \mathcal{H}$ is a Hilbert space, $m = 0$ and $\eta(x, y) = x - y$, $\forall\, x, y \in \mathcal{H}$, then Definition 3.1 reduces to the definition of M-monotone operators (Sun et al. (2008)).

Theorem 3.1 Let $A, B : E \to E$, $\eta : E \times E \to E$ and $H : E \times E \to E^*$ be single-valued mappings and $H(A, B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : E \to 2^{E^*}$ be an $(H(\cdot,\cdot), \eta)$-monotone operator with respect to A and B. If $\langle x - y, \eta(u, v)\rangle \geq 0$ holds for all $(v, y) \in$ Graph(M), where Graph(M) $= \{(a, b) \in E \times E : b \in M(a)\}$, then $(u, x) \in$ Graph(M).

Proof Since M is $(H(\cdot,\cdot), \eta)$-monotone with respect to A and B, we know that $(H(A, B) + \rho M)(E) = E^*$ holds for all $\rho > 0$ and so there exists $(u_1, x_1) \in$ Graph(M) such that

$$H(Au, Bu) + \rho x = H(Au_1, Bu_1) + \rho x_1.$$

Since $H(A, B)$ is α-strongly η-monotone with respect to A, β-relaxed monotone with respect to B and $\alpha > \beta$, we have

$$\begin{aligned} 0 &\leq \rho\langle x - x_1, \eta(u, u_1)\rangle \\ &= -\langle H(Au, Bu) - H(Au_1, Bu_1), \eta(u, u_1)\rangle \\ &= -\langle H(Au, Bu) - H(Au_1, Bu), \eta(u, u_1)\rangle - \langle H(Au_1, Bu) - H(Au_1, Bu_1), \eta(u, u_1)\rangle \\ &\leq -(\alpha - \beta) \parallel u - u_1 \parallel^2 \leq 0. \end{aligned}$$

This implies that $u = u_1$ and $x = x_1$. Thus $(u, x) = (u_1, x_1) \in$ Graph(M). This completes the proof. □

Theorem 3.2 Let $A, B : E \to E$, $\eta : E \times E \to E$ and $H : E \times E \to E^*$ be single-valued mappings and $H(A, B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : E \to 2^{E^*}$ be an $(H(\cdot,\cdot), \eta)$-monotone operator with respect to A and B. Then the operator $(H(A, B)+\rho M)^{-1}$ is single-valued for $0 < \rho < \frac{r}{m}$, where $r = \alpha - \beta$.

Proof. For any given $u^* \in E$, let $\forall\, u, v \in (H(A, B) + \rho M)^{-1}(u^*)$. It follows that

$$-H(Au, Bu) + u^* \in \rho M(u) \;\; and \;\; - H(Av, Bv) + u^* \in \rho M(v).$$

Since $M : E \to 2^{E^*}$ is an $(H(\cdot,\cdot), \eta)$-monotone operator with respect to A and B and $H(A, B)$ is α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$, we have

$$\begin{aligned}
-m \parallel u-v \parallel^2 &\le \frac{1}{\rho}\langle(-H(Au,Bu)+u^*)-(-H(Av,Bv)+u^*),\eta(u,v)\rangle \\
&= -\frac{1}{\rho}\langle H(Au,Bu)-H(Av,Bv),\eta(u,v)\rangle \\
&= -\frac{1}{\rho}\langle H(Au,Bu)-H(Av,Bu),\eta(u,v)\rangle \\
&\quad -\frac{1}{\rho}\langle H(Av,Bu)-H(Av,Bv),\eta(u,v)\rangle \\
&\le -\frac{1}{\rho}(\alpha-\beta) \parallel u-v \parallel^2 = -\frac{r}{\rho} \parallel u-v \parallel^2 .
\end{aligned}$$

This show that

$$m\rho \parallel u-v \parallel^2 \ge r \parallel u-v \parallel^2 .$$

If $u \neq v$, then $\rho \ge \frac{r}{m}$ contradicts with $0 < \rho < \frac{r}{m}$. Thus $u = v$, that is, $(H(A,B)+\rho m)^{-1}$ is singe-valued. The proof is completed. □

Remark 3.2 Theorem 3.1 and Theorem 3.2 improve the similar conclusions (see Sun et al.(2008), Huang and Fang (2003), Zhang (2007), Feng and Ding (2009)).

Base on Theorem 3.2, we can define the generalized resolvent operator $R^{M,\eta}_{H(\cdot,\cdot),\rho}$ associated with an $(H(\cdot,\cdot),\eta)$-monotone mapping M as follows.

Definition 3.2 Let $A, B : E \to E$, $\eta : E \times E \to E$ and $H : E \times E \to E^*$ be single-valued mappings and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : E \to 2^{E^*}$ be an $(H(\cdot,\cdot),\eta)$-monotone operator with respect to A and B. Then the general resolvent operator $R^{M,\eta}_{H(\cdot,\cdot),\rho} : E \to E$ is defined by

$$R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) = (H(A,B)+\rho M)^{-1}(u), \quad \forall\, u \in E.$$

Remark 3.3 The general resolvent operators associated with $(H(\cdot,\cdot),\eta)$-monotone operators include as special cases the corresponding resolvent operators associated with maximal monotone operators (Zeidler (1985)), η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, A-monotone mappings, (A,η)-monotone operators, G-η-monotone operators, M-monotone operators, H-monotone operators, A-monotone operators and H-η-monotone operators, respectively.

Theorem 3.3 Let $A, B : E \to E$, $H : E \times E \to E^*$ be single-valued mappings, $\eta : E \times E \to E$ be τ-Lipschitz continuous and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : E \to 2^{E^*}$ be an $(H(\cdot,\cdot),\eta)$-monotone operator with respect to A and B. Then the resolvent operator $R^{M,\eta}_{H(\cdot,\cdot),\rho} : E \to E$ is $\frac{\tau}{r-\rho m}$–Lipschitz continuous for $0 < \rho < \frac{r}{m}$, where $r = \alpha - \beta$, that is,

$$\parallel R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) - R^{M,\eta}_{H(\cdot,\cdot),\rho}(v) \parallel \le \frac{\tau}{r-\rho m} \parallel u-v \parallel, \forall\, u,v \in E.$$

Proof Let $u, v \in H$ be any given points, it follows from (3.2) that

$$R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) = (H(A,B)+\rho M)^{-1}(u)$$

and

$$R^{M,\eta}_{H(\cdot,\cdot),\rho}(v) = (H(A,B)+\rho M)^{-1}(v).$$

This implies that

$$\frac{1}{\rho}(u - H(A(R^{M,\eta}_{H(\cdot,\cdot),\rho}(u)), B(R^{M,\eta}_{H(\cdot,\cdot),\rho}(u))) \in M(R^{M,\eta}_{H(\cdot,\cdot),\rho}(u)),$$

$$\frac{1}{\rho}(v - H(A(R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)), B(R^{M,\eta}_{H(\cdot,\cdot),\rho}(v))) \in M(R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)).$$

For the sake of brevity, let $z_1 = R^{M,\eta}_{H(\cdot,\cdot),\rho}(u)$ and $z_2 = R^{M,\eta}_{H(\cdot,\cdot),\rho}(v)$.
Since M is m-relaxed η-monotone, we get

$$
\begin{aligned}
-m \| z_1 - z_2 \|^2 &\leq \frac{1}{\rho}\langle u - H(Az_1, Bz_1) - (v - H(Az_2, Bz_2)), \eta(z_1, z_2)\rangle \\
&= \frac{1}{\rho}\langle u - v - (H(Az_1, Bz_1) - H(Az_2, Bz_2)), \eta(z_1, z_2)\rangle.
\end{aligned}
$$

From the above inequality and the conditions in the Theorem 3.3, we have

$$
\begin{aligned}
\tau \| u - v \| \cdot \| z_1 - z_2 \| &\geq \| u - v \| \cdot \| \eta(z_1, z_2) \| \geq \langle u - v, \eta(z_1, z_2)\rangle \\
&\geq \langle H(Az_1, Bz_1) - H(Az_2, Bz_2), \eta(z_1, z_2)\rangle - \rho m \| z_1 - z_2 \|^2 \\
&\geq \langle H(Az_1, Bz_1) - H(Az_2, Bz_1), \eta(z_1, z_2)\rangle \\
&\quad + \langle H(Az_2, Bz_1) - H(Az_2, Bz_2), \eta(z_1, z_2)\rangle - \rho m \| z_1 - z_2 \|^2 \\
&\geq (\alpha - \beta - \rho m) \| z_1 - z_2 \|^2 = (r - \rho m) \| z_1 - z_2 \|^2 .
\end{aligned}
$$

Hence

$$\| R^{M,\eta}_{H(\cdot,\cdot),\rho}(u) - R^{M,\eta}_{H(\cdot,\cdot),\rho}(v) \| \leq \frac{\tau}{r - \rho m} \| u - v \|, \forall u, v \in E.$$

This completes the proof. □

4. An application for solving a generalized mixed variational inclusion

In this section, we shall study a new generalized mixed variational inclusion involving $(H(\cdot,\cdot),\eta)$-monotone operators in Banach spaces and construct an iterative algorithm for approximating the solution of this variational inclusion by using the resolvent operator technique.

Throughout the rest of the paper, unless otherwise stated, let E be a real Banach space with the dual space E^* and the norm $\| \cdot \|$, $\langle \cdot, \cdot \rangle$ be the dual pair between E and E^*, $CB(E)$ be the family of all bounded closed subsets of E, $J : E \to 2^{E^*}$ be the normalized duality mapping on E defined by

$$J(x) = \{f^* \in E^* \colon \langle x, f^*\rangle = \| f^* \| \| x \|, \| f^* \| = \| x \|\}, \quad \forall\, x \in E$$

and $J^* : E^* \to E^{**}$ be the normalized duality mapping on E^* defined by

$$J^*(y) = \{f \in E^{**} : \langle f, y\rangle = \| y \| \| f \|, \ \| f \| = \| y \|\}, \ \forall y \in E^*,$$

where E^{**} is a dual space of E^*.

We observe that $E = \mathbb{H}$ is a Hilbert space, then J and J^* are the identity mappings on $\mathbb{H}$. In the sequel, j and j^* denote a selection of J and J^*, respectively.

Let $G : E \to CB(E)$, $S : E \to CB(E)$ and $T : E \to CB(E)$ be set-valued mappings, and let $N : E \times E \to E^*$, $p, g \colon E \to E$, $A, B : E \to E$, $H : E \times E \to E^*$, $\eta : E \times E \to E$ and $F : E \times E \times E \to E^*$ be single-valued mappings. Let $M : E \times E \to 2^{E^*}$ be a set-valued mapping such that for each fixed $z \in G(E)$, $M(\cdot, z) : E \to 2^{E^*}$ is an $(H(\cdot,\cdot),\eta)$-monotone operator with respect to A and B and $(g - p)(E) \bigcap dom(M(\cdot, z)) \neq \emptyset$. We consider the following generalized mixed variational inclusion: for given $\omega \in E^*$ find $u \in E$, $x \in S(u)$, $y \in T(u)$ and $z \in G(u)$ such that

$$\omega \in F(u, u, z) + N(x, y) + M((g - p)(u), z) \tag{1}$$

Special cases

(I) If $F = p = \omega = 0$ and $E = \mathbb{H}$ is a Hilbert space, then the problem (1) reduces to a generalized mixed quasi-variational-like inclusion with $(H(\cdot,\cdot),\eta)$-monotone operators in a Hilbert space: find $u \in E$, $x \in S(u)$, $y \in T(u)$ and $z \in G(u)$ such that

$$0 \in N(x, y) + M(g(u), z). \tag{2}$$

If M is H-monotone in the first argument, then the problem (2) was introduced and studied by Zeng (2007).

From Definition 3.2, we can obtain the following conclusion.

Lemma 4.1 Let E, ω, F, N, T, S, p, g, H, η, G and M be same as in the problem (1). Then (u, x, y, z) is a solution of the problem (1) if and only if $(u, x, y, z,)$ satisfies the following relation

$$(g-p)(u) = R^{H(\cdot,\cdot),\eta}_{M(\cdot,z),\rho}(H(A((g-p)(u)), B((g-p)(u))) + \rho\omega - \rho N(x, y) - \rho F(u, u, z)). \tag{3}$$

where $u \in E$, $x \in S(u)$, $y \in T(v)$, $z \in G(u)$ $R^{H(\cdot,\cdot),\eta}_{M(\cdot,z),\rho}$=$(H(A, B) + \rho M(\cdot, z))^{-1}$ and $\rho > 0$ are constants.
Remark 4.1 The equality (3) can be written as

$$z = H(A((g-p)(u)), B((g-p)(u))) + \rho\omega - \rho N(x, y) - \rho F(u, u, z), \quad (g-p)(u) = R^{M,\eta}_{H(\cdot,\cdot),\rho}(z),$$

where $\omega \in E^*$ is any given element and $\rho > 0$ is a constant. By Nadler (1969), we know that this formulation enables us to suggest the following iterative algorithm.

Algorithm 4.1 Step 1. For given $\omega \in E^*$ and $\rho > 0$, choose $u_0 \in E$, $x \in S(u_0)$, $y \in T(v_0)$ and $z \in G(u_0)$.

Step 2. Let

$$(g-p)(u_{n+1}) = R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_n),\rho}(H(A((g-p)(u_n)), B((g-p)(u_n))) + \rho\omega - \rho N(x_n, y_n) - \rho F(u_n, u_n, z_n)) + e_n. \tag{4}$$

Step 3. Choose $x_{n+1} \in S(u_{n+1})$, $y_{n+1} \in T(u_{n+1})$ and $z_{n+1} \in G(u_{n+1})$ such that

$$\begin{cases} \| x_{n+1} - x_n \| \le (1 + \frac{1}{n+1})\widetilde{H}(S(u_{n+1}), S(u_n)), \\ \| y_{n+1} - y_n \| \le (1 + \frac{1}{n+1})\widetilde{H}(T(u_{n+1}), T(u_n)), \\ \| z_{n+1} - z_n \| \le (1 + \frac{1}{n+1})\widetilde{H}(G(u_{n+1}), G(u_n)). \end{cases} \tag{5}$$

Step 4. Choose errors $\{e_n\} \subset E$ to take into account a possible inexact computation such that

$$\sum_{j=1}^{\infty} \| e_j - e_{j-1} \| \varpi^{-j} < \infty, \quad \forall \varpi \in (0, 1). \lim_{n\to\infty} e_n = 0. \tag{6}$$

Step 5. If $x_{n+1} \in S(u_{n+1})$, $y_{n+1} \in T(v_{n+1})$ and $z_{n+1} \in G(v_{n+1})$ satisfy (5) to sufficient accuracy, stop; otherwise, set $n := n+1$ and return to Step 2.

Definition 4.1 A set-valued mapping $A : E \to CB(E)$ is said to be $\widetilde{H}$-Lipschitz continuous if there exists a constant $\mathfrak{L} > 0$ such that

$$\widetilde{H}(A(x), A(y)) \le \mathfrak{L} \| x - y \|, \quad \forall\, x, y \in E.$$

Definition 4.2 Let $F : E \times E \times E \to E^*$ be a single-valued mapping. F is said to be

(1) ξ_1-Lipschitz continuous in the first argument if there exists some constant $\xi_1 > 0$ such that

$$\| F(u_1, \cdot, \cdot) - F(v_1, \cdot, \cdot) \| \le \xi_1 \| u_1 - v_1 \|, \quad \forall\, u_1,\ v_1 \in E;$$

(2) ξ_2-Lipschitz continuous in the second argument if there exists some constant $\xi_2 > 0$ such that

$$\| F(\cdot, u_2, \cdot) - F(\cdot, v_2, \cdot) \| \le \xi_2 \| u_2 - v_2 \|, \quad \forall\, u_2, v_2 \in E;$$

(3) ξ_3-Lipschitz continuous in the third argument if there exists some constant $\xi_3 > 0$ such that

$$\| F(\cdot, \cdot, u_3) - F(\cdot, \cdot, v_3) \| \le \xi_3 \| u_3 - v_3 \|, \quad \forall\, u_3, v_3 \in E.$$

Definition 4.3 Let $N : E \times E_2 \to E^*$ be a single-valued mapping. N is said to be

(1) $\bar{\alpha}$-Lipschitz in the first argument if there exists some constant $\bar{\alpha} > 0$ such that

$$\| N(u_1, v') - N(u_2, v') \| \le \bar{\alpha} \| u_1 - u_2 \|, \quad \forall\, u_1, u_2 \in E,\ v' \in E;$$

(2) $\bar{\beta}$-Lipschitz in the second argument if there exists some constant $\bar{\beta} > 0$ such that

$$\| N(u', v_1) - N(u', v_2) \| \le \bar{\beta} \| v_1 - v_2 \|, \quad \forall\, v_1, v_2 \in E,\ u' \in E.$$

Theorem 4.1 Let $G : E \to CB(E)$, $S : E \to CB(E)$ and $T : E \to CB(E)$ be set-valued mappings, and let $N : E \times E \to E^*$, A, B, p, $g : E \to E$, $H : E \times E \to E^*$, $\eta : E \times E \to E$, and $F : E \times E \times E \to E^*$ be single-valued mappings. Let

$M : E \times E \to 2^{E^*}$ be a set-valued mapping such that for each fixed $z \in G(E)$, $M(\cdot, z) : E \to 2^{E^*}$ is an $(H(\cdot,\cdot),\eta)$-monotone mapping with respect to A and B, and $(g-p)(E) \bigcap dom(M(\cdot,z)) \neq \emptyset$. Furthermore, suppose the following conditions are satisfied:

(i) $H(A,B)$ is α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$, and ϵ_1-Lipschitz continuous with respect to A and ϵ_2-Lipschitz continuous with respect to B;

(ii) S, T and G are $\widetilde{H}$-Lipschitz continuous with constants l_1, l_2 and l_3, respectively;

(iii) $\eta : E \times E \to E$ is τ-Lipschitz continuous and N is $\bar{\alpha}$-Lipschitz continuous in the first argument and $\bar{\beta}$-Lipschitz continuous in the second argument;

(iv) $g-p$ is s-Lipschitz continuous, and $g-p-I$ is ς-strongly accretive, where I denotes the identity mapping on E;

(v) F is ξ_j-Lipschitz continuous in the j-th argument for $j = 1,2,3$;
In addition, if there are constants $\mu > 0$ such that

$$\| R^{H(\cdot,\cdot),\eta}_{M(\cdot,z),\rho}(u) - R^{H(\cdot,\cdot),\eta}_{M(\cdot,\bar{z}),\rho}(u) \| \leq \mu \| z - \bar{z} \|, \qquad \forall\ (z,\bar{z}) \in E \times E, u \in E^*, \tag{7}$$

and there exist constants $0 < \rho < \frac{r}{m}$, where $r = \alpha - \beta$, such that

$$\tau\sqrt{(\epsilon_1+\epsilon_2)^2 s^2 + 2\rho\bar{\alpha}l_1((\epsilon_1+\epsilon_2)s + \rho\bar{\alpha}l_1)} + \tau\rho(\bar{\beta}l_2 + \xi_1 + \xi_2 + \xi_3 l_3) + \mu l_3 < (r-\rho m)\sqrt{2\varsigma+1} \tag{8}$$

Then the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 4.1 converge strongly to u^*, x^*, y^* and z^*, respectively, and (u^*,x^*,y^*,z^*) is a solution of the problem (1).

Proof Since $g-p-I$ is ς-strongly monotone, by Lemma 2.1, we have the following estimate:

$$\begin{aligned}
&\| u_{n+2} - u_{n+1} \|^2 \\
=&\| (g-p)(u_{n+2}) - (g-p)(u_{n+1}) + u_{n+2} - u_{n+1} \\
&- ((g-p)(u_{n+2}) - (g-p)(u_{n+1})) \|^2 \\
\leq&\| (g-p)(u_{n+2}) - (g-p)(u_{n+1}) \|^2 \\
&- 2\langle (g-p-I)(u_{n+2}) - (g-p-I)(u_{n+1}), j(u_{n+2}-u_{n+1})\rangle \\
\leq&\| (g-p)(u_{n+2}) - (g-p)(u_{n+1}) \|^2 - 2\varsigma \| u_{n+2} - u_{n+1} \|^2,
\end{aligned} \tag{9}$$

which implies that

$$\| u_{n+2} - u_{n+1} \| \leq \frac{1}{\sqrt{2\varsigma+1}} \| (g-p)(u_{n+2}) - (g-p)(u_{n+1}) \| . \tag{10}$$

Now, by using (4), (7) and Theorem 3.3, we have

$$\begin{aligned}
&\| (g-p)(u_{n+2}) - (g-p)(u_{n+1}) \| \\
=&\| R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_{n+1}),\rho}(H(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))) + \rho\omega - \rho N(x_{n+1},y_{n+1}) \\
&- \rho F(u_{n+1},v_{n+1},z_{n+1})) + e_{n+1} - (R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_n),\rho}(H(A((g-p)(u_n)), B((g-p)(u_n))) \\
&+ \rho\omega - \rho N(x_n,y_n) - \rho F(u_n,v_n,z_n)) + e_n) \| \\
\leq&\| R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_{n+1}),\rho}(H(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))) + \rho\omega - \rho N(x_{n+1},y_{n+1}) \\
&- \rho F_1(u_{n+1},v_{n+1},z_{n+1})) - R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_n),\rho}(H(A((g-p)(u_n)), B((g-p)(u_n))) \\
&+ \rho\omega - \rho N(x_n,y_n) - \rho F(u_n,v_n,z_n)) \| + \| e_{n+1} - e_n \| \\
\leq&\| R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_{n+1}),\rho}(H(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))) + \rho\omega - \rho N(x_{n+1},y_{n+1}) \\
&- \rho F(u_{n+1},v_{n+1},z_{n+1})) - R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_{n+1}),\rho}(H(A((g-p)(u_n)), B((g-p)(u_n))) \\
&+ \rho\omega - \rho N(x_n,y_n) - \rho F(u_n,v_n,z_n)) \| \\
&+ \| R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_{n+1}),\rho}(H(A((g-p)(u_n)), B((g-p)(u_n))) + \rho\omega - \rho N(x_n,y_n) - \rho F(u_n,v_n,z_n)) \\
&- R^{H(\cdot,\cdot),\eta}_{M(\cdot,z_n),\rho}(H(A((g-p)(u_n)), B((g-p)(u_n))) + \rho\omega - \rho N(x_n,y_n) \\
&- \rho F(u_n,v_n,z_n)) \| + \| e_{n+1} - e_n \| \\
\leq& \frac{\tau}{r-\rho m}[\| (H(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))) - H(A((g-p)(u_n)), B((g-p)(u_n))) \\
&- \rho(N(x_{n+1},y_{n+1}) - N(x_n,y_{n+1})) \| + \rho \| F(u_{n+1},v_{n+1},z_{n+1}) - F(u_n,v_n,z_n) \| \\
&+ \rho \| N(x_n,y_{n+1}) - N(x_n,y_n) \|] + \| e_{n+1} - e_n \| + \mu \| z_{n+1} - z_n \|
\end{aligned} \tag{11}$$

Since $g-p$ and $H(\cdot,\cdot)$ are Lipschitz continuous, by using Lemma 2.1, we obtain

$$\begin{aligned}
&\| H(A((g-p)(u_{n+1})),B((g-p)(u_{n+1})))-H(A((g-p)(u_n)),B((g-p)(u_n)))\\
&-\rho(N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}))\|^2\\
\le&\| H(A((g-p)(u_{n+1})),B((g-p)(u_{n+1})))-H(A((g-p)(u_n)),B((g-p)(u_n)))\|^2\\
&-2\rho\langle N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}),\\
&j_1^*(H(A((g-p)(u_{n+1})),B((g-p)(u_{n+1})))-H(A((g-p)(u_n)),B((g-p)(u_n)))\\
&-\rho(N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})))\rangle\\
\le&\| H(A((g-p)(u_{n+1})),B((g-p)(u_{n+1})))-H(A((g-p)(u_n)),B((g-p)(u_n)))\|^2\\
&+2\rho\| N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})\|\\
&[\| H(A((g-p)(u_{n+1})),B((g-p)(u_{n+1})))-H(A((g-p)(u_n)),B((g-p)(u_n)))\|\\
&+\rho\| N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})\|]\\
\le&(\epsilon_1+\epsilon_2)^2s^2\| u_{n+1}-u_n\|^2+2\rho\| N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})\|\\
&[(\epsilon_1+\epsilon_2)s\| u_{n+1}-u_n\|+\rho\| N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})\|].
\end{aligned}\tag{12}$$

Now, by Algorithm 4.1, and conditions (ii)-(iii) and (v), we get

$$\begin{aligned}
&\| N(x_{n+1},y_{n+1})-N(x_n,y_{n+1})\|\le\bar{\alpha}\| x_{n+1}-x_n\|\\
&\le\bar{\alpha}(1+\frac{1}{n+1})\widetilde{H}(S(u_{n+1}),S(u_n))\le\bar{\alpha}(1+\frac{1}{n+1})l_1\| u_{n+1}-u_n\|,
\end{aligned}\tag{13}$$

$$\begin{aligned}
&\| N(x_n,y_{n+1})-N(x_n,y_n)\|\le\bar{\beta}\| y_{n+1}-y_n\|\\
&\le\bar{\beta}(1+\frac{1}{n+1})\widetilde{H}(T(u_{n+1}),T(u_n))\le\bar{\beta}(1+\frac{1}{n+1})l_2\| u_{n+1}-u_n\|,
\end{aligned}\tag{14}$$

$$\| z_{n+1}-z_n\|\le(1+\frac{1}{n+1})\widetilde{H}(G(u_{n+1}),G(u_n))\le(1+\frac{1}{n+1})l_3\| u_{n+1}-u_n\|,\tag{15}$$

$$\begin{aligned}
&\| F(u_{n+1},u_{n+1},z_{n+1})-F(u_n,u_n,z_n)\|\\
&\le\xi_1\| u_{n+1}-u_n\|+\xi_2\| u_{n+1}-u_n\|+\xi_3\| z_{n+1}-z_n\|\\
&\le(\xi_1+\xi_2+(1+\frac{1}{n+1})\xi_3l_3)\| u_{n+1}-u_n\|.
\end{aligned}\tag{16}$$

From (12) and (13), we get

$$\begin{aligned}
&\| H(A((g-p)(u_{n+1})),B((g-p)(u_{n+1})))-H(A((g-p)(u_n)),B((g-p)(u_n)))\\
&-\rho(N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}))\|^2\\
\le&[(\epsilon_1+\epsilon_2)^2s^2+2\rho\bar{\alpha}(1+\frac{1}{n+1})l_1((\epsilon_1+\epsilon_2)s+\rho\bar{\alpha}(1+\frac{1}{n+1})l_1)]\| u_{n+1}-u_n\|^2.
\end{aligned}\tag{17}$$

By (11)-(17), we obtain

$$\begin{aligned}
&\|(g-p)(u_{n+2})-(g-p)(u_{n+1})\|\\
\le&\frac{\tau}{r-\rho m}[\|(H(A((g-p)(u_{n+1})),B((g-p)(u_{n+1})))-H(A((g-p)(u_n)),B((g-p)(u_n)))\\
&-\rho(N(x_{n+1},y_{n+1})-N(x_n,y_{n+1}))\|+\rho\| F(u_{n+1},v_{n+1},z_{n+1})-F(u_n,v_n,z_n)\|\\
&+\rho\| N(x_n,y_{n+1})-N(x_n,y_n)\|]+\| e_{n+1}-e_n\|+\mu\| z_{n+1}-z_n\|\\
\le&[\frac{\tau}{r-\rho m}\sqrt{(\epsilon_1+\epsilon_2)^2s^2+2\rho\bar{\alpha}(1+\frac{1}{n+1})l_1((\epsilon_1+\epsilon_2)s+\rho\bar{\alpha}(1+\frac{1}{n+1})l_1)}\\
&+\frac{\tau\rho}{r-\rho m}(\bar{\beta}(1+\frac{1}{n+1})l_2+\xi_1+\xi_2+(1+\frac{1}{n+1})\xi_3l_3)+\mu(1+\frac{1}{n+1})l_3]\| u_{n+1}-u_n\|\\
&+\| e_{n+1}-e_n\|.
\end{aligned}\tag{18}$$

Hence

$$\begin{aligned}
&\| u_{n+2} - u_{n+1} \| \\
&\leq \frac{1}{\sqrt{2\varsigma+1}} \| (g-p)(u_{n+2}) - (g-p)(u_{n+1}) \| \\
&\leq \frac{1}{\sqrt{2\varsigma+1}} [\frac{\tau}{r-\rho m} \sqrt{(\epsilon_1+\epsilon_2)^2 s^2 + 2\rho\bar{\alpha}(1+\frac{1}{n+1})l_1((\epsilon_1+\epsilon_2)s + \rho\bar{\alpha}(1+\frac{1}{n+1})l_1)} \\
&+ \frac{\tau\rho}{r-\rho m}(\bar{\beta}(1+\frac{1}{n+1})l_2 + \xi_1 + \xi_2 + (1+\frac{1}{n+1})\xi_3 l_3) + \mu(1+\frac{1}{n+1})l_3] \| u_{n+1} - u_n \| \\
&+ \frac{1}{\sqrt{2\varsigma+1}} \| e_{n+1} - e_n \| \leq \Lambda_{n+1} \| u_{n+1} - u_n \| + \frac{1}{\sqrt{2\varsigma+1}} \| e_{n+1} - e_n \|,
\end{aligned} \tag{19}$$

where $\Lambda_{n+1} = \frac{1}{\sqrt{2\varsigma+1}}[\frac{\tau}{r-\rho m} \sqrt{(\epsilon_1+\epsilon_2)^2 s^2 + 2\rho\bar{\alpha}(1+\frac{1}{n+1})l_1((\epsilon_1+\epsilon_2)s + \rho\bar{\alpha}(1+\frac{1}{n+1})l_1)}$
$+ \frac{\tau\rho}{r-\rho m}(\bar{\beta}(1+\frac{1}{n+1})l_2 + \xi_1 + \xi_2 + (1+\frac{1}{n+1})\xi_3 l_3) + \mu(1+\frac{1}{n+1})l_3]$.

Let
$\Lambda = \frac{1}{\sqrt{2\varsigma+1}}[\frac{\tau}{r-\rho m} \sqrt{(\epsilon_1+\epsilon_2)^2 s^2 + 2\rho\bar{\alpha}l_1((\epsilon_1+\epsilon_2)s + \rho\bar{\alpha}l_1)}$
$+ \frac{\tau\rho}{r-\rho m}(\bar{\beta}l_2 + \xi_1 + \xi_2 + \xi_3 l_3) + \mu l_3]$ and $\gamma = \frac{1}{\sqrt{2\varsigma+1}}$. Then we know $\Lambda_n \to \Lambda$ as $n \to \infty$.

By (8), we know that $0 < \Lambda < 1$ and hence there exist $n_0 > 0$ and $\Lambda_0 \in (0, 1)$ such that $\Lambda_{n+1} \leq \Lambda_0$ for all $n \geq n_0$. Therefore, by (19) we have

$$\| u_{n+2} - u_{n+1} \| \leq \Lambda_0 \| u_{n+1} - u_n \| + \gamma \| e_{n+1} - e_n \|, \quad \forall n \geq n_0. \tag{20}$$

(20) implies that

$$\| u_{n+1} - u_n \| \leq \Lambda_0^{n-n_0} \| u_{n_0+1} - u_{n_0} \| + \gamma \sum_{j=1}^{n-n_0} \Lambda_0^{j-1} t_{n-(j-1)}, \tag{21}$$

where $t_{n+1} = \| e_{n+1} - e_n \|$ for all $n \geq n_0$. Hence, for any $m \geq n > n_0$, we have

$$\begin{aligned}
\| u_m - u_n \| &\leq \sum_{k=n}^{m-1} \| u_{k+1} - u_k \| \\
&\leq \sum_{k=n}^{m-1} \Lambda_0^{k-n_0} \| u_{n_0+1} - u_{n_0} \| + \gamma \sum_{k=n}^{m-1} \Lambda_0^k [\sum_{j=1}^{k-n_0} \frac{t_{k-(j-1)}}{\Lambda_0^{k-(j-1)}}].
\end{aligned}$$

Since $\sum_{j=1}^{\infty} \| e_j - e_{j-1} \| \varpi^{-j} < \infty, \quad \forall \varpi \in (0,1)$ and $0 < \chi_0 < 1$, it follows that $\| u_m - u_n \| \to 0$ as $n \to \infty$, and so u_n is a Cauchy sequence in E. Thus, there exists $u^* \in E$ such that $u_n \to u^*$ as $n \to \infty$. By Algorithm 4.1 and the Lipschitz continuity of S, T and G, we get

$$\begin{cases}
\| x_{n+1} - x_n \| \leq (1+\frac{1}{n+1})\widetilde{H}(S(u_{n+1}), S(u_n)) \leq (1+\frac{1}{n+1})l_1 \| u_{n+1} - u_n \|, \\
\| y_{n+1} - y_n \| \leq (1+\frac{1}{n+1})\widetilde{H}(T(u_{n+1}), T(u_n)) \leq (1+\frac{1}{n+1})l_2 \| v_{n+1} - v_n \|, \\
\| z_{n+1} - z_n \| \leq (1+\frac{1}{n+1})\widetilde{H}(G(u_{n+1}), G(u_n)) \leq (1+\frac{1}{n+1})l_3 \| u_{n+1} - u_n \|,
\end{cases} \tag{22}$$

It follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all Cauchy sequence. Thus, there exist x_1, y_1, z_1, x_2, y_2 and z_2 such that $x_n \to x^*$, $y_n \to y^*$, and $z_n \to z^*$, as $n \to \infty$. Next, we will show that $x^* \in S(u^*)$. Noting $x_n \in S(u_n)$, we have

$$\begin{aligned}
d(x^*, S(u^*)) &\leq \| x^* - x_n \| + d(x_n, S(u^*)) \\
&\leq \| x^* - x_n \| + \widetilde{H}(S(u_n), S(u^*)) \\
&\leq \| x^* - x_n \| + l_1 \| u_n - u^* \| \to 0 \qquad (n \to \infty).
\end{aligned}$$

Since $S(u)$ is closed, it implies $x^* \in S(u^*)$. Similarly, one can show that $y^* \in T(u^*)$ and $z^* \in G(u^*)$.
By the condition (7), Theorem 3.3 and the continuity of all mappings, letting $n \to \infty$ in (4), we obtain

$$(g-p)(u^*) = R^{H(\cdot,\cdot),\eta}_{M(\cdot,z^*),\rho}(H(A((g-p)(u^*)), B((g-p)(u^*))) + \rho\omega - \rho N(x^*, y^*) - \rho F(u^*, u^*, z^*)). \tag{23}$$

where $u^* \in E$, $x^* \in S(u^*)$, $y^* \in T(v^*)$, $z^* \in G(u^*)$ $R^{H(\cdot,\cdot),\eta}_{M(\cdot,z^*),\rho} = (H(A, B) + \rho M(\cdot, z^*))^{-1}$ and $\rho > 0$ are constants.
By Lemma 4.1, (u^*, x^*, y^*, z^*) is a solution of the problem (1). This completes the proof. □

Remark 4.1 By Algorithm 4.1 and Theorem 4.1, it is easy to obtain the convergence results for iterative algorithms for special cases of the problem (1), We omit them here. We emphasize that the existence result and algorithm of solutions for the problem (1) are given in general Banach spaces without uniform smoothness and the set-valued mappings that may not be monotone or accretive.

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