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On Dynamical Systems for Transport Logistic and Communications

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Abstract

In this paper a discrete dynamical system is considered . There is a dial with N positions (vertices) and M particles. Particles are located in vertices. Each particle moves, at every time unit, in accordance with its plan. The plan is logistics, given through a real number which belongs to the segment $[0, 1]$. The number is represented in positional numeral system with base N equal to the number of vertices. A competition takes place if particles must move in opposite directions simultaneously. A rule of competition resolution is given. Systems characteristics are investigated for sets of rational and irrational plans. Some algebraic constructions are introduced for this purpose. Probabilistic analogues (random walks) are also considered.

Keywords: discrete dynamical systems; Markov process; number theory; ergodic theory

1. Introduction

A dynamical system, named bipendulum, was introduced and investigated in (Kozlov, Buslaev, & Tatashev, 2015c). This system is the simplest version of the transport-logistic problem. The geometric interpretation of this system is the following. Each particle is located in one of two vertices. A channel connects the vertices. Particles must move or not move at each discrete time instant in accordance with plans. Delays are due to the condition that the particles cannot move towards each other. A competition takes place if two particles attempt to move in opposite directions. A rule of competitions resolution is given. The implementation of plans of losing competitions particles is delayed.

Plans of particles are given by binary representation of numbers which belong to the segment $[0, 1]$. A rational number determines a plan, if the representation of this number is a periodic positional fraction or a common fraction. In the first of these cases, we assume that the particle plan is written on the tape, and the particle reads digits of the plan successively. The transition to implementation of the next digit of the plan is equivalent to the following. Digits of plans are shifted to the left onto a position, and the digit to the left of the point is excluded. Therefore this transition is equivalent to multiplication of a number by 2 and exclusion of the integral part. This operation is called the Bernoulli shift (Schuster, 1984). Let us consider a set of plans defined by proper common fractions with the same denominator N. An algebra W_N of elements of this set with respect to the operation of Bernoulli shift is introduced in (Kozlov, Buslaev, & Tatashev, 2015c), (Buslaev, & Tatashev, 2016). The algebra W_N, depending on N, is represented uniquely as a sum of connected components which are subalgebras W_{N_i}, where N_i are divisors of N. In this case, dynamics of particles can be interpreted as motion of particles on the subalgebra graph, where weights of arcs generate logistics. The concept of particle tape velocity was introduced. Thus this algebra is an important component of the dynamical system called bipendulum.

A system with any number of vertices and particles is considered in this paper. The system is called real-valued pendulum. The plan of each particle is given by a number which belongs to the segment $[0, 1]$. This number is represented in the numeral positional system with the base equals to the number of vertices. If particles try to move towards each other, then delays take place as in the case of bipendulum. Transition to the next digit of the plan is equivalent to applying of the generalized Bernoulli shift operation to the number which determines the plan. The generalized Bernoulli shift is defined as follows. The number is multiplied by N and the integral part is excluded. The base of positional system is equal to the number of vertices. Thus algebra of proper fractions generate real-valued $N-$pendulum as Bernoulli algebra is the abstract basis for bipendulum.

Some discrete dynamical systems with symmetric periodic structures were introduced and investigated in (Kozlov, Buslaev, Tatashev & Yashina ,2014) (Kozlov, Buslaev & Tatashev, 2015b), (Kozlov, Buslaev & Tatashev, 2015a).

2. Formulation of Problem

In Section 2, the problem of formulation is given in terms of dynamical systems.

2.1. Particles Dislocation

Suppose there are N vertices $V_0, \dots, V_{N-1}$ and M particles $P_1, \dots, P_M$, Figure 1. Each particle is in one of N vertices at every moment of time. There are logistic plans of particles dislocations. Each plan is a real number $a^{(j)}$ belonging to the segment $[0, 1]$. The plan of a particle determines a sequence of indexes of vertices in which the particle must be located at successive moments of time. This sequence is given by N-ary representation of the number $a^{(i)}$. The vertices are connected by channels. If two particles try to move towards each other, then a competition takes place. In this case only particles, trying to move in one of two directions, realize the attempt. This direction is chosen in accordance with a given rule. Implementation of plans of losing competition particles is delayed. The main essence of this system is total planning of particles dislocation.

The system state can be described for one of special cases with a binary matrix. The rows of the matrix correspond to particles. Each row contains a single "one". The index of the column, containing this "one", is equal to the index of the vertex, containing the particle at present time. This index is equal to one of numbers $0, 1, 2, \dots, N-1$, Figure 1.

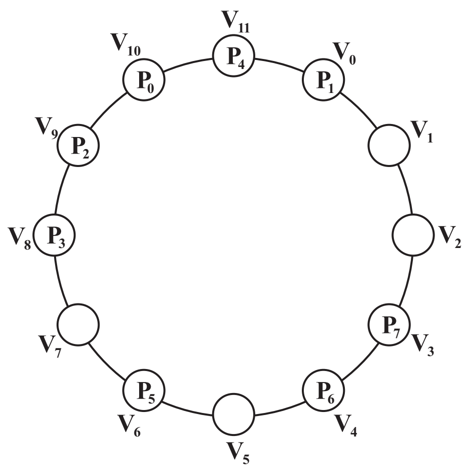

Figure 1. Round table and dislocation of particles

2.2. Plan Logistics

A real number $a^{(i)}$ is given. This number is called the plan of the particle P_i , $i = 1, 2, \dots, M$. This number is represented in N-ary system

$$a^{(i)} = 0.a_1^{(i)} a_2^{(i)} \dots a_k^{(i)} \dots, \tag{1}$$

where each digit value is equal to one of the numbers $0, 1, \dots, N-1$. We assume that the number $a^{(i)}$ is recorded on the tape of the particle P_i, $i = 1, \dots, M$. Each particle reads a digit recorded on its tape at every discrete time moment $T = 1, 2, \dots$. This digit determines the index of the vertex such that the particle tries to occupy this vertex, Figure 2.

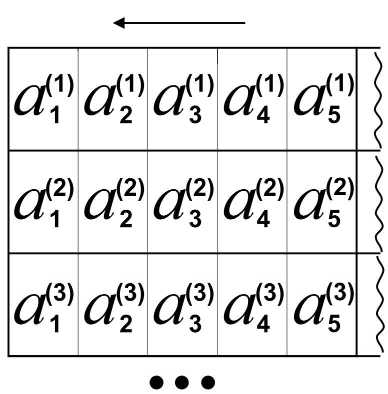

Figure 2. Turing tapes and plan logistics

2.3. Random Walk Logistics

Random walk is considered as an alternative for the logistics plan. In the case of random walk, the next planned dislocation of particle $P_i(T+1)$, $i = 1, \dots, M$, will be determined, when $P_i(T)$ has been realized. Each digit of the number $a^{(i)}$ is played before the particle reads this digit, and the value of the digit is equal to j with probability $p_{i,j}$, $j = 0, \dots, N-1$; the numbers $p_{i,j}$ are given, $0 < p_{i,j} < 1$, $p_{i,0} + p_{i,1} + \cdots + p_{i,N-1} = 1$, $1 \le i \le M$. The finally configuration is formed if possible competitions have been taken into account.

2.4. Rules of Activity and Competition for Plan Logistics

The particle P_i must be located in the vertex $V_{a_{jT}}$ in accordance with the plan of this particle, at the moment T, $i = 1, \dots, M$, $T = 1, 2, \dots$ If the particle P_i is in the vertex $V_{a^{(i)}_T}$ at the moment T, then at the moment $T+1$, $T = 0, 1, 2, \dots$, this particle will be in the vertex $V_{a^{(i)}_{T+1}}$ except the case of competition. *A competition takes place if*, at present moment of time, there are particles which try to come from the vertex V_k to the vertex V_l, and particles which try to come from the vertex V_l to the vertex V_k, $k \ne l$, $0 \le k, l \le N-1$. If $s_{k,l}$ particles try to move from the vertex V_k to the vertex V_l and $s_{l,k}$ particles try to move from the vertex V_l to the vertex V_k, then all particles, trying to move from the vertex V_k to the vertex V_l, win the competition with probability, Figure 3,

$$\frac{s_{k,l}}{s_{k,l} + s_{l,k}}. \tag{2}$$

If a competition takes place at the moment of time T, then all competing particles, winning the competition, will be, at the moment of time $T+1$, in the vertex, determined by the plan, and losing particles do not move. The tape of winning particle will read next digit at the moment of time $T+1$. The losing particle will read at the moment of time $T+1$ the digit corresponding the moment of time T. Initial conditions are given. These conditions determine the location of particles at the moment of time $T = 0$.

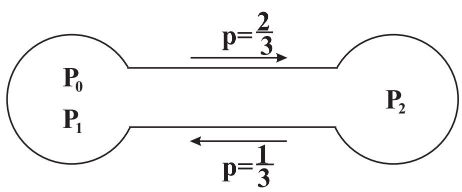

Figure 3. Competition resolution

2.5. Quantitative and Qualitative Characteristics

Denote by $D_i(T)$ the number of transitions of the particle P_i tape during the time interval $[0; T]$, $i = 1, \dots, M$; $T = 1, 2, \dots$; $H(t)$ is the number of competitions during the time interval $(0; T]$; $H_i(t)$ is the number of lost competitions of the particle P_i during time interval $(0; T]$, $T > 0$,

$$H_1(T) + H_2(T) + \cdots + H_M(T) = H(T).$$

The limit

$$w_i = \lim_{T\to\infty} \frac{D_i(T)}{T}, \; i = 1, \dots, M, \tag{3}$$

is called *the velocity of the particle P_i tape,* $i = 1, \dots, M$, if this limit exists.

The limit

$$h = \lim_{T\to\infty} \frac{H(T)}{T}, \; i = 1, \dots, M, \tag{4}$$

is called *the intensity of competitions* if this limit exists.

The limit

$$h_i = \lim_{T\to\infty} \frac{H_i(T)}{T}, \; i = 1, \dots, M, \tag{5}$$

is called the intensity of lost particle P_j competitions, if this limit exists.

The limits (3)–(5) depend on the process realization. These limits can exist or not exist depending on the realization.

The system is in *the state of synergy* after a moment of time T_{syn} if, after the moment of time T_{syn}, no competitions take place.

3. Rational Logistic Plans

In Section 3, we introduce concepts related to real-valued pendulum, and prove theorems about particles tapes velocities.

If the number $a^{(i)}$ is rational, then this number can be represented as a periodic fraction

$$a^{(i)} = a_1^{(i)} a_2^{(i)} \dots a_{l_i}^{(i)} \left(a_{l_i+1}^{(i)} a_{l_i+2}^{(i)} \dots a_{l_i+m_i}^{(i)} \right),$$

where l_j is the length of the aperiodic part of the number $a^{(i)}$ representation, and m_i is the length of the repeating part of the representation, $i = 1, \dots, M$.

3.1. Extreme Rational Bipendulum

Consider the following example. Suppose $N = M = 2$; $a^{(1)}$ and $a^{(2)}$ are the numbers 1/3 and 1/5 respectively. This numbers are represented in the binary system as periodic fractions

$$a^{(1)} = \frac{1}{3} = 0.(01),\ a^{(2)} = 0.(0011).$$

Lemma 1, (Kozlov, Buslaev, & Tatashev, 2015c). *Suppose*

$$a^{(1)} = 0.(01),\ a^{(2)} = 0.(0011).$$

Then limits (3), (4) and (5) exist with probability 1, and

$$h = \frac{2}{5},\ h_1 = h_2 = \frac{1}{5},\ w_1 = w_2 = \frac{4}{5}.$$

It is proved in (Kozlov, Buslaev, & Tatashev, 2015c) that, if $N = 2$, $M = 2$, i.e., in the case of bipendulum, the velocity of particles is not less than $\frac{4}{5}$. We shall prove the following. If $N \geq 3$, then the velocity of particles can be equal to $\frac{3}{4}$.

Lemma 2. *Suppose $N = 3$, $M = 2$.*

$$a^{(1)} = 0.(012),\ a^{(2)} = 0.(021).$$

Then $h = \frac{1}{2}$, $h_1 = h_2 = \frac{1}{4}$, $w_1 = w_2 = \frac{3}{4}$.

Proof. The system is a Markov chain (Feller, 1970), (Kemeny, & Snell, 1976), (Borovkov, 1986) with states (i_1, i_2), $0 \leq i_1, i_2 \leq 2$, where i_j is a digit such that the particle P_j, $j = 1, 2$, reads this digit. There are 9 states

$$S_0 = (0,0),\ S_1 = (0,1),\ S_2 = (0,2),\ S_3 = (1,0),\ S_4 = (1,1),$$

$$S_5 = (1,2),\ S_6 = (2,0),\ S_7 = (2,1),\ S_8 = (2,2).$$

States S_2, S_3, and S_7 are inessential states of the chain. The other 6 states form a set of essential states. This set is divided into two subsets:

$$G_1 = \{S_0, S_4, S_8\},\ G_2 = \{S_1, S_5, S_6\}.$$

The chain goes from any state of the set G_1 to a state of the set G_2, and from any state of the set G_2 to a state of the set G_1. If the system is in a state of the set G_1, then there is no competition. If the system is in a state of the set G_2, then there is a competition. Thus, $h = \frac{1}{2}$, $w_1 = w_2 = \frac{3}{4}$.

Theorem 1. *Suppose $M = 2$. Then the greatest possible value of the competitions intensity is equal to $\frac{1}{2}$, and the least possible value of the tape velocity is equal to $\frac{3}{4}$.*

Proof. It follows from Lemma 2 that, if $M = 2$, then plans of particles can be given such that $h = \frac{1}{2}$, $v = \frac{3}{4}$. If $M = 2$, then plans cannot be given such that the competitions intensity is more than $\frac{1}{2}$. Indeed, if there is a competition at moment of time T, then there will be no competition at moment of time $T + 1$. Theorem 1 has been proved.

3.2. Phase Pendulums

Suppose $N = M = 2$, and we define the plan $a^{(2)}$, shifting plan $a^{(1)}$ for a fixed number of positions.

Lemma 3. *Suppose*

$$a^{(i)} = 0.\left(a_1^{(i)} a_2^{(i)} \dots a_m^{(i)} \right),\ i = 1, 2,$$

and we get the plan $a^{(2)}$, shifting plan $a^{(1)}$ onto c positions $(0 \le c \le m-1)$ to the left, i.e.,

$$a^{(2)}_{k+c} = a^{(1)}_k,\ k = 1, 2, \ldots, m,$$

where addition is meant modulo m. Then the system comes to the state of synergy after a time interval with finite expectation.

Proof. The system is a Markov chain (Feller, 1970), (Kemeny, & Snell, 1976), (Borovkov, 1986) with m^2 states. The system comes to the state of synergy with a positive probability, and the system cannot leave the state of synergy. Therefore a state is essential if and only if the system is in the state of synergy. The Markov chain comes from any inessential state to the set of essential states after a time interval with finite expectation. Lemma 3 has been proved.

Theorem 2. *Suppose*

$$a^{(i)} = 0.a^{(i)}_1 \ldots a^{(i)}_{l_i}(a^{(i)}_{l_i+1}a^{(i)}_{l_i+2} \ldots a^{(i)}_{l_i+m}),\ i = 1, 2,$$

and we get the repeating part of $a^{(2)}$, shifting the repeating part of $a^{(1)}$ onto a fixed number of positions. Then the system comes to the state of synergy after a time interval with finite expectation.

Proof. Both particles will read digits of repeating parts of their plans after a time interval with finite expectation. Theorem 2 follows from Lemma 3.

4. Functional Interpretation of Bipendulum

In Section 4, we consider functional representation of plans. We define variation of proper fraction .

Suppose the binary representation of the number j/K is

$$\frac{j}{K} = 0.a_1 \ldots a_l(a_{l+1} \ldots a_{l+m}).$$

Suppose

$$\theta(i) = \begin{cases} 0,\ a_i = a_{i+1}, \\ 1,\ a_i \neq a_{i+1} \end{cases}$$

The *variation* of the representation of the number $\frac{j}{K}$ is defined as

$$V(j/K) = \frac{1}{m}\sum_{i=l+1}^{l+m} \theta(i),$$

where the addition, in indexes, is meant modulo m. The limit

$$Var(a) = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \theta(i)$$

is called the *variation of the binary representation* of a real number $a \in [0, 1)$, $a = 0, a_1a_2 \ldots a_k \ldots$ if this limit exists. It is evident that $V(a) = Var(a)$ for rational number a. Suppose

$$Var_{\min} = \min\left(Var(a^{(1)}), Var(a^{(2)})\right).$$

Theorem 3. *Suppose $N = M = 2$. Then inequalities are true*

$$h \le 2Var_{\min}(2 + Var_{\min})^{-1},$$

$$v_1 = v_2 \ge 2(2 + Var_{\min})^{-1}.$$

Proof. If $\theta(i) = 0$ and a particle reads digit a_i at time T, then the particle will be read digit a_{i+1}, $i = 1, 2, \ldots$, at time $T + 1$. If $\theta(i) = 1$, the particle reads digit a_i at time T, and no competition takes place, then the particle will read digit a_{i+1} at time $T + 1$, $i = 1, 2, \ldots$. Suppose $\theta(i) = 1$, the particle reads digit a_i at time T, and there is a competition at this time. Then, at time $T + 1$, with probability $\frac{1}{2}$, the particle reads digit a_i, and with probability $\frac{1}{2}$, the particle reads digit a_{i+1}. Therefore, the average time of particle going from digit a_i to digit a_{i+1} is not greater than

$$(1 - Var_{\min}) \cdot 1 + Var_{\min} \cdot \frac{3}{2} = 1 + \frac{Var_{\min}}{2}.$$

Thus, $v_1 = v_2 \geq ((2 + Var_{min})/2)^{-1} = 2(2 + Var_{min})^{-1}$, $h = 2(1 - v_1) \leq 2Var_{min}(2 + Var_{min})^{-1}$. Theorem 3 has been proved.

Corollary 2. *Suppose $N = M = 2$. Then the following inequalities are true*

$$h_1 = h_2 \leq \begin{cases} 2Var_{\min}/(1 + Var_{\min}), & Var_{\min} \leq 2/3, \\ 1/2, & Var_{\min} \geq 2/3, \end{cases}$$

$$v_1 = v_2 \geq \begin{cases} 2/(2 + Var_{\min}), & Var_{\min} \leq 2/3, \\ 3/4, & Var_{\min} \geq 2/3. \end{cases}$$

5. Generalized Bernoulli Algebras on Proper Fractions

5.1. The Concept of Generalized Bernoulli Algebra

Algebras of proper common fractions with respect to the operation, called Bernoulli shift, (Schuster, 1984), were considered in (Buslaev, Tatashev, 2016). If this operation is applied to a fraction, the fraction is multiplied by 2, and the integral part of the product is excluded. We introduce algebras with operation of Bernoulli shift. If this operation is applied, then the fraction is multiplied by the natural number N, and the integral part is excluded. Consider the set of proper fractions

$$G_K = \left\{ \frac{0}{K}, \frac{1}{K}, \frac{2}{K}, \ldots, \frac{K-1}{K} \right\}$$

with a unary operation

$$FB^N(x) = Nx - [Nx],$$

where $[Nx]$ is the integral part of Nx.

Suppose W_K^N is the algebra with the operation FB^N on the set G_K. The number i/K $(0 \leq i \leq K-1)$ generates an algebra. Denote by $W_K^N(i)$ the subalgebra generated by the element i/K, $i = 0, 1, \ldots, K-1$.

Elements i/K and j/K are called *communicating* with each other if $j/K \in W_K^N(i)$ and $i/K \in W_K^N(j)$.

The element i/K is called *inessential* if there exists an element j/K such that $j/K \in W_K^N(i)$ and $i/K \notin W_K^N(j)$. The other elements are *essential.*

The set of essential elements can be divided into disjoint sets such that any two elements of the same set communicate with each other. These sets are called *classes of communicating essential elements* or *orbits.* If an element i/N satisfies equation

$$FB^N\left(\frac{i}{K}\right) = i/K,$$

then this element is called *absorbing.*

If i/K and j/K communicate with each other, then $W_K^N(j) = W_K^N(i)$, $i, j = 1, 2, \ldots, K-1$. If $j/K \neq W_K^N(i)$ and $i/K \neq W_K^N(j)$, then $W_K^N(i)$ and $W_K^N(j)$ are disjoint subalgebras.

Identifying subalgebras $W_K^N(i)$ and $W_K^N(j)$ with each other for $W_K^N(i) = W_K^N(j)$, we divide the algebra W_K^N into disjoint subalgebras. Each of these subalgebras contains a class of communicating essential elements and a set of inessential elements. This set of inessential elements can be empty. These subalgebras are called *connected components of the algebra.*

Consider a proper fraction i/K, $K \geq 1$, $0 \leq i < K$. The following theorem allows to find the N-ary representation of this fraction:

$$i/K = \sum_{j=1}^{\infty} a_j \cdot N^{-j},$$

where a_j equals one of numbers $0, 1, \ldots, N-1$, $j = 1, 2, \ldots$

Theorem, (Broido, Ilyina, 2006). *Suppose*

$$b_0 = i/K, \; b_j = FB^N(b_{j-1}), \; j = 1, 2, \ldots$$

Then $a_j = [Nb_{j-1}]$.

This theorem gives an approach which allows to find the N-ary representation of proper fraction $b_0 = \frac{j}{K}$. The value of the first digit a_1 of the representation equals the integral part of Nb_0. The fractional part of the product is the value of b_1. The value of the digit a_j equals the integral part of the product b_{j-1} and N. The fractional part of this product equals the value of b_j, $j = 1, 2, \dots$ Suppose numbers l ($l \geq 1$) and m ($m \geq 1$) are such that $b_l = b_{l+m}$, and there are no numbers i and j such that $i, j < l + m$ and $b_i = b_j$. Then we have $a_i = a_{i+m}$ for any $i \geq l$. Hence the N-ary representation of the proper fraction s/K contains an aperiodic part with length $l - 1$ and a repeating part with length m. We write

$$\frac{s}{K} = 0.a_1 \dots a_{l-1}(a_l \dots a_{l+m-1}).$$

We have formulated an approach to find the binary representation of a proper fraction. This approach is similar to the approach of converting the decimal representation of a proper fraction to the N-ary representation, (Broido & Ilyina, 2006). It is easy to reconstruct the trajectory of the element s/K of the algebra W_K^N if we know the N-ary representation of the number s/K.

Some problems of number theory, related to N-ary representation of numbers, are considered in (Uteshev, Cherkasov, Shaposhnikov, 2001).

5.2. Absorbing Elements of Algebra W_K^N

We shall prove theorems about absorbing elements of algebras W_K^N.

Lemma 4. *If $N = 2$, then the element $\frac{0}{K}$ is a unique absorbing element of the algebra W_K^2.*

Proof. It is obvious that $\frac{0}{K}$ is an absorbing element of the algebra W_K^N for any N and K. Since, for $i = 1, \dots, K - 1$,

$$\frac{i}{K} < \frac{2i}{K} < \frac{i}{K} + 1,$$

then there exist no other absorbing elements of the algebra W_K^2.

We shall give some examples. These examples show following. If $N \geq 3$, there exist algebras which do not contain elements not equal to $\frac{0}{N}$, and algebras which contain absorbing elements not equal to $0/N$.

Example 1. Algebra W_5^3 contains an absorbing element $\frac{0}{N}$ and an orbit with period 4.

Example 3. Algebra W_6^3 contains two absorbing elements $\frac{0}{6}$ and $\frac{3}{6}$. The other elements of the algebra are inessential.

Lemma 5. *Suppose $N = K + 1$. Then all elements of the subalgebra W_K^N are absorbing.*

Proof. If $N = K + 1$, then we have

$$FB^N\left(\frac{j}{K}\right) = (K + 1)jK^{-1} - j = j/K,\ j = 0, 1, \dots, K - 1.$$

5.3. Structure of the Algebra W_K^N, Where K is Coprime to N

In Subsection 5.3, we prove theorems about generalized Bernoulli algebras in the case of coprime K and N.

5.3.1 Preliminary Definitions and Results

Suppose the canonical representation of the number K, $K \geq 2$, has the form

$$K = p_1^{s_1} \dots p_l^{s_l}, \tag{6}$$

where $2 \leq p_1 < \dots < p_l$ are prime numbers, $s_1, \dots, s_l$ are natural numbers. We write $K = \overline{p}^{\overline{s}}$, where $\overline{p} = (p_1, \dots, p_l)$, $\overline{s} = (s_1, \dots, s_l)$. Denote by $E(K)$ the Euler's function (Buchstab, 1972),

$$E(K) = \left(p_1^{s_1} - p_1^{s_1-1}\right) \dots \left(p_l^{s_l} - p_l^{s_l-1}\right).$$

Euler's theorem on numbers, (Buchstab, 1972). *The value of $E(K)$ is equal to the number of positive integers that are coprime to N and less than K*

This function can be also represented as

$$E(K) = K\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_l}\right).$$

Euler's theorem on divisibility, (Buchstab, 1972). *Suppose N and K are coprime natural numbers. Then K is a divisor of $N^{E(K)} - 1$.*

Fermat's little theorem is a special case of Euler's theorem on divisibility.

Fermat's little theorem, (Buchstab, 1972). *If p is prime, and p is not a divisor of N, then p is a divisor of $N^{p-1} - 1$.*

Denote by $LCM(a_1, \ldots, a_L)$ the least common multiple of numbers $a_1, \ldots, a_l$. Suppose K has the form (6).

Generalized Euler's function is the function $e(K)$, $e(1) = 1$,

$$e(K) = LCM\left(p_1^{s_1-1}(p_1 - 1), \ldots, p_l^{s_l-1}(p_l - 1)\right).$$

It is obvious that any divisor of $e(K)$ is a divisor of $E(K)$.

The generalized Euler's divisibility theorem, (Buchstab, 1972). *Let N and K be coprime positive integers. Then K is a divisor of $N^{E(K)} - 1$.*

According to this theorem, if N and K are coprime, then there exists a positive number l such that K is a divisor of $N^l - 1$. *The smallest positive integer l such that K is a divisor of $N^l - 1$ is called the order of K modulo N, (Buchstab, 1972). Denote by $m(K, N)$ the order of K modulo N.*

The generalized Euler's order theorem, (Buchstab, 1972). *Let N and K be coprime positive integers. Then $m(K, N)$ is a divisor of $e(K)$.*

The elements of the algebra W_K^N can be divided into two classes. They are *the class of noncancelable fractions and the class of cancelable fractions*

$$W_K^N = W_K^{N,copr[ime]} + W_K^{N,canc[elable]}.$$

Consider the set of all elements j/K of the algebra W_K^N such that j/K is a noncancelable fraction. The number of these elements is equal to the number of positive integers less than K and coprime to N, i.e., this number is equal to Euler's function. Suppose

$$\delta(d, E(K)) = \begin{cases} \frac{E(K)}{m(K,N)}, \ d = m(K, N), \\ 0, \ d \neq m(K, N). \end{cases} \tag{7}$$

In accordance with section 5.1, an orbit of period d is a set of elements of the algebra W_K^N

$$\frac{j_1}{K}, \frac{j_2}{K}, \ldots, \frac{j_d}{K},$$

$0 \leq j_i \leq K - 1$, $i = 1, \ldots, d$ such that

$$\frac{j_{i+1}}{K} = FB^N\left(\frac{j_i}{K}\right) \neq \frac{j_1}{K}, \ i = 1, \ldots, d - 1,$$

$$\frac{j_1}{K} = FB^N\left(\frac{j_d}{K}\right).$$

Denote by $O(K, d)$ *the number of orbits with period d* ($d \geq 1$).

5.3.2 Simple Case

Suppose $K = p^s$, where $p \geq 2$ is a prime number, the number p is not a divisor of K, s is a natural number,

$$E(K) = p^s - p^{s-1}.$$

Theorem 4. *The following formula is true*

$$O(K, d) = \sum_{r=1}^{s} \delta(d, E(p^r)) = \sum_{r=1}^{s} \delta(m(p^r), E(p^r)).$$

The proof is based on Lemma 6.

Lemma 6. *Suppose $r \leq s$, j/p^r is an element of the algebra $W_{p^r}^N$ such that it is a noncancelable fraction. Then this element belongs to an orbit of period $m(p^r)$. The set of cancelable fractions of the algebra $W_{p^r}^N$ is a subalgebra isomorphic to the algebra $W_{p^{r-1}}^N$.*

Proof. Suppose the number a satisfies the equation

$$N^m \cdot \frac{j}{K} = \frac{j}{K} + a.$$

The number a is natural if and only if the number K is a divisor of $N^m - 1$. We take into account that j and K are coprime, and $j < K$. Applying to a noncancelable element the generalized Bernoulli shift operation $m(K, N)$ times, we obtain the same element. If the generalized Bernoulli shift is applied to this element less than $m(K, N)$ times, then the same element cannot be obtained. It is obviously that the subalgebra of cancelable fractions of the algebra $W^N_{p^r}$ and the algebra $W^N_{p^{r-1}}$ are isomorphic to each other. Lemma 6 has been proved.

5.3.3 General Case

Suppose K is coprime to N.

Lemma 7. *Any element j/K of the algebra W^N_K belongs to an orbit, and, if $j/K \in W^{N,copr}_K$, i.e., the element j/K is a noncancelable fraction, then the period of this orbit equals $m(K, N)$.*

Proof. Apply the Bernoulli shift operation to the element j/K i times. We obtain the same element if and only if the equation

$$N^i \cdot \frac{j}{K} = \frac{j}{K} + a, \tag{8}$$

contains natural number a. If $i = e(K)$ in (8), then the number a is natural. We take into account that K is coprime to N. Hence there exists an orbit with period not greater than $e(K)$ such that this orbit contains the element j/K. If the fraction j/K is a noncancelable fraction, then $i = m(K, N)$ is the smallest value of i such that the number a is a natural number. From this Lemma 7 follows.

Lemma 8. *Let K be a natural number, and K is coprime to N. Then two elements $a \in W^{N,copr}_K$, $b \in W^{N,canc}_K$ of the algebra W^N_K cannot belong to the same orbit.*

Proof. If an element is cancelable fraction, and we apply the generalized Bernoulli shift operation to this element any times, then we can obtain no noncancelable fraction. From this, Lemma 8 follows.

Denote by $\delta_{d,K}$ the number of orbits with period d in the subalgebra $W^{N,copr}_K$. In accordance with Lemmas 7 and 8,

$$\delta_{m(p^r),p^r} = \frac{E(p^r)}{m(p^r, N)}.$$

If $d \neq m(p^r, N)$, then $\delta_{d,p^r} = 0$.

Suppose $R = R(N)$ is the set of vectors $r = (r_1, \ldots, r_l)$ with integer nonnegative numbers, $0 \le r_1 \le s_1, \ldots, 0 \le r_l \le s_l$, and at least one of numbers $r_1, \ldots, r_l$ is positive.

Theorem 5. *Suppose $O(K, d)$ is the number of orbits with period d ($d \ge 1$) in algebra W^N_K. Then the following formula is true*

$$O(K, d) = \sum_{\bar{r} \in R} \delta(d, \bar{p}^{\bar{r}}), \tag{9}$$

where $\delta(d, p^r)$ is calculated in accordance with (7).

Proof. If $r_1, r_2 \in R(K)$, then, for $K_1 = \bar{p}^{\bar{r}_1}$, $K_2 = \bar{p}^{\bar{r}_2}$, $K_1 \neq K_2$ we have

$$W^{N,copr}_{K_1} \cap W^{N,copr}_{K_2} = \emptyset.$$

Therefore no orbit can belong to both $W^{N,copr}_{K_1}$ and $W^{N,copr}_{K_2}$. From this fact and Lemmas 7 and 8, Theorem 5 follows.

5.4. Algebra W^N_K in the Case of $K = N^h K_1$, Where K_1 and N are Coprime

In the subsection 5.4, we assume that K equals K_1 multiplied by N^h, where K_1 and N are coprime. We shall prove a theorem about the form of generalized of Bernoulli algebras.

Theorem 6. *Let h be a natural number,*

$$K = N^h K_1,$$

where $K_1 = p_1^{s_1} \ldots p_l^{s_l}$, $p_1 < \cdots < p_L$ are prime numbers, and none of these numbers is not a divisor of N. Then the following is true.

1) Algebras W^N_K and W^N_{K/N^h} contain the same number of orbits with each period.

2) The algebra W^N_K contains K_1 essential elements. The other $K - K_1$ elements are nonessential elements.

3) The graph of algebra W^N_K contains, as its subgraphs, K_1 trees, and the depth of each of these tree equals h. Each of K_1 essential elements of the algebra W^N_K is the root of precisely one tree, and the degree of the root equals 1. The degree of *other vertices, for any tree, is equal to N except the vertices of the level h. Any tree contains N^h elements. One of these elements is the root of the tree, and there are N^{i-1} elements that are vertices of the level i for any $i = 1, \dots, h$.*

The proof of Theorem 6 is similar to proofs of Theorems 5 – 7 in (Buslaev, & Tatashev, 2016), where the case of $N = 2$ was considered.

5.5. Algebras W_K^N and W_d^N, Where d is a Divisor of N

Consider algebras W_K^N and W_d^N, where d is a divisor of the number N. In Subsection 5.5, we prove theorems that alow to compare the algebras W_K^N and W_d^N.

Theorem 7. *Let d be a divisor of the number K. If the element i/d of the algebra W_d^N belongs to the orbit with period m, then the element $\frac{K}{d}\frac{i}{K}$ of the algebra W_K^N belongs to an orbit with period m. If the elements i/d and j/d of the algebra W_d^N belong to the same orbit, then elements $\frac{K}{d}\frac{i}{K}$, $\frac{K}{d}\frac{j}{K}$ of the algebra W_K^N also belong to the same orbit. If the elements i/d and jd of the algebra W_d^N belong to different orbits, then the elements $\frac{K}{d}\frac{i}{K}$ and $\frac{K}{d}\frac{j}{K}$ of the algebra W_K^N also belong to the different orbits. If the element i/d of the algebra W_d^N is inessential, then the element $\frac{K}{d}\frac{i}{K}$ of the algebra W_K^N is also inessential. Elements $\frac{K}{d}\frac{i}{K}$ and $\frac{K}{d}\frac{j}{K}$ of the algebra W_K^N belong to the same connected component if and only if the elements i/d and j/d of the algebra W_d^N also belong to the same connected component.*

Proof. The value of the element $\frac{K}{d}\frac{i}{K}$ of the algebra W_K^N and the value of element i/d of the algebra W_d^N are the same for any $i = 1, 2, \dots, d-1$. From this, Theorem 7 follows.

Corollary 3. *Let d be a divisor of the number K. Then the number of orbits with period m, the number of connected components and the number of elements of the algebra W_K^N are not less than in the algebra W_d^N.*

6. Rational Pendulums and Generalized Bernoulli Algebras

In section 6 we consider connection between generalized Bernoulli algebras and real valued pendulums. We show that the algebraic construction of rational logistic plans has an effect on properties of real valued pendulums.

6.1. Generalized Bernoulli Algebra and Real Valued Pendulum

Consider a real valued pendulum with rational particles plans given by the rational numbers $a^{(1)} = s_1/k_1$, $a^{(2)} = s_2/k_2$, $0 \le s_i < k_i,\ i = 1, 2$. Suppose k is the least common multiple of numbers k_1, k_2, and

$$a^{(i)} = \frac{j_i}{k},\ 0 \le j_i < k,\ i = 1, 2. \tag{10}$$

The generalized Bernoulli algebra W_k^N contains all subalgebras (orbits), satisfying initial conditions (10).

6.2. Subalgebras, Orbits, and Velocity of Turing Tapes

To each initial condition $a^{(i)}$, $i = 1, \dots, M$, assign vector of elements $b^{(i)}$, $i = 1, \dots, M$, where $a^{(i)} = b^{(i)}$ if $a^{(i)}$ is an element of the orbit of a subalgebra, or the first element of the orbit of the subalgebra on the path if $a^{(i)}$ is an inessential element.

Lemma 9. *Suppose $M \ge 2$, and $a^{(i)}$, $i = 1, \dots, M$, are plans. Then, with probability 1, numerical characteristics of the real valued pendulum are the same as in the case of plans $b^{(i)}$, $i = 1, \dots, M$, and the elements of the pendulum, belonging to the same orbit, can be not taken into account.*

The proof is evident.

Theorem 8. *Suppose $M \ge 2$, and plans of particles $a^{(i)}$, $i = 1, \dots, M$, belong to the same subalgebra of the algebra W_k^N. Then the system comes to the state of synergy after a time interval with finite expectation.*

Proof. Suppose $M = 2$. We can get the periodic part of plan $a^{(2)}$, shifting the periodic part of the plan $a^{(1)}$ onto a fixed number of positions. There will be a finite number on competitions before the the system comes to the state of synergy, or the periodic parts of the plan $a^{(1)}$ and $a^{(2)}$ coincide after a time interval with finite expectation, and the system also come to the state of synergy. From this the theorem follows.

In the general case, each particle will read digits of periodic parts of plans after an instant with finite expectation. Suppose that, from time T_1, the set of competing particles, containing the particle with the least index, wins each competition. The particle P_1 does not lose competitions. The particle P_2 can lose competitions with the particle P_1. However there can be no more than $d-1$ competitions of these particles. The particle P_2 cannot compete with the particle P_1 after a finite instant T_2. Similarly, there are $T_3, \dots, T_M$ ($T_1 \le \dots \le T_M$) such that, after time T_i none of particles $P_1, \dots, P_i$ can compete with any other particle of this set. There will be no competitions after time T_M. Thus, with positive probability, the system comes to the state of synergy after time bounded above. From this, the theorem follows.

6.3. The Velocity of Turing Tapes with Plans from Different Subalgebras

Suppose there 2 particles and plans of particles belong to different subalgebras.

Particles will read repeating parts of their plans after a time interval with finite expectation. Therefore it is sufficient to know to which subalgebras plans belong.

Let (i_1, i_2) be the system state such that the particle P_j reads the i_j-th digit of the period, $j = 1, 2$.

We introduce Markov chain $X(t')$ obtained from the original chain as follows. The process is considered only at instants t' such that competitions take place at these instants. New time scale is obtained from the original chain, excluding all instants such that there are not any competitions at these instants.

We can find transition probabilities between all states. There are no more than two positive elements at each row of the transition probability matrix. Break of the chain is the state of synergy. The time of going to the next state is calculated. This time depends on what state will be the next state. The problem of calculation of transition probabilities is connected with analysis of linear Diophantine equations with two variables. When we have found to what class of communicating essential states the initial state belongs, we find the system of equations for steady state probabilities and calculate the steady probabilities. When we have found steady state probabilities, we calculate the average duration between competitions, and calculate the velocity of particles tapes.

Following theorems give conditions of synergy in the case such that the plans of particles belong to different algebras.

We introduce the operation $+,*$ of bitwise addition and multiplication of binary representations of rational plans $a^{(i)}$. Then the equality $a^{(1)} * a^{(2)} = a^{(1)}$ is true if the set of positions with 'ones' of number $a^{(1)}$ belongs to the set of positions with 'ones' of the number $a^{(2)}$. Denote this relation by $a^{(1)} \leq a^{(2)}$. From $a^{(1)} + a^{(2)} = 1$ it follows that $a^{(1)} * a^{(2)} = 0$.

Theorem 9. *Suppose $M = 2$, $N = 2$*

$$a^{(1)} + a^{(2)} = 1.$$

Then the system comes to the state of synergy after a time interval with finite expectation.

Proof. Suppose

$$a^{(j)} = 0.(a_1^{(j)} \dots a_d^{(j)}),\ j = 1, 2;$$

$b^{(1)} = 0.(b_1^{(1)} \dots b_d^{(1)})$, where $b_i^{(1)} = a_{i+1}^{(1)}$, $i = 1, \dots, d$ (addition is meant modulo d), i.e., we get the plan $b^{(1)}$, shifting the plan $a^{(1)}$ onto a position to left. Consider bipendulum with plans $b^{(1)}$ and $a^{(2)}$. If $a_i^{(2)} = a_{i+1}^{(2)}$ (addition is meant modulo d), then the first competition cannot take place at time $T = i$, $i = 1, \dots, d$. If $a_i^{(2)} \neq a_{i+1}^{(2)}$, then $b_i^{(1)} = a_{i+1}^{(1)} = a_i^{(2)}$, and, therefore, the first competition also does not take place at time $T = i$. Thus the bipendulum is in the state of synergy from the initial instant. From this and Lemma 3, it follows that the original bipendulum comes to the state of synergy after a time interval with finite expectation. Theorem 9 has been proved.

Theorem 10. *Suppose $M = 2$, $N = 2$*

$$a^{(1)} \leq a^{(2)}.$$

Then the system comes to the state of synergy after a time interval with finite expectation.

The proof is evident.

Remark 1. The following example shows that the synergy can take place for plans given by fractions with coprime denominators.

Suppose $N = M = 2$,

$$a^{(1)} = \frac{1}{7} = 0.(001),\ a^{(2)} = \frac{4}{9} = 0.(011100).$$

The number $a^{(1)}$ can be represented in the form $a^1 = 0.(001001)$. Therefore no competition can take place at the first 6 steps. Hence no competition can take place in the future.

Remark 2. The following example shows the following. It is possible that plans of all particles are given by fractions with the same denominator such that this denominator is a prime number, and the system cannot come to the state of synergy.

Suppose $N = M = 2$,

$$a^{(1)} = \frac{3}{73} = 0.(000010101),\ a^{(2)} = \frac{13}{73} = 0.(001011011).$$

The system cannot come to the state of synergy.

Indeed, if we shift the repeating part of a plan onto any number of positions, then a competition takes place after a finite time.

6.4. Interpretation of Real Valued Pendulum in Terms of Bernoulli Algebra

We can assume, in the general case, that there are N vertices and M particles, then the plans of the particles P_j, are given by fractions such that these fractions generate orbits. This means that the process of cyclic shift of the periodic sequence

$$a^{(j)} = \frac{s_j}{k_j},\ 0 \le s_j < k_j,\ j = 1, \dots, M,$$

takes place.

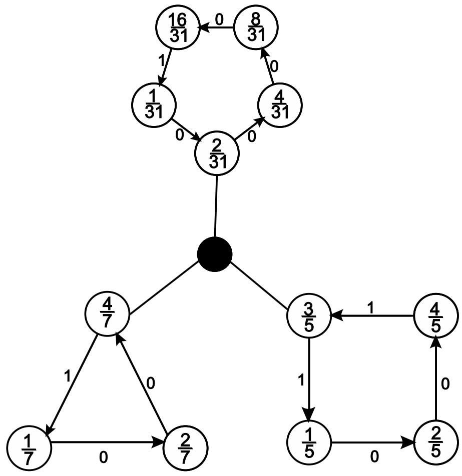

Figure 4. Bipendilum with orbits, $M = 3, N = 2$

We introduce the system such that this system is equivalent to real valued pendulum, Figures 4, 5. There are M particles $P_1, \dots, P_M$ and k vertices, where k is the least common multiple of numbers $k_1, \dots, k_M$. Each vertex corresponds to an element of the algebra W_k^N. The initial state is given such that the particle P_j is in the vertex, corresponding to an element of the subalgebra generated by the element $a^{(j)} = \frac{s_j}{k_j}$, $0 \le s_j < k_j$, $j = 1, \dots, M$. The vertex such that this vertex corresponds to the element with representation

$$\alpha = 0.(a_1, \dots, a_m)$$

is labeled with the value of digit a_m. At each discrete instant, the particle located in the vertex, corresponding to the element x, goes to the vertex corresponding to the element $y = FB^N(x)$, if no competition takes place. A competition takes place if, for some i, j $(0 \le i, j \le N-1)$, there are particles, trying to come from a vertex with the label i to a vertex with the label j, and particles, trying to come from a vertex with the label j to a vertex with the label i. If a competition takes place, then

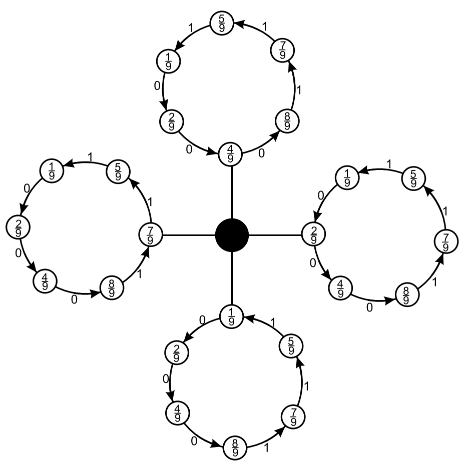

Figure 5. Phase bipendilum,$M = 4, N = 2$

only particles, winning the competition, move. Particles, losing the competition, do not move. Each particle moves along an orbit of the subalgebra W_K^N. If, at present time T, the particle is located in the vertex, corresponding to the element

$\alpha = 0.(a_1, \ldots, a_m)$, then, at this instant, in the real valued pendulum, the representation of this element is written on the particle tape. At this instant, the particle of the real valued pendulum reads the digit a_1, and therefore tries to come to the vertex V_{a_1}. The preceding digit that the particle read is a_m. Therefore the particle is located in the vertex V_{a_m} at present time.

The velocity of particle, in the introduced system, and the velocity of particle tape, in the real valued pendulum, are the same.

7. Random Walks Real Valued Pendulums

Suppose that, at each time T, we choose the index of the vertex such that the particle tries to come to this vertex. The particle must be located in the same vertex at next instant with probability q, $0 < q < 1$. Any other vertex is chosen with probability $0 < p < 1$, $(N-1)p + q = 1$. The attempt of particle to come to another vertex is realized if no competition takes place, or the particle wins the competition. Competitions take place in the cases such that in these cases competitions take place in the real valued pendulum with planned movement. Rules of competitions resolution are the same. Particles, losing a competition, do not move, and do not repeat the attempt. The index of the next vertex is chosen again for each particle.

Theorem 11. *Suppose M=2. Then, with probability 1, the intensity of competitions in random pendulum equals*

$$\frac{p^2(1-q^2-(N-1)p^2)}{1+2pq-q^2}.$$

Proof. Consider a Markov chain with two states. The chain is in the state G_1 if both particles are in the same vertex. The chain is in the state G_2 if particles are in different vertices.

Denote by p_{ij} the probability of the transition from the state G_i to the state G_j, $i, j = 1, 2$.

If the chain is in the state G_1, then there is no competition. With probability $q^2 + (N-1)p^2$, the particles will be located in the same vertex at time $T+1$. With probability $q^2 + (N-1)p^2$, the particles will be located in different vertices at time $T+1$. Therefore, we have

$$p_{11} = q^2 + (N-1)p^2, \ p_{12} = 1 - q^2 - (N-1)p^2.$$

If the chain is in the state G_2 at the time T, then with probability p^2 there is a competition and the chain will be in the state G_1 at time $T+1$. With probability $(N-2)p^2$, particles come to the same vertex, and this vertex is not any vertex containing one of these particles. The system will be in state G_1 in this case. With probability $2pq$, one of particles does not change vertex, and the other particle comes to the same vertex. In this case, the chain also comes to the state G_1. Hence,

$$p_{21} = (N-1)p^2 + 2pq, \ p_{22} = 1 - (N-1)p^2 - 2pq.$$

The chain consists of two communicating states. There exist steady state probabilities. Denote by p_i the steady probability of the state G_i, $i = 1, 2$. Steady probabilities satisfy the equation

$$(1 - q^2 - (N-1)p^2)p_1 = ((N-1)p^2 + 2pq)p^2,$$

$$p_1 + p_2 = 1.$$

Solution of this system is

$$p_1 = \frac{(N-1)p^2 + 2pq}{1+2pq-q^2}, \ p_2 = \frac{1-q^2-(N-1)p^2}{1+2pq-q^2}.$$

If the chain is in the state G_1 at present time, then there is no competition. If the chain is in the state G_2, then a competition takes place with probability p^2. The steady state probability of competition at current time equals

$$p^2 p_2 = \frac{p^2(1-q^2-(N-1)p^2)}{1+2pq-q^2}.$$

Suppose $H(T)$ is number of competitions in the time interval $(0, T)$. With probability 1, $\frac{H(T)}{T}$ tends as $T \to \infty$ to the steady probability of a competition at present time, (Borovkov, 1984). From this, the theorem follows.

Suppose $p = q = \frac{1}{N}$; then, from Theorem 11, we have the following statement.

Corollary 1. *Suppose each next vertex is chosen equiprobably. Then, with probability 1, the intensity of competitions in random pendulum equals* $\frac{N-1}{N(N^2+1)}$.

If $N = 2$, and $p = q = \frac{1}{2}$, in accordance with Theorem 11, we have $h = \frac{1}{2}$. This value of competitions intensity, in the case of bipendulum, has been found in (Kozlov, Buslaev, & Tatashev, 2015c), (Buslaev, & Tatashev, 2015).

8. Computer Simulation of Irrational Pendulums

8.1. Random Sequences of Digits and Random Pendulum

A number is called *normal* to base N if every sequence of k consecutive digits of N-ary representation appears with limiting probability N^{-k}, (Becher & Figueira, 2002). As it is noted in (Becher & Figueira, 2002), there exists widely conjectured opinion that the fundamental constants, like $\sqrt{2}$, π, e, are normal to every $q \geq 2$. However this statement has not been proved. The same hypothesis is also supposed to be true (Becher & Figueira, 2002) on the irrational algebraic numbers.

In (Kolmogorov, & Uspenskii, 1987), it was considered in what case a sequence of digits in N-ary representation of a real number can be regarded as if the value of every next digit is chosen with probability $1/N$ independent of the other digits of this plan and other plans. It is assumed that the number is defined with aid of an algorithm. In (Kolmogorov, & Uspenskii, 1987), approaches have been developed that allow to characterize the sequences, behaving as random sequences. However this problem is very hard to solve, and no exhaustive solution of this problem has been found.

It is obvious that normality of a digits sequence to any base is a necessary condition for the sequence can be regarded as a random sequence.

Suppose that a sequence of digits in N-ary representation of real valued number plans can be regarded as if every next digit is chosen with probability $1/N$ independent of the other digits of this plan and other plans. Behavior of this real valued pendulum is similar to the behavior of the random valued pendulum, considered in Section 7, except the following. Particles, losing competitions, try to come to the same vertex again. However the intensity of competitions will be the same as in the case of random pendulum. A stochastic process can be introduced for this real valued pendulum. This stochastic process is similar to the Markov chain considered in the proof of Theorem 11. Transition probabilities of this stochastic process do not depend on that whether a competition takes place at preceding instant or not, and, hence, the stochastic process is a Markov chain. Thus, if $M = 2$, then, in accordance with Corollary 3, the intensity of competitions equals

$$h = \frac{N-1}{N(N^2+1)},$$

and the particles tapes velocity equals

$$w_1 = w_2 = 1 - \frac{h}{2} = \frac{2N^3 + N + 1}{2N(N^2+1)}.$$

If $N = 2$, i.e., in the case of bipendulum, we have

$$h = \frac{1}{10},\ w_1 = w_2 = \frac{19}{20}.$$

We shall describe results of simulation experiments. In these experiments plans was given with aid of well-known irrational constants.

8.2. Irrational Plans

Rational plans are related to periodic conditions and irrational plans are related to chaotic conditions of movement.

If plans of particles are irrational numbers, then the system cannot be described with finite Markov chain.

Simulation experiments have been implemented. The plans were irrational numbers such as $\sqrt{2}(mod\ 1)$, $\sqrt{3}(mod\ 1)$, $\pi - 3$, $\sqrt{5}(mod\ 1)$, Figure 6.

8.3. Chaotic Behavior of the System in the Case of Irrational Plans

Let us describe results of experiments in the case $N = M = 2$ (bipendulum). The results of experiments show that the velocity of particles is equal to $\frac{19}{20}$ as in the case of the chaotic bipendulum. The simulation experiments are stable in the case of rational plans. The simulation experiments can be unstable in the case of irrational plans.

8.4. Phase Pendulums

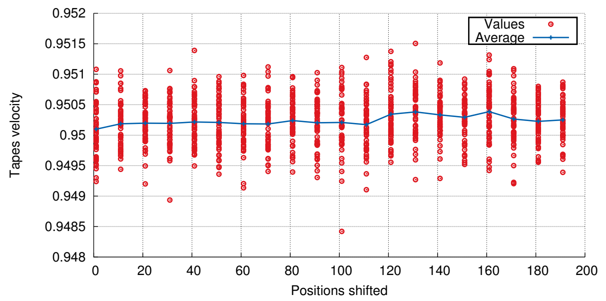

Figure 6. Logistic pendulum $\sqrt{2}(\mathrm{mod}\,1) - \sqrt{3}(\mathrm{mod}\,1)$

Let us get the plan a_2, shifting plan a_1 onto a fixed number of positions. If plans are rational numbers, then the system comes to the state of synergy after a time interval with a finite expectation. If plans are irrational numbers, then the system comes to the state of synergy, with probability 1, after a time interval interval T_{syn} with infinite expectation.

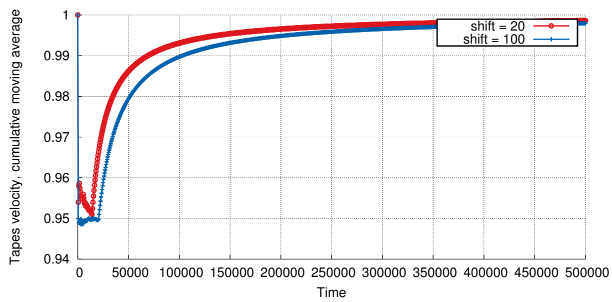

Figure 7. Phases shift $\sqrt{2}$ mod 1

The behavior of the bipendulum has been investigated with plans $a_1 = \sqrt{2}$ and a_2 such that we get a_2, shifting a_1 onto a fixed number of positions. The dependence of the average velocities on the time interval $(0, T)$ is shown in Figure 7. We suppose that the system comes to the state of synergy with probability 1 for a time interval. However the duration of this interval can be large.

9. Comments, and Further Research

(9.1) General transport-logistical problem is described in article (Buslaev, & Tatashev, 2015).

(9.2) Rational pendulum as $N = M = 2$ (bipendulum) is introduced considered in (Kozlov, Buslaev, & Tatashev, 2015c).

(9.3) Algebras with rational logistical plans are described in (Buslaev, & Tatashev, 2016). The Bernoulli algebra divides the triangle of rational numbers into subalgebras. If the plans belong to the same algebra, then the tape velocity of the bipendulum equals 1 *(the synergy)*.

(9.4) There is remarkable distinction between behavior of rational pendulums and behavior of irrational pendulums. In particular, the estimation of irrational bipendulum velocity over half a million characters is equal to the random walk bipendulum velocity.

(9.5) Irrational phase bipendulum converges also to synergy but with remarkably lower velocity.

(9.6) It is interesting to find exact lower bound of a real valued pendulum numeric characteristic when quantities of vertices and particles are equal to each other. In our paper, this problem has been solved in the case of 2 particles and any number of vertices.

(9.7) It is interesting to find the intensity competitions in the case of random pendulum with any number of particles. In our paper, this problem has been solved in the case of 2 particles.

(9.8) It is proved from random walk theory that on two dimension lattice any point is reached with 100% probability during infinite expectation time. In case of a random walk on a lattice with dimension more than two, the particle returns to given point with probability less than one. We may assume that, with positive probability, phase irrational pendulum with more than two particles is not synchronized for finite time.

(9.9) We use constructive approaches to determine irrational numbers. Problems of algorithmic construction of irrational numbers was considered in (Becher & Figueira, 2002), (Kolmogorov, & Uspenskii, 1987).

(9.10) Limits (1) –(3) do not exist for some irrational bipendulum .

(9.11) The model, considered in our paper, can be interpreted as a traffic model.

(9.12) We introduce the following classification of plans. The plan can be given by a fraction without non-repeating part in the simplest case. In more general case, there are both a non-repeating part and repeating part in the fraction which determines the plan. We use the concept of normal numbers to classify irrational plans. Normality of plans is not a sufficient condition for the particles velocity to equal $\frac{2N^3+N+1}{2N(N^2+1)}$. The sequences of plan symbols must seem independent.

(9.13) The dynamical system, studied in our paper, is similar to a physical system of interacting pendulums. This physical system is considered in (Mandelstam, 1972).

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