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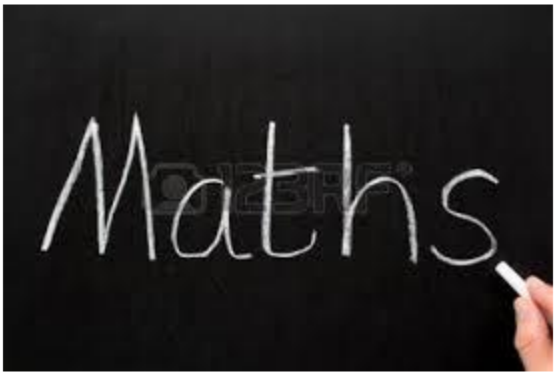
Practical Approaches to Developing **Mental Maths Strategies** for Multiplication and Division



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Contents

Introduction

This booklet is intended to support teachers in developing multiplication and division mental maths strategies in their classrooms. It has been designed to accompany the PDST Mental Maths workshops.

The booklet explores the key properties of number and number relationships relating to multiplication and division and outlines practical approaches to developing an understanding of these. It also explores background knowledge for teachers and fundamental facts in relation to mental maths.

A range of concrete, practical activities that will support pupils in their development of multiplication and division mental maths strategies are also outlined. Also included is a suggested alternative strategic sequence through which pupils may acquire their multiplication facts.

Finally, a selection of engaging and enjoyable activities to consolidate learning and provide opportunities for pupils to master multiplication and division facts is included.

Background Knowledge for Teachers

Stages of Progression

Arthur Baroody [1] identifies three stages through which children progress in order to acquire the basic facts of addition, subtraction, multiplication and division:

1. Counting Strategies: using object counting (for example blocks or fingers) or verbal counting to determine the answer. For example, with 3 × 7 pupil starts with 7 and skip counts on verbally 7, 14, 21.
2. Reasoning Strategies: using known information to logically determine an unknown combination. For example, with 3 × 7 pupil knows that double 7 is 14 and one more set of 7 is 21.
3. Mastery: efficient (fast and accurate) production of answers. For example, with 3 × 7, pupil quickly responds, *'It's 21; I just know it.'*

Three different ways of thinking about multiplication are:

- As repeated addition, for example 3 + 3 + 3 + 3
- As an array, for example four rows of three objects
- As a scaling factor, for example, making a line 3 cm long four times

(Crown: 2010, p.51)

Multiplication Symbol [2]

The use of the multiplication sign can cause difficulties. Strictly, 3 × 4 means four threes or 3 + 3 + 3 + 3. Read correctly, it means 3 multiplied by 4. However, colloquially it is read as '3 times 4', which is 4 + 4 + 4 or three fours. Fortunately, multiplication is commutative: 3 × 4 is equal to 4 × 3, so the outcome is the same. It is also a good idea to encourage children to think of any product either way round, as 3 × 4 or as 4 × 3, as this reduces the facts that they need to remember by half.

[1] Baroody, A (2006) *Why Children Have Difficulties Mastering the Basic Number Combinations and how to help them* p.22

[2] Crown (2010) *Teaching Children to Calculate Mentally* p.51

Known Facts [3]

A useful link between multiplication and addition allows children to work out new facts from facts that they already know. For example, the child who can work out the answer to 8 × 6 (six eights) by recalling 8 × 5 (five eights) and then adding 8 will, through regular use of this strategy, become more familiar with the fact that 8 × 6 is 48.

Teaching Multiplication and Division

Multiplication and Division are often taught separately, with multiplication preceding division. However division and multiplication are inverse operations. Every multiplication calculation can be replaced by equivalent division calculations and vice versa.[4] Therefore it is important to combine multiplication and division soon after multiplication has been introduced in order to help pupils see how they are related[5]

There are two concepts of division:

- The partition or fair-sharing idea, such as sharing 20 sweets among 4 children.
- The measurement or repeated subtraction concept. If you have €80 and you spend €5 per day, how long will your money last?[6]

Dealing with Remainders[7]

Real life problems often result in remainders. A remainder can be dealt with in a number of ways.

- The remainder is **discarded**, leaving a smaller whole-number answer.
- The remainder can "**force**" the answer to the next highest whole number.
- The answer is **rounded** to the nearest whole number for an approximate result.

Addressing what to do with remainders must be central to teaching about division.

[3] Crown (2010) *Teaching Children to Calculate Mentally* p.51
[4] Crown (2010) *Teaching Children to Calculate Mentally* p.51
[5] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.157
[6] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). *Elementary and Middle School Mathematics Teaching Developmentally.* 8th edn. Pearson: Allyn and Bacon p. 160
[7] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.157

Division using Arrays[8]

Using an array for a division problem such as 200 ÷ 8 can help pupils think about the known and the unknown components in division. The array can be used to show how to build up to the dividend when solving a division problem with the Think Multiplication or Partial Quotient strategies. Using 200 ÷ 8 we can account for partial areas of (8 × 10) + (8 × 10) + (8 × 5), until we have a total area of 200 (Parrish 2010: p.235).

	10	10	5
8	8 × 10 = 80	8 × 10 = 80	8 × 5 = 40

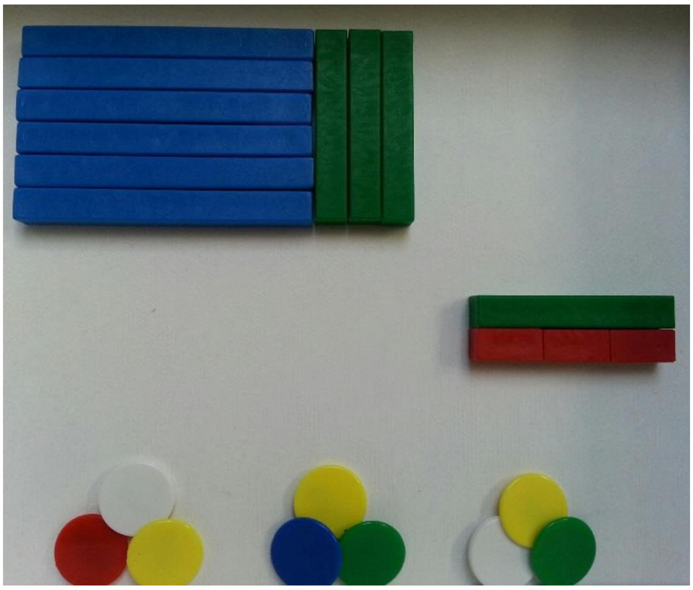
An array model is as important to multiplication and division as the number line model is to addition and subtraction. The visual representation of rows and columns help pupils as they develop their proportional reasoning. Like the part/whole box model for addition and subtraction, the array identifies the parts (factors) and the whole (total area of the product) and can be used to demonstrate and prove pupil strategies.

(Parrish 2010: p.233)

[8] Parrish, S. (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.235

Linear, Area and Set Model for Multiplication and Division

This manual advocates the linear, area and set models for developing concepts for multiplication and division calculations.



The **area (array)** model may appeal to spatial learners.

The **linear** model is compatible with logical and spatial learning styles.

Finally the **set** model allows for tangible and kinaesthetic learning experiences related to understanding multiplication and division.

All models allow for the use of manipulatives and concrete materials and transfer to both the pictorial and the abstract representations. This myriad of learning experiences for the development of the same concept means that different learning styles and abilities are catered for as well as providing repeated opportunities to consolidate learning in a fun and interactive way.

Pupils may be familiar with the concept of multiplication as repeated addition using set models and number lines, however to move pupils to more multiplicative ways of thinking the (area) array model is extremely important for both multiplication and division. An array is the arrangement of things in rows and columns, such as a rectangle of square tiles or blocks (or Cuisenaire). Pupils benefit from activities with models to focus on the meaning of the operation and the associated symbolism (adapted from Van de Walle, 2013, p. 162).

Possible Pupil Misconceptions

Zero

From pupils' experiences of multiplication, they will often notice that when whole numbers are multiplied, what results is a bigger number. Similarly, when whole numbers are divided, a smaller answer occurs. However, when numbers are multiplied by zero, the result is very different.

Rote learning a rule such as 'when multiplying or the answer will always be zero' is **not** recommended. Instead, try posing meaningful problems to pupils and discussing the results. For example, 'How many grams of fat are there in 7 servings of celery, if the celery has 0 grams of fat in each serving?'[9]

A concrete approach to exploring the zero property of multiplication would involve asking pupils to model 6 × 0 or 0 × 8 with an array. Similarly, exploring number line problems involving zero are very worthwhile. For example, what would 5 hops of zero be? What would 0 hops of 5 be?[10]

It is important to avoid an arbitrary rule in relation to division by zero. Some pupils are simply told '*Division by zero is not allowed'* without any understanding as to why this is the case. Instead, pose problems to be modelled that involve zero. For example, 'Take thirty counters. How many sets of zero can be made?' Or, 'Put twelve blocks in zero equal groups. How many in each group?'[11]

Language of Division

Using the phrase *'goes into'* when discussing division is often very confusing for pupils and can lead to needless confusion. The phrase carries little meaning about division concepts, especially in

[9] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.160

[10] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.160

[11] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.161

connection with fair sharing or partitioning. This *'guzzinta'* terminology[12] is simply engrained in adult parlance. Its use is not recommended.

[12] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.157

Uses of Mental Calculation

The following six aspects of mathematics require the use of mental calculation[13]

Recalling Facts:	Applying Facts:
• What is 3 add 7? • What is 6 × 9? • How many days are there in a week?... in four weeks? • What fraction is equivalent to 0.25? • How many minutes in an hour? ... in six hours?	• Tell me two numbers that have a difference of 12. • If 3 × 8 is 24, what is 6 × 0.8? • What is 20% of €30? • What are the factors of 42? • What is the remainder when 31 is divided by 4?

Hypothesising or Predicting:	Designing and Comparing Procedures:
• The number 6 is 1 + 2 + 3, the number 13 is 6 + 7. Which numbers to 20 are the sum of consecutive numbers? • Roughly, what is 51 times 47? • On a 1 to 9 keypad, does each row, column and diagonal sum to a number that is a multiple of 3?	• How might we count a pile of sticks? • How could you subtract 37 from 82? • How could we test a number to see if is divisible by 6? • How could we find 20% of a quantity? • Are these all equivalent calculations: 34 – 19; 24 – 9; 45 – 30; 33 – 20; 30 – 15?

Interpreting Results:	Applying Reasoning:
• So what does that tell us about numbers that end in 5 or 0? • Double 15 and double again; now divide your answer by 4. What do you notice? Will this always work? • I know 5% of a length is 2 cm. What other percentages can we work out quickly?	• The seven coins in my purse total 23c. What could they be? • In how many different ways can four children sit at a round table? • Why is the sum of two odd numbers always even?

[13] Crown (2010) *Teaching Children to Calculate Mentally*.p.22

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Fundamental Facts for Multiplication and Division

The Commutative Property

Numbers can be multiplied in any order for example 7 × 12 = 12 × 7. This property is quite useful in problem solving, mastering basic facts and in mental maths. Therefore, it is important to spend some time helping pupils to construct the relationship. The **array model** is the most useful for demonstrating the commutative property.[14] This relationship applies to multiplication but not to division. In division, order does matter, for example, 6 ÷3 ≠ 3÷6.

The Associative Property

When three or more numbers are multiplied together, they can be multiplied in any order. It is a useful property for children to understand as it allows them to multiply combinations of numbers they know. In practice, two of the numbers have to be multiplied together or associated first, and then a third number is multiplied to the associated pair for example 7 × 5 × 3 = (7 × 5) × 3 = 7 × (5 × 3) = (7 × 3) × 5

The Distributive Property

The distributive property of multiplication over addition, refers to the idea that either one of the two factors in a product can be split, (decomposed) into two or more parts and each part multiplied separately and then added. The result is the same as when the original factors are multiplied. For example, 9 × 6 is the same as (5 × 6) + (4 × 6). The 9 can be split or partitioned into 5 and 4, the concept involved is very useful in relating one basic fact to another, and it is also involved in the development of two-digit computation.[15] (See p. 21 for practical activities to develop further understanding and application of this property).

Inverse Relationship

Every multiplication calculation can be replaced by an equivalent (corresponding/matching) division calculation and vice versa for example 5 × 7 = 35 implies 35 ÷ 5= 7 and 35 ÷ 7= 5[16]

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[14] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.160

[15] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.161

[16] Crown (2010) *Teaching Children to Calculate Mentally.*p.51

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Teaching and Learning

Suggested Multiplication & Division Strategies

- Skip Counting
- Doubles 2s, 4s, 8s
- Doubling & Halving
- Repeated Addition/Subtraction
- Partitioning by Place Value
- Break Down Factors/ Factorisation
- Rounding & Compensating
- Using Known Facts
- Multiplying/Diviiding by multiples of ten
- Partial Products/Partial Quotients
- **Division**
  - Think multiplication
  - Proportional Adjustment/Reasoning

It is important to note that although pupils may develop some of these strategies earlier than others, they are not incremental and strategies can be used interchangeably (and sometimes in conjunction with each other) based on a number of factors e.g.:

- The age of the pupil
- The numbers involved
- Learning style etc.

Instructional Framework

Table 1.1 on the following page illustrates a framework for advancing mathematical thinking. Although it does not explicitly refer to concrete materials or manipulatives, the use of these are often a prerequisite for developing mathematical thinking and can be used as a stimulus for this type of classroom discourse.

Table 1.1 Strategies for Supporting and Developing Mathematical Thinking

Eliciting	Supporting	Extending
Facilitates pupils' responding Elicits many solution methods for one problem from the entire class *e.g. "Who did it another way?; did anyone do it differently?; did someone do it in a different way to X?; is there another way of doing it?"* Waits for pupils' descriptions of solution methods and encourages elaboration Creates a safe environment for mathematical thinking *e.g. all efforts are valued and errors are used as learning points* Promotes collaborative problem solving *Orchestrates classroom discussions* Uses pupils explanations for lesson's content Identifies ideas and methods that need to be shared publicly *e.g. "John could you share your method with all of us; Mary has an interesting idea which I think would be useful for us to hear."*	*Supports describer's thinking* Reminds pupils of conceptually similar problem situations Directs group help for an individual student through collective group responsibility Assists individual pupils in clarifying their own solution methods *Supports listeners' thinking* Provides teacher-led instant replays *e.g. "Harry suggests that ...; So what you did was ...; So you think that ...".* Demonstrates teacher-selected solution methods without endorsing the adoption of a particular method *e.g. "I have an idea ...; How about ...?; Would it work if we ...?; Could we ...?".* *Supports describer's and listeners' thinking* Records representation of each solution method on the board Asks a different student to explain a peer's method *e.g. revoicing*	*Maintains high standards and expectations for all pupils* Asks all pupils to attempt to solve difficult problems and to try various solution methods *Encourages mathematical reflection* Facilitates development of mathematical skills as outlined in the PSMC for each class level *e.g. reasoning, hypothesising, justifying, etc.* Promotes use of learning logs by all pupils *Goes beyond initial solution methods* Pushes individual pupils to try alternative solution methods for one problem situation Encourages pupils to critically analyse and evaluate solution methods *e.g. by asking themselves "are there other ways of solving this?; which is the most efficient way?; which way is easiest to understand and why?".* Encourages pupils to articulate, justify and refine mathematical thinking *Revoicing can also be used here* Uses pupils' responses, questions, and problems as core lesson including student-generated problems *Cultivates love of challenge*

This is adapted from Fraivillig, Murphy and Fuson's (1999) Advancing Pupils' Mathematical Thinking (ACT) framework.

Classroom Culture

Creating and maintaining the correct classroom culture is a pre-requisite for developing and enhancing mathematical thinking. This requires the teacher to:

- cultivate a 'have a go' attitude where all contributions are valued;
- emphasise the importance of the process and experimenting with various methods;
- facilitate collaborative learning through whole-class, pair and group work;
- praise effort;
- encourage pupils to share their ideas and solutions with others;
- recognise that he/she is not the sole validator of knowledge in the mathematics lesson;
- ask probing questions;
- expect pupils to grapple with deep mathematical content;
- value understanding over 'quick-fix' answers; and
- use revoicing[17] (reformulation of ideas) as a tool for clarifying and extending thinking.

In this type of classroom pupils are expected to:

- share ideas and solutions but also be willing to listen to those of others; and
- take responsibility for their own understanding but also that of others.

[17] Revoicing is 'the reporting, repeating, expanding or reformulating a student's contribution so as to articulate presupposed information, emphasise particular aspects of the explanation, disambiguate terminology, align students with positions in an argument or attribute motivational states to students' (Forman & Larreamendy-Joerns, 1998, p.106).

Problems and Solutions in Supporting and Developing Mathematical Thinking

Table adapted from Chapin, O'Connor and Anderson (2013) Classroom Discussions In Math: A Teacher's Guide for Using Talk Moves to Support the Common Core and More. Math Solution

Problem	Solution
Only one pupil volunteers to talk – the pupil who always has the right answer	Try the **Eliciting** moves: *wait time, turn and talk* or *stop and jot*, after *turn and talk* or *stop and jot*, ask '*will you share that with the class?'*
Problem	**Solution**
The pupil's answer is wrong, and the teacher isn't clear on what the pupil is thinking	Try the **Eliciting** move *say more….*
Problem	**Solution**
What the pupil is trying to say just gets more confusing as he or she attempts to say more	Try the **Supporting** move revoicing, *so you are saying?*
Problem	**Solution**
The teacher isn't sure that the other pupils have heard and understood a pupil's idea	Try the **Supporting** move revoicing, *who can revoice…?*
Problem	**Solution**
The teacher feels that a pupil needs to take his or her reasoning deeper. How can this happen?	Try one of the **Extending** moves, *why do you think that?*
Problem	**Solution**
Other pupils may be tuning out as one pupil focuses deeper on his or her reasoning. How can the teacher help everyone deepen their own understanding?	Try the **Supporting** move revoicing and say, *'who can put that into their own words?'*
Problem	**Solution**
The teacher needs to engage pupils beyond simply listening and repeating	Try the **extending** move, *do you agree or disagree…and why?*
Problem	**Solution**
How do we invite pupils in when we sense that the conversation is clear enough and there is enough common ground to move forward?	Try the **extending** move, *who can add on?*

Possible Resources

Counters	Counting Stick
Interlocking Cubes	Multiplication Grid
Coins	Arrays
100 squares	99 squares
Place Value Arrow Cards	Cuisenaire Rods
Empty Number Lines	Ten Frames
Interactive Whiteboard	Dice with various numbers of faces
Playing Cards	Calculators
Dominoes	Target Boards
Digit Cards	Number Fans
Individual white boards	Dotted/Grid/Square Paper

Key Teaching Principles for Mental Maths[18]

- Encourage children to share their mental methods.
- Encourage children to choose efficient strategies.
- Encourage children to use informal jottings to keep track of the information they need when calculating.
- Commit regular time to teaching mental calculation strategies.
- Provide practice time with frequent opportunities for children to use one or more facts that they already know to work out more facts.
- Introduce practical approaches and jottings, with models and images children can use, to carry out calculations as they secure mental strategies.
- Encourage children in discussion when they explain their methods and strategies to you and their peers.
- Ensure that children can confidently add and subtract and multiply and divide any pair of two-digit numbers mentally, using jottings to help them where necessary.
- Teach a mental strategy explicitly but in addition invite children to suggest an approach and to explain their methods of solution to the rest of the class.

[18] Adapted from Crown (2010)

- Hands on learning is important.
- Provide suitable equipment for children to manipulate and explore how and why a calculation strategy works. That helps them to describe and visualise the method working.
- Encourage children to discuss their mistakes and difficulties in a positive way so that they learn from them.

Assessment

- A 'mental test' can help children to monitor changes in their own performance over time. The traditional mental arithmetic test involves a set of unseen questions. A worthwhile alternative is to give children examples of the type of questions 10 minutes in advance, so that they can think about the most efficient way to answer the questions. The purpose of this preparation time is not to try to commit answers to memory but to sort the questions into those they 'know' the answer to, and those that they need to figure out. Pairs of children can talk about their 'figuring out' methods and after the test the whole class can spend some time discussing the strategies they used. [19]
- Collecting the questions, then giving children the test with the questions in a random order, also encourages attention to strategies. The same test can be used at a different time for children to try to beat their previous score. [20]
- Don't use lengthy timed tests. Pupils get distracted by the pressure and abandon their reasoning strategies. They can lead to pupil anxiety, which does not support mathematical learning. If there is any purpose for a timed test of basic facts it may be for diagnosis – to determine which combinations are mastered and which remain to be learned. Even for diagnostic purposes timed tests should only occur once every couple of weeks. [21]

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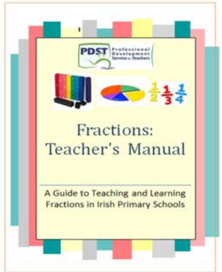
[19] Crown (2010) *Teaching Children to Calculate Mentally* p.19

[20] Crown (2010) *Teaching Children to Calculate Mentally* p.19

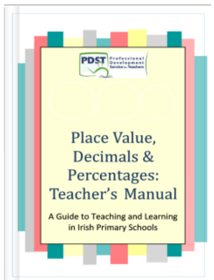
[21] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.184

Teacher Manuals for Supporting and Developing Mathematical Thinking

The instructional framework for supporting and developing pupils' mathematical thinking is described in detail in the following three PDST Manuals. Click on each image to access a free e-copy of the manuals.

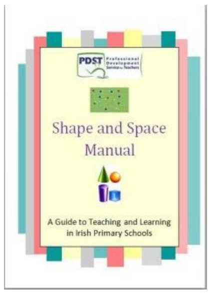


- *Fractions: Teacher's Manual, A Guide to Teaching and Learning Fractions in Irish Primary Schools*

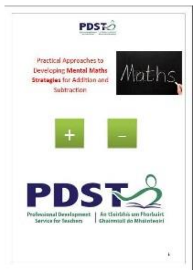


- *Place Value, Decimals and Percentages: Teacher's Manual, A Guide to Teaching and Learning in Irish Primary Schools*

- *Shape and Space: Teacher's Manual, A Guide to Teaching and Learning in Irish Primary Schools*



- *Mental Maths: Practical Approaches to Developing Mental Maths Strategies for Addition and Subtraction*



Practical Strategies to Develop Multiplication and Division Properties

Although mental maths is often used in the abstract, a solid foundation in number properties is central to pupils' success in developing and applying mental maths strategies. The following activities that explore the development of number properties are included as pre-requisites for the development of mental maths strategies.

Commutative Property

There are a number of concrete ways of constructing the commutative property of multiplication with pupils:

Cuisenaire Rods [22] [23]

- How can you show $2 \times 4 = 4 \times 2$?
- Make a purple rectangle consisting of 2 purple rods.
- Place the purple rectangle on centimetre-squared paper to show that the area is 8 sq cm.

- Make a red rectangle consisting of 4 red rods.
- Place the red rectangle on cm-squared paper.
- Each rectangle covers the same area.
- Place the purple rectangle on top of the red rectangle to show that 2×4 and 4×2 are exactly equal.

- Now try these: **$3 \times 9 = 9 \times 3$**

How could we use our Cuisenaire rods to work out this sum? What do you notice when you use the blue rods? Can you try this sum with green rods? What happens if you put the blue rods over the green rods? Does anyone know why multiplication works in both directions?
Did your answer change?
Let's try that with some other sums to check if it always works. What does that tell us about the way we can multiply two numbers? We have to put our materials away soon. Can you suggest a way of recording what we discovered? Record what you have learned in your learning log.

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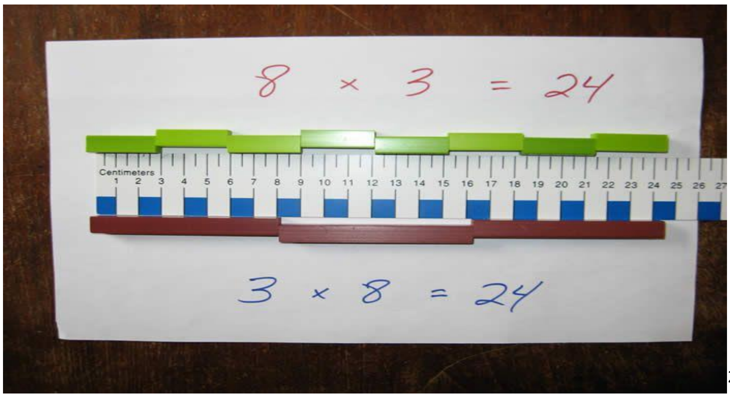
[22] Western Education and Library Board, Curriculum Advisory & Support Service Multiplication, Division and Fractions with Cuisenaire. Available at: cass.welbni.org/.../46/79_6_P3%20and%20%20P4%20Resources.doc.
[23] Image: http://highhillhomeschool.blogspot.ie/2014/02/multiplication-and-square-numbers-with.html



[24]

In addition, a **linear model** to represent the same question ensures pupils have multiple experiences with the same calculation which helps to consolidate understanding of the commutative property of multiplication.

ICT Opportunities
[Interactive Cuisneaire]

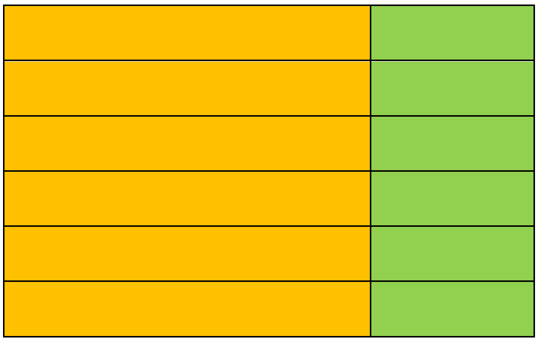
Distributive property of multiplication[25]

Cuisenaire Rods

How would you show the multiplication sum 6 × 13 using Cuisenaire rods?

10 3

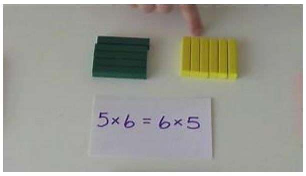
6



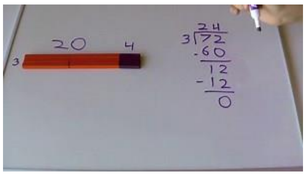
[24] Image: http://www.maththings.net/MultiplicationandDivision.htm
[25] Western Education and Library Board, Curriculum Advisory & Support Service Multiplication, Division and Fractions with Cuisenaire. Available at: cass.welbni.org/.../46/79_6_P3%20and%20%20P4%20Resources.doc.

6 x 13 sounds like a tricky sum, I wonder is there any way we could break it down? How could we partition 13? Yes 10 and 3. What about showing 10 x 6 using rods? That makes 60. Now can you show the other factor 3 x 6 using rods? That makes 18. What do you get if you put them together? Yes 78. Are there any other ways to show this sum? Find all the ways to show this sum using Cuisenaire.

The following video shows how Cuisenaire Rods can be further used to develop multiplication concepts:



The following video shows how Cuisenaire Rods can be used effectively to develop division concepts:

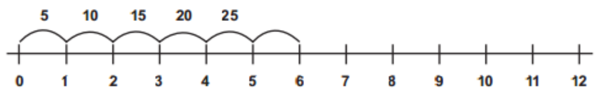


The following video shows how Cuisenaire Rods can be used to effectively to explore the Distributive property:

Practical Activities for developing Multiplication & Division Strategies

Skip counting[26]

Students who can skip count by rote can use skip counting sequences to work out answers to multiplication facts. For example, to work out 6 × 5, students can count in 5s along six of their fingers or along six jumps on a number line.



The clock face

Pupils who are familiar with the concept of five-minute intervals on an analogue clock face and are competent at skip counting in 5s will find the clock a useful mnemonic for practising the 5s facts. Discuss the relationship between the numbers on the clock and the minutes they represent.

Counting Stick

The counting stick can be used to model skip counting and scaffold children's understanding of multiplication facts. Using facts the children know such as × 10 can help to provide anchor points and provide encouragement. Halving the 10 fact will produce the × 5 fact. Doubles can also be used for × 2 and × 4 and × 8, leaving less for children to memorise as instead they are applying known facts to unknown facts. Initially post-it notes in whatever multiplication fact you are using can be used to scaffold the children, and removed as they gain fluency in a particular set of facts. (See p.58 for video footage of the counting stick in use for multiplication facts)

[26] Mental Computation Strategies, addition, subtraction, multiplication and division. Available at: http://www.currumbiss.eq.edu.au/Restricted/Currumbin%20numeracy/MentalComp.pdf

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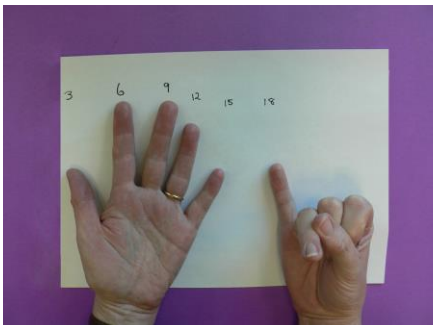
See p.57 for more counting strategies for multiplication and division.

Skip Counting for Division

Linking with multiplication, a strategy for division is to skip count to the number being shared and use fingers to record the count. This example shows skip counting on fingers to find 18 ÷ 3. There are six fingers, so 18 ÷ 3 = 6 [27]



Multiplication Strategy: Repeated Addition

Repeated Addition is regarded as an entry-level strategy for multiplication, therefore caution is advised in using this strategy, as we want pupils to move towards more advanced multiplicative ways of thinking.

(Parrish: 2010, p.265)

Repeated Addition builds on the concept of skip counting. Here pupils can add groups together to produce a total. Real life word problems can be a good context to help engage pupils as well as

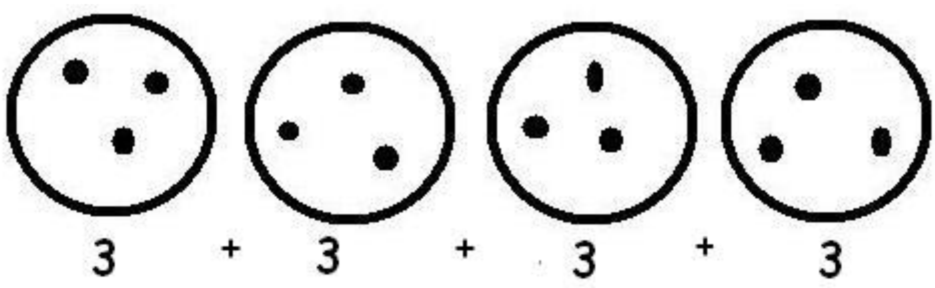
[27] Mental Comp.pdf p.44

attaching meaning to the numbers in the problem and providing multiple access points for different pupils. [28]

Set Model for Repeated Addition

For example, to solve the problem **3x4**, a pupil can represent concretely or pictorially an image of 4 plates with 3 sweets on each. They may skip count in 3s to make 12, or use addition knowledge to add

3 + 3 = 6,   6 +3 = 9,   9 + 3 =12



[29]

This represents a set model approach to working out multiplication calculations, using repeated addition knowledge.

Can anyone think of a way we could use our plates and counters to work out this problem? What facts do we know? Good idea we could put 3 sweets on each plate. How many plates do we need? Yes 4 plates, because 3 × 4 means 4 groups of 3. What could you do to work out the total number of sweets? Yes you could add them together. Another way of saying this is to use our skip counting and count up in threes. '3, 6, 9, 12', so there are 12 sweets in total or 4 groups of 3 makes 12.

[28] Parrish, S. (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.237

[29] Image: http://www.abc123kidz.com/images/arrayrepadd.JPG

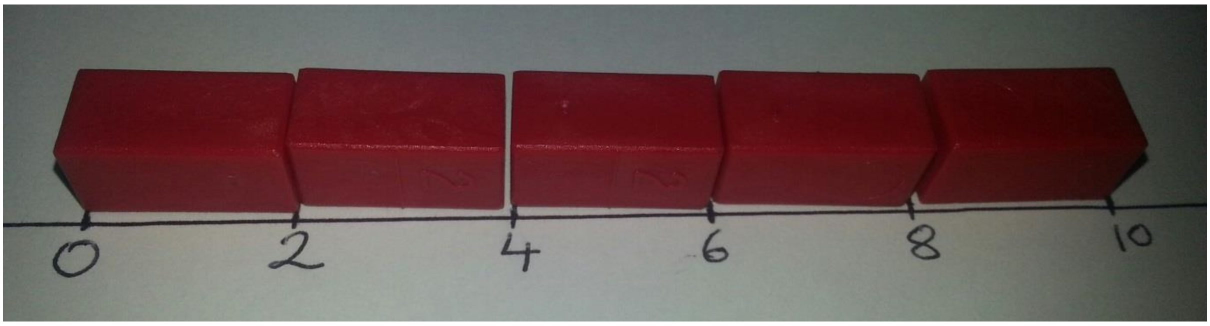
Repeated Subtraction for Division

Similar to **Repeated Addition for Multiplication** above, **Repeated Subtraction** is an entry-level strategy for division and will naturally occur when pupils are presented with initial division problems. Pupils should be encouraged to move towards multiplicative ways of thinking rather than a removal approach to division.

(Parrish: 2010, p.287)

For example, I have 10 stickers to use before the end of class. If I give out 2 stickers for each correct answer, how many children can get a sticker? $10 \div 2 = ?$

Pupils may suggest 10 -2 -2 -2 -2 -2 (repeated subtraction)



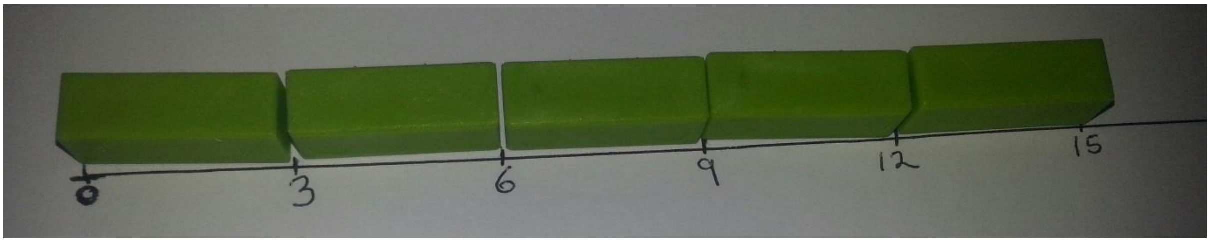
Scaffold pupils' thinking to making groups of 2 which links with their prior multiplication knowledge. 5 groups of 2 makes 10, so $5 \times 2 = 10$, therefore $10 \div 2 = 5$

Use the Cuisenaire to show all the stickers I have, that's 10, what could you do then to show teacher giving them away? You could take off a two rod each time. How many times could you do that? So five children got two stickers each. What other way could we show the children getting the stickers? We could show five groups of 2. What's another way to say that? Yes 2 x 5 = 10. Another way to say that is 10 ÷ 2 = 5. Let's take a photograph of our work. Could you draw this problem in different ways in your learning log?

Division as Repeated Subtraction (Sharing/Dealing out) - Linear Model

In a similar way Cuisenaire rods can be used in a linear model to explore the concept of division (as sharing out), and to explore the links between multiplication and division. The number line is an effective way of modelling division as repeated subtraction, or grouping.[30]

For example: I have 15 sweets to share between 3 children, how many will they get each? 15 ÷ 3



When the moves are recorded as an abstract division sentence, as in 15 ÷ 3 = 5 it is important to discuss what each number in the equation represents.

Do the children appreciate that division is a quick way of doing repeated subtraction?

[30] Draft Multiplication Strategies WELB CASS Numeracy Team p 14

Can you show all the sweets using Cuisenaire Rods? Which rods did you use? So 5 of the green rods makes 15. What's another way to say this? 5 × 3 = 15. How could you show sharing the sweets by taking away rods? Could you find a way to record that? 15 - 3 = 12, - 3 = 9, - 3 = 6, - 3 = 3, - 3 = 0

15 – 3 – 3 – 3 – 3 – 3 = 0

3 + 3 + 3 + 3 + 3 = 15

5 × 3 = 15 or 15 ÷ 3 = 5 [31]

Division Arrays

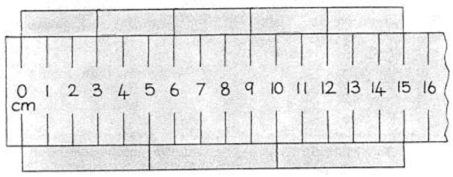
Use of arrays to advance pupils towards the use of more effective strategies for division, will lead to less dependence on the repeated subtraction strategy. See the following Virtual Manipulative to explore this area model to develop understanding.

ICT Opportunities
[Division Arrays]

[31] Draft Multiplication Strategies WELB CASS Numeracy Team p. 14

Division as inverse multiplication with Cuisenaire Rods[32]

In a similar way Cuisenaire rods can be used to explore the concept of division, and to explore the links between multiplication and division. The number line is an effective way of modelling division as repeated subtraction, or grouping.



How many 3-rods (how many 'threes') are needed to make 15?

Can we line up the rods to make 5 x 3 = 15?

Can we take each 3-rod away, one at a time, to make

15 - 3 = 12, - 3 = 9, - 3 = 6, - 3 = 3, - 3 = 0

When the moves are recorded as an abstract division sentence, as in 15 ÷ 3 = 5 it is important to discuss what each number in the equation represents.

Do the children appreciate that division is a quick way of doing repeated subtraction?

15 – 3 – 3 – 3 – 3 – 3 = 0

3 + 3 + 3 + 3 + 3 = 15

5 x 3 = 15

15 ÷ 3 = 5

[32] Draft Multiplication Strategies WELB CASS Numeracy Team

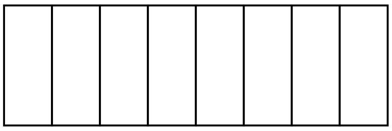
Division as inverse multiplication- Area

- How could we show 24 divided by 8?
- Explore different ways of expressing this

E.g.: 24 buns shared between 8 children gives each child 3 buns

24 euros shared between 8 girls means that each girl gets 3 euros…

Use rods to show 24 ÷ 8 = 3

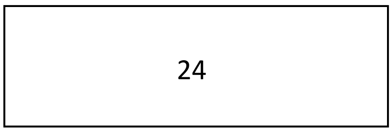


1 group of 3 = 3

2 groups of 3 = 6

3 groups of 3 = 9

8 groups of 3 = 24



8

3 | 24

Point out the connection between multiplication and division, but more importantly, the connection between the concrete and the abstract.

36 ÷ 4 = ☐

Explore different ways of expressing this, in words and using rods.

Multiplication & Division Fact Families

From each of the following number trios make a multiplication and division fact family.

7 28 4

63 7 9

10 9 90

72 8 9

7 49 7

180 9 20

Multiplication Strategy: Doubles

Pupils should be encouraged to think about doubling if any of the numbers in a fact is a 2, 4 or 8. For example, to calculate 28 × 4, pupils can work out 28 × 2 and double the answer. Similarly, to calculate 15 × 8, pupils can work out 15 × 2, double the answer and double it again, enabling pupils to work out 4s and 8s calculations easily from their doubles knowledge.

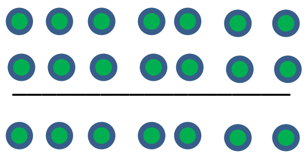
Facts that have 2 as a factor are equivalent to the addition doubles and should already be known by pupils who have their addition facts.[1]

(Van de Walle: 2013, p.178).

Double and one more set[33]

The double and one more set strategy is a way to think of facts with one factor of 3. With an array or a set picture, the double part can be circled, and it is clear that there is one more set.

3 x 7 =



7 Double 7

7

7 One more 7

Double 7 is 14

One more 7 is 21.

Pupils should be enabled to see the relationship between doubles and one set more or one set less, for example 21 × 3 can be calculated by doubling 21 and adding one more set of 21.

[33] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). Elementary and Middle School Mathematics Teaching Developmentally. 8th edn. Pearson: Allyn and Bacon. p.183/184

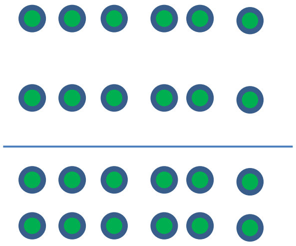
Double and Double Again[34]

The double and double again strategy is applicable to all facts with 4 as one of the factors. Remind pupils that the idea works when 4 is the second factor as well as the first.

4 x 6 =

Double 6 = 12

Double again = 24.



It is imperative that pupils completely understand and use the commutative property. This can be visualised by using arrays. For example, 2 x 8 can be described as 2 rows of 8 or 8 rows of 2. Understanding the commutative property cuts the basic facts to be memorised in half! Multiplication arrays are a useful model for developing an understanding of this property.

ICT Opportunities

Multiplication Arrays

Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). . 8th edn. Pearson: Allyn and Bacon. P.183/184

Multiplication Strategy: Known Facts

Numerous facts and calculations can be derived by relating the calculation to an already known fact or *helping* fact. For example if pupils know their doubles facts, they can use these known facts to generate facts for × 3 and × 4 [35]

Pupils should be encouraged to represent doubles and near doubles calculations using array models with materials such as counters and squared paper.

Multiples of 10 (links to facts for doubles, known facts and counting stick activities)

Being able to multiply by 10 and multiples of 10 depends on an understanding of place value and knowledge of multiplication and division facts. This ability is fundamental to being able to multiply and divide larger numbers.[36]

A common pupil-generated strategy is to use the 10s facts to work out the 5s facts (by doubling and halving).

For 5 × 6 'think double 5 and halve the six to make 10 by 3'.

For 7 × 5 'think double 5 to make 7 by 10, then halve the answer (half of 70 is 35).[37]

ICT Opportunities

[Making Arrays]

Rounding and Compensating Strategy[38]

Children and adults look for ways to manipulate numbers so that the calculations are easy. For example-see next page.

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[35] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.184

[36] Crown (2010) *Teaching Children to Calculate Mentally*. P. 63

[37] Mental Computation Strategies, addition, subtraction, multiplication and division. Available at: http://www.currumbiss.eq.edu.au/Restricted/Currumbin%20numeracy/MentalComp.pdf p. 35

[38] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.227

27 × 4	**17 × 70**
27 + 3 = 30	**17 + 3 = 20**
30 × 4 = 120	20 × 70 = 1400
3 × 4 = 12 –	3 × 70 = 210 –
108	1190

Rounding and Compensation for division involves finding a number that is close to the total, and working from that number to find an answer.

Pose the following division problem to pupils:

Sarah uses eight bus tickets every week to travel around town. She wins 152 bus tickets in a radio competition. How long will they last her?[39]

What strategy did you use to work this out? It's quite difficult to do the operation 152 ÷ 8 in your head. Can you change the numbers to make them easier or friendlier to work with? Would it help to round the big number? What could you round it to? Up to the nearest ten? Okay, so that would make 160. It's easier to divide 160 by 8. So that gives us 20. Is that our final answer? Why not? What do we need to do to work out our final answer? Why? So our final answer is 20 weeks – 1 week (8 tickets).

Further division examples to apply this strategy:[40]

343 ÷ 7
198÷ 9
1194 ÷ 6
686 ÷ 7
1764 ÷ 18

[39] http://nzmaths.co.nz/resource/multiplication-and-division-pick-n-mix-1
[40] http://nzmaths.co.nz/resource/multiplication-and-division-pick-n-mix-1

Making Friendly Numbers Strategy[41]

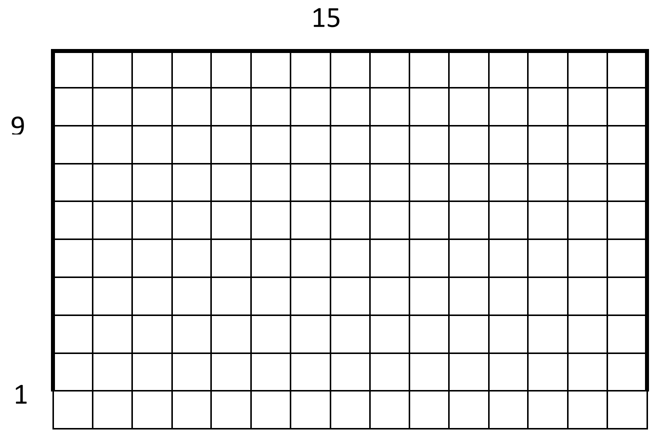
This strategy is closely related to Rounding and Compensating.

Often a multiplication problem can be made easier by changing one of the factors to a friendly or landmark number. Pupils who are comfortable multiplying by multiples of ten will often adjust factors to allow them to take advantage of this strength.

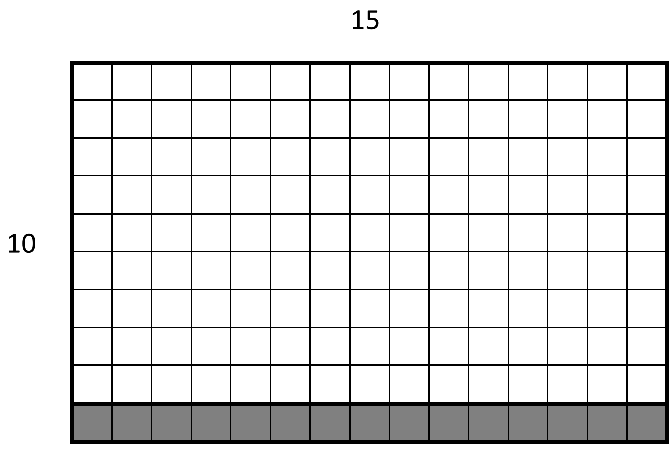
9 × 15
+1 (group of 15)
10 × 15 = 150

150 – 15 = 135

With this strategy, notice that not one, but one group of 15 was added. This is a very important distinction for pupils and one that comes as they develop multiplicative reasoning.



Since one extra group of 15 was added, it now must be subtracted. The initial problem was 9 × 15, but it was changed to 10 × 15, which resulted in an area of 150 squares.



The extra group of 15 is subtracted from the total area to represent the product for 9 × 15.

[41] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.247

Try to pose number calculations as word problems whenever possible to encourage the acquisition, development and application of problem solving skills.

Partitioning Strategies[42]

Remember: All partitioning strategies rely on the distributive property. (See p.11 for further information on this property).

(Van de Walle et al: 2010, p.227)

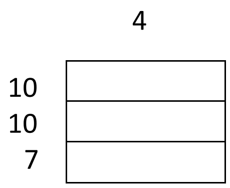
Children break up numbers in a variety of ways that reflect an understanding of base-ten concepts. The following examples illustrate four of the possible partitions for multiplication and their corresponding division application.

By Decades

27 × 4	**108 ÷ 4**
4 × 20 = 80	100 ÷ 4 = 25
4 × 7 = 28 +	8 ÷ 4 = 2 +
108	27

By Tens and Ones

27 × 4
10 × 4 = 40
10 × 4 = 40
7 × 4 = 28 +
108



The array supports pupils in modelling this partitioning strategy

[42] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2010). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.227

Partitioning the Multiplier – Multiplication only

It is also possible to partition the multiplier only in a multiplication problem. For example, 32 × 5 = (32 × 3) + (32 × 2) = 66 + 44 = 110. It is important that pupils discover that this strategy is applicable to multiplication problems only and not to problems involving division.

Partial Products[43]

Partial Products is based on the distributive property and keeps place value intact. It closely resembles the standard vertical algorithm for multiplication. When pupils understand that the factors in a multiplication problem can be decomposed or broken apart into addends, this allows them to use smaller problems to solve more difficult ones. [44]

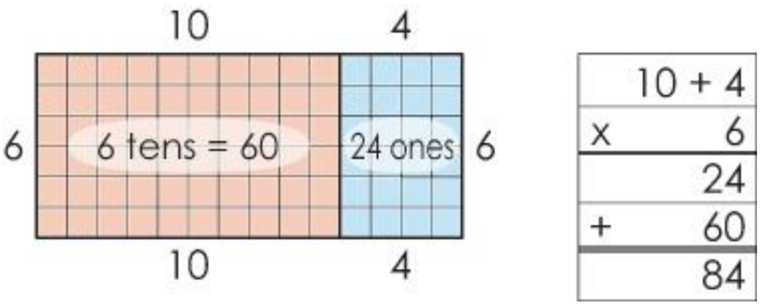
See p.21 for further examples of using Cuisenaire to model the distributive property of multiplication.

Array Model

The array model is an excellent way to help students think about multiplying when breaking factors apart and providing proof for their reasoning. [45]

This example demonstrates changing 6 × 14 to (6 × 10) + (6 × 4) using an array model.

[43] Parrish, S. (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.248
[44] Parrish, S. (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.248
[45] Parrish, S. (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.250



[46]

Open Array

Once children have had plenty of experience modelling the distributive properties of multiplication using Cuisenaire or blocks, they should be encouraged to draw their models on squared paper to maintain the integrity of the size of the arrays. Further abstraction of this concept will enable them to use open arrays, which means drawing a rectangle (not necessarily to scale) to show how the partial products or distributed factors still multiply to give the same product or answer. For example, to solve the problem **8 × 25**, pupils can show how breaking up the problem into smaller parts and multiplying **(8 × 20) + (8 × 5)** will yield the total area of the rectangle.

Has anyone any ideas how we could draw out this problem? We could use squared paper or we also know we can simply draw a rectangle. What factor could we put on the short side of the rectangle? Yes 8. Could we break 25 into factors to make an easier calculation? Yes 20 and 5. Can you show that now on your rectangles? What strategies can you use to work out 8 × 20? Now what's 8 × 5? What do we do with then? Well done, we add them to get the total area of the rectangle which is 200.

[46] http://illuminations.nctm.org/uploadedImages/Content/Lessons/Images/3-5/2x1.jpg

	20	5
8	**8 × 20**	**8 × 5**

When pupils have had experience using Cuisenaire or other materials to model partial product calculations concretely, as well as representing these pictorially on squared paper and through open arrays, they will then be ready to move towards a more abstract representation of the partial products calculations, which is closely related to the standard algorithm for multiplication.

12 × 15 [47]

Horizontal	Vertical
12 × 15	15
12 × (10 + 5)	× 12
12 × 10 = 120	120 (12 × 10)
12 × 5 = 60	+ 60 (12 × 5)
120 + 60 = 180	180

Whether the problem is written horizontally or vertically, the fidelity of place value is kept.

In this example the 15 is thought about as (10 + 5) while the 12 is kept whole.

Further possibilities include breaking up the 12 and keeping 25 whole

(4 + 4 + 4) x 15 **or** breaking both factors apart

(10 + 2) x (10 + 5)

[47] Parrish, S. (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.249

Division Strategy: Partial Quotients[48]

Like the Partial Products strategy for multiplication, this strategy maintains place value and mathematically correct information for students. It allows them to work their way toward the quotient by using friendly multipliers such as tens, fives, and twos without having to immediately find the largest quotient. As the student chooses larger multipliers, the strategy becomes more efficient. [49]

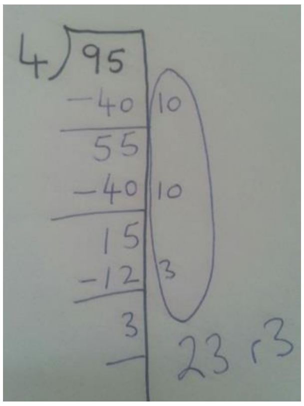
```
384 ÷ 16

        024
     16√384
       -160      (10)
        224
       -160      (10)
         64
        -32       (2)
         32
        -32      +(2)
          0       24
```

When learning the procedure for the standard algorithm, pupils are often told that 16 cannot go into 3 (300) which is incorrect; 16 can divide into 3, but it would result in a fraction. With the Partial Quotients strategy the "3" maintains its value of 300 and can certainly be divided by 16.

As the pupil works, he keeps track of the partial quotients by writing them to the side of the problem. When the problem is solved, the partial quotients are totalled and the final answer is written over the dividend.

Example A demonstrates using friendly 10s and 2s to solve the problem. As the 10s and 2s are recorded to the side of the problem, they represent 10 x 16 and 2 x 16.

[48] Parrish, S. (2010*). Number Talks Helping Children Build Mental Math and Computation Strategies* p.259
[49] Parrish, S. (2010*). Number Talks Helping Children Build Mental Math and Computation Strategies* p.259



Let's try 95 ÷ 4. Could you estimate first? Let's use our multiplication facts to help us work it out. What about 4 × 10? Write the partial quotients along the side. Now we could subtract 40 from the total. What would you try next? Is there enough left for 4 × 10. Record it along the side. Yes, now we have 15 left, count up in 4s, 3 × 4 = 12. Add up the partial quotients along the side. What does that give us? 23 r 3

Doubling/Halving[50]

Historically, multiplication was carried out by a process of doubling and halving. Most people find doubles the easiest facts to remember, and they can be used to simplify other calculations. Sometimes it can be helpful to halve one of the numbers in a multiplication calculation and double the other.[51]

Doubles Dominoes[52]

This needs a set of dominoes in which, for example, 7 × 2, 2 × 7, 7 + 7 and 14 can be matched. Watch to see which facts the children can recall quickly.

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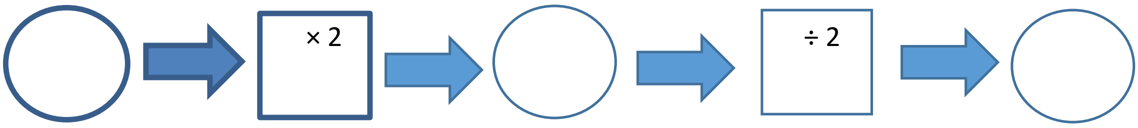
[50] Crown (2010) *Teaching Children to Calculate Mentally* p.61
[51] Crown (2010) *Teaching Children to Calculate Mentally* p.61
[52] Crown (2010) *Teaching Children to Calculate Mentally* p.61

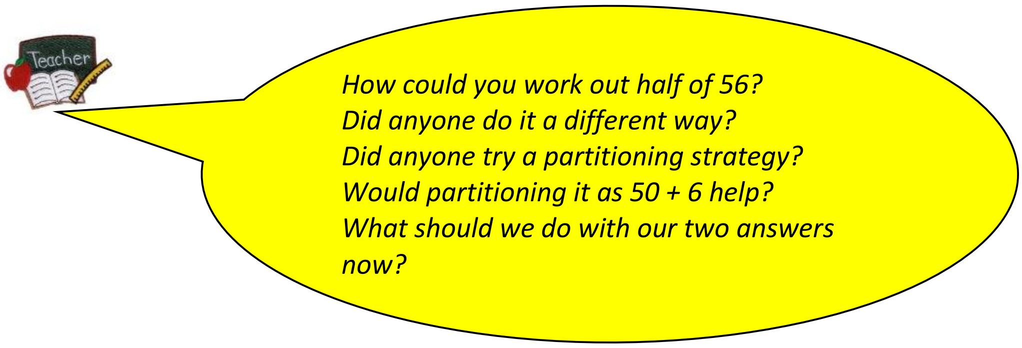
Doubling and halving function machines[53]



Ask one child to choose a number and another to choose whether to use the 'doubling' or 'halving' machine. Then ask a third child to say how the number is transformed by the machine.

Halve a two-digit number[54]

Ask children to halve a two-digit number such as 56. Discuss ways in which they might work it out. If not suggested by the children, lead them to discover that it's possible to partition 56 as 50 + 6 and to work out halve of 50 and halve of 6 and then add these together.



Ask children to suggest an even two-digit number and challenge other children to find a way of halving it. Some children may be able to halve odd numbers, for example, half of 47 is 23 and a half.

An extension of this would be to ask children to halve three-digit numbers, using the same partitioning strategy. For example, half of 364 would be half of (300 + 60 + 4) = 150 + 30 + 2.

Keep Doubling [55]

Start with a small number, for example 2, 3, 5 or 7. Start doubling it by going round the class. How far can you go?

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[53] Crown (2010) *Teaching Children to Calculate Mentally* p.61
[54] Crown (2010) *Teaching Children to Calculate Mentally* p.61
[55] Crown (2010) *Teaching Children to Calculate Mentally* p.61

This is a chance to try doubling larger numbers. Ask children to explain how they worked out their double.

Doubling and Halving Number Chains [56]

Ask someone to choose a number. Say that the rule is: ‘If the number is even, halve it, if it is odd, add 1 and halve it.’ Go round the class generating the chain. Write all numbers in the chain on the board.

14 → 7 → 4 → 2 → 1

Ask for a new starting number. Continue as before.

> Number chains can be quite intriguing as it is usually not possible to guess what will happen. As more and more starting numbers are chosen, the chains can build up to a complex pattern. For example, the starting number 8 joins the chain above at 4; the starting number 13 joins the chain at 7. A starting number of 23, for example, goes to 12, then 6, then 3, then joins the chain at 2.
>
> (Crown: 2010, p.62)

Money and Percentage doubling and halving [57]

When finding 20% of an amount, say €5.40, discuss how it is easier to first find 10% and then double. 10% of €5.40 is 54c so 20% of €5.40 is €1.08.

Ask children how they would find 5% of €5.40. Then get them to work out 15% of €5.40.

[56] Crown (2010) *Teaching Children to Calculate Mentally* p.62
[57] Crown (2010) *Teaching Children to Calculate Mentally* p.62

> Encourage children to use a range of methods for working out other percentages. For example, they might find 15% of €15.40 by finding 10% then halving that to find 5% and adding the two together. Or, having found 5%, they might multiply that result by 3. They could work out 17.5% by finding 10%, 5% and 2.5% and adding all three together.
>
> (Crown: 2010, p.62)

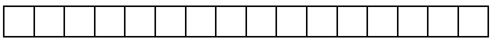
Arrays [58]

When children are provided with opportunities to build arrays that have the same area and study the patterns of the dimensions, they often notice a relationship that occurs between the factors or dimensions of the arrays. Consider the number 16. The following are all the possible arrays that could be built:

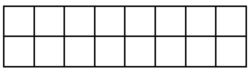
1 × 16
2 × 8
4 × 4
8 × 2
16 × 1

Although the factors change each time, the area remains the same. Ask children to look for patterns. What do you notice about the numbers on the left/on the right? When the factors on the left double, the factor on the right halves.

This concept can be explored concretely by physically building up the arrays and observing what happens to the rows each time a new array is built. Grid paper, geoboards or cubes can be used to build arrays.

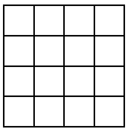


When 1 × 16 is halved, the number of rows doubles and the number of columns halve, resulting in 2 × 8:



[58] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.250

When the 2 × 8 is halved, the number of rows doubles and the number of columns halve, resulting in a 4 × 4:



Doubling and halving can be continued until a 16 × 1 is reached.

Use grid paper to create arrays easily and quickly.

Pose the following problem to the children:

At the music festival there are 32 schools with 25 students in each choir, how many students are there altogether in the choirs? [59]

32 x 25
↓ ÷4 ↓ x4
8 x 100

Doubling and halving would also be useful in solving this problem:

32 x 25
↓ ÷2 ↓ x2
16 x 50

What could you multiply one of these numbers by to make it easier to work with?
What would you then need to do to the other number?
Why is this strategy useful for this problem?
What knowledge helps you to solve a problem like this?

[59] http://nzmaths.co.nz/resource/multiplication-and-division-pick-n-mix-1

The doubling and halving strategy can be extended to include trebling and thirding. Posing problems where this can be applied will allow pupils the opportunity explore this. For example, 300 × 180 = 900 × 60

Proportional Reasoning/Adjustment for division[60]

As pupils become stronger with their understanding of factors, multiples and fractional reasoning, they may look at division from a proportional reasoning perspective. Based upon the pupils' experiences with doubling and halving to solve multiplication problems, they can be lead to explore if the same approach will work for division.

Proportional reasoning can be applied in division problems if both the dividend and divisor can be divided by the same amount to create a simpler problem. If the dividend and divisor share common factors, then the problem can be simplified and the answer can be calculated mentally.

Not all problems lend themselves to doubling and halving or proportional reasoning. This would be an important area for pupils to investigate.

[60] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.259

Breaking Factors into Smaller Factors and Factorisation[61]

Breaking factors into smaller factors instead of addends can be a very effective and efficient strategy for multiplication. The associative property is at the core of this strategy. It is a powerful mental strategy – especially when problems become larger and one of the factors can be changed to a one-digit multiplier.

12 × 25

(4 × 25) + (4 × 25) + (4 × 25)

100 + 100 + 100 = 300

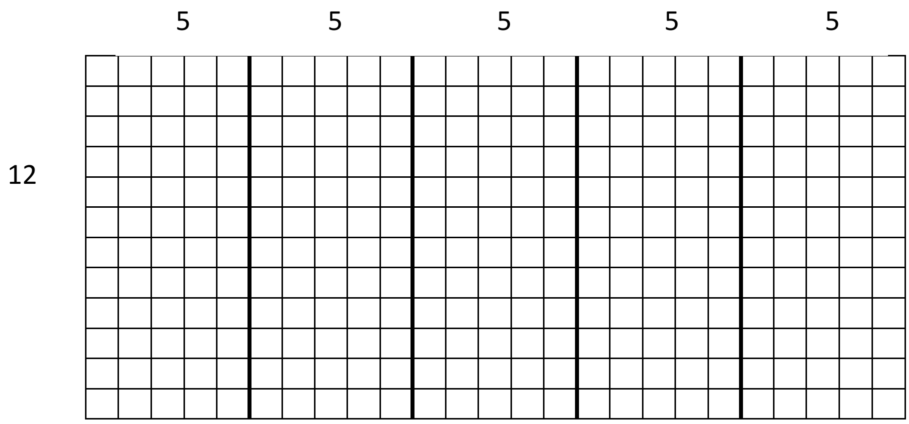
Pupils may approach a problem such as 12 × 25 by breaking the 12 into 3 groups of 4

(4 × 25) + (4 × 25) + (4 × 25) = 3 × (4 × 25)

12 × 25 = 3 × (4 × 25)

Help pupils to connect their thinking to the associative property by recording the problem as 3 × (4 × 25). Encourage them to discuss whether 12 × 25 is the same as 3 × 4 × 25. This is one way to begin making a bridge into factors and using the associative property

Using an array model will assist pupils in visualising how the associative property works with 12 × 25. This model shows how the problem can be represented as five groups of 12 × 5 or (12 × 5) × 5



[61] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.252

12 × (5 × 5) = (12 × 5) × 5

60 × 5 = 300

We can also use the associative property and knowledge about factorisation to think if 25 as 5 × 5.

Factorisation

The Factorisation strategy involves using factors to simplify the division problem.

Pose the following division problem to explore this strategy:

Stephanie has 486 marbles to share among 18 of her classmates. How many marbles will each class mate get? [62]

What strategy might you use to solve this problem? Could factors help you? What are the factors of 18? Can you multiply three numbers to make 18? So you could use 3 × 3 × 2? So how could you use this information to work out 486 ÷ 18? Does it matter which factor we divide by first? Why not? So can you calculate your answer now?

Factorisation allows opportunities to revisit the associative property. See p.11 for background information on this property.

Further practice at exploration and development of this strategy can be gained through the following:[63]

532 ÷ 8 (÷2, ÷2, ÷2)

348 ÷ 12 (÷2, ÷2, ÷3)

4320 ÷ 27 (÷3, ÷3, ÷3)

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[62] http://nzmaths.co.nz/resource/multiplication-and-division-pick-n-mix-1

[63] http://nzmaths.co.nz/resource/multiplication-and-division-pick-n-mix-1

135 × 12 (×2, ×2, ×3)
43 × 8 (×2, ×2, ×2)
27 × 16 (×2, ×2, ×2, ×2)

Think Multiplication for Division[64]

Similar to the scaffolding Parrish suggests for moving pupils towards multiplicative thinking, the strategy called **Think Multiplication** works by encouraging pupils to turn unknown division calculations into known multiplication facts.

> Similar to the Adding Up strategy for subtraction, the Multiplying Up [Think Multiplication for Division] strategy provides access to division by building on the pupil's strength in multiplication. Pupils realise they can also multiply up to reach the dividend. This is a natural progression as they become more confident in their use and understanding of multiplication and its relationship to division. Initially, pupils may rely on using smaller factors and multiples which will result in more steps. This can provide an opportunity for discussions related to choosing efficient factors with which to multiply.
>
> (Parrish: 2010, p.258)

Understanding the inverse relationship between multiplication and division is essential. For 15 ÷ 3, think *'3 "whats" are 15 (3 × □= 15). 3 fives are 15, so 15 ÷ 3 is 5'.*

[64] Mental Comp.pdf p.44

How would we solve 6 ÷ 3? How many red rods would be the same as the dark green rod? Could we try counting up the red rods?
Can anyone think of another way of saying that number sentence? Can you record this in your learning log?

This strategy allows pupils to build on multiplication problems that are comfortable and easy to use such as multiplying by tens and twos.

(Parrish: 2010, p.258)

Area Model[65]

384 ÷ 16

10 × 16 = 160

10 × 16 = 160

2 × 16 = 32

2 × 16 = 32

10 + 10 + 2 + 2 = 24

24 × 16 = 384

	10	10	2	2
16	16 × 10 = 160	16 × 10 = 160	16 × 2 = 32	16 × 2 = 32

Here, the open array (area model) can be used to model the student's strategy and link the operations of multiplication and division and is similar to the Partial Products strategy for multiplication discussed earlier.

65 Parrish, S. (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.258

Towards Fluency and Consolidation in Applying Strategies

Unless otherwise referenced, the following games came from *Mental Comprehension Strategies Addition, Subtraction, Multiplication & Division* accessed via http://www.currumbiss.eq.edu.au/Restricted/Currumbin%20numeracy/MentalComp.pdf

Remember: these games are to develop fluency in strategy application and number fact application. They should be played **following** focussed reasoning strategy development and **not** in place of strategy development. Wherever possible, as pupils engage in these games, they should be encouraged to share the strategies they used to calculate. Challenge pupils to apply their strategy knowledge by including opportunities to calculate with bigger numbers, rather than staying within 100.

Beat the calculator

Work in pairs. One player uses a calculator and the other uses mental computation to answer each fact. Partners compete to answer the fact first.

Card game

Picture cards are worth 10. All other cards hold their face value. Place all the cards in a pile. Take turns to turn over the top two cards. Players multiply the cards by each other. The first player to give the correct answer keeps the cards. The player with the most cards at the end of the game is the winner. Encourage pupils to share their strategies for calculating.

Snap

Provide a set of fact cards and answer cards into a pile. Share the cards equally between the players. Players take turns to turn over their top card. If the cards are a match, players ‘snap’ by placing their hand on top of the pile of cards. The player with the most cards at the end of the game is the winner.

Concentration

Provide a set of fact cards and matching answer cards. Place the cards face down in rows. Players take turns to turn over two cards to try to match a fact card with an answer card. They keep matching pairs. The player with the most pairs when all the cards have been matched is the winner.

Go fish

Provide a set of fact cards and matching answer cards. Deal five cards to each player. Players take turns to ask each other for a matching fact or answer card. The player with the most pairs at the end of the game is the winner.

For example, If they are holding a card that says 4 x 5, they would say Do you have a 20? If they are holding a 20 they might ask, Do you have a 4 multiplied by 5/5 multiplied by 4/2 multiplied by 10/10 multiplied by 2/double 10?

Strategy dice

Roll two 10-sided dice. Score one point for each time the designated strategy is rolled,
e.g., if the strategy is 'doubles', a player who rolls a two and any other number scores a point.

Shoot the sheriff

Two players stand back-to-back. As the caller counts slowly to three, they take three steps away from each other. On the count of three, the caller says a fact and the players turn and 'fire' (say) the answer. The first player to fire the correct answer is the winner. (The winner could also be decided by best of three.) The winner remains in the game and the loser chooses who will compete against the winner next.

Relations race

Players stand at a 'starting line'. A caller says multiplication fact and the players call out the related division fact (e.g. the caller says 'double 6 is 12' and the players call '12 divided by 2 is 6'). The first player to call the related fact takes a step forward. The winner is the player who is furthest ahead after a given time or number of calls.

One step at a time

Players stand at a 'starting line'. The caller says a fact/problem and the players call the answer. The first player to call the correct answer takes a step forward. The winner is the player who is furthest ahead at the end of the game.

Race around the hundred board

Players choose a marker and place it on the 0 square of a hundred board. Players take turns to throw two dice (concrete or onscreen), say the associated fact, and then move that many squares. The winner is the first player to reach 100.

Note: The game can be made more complex by adding rules.

For example, if a player lands on a number ending in 0, they have to go back two squares

Show me the money

This game gives the opportunity to answer facts and practise swapping €2 coins for the equivalent amount of €1 coins.

Find €10 in coins (€1 and €2) for each player and place in a 'bank'. Find and shuffle the cards for the facts that are being practised and place them face down in a pile. Players take turns to turn over the top card and answer the fact. They take €1 from the bank each time they answer a fact correctly. Players may need to swap two €1 coins for a €2 coin to do this. Discuss with players how to solve this problem. The winner is the player with the most money at the end of the game.

Target practice

The number of rounds and how the winner is chosen should be decided before the game begins, e.g., the player with the largest score after five rounds is the winner.

Create a large target showing numbers to 10 (e.g. draw a target on large sheets of paper/draw in chalk on cement/modify a commercially produced target). Throw two small soft markers onto the target and multiply the two numbers. The answer is that player's score for that round.

Note: If one marker doesn't land on a number, throw again. If both markers land on the same number, the answer will be zero.

Number facts competition with dominoes

Each player places half a set of dominoes face down in front of them. One student acts as referee. Players take turns to turn one of their dominoes face up so other players can see it. They multiply the number of spots on each end of the domino together. The first player to say the correct answer keeps the domino. If the referee decides it was a tie, the domino is removed from play. The player with the most dominoes at the end of the game is the winner.

Around the world

Two students stand while the rest of the class (the 'world') remains seated. A caller says a number fact and both students call out an answer. The first student to answer correctly is the winner and remains standing while the other sits down. The next person in line stands up to compete against the winner. Play continues until every student has competed. The winner is the last student standing. In the event of a tie, call another fact.

Salute! [66]

Pupils work in groups of three, with each one getting a deck of cards (omitting face cards and using aces as ones). Two pupils draw a card without looking at it and place it on their forehead facing outward (so the others can see it). The pupil with no card gives the product) of both numbers. The first of the other two pupils to correctly say what number is on their forehead 'wins' the card. Give a value of 10 to picture cards to provide opportunities to build fluency in these important facts.

[67]The Range Game

For two players (a third pupil can pick starting number and moderate). Select a target range within which you want pupils to aim for on the calculator. The rules of either multiplication or division only operations are agreed. The teacher or third pupil enters the starting number in the calculator. After the first or second turn decimal factors are usually required. This provides excellent understanding of multiplication or division by decimals. For example, a sequence for a target range of **262 to 265** might be like this:

Start with 63.

Player 1 x 5 = → 315 (too high)
Player 2 x 0.7 = → 220.5 (too low)
Player 1 x 1.3 = → 286.65 (too high)
Player 2 x 0.9 = → 257.985 (too low)
Player 1 **x** 1.03 = → 265.72455 (very close) *What would you press next…?*

[66] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). *Elementary and Middle School Mathematics Teaching Developmentally.* 8th edn. Pearson: Allyn and Bacon p.185

[67] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). *Elementary and Middle School Mathematics Teaching Developmentally.* 8th edn. Pearson: Allyn and Bacon p.225

Jump to It [68]

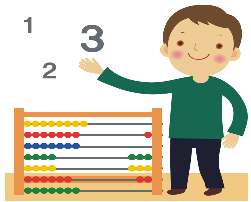
This activity focuses on division. Pupils begin with a start number and estimate how many times they will add that start number to reach the goal. The numbers can vary to meet the needs and experiences of your pupils, here are some to get started.

Jump Number	Goal	Estimate of Jumps	Was Estimate Reasonable?
5	72		
11	97		
7	150		
14	135		
47	1200		

To check estimates on the calculator, pupils can enter 0 + (jump number) and press =once for every estimated jump, or multiply (jump number) x (estimate of jumps). Pupils with learning disabilities may need to have a number line close by. Then they can mark their goal number with a sticky dot and use another colour dot to mark their first estimate. This will support them in the process of deciding whether they need to lower or raise their estimation of the number of jumps.

[68] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). *Elementary and Middle School Mathematics Teaching Developmentally.* 8th edn. Pearson: Allyn and Bacon p.254

Counting Activities using Multiplication and Division



These counting sessions should have:

- A lively pace
- Enthusiastic participation
- Two or three different short focussed activities (variety will maintain interest)
- Physical activity
- Choral response
- Individual response

Pupils should develop the ability to:

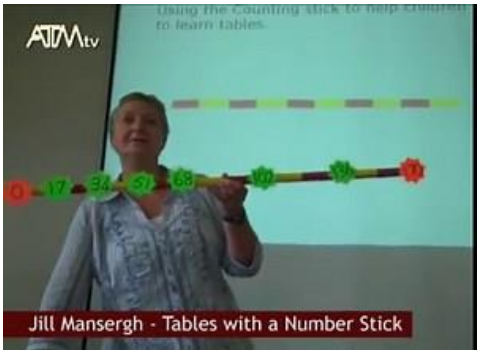
- Count forwards in different multiples of numbers
- Count backwards in different multiples of numbers
- Count forwards/backwards from different starting points.

Some suggested activities include:

Counting Stick

The counting stick can be used to model skip counting and scaffold children's understanding of multiplication facts. Using facts the children know such as × 10 can help to provide anchor points and provide encouragement. Halving the 10 fact will produce the × 5 fact. Doubles can also be used for × 2 and × 4 and x 8, leaving less for children to memorise as instead they are applying known facts to unknown facts. Initially post it notes in whatever × fact you are using can be used to scaffold the children, and removed as they gain fluency in a particular set of facts. (This links with another strategy, doubles below.

The following video shows the counting stick being used to learn the 17 times tables!



Rhythm Counting

Use actions such as: clapping, slapping, tapping. Pupils chant times tables in time with the rhythm. Musical instruments such as claves can work well to help establish a rhythm for counting in this way.

Live Number Line

Pupils are given large cards with a multiple of a particular number on them and are asked to line up in order of the sequence. Teacher/pupil then asks other pupils to swop with those in the line using differentiated instruction such as 'Swap with the person holding double 7. 'Swap with the person holiday 6 × 7' 'Swap with the person holding a multiple which is an odd number between 40 and 50.'

The Sound of a Number Game for Multiplication (also known as Counting Can)

Teacher tells the children what multiplying sequence they are focusing on e.g. × 6. Pupils close their eyes and for each cube dropped into the tin, the pupils count silently in multiples of 6 in their heads. As pupils get more proficient at this skill, the teacher can begin at different numbers, e.g. 24 and then pupils count on for each cube added. When the teacher stops adding cubes, the questions can include: 'what multiple of 6 did we get to?', 'If we got to 66 how many cubes must be in the tin?' This can be extended by giving the pupils the total and then removing some cubes and asking the pupils: 'I removed 3 cubes, what's the total in the tin now?', 'what's the value of the cubes in my hand?'

Stand and Sit

Pupils stand and then sit while saying a particular multiplication sequence, for example for multiples of 4, stand when you say a multiple of 10.

Clap and Snap

Count forwards clapping in time, then count backwards snapping fingers in time.

Stamp and Tap

Pupils find a space facing the board. Count forwards in multiples of a particular number stamping feet in time. Stop at required number word and turn in opposite direction. Now count back tapping their shoulders in time. (Do this without pausing!)

Show me

Teacher shows flash cards with different products. Pupils show two possible factors for that product using number fans or individual whiteboards.

Count Around

Pupils stand in a circle and count around in multiples of a particular number, each child saying the next number in the sequence. Start counting at first multiple, pupil who says an even number sits down. Keep going until only one pupil is standing. (This can be varied using shorter/longer sequences, using different starting/finishing points, doing it backwards, etc.)

Counting Choir

Divide class into 3 groups. Teacher is in the role of conductor, with a baton. Teacher begins to count in multiples of a particular number and then points the baton at one group to continue counting in unison. Teacher then points the baton at a different group who continue the sequence.

Target Boards

The Target Board is a very effective and versatile resource for mental/oral maths which can be placed on the whiteboard or wall. Each target board is a collection of numbers. When using target boards encourage pupils to share their thinking and explain their mental methods. This helps pupils to realise there is more than one way to solve a problem. Explaining how you worked out something is a powerful way of learning. For example: (Adapted from Bird, 1999, p.12)

- Tell me two numbers that have a product of 16.
- Double each number. Write down your answers. Discuss your strategy for doubles with your partner.
- Divide each number by 3 and write down the remainder.
- Multiply each number by 10. Is there a quick way to do this? Why does this rule work?
- Divide each number by 10. Do you always get a whole number? Why?

Possible Sequence for Developing Multiplication Facts[69]

(Adapted from First Steps in Mathematics, Book 2, Number)

A traditional approach to learning multiplication facts is to learn to chant through facts in order- 2s, 3s, 4s etc. This approach is not particularly helpful and places a heavy load on memory. It impedes strategy development, precludes pupils from developing and applying the commutative property and prevents pupils from extracting the required fact from a whole set of facts. For example, in order to recall 7 × 8, many pupils need to chant the whole 7 table from the beginning.

It is much more likely that pupils will remember basic facts if they practise them in a strategic way, using clusters of facts.

The following sequence could be followed:

Build up the facts 5 × 5

Start with twos (doubles), fours (double doubles) and fives (because of the easy patterns and the links to our fingers) and then the threes (one set more than the double). The commutative property reduces the number of facts from 25 to 15 and if the ones are removed, there are only ten to remember. Use a five by five table to illustrate to pupils that by learning just these ten facts, they end up with 25.

5					25
4				16	20
3			9	12	15
2		4	6	8	10
1	1	2	3	4	5
×	**1**	**2**	**3**	**4**	**5**

[69] Government of Australia, Dept. of Education & Training (2005), *First Steps in Mathematics, Book 2, Number, Understanding Operations, Calculate, Reason About Number Patterns* p.190

The ones and twos

Focus on the doubles strategy and encourage pupils build up their already acquired facts by doubling each fact. Doubling 3s facts leads to the 6s.

The tens

Focus on counting in tens. Provide opportunities for pupils to explore patterns and link this to place value concepts.

The squares

Many teachers find that explorations of patterns of squares help pupils to learn the square number facts 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. Drawing/making square numbers and linking it to the concept of area allows pupils to concretely explore the patterns and thus more readily recall square number facts.

At this stage pupils have now developed the facts from 1 × 1 to 5 × 5 and the facts involving one, two, ten and the squares. Have pupils generate a two-way table in which they record the multiplication facts they now know.

10	10	20	30	40	50	60	70	80	90	100
9	9	18							81	
8	8	16						64		
7	7	14					49			
6	6	12				36				
5	5	10	15	20	25	30	35	40	45	50
4	4	8	12	16	20	24	28	32	36	40
3	3	6	9	12	15	18	21	24	27	30
2	2	4	6	8	10	12	14	16	18	20
1	1	2	3	4	5	6	7	8	9	10
×	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

There are various orders in which the remaining 'facts' can be developed. One possible sequence would be as follows:

The five facts

Build up the five facts to 5 × 10, noting the relationship to the ten facts and the pattern in the units place. Add these to the table and engage pupils in activities to master these facts.

The fours facts and eight facts to 8 × 8

Have pupils connect their doubles knowledge to identify double doubles (fours). Add these facts to the table. Move on to connect the fours to eights (double fours). For example, if you know 4 × 3, you can double your answer to calculate8 × 3. Add these facts to the table. Apply the commutative property to add the remaining corresponding fours and eights facts. Initially, when practising these facts, pupils should be given plenty of time to apply their doubles strategies. Gradually they will build up speed to reach mastery of these facts.

The three and six facts

There are three additional facts (3 × 6, 3 × 7, 3 × 9) and add these to the table. Pupils may apply strategies such as one set more/less than the corresponding doubles fact to calculate these. For example, to work out 3 × 9, the pupil knows that 2 × 9 is 18, so 3 × 9 is one more set of 9 added to 18, which makes 27.

Double the above facts to establish the sixes or use other known facts. For example if 3 × 7 is 21, then 6 × 7 is 42. Or by knowing 6 × 6, I need to only add one for set of 6 to 36 to get the answer to 6 × 7.

The nine facts

Use knowledge of threes and sixes to add the additional two nine facts and their commutative partners. It is worthwhile to revisit the nines facts to link them to tens less one so that pupils can see that 9 × 7 is ten sevens less seven. Again, in the initial stages, allowing pupils time to apply these strategies during practice is important. Gradually, pupils will increase their speed to reach mastery of these facts.

The seven facts

At this stage all sevens facts are known!

Further Examples for Exploring and Embedding Strategies

The following numerical problems will provide opportunities for pupils to deepen their initial understanding of each strategy and to become more efficient in its application.

Doubles/Near Doubles

7 x 20	4 x 28	9 x 200	10 x 200	11 x 4
12 x 20	13 x 200	15 x 200	17 x 2000	20 x 2000
80 ÷ 4	120 ÷ 4	1200 ÷ 4	1600 ÷ 4	2800 ÷ 4

Doubling and Halving

12 x 4	14 x 4	35 x 12	160 x 4	12 x 14
18 x 16	12 x 16	52 x 18	5 x 64	3.2 x 20
16 x 35	40 x 50	15 x 18	25 x 160	25 x 280
125 x 84	3 x 160	360 x 50	18 x 400	30 x 500

Proportional Adjustment/Reasoning[70]

496 ÷ 8	400 ÷ 16	720 ÷ 36	360 ÷ 18	800 ÷ 40
1000 ÷ 8	384 ÷ 16	288 ÷ 12	184 ÷ 8	308 ÷ 28

Partitioning Strategies[71]

28 x 5	16 x 8	23 x 9	90 x 9	92 x 8
52 x 9	13 x 21	123 x 4	260 x 21	242 x 7
5.3 x 4	3.5 x 8	4.7 x 6	616 ÷ 4	330 ÷ 15
505 ÷ 5	720 ÷ 9	816 ÷ 8	369 ÷ 3	490 ÷ 7

[70] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies p.299*

[71] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.238

Partial Products[72]

4 x 7	8 x 9	6 x 5	4 x 30	5 x 150
8 x 25	8 x 45	6 x 125	4 x 115	6 x 532
8 x 112	8 x 256	6 x 325	13 x 15	14 x 16
15 x 33	15 x 11	25 x 14	35 x 24	16 x 22

Partial Quotients[73]

40 ÷ 4	64 ÷ 4	48 ÷ 3	54 ÷ 3	92 ÷ 3
80 ÷ 5	75 ÷ 5	300 ÷ 3	420 ÷ 3	496 ÷ 4
120 ÷ 6	180 ÷ 6	348 ÷ 6	124 ÷ 4	235 ÷ 5
256 ÷ 4	496 ÷ 8	900 ÷ 3	852 ÷ 3	275÷ 25
240 ÷ 12	360 ÷ 12	372÷ 12	368 ÷ 12	525 ÷ 35
600 ÷ 15	195 ÷ 13	675 ÷ 25	500 ÷ 20	540 ÷ 15

Rounding and Compensating[74]

3 x 98	79 x 5	€7.95 x 25	€11.98 x 12	€4.96 x 5
52 x 9	84 ÷ 3	91 x 6	598 x 6	19.95 x 11
192 x 4	26 x 32	9998 x 7	548 x 3	133 ÷ 7

Break down Factors and Factorisation [75]

6 x 4	6 x 8	12 x 4	9 x 4	5 x 12
6 x 9	4 x 8	8 x 5	15 x 4	25 x 8
8 x 35	24 x 9	16 x 8	32 x 8	12 x 25
16 x 35	16 x 45`	72 x 15	12 x 15	36 x 15
18 x 35	16 x 25	24 x 15	12 x 35	14 x 25

[72] Parrish, S. (2010). Number Talks Helping Children Build Mental Math and Computation Strategies p.248, 274-275

[73] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p.258-259, 290- 292

[74] Van de Walle, J., Karp, K.S. & Bay-Williams, J.M. (2013). *Elementary and Middle School Mathematics Teaching Developmentally.* 7th edn. Pearson p.238

[75] Parrish, S (2010). *Number Talks Helping Children Build Mental Math and Computation Strategies* p. 252, 283 - 285

Division as Think Multiplication/Multiplying Up [76]

56 ÷ 4	79 ÷ 5	68 ÷ 3	96 ÷ 3	38 ÷ 2	85 ÷ 5
48 ÷ 4	72 ÷ 4	99 ÷ 6	453 ÷ 3	999 ÷ 4	500 ÷ 4
215 ÷ 4	960 ÷ 3	792 ÷ 8	536 ÷ 6	484 ÷ 4	836 ÷ 7
900 ÷ 50	755 ÷ 35	840 ÷ 25	658 ÷ 15	756 ÷ 24	498 ÷ 15
699 ÷ 17	321 ÷ 21	825 ÷ 17			

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