# A term of length 4,523,659,424,929

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**Abstract** Bourbaki suggest that their definition of the number 1 runs to some tens of thousands of symbols. We show that that is a considerable under-estimate, the true number of symbols being that in the title, not counting 1,179,618,517,981 disambiguatory links.

## 1: Introduction

Bourbaki, the self-perpetuating French group of mathematicians, are ill at ease with logic and foundations. Some signs of that are given in my essay *The Ignorance of Bourbaki* [M]; Leo Corry has remarked in his book [Co], pp 318-9, that Bourbaki do not in their later volumes use the system they have so carefully set out in their Volume I; indeed, there was, according to copies of *La Tribu* unearthed by Corry, ([C], page 319 and footnote 63 on page 320) a considerable debate within Bourbaki as to whether the volume on *La théorie des Ensembles* should be written at all.

That volume, indeed, has met criticism. Serre has remarked that few logicians like it. Godement, in the first hundred pages of his text *Algèbre*, though he follows Bourbaki's exposition of logic and set theory, tells his readers to eschew formal reasoning. Another French scholar is quoted in Chouchan [Ch], page 124:

> *Jacques Roubaud parle même de l'effroyable premier livre sur la théorie des ensembles: un vrai désastre, auquel le monde a renoncé depuis longtemps. ... On ne se rend pas compte que c'est une présentation souvent fallacieuse.*

We shall suggest that the difficulties experienced by Bourbaki, in the normally straightforward task of setting up an axiomatic system of set theory as a basis for mathematics, are the consequence of an injudicious choice for their underlying logical formalism. In the present chapter we make this point:

1·0 Proposition *Bourbaki's abbreviated structuralist definition of the number 1, when expanded into the primitive symbolism of the first edition of La Théorie des Ensembles, comprises 4,523,659,424,929 symbols together with 1,179,618,517,981 links between certain of those symbols.*

That definition is quoted in the next paragraph. In §2 we review Bourbaki's syntax; §§3–6 give the details of the calculation of the length of that formula, using the formalism of the original edition; in the seventh section we remark that its length is vastly increased by the formalism of the 1970 edition. Some brief comments on the psychological significance of these arithmetical freaks will be found in the final section.

### Bourbaki's abbreviated definition of 1

Chapters I and II of Bourbaki's *Théorie des Ensembles* were published in 1954, and Chapter III in 1956. Among the primitive signs used was a reverse C, standing presumably for "couple", to denote the ordered pair of two objects. Being typographically unable to reproduce that symbol, we use instead the symbol $\bullet$. With that change, the footnote on page 55 of Chapter III reads

> *Bien entendu, il ne faut pas confondre le terme mathématique désigné (chap. I, §1, n$^{\text{o}}$ 1) par le symbole "1" et le mot "un" du langage ordinaire. Le terme désigneé par "1" est égal, en vertu de la définition donnée ci-dessus, au terme désigné par le symbole*
>
> $$\tau_Z((\exists u)(\exists U)(u = (U, \{\varnothing\}, Z) \text{ et } U \subset \{\varnothing\} \times Z \text{ et } (\forall x)((x \in \{\varnothing\}) \implies (\exists y)((x, y) \in U))$$
> $$\text{et } (\forall x)(\forall y)(\forall y')(((x, y) \in U \text{ et } (x, y') \in U) \implies (y = y')) \text{ et } (\forall y)((y \in Z) \implies (\exists x)((x, y) \in U)))).$$
>
> *Une estimation grossière montre que le terme ainsi désigné est un assemblage de plusieurs dizaines de milliers de signes (chacun de ces signes étant l'un des signes $\tau$, $\square$, $\vee$, $\neg$, $=$, $\in$, $\bullet$).*

### Acknowledgments

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## 2: Bourbaki's syntax

Bourbaki use the Hilbert operator but write it as $\tau$ rather than $\varepsilon$, which latter is visually too close to the sign $\in$ for the membership relation.* Bourbaki use the word *assemblage*, or, in their English translation, *assembly*, to mean a finite sequence of signs or letters, the signs being $\tau$, $\square$, $\vee$, $\neg$, $=$, $\in$ and $\bullet$.

The substitution of the assembly $A$ for each occurrence of the letter $x$ in the assembly $B$ is denoted by $(A|x)B$.

Bourbaki use the word *relation* to mean what in English-speaking countries is usually called a well-formed formula.

2·0 The rules of formation for $\tau$-terms are these:

let $R$ be an assembly and $x$ a letter; then the assembly $\tau_x(R)$ is obtained in three steps:

(2·0·0) form $\tau R$, of length one more than that of $R$;
(2·0·1) link that first occurrence of $\tau$ to all occurrences of $x$ in $R$
(2·0·2) replace all those occurrences of $x$ by an occurrence of $\square$.

In the result $x$ does not occur. The point of that is that there are no bound variables; as variables become bound (by an occurrence of $\tau$,) they are replaced by $\square$, and those occurrences of $\square$ are linked to the occurrence of $\tau$ that binds them.

The intended meaning is that $\tau_x(R)$ is some $x$ of which $R$ is true.

Certain assemblies are *terms* and certain are *relations.* These two classes are defined by a simultaneous recursion, presented in Godement [G] thus:

T1: every letter is a term
T2: if $A$ and $B$ are terms, the assembly $\bullet AB$ , in practice written $(A, B)$, is a term.
T3: if $A$ and $T$ are terms and $x$ a letter, then $(A|x)T$ is an term.
T4: if $R$ is a relation, and $x$ a letter, then $\tau_x(R)$ is an term.
R1: If $R$ and $S$ are relations, the assembly $\vee RS$ is a relation; in practice it will be written $(R \vee S)$.
R2: $\neg R$ is a relation if $R$ is.
R3: if $R$ is a relation, $x$ a letter, and $A$ a term, then the assembly $(A|x)R$ is a relation.
R4: If $A$ and $B$ are terms, $=AB$ is a relation, in practice written $A = B$.
R5: If $A$ and $B$ are terms, the assembly $\in AB$ is a relation, in practice written $A \in B$.

That is all.

2·1 Remark Clauses T3 and R3 are, as pointed out to me by Solovay, redundant — if omitted, they can be established as theorems — and were added to Bourbaki's original definition by Godement, presumably for pedagogical reasons.

2·2 Remark Note that every term begins with a letter, $\bullet$ or $\tau$; every relation begins with $=$, $\in$, $\vee$, or $\neg$. Hence no term is a relation.

Quantifiers are introduced as follows:

2·3 Definition $(\exists x)R$ is $(\tau_x(R) \mid x)R$;

2·4 Definition $(\forall x)R$ is $\neg(\exists x)\neg R$.

* The possible significance of the choice of the letter $\tau$ is to be discussed in Chapter II of my *Danish Lectures.*

## 3: Some calculations

3·0 Definition We write $\ell h(A)$ for the length of an assembly $A$, not counting any links that are there; $oc(x, A)$ for the number of occurrences in $A$ of the letter $x$, and $\lambda(A)$ for the number of links in $A$, which will equal the number of occurrences of $\Box$.

3·1 Proposition *If $R$ is of length $r$, $\tau_x R$ is of length $r+1$.*

*Proof :* We have added a $\tau$, and replaced each $x$ by a $\Box$.

3·2 Proposition $\lambda(\tau_x(R)) = oc(x, R) + \lambda(R)$.

3·3 Proposition *If $x$ has $m$ occurrences in $R$ and $y$ (distinct from $x$) has $k$ occurrences in $R$, then in $\tau_x(R)$, $x$ has no occurrences and $y$ has $k$ occurrences.*

3·4 Proposition *If $x$ has $m$ occurrences in $R$ and $y$ (distinct from $x$) has $k$ occurrences in $R$, then in each of the formulæ $(\exists x)R$ and $(\forall x)R$, $x$ has no occurrences and $y$ has $(m+1)k$ occurrences.*

*Proof :* There are the original $k$ occurrences of $y$, and each of the $m$ $x$'s has been replaced by $\tau_x(R)$, in each of which $y$ has $k$ occurrences. ⊣ (3·4)

3·5 Proposition *If $R$ is of length $r$ and has $m$ occurrences of $x$, then the length of $(\exists x)R$ is $r(m+1)$*

*Proof :* Each replacement of $x$ by $\tau_x(R)$ has increased the length by $r$. ⊣ (3·5)

3·6 Proposition *If $R$ is of length $r$ and has $m$ occurrences of $x$, then the length of $(\forall x)R$ is $(r+1)(m+1)+1$.*

*Proof :* The formula is $\neg(\exists x)\neg R$ and $\neg R$ is of length $r+1$ and has $m$ occurrences of $x$. ⊣ (3·6)

3·7 Proposition *If $x$ has $m$ occurrences in $R$, and $R$ has $\ell$ links, $\lambda((\exists x)R) = \lambda((\forall x)R) = m(\ell + m) + \ell$.*

*Proof :* $\lambda(\tau_x(R)) = \ell + m$, by Proposition 3·2. Each occurrence of $x$ in $R$ is replaced by one of $\tau_x(R)$, and then there are the $\ell$ original links in $R$. ⊣ (3·7)

3·8 Remark A curiosity of this syntax, not needed for our present calculations, is that two trivially equivalent formulæ might have markedly different lengths. Thus if $R$ has 2 occurrences of $x$, 5 of $y$ and 3 of $z$, and is of length 50, the formula $(\exists x)(\exists y)R$ will be of length 3900, with 234 occurrences of $z$, whereas the formula $(\exists y)(\exists x)R$ will be of length 2400 with 144 occurrences of $z$.

## 4: Parsing that term

We begin by repeating Bourbaki's abbreviated term in open display, with $y'$ replaced by $z$:

$$\begin{aligned}\tau_Z\Bigg((\exists u)(\exists U)\\
&\Big(u = (U, \{\varnothing\}, Z) \text{ et}\\
&U \subset \{\varnothing\} \times Z \text{ et}\\
&(\forall x)\Big((x \in \{\varnothing\}) \implies (\exists y)\big((x,\, y) \in U\big)\Big) \text{ et}\\
&(\forall x)(\forall y)(\forall z)\Big(\big((x,\, y) \in U \text{ et } (x,\, z) \in U\big) \implies (y = z)\Big) \text{ et}\\
&(\forall y)\Big((y \in Z) \implies (\exists x)\big((x,\, y) \in U\big)\Big)\Big)\Bigg).\end{aligned}$$

That is of the form

$$\tau_Z\Bigg((\exists u)(\exists U)\Big(A(u, U, Z) \text{ et } B(U, Z) \text{ et } C(U) \text{ et } D(U) \text{ et } E(U, Z)\Big)\Bigg).$$

## 5: The clauses

5·0 $E(U,Z)$ is of the form

$$(\forall y)\big(\vee\neg \in yZ \in \bullet\tau_x \in \bullet\Box_x yUyU\big)$$

If we write

$$e(y,U,Z) \ =_{\mathrm{df}} \ \vee\neg \in yZ \in \bullet\tau_x \in \bullet\Box_x yUyU$$

we see by inspection that $e$ has 15 symbols (not counting subscripts, which represent links) and 1 link, 3 occurrences of $y$, 1 of $Z$ and 2 of $U$.

Hence $E(U,Z)$ has (15 + 1)(3 + 1) + 1 = 65 symbols, among them 4 occurrences of Z and 8 of U; and 3(3+1) + 1 = 13 links.

5·1 $D(U)$ is of the form $(\forall x)(\forall y)(\forall z)d(U,x,y,z)$, where $d(U,x,y,z)$ is

$$\vee\neg\neg \vee \neg \in \bullet xyU\neg \in \bullet xzU = yz;$$

by inspection $d$ is of length 19, with 2 occurrences of $U$, 2 occurrences of $x$, 2 occurrences of $y$ and 2 occurrences of $z$, and no links.

Hence $(\forall z)d$ is of length $(19+1)(2+1)+1 = 61$, and has 6 occurrences of $x$, 6 of $y$ and 6 of $U$, and 4 links; $(\forall y)(\forall z)d$ is of length $(61+1)(6+1)+1 = 435$, and has 42 occurrences of $x$, and of $U$, and $6(4+6)+4 = 64$ links; finally $D(U)$ which is $(\forall x)(\forall y)(\forall z)$ is of length $(435+1)(42+1)+1 = 18749$, and has $43 \times 42 = 1806$ occurrences of $U$, and $42(64+42)+64 = 4516$ links.

5·2 REMARK According to the footnote on page E.II 6, $\varnothing$ is

$$\tau\neg\neg\neg \in \tau\neg\neg \in \Box\Box\Box,$$

or, with the links indicated by subscripts,

$$\tau_x\neg\neg\neg \in \tau_y\neg\neg \in \Box_y\Box_x\Box_x.$$

So it has 3 links and 12 symbols.

$\{x\}$ is the term $\tau_y\forall z(z \in y \iff z = x)$ (slightly simplified from the actual definition as $\{x,\, x\}$).
$z \in y \iff z = x$ is

$$\neg \vee \neg \vee \neg \in zy = zx\neg \vee \neg = zx \in zy$$

which has 20 symbols, with 4 occurrences of $z$, 2 of $y$, and 2 of $x$, and 0 links.

Call that $f(x,y,z)$. $(\forall z)f(x,y,z)$ therefore is of length $(20+1)(4+1)+1 = 106$, with 10 occurrences of $y$ and 10 of $x$, and 16 links.

Therefore $\{x\}$ is of length 107, with 10 occurrences of $x$ and 26 links. Replacing each $x$ by $\varnothing$, we find:

5·3 PROPOSITION $\{\varnothing\}$ *is of length* $97+120 = 217$, *with 56 links.*

5·4 $C(U)$ is of the form $(\forall x)c(x,U)$, where $c(x,U)$ is

$$\Big((x \in \{\varnothing\}) \implies (\exists y)\big((x,\, y) \in U\big)\Big),$$

that is,

$$\vee\neg \in x\{\varnothing\} \in \bullet x\tau_y \in \bullet x\Box_y UU,$$

so replacing $\{\varnothing\}$ by its expansion into 217 symbols, we see that $c(x,U)$ is of length 231 symbols, with 3 occurrences of $x$ and 2 of $U$, and 57 links. Hence $C(U)$ is of length $(232 \times 4)+1 = 929$ symbols, with $3(57+3)+57 = 237$ links, and 8 occurrences of $U$.

5·5 The Cartesian product is defined by Bourbaki on page E.II.3. Expanding their notation for the class-forming operator, $\{w \mid R\}$, we have

$$X \times Y = \tau_w(\forall z)\big((z \in w) \iff (\exists x)(\exists y)(z = (x, y) \;\&\; x \in X \;\&\; y \in Y)\big).$$

Write $b_0(x, y, z, X, Y)$ for $(z = (x, y) \;\&\; x \in X \;\&\; y \in Y)$.

$\mathfrak{A} \;\&\; \mathfrak{B}$ is $\neg \vee \neg\mathfrak{A}\neg\mathfrak{B}$, so $(\mathfrak{A} \;\&\; \mathfrak{B}) \;\&\; \mathfrak{C}$ is $\neg \vee \neg\neg \vee \neg\mathfrak{A}\neg\mathfrak{B}\neg\mathfrak{C}$.

So taking $\mathfrak{A}$ to be $z = (x, y)$, $\mathfrak{B}$ to be $x \in X$ and $\mathfrak{C}$ to be $y \in Y$, we have $b_0(x, y, z, X, Y)$ is

$$\neg \vee \neg\neg \vee \neg = z \bullet xy \neg \in xX \neg \in yY,$$

which by inspection is of length 19, with 2 occurrences each of $x$ and $y$, 1 each of $z$, $X$, and $Y$, and 0 links.

Therefore $(\exists y)b_0$ is of length $19 \times 3 = 57$, with 6 occurrences of $x$, 3 each of $z$, $X$, and $Y$, and 4 links; $(\exists x)(\exists y)b_0$ is of length $57 \times 7 = 399$, with 21 occurrences each of $z$, $X$ and $Y$, and 64 links.

Call that $b_1(z, X, Y)$.

$\mathfrak{A} \iff \mathfrak{B}$ is $\neg \vee \neg \vee \neg\mathfrak{A}\mathfrak{B}\neg \vee \neg\mathfrak{B}\mathfrak{A}$, so writing $b(w, z, X, Y)$ for $z \in w \iff b_1(z, X, Y)$, we see that $b$ is of length $8 + (2 \times 3) + (2 \times 399) = 812$, with 2 occurrences of $w$, 44 occurrences of $z$, 42 occurrences each of $X$ and $Y$, and 128 links.

Therefore $(\forall z)b$ is of length $(813 \times 45) + 1 = 36586$, with $42 \times 45 = 1890$ occurrences each of $X$ and $Y$, 90 occurrences of $w$, and $44(128 + 44) + 128 = 7696$ links. So we conclude that

5·6 Proposition *The term $X \times Y$ is of length 36587, has $90 + 7696 = 7786$ links, and has 1890 occurrences each of $X$ and $Y$.*

Now we have seen that $\{\varnothing\}$ is of length 217, with 56 links. Replacing each occurrence of $X$ by one of $\{\varnothing\}$, we increase the length by $1890 \times 216 = 408240$, and add $56 \times 1890 = 105840$ links. So we conclude that:

5·7 Corollary *The term $\{\varnothing\} \times Z$ has 1890 occurrences of $Z$ in it, and is of length 444827, with $7786 + 105840 = 113626$ links.*

5·8 The formula $U \subset V$ is $(\forall s)((s \in U) \implies (s \in V))$; $((s \in U) \implies (s \in V))$ is

$$\vee \neg \in sU \in sV,$$

which is of length 8, with 2 occurrences of $s$, one each of $U$ and $V$, and no links; hence $U \subset V$ is of length $9 \times 3 + 1 = 28$, with 3 occurrences each of $U$ and $V$, and 4 links.

We conclude that the formula $B(U, Z) \iff_{\text{df}} U \subset \{\varnothing\} \times Z$ is, when we replace each $V$ by $\{\varnothing\} \times Z$, of length $28 + (3 \times 444826) = 1{,}334{,}506$, with 3 occurrences of $U$, 5670 occurrences of $Z$ and $4 + (3 \times 113626) = 340882$ links.

5·9 The triple $(X, Y, Z)$ is defined to be $((X, Y), Z)$; in other words $\bullet\bullet XYZ$; so $u = (X, Y, Z)$ is $= u\bullet\bullet XYZ$, of length 7, with one occurrence each of $u$, $X$, $Y$, $Z$ and no link; so $u = (U, \{\varnothing\}, Z)$ is of length 223, with one occurrence each of $u$, $U$, and $Z$, and 56 links.

## 6: The grand total

The formula $\mathfrak{A} \,\&\, \mathfrak{B}$ is $\neg\vee\neg\mathfrak{A}\neg\mathfrak{B}$, so $((((\mathfrak{A} \,\&\, \mathfrak{B}) \,\&\, \mathfrak{C}) \,\&\, \mathfrak{D}) \,\&\, \mathfrak{E})$ is $\neg\vee\neg\neg\vee\neg\neg\vee\neg\neg\vee\neg\mathfrak{A}\neg\mathfrak{B}\neg\mathfrak{C}\neg\mathfrak{D}\neg\mathfrak{E}$, which is 16 symbols plus those in $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{E}$.

So adding up: the number 1 is of the form $\tau_Z((\exists u)(\exists U)F(u, U, Z))$ where $F$ is the conjunction of $A$ to $E$. We have this table:

| formula | length | u | U | Z | links |
|---|---|---|---|---|---|
| A | 223 | 1 | 1 | 1 | 56 |
| B | 1334506 | 0 | 3 | 5670 | 340882 |
| C | 929 | 0 | 8 | 0 | 237 |
| D | 18749 | 0 | 1806 | 0 | 4516 |
| E | 65 | 0 | 8 | 4 | 13 |
| total: | 1354472 | 1 | 1826 | 5675 | 345704 |

From that we conclude that $F$ is a formula of length 1,354,488, with 1 occurrence of $u$, 1826 occurrences of $U$, 5675 occurrences of $Z$, and 345,704 links.

Hence $(\exists U)F$ is of length $1354485 \times 1827 = 2474649576$, with 1827 occurrences of $u$, $1827 \times 5675 = 10368225$ occurrences of $Z$, and $1826(345704 + 1826) + 345704 = 634,935,484$ links; and $(\exists u)(\exists U)F$ is of length $2474649576 \times 1828 = 4523659424928$ with $1828 \times 1827 \times 5675 = 18953115300$ occurrences of $Z$, and $1827(634935484 + 1827) + 634935484 = 1,160,665,402,481$ links.

Finally the term denoting the number 1 is of length one more than that, namely 4,523,659,424,929, with 18,953,115,300 + 1,160,665,402,481 = 1,179,618,517,981 links, without which, of course, the formula would be unreadable.

## 7: The later edition

In the combined 1970 edition of chapters I to IV, Bourbaki revert to the definition familiar to set theorists of the ordered pair of $x$ and $y$ as $\{\{x\}, \{x, y\}\}$. The corresponding footnote, on page E III 24 of that edition, is almost identical to the original, the only differences being the omission of a primitive symbol (the reverse C) for ordered pair, and the reference to Chapter I appearing more simply as (I, p.15).

Though there are good reasons for that change, it would mean, given the commitment of Bourbaki to the $\tau$ operator, an enormous increase in the number of symbols in their definition of the term 1, for $\bullet xy$, instead of being of length 3 with one occurrence each of $x$ and $y$, and no link, will be of length 4,545, with 336 occurrences of $x$, 196 occurrences of $y$ and 1,114 links. $X \times Y$ will now be of length roughly $3{\cdot}1845912 \times 10^{18}$, with $1{\cdot}15067 \times 10^{18}$ links, and $6{\cdot}982221 \times 10^{14}$ occurrences each of $X$ and of $Y$, and a program in Allegro Common Lisp written by Solovay yields these exact figures:

7·0 Proposition *If the ordered pair $(x, y)$ is introduced by definition rather than taken as a primitive, the term defining 1 will have 2409875496393137472149767527877436912979508338752092897 symbols, with 87188023373394906994618280491091227472430953034182177 links.*

At 80 symbols per line, 50 lines per page, 1,000 pages per book, the shorter version would occupy more than a million books, and the longer, $6 \times 10^{47}$ books. I believe that the approach customary among set-theorists, whereby one takes the class-forming operator as a primitive symbol, and defines

$$0 =_{\mathrm{df}} \{x \mid \neg\, x = x\} \qquad \text{and} \qquad 1 =_{\mathrm{df}} \{x \mid x = 0\},$$

is simpler.

## 8: Higher cardinals

8·0 REMARK (Solovay) Bourbaki's definition of 1 as given will not generalise satisfactorily to higher cardinals as it omits a clause stating that the function $u$ with graph $U$ is 1-1. If one seeks an *ad hoc* definition of the number 1 in Bourbaki's dialect of set theory, a shorter one — perhaps *the* shortest ? — would be $\tau_Z\big(\exists x(x \in Z \text{ et } \forall y(y \in Z \implies y = x))\big)$, which runs to 176 symbols with 56 links.

8·1 REMARK Rough calculations suggest that for large $n$, a Bourbachiste definition of $n$ as *some object $a$ for which $\exists x_1 \ldots \exists x_n\ a = \{x_1, \ldots, x_n\}$, with the $x$'s all distinct* will have over $n \exp(2 \exp(n+1))$ symbols, whereas von Neumann's definition of $n$ as the set of all $m$ less than $n$, when written in the symbolism of a standard set-theoretic formalism with quantifiers taken as primitive, has $O(2 \exp n)$ symbols, whilst a yet shorter definition of von Neumann's $n$ would be *the union of the class of all $b$ such that* ((*$b$ is a transitive set of transitive sets*) *and* ($\exists x_1 \ldots \exists x_n\ b = \{x_1, \ldots x_n\}$ *with the $x$'s all distinct*)), which has $O(n^2)$ symbols; I learn from Solovay that that can be improved to $O(\log n \log \log n)$ or even $O(\log n)$ with "recycling of variables".

## 9: Discussion

Early reviewers such as Mostowski wrote that Bourbaki's chosen foundations were "cumbersome"; I had not realised to what extent till I read a footnote in Bourbaki, reproduced in Godement, saying that the term for the number 1 would take some tens of thousands of signs to write out in full. I thought, "That must be false, surely only a couple of hundred;" and then the truth emerged.

I see in the hopeless unwieldiness of their system of logic, with its remarkable explosion in the length of formulæ, a possible explanation of the psychological stress suffered by some readers of Bourbaki. What will happen to a young innocent who decides to learn mathematics by reading Bourbaki, and to start with Volume I ? It will tie him in knots. Either he will shut the book in disgust, or he will persevere and then he will be paralysed by the mental effort required to disentangle the formalism.

Bourbaki themselves took the first course: as remarked by Corry, they shied away from their own foundations. I expect that they came to the conclusion that logic is crazy — they had to conclude that to protect their sanity; but were they aware that the picture of logic they were giving to their disciples is merely a grotesque distortion and diminution of that subject ? Is it too fanciful to see here, in this choice of formalism, with its unintuitive treatment of quantifiers, the reason for the phenomenon (which many mathematicians in various European countries have drawn to my attention whilst beseeching me not to betray their identity, lest the all-powerful Bourbachistes take revenge by depriving them progressively of research grants, office facilities and ultimately of employment) that where the influence of Bourbaki is strong, support for logic is weak ? How does one get the message across, to those who have accepted the Bourbachiste gospel, that logicians are actually not interested in a formal system of such purposeless prolixity, still less do they advocate it as the proper intellectual framework for doing mathematics ?

As mentioned above, I have in preparation a fuller discussion of this topic. Other chapters of *Danish Lectures* will discuss the foundational exposition in the first hundred pages of Godement's 600-page text Algebra, Dieudonné's essay on the philosophy of Bourbaki, sundry writings of other members of the Bourbaki school, and, what is at the core of Bourbaki's mistreatment of logic, the efforts of Hilbert and his school to use his $\varepsilon$ operator to establish the consistency and completeness of mathematics.

For really the débâcle is hardly Bourbaki's fault. The founding fathers, with Chevalley in the lead, were keen to introduce Hilbertian standards of rigour to France; and in their youthful enthusiasm they swallowed the Hilbertian promise of a complete and consistent mathematical system hook, line and sinker. Their formalism rests on a device of Hilbert with which he pursued the chimera of consistency proofs. In the analysis of a *complete* system, the $\varepsilon$-operator might make sense, but in an *incomplete* system, such as Gödel showed all moderately expressive reasonable fragments of mathematics to be, it becomes very tortuous, and not something to place at the centre of a serious exposition of mathematical truth.

## References


[B0] N. Bourbaki, Fascicule des Resultats, 1939.

[B1] N. Bourbaki, Théorie des Ensembles, first edition: Chapters 1, 2 1954; Chapter 3, 1956; Chapter 4, 1957. Hermann, Paris.

[B2] N. Bourbaki, Théorie des Ensembles, 1970. Hermann, Paris.

[B3] English edition: Theory of Sets, 1968. Hermann, Paris.

[Ch] M. Chouchan, *Nicolas Bourbaki, Faits et Légendes*, Editions du Choix, Argenteuil, 1995.

[Co] L. Corry, *Modern Algebra and the Rise of Mathematical Structures*, Birkhäuser, 1996.

[G] R. Godement, *Cours d'Algèbre*, 1963, revised 1966; published in English as *Algebra*, 1968.

[M] A. R. D. Mathias, The Ignorance of Bourbaki, in *Mathematical Intelligencer* **14** (1992) 4–13 MR 94a:03004b, and also in *Physis Riv. Internaz. Storia Sci (N.S.)* **28** (1991) 887–904 MR 94a:03004a; available in a Hungarian translation by András Racz as *Bourbaki tévútjai*, in *A Természet Világa*, 1998, III. különszáma.