The Effect of Caustics in Acoustic Inverse Scattering Experiments

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THE EFFECT OF CAUSTICS IN ACOUSTIC INVERSE SCATTERING EXPERIMENTS

by

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ABSTRACT

Most inversion techniques described in the literature rely on the validity of ray tracing, which breaks down in the presence of caustics. The linearized acoustic inverse problem with constant reference velocity is analyzed in order to quantify the effects of a caustic in a probing wavefront on the scattered signal.

When the sound velocity is perturbed by a localized, unidirectional, high frequency inhomogeneity, the surprising result obtained is that the energy in the scattered field is spread out if the perturbation is located on the caustic. This spreading of energy allows the construction of an oscillatory integral representation of the scattered field, which has the same form, whether or not an incident caustic is present. On the other hand, a sequence of localized high frequency sound velocity perturbations is constructed such that the size of the scattered signal relative to the size of the inhomogeneity becomes arbitrarily large as the support of the perturbation approaches the caustic.

In regions where there are no caustics, a general inverse operator is found for smoothly varying reference velocities. This operator is shown to be equivalent to an inverse operator constructed by Beylkin (1985).

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Introduction

Inverse scattering problems are central in many data-processing technologies, including tomography, radar tracking, seismology, and ultra-sonic testing. In any inverse scattering problem, if the medium is inhomogeneous, then wavefronts generally will become singular. The variation in wave velocity causes the wavefronts to fold over on themselves, as in Figure 1.1. This produces an imperfect focusing effect called a caustic. In this case, the relationship between the data and the scatterer is not well understood at present. In order to begin an investigation of this relationship, we will concentrate our efforts on the linearized acoustic inverse problem, which is directly related to many of the problems listed above.

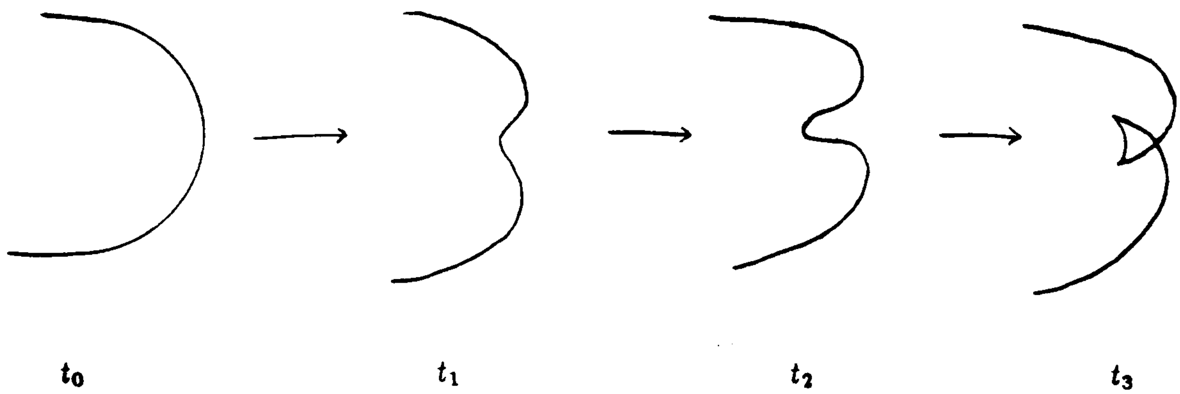
The acoustic wave equation, or forward problem is:

$$\frac{1}{c^2(\mathbf{x})}u_{tt} - \nabla^2 u = 0 \quad \text{in } \Re^n \tag{1}$$

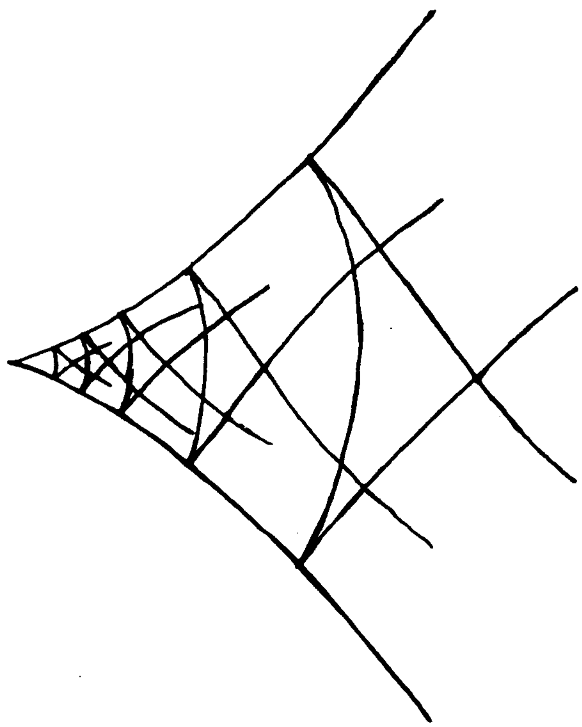
with some initial-boundary data. The inverse problem can be stated as follows:

Let $S[c] = u$ on (some hypersurface in $\Re^n$) $\times [0, T]$. Then

given S_{measured}, find $c(\mathbf{x})$ such that $S[c] = S_{\text{measured}}$.



(a)



(b)

Figure 1.1. (a) An inhomogeneous medium causes a wavefront to fold over on itself. (b) The caustic is the envelope of the singular wavefronts.

Linearization of the inverse problem is accomplished by a perturbation technique called the Born approximation. The wavespeed $c(\mathbf{x})$ is assumed to be a small perturbation about a smooth background velocity, which implies a perturbation of the acoustic wave field u about a reference field

$$\begin{aligned} c &= c_0 + \delta c \\ u &= u_0 + \delta u \,. \end{aligned}$$

Then the forward perturbational problem can be expressed as

$$\begin{aligned} \tfrac{1}{c_0^2} u_{0_{tt}} - \nabla^2 u_0 &= 0 \\ \tfrac{1}{c_0^2} \delta u_{tt} - \nabla^2 \delta u &= \tfrac{2\delta c}{c_0^3} u_{0_{tt}} \,. \end{aligned} \tag{2}$$

The linearized inverse problem is stated as follows:

> Given c_0, let $\delta S = DS[c_0]\delta c = \delta u$ on (some hypersurface in $\Re^n$) $\times$ $[O,T]$. Then given $\delta S_{\text{measured}}$, find $\delta c(\mathbf{x})$ such that $DS[c_0]\delta c = \delta S_{\text{measured}}$.

Most inversion techniques in practice today are based on the method of geometrical optics. The behavior of a wavefront spreading into an undisturbed region is obtained by propagating discontinuities along characteristics, or rays. The theory is valid only if the rays fill the entire region of interest and do not cross. As in Figure 1.2, a caustic is formed by the envelope of crossing rays, making the geometrical optics method, and hence the inversion technique, invalid.

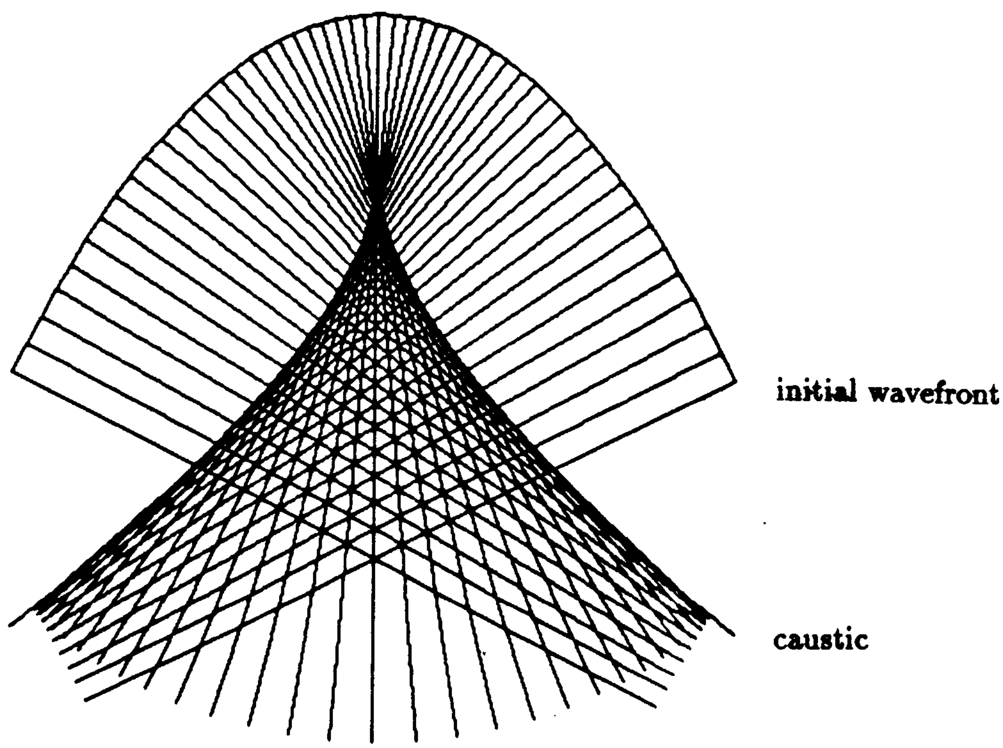


Figure 1.2. A cusped caustic formed by a convex wavefront in a homogeneous medium.

Even though a caustic is defined by the invalidity of the geometrical optics method, it is a real physical phenomenon where the amplitude becomes very large. In fact, Kulkarney and White [2] have shown that in a medium with small random perturbations of a constant wavespeed, every ray passes through a caustic with probability one in a propagation length scale of order $\mathcal{O}(\sigma^{-\frac{2}{3}})$, where the standard deviation of propagation speed fluctuations is $\mathcal{O}(\sigma)$.

It makes sense that caustics should hold some information about the medium and perhaps even "illuminate" it. What has not been done is to quantify this information and take it into account when solving the inverse problem. Perhaps Morawetz [3, p. 331] said it best. "There is as much information about the sound speed carried in a caustic as in a smooth ray pattern. The problem is to extract it."

Since all methods available today for solving the linearized inverse problem ignore the presence of caustics, it is natural to ask what effect a caustic in a probing wavefront has on the scattered signal, $\delta u|_{\text{surface}}$. Even further, we want to know if it is possible to represent the scattered field with methods that are available today. Ultimately the goal is to find an inverse operator for that representation.

These issues are addressed in the following manner: In Chapter 2, we review the available literature, including a modification to the geometrical optics method to allow solutions near caustics. In Chapter 3, we analyze the problem of a convex initial wavefront moving into a region with constant background velocity. The initial wavefront forms a caustic and the perturbation in wavespeed is taken to be localized, high frequency and unidirectional in order to isolate the results. The surprising result obtained is that if the perturbation is located on the caustic in the background field, the scattered signal is not amplified; it is in fact spread out. The problem with this result arises in taking the perturbation to be more general than unidirectional. In Chapter 4 a sequence of perturbations is constructed so that when there is a caustic present in the incident field, the size of the scattered signal, δu_n, relative to that of the scatterer, δc_n, becomes infinite as δc_n approaches the caustic. That is

$$\frac{||\delta u_n||^2}{||\delta c_n||^2} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty \, .$$

It is also shown that if there are no caustics present, then $\frac{||\delta u_n||^2}{||\delta c_n||^2}$ remains bounded. In Chapter 5 we derive an inverse operator for the representation of the scattered signal obtained in Chapter 3, and in Chapter 6 this inverse is shown to be related to the inverse operator constructed by Beylkin in [13].

CHAPTER 2

Background

2.1 Geometrical Optics

Geometrical optics is a method for obtaining the behavior of a wavefront spreading into an undisturbed region by propagating discontinuities along rays. For an excellent discussion of this method, see Whitham [1].

The wave equation is solved asymptotically near the wavefront by means of an ansatz of the form

$$u(\mathbf{x},t) \sim \sum_{j=1}^{\infty} a_j(\mathbf{x},t) f_j(\phi(\mathbf{x},t))$$

where $f_j' = f_{j-1}$.

The wavefront is given by $\phi(\mathbf{x},t) = 0$, and it is assumed that u is identically zero in $\phi(\mathbf{x},t) < 0$. For us, f_j has either the form

$$f_j(\phi) = H_{m+j}(\phi) = \begin{cases} \frac{\phi^{m+j}}{(m+j)!}\ , & \phi > 0 \\ 0\ , & \phi < 0 \end{cases}$$

for a wavefront expansion, where $H_0(\phi)$ is the Heaviside function, or $f_j(\phi) = \frac{e^{ik\phi}}{(ik)^j}$ for a high frequency approximation.

Substitution of the ansatz into the wave equation (1) and setting the coefficients of the f_j equal to zero gives equations for the phase ϕ and the amplitudes a_j:

$$|\nabla\phi|^2 = \tfrac{1}{c_0^2}\phi_t^2 \qquad \text{(eikonal equation)}$$

$$\tfrac{2}{c_0^2}\phi_t a_{1t} - 2\nabla\phi\cdot\nabla a_1 + (\tfrac{1}{c^2}\phi_{tt} - \nabla^2\phi)a_1 = 0 \qquad (1^{st} \text{ transport equation})$$

The eikonal equation is solved using Hamilton-Jacobi theory, which is a method of characteristics for nonlinear equations. The solution is obtained by solving the following system of O.D.E.s:

$$\mathbf{p} = \nabla\phi \qquad \tfrac{d\mathbf{x}}{d\sigma} = \mathbf{p} \qquad \tfrac{d\mathbf{p}}{d\sigma} = -\tfrac{\phi_t^2}{c_0^3}\nabla c_0(\mathbf{x})$$

$$\tau = \phi_t \qquad \tfrac{dt}{d\sigma} = -\tfrac{\tau}{c_0^2(\mathbf{x})} \qquad \tfrac{d\tau}{d\sigma} = 0$$

$$\tfrac{d\phi}{d\sigma} = 0$$

In this case ϕ is constant along rays. The amplitude $a_1(\mathbf{x}, t)$ is also given by the solution of an O.D.E. along the rays:

$$\begin{aligned}\frac{da_1}{d\sigma} &= a_t\frac{dt}{d\sigma} + \nabla a \cdot \frac{dx}{d\sigma}\\ \frac{da_1}{d\sigma} &= \left(\frac{1}{c_0^2}\phi_{tt} - \nabla^2\phi\right) a_1 \,.\end{aligned}$$

These equations can always be solved as long as the rays cover the entire region of interest and do not cross. If the rays tend to focus, then the geometrical optics approximation incorrectly predicts that as the ray density becomes infinite, so does the amplitude. The envelope of these rays is called a caustic (see Figure 1.2).

To see this, multiply the transport equation by a_1:

$$a_1\left[\frac{2}{c_0^2}\phi_t a_{1t} - 2\nabla\phi\cdot\nabla a_1 + (\frac{1}{c_0^2}\phi_{tt} - \nabla^2\phi)a_1\right] = 0 \qquad (3)$$

This gives the space-time divergence

$$\frac{\partial}{\partial t}(\frac{1}{c_0^2}\phi_t a_1^2) + \nabla \cdot (-a_1^2 \nabla \phi) = 0 \ .$$

Consider a slim tube formed by rays emanating from the initial wavefront in some disk D_0 centered at a point $\mathbf{x}_0$ and going to the wavefront at some later time t. Let D_t be the slice of rays at time t. (See Figure 2.1.) Integrating (3) over the volume R_t formed by the rays bounded by D_0 and D_t and using the divergence theorem, we get

$$\int_{\partial R_t} (\frac{a_1^2}{c_0^2}\phi_t, -a_1^2 \nabla \phi) \cdot \mathbf{n} ds = 0 \ .$$

On D_0 the normal is $\mathbf{n} = (-1, 0)$; on D_t, $\mathbf{n} = (1, 0)$; and on the sides $\mathbf{n}$ is perpendicular to the rays, which are parallel to the vector $(\frac{1}{c_0^2}\phi_t, -\nabla\phi)$. Thus

$$\int_{D_t} \frac{a_1^2}{c_0^2}\phi_t ds - \int_{D_0} \frac{a_1^2}{c_0^2}\phi_t ds = 0 \ . \tag{4}$$

Suppose $\chi(\mathbf{X}, t)$ maps a point on the initial wavefront to a point $\mathbf{x}$ at time t. Let $J(\mathbf{X}, t) = |\nabla \chi|$ be the Jacobian of this map. Then

$$\int_{D_0} \left[\frac{a_1^2(\chi(\mathbf{X}, t), t)}{c_0^2(\chi(\mathbf{X}, t))} \phi_t(\chi(\mathbf{X}, t), t) J(\mathbf{X}, t) - \frac{a_1^2(\mathbf{X}, 0)}{c_0^2(\mathbf{X})} \phi_t(\mathbf{X}, 0) \right] ds = 0 \ .$$

Since this holds for any disk D_0 on the initial wavefront, then as long as the integrand is continuous,

$$a_1(\chi(\mathbf{X}, t), t) = a_1(\mathbf{X}, 0) \frac{c_0(\chi(\mathbf{X}, t))}{c_0(\mathbf{X})} J^{-\frac{1}{2}}(\mathbf{X}, t) \ . \tag{5}$$

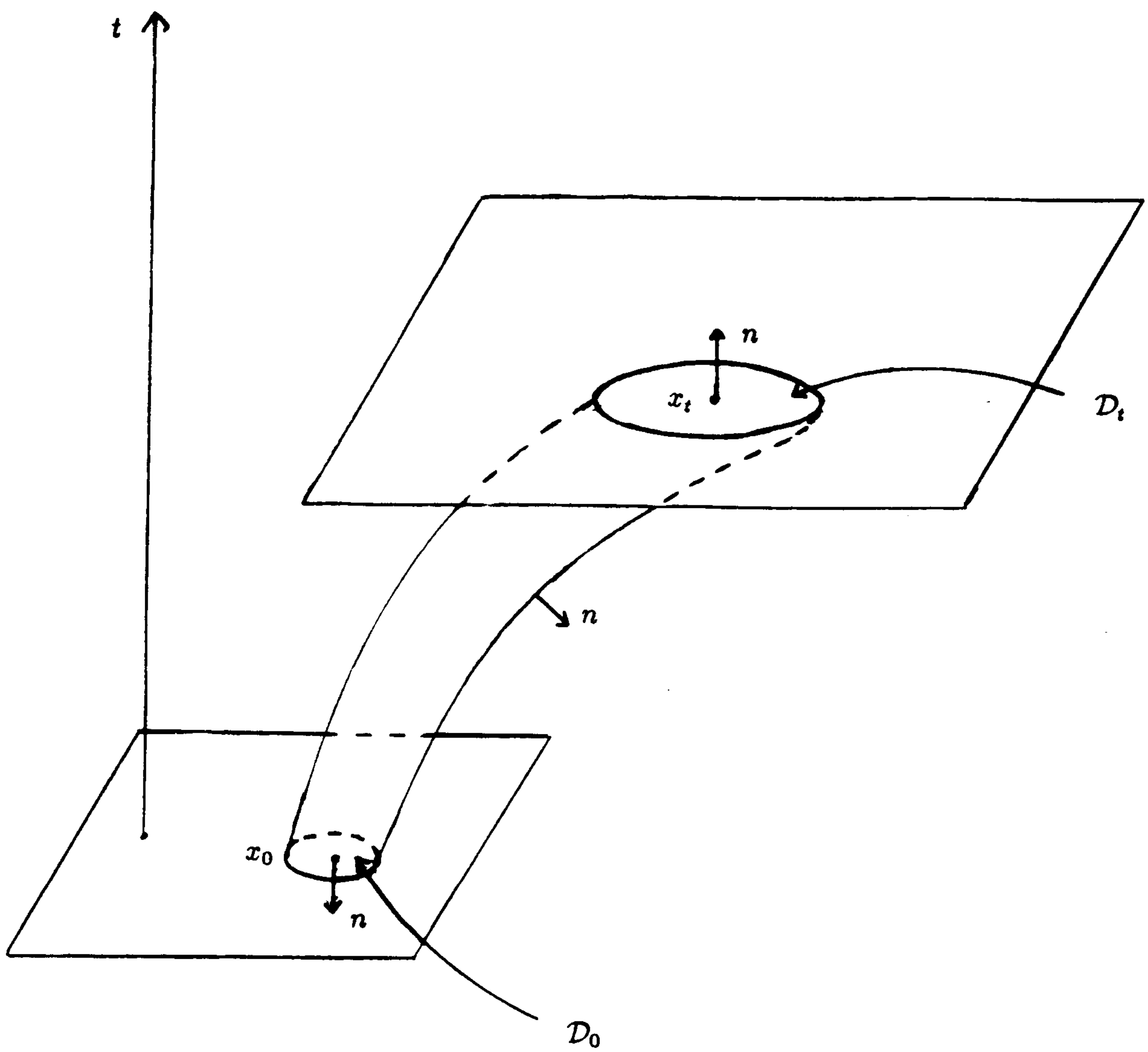


Figure 2.1. R_t is the slim tube of rays going from the initial wavefront in the disk D_0 to the wavefront at some later time t.

When $\chi(\mathbf{X},t)$ becomes singular, J approaches zero and (5) is no longer valid. But expanding (4) to first order about $\mathbf{X}_0$, we get

$$\frac{a_1^2(\mathbf{x}_0)}{c_0^2(\mathbf{x}_0)}\Delta D_t = \frac{a_1^2(\mathbf{X}_0)}{c_0^2(\mathbf{X}_0)}\Delta D_0 \ ,$$

where ΔD_t and ΔD_0 are the infinitesimal areas of the disks D_t and D_0. This is actually a statement that energy flux along the ray tube is constant. It is clear that when ΔD_t approaches zero, $a_1^2(\mathbf{x})$ approaches infinity. The ray tube collapses, and the theory is no longer valid.

2.2 Caustics

Although geometrical optics fails at a caustic, it can be altered to produce valid results. Expanding on the example of Keller [4], Ludwig [5] has constructed uniform asymptotic expansions for solutions of linear hyperbolic equations at a caustic. For the simple case of a smooth caustic and the reduced wave equation

$$\nabla^2 u + \omega^2 u = 0,$$

the ansatz is changed to a linear combination of the Airy function and its derivative. The structure is such that on one side of the caustic the original oscillatory expansion is given by geometrical optics (illuminated side), and on the other side the solution is exponentially decaying (dark side). The transition between the two sides is smooth with asymptotically larger amplitude along the caustic.

We show here the motivation for the construction since it will also motivate some of our own analysis. An appropriate ansatz for the leading order asymptotic solution of the reduced wave equation is

$$u(\mathbf{x}) = a(\mathbf{x})e^{i\omega\phi(\mathbf{x})} ,$$

where ω is large.

The eikonal and transport equations are respectively

$$|\nabla\phi|^2 = 1$$

$$2\nabla\phi \cdot \nabla a + a\nabla^2\phi = 0 \quad .$$

But at a caustic, the local ray density goes to zero, the amplitude goes to infinity, and the geometrical optics solution becomes invalid.

If, however, the initial wavefront were decomposed into plane waves, each plane wave could propagate into the region without forming a caustic since the wavespeed is identically one. This suggests an ansatz of the form

$$u(\mathbf{x}) = \int a(\mathbf{x},\theta)e^{ik\phi(\mathbf{x},\theta)}d\theta, \tag{6}$$

where $\phi(\mathbf{x},\theta) =$ constant describes a plane.

Ordinarily, the method of stationary phase can be used to get an asymptotic representation for u. If the stationary points defined by

$$\frac{\partial\phi}{\partial\theta}(\mathbf{x},\theta) = 0$$

are given by θ_j, then

$$u(\mathbf{x}) \sim \sum_j \left(\frac{2\pi}{\omega|\phi_{\theta\theta}(\mathbf{x},\theta_j)|}\right)^{\frac{1}{2}} e^{\pm i\frac{\pi}{4}} a(\mathbf{x},\theta_j(\mathbf{x}))e^{i\omega\phi(\mathbf{x},\theta_j)}$$
$$= 0\,(\omega^{-\frac{1}{2}})$$

as long as $\phi_{\theta\theta}(\mathbf{x},\theta_j) \neq 0$. If $\phi_\theta(\mathbf{x},\theta_j) = \phi_{\theta\theta}(\mathbf{x},\theta_j) = 0$ and $\phi_{\theta\theta\theta}(\mathbf{x},\theta_j) \neq 0$, then similar analysis yields $u(\mathbf{x}) \sim 0(\omega^{-\frac{1}{3}})$. If two stationary points coalesce, then $\phi_{\theta\theta}$ approaches zero, and in the region between $\phi_{\theta\theta} = 0$ and $\phi_{\theta\theta} \neq 0$, the method of stationary phase breaks down. This transition zone corresponds to the region near a caustic. Notice in Figure 2.2 that each stationary point corresponds to a contribution from a ray. Each point to the right of the caustic is the intersection of two rays. Along the caustic, each point corresponds to only one ray. No rays penetrate to the left of the caustic. This region is called a shadow zone.

The simplest function modelling this phenomenon is given by Chester, Friedman, and Ursell [6]. They showed that $\xi(\mathbf{x},\theta)$, $\gamma(\mathbf{x})$, and $\rho(\mathbf{x})$ can be found so that

$$\phi(\mathbf{x},\theta) = \gamma + \rho\xi - \frac{1}{3}\,\xi^3\,,$$

provided that ϕ is analytic. With this change of variables, the integral (6) becomes

$$u(\mathbf{x}) = e^{i\omega\gamma}\int a(\mathbf{x},\theta(\xi))\frac{d\theta}{d\xi}e^{i\omega(\rho\xi-\frac{1}{3}\,\xi^3)}d\xi\ .$$

The stationary points are given by

$$\frac{d}{d\xi}(\rho\xi - \frac{1}{3}\,\xi^3) = \rho - \xi^2 = 0\ .$$

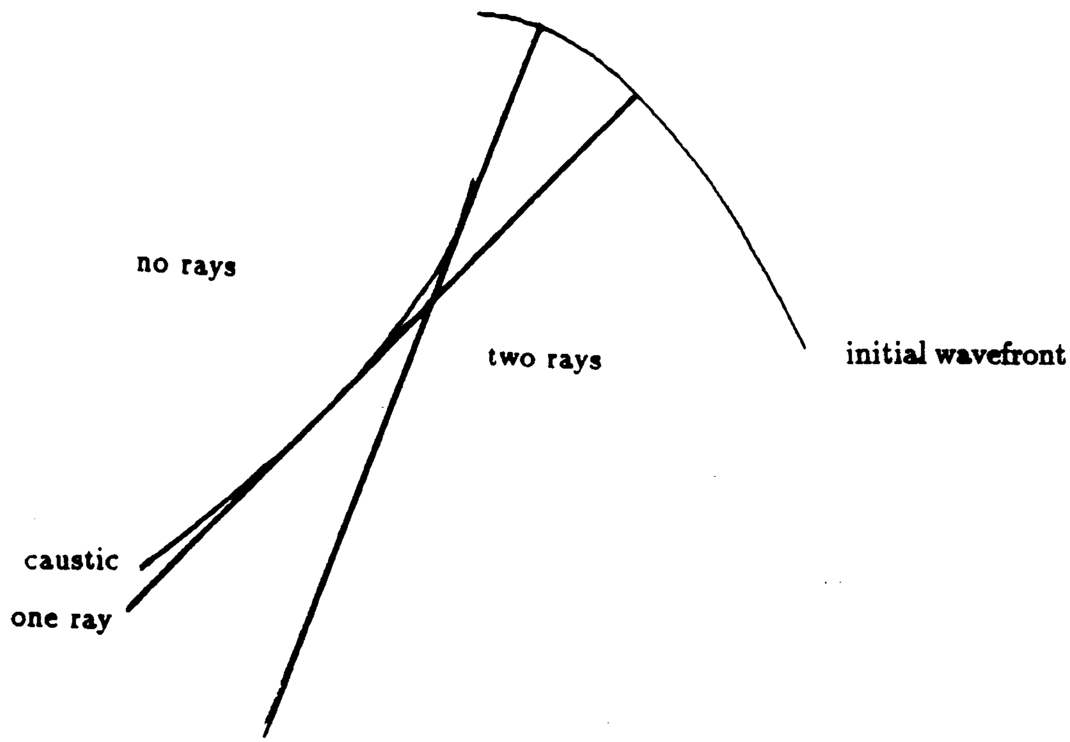


Figure 2.2. Each stationary point corresponds to a contribution from a ray.

As $\pm\sqrt{\rho} \to 0$, the second derivative also goes to zero, but the third derivative is constant.

By expanding $a(\mathbf{x}, \theta(\xi))\frac{d\theta}{d\xi}$ in a Taylor series

$$a(\mathbf{x}, \theta)\frac{d\theta}{d\xi} = a_0 + a_1\xi + (\xi^2 - \rho)Q(\xi)$$

and integrating by parts one can show that

$$\begin{aligned} u(\mathbf{x}) &= 2\pi e^{i\omega\gamma(\mathbf{x})}\left[\frac{a_0(\mathbf{x})}{\omega^{\frac{1}{3}}}\mathcal{A}i\left(-\omega^{\frac{2}{3}}\rho(\mathbf{x})\right) + \frac{a_1(\mathbf{x})}{i\omega^{\frac{2}{3}}}\mathcal{A}i'\left(-\omega^{\frac{2}{3}}\rho(\mathbf{x})\right)\right] \\ &\quad + 0\left(\frac{e^{i\omega\gamma}}{\omega}\right), \end{aligned}$$

where

$$\mathcal{A}i(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t\xi - \frac{1}{3}\xi^3)}d\xi$$

is an Airy function. This provides a new ansatz

$$u(\mathbf{x}) = e^{i\omega\gamma(\mathbf{x})}\left[a_0(\mathbf{x})\mathcal{A}i\left(-\omega^{\frac{2}{3}}\rho(\mathbf{x})\right) + \frac{a_1(\mathbf{x})}{i\omega^{\frac{1}{3}}}\mathcal{A}i'\left(\omega^{\frac{2}{3}}\rho(\mathbf{x})\right)\right]. \tag{7}$$

Substituting (7) into $\nabla^2 u + \omega^2 u = 0$ and collecting like powers of ω gives equations for $\gamma(\mathbf{x})$, $\rho(\mathbf{x})$, $a_0(\mathbf{x})$, and $a_1(\mathbf{x})$ corresponding to eikonal and transport equations. It can be shown that depending on the sign of $\rho(\mathbf{x})$, the eikonal equation is hyperbolic (illuminated region), elliptic (shadow region), and parabolic (caustic). (See Figure 2.3.)

In the illuminated region, the original geometrical optics approximation is still valid. Near the caustic, the new Airy function representation is appropriate. If the

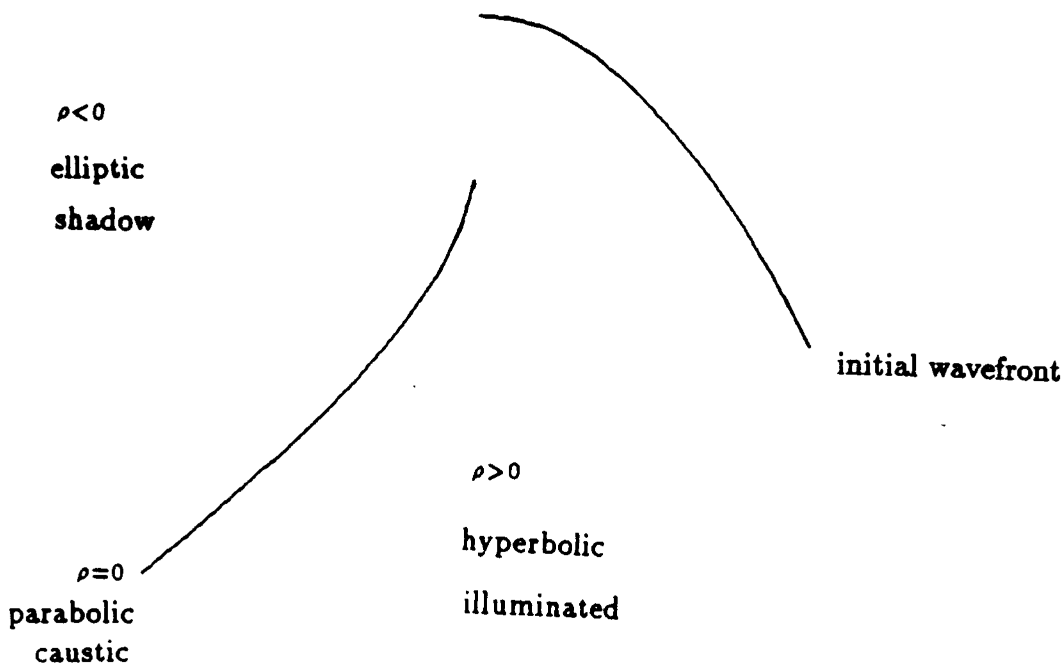


Figure 2.3. Depending on the sign of $\rho(\mathbf{x})$, the eikonal equation is hyperbolic, elliptic, or parabolic.

caustic is analytic and a_1 is analytic on the caustic, then γ, a_0, a_1 and ρ can be continued into the shadow region, and the solution can be interpreted in terms of complex rays.

There are similar results for cusped caustics, linear systems, and progressing wave expansions using generalized Airy functions or solutions to the Tricomi equation

$$\partial x^2 f(x,y) = x \partial y^2 f(x,y) \; .$$

Kravstov [7,8] has also developed uniform asymptotic solutions of hyperbolic equations, in particular, Maxwell's equations. Stickler et al. [9] have applied Ludwig's progressing wave ansatz to a point source problem.

2.3 Inverse Problems

There are many methods available for solving linearized inverse problems. Clayton and Stolt [10] use the WKBJ method to asymptotically obtain the Green's operators for the reference field. Then a comparison of the Green's operators to the kernel of a Fourier transform facilitates the relation of the data field to the medium parameters. Of course, the WKBJ method is invalid in the vicinity of turning points, which correspond to points on a caustic. Cohen and Bleistein [11] use a family of specific reference fields, that they call probes, to solve an integral equation of the form

$$\int u_0 v \alpha(r) dV = \Theta$$

for the medium parameter $\alpha(r)$. Stacking and migration methods are also based on ray theory. An illuminating exposition on the connection between geometrical optics and migration methods is found in a paper by Lailly [12].

Beylkin [13] gives a rigorous derivation of migration which he defines as a discontinuity imaging technique. He linearizes the inverse problem for the Helmholtz equation with a point source. That is,

$$\begin{aligned}
[\nabla^2 + k^2 n^2(\mathbf{x})]v &= \delta(\mathbf{x} - \eta) \\
n^2(\mathbf{x}) &= n_0^2(\mathbf{x}) + f(\mathbf{x}) \\
v &= v^{in} + v^{sc}
\end{aligned}$$

gives

$$\begin{aligned}
[\nabla^2 + k^2 n_0^2]v^{in} &= \delta(\mathbf{x} - \eta) \\
[\nabla^2 + k^2 n_0^2]v^{sc} &= -k^2 f(\mathbf{x}) v^{in} .
\end{aligned}$$

Geometrical optics is used to find the Green's function for v^{in} and for v^{sc}:

$$\begin{aligned}
v^{in} &\sim \alpha(k) A^{in}(\mathbf{x}, \eta) e^{ik\phi^{in}(\mathbf{x},\eta)} \\
v^{out} &\sim \alpha(k) A^{out}(\mathbf{x}, \xi) e^{ik\phi^{out}(\mathbf{x},\xi)} .
\end{aligned}$$

Then the scattered field is represented as

$$v^{sc}(k,\xi,\eta) = Wf(k,\xi,\eta) = (-ik)^{n-1}\int_X e^{ik\phi(x,\xi,\eta)}a(\mathbf{x},\xi,\eta)f(\mathbf{x})d\mathbf{x},$$

where $\phi = \phi^{in} + \phi^{out}$, $a = A^{in}A^{out}$, and η and ξ represent source and receiver positions.

The solution is shown to be related to the Fourier transform of a generalized Radon transform (GRT)

$$v(k,\xi,\eta) = (-ik)^{n-1}\widehat{Rf}(k,\xi,\eta)\ ,$$

where

$$\begin{aligned} Rf(t,\xi,\eta) &= \int f(\mathbf{x})a(x,\xi,\eta)\delta(t-\phi(\mathbf{x},\xi,\eta))d\mathbf{x}, \quad \text{for} \quad t \geq 0 \\ Rf(t,\xi,\eta) &= 0, \quad \text{for} \quad t \leq 0\ . \end{aligned}$$

This is a generalized Radon transform since $\{\phi = 0\}$ can be any hypersurface, not just a plane.

The inverse is constructed from the dual to the GRT, which is called a generalized backprojection operator. The operator

$$\begin{aligned} Ff(\mathbf{y}) = R^*\mathcal{F}^+Wf(\mathbf{y}) &= \frac{1}{(2\pi)^n}\int_0^\infty \int_{\partial X}\int_X e^{ik\Phi(\mathbf{x},\mathbf{y},\xi,\eta)}A(\mathbf{x},\mathbf{y},\xi,\eta) \\ &\quad \times f(\mathbf{x})d\mathbf{x}d\xi k^{n-1}dk \end{aligned}$$

is shown to be asymptotic to the identity operator. The operator

$$\mathcal{F}^+v(t) = \frac{t^{n-1}}{(2\pi)^n}\int_0^\infty v(k)e^{-ikt}dk$$

is applied since v^{sc} is needed in the time domain. Also,

$$\Phi(\mathbf{x},\mathbf{y},\xi,\eta) = \phi(\mathbf{x},\xi,\eta) - \phi(\mathbf{y},\xi,\eta) .$$

Since the Taylor series for Φ is

$$\Phi = \nabla_y \phi(\mathbf{y},\xi,\eta) \cdot (\mathbf{x}-\mathbf{y}) + \mathcal{O}(|\mathbf{x}-\mathbf{y}|^2) ,$$

the following change of variables makes $Ff(y)$ look almost like the identity operating on $f(y)$:

$$\begin{aligned} \mathbf{p} &= k\nabla_y \phi(\mathbf{y},\xi,\eta) \\ d\mathbf{p} &= k^{n-1} h(\mathbf{y},\xi) d\xi dk . \end{aligned}$$

Then

$$Ff(y) \sim (I_{\partial x_\eta^0} f)(y) = \frac{1}{(2\pi)^n} \int_{\Omega_\eta(y)} e^{i\mathbf{p}\cdot\mathbf{y}} \hat{f}(\mathbf{p}) d\mathbf{p} ,$$

where $\Omega_\eta(\mathbf{y})$ is the part of the region where no caustics are present. Of course, it is critical to this method that

$$h(\mathbf{y},\xi) = \det \begin{bmatrix} \phi_{y_1} & \phi_{y_2} & \cdots & \phi_{y_n} \\ \phi_{y_1\xi_1} & \phi_{y_2\xi_1} & \cdots & \phi_{y_n\xi_1} \\ \vdots & \vdots & & \vdots \\ \phi_{y_1\xi_{n-1}} & \phi_{y_2\xi_{n-1}} & \cdots & \phi_{y_n\xi_{n-1}} \end{bmatrix} \tag{8}$$

be non-zero. This is equivalent to the condition that there be no caustics present. In fact, Beylkin uses a cutoff function to ensure that $h(\mathbf{y},\xi) \neq 0$. This is why $\Omega_\eta(\mathbf{y})$

is not the whole region. The migration scheme is then given by

$$f_{mig} \sim Re\ I_{\partial x^0} f\ .$$

Only the discontinuities of f are recovered since $Re\ I_{\partial x^0} f$ is just the leading order asymptotic behavior.

CHAPTER 3

Plane Wave Velocity Perturbation

The first step in solving an inverse problem is making sure we can solve the forward problem associated with it. A potential problem is that in the right hand side of the perturbational equation is the product of δc and $u_{0_{tt}}$. One might expect that a perturbation located on a caustic in the reference field would be illuminated. That is, it would have an effect on the scattered field. It is not clear that the trace of the scattered field can be represented using geometric optics or the methods of Ludwig.

In this chapter we analyze an appropriate problem in order to investigate the effect on the scattered signal of having a caustic in the reference wavefield. We decompose initial data into plane waves as is suggested by Ludwig's construction. Then geometrical optics is used to propagate each plane wave through the perturbation. The plane waves are put back together again, and caustics are located by the failure of a stationary phase calculation to reduce the number of integration variables.

3.1 An Appropriate Problem

To begin this investigation, we analyze the simplified problem of a strictly convex wavefront moving into a two-dimensional region where the reference wavespeed is constant, $c_0 = 1$. Since the wavefront is convex, a cusped caustic will develop even though the region is homogeneous. We take δc to be a localized high frequency perturbation in some direction $\hat{\mathbf{k}}$,

$$\delta c = \chi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $\chi(\mathbf{x})$ is an envelope function. We will later discuss what happens when solutions are summed over $\mathbf{k}$. See Figure 3.1 for an illustration.

Next, we choose the form of the singularity in the initial data. The fundamental solution of the wave equation in 2-D with $U \equiv 0$ for $t < 0$ and $c \equiv 1$ is

$$U(\mathbf{x}, t) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2 - x^2}}H(t - x),$$

which suggests using the following initial data:

$$U_0(\mathbf{x}, 0) = \alpha(\mathbf{x})\frac{1}{\sqrt{\psi(\mathbf{x})}}H(-\psi(\mathbf{x}))$$

where $\psi(\mathbf{x}) = 0$ describes the initial wavefront, and $\alpha(\mathbf{x})$ is smooth, with compact support. The initial value $\frac{\partial U}{\partial t}(\mathbf{x}, 0)$ is determined by specifying that it be a single progressing wave, to leading order.

To be assured that a simple caustic will occur, we assume that the incident

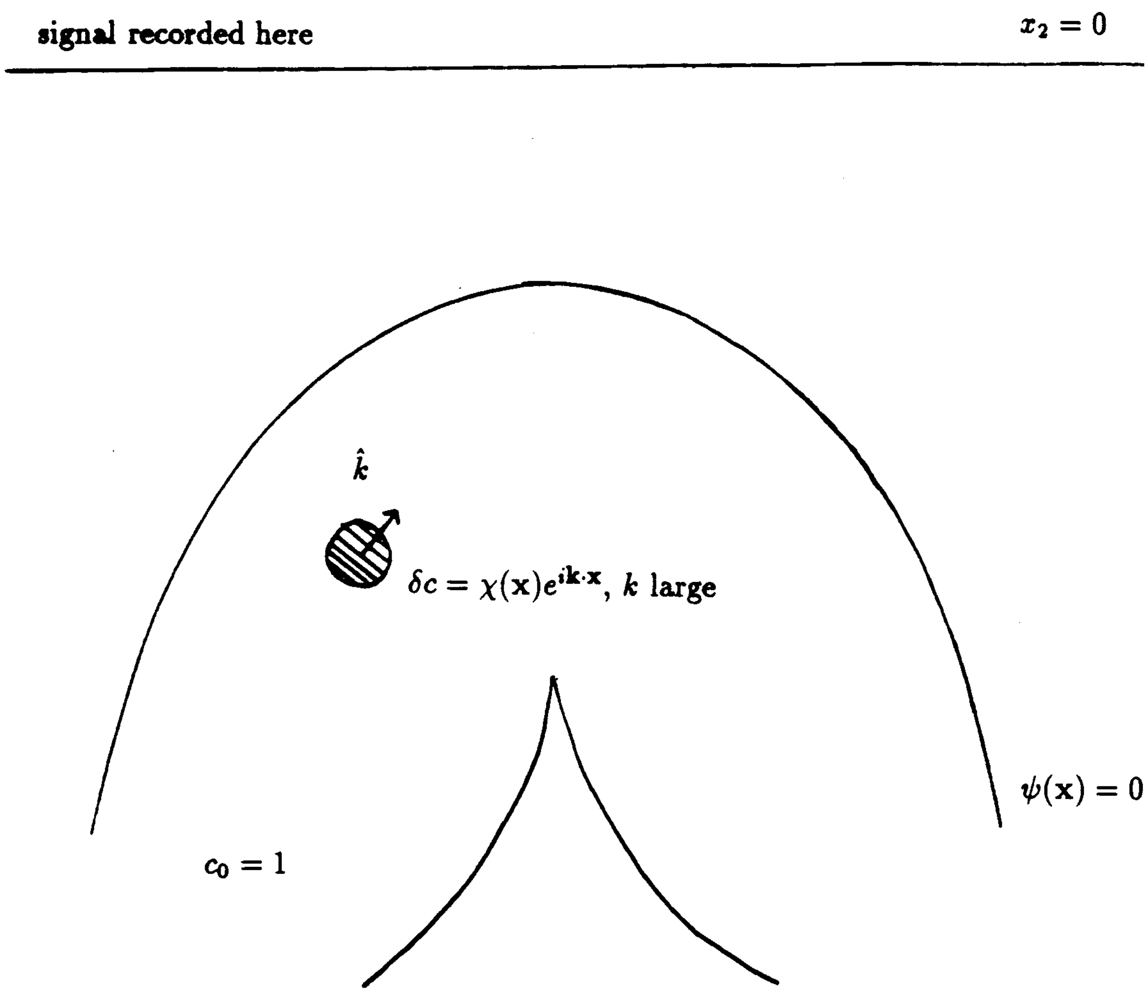


Figure 3.1. The problem to be analyzed.

wavefront is strictly convex. To simplify calculations, we also assume that $\psi(\mathbf{x})$ can be described by its graph, $\psi(\mathbf{x}) = x_2 - \Gamma(x_1)$.

3.2 Decompose Initial Data into Plane Waves

Since the geometrical optics representation breaks down near a caustic, we decompose the initial data into plane wave components, each of which can be propagated through the region containing the perturbation δc without forming a caustic:

$$U(\mathbf{x},0) = U_0(\mathbf{x},0) = \int_{|\theta|=1} d\theta u_0(\mathbf{x},0;\theta) \; .$$

Then

$$\delta U(\mathbf{x},t) = \int_{|\theta|=1} d\theta \delta u(\mathbf{x},t;\theta)$$

where

$$\begin{aligned} &\delta u_{tt} - \nabla^2 \delta u = 2\chi(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} u_{0_{tt}} \\ &\delta u = \delta u_t = 0, \quad \text{for} \quad t < 0 \; . \end{aligned} \tag{9}$$

To accomplish the decomposition we write the initial data as the inverse of its Fourier transform:

$$U(\mathbf{x},0) = \int_{|\theta|=1} d\theta \int_0^\infty d\omega\, \omega e^{i\omega\theta\cdot\mathbf{x}} F(U(\mathbf{x},0)) \tag{10}$$

where

$$F(U(\mathbf{x},0)) = \int_{\Re^2} d\mathbf{y}\; e^{-i\omega\theta\cdot\mathbf{y}} \alpha(\mathbf{y}) \frac{1}{\sqrt{\psi(\mathbf{y})}} H(-\psi(\mathbf{y}))$$

is the Fourier transform of $U(\mathbf{x},0)$. Making the change of variables $\eta = (y_1, -\psi(y)) = (y_1, \Gamma(y_1) - y_2)$ and evaluating $H(\eta_2)$ gives

$$F(U(\mathbf{x},0)) = \int_{-\infty}^{\infty} d\eta_1 \int_0^{\infty} d\eta_2 i\alpha(\eta_1, \Gamma(\eta_1) - \eta_2)\frac{1}{\sqrt{\eta_2}}e^{-i\omega\theta\cdot(\eta_1,\Gamma(\eta_1)-\eta_2)}.$$

By writing $\alpha(\eta_1, \Gamma(\eta_1) - \eta_2) = \alpha(\eta_1, \Gamma(\eta_1)) + \eta_2\frac{\alpha(\eta_1,\Gamma(\eta_1)-\eta_2)-\alpha(\eta_1,\Gamma(\eta_1))}{\eta_2}$ and invoking the assumption that α and $\frac{\partial\alpha}{\partial\eta_2}$ have compact support in η_2, $F(U(\mathbf{x},0))$ can be approximated by

$$F(U(\mathbf{x},0)) =$$
$$\int_{-\infty}^{\infty} d\eta_1 ie^{-i\omega\theta\cdot(\eta_1,\Gamma(\eta_1))}\left[\alpha(\eta_1,\Gamma(\eta_1))\int_0^{\infty} d\eta_2\frac{1}{\sqrt{\eta_2}}e^{i\omega\theta_2\eta_2} + 0(\frac{1}{\omega})\right].$$

The η_2 integral is evaluated by using Cauchy's integral theorem to write it as

$$\int_0^{\infty} d\eta_2\frac{1}{\sqrt{\eta_2}}e^{i\omega\theta_2\eta_2} = \int_0^{sgn\theta_2 i\infty} d\eta_2\frac{1}{\sqrt{\eta_2}}e^{i\omega\theta_2\eta_2},$$

then rotating the contour of integration from the real η_2 axis to the positive or negative imaginary η_2 axis depending on the sign, $sgn\ \theta_2$. Then

$$F(U(\mathbf{x},0)) = \int_{-\infty}^{\infty} d\eta_1 ie^{-i\omega\theta\cdot(\eta_1,\Gamma(\eta_1))}\alpha(\eta_1,\Gamma(\eta_1))e^{sgn\theta_2 i\frac{\pi}{4}}\sqrt{\frac{\pi}{\omega|\theta_2|}} + 0(\frac{1}{\omega}).$$

The η_1 integral is then evaluated asymptotically using the method of stationary phase. Letting η_1^* represent the stationary point defined by $\Gamma'(\eta_1^*) = -\frac{\theta_1}{\theta_2}$, we have

$$F(U(\mathbf{x},0)) \sim i\alpha(\eta_1^*,\Gamma(\eta_1^*))\frac{\pi}{\omega|\theta_2|}\sqrt{\frac{2}{\Gamma''(\eta_1^*)}}e^{-i\omega\theta\cdot(\eta_1^*,\Gamma(\eta_1^*))}.$$

Notice that to each point on the initial wavefront $\psi(\eta) = 0$ is associated a direction θ, which is the unit normal to $\psi(\eta) = 0$ at that point. That is,

$$\theta = \frac{\nabla\psi}{|\nabla\psi|}(\eta_1^*) \tag{11}$$

since the stationary point is given by $\theta \cdot (1, \Gamma'(\eta_1^*)) = 0$ and $\psi(\eta) = \eta_2 - \Gamma(\eta_1)$.

Substituting the asymptotic evaluation of $F(U(\mathbf{x}, 0))$ into the expression (10) for $U(\mathbf{x}, 0)$ gives

$$U(\mathbf{x}, 0) \sim \int_{|\theta|=1} d\theta i\alpha(\eta_1^*, \Gamma(\eta_1^*)) \frac{\pi}{|\theta_2|} \sqrt{\frac{2}{\Gamma''(\eta_1^*)}} \int_0^\infty dw \; e^{i\omega\theta\cdot(\mathbf{x}-(\eta_1^*,\Gamma(\eta_1^*)))}$$

The limits of integration of the ω integral can be extended to include the negative ω axis by changing the sign of both ω and θ. Then

$$\int_{-\infty}^\infty d\omega \; e^{i\omega\theta\cdot(\mathbf{x}-(\eta_1^*,\Gamma(\eta_1^*)))} = \delta(\mathbf{x}\cdot\theta - (\eta_1^*, \Gamma(\eta_1^*))\cdot\theta),$$

and the initial data is decomposed into plane wavefronts with a δ-function singularity,

$$U(\mathbf{x}, 0) \sim \int_{|\theta|=1} d\theta i\alpha(\eta_1^*, \Gamma(\eta_1^*)) \frac{\pi}{|\theta_2|} \frac{1}{\sqrt{2\Gamma''(\eta_1^*)}} \delta(\mathbf{x}\cdot\theta - (\eta_1^*, \Gamma(\eta_1^*))\cdot\theta).$$

Let $\tau(\theta) = -(\eta_1^*, \Gamma(\eta_1^*))\cdot\theta$. This is the distance from the line $\mathbf{x}\cdot\theta = 0$ to the tangent to the curve $\psi(\mathbf{x}) = 0$ at $(\eta_1^*, \Gamma(\eta_1^*))$ and can be considered as a time delay when solving (9). (See Figure 3.2.) To keep the notation simple we will drop $\tau(\theta)$ until we obtain $\delta u(\mathbf{x}, t; \theta)$, at which point we will shift $t \to t - \tau(\theta)$.

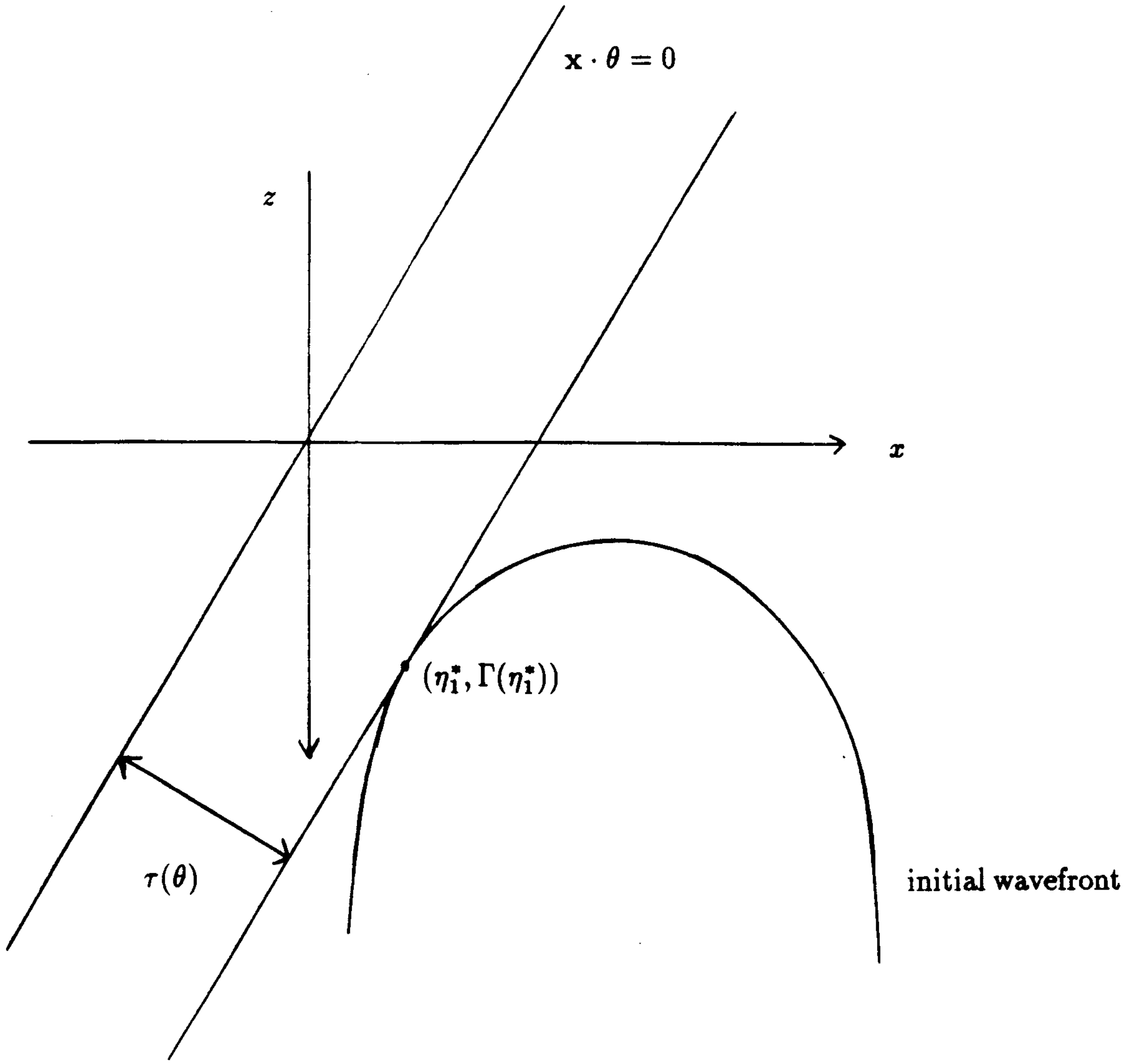


Figure 3.2. The time delay $\tau(\theta)$ is the distance from the line $\mathbf{x} \cdot \theta = 0$ to the tangent to the curve $\psi(\mathbf{x}) = 0$ at $(\eta_1^*, \Gamma(\eta_1^*))$.

3.3 Geometrical Optics Approximation

We can now solve each plane wave problem. Clearly, the reference field u_0 is of the form $u_0(x,t;\theta) = a(\theta)\delta(t - x\cdot\theta)$. Then a progressing plane wave representation can be derived for the perturbational field δu, where

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u = 2\chi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\partial^2}{\partial t^2}u_0 \tag{12}$$

This representation will be valid up to the time that caustics form in the perturbational field.

The calculation of δu is very similar to a calculation done by Symes and Santosa in [13]. We guess that δu has the expansion

$$\delta u(\mathbf{x},t;\theta) =$$

$$b_1(\mathbf{x},t)\delta'(t-\phi(\mathbf{x})) + b_2(\mathbf{x},t)\delta(t-\phi(\mathbf{x})) + b_3(\mathbf{x},t)H(t-\phi((\mathbf{x})),$$

where $\phi(x)$ is the incident phase $\phi(\mathbf{x}) = \mathbf{x}\cdot\theta$.

Substitution of this expression into (12) with $|\nabla\phi|^2 = |\theta|^2 = 1$ gives

$$\begin{aligned}
& (2b_{1t} + 2\nabla b_1\cdot\nabla\phi + b_1\nabla^2\phi - 2\chi(x)e^{i\mathbf{k}\cdot\mathbf{x}}a)\delta''(t-\phi) \\
+ \quad & (2b_{2t} + 2\nabla b_2\cdot\nabla\phi + b_2\nabla^2\phi + b_{1tt} - \nabla^2 b_1)\delta'(t-\phi) \\
+ \quad & (2b_{3t} + 2\nabla b_3\cdot\nabla\phi + b_3\nabla^2\phi + b_{2tt} - \nabla^2 b_2)\delta(t-\phi) \\
+ \quad & (b_{3tt} - \nabla^2 b_3)H(t-\phi) = 0
\end{aligned}$$

or $f_1\delta''(t-\phi)+f_2\delta'(t-\phi)+f_3\delta(t-\phi)+f_4H(t-\phi)=0$. For this to hold, the following conditions on the coefficients f_1-f_4 must be satisfied:

$$\left.\begin{array}{l} f_1=0 \\ 2f_{1t}-f_2=0 \\ f_{1tt}-2f_{2t}+f_3=0 \end{array}\right\} on\ t=\phi \qquad (13)$$
$$f_4=0\ in\ t>\phi\,.$$

The first condition in (13) gives an equation for $b_1(\mathbf{x},t)$ on the incident wavefront, $t=\phi(\mathbf{x})$:

$$2b_{1t}+2\nabla b_1\cdot\nabla\phi+b_1\nabla^2\phi=2\chi(x)e^{i\mathbf{k}\cdot\mathbf{x}}a. \qquad (14)$$

Since the high frequency perturbation, $\delta c=\chi(x)e^{i\mathbf{k}\cdot\mathbf{x}}$, causes energy to be reflected, we assume that $b_1(\mathbf{x},t)$ has an asymptotic expansion of the form

$$b_1(\mathbf{x},t)\sim\sum_{n=1}^{\infty}\frac{b_1^n(\mathbf{x})}{(ik)^n}e^{ik\phi_r(\mathbf{x},t)}, \qquad (15)$$

where ϕ_r is the reflected phase. Upon substituting (15) into (14) and matching powers of k^{-1}, the 0(1) term obtained is

$$2b_1^1\phi_{r_t}+2b_1^1\nabla\phi_r\cdot\nabla\phi=2\chi ae^{i(\mathbf{k}\cdot\mathbf{x}-k\phi_r)}$$
$$on\ \ t=\phi(\mathbf{x})\,. \qquad (16)$$

This implies that

$$k\phi_r(\mathbf{x},\phi(\mathbf{x}))\ =\ \mathbf{k}\cdot\mathbf{x} \qquad (17)$$

$$\begin{aligned}\nabla\phi\cdot\nabla[k\phi_r(\mathbf{x},\phi(\mathbf{x}))] &= \nabla\phi\cdot\nabla[\mathbf{k}\cdot\mathbf{x}] \\ \nabla\phi_r\cdot\nabla\phi+\phi_{r_t}|\nabla\phi|^2 &= \hat{\mathbf{k}}\cdot\nabla\phi. \qquad (18)\end{aligned}$$

Note that the initial condition together with the eikonal equation implies Snell's equal angle law of reflection. From (17),

$$\frac{\hat{\mathbf{k}}}{\phi_{r_t}}=\frac{\nabla\phi_r}{\phi_{r_t}}+\nabla\phi\,.$$

That is, the vector $\frac{\hat{k}}{\phi_{r_t}}$ is the sum of two unit vectors since $|\nabla\phi_r|^2=\phi_{r_t}^2$ and $|\nabla\phi|^2=1$. The physical ray $\frac{dx}{dt}=-\frac{\nabla\phi_r}{\phi_{r_t}}$ is given by the Hamilton-Jacobi equations. Then

$$-\frac{\hat{\mathbf{k}}}{\phi_{r_t}}=\frac{dx}{dt}-\nabla\phi\,,$$

which is precisely Snell's law, as can easily be seen from Figure 3.3.

Using (18) in (16) and $\phi(\mathbf{x})=\mathbf{x}\cdot\theta$ gives

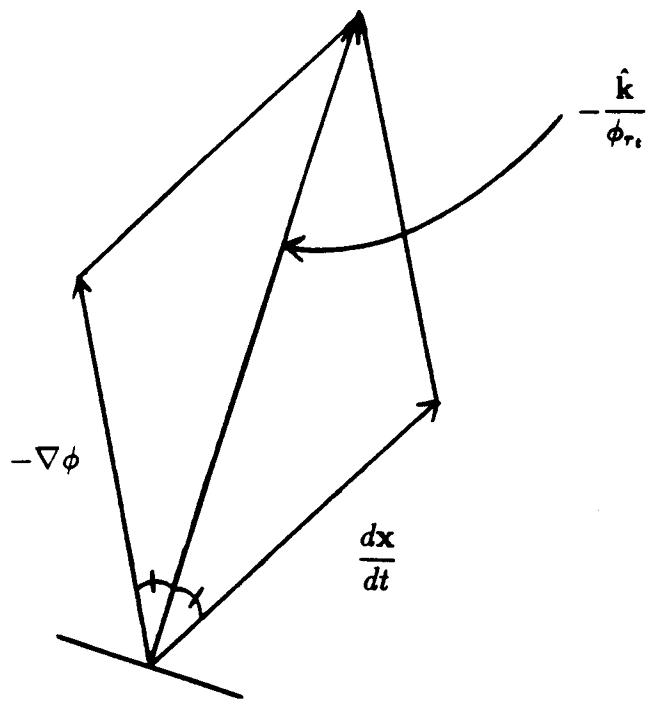
$$b_1^1=\frac{\chi a}{\hat{\mathbf{k}}\cdot\nabla\phi}=\frac{\chi a}{\hat{\mathbf{k}}\cdot\theta}\,.$$

The second condition in (13) gives the following differential equation for $b_2(\mathbf{x},t)$ on the incident wavefront:

$$\begin{aligned}&2b_{2_t}+2\nabla b_2\cdot\nabla\phi+b_2\nabla^2\phi= \\ &\qquad 3b_{1_{tt}}+4\nabla b_{1_t}\nabla^2\phi+\nabla^2 b_1 \quad \textit{on } t=\phi(x)\,. \qquad (19)\end{aligned}$$

Assuming that $b_2(\mathbf{x},t)$ can be represented by

$$b_2(\mathbf{x},t)\sim\sum_{n=0}^{\infty}\frac{b_2^n(\mathbf{x})}{(ik)^n}e^{ik\phi_r(\mathbf{x},t)},$$



$$-\frac{\hat{\mathbf{k}}}{\phi_{r_t}} = \frac{d\mathbf{x}}{dt} - \nabla\phi$$

Figure 3.3. Snell's equal angle law of reflection.

and substituting this expression into (19) yields

$$2b_2^0\phi_{r_t} + 2b_2^0\nabla\phi_r\cdot\nabla\phi = 3b_1^1\phi_{r_t}^2 + 4b_1^1\phi_{r_t}\nabla\phi_r\cdot\nabla\phi + b_1^1|\nabla\phi_r|^2$$

as the highest order term, $\mathcal{O}(k)$. We use the eikonal equation for the reflected phase, $\phi_{r_t}^2| = |\nabla\phi_r^2|$, which comes out of the fourth condition in (13) and will be derived later in this section. Then

$$b_2^0 = 2b_1^1\phi_{r_t}(\mathbf{x},\phi(\mathbf{x})) = \frac{2\chi a}{\hat{\mathbf{k}}\cdot\theta}\phi_{r_t}(\mathbf{x},\phi(\mathbf{x})).$$

The third condition in (13) gives an equation for $b_3(\mathbf{x},t)$ on $t=\phi(\mathbf{x})$:

$$\begin{aligned} 2b_{3_t} + 2\nabla b_3\cdot\nabla\phi + b_3\nabla^2\phi &= -b_{1_{ttt}} - 2\nabla b_{1_{tt}}\cdot\nabla\phi - b_{1_{tt}}\nabla^2\phi - \nabla^2 b_{1_t} \\ &\quad + b_{2_{tt}} + 2\nabla b_{2_t}\cdot\nabla\phi + b_{2_t}\nabla^2\phi + \nabla^2 b_2 \\ &\quad\quad on\ t=\phi(\mathbf{x})\ . \end{aligned} \tag{20}$$

Substituting the expansion

$$b_3(\mathbf{x},t) = \sum_{n=-1}^{\infty}\frac{b_3^n(\mathbf{x},t)}{(ik)^n}e^{ik\phi_r(\mathbf{x},t)} \tag{21}$$

into (20) yields the following as the highest order, $\mathcal{O}(k^2)$, term:

$$\begin{aligned} 2b_3^{-1}\phi_{r_t} + 2b_3^{-1}\nabla\phi_r\cdot\nabla\phi &= -b_1^1\phi_{r_t}^3 - 2b_1^1\phi_{r_t}^2\nabla\phi_r\cdot\nabla\phi - b_1^1|\nabla\phi_r|^2 \\ &\quad + b_2^0\phi_{r_t}^2 + 2b_2^0\phi_{r_t}\nabla\phi_r\cdot\nabla\phi + b_2^0|\nabla\phi_r|^2\ . \end{aligned}$$

Again using $\phi_{r_t}^2 = |\nabla\phi_r|^2$ we get

$$\begin{aligned} b_3^{-1} &= -b_1^1\phi_{r_t} + b_2^0\phi_{r_t} \quad on\ t = \phi(\mathbf{x}) \\ &= \frac{\chi a}{\hat{\mathbf{k}}\cdot\theta}\phi_{r_t}^2(\mathbf{x}, \phi(\mathbf{x}))\ . \end{aligned} \tag{22}$$

Finally, the fourth condition in (13) is the wave equation for $b_3(\mathbf{x}, t)$ in the light cone $t > \phi(\mathbf{x})$:

$$b_{3_{tt}} - \nabla^2 b_3 = 0\ . \tag{23}$$

Using the above expansion for $b_3(\mathbf{x}, t)$ in (23), we get from the $0(k^3)$ and $0(k^2)$ terms the eikonal and first transport equations:

$$\phi_{r_t}^2 - |\nabla\phi_r|^2 = 0 \tag{24}$$

$$2\phi_{r_t} b_{3_t}^{-1} - 2\nabla b_3^{-1}\cdot\nabla\phi_r + (\phi_{r_{tt}} - \nabla^2\phi_r)b_3^{-1} = 0, \quad t > \phi(\mathbf{x})\ . \tag{25}$$

The solution to the eikonal equation (24) with the initial condition $k\phi_r(\mathbf{x}, \phi(\mathbf{x})) = \mathbf{k}\cdot\mathbf{x}$, where $\phi(\mathbf{x}) = \mathbf{x}\cdot\theta$, is easily found to be

$$\phi_r(\mathbf{x}, t) = \left(\hat{\mathbf{k}}, -\frac{\theta}{2\hat{\mathbf{k}}\cdot\theta}\right)\cdot\mathbf{x} + \frac{1}{2\hat{\mathbf{k}}\cdot\theta}t.$$

Then the transport equation (25) reduces to

$$\phi_{r_t} b_{3_t}^{-1} - \nabla b_3^{-1}\cdot\nabla\phi_r = 0.$$

Since $\frac{db_3^{-1}}{d\sigma} = b_{3_t}^{-1}\frac{dt}{d\sigma} + \nabla b_3^{-1}\cdot\frac{dx}{d\sigma}$, b_3^{-1} is constant along rays described by

$$\begin{aligned} \frac{dx}{d\sigma} &= \nabla\phi_r = \hat{\mathbf{k}} - \frac{\theta}{2\hat{\mathbf{k}}\cdot\theta} \\ \frac{dt}{d\sigma} &= -\phi_{r_t} = -\frac{1}{2\hat{\mathbf{k}}\cdot\theta} \end{aligned}$$

or $\frac{dx}{dt} = \theta - 2\hat{\mathbf{k}} \cdot \theta \hat{\mathbf{k}}$.

On the incident wavefront, b_3^{-1} is given by (22) so that

$$b_3^{-1}(\mathbf{x}, t) = \frac{\chi(\mathbf{x}_0)}{4(\hat{\mathbf{k}} \cdot \theta)^3} a(\theta) \quad on \quad \mathbf{x} = \mathbf{x}_0 - (2\hat{\mathbf{k}} \cdot \theta \hat{\mathbf{k}} - \theta)(t - \theta \cdot \mathbf{x}_0) \ .$$

Then, shifting the time variable t to incorporate the time delay $\tau(\theta)$ from the plane wave decomposition, the perturbational field can be represented as follows:

$$\begin{aligned} \delta u(\mathbf{x}, t; \theta) \quad \sim \quad & e^{ik[(\hat{\mathbf{k}} - \frac{\theta}{2\hat{\mathbf{k}} \cdot \theta}) \cdot \mathbf{x} + \frac{1}{2\hat{\mathbf{k}} \cdot \theta}(t - \tau(\theta))]} \\ & \times \left\{ -\frac{\chi(\mathbf{x}) a(\theta) i}{k\hat{\mathbf{k}} \cdot \theta} \delta'(t - \tau(\theta) - \mathbf{x} \cdot \theta) \right. \\ & + \frac{\chi(\mathbf{x}) a(\theta)}{(\hat{\mathbf{k}} \cdot \theta)^2} \delta(t - \tau(\theta) - \mathbf{x} \cdot \theta) \\ & \left. + \ \frac{\chi(\mathbf{x}_0) a(\theta) ik}{4(\hat{\mathbf{k}} \cdot \theta)^3} H(t - \tau(\theta) - \mathbf{x} \cdot \theta) \right\} \end{aligned}$$

where $\mathbf{x} = \mathbf{x}_0 - (2\hat{\mathbf{k}} \cdot \theta \hat{\mathbf{k}} - \theta)(t - \tau(\theta) - \mathbf{x}_0 \cdot \theta)$ (or $x_0 = (I + (2\hat{\mathbf{k}} \cdot \theta \hat{\mathbf{k}} - \theta) \otimes \theta)^{-1}(\mathbf{x} + (2\hat{\mathbf{k}} \cdot \theta \hat{\mathbf{k}} - \theta)t)$).

3.4 Sum the Plane Waves

After summing over the contributions from each plane wave, the perturbational field at the surface $x_2 = 0$ is

$$\begin{aligned} \delta U\big|_{x_2=0} = \int_{|\theta|=1} d\theta \delta u \ \big|_{x_2=0} \sim \\ \int_{|\theta|=1} d\theta \frac{ik}{4(\hat{\mathbf{k}} \cdot \theta)^3} \chi(\mathbf{x}_0) a(\theta) e^{ik\phi_r(\mathbf{x}, t - \tau(\theta))} \ \big|_{x_2=0} \end{aligned}$$

since the incident wavefront moves away from $x_2 = 0$.

In order to evaluate the integral it is convenient to change variables. There is a natural change of variables given by the correlation (11) between a direction θ and a point on the initial wavefront. We let $\mathbf{z}(z_1) = (z_1, \Gamma(z_1))$, where $\psi(\mathbf{z}) = 0$. Then $\theta(z_1) = \frac{\nabla\psi(\mathbf{z})}{|\nabla\psi(\mathbf{z})|} = (1 + \Gamma'^2(z_1))^{-\frac{1}{2}}(-\Gamma'(z_1), 1)$ and

$$\tau(\theta(z_1)) = -\mathbf{z}(z_1) \cdot \theta(z_1) = (1 + \Gamma'^2(z_1))^{-1/2}(-\Gamma(z_1) + z_1\Gamma'(z_1)].$$

The curve $\psi(\mathbf{z}) = 0$ is strictly convex so that the map $\theta \mapsto z_1$ is invertible. This gives

$$\begin{aligned} \delta U \mid_{x_2=0} \quad \sim \quad & -ik \int_{-\infty}^{\infty} dz_1 \chi(\mathbf{x}_0(\theta(z_1))) \\ & \frac{a(\theta)}{4\hat{\mathbf{k}} \cdot \theta} \mid_{x_2=0} \frac{\Gamma''(z_1)}{1 + \Gamma'^2(z_1)} e^{ik\tilde{\phi}_r(z_1;x_1,t,\hat{k})} \end{aligned} \tag{26}$$

where

$$\begin{aligned} \tilde{\phi}_r(z_1; x_1, t, \hat{k}) \quad &= \quad \phi_r(\mathbf{x}, t - \tau(\theta(z_1)))|_{x_2=0} \\ &= \quad \frac{1}{2}(\hat{k}_2 - \hat{k}_1\Gamma'(z_1))^{-1}[(1 + \Gamma'^2(z_1))^{\frac{1}{2}}t \\ & \qquad + \Gamma(z_1) - z_1\Gamma'(z_1) + x_1(\Gamma(z_1) + 2(\hat{k}_2 - \hat{k}_1\Gamma')\hat{k}_1)] \ . \end{aligned}$$

3.5 Locate Caustics

Caustics occur where the stationary phase expansion of (26) is invalid, which is where the phase $\tilde{\phi}_r(z_1)$ has degenerate stationary points. Using

$$\begin{aligned}\frac{d\tilde{\phi}_r}{dz_1} &= \frac{\Gamma''}{2(\hat{k}_2 - \hat{k}_1\Gamma')}\left[2\hat{k}_1\tilde{\phi}_r + (1+\Gamma'^2)^{-1/2}\Gamma' t - z_1 + x_1(1-2\hat{k}_1^2)\right] \\ &= 0\end{aligned}$$

gives

$$\frac{d^2\phi_r}{dz_1^2} = \frac{\Gamma''}{2(\hat{k}_2 - \hat{k}_1\Gamma')}\left[\frac{\Gamma''}{(1+\Gamma'^2)^{3/2}}t - 1\right].$$

Thus the caustic locus of the perturbational field on the line $x_2 = 0$ is

$$t = \frac{1}{\Gamma''(z_1)}(1+\Gamma'^2(z_1))^{3/2}. \qquad (27)$$

Note that it is independent of the perturbation direction $\hat{\mathbf{k}}$ and the receiver location x_1. Also notice that the right hand side is the radius of curvature of $\psi(\mathbf{z}) = 0$. This leads one to think that (27) should also represent the caustic locus of the incident wavefront. This is verified by studying the map $(z_1, t) \mapsto (\tilde{z}_1, \tilde{z}_2)$ by which points on the initial wavefront are mapped along rays to the wavefront at a later time t. That is,

$$(\tilde{z}_1, \tilde{z}_2) = (z_1, \Gamma(z_1)) + t(1+\Gamma'^2(z_1))^{-1/2}(-\Gamma'(z_1), 1).$$

A caustic is found by looking for where this map is singular. The Jacobian is

$$\det \nabla_{(z_1,t)}(\tilde{z}_1, \tilde{z}_2) = (\Gamma'^2(z_1)+1)^{-1}[(\Gamma'^2(z_1)+1)^{3/2} - t\Gamma''(z_1)],$$

which is zero precisely when

$$t = \frac{1}{\Gamma''(z_1)}(1 + \Gamma'^2(z_1))^{3/2}.$$

Thus, if there is a caustic in the perturbational field at the surface $x_2 = 0$, then the distance along a ray from the incident wavefront to the incident caustic must equal the distance along a ray from the incident wavefront to the perturbation plus the distance along the corresponding reflected ray to the perturbational caustic. This can only happen when the perturbation $\delta c = \chi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}$ is located above the incident caustic. (See Figure 3.4.) In that case, it is possible to represent the scattered field at the surface using the methods of Ludwig.

If the perturbation is located near the incident caustic, the only caustic present in the perturbational field is that piece of the incident caustic. Instead of amplifying the perturbational field, the caustic causes the reflected rays to be spread out. (See Figure 3.5.) If the perturbation is below the incident caustic, there will be no caustic at all in the perturbational field. In fact, the incident wavefront is concave there, so that the reflected wavefront is also concave. (See Figure 3.6.) For the cases above, the scattered field at the surface can be represented by a geometrical optics expansion. In any case, δU can be represented using known methods, even if the perturbation sits on a caustic.

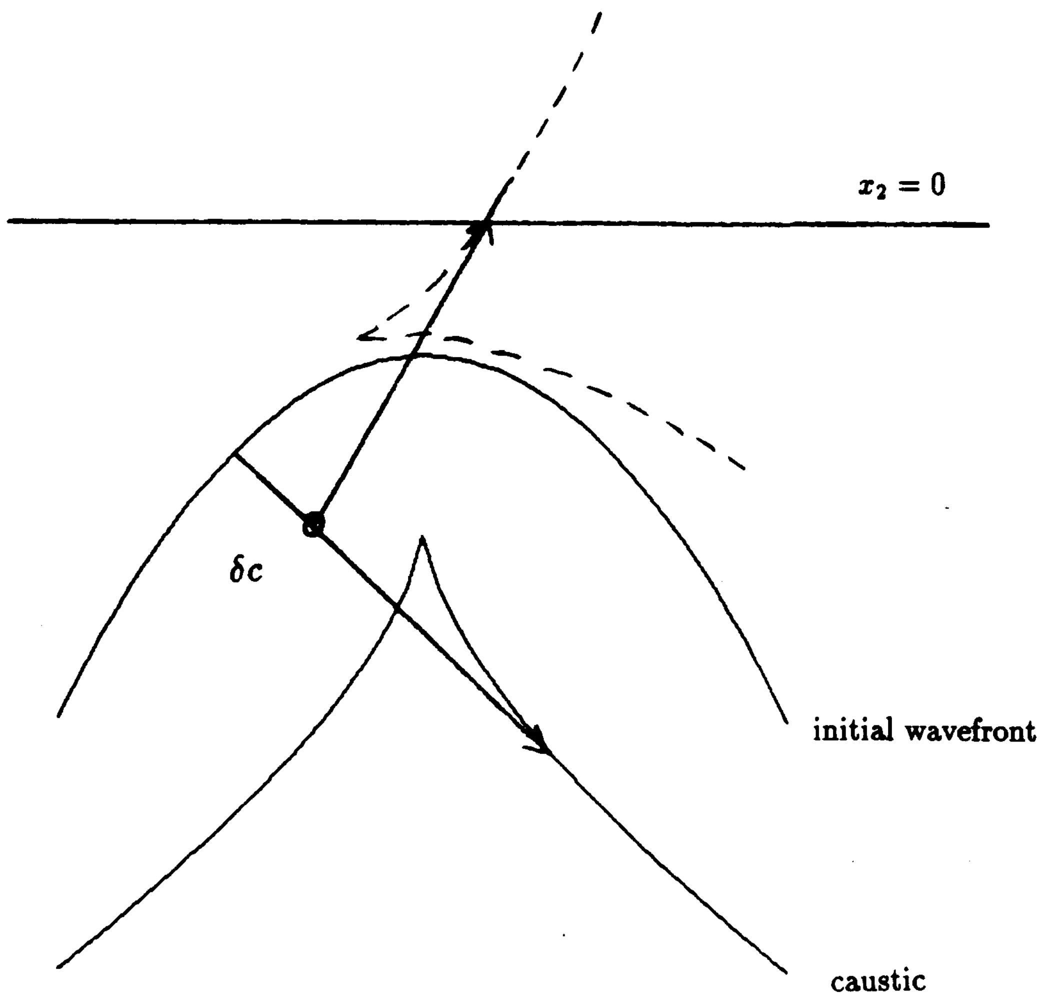


Figure 3.4. The perturbation is located above the caustic in the reference field.

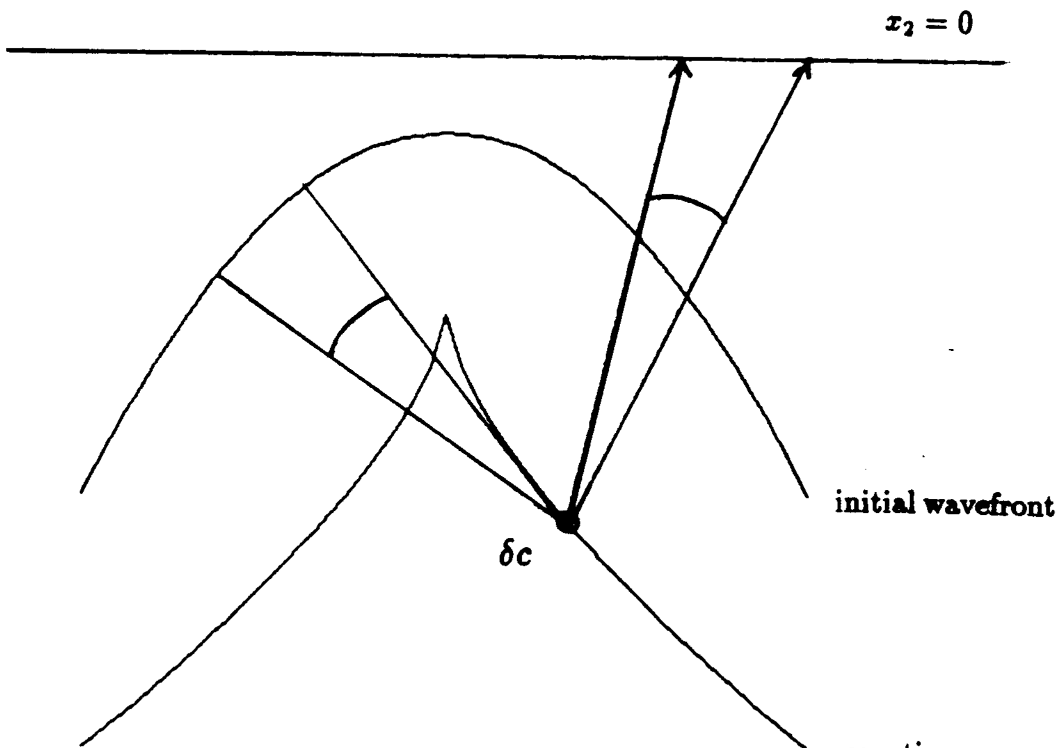


Figure 3.5. The perturbation is located on the caustic in the reference field.

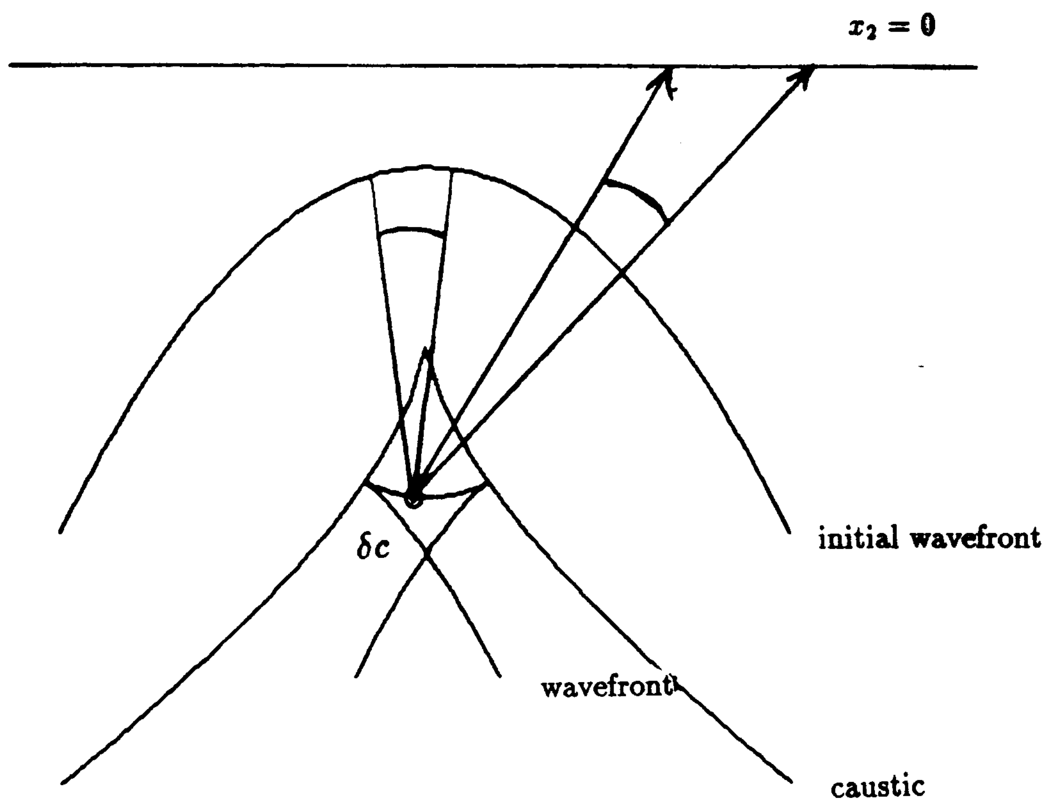


Figure 3.6. The perturbation is located below the caustic in the reference field.

3.6 Integral Operator Representation for the Scattered Signal

If there are no caustics in the perturbational field, then the method of stationary phase can be used to asymptotically represent the integral in δU (26). Without loss of generality, we can assume that there is only one stationary point. We then get the following form for the scattered signal:

$$\delta U|_{x_2=0} \sim A(\mathbf{k}, x_1, t) e^{i\phi_r(\mathbf{k}, x_1, t)} ,$$

where

$$A(\mathbf{k}, x_1, t) = (ik)^{\frac{1}{2}} \chi(\mathbf{x}_0(\theta(z_1^*))) \frac{a(\theta(z_1^*))}{4\hat{\mathbf{k}} \cdot \theta(z_1^*)} \frac{\Gamma''(z_1^*)}{1 + \Gamma'^2(z_1^*)} \bigg|_{x_2=0}$$

and $\phi_r(\mathbf{k}, x_1, t) = k\tilde{\phi}_r(z_1^*, x_1, t, \hat{\mathbf{k}})$.

To generalize this formulation to general perturbations, δc, we can take

$$\delta c = \chi(\mathbf{x}) \widehat{\delta c}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

to be one localized Fourier component of a general velocity perturbation $\delta c(\mathbf{x})$. Then the previous analysis will still be valid.

Since δc occurs only in the right hand side of the first transport equation,

$$2b_{1_t} + 2\nabla b_1 \cdot \nabla\phi + b_1 \nabla^2 \phi = 2\chi(\mathbf{x}) \widehat{\delta c}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} a$$

we can make a simple modification to account for more general perturbations. We need only change the ansatz for b_1 to account for a factor of $\widehat{\delta c}(\mathbf{k})$:

$$b_1(\mathbf{x},t) \sim \sum_{n=1}^{\infty} \frac{b_1^n(\mathbf{x})\widehat{\delta c}(\mathbf{k})}{(ik)^n} e^{ik\phi_r(\mathbf{x},t)} .$$

The leading order term in b_1 is

$$b_1^1 = \frac{\chi a}{4\hat{\mathbf{k}}\cdot\theta}\widehat{\delta c}(\mathbf{k}) .$$

Similar expressions can be obtained for b_2 and b_3. The expression for the scattered field will change only by a factor of $\widehat{\delta c}(\mathbf{k})$:

$$\delta U|_{x_2=0} \sim A(\mathbf{k},x_1,t)e^{i\phi_r(\mathbf{k},x_1,t)}\widehat{\delta c}(\mathbf{k}) .$$

Summing over $\mathbf{k}$ gives the trace of the perturbational field δW due to a localized velocity perturbation $\delta c(\mathbf{z})$,

$$\delta W|_{x_2=0} \sim \int d\mathbf{k} A(\mathbf{k},x_1,t)e^{i\phi_r(\mathbf{k},\mathbf{x}_1,t)}\widehat{\delta c}(\mathbf{k}) .$$

We can write $\widehat{\delta c}(\mathbf{k})$ as a Fourier integral,

$$\delta W|_{x_2=0} \sim \int d\mathbf{k}\ A(\mathbf{k},x_1,t)e^{i\phi_r(\mathbf{k},\mathbf{x}_1,t)} \int d\mathbf{z}\ \delta c(\mathbf{z})e^{-i\mathbf{k}\cdot\mathbf{z}} .$$

Then the scattered signal can be represented by the following Fourier integral operator acting on the velocity perturbation $\delta c(\mathbf{z})$:

$$\begin{aligned} \delta W|_{x_2=0} \sim\ & [P\delta c](x_1,t) = \\ & \int\int d\mathbf{k}d\mathbf{z} A(\mathbf{k},x_1,t)e^{i[\phi_r(\mathbf{k},x_1,t)-\mathbf{k}\cdot\mathbf{z}]}\delta c(\mathbf{z}) . \end{aligned}$$

CHAPTER 4

An Example of Anomalous Scattering Strength from a Caustic

In the previous chapter it was determined that if a high frequency, unidirectional perturbation were located near a caustic in the incident field, the energy would be spread out instead of focused. This result does not extend to general perturbations, δc. It was expected that the product of δc and $u_{0_{tt}}$ in the right hand side of the perturbational problem

$$\delta u_{tt} - \nabla^2 \delta u = 2\delta c \, u_{0_{tt}}$$

would cause the scattered signal to be illuminated if δc were located near a caustic in the reference field u_0. This is, in fact, the case. Caustics in a probing wavefield do cause velocity perturbations to scatter signals more strongly than a smooth ray pattern would.

In order to quantify this statement, we find a sequence of general perturbations δc that generate $\mathcal{O}(1)$ scattered signals in a smooth reference field and increasingly larger signals in the presence of a caustic. That is, $\frac{|\delta u|}{|\delta c|} \to \infty$ as δc moves closer and closer to the caustic. This example illustrates the fact that even if there is no caustic in the perturbational field, the scattered signal can be anomalously large.

The construction of this example is best described by the pictures in Figure 4.1. A perturbation δc is constructed so that the incident wavefront will be reflected as a plane wave. The perturbation is localized to lie within a ray tube in the incident field. As δc moves closer to the caustic, the reflected ray tube gets smaller. That is, the same amount of energy is reflected into a decreasing area.

We first construct the scattered field due to a velocity perturbation of the form

$$\delta c = \chi(x,z)e^{ik\gamma(x,z)} .$$

The incident field at $t=0$ is taken to be

$$u_0(x,z,0) = \alpha(x,z)H(-\psi(x,z)) ,$$

which gives the reference field

$$u_0(x,z,t) \sim a(x,z,t)H(t-\phi(x,z)) .$$

The phase ϕ satisfies the eikonal equation $|\nabla\phi|^2 = 1$, and the amplitude $a(x,z,t)$ is given by the transport equation

$$2\nabla a \cdot \nabla\phi + a\nabla^2\phi = 0 .$$

The Heaviside singularity $H(-\psi)$ is chosen rather than the point source singularity $\frac{1}{\sqrt{\psi}}H(-\psi)$ because the construction is simpler. In fact, it parallels the construction in Chapter 3.

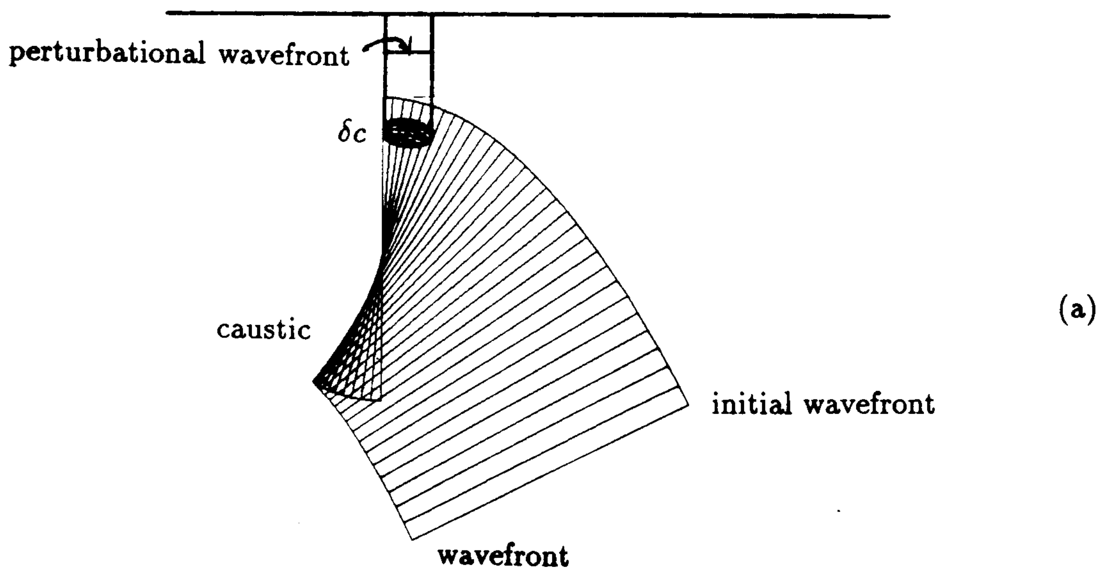
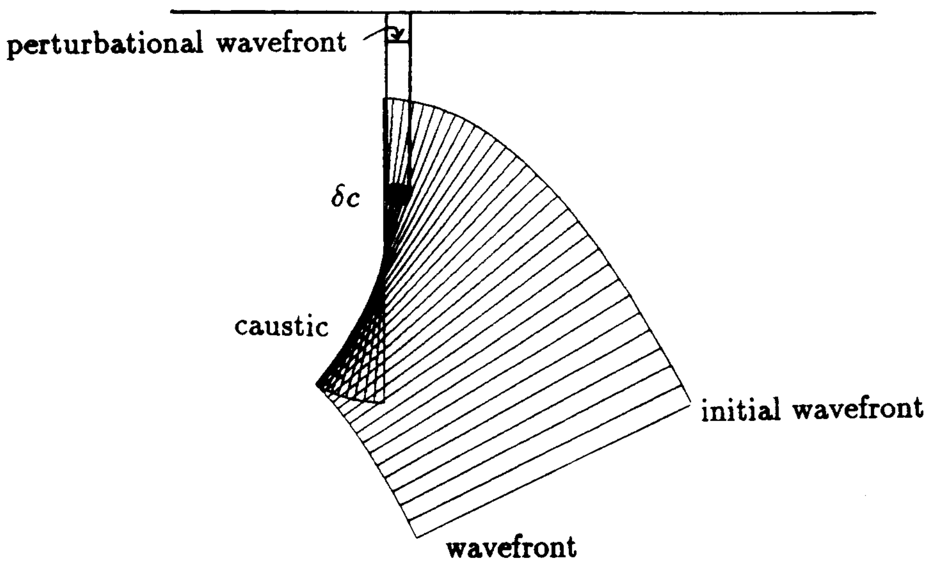


Figure 4.1. As the perturbation is placed closer and closer to the caustic, the reflected ray tube gets smaller and smaller.



(b)

Figure 4.1. As the perturbation is placed closer and closer to the caustic, the reflected ray tube gets smaller and smaller.

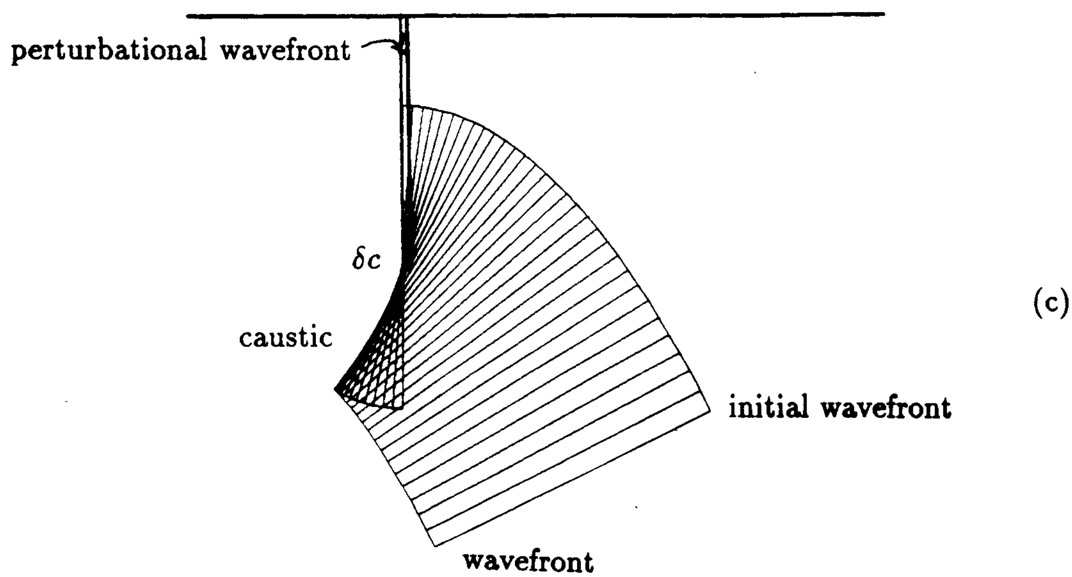


Figure 4.1. As the perturbation is placed closer and closer to the caustic, the reflected ray tube gets smaller and smaller.

The scattered field has the form

$$\delta u(x,z,t) = b_1(x,z,t)\delta(t-\phi(x,z)) + b_2(x,z,t)H(t-\phi(x,z))$$

where

$$\begin{aligned} b_1(x,z,t) &= \sum_{j=1}^{\infty} \frac{b_1^j(x,z)}{(ik)^k} e^{ik\phi_r(x,z,t)} \\ b_2(x,z,t) &= \sum_{j=0}^{\infty} \frac{b_2^j(x,z)}{(ik)^j} e^{ik\phi_r(x,z,t)} . \end{aligned}$$

The phase must satisfy the initial condition

$$\phi_r(x,z,\phi(x,z)) = \gamma(x,z) \tag{28}$$

and the eikonal equation

$$|\nabla\phi_r|^2 = \phi_{r_t}^2 .$$

We want to construct the perturbations $\delta c = \chi(x,z)e^{ik\gamma(x,z)}$ in such a way that the reflected phase ϕ_r is a plane wave that moves in the $-z$ direction. That is, we want

$$\phi_r(x,z,t) = t + z .$$

This automatically satisfies the eikonal equation. In order to satisfy the initial condition (28), we must take

$$\gamma(x,z) = \phi(x,z) + z .$$

The coefficient b_1^1 satisfies the transport equation

$$2b_1^1\phi_{r_t} + 2b_1^1\nabla\phi_r \cdot \nabla\phi = 2\chi a e^{ik(\gamma-\phi_r)} . \tag{29}$$

Taking the gradient of the initial condition (28) with $\gamma(x,z)=\phi(x,z)+z$ gives

$$\nabla\phi_r(x,z,\phi(x,z))+\phi_{r_t}(x,z,\phi(x,z))\nabla\phi(x,z)=\nabla\phi(x,z)+(0,1)$$

and

$$\nabla\phi_r\cdot\nabla\phi+\phi_{r_t}|\nabla\phi|^2=|\nabla\phi|^2+\phi_z\ .$$

But $|\nabla\phi|^2=1$ so that

$$\nabla\phi_r\cdot\nabla\phi+\phi_{r_t}=1+\phi_z\ . \tag{30}$$

Using (30) in (29), we obtain

$$b_1^1=\frac{\chi a}{1+\phi_z}\quad\text{on}\quad\phi(x,z)=t\ .$$

Similarly,

$$b_2^0=2b_1^1\phi_{r_t}(x,z,\phi(x,z))=\frac{2\chi a}{1+\phi_z}\quad\text{on}\quad\phi(x,z)=t\ .$$

The coefficient b_2 satisfies the wave equation in the light cone

$$b_{2_{tt}}-\nabla^2 b_2=0\quad\text{in}\quad t>\phi(x,z)\ . \tag{31}$$

Substituting the above expansion for b_2 in (31) gives the eikonal equation for the reflected phase and the following transport equation for b_2^0:

$$\phi_{r_t}b_{2_t}^0-\nabla b_2^0\cdot\nabla\phi_r=0\quad\text{in}\quad t>\phi(x,z)\ .$$

Then b_2^0 is constant along the rays described by

$$\begin{aligned}\frac{d\mathbf{x}}{d\sigma}&=\nabla\phi_r=(0,1)\\ \frac{dt}{d\sigma}&=-\phi_{r_t}=-1\end{aligned}$$

or $\frac{d\mathbf{x}}{dt} = (0, -1)$, where $\mathbf{x} = (x, z)$. Then the reflected field at the surface can be written as

$$\delta u(x, 0, t) \sim \frac{\chi(x_0, z_0) a(x_0, z_0)}{1 + \phi_z(x_0, z_0)} e^{ik(t+z)} \Big|_{z=0}$$

where the rays are given by

$$(x, z) = (x_0, \phi(x_0, z_0) - t) ,$$

and (x_0, z_0) is on the incident wavefront $\phi(x_0, z_0) = c$. Notice that the amplitude of the reflected field is constant along reflected rays. This construction is valid only above the incident caustic, but this is sufficient for present purposes.

To form the sequence, let each element of $\{\delta c_n\}$ be a localized high frequency perturbation of the form

$$\delta c_n = \chi_n(x, z) e^{ik\gamma(x,z)} ,$$

where

$$\gamma(x, z) = \phi(x, z) + z .$$

In order to describe the characteristic function $\chi_n(x, z)$, let $\mathcal{T}$ be a slim ray tube enclosed by two incident rays. The cross sectional area of the ray tube goes to zero as the rays approach the caustic (see Figure 4.2). Let r be the length of the bisector of the two bounding rays between the initial wavefront and the crossing of the rays, as in Figure 4.3. Then let $\chi_n(x, z)$ be equal to one on a disk $\mathcal{D}_n$ that is centered at a point on the bisector a distance $r - \frac{r}{n}$ from the incident wavefront. This disk

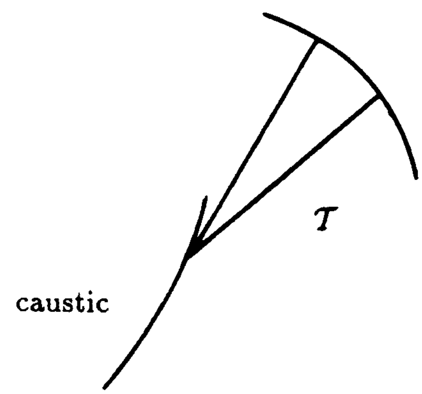


Figure 4.2. $\mathcal{T}$ is a slim ray tube enclosed by two incident rays.

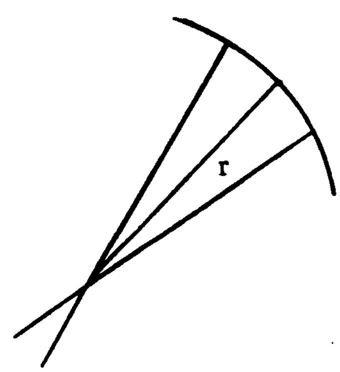


Figure 4.3. The length of the bisector of the two bounding rays between the initial wavefront and the crossing of the rays is r.

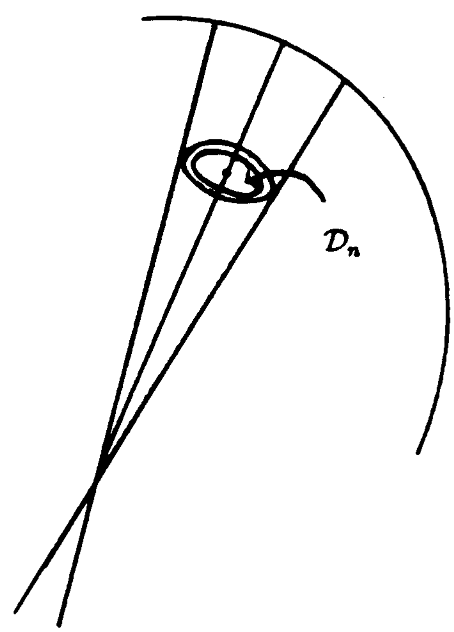


Figure 4.4. The envelope function $\chi_n(\mathbf{x})$ is one on the disk $\mathcal{D}_n$.

should be contained strictly within the ray tube, and $\chi_n(x)$ should be zero outside $\mathcal{T}$ (see Figure 4.4). As n gets larger, $\mathcal{D}_n$ moves closer to the caustic.

The relative size of the scattered signal to the perturbation is given in terms of the L^2 norms:

$$\begin{aligned} \|\delta c_n\|^2 &= \int\!\!\int dx_0 dz_0 |\chi_n(x_0, z_0)|^2 = \|\chi_n\|^2 \\ \|\delta u_n\|^2 &= \int\!\!\int dx\; dt \left|\chi_n(x_0, z_0) \frac{a(x_0, z_0)}{1 + \phi_z(x_0, z_0)}\bigg|_{z=0}\right|^2 . \end{aligned} \tag{32}$$

The time that it takes to get from the initial wavefront to the reflector then to the surface is $\phi(x_0, z_0) + z_0$ so that points (x_0, z_0) are mapped to (x, t) by

$$(x, t) = (x_0, \phi(x_0, z_0) + z_0) \quad \text{at} \quad z = 0 .$$

This is illustrated in Figure 4.5. Making this change of variables in (32) changes the amplitude only by the Jacobian factor $J = (1 + \phi_z)$. Then

$$\begin{aligned} \|\delta u_n\|^2 &= \int\!\!\int dx_0 dz_0 \left|\chi_n(x_0, z_0) \frac{a(x_0, z_0)}{\sqrt{1 + \phi_z(x_0, z_0)}}\right|^2 \\ &= \left\|\frac{\chi_n a}{\sqrt{1 + \phi_z}}\right\|^2 . \end{aligned}$$

Since ϕ satisfies the eikonal equation $|\nabla\phi|^2 = 1$ and $\phi_z > 0$, the factor $(1+\phi_z)^{-\frac{1}{2}}$ is $\mathcal{O}(1)$. But as the incident rays get closer to the caustic, the amplitude $a(x_0, z_0)$ approaches infinity. This was shown in Chapter 2 by integrating the transport equation over the ray tube, which collapses. Thus

$$\frac{\|\delta u_n\|^2}{\|\delta c_n\|^2} \to \infty \quad \text{as} \quad n \to \infty .$$

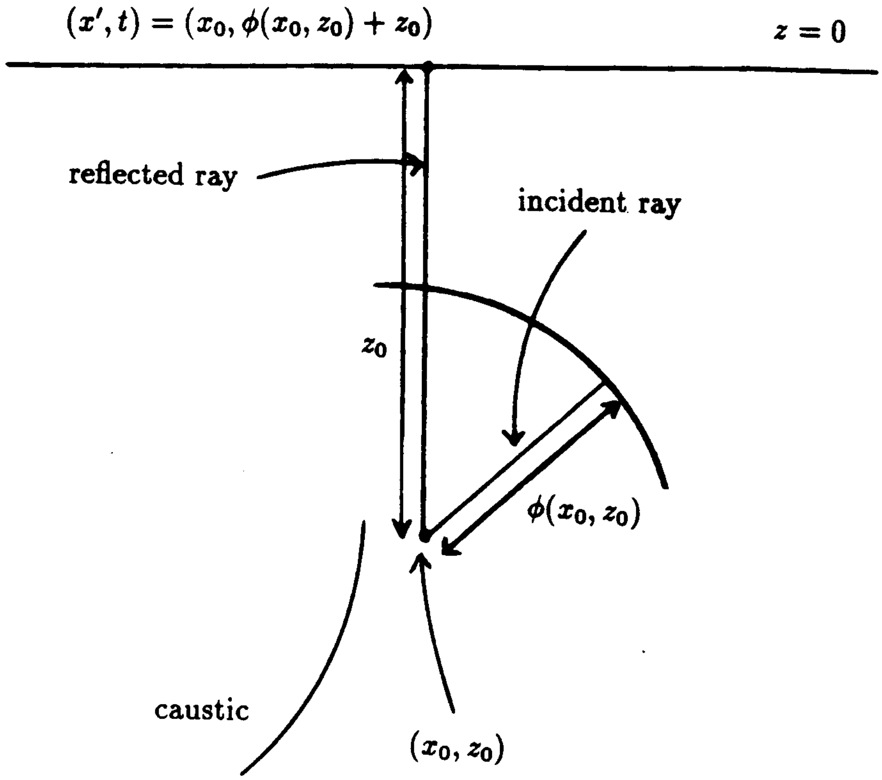


Figure 4.5. Points (x_0, z_0) are mapped along reflected rays to (x', t) at $z = 0$.

If there is no caustic in the incident wavefield then $\frac{||\delta u||^2}{||\delta c||^2}$ is always bounded since the amplitude remains bounded.

An incident point source field $\frac{1}{\sqrt{\psi}}H(-\psi)$ could have been used instead of the Heaviside field, $H(-\psi)$. Under these circumstances, the conclusion would be that

$$\frac{||k^{-\frac{1}{2}}\,\delta u||}{||\delta c||} \quad \text{remains bounded} \tag{33}$$

as long as the support of δc stays away from the incident caustic, but approaches infinity if the support of δc approaches the caustic.

This is verified by a result of Rakesh [16], which states that if there are no caustics present, then

$$\left\| \delta u|_{x_n=0} \right\|_{s-(n-1)/2} \leq \text{constant}||\delta c||_s \,. \tag{34}$$

The s-norm, $||\cdot||_s$, is the Sobolev norm defined by

$$||f||_s = \left(\int_{\mathbb{R}^n} d\mathbf{k}(1+k^2)^s|\hat{f}(\mathbf{k})|^2 \right)^{\frac{1}{2}} \,.$$

In two dimensions, (34) can be written as

$$\left\| \delta u|_{x_2=0} \right\|_{s-\frac{1}{2}} \leq \text{constant}||\delta c||_s \,,$$

which says that the $s - \frac{1}{2}\,^{th}$ derivative of δu is the same size as the s^{th} derivative of δc. Since the factor of $k^{-\frac{1}{2}}$ in (33) acts like a $(-\frac{1}{2})^{th}$ derivative, the results (33) and (34) are consistent.

CHAPTER 5

The Inverse Operator

5.1 The Construction of the Inverse

In Chapter 3 we found the following Fourier integral operator representation for the scattered signal due to a general perturbation $\delta c(\mathbf{z})$:

$$\begin{aligned} \delta W|_{x_2=0} &\sim [P\delta c](x_1,t) \\ &= \int\int d\mathbf{k}d\mathbf{z}A(\mathbf{k},x_1,t)e^{i[\phi_r(\mathbf{k},x_1,t)-\mathbf{k}\cdot\mathbf{z}]}\delta c(\mathbf{z}) \ . \end{aligned} \tag{35}$$

This representation is valid away from any caustics in the perturbational field. It is, in fact, valid even if there are caustics in the incident field. The goal of this chapter is to find an inverse operator for this representation. That is, we will find an inverse of the forward map

$$DS[c_0]\delta c = \delta W|_{x_2=0} \ .$$

Let P_1 be the operator defined by

$$P_1\delta c(x_1,t) = \int\int d\mathbf{k}_1 d\mathbf{z}A_1(\mathbf{k}_1,x_1,t)e^{i[\phi_r(\mathbf{k}_1,x_1,t)-\mathbf{k}_1\cdot\mathbf{z}]}\delta c(\mathbf{z}) \ .$$

The operators P and P_1 are the same; the subscript 1 is used only for clarification.

We intend to find an operator P_2 that is close to an inverse operator. To define what close means, we must introduce a parameter τ that describes the roughness of δc. Recall that we assumed in the geometrical optics calculations that δc acted as a reflector. This is a statement that δc is rough. One way of viewing this is if the velocity perturbation can be written in the form

$$\delta c(\mathbf{z}) = \alpha(\mathbf{z}) e^{i\tau\gamma(\mathbf{z})} .$$

As $\tau \rightarrow \infty$, δc oscillates more and more rapidly and thus becomes very rough. Assuming that $\delta c(\mathbf{z})$ can be represented in this way and $\alpha(z)$ has compact support in $\mathbf{z}$, we will look for an operator P_2 such that

$$P_2 P_1 \delta c(\mathbf{z}) \sim I \delta c(\mathbf{z}) \quad \text{as} \quad \tau \rightarrow \infty .$$

Define P_2 as

$$P_2 f(\mathbf{y}) = \int\int d\mathbf{k}_2 d\mathbf{x} A_2(\mathbf{k}_2, \mathbf{x}) e^{-i[\phi(\mathbf{k}_2,\mathbf{x}) - \mathbf{k}_2 \cdot \mathbf{y}]} f(\mathbf{x}) . \tag{36}$$

P_2 is similar to the adjoint operator to P_1. Then, letting $\tilde{\mathbf{x}}$ represent (x_1, t),

$$\begin{aligned} P_2 P_1 \delta c(\mathbf{y}) &= \int\int\int\int d\mathbf{k}_1 d\mathbf{k}_2 d\tilde{\mathbf{x}} d\mathbf{z} e^{i[\phi(\mathbf{k}_1,\tilde{\mathbf{x}}) - \mathbf{k}_1 \cdot \mathbf{z} - \phi(\mathbf{k}_2,\tilde{\mathbf{x}}) + \mathbf{k}_2 \cdot \mathbf{y}]} \\ &\quad \times A_1(\mathbf{k}_1, \tilde{\mathbf{x}}) A_2(\mathbf{k}_2, \tilde{\mathbf{x}}) \delta c(\mathbf{z}) . \end{aligned}$$

Since $\delta c(\mathbf{y})$ can be written as the inverse of its Fourier transform,

$$\delta c(\mathbf{y}) = \int\int d\mathbf{k} d\mathbf{z} e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{z})} \delta c(\mathbf{z}) , \tag{37}$$

the idea is to make $P_2P_1\delta c(\mathbf{y})$ look like (37). In order to accomplish this, two of the integrals in $P_2P_1\delta c(\mathbf{y})$ must be eliminated. But a large parameter in the phase function is necessary to perform a stationary phase calculation. Therefore, we make the change of variables

$$\mathbf{k}_i \to \tau\mathbf{k}_i, \quad i = 1, 2 \ .$$

The amplitude in A_1 was shown in Chapter 3 to be homogeneous of order $\frac{1}{2}$. Then assuming that A_2 is homogeneous of order ℓ, we have

$$\begin{aligned} P_2P_1\delta c(\mathbf{y}) &= \int\int\int\int d\mathbf{k}_1 d\mathbf{k}_2 d\tilde{\mathbf{x}} d\mathbf{z} \tau^{4+\frac{1}{2}+\ell} e^{i\tau\Phi(\mathbf{k}_1,\mathbf{k}_2,\tilde{\mathbf{x}},\mathbf{y},\mathbf{z})} \\ &\quad \times A_1(k_1,\tilde{\mathbf{x}})A_2(k_2,\tilde{\mathbf{x}})\delta c(\mathbf{z}) \ , \end{aligned}$$

where $\Phi(\mathbf{k}_1,\mathbf{k}_2,\tilde{\mathbf{x}},\mathbf{y},\mathbf{z}) = \phi(\mathbf{k}_1,\tilde{\mathbf{x}}) - \mathbf{k}_1\cdot\mathbf{z} - \phi(\mathbf{k}_2,\tilde{\mathbf{x}}) + \mathbf{k}_2\cdot\mathbf{y}$.

We intend to do a stationary phase calculation on the $(\mathbf{k}_2,\tilde{\mathbf{x}})$ variables. Therefore, the error term will be expressed as an integral over the $(\mathbf{k}_1,\mathbf{z})$ variables of a function that is $\mathcal{O}(\tau^{-1})$. In order to ensure that the entire integral is $\mathcal{O}(\tau^{-1})$ and can be considered asymptotically of lower order than the leading term, we do the following calculation. The integral $P_2P_1\delta c(\mathbf{y})$ can be rewritten as

$$P_2P_1\delta c(\mathbf{y}) = \int\int d\mathbf{k}_1 d\mathbf{z} \tau^{4+\frac{1}{2}+\ell} a(\mathbf{k}_1,\mathbf{z}) e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} \alpha(\mathbf{z}) e^{i\tau\gamma(\mathbf{z})}$$

where

$$a(\mathbf{k}_1,\mathbf{z}) = \int\int d\mathbf{k}_2 d\tilde{\mathbf{x}} A_1(\mathbf{k}_1,\tilde{\mathbf{x}})A_2(\mathbf{k}_2,\tilde{\mathbf{x}}) e^{i\tau[\phi(\mathbf{k}_1,\tilde{\mathbf{x}})-\phi(\mathbf{k}_2,\tilde{\mathbf{x}})+\mathbf{k}_2\cdot\mathbf{y}]} \ .$$

Then, letting $b(\mathbf{k}_1, \mathbf{z}) = \alpha(\mathbf{z})a(\mathbf{k}_1, \mathbf{z})$ and $q = 4 + \frac{1}{2} + \ell$,

$$\begin{aligned} P_2P_1\delta c(\mathbf{y}) &= \int\int d\mathbf{k}_1 d\mathbf{z} \tau^q b(\mathbf{k}_1, \mathbf{z}) e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} e^{i\tau\gamma(\mathbf{z})} \\ &= \int\int_{k_1\leq 1} d\mathbf{k}_1 d\mathbf{z} \tau^q b(\mathbf{k}_1, \mathbf{z}) e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} e^{i\tau\gamma(\mathbf{z})} \\ &\quad + \int\int_{k_1\geq 1} d\mathbf{k}_1 d\mathbf{z} \tau^q b(\mathbf{k}_1, \mathbf{z}) e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} e^{i\tau\gamma(\mathbf{z})} . \end{aligned}$$

The amplitude $b(\mathbf{k}_1, \mathbf{z})$ has compact support in $\mathbf{z}$ since $\alpha(\mathbf{z})$ does. Therefore, the integral over the region $k_1 \leq 1$ is bounded by something that is $\mathcal{O}(\tau^q)$,

$$\int\int_{k_1\leq 1} d\mathbf{k}_1 d\mathbf{z} \tau^q b(\mathbf{k}_1, \mathbf{z}) e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} e^{i\tau\gamma(\mathbf{z})} \leq \text{constant } \mathcal{O}(\tau^q) .$$

For the infinite integral, we write

$$e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} = \frac{\mathbf{k}_1}{i\tau k_1^2} \cdot \nabla_z e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} ,$$

then do an integration by parts in $\mathbf{z}$:

$$\begin{aligned} &\int\int_{k_1\geq 1} d\mathbf{k}_1 d\mathbf{z} \tau^q b(\mathbf{k}_1, \mathbf{z}) e^{i\tau\gamma(z)} \frac{\mathbf{k}_1}{i\tau k_1^2} \cdot \nabla_z e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} \\ = \; & -\int\int_{k_1\geq 1} d\mathbf{k}_1 d\mathbf{z} \tau^q \frac{\mathbf{k}_1}{k_1^2} \cdot \left[\nabla_z\gamma(\mathbf{z}) b(\mathbf{k}_1, \mathbf{z}) + \frac{1}{i\tau}\nabla_z b(\mathbf{k}_1, \mathbf{z})\right] e^{i\tau\gamma(\mathbf{z})} e^{i\tau\mathbf{k}_1\cdot\mathbf{z}} . \end{aligned}$$

Thus, one integration by parts pulls in a factor of k_1^{-1} without changing the order in τ. Two more applications of integration by parts provides a factor of k_1^{-3} ensuring the absolute convergence of the integral, again without changing the order in τ.

It is possible now to estimate the integral $P_2P_1\delta c(\mathbf{y})$ as

$$\begin{aligned} P_2P_1\delta c(\mathbf{y} &\leq \tau^q \int\int d\mathbf{k}_1 d\mathbf{z} \left| b(\mathbf{k}_1, \mathbf{z}) e^{i\tau\mathbf{k}_1\cdot z} e^{i\tau\gamma(\mathbf{z})} \right| \\ &= \mathcal{O}(\tau^q) . \end{aligned}$$

Now stationary phase can be performed on the $(\mathbf{k}_2, \tilde{\mathbf{x}})$ variables. The stationary points are given by

$$\begin{aligned} \nabla_{k_2}\Phi &= \mathbf{y} - \nabla_{k_2}\phi(\mathbf{k}_2, \tilde{\mathbf{x}}) = 0 & (38) \\ \nabla_{\tilde{x}}\Phi &= \nabla_{\tilde{x}}\phi(\mathbf{k}_1, \tilde{\mathbf{x}}) - \nabla_{\tilde{x}}\phi(\mathbf{k}_2, \tilde{\mathbf{x}}) = 0 \,. & (39) \end{aligned}$$

The second equation (39) is satisfied by $\mathbf{k}_2 = \mathbf{k}_1$. The first equation (38) specifies some stationary points $(\mathbf{k}_1, \tilde{\mathbf{x}}_j^*)$. As long as $\mathbf{k}_2 = \mathbf{k}_1$ is the only solution of (39) then the following calculation of the approximate inverse can be done. It doesn't matter how many points $\tilde{\mathbf{x}}_j^*$ there are. For simplicity of notation we will assume that there is only one; call it $\tilde{\mathbf{x}}^*$. In any case, the implicit function theorem ensures that the stationary points will be locally unique as long as

$$\det |\nabla_{k_1}\nabla_{\tilde{x}}\phi(\mathbf{k}_1, \tilde{\mathbf{x}}^*)| \neq 0 \,.$$

This mixed Hessian

$$H = \begin{bmatrix} \phi_{tk_1} & \phi_{tk_2} \\ \phi_{x_1k_1} & \phi_{x_1k_2} \end{bmatrix}$$

is singular precisely at a caustic and corresponds to the singularity of Beylkin's matrix (8). If H is non-singular, the order in k of the amplitude $A_1(\mathbf{k}_1, \mathbf{x}', t)$ determines the size of the operator P_1 [15].

Notice that condition (39) concerns the ray direction vector

$$\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial t}\right) = \left(\frac{dx_1}{d\sigma}, \frac{dx_2}{d\sigma}, \frac{dt}{d\sigma}\right) \,.$$

Thus, the assumption that the solution of (39) is globally unique is a statement that the direction of the rays at the surface change when the direction of the perturbation changes. For an illustration when the background velocity is constant, see Figure 5.1.

At the point $(\mathbf{k}_2, \tilde{\mathbf{x}}) = (\mathbf{k}_1, \tilde{\mathbf{x}}^*)$, the second derivatives of Φ are given by

$$\begin{aligned}
\nabla_{k_2}\nabla_{k_2}\Phi|_{(\mathbf{k}_1,\tilde{\mathbf{x}}^*)} &= -\nabla_{k_2}\nabla_{k_2}\phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*) \\
\nabla_{\tilde{x}}\nabla_{\tilde{x}}\Phi_{(\mathbf{k}_1,\tilde{\mathbf{x}}^*)} &= \nabla_{\tilde{x}}\nabla_{\tilde{x}}\phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*) - \nabla_{\tilde{x}}\nabla_{\tilde{x}}\phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*) = 0 \\
\nabla_{k_2}\nabla_{\tilde{x}}\Phi|_{(\mathbf{k}_1,\tilde{\mathbf{x}}^*)} &= -\nabla_{k_2}\nabla_{\tilde{x}}\phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*)\,.
\end{aligned}$$

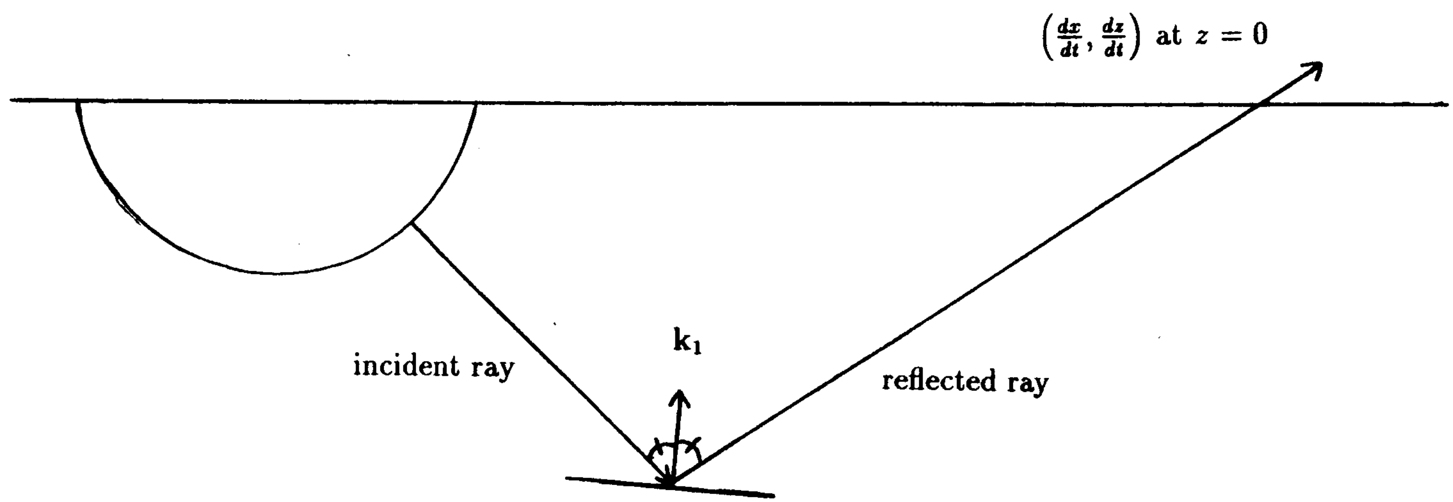
Therefore $\det[\nabla_{(k_1,\tilde{x})}\nabla_{(k_1,\tilde{x})}\Phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*)] = -\det[\nabla_{k_1}\nabla_{\tilde{x}}\phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*)]$, and

$$\begin{aligned}
P_2P_1\delta c(\mathbf{y}) &= \int\!\!\int d\mathbf{k}_1 d\mathbf{z} \tau^{2+\frac{1}{2}+\ell}(2\pi)^2 A_1(\mathbf{k}_1,\tilde{\mathbf{x}}^*)A_2(\mathbf{k}_1,\tilde{\mathbf{x}}^*)\delta c(\mathbf{z}) \\
&\quad \times \frac{e^{-i\frac{\pi}{4}\sum sgn\lambda_j}}{\sqrt{\det\nabla_{k_2}\nabla_{\tilde{x}}\phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*)}}\, e^{i\tau(\mathbf{y}-\mathbf{z})\cdot\mathbf{k}_1} \\
&\quad + \int\!\!\int d\mathbf{k}_1 d\mathbf{z} e^{i\tau(\mathbf{y}-\mathbf{z})\cdot\mathbf{k}_1}\mathcal{O}(\tau^{2+\frac{1}{2}+\ell-1})\,,
\end{aligned}$$

where the λ_j are eigenvalues of $\nabla_{(k_2,\tilde{x})}\nabla_{(k_2,\tilde{x})}\Phi(\mathbf{k}_1,\tilde{\mathbf{x}}^*)$.

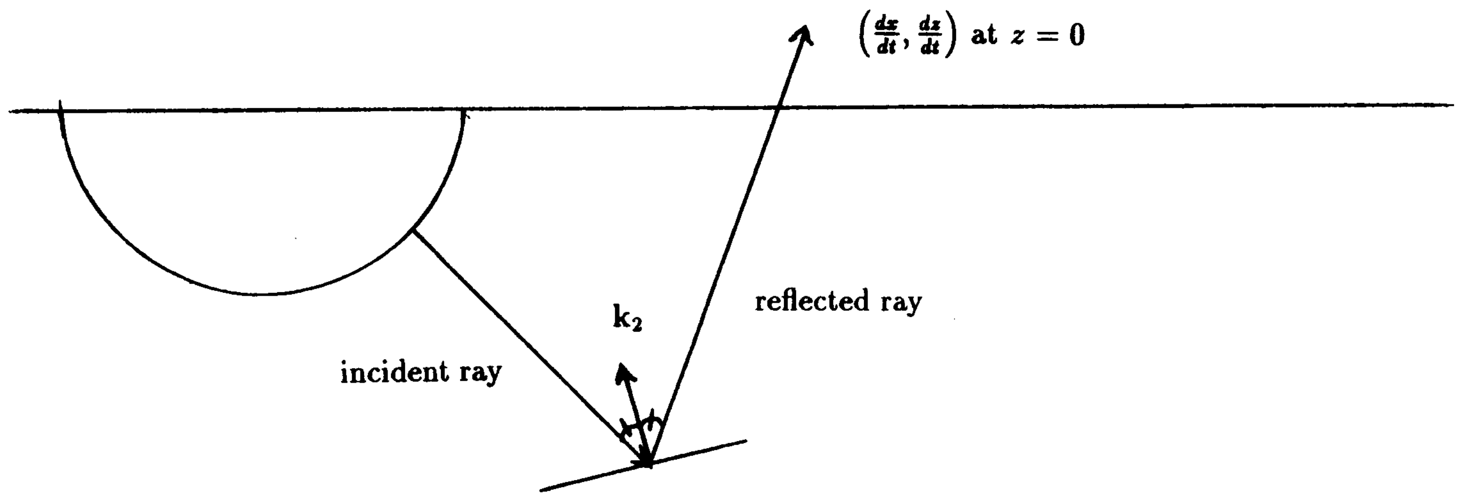
Lettingl $A_2(\mathbf{k}_2,\tilde{\mathbf{x}}) = \frac{1}{(2\pi)^2}\,[A_1(\mathbf{k}_2,\tilde{\mathbf{x}})]^{-1}\, e^{i\frac{\pi}{4}\sum sgn\lambda_j}\sqrt{\det\nabla_{k_2}\nabla_{\tilde{x}}\phi(\mathbf{k}_2,\tilde{\mathbf{x}})}$ and $\ell = -\frac{1}{2}$, we have

$$\begin{aligned}
P_2P_1\delta c(\mathbf{y}) &= \int\!\!\int d\mathbf{k}_1 d\mathbf{z}\tau^2\delta c(\mathbf{z}) e^{i\tau(\mathbf{y}-\mathbf{z})\cdot\mathbf{k}_1} \\
&\quad + \int\!\!\int d\mathbf{k}_1 d\mathbf{z}\ \delta c(\mathbf{z}) e^{i\tau(\mathbf{y}-\mathbf{z})\cdot\mathbf{k}_1}\mathcal{O}(\tau)\,.
\end{aligned}$$



(a) $\delta c = \chi(\mathbf{x}) e^{i\mathbf{k}_1 \cdot \mathbf{x}}$

Figure 5.1. The direction of the reflected ray at the surface changes if the direction of the perturbation changes.



(b) $\delta c = \chi(\mathbf{x})e^{i\mathbf{k}_2\cdot\mathbf{x}}$

Figure 5.1. The direction of the reflected ray at the surface changes if the direction of the perturbation changes.

Changing variables again

$$\tau \mathbf{k}_i \to \mathbf{k}_i$$

gives

$$P_2 P_1 \delta c(\mathbf{y}) = \int\int d\mathbf{k}_1 d\mathbf{z}\, \delta c(\mathbf{z}) e^{i(\mathbf{y}-\mathbf{z})\cdot\mathbf{k}_1} + \int\int d\mathbf{k}_1 d\mathbf{z}\, \delta c(\mathbf{z}) e^{i(\mathbf{y}-\mathbf{z})\cdot\mathbf{k}_1} \mathcal{O}(\tau^{-1}) \, .$$

As we saw above, the error term is $\mathcal{O}(\tau^{-1})$; therefore

$$P_2 P_1 \delta c(\mathbf{y}) \sim I \delta c(\mathbf{y}) \, .$$

5.2 An Extension for 3-D Varying Background Velocity

Note that the representation of the trace of the perturbational field δU found in Section 5.1 (35) is general enough to include the case of a general initial wavefront moving into a medium with a smoothly varying background velocity $c_0 = c_0(\mathbf{x})$. In this case the reference field has the form

$$u_0 = a(\mathbf{x}, t)\delta(t - \phi(\mathbf{x})) \, .$$

Then the reflected phase $\phi_r(\mathbf{k}, \mathbf{x}, t)$ satisfies the eikonal equation

$$\frac{1}{c_0^2(\mathbf{x})}\phi_{r_t}^2 - |\nabla \phi_r|^2 = 0 \, ,$$

$$\phi_r(x, \phi(x)) = \hat{\mathbf{k}} \cdot \mathbf{x} \, ,$$

and the amplitude A solves the transport equation

$$2\phi_{r_t} A_t - 2\nabla A \cdot \nabla \phi_r + (\phi_{r_{tt}} - \nabla^2 \phi_r) A = 0, \quad t > \phi(\mathbf{x})$$

$$A = \frac{ik\chi(\mathbf{x})a(\mathbf{x},t)}{c_0^3(\mathbf{x})\hat{\mathbf{k}}\cdot\nabla\phi}\,\widehat{\delta c}\,(\mathbf{k})\phi_{r_t}^2(\mathbf{x},t) \quad on \quad t = \phi(\mathbf{x})\ .$$

To extend the calculation to three dimensions, note that the vector $\mathbf{x}$ can be n-dimensional

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad n > 1\ .$$

Let $\mathbf{x}' = (x_1, x_2, \ldots, x_{n-1})$. Then the forward map can be expressed as

$$\delta W|_{x_n=0} \sim P_1\delta c(\mathbf{x}', t) = \int\int d\mathbf{k}_1 d\mathbf{z} A_1(\mathbf{k}_1, \mathbf{x}', t)e^{i[\phi_r(\mathbf{k}_1,\mathbf{x}',t)-\mathbf{k}_1\cdot\mathbf{z}]}\delta c(\mathbf{z})\ .$$

The amplitude A_1 is homogeneous of order m, where m depends on the dimension n and the initial singularity. Furthermore, the inverse operator P_2 can be defined as

$$P_2\, f(\mathbf{y}) = \int\int d\mathbf{k}_2 d\tilde{\mathbf{x}} A_2(\mathbf{k}_2, \tilde{\mathbf{x}})e^{-i[\phi(\mathbf{k}_2,\tilde{\mathbf{x}})-\mathbf{k}_2\cdot\mathbf{y}]}f(\tilde{\mathbf{x}})\ ,$$

where A_2 is defined as in Section 5.1.

CHAPTER 6

The Relationship to Beylkin's Inverse

Although the inverse operator (36) derived in Chapter 5 appears to be different from Beylkin's inverse (8) described in Chapter 2, they should be related if the initial wavefronts are the same. The differential equation that Beylkin treats is the reduced wave equation, which is related to the time dependent wave equation by the Fourier transform. In this chapter we show that the two forward operators are equivalent up to amplitude. It then follows that the inverses are also equivalent.

The integral operator representation for the scattered signal generalized to n dimensions is

$$[P_1\delta c](\mathbf{x}',t) = \int\int d\mathbf{k}_1\, d\mathbf{z} A_1(\mathbf{k}_1,\mathbf{x}',t) e^{ik_1[\phi_r(\hat{\mathbf{k}}_1,\mathbf{x}',t)-\hat{\mathbf{k}}_1\cdot\mathbf{z}]}\delta c(\mathbf{z})\ . \tag{40}$$

The scattered field solves the perturbational wave equation

$$\begin{aligned} \frac{1}{c_0^2}\delta u_{tt} - \nabla^2\delta u &= \frac{2\delta c}{c_0^3} u_{0_{tt}} \\ \delta u &= \delta u_t = 0 \quad \text{for} \quad t < 0, \end{aligned}$$

where the incident wave field u_0 satisfies

$$\frac{1}{c_0^2} u_{0_{tt}} - \nabla^2 u_0 = 0$$

$$u_0(\mathbf{x},0) \quad = \quad g(\psi(\mathbf{x}))\ ,$$

and $g(\psi(\mathbf{x}))$ is singular at $\psi(\mathbf{x}) = 0$.

The Fourier transform of the integral operator representation for the scattered signal found in Beylkin [13] is

$$\mathcal{F}v^{sc}(\xi,t) = \int\int d\mathbf{x}d\omega \frac{\omega^{n-1}}{(2\pi)^n} f(\mathbf{x})a(\mathbf{x},\xi)e^{i\omega(\phi(\mathbf{x},\eta)+\phi(\mathbf{x},\xi)-t)}\ . \tag{41}$$

The scattered signal v^{sc} solves the reduced wave equation

$$\omega^2 n_0^2 v^{sc} + \nabla^2 v^{sc} = -\omega^2 f(\mathbf{x}) v^{in}$$

$$v^{sc} = 0\ , \quad x \ \text{large}\ ,$$

where v^{in} satisfies

$$\omega^2 n_0^2 v^{in} + \nabla^2 v^{in} = \delta(x-\eta)$$

$$v^{in} = 0\ , \quad x \ \text{large}\ .$$

The scattered field v^{sc} is solved by the use of the Green's function

$$\omega^2 n_0^2 v^{out} + \nabla^2 v^{out} = \delta(\mathbf{x}-\xi)\ .$$

If the initial wavefront $u_0(\mathbf{x},0)$ were due to a point source, then the integral operator representations of the scattered field, (40) and (41), should be very closely related. In order to exhibit this relationship, we first analyze the representation

(40) shown in Chapter 5. We can split the integral over $\mathbf{k}_1$ into an integral over the unit vector $\hat{\mathbf{k}}_1$ and an integral over the length k_1. Then

$$P_1\delta c(\mathbf{x}',t) = \int\int d\hat{\mathbf{k}}_1 d\mathbf{z} \tilde{A}_1(\hat{\mathbf{k}}_1,\mathbf{x}',t)\delta c(\mathbf{z}) \int dk_1\ k_1^{m+n-1} e^{ik_1[\phi_r(\hat{\mathbf{k}}_1,\mathbf{x}',t)-\hat{\mathbf{k}}_1\cdot\mathbf{z}]}$$

if $A_1(\mathbf{k}_1,\mathbf{x}',t)$ is homogeneous of order m.

Since the Fourier transform of the j^{th} derivative of $\delta(x)$ is $-(-ik)^j$,

$$P_1\delta c(\mathbf{x}',t) =$$
$$-\int\int d\hat{\mathbf{k}}_1 d\mathbf{z}\tilde{A}_1(\hat{\mathbf{k}}_1,\mathbf{x}',t)\delta c(\mathbf{z})(-i)^{1-m-n}\delta^{(m+n-1)}(\phi_r(\hat{\mathbf{k}}_1,\mathbf{x}',t)-\hat{\mathbf{k}}_1\cdot\mathbf{z})\ ,$$

where $\delta^{(m+n-1)}$ is the $(m+n-1)^{th}$ derivative of $\delta(\mathbf{x})$.

This representation of the scattered signal is an integral of the form

$$I = \int d\mathbf{x}\ a(\mathbf{x})\delta^{(m+n-1)}(\phi(\mathbf{x}))\ .$$

Changing variables to

$$\mathbf{y} = (y_1,\ldots,y_n) = (x_1,\ldots,x_{n-1},\phi(\mathbf{x}))$$

gives the Jacobian factor

$$J = \left[\frac{\partial\phi}{\partial x_n}(y_1,\ldots,y_{n-1}), x_n(\mathbf{y}))\right]^{-1}\ .$$

The solution of

$$y_n - \phi(y_1,\ldots,y_{n-1},x_n) = 0$$

gives $x_n = x_n(y_n\,; y_1, \ldots, y_{n-1})$ since $\frac{\partial\phi}{\partial x_n}$ is never zero. Therefore

$$I = \int d\mathbf{y} \frac{a(y_1, \ldots, y_{n-1}, x_n(\mathbf{y}))}{\frac{\partial\phi}{\partial x_n}(y_1, \ldots, y_{n-1}, x_n(\mathbf{y}))}\, \delta^{(m+n-1)}(y_n)$$

which is reduced to

$$\begin{aligned} I &= \int_{y_n=0} d\mathbf{y} \left(\frac{\partial}{\partial y_n}\right)^{m+n-1} \left[\frac{a(y_1, \ldots, y_{n-1}, x_n(\mathbf{y}))}{\frac{\partial\phi}{\partial x_n}(y_1, \ldots, y_{n-1}, x_n(\mathbf{y}))}\right] \\ &= \int_{\phi=0} d\mathbf{x}\, w(\mathbf{x}) \; . \end{aligned}$$

Thus the scattered signal can be written as

$$P_1 \delta c(\mathbf{x}', t) = \int\!\!\int_{\phi_r(\hat{\mathbf{k}}_1, \mathbf{x}', t) = \hat{\mathbf{k}}_1 \cdot \mathbf{z}} d\hat{\mathbf{k}}_1 d\mathbf{z}\, w(\hat{\mathbf{k}}_1, \mathbf{x}', t, \mathbf{z}) \; .$$

The reflected phase $\phi_r(\hat{\mathbf{k}}_1, \mathbf{x}, t)$ is constant along rays, and if there are no caustics present, then the rays do not cross. Thus, given $(\hat{\mathbf{k}}_1, \mathbf{x}', t)$, there is a unique point $\mathbf{z}$ for which

$$\phi_r(\hat{\mathbf{k}}_1, \mathbf{x}', t) = \hat{\mathbf{k}}_1 \cdot \mathbf{z}$$

holds. The $\mathbf{z}$ that satisfies this is specified by the initial condition for the reflected phase

$$\phi_r(\hat{\mathbf{k}}_1, \mathbf{z}, \phi(\mathbf{z})) = \hat{\mathbf{k}}_1 \cdot \mathbf{z}$$

and is the reflection point of the ray which passes through $(\mathbf{x}', t)$ (see Figure 6.1). Since this is an equation in $\hat{\mathbf{k}}_1$ and $\mathbf{z}$ only, it specifies $\hat{\mathbf{k}}_1$ as a function of $\mathbf{z}$. Thus (42) is an integral over the normal vectors $\hat{\mathbf{k}}_1$, which parameterize the constant

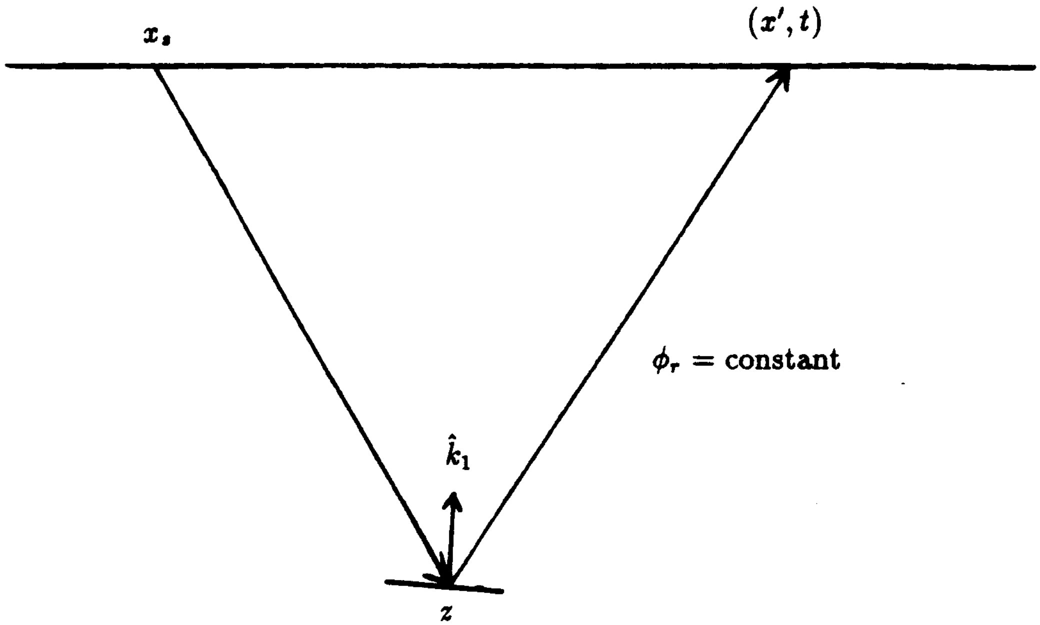


Figure 6.1. The point $\mathbf{z}$ is the reflection point of the ray which passes through $(\mathbf{x}', t)$.

travel time ellipse $t_1 + t_2 =$ constant (see Figure 6.2). If the reference velocity is constant, then the curve $t_1 + t_2 =$ constant is an ellipse, otherwise it is ellipse-like.

Beylkin's representation for the scattered field can similarly be written as an integral over a constant phase surface,

$$v^{sc}(\xi, t) = \int_{\phi(\mathbf{x},\eta)+\phi(\mathbf{x},\xi)=t} d\mathbf{x}\, w(\mathbf{x}, \xi, \eta) \ .$$

The phase ϕ satisfies the eikonal equation

$$|\nabla \phi|^2 = n_0^2$$

which is solved by the Hamilton-Jacobi equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{p} \qquad\qquad \dot{\phi} = n_0^2 \\ \dot{\mathbf{p}} &= -2n_0 \nabla n_0 \ . \end{aligned}$$

Then

$$\phi = \int_{rays} n_0^2 \ ,$$

where the rays are given by $\dot{\mathbf{x}} = \mathbf{p}$. Thus $\phi(\mathbf{x}, \eta)$ is the geodesic distance along the incident ray from the source point η on the surface to the interior point $\mathbf{x}$, and $\phi(\mathbf{x}, \xi)$ is the geodesic distance along the reflected ray from the point $\mathbf{x}$ to the receiver position ξ (see Figure 6.3). Therefore, v^{sc} is an integral over the same constant travel time ellipse as δu. That is, up to amplitudes, the solutions to the forward problems δu and v^{sc} are equivalent. It is expected that the amplitudes will also prove to be equivalent.

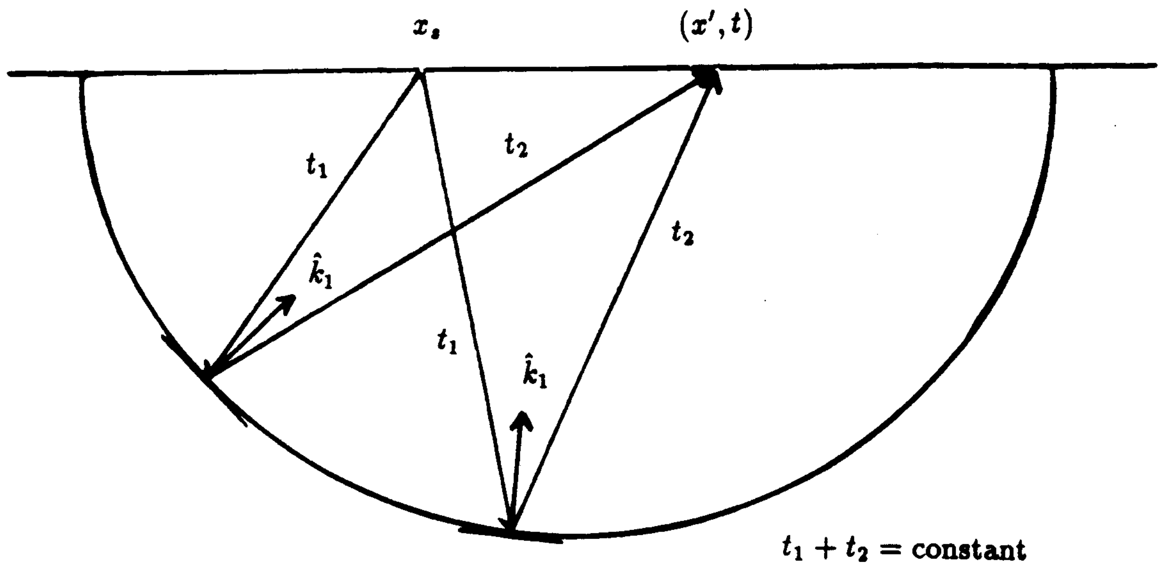


Figure 6.2. The vectors $\hat{\mathbf{k}}_1$ parameterize the constant travel time ellipse $t_1 + t_2 =$ constant.

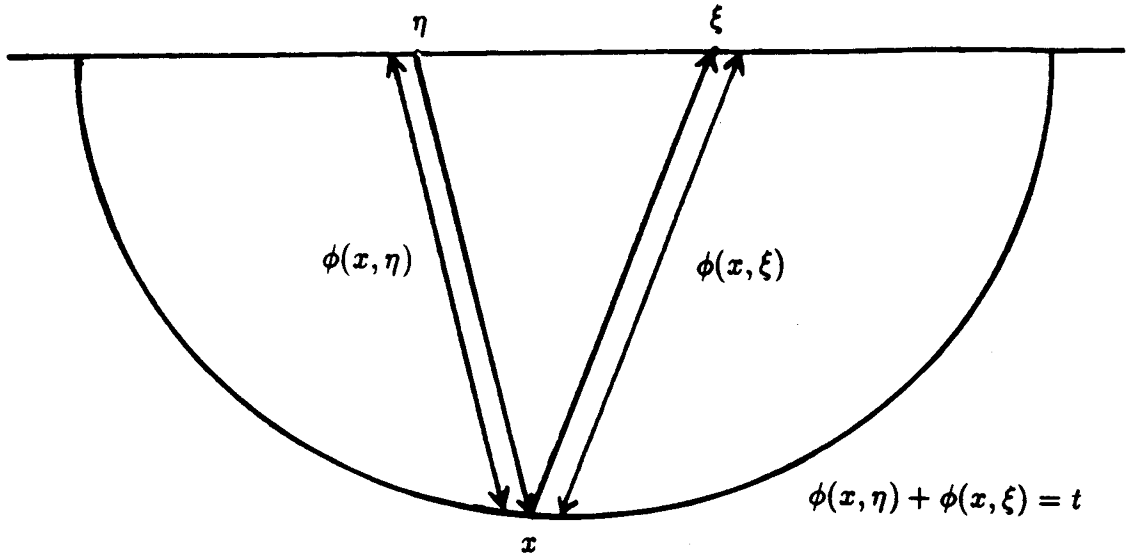


Figure 6.3. The phases $\phi(\mathbf{x},\eta)$ and $\phi(\mathbf{x},\xi)$ are the geodesic distances along the incident and reflected rays, respectively.

Since the inverse operators for both δu and v_{sc} are closely related to the adjoint operators, the equivalence relation found for the forward maps extends to the inverses. The inverse operators for both δu and v^{sc} are also integrals over constant phase surfaces. In this case, they are constant depth hyperbolas with integration over arrival times (see Figure 6.4).

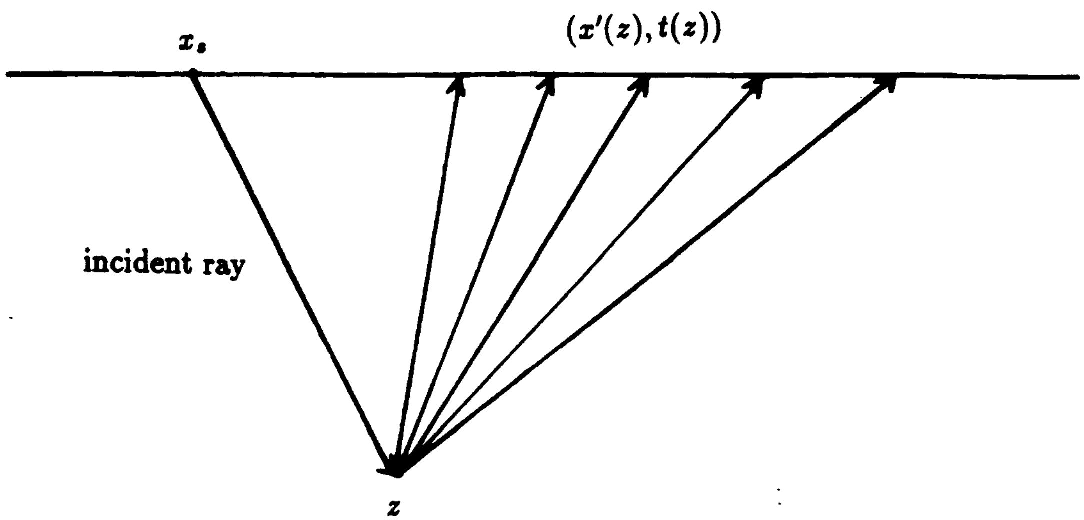


Figure 6.4. The inverse operators are integrals over constant depth surfaces parameterized by arrival times.

REFERENCES

1. G.B. Whitham [1974]. *Linear and Nonlinear Waves*, Wiley-Interscience, pp. 235-247.

2. V.A. Kulkarney and B.S. White [1982]. Focusing of waves in turbulent inhomogeneous media, *Phys. Fluids* Vol. 25 (10).

3. C.S. Morawetz [1981]. A formulation for higher dimensional inverse problems for the wave equation, *Comp. and Math. with Appls.*, Vol. 7, pp. 319-331.

4. J.B. Keller [1958]. A geometrical theory of diffraction, *Calculus of Variations and its Applications*, Proc. Symposia Appl. Math., Vol. 8, pp. 27-52, McGraw-Hill, New York.

5. D. Ludwig [1966]. Uniform asymptotic expansions at a caustic, *Comm. on Pure and Appl. Math.*, Vol. 19, pp. 215-250.

6. C. Chester, B. Friedman, and F. Ursell [1957]. An extension of the method of steepest descents, *Proc. Camb. Phil. Soc.*, vol. 54, pp. 599-611.

7. Y.A. Kravstov [1964]. A modification of the geometrical optics method, *Radiofizika*, Vol. 7, pp. 664-673.

8. Y.A. Kravstov [1964]. Asymptotic solutions of Maxwell's equations near a caustic, *Radiofizika*, Vol. 7, pp. 1049-1056.

9. D.C. Stickler, D.S. Ahluwalia and L. Ting [1981]. Application of Ludwig's uniform progressing wave ansatz to a smooth caustic, *J. Acoust. Soc. Am.*, Vol. 69 (6), pp. 1673-1681.

10. R.W. Clayton and R.H. Stolt 1981]. A Born-WKBJ inversion method for acoustic reflection data, *Geophysics*, Vol. 46 (11), pp. 1559-1567.

11. J.K. Cohen and N. Bleistein [1977]. An inverse method for determining small variations in propagation speed, *SIAM J. Appl. Math.*, Vol. 32 (4), pp. 784-799.

12. P. Lailly [1983]. Migration methods: partial but efficient solutions to the seismic inverse problem, *Inverse Problems of Acoustic and Elastic Waves*, pp. 182-214, SIAM, Philadelphia.

13. G. Beylkin [1985]. Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform, *J. Math. Phys.*, Vol. 26 (1), pp. 99-108.

14. F. Santosa and W.W. Symes [1988]. High-frequency perturbational analysis of the surface point-source response of a layered fluid, *J. Comp. Phys.*, vol. 74 (2), pp. 318-381.

15. J.J. Duistermaat [1973]. Fourier integral operators, Courant Institute Lecture Notes, New York, NY.

16. Rakesh [1986]. A coefficient determination problem for the wave equation, Ph.D. thesis, Cornell University.