Chapter 7

Advanced Integration Techniques

Before introducing the more advanced techniques, we will look at a shortcut for the easier of the substitution-type integrals. Advanced integration techniques then follow: integration by parts, trigonometric integrals, trigonometric substitution, and partial fraction decompositions.

7.1 Substitution-Type Integration by Inspection

In this section we will consider integrals which we would have done earlier by substitution, but which are simple enough that we can guess the approximate form of the antiderivatives, and then insert any factors needed to correct for discrepancies detected by (mentally) computing the derivative of the approximate form and comparing it to the original integrand. Some general forms will be mentioned as formulas, but the *idea* is to be able to compute many such integrals without resorting to writing the usual u-substitution steps.

Example 7.1.1 *Compute* $\int \cos 5x\,dx$.

Solution: We can anticipate that the **approximate form**[1] *of the answer is* $\sin 5x$*, but then*

$$\frac{d}{dx}\sin 5x = \cos 5x \cdot \frac{d}{dx}(5x) = \cos 5x \cdot 5 = 5\cos 5x.$$

Since we are looking for a function whose derivative is $\cos 5x$*, and we found one whose derivative is* $5\cos 5x$*, we see that our candidate antiderivative* $\sin 5x$ *gives a derivative with an extra factor of* 5*, compared with the desired outcome. Our candidate antiderivative's* **derivative** *is 5 times too large, so this candidate antiderivative,* $\sin 5x$ *must be 5 times too large. To compensate and arrive at a function with the proper derivative, we multiply our candidate* $\sin 5x$ *by* $\frac{1}{5}$*. This give us a new candidate antiderivative* $\frac{1}{5}\sin 5x$*, whose derivative is of course* $\frac{1}{5}\cos 5x \cdot 5 = \cos 5x$*, as desired. Thus we have*

$$\int \cos 5x\,dx = \frac{1}{5}\sin 5x + C.$$

It may seem that we wrote more in the example above than with the usual u-substitution method, but what we wrote could be performed mentally without resorting to writing the details.

In future sections, an integral such as the above may occur as a relatively small step in the execution of a more advanced and more complicated method (perhaps for computing a much more difficult integral). This section's purpose is to point out how such an integral can be quickly dispatched, to avoid it becoming a needless distraction in the more advanced methods.

[1] In this section, by *approximate form* we mean a form which is correct except for **multiplicative constants**.

7.1.1 The Method

The method used in all the examples here can be summarized as follows:

1. Anticipate the form of the antiderivative by an *approximate form* (correct up to a multiplicative constant).
2. Differentiate this approximate form and compare to the original integrand function;
3. If Step 1 is correct, i.e., the approximate form's derivative differs from the original integrand function by a multiplicative constant, insert a compensating, reciprocal multiplicative constant into the approximate form to arrive at the actual antiderivative;
4. For verification, differentiate the answer to see if the original integrand function emerges.

For instance, some general formulas which should be quickly verifiable by inspection (that is, by reading and mental computation rather than with paper and pencil, for instance) follow:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C, \tag{7.1}$$

$$\int \cos kx\,dx = \frac{1}{k}\sin kx + C, \tag{7.2}$$

$$\int \sin kx\,dx = -\frac{1}{k}\cos kx + C, \tag{7.3}$$

$$\int \sec^2 kx\,dx = \frac{1}{k}\tan kx + C, \tag{7.4}$$

$$\int \csc^2 kx\,dx = -\frac{1}{k}\cot kx + C, \tag{7.5}$$

$$\int \sec kx\tan kx\,dx = \frac{1}{k}\sec kx + C, \tag{7.6}$$

$$\int \csc kx\cot kx\,dx = -\frac{1}{k}\csc kx + C, \tag{7.7}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C. \tag{7.8}$$

Example 7.1.2 *The following integrals can be computed with u-substitution, but also are computable by inspection:*

- $\displaystyle\int e^{7x}\,dx = \frac{1}{7}e^{7x} + C,$
- $\displaystyle\int \frac{1}{5x-9}\,dx = \frac{1}{5}\ln|5x-9| + C,$
- $\displaystyle\int \sin 5x\,dx = -\frac{1}{5}\cos 5x + C,$
- $\displaystyle\int \cos\frac{x}{2}\,dx = 2\sin\frac{x}{2} + C,$
- $\displaystyle\int \sec^2 \pi x\,dx = \frac{1}{\pi}\tan \pi x + C,$
- $\displaystyle\int \csc 6x\cot 6x\,dx = -\frac{1}{6}\csc 6x + C.$

While it is true that we can call upon the formulas (7.1)–(7.8), the more flexible strategy is to anticipate the form of the antiderivative and adjust accordingly. For instance, we have the following antiderivative form, written two ways:

$$\int \frac{1}{u}\,du = \ln|u| + C,$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

(As usual, the second form is the same as the first where $u = f(x)$.) So when we see an integrand which is a fraction with the numerator being the derivative of the denominator except for multiplicative constants, we know the antiderivative will be, approximately, the natural log of the absolute value of that denominator.

Example 7.1.3 *Consider* $\displaystyle\int \frac{x}{x^2+1}\,dx$

Here we see that the derivative of the denominator of the integrand is present—up to a multiplicative constant—in the numerator. Our candidate approximate form can then be given by $\ln|x^2+1| = \ln(x^2+1)$. *Now we differentiate to see what constant factor we need to insert to get the correct derivative:*

$$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}.$$

To correct for the extra factor of 2 *and thus get the correct derivative, we insert the factor* $\frac{1}{2}$*:*

$$\frac{d}{dx}\left[\frac{1}{2}\ln(x^2+1)\right] = \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x = \frac{x}{x^2+1},$$

as desired. Thus

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C.$$

To be sure, a quick mental check by differentiation verifies the answer.

Of course there are many other forms. Recall we had many other integration formulas, as in Subsection 6.6.1, page 578. For instance it is not difficult to see, or check, that

$$\int \frac{1}{u^2+1}\,du = \tan^{-1}u + C \qquad \Longrightarrow \qquad \int \frac{1}{a^2x^2+1}\,dx = \frac{1}{a}\tan^{-1}(ax) + C.$$

Example 7.1.4 *For instance, we have the following integral computations, which can be seen by relatively easily taking derivatives of the presented antiderivatives.*

- $\displaystyle\int \frac{1}{9x^2+1}\,dx = \frac{1}{3}\tan^{-1}3x + C,$
- $\displaystyle\int \frac{1}{\sqrt{1-3x^2}}\,dx = \frac{1}{\sqrt{3}}\sin^{-1}\left(\sqrt{3}x\right) + C,$
- $\displaystyle\int \sec 3x\,dx = \frac{1}{3}\ln|\sec 3x + \tan 3x| + C,$
- $\displaystyle\int \tan 2x\,dx = \frac{1}{2}\ln|\sec 2x| + C.$

The method can be used for somewhat more complicated integrals as well, though there does come a point where it seems more natural to simply execute the full substitution method, which is more "constructive" than our method here. However, our "approximate and correct" (read as verbs) method here can be reasonably employed on still more complicated integrals.

Example 7.1.5 *Consider* $\displaystyle\int \frac{1}{\sqrt{5x-9}}\,dx.$

Of course this can be rewritten $\int (5x-9)^{-1/2}\,dx$. *Now it is crucial that a complete substitution,* $u = 5x-9 \implies du = 5\,dx$, *etc., would show that* du *and* dx *agree except for a multiplicative constant, so we know that the integral—up to a multiplicative constant—is of approximate form* $\int u^{-1/2}\,du$, *which calls for the power rule.*

The approximate form of the antiderivative is thus $u^{1/2} = (5x-9)^{1/2}$, *which we write in* x *and then differentiate,*

$$\frac{d}{dx}(5x-9)^{1/2} = \frac{1}{2}(5x-9)^{-1/2}\cdot 5,$$

which has **extra** *factors (compared to our original integrand) of collectively* $\frac{5}{2}$*. To cancel their effects we include a factor* $\frac{2}{5}$ *in our actual, reported antiderivative. Thus*

$$\int \frac{1}{\sqrt{5x-9}}\,dx = \frac{2}{5}(5x-9)^{1/2} + C = \frac{2}{5}\sqrt{5x-9} + C.$$

Note that a quick derivative computation, albeit involving a (simple) chain rule, gives us the correct function $1/\sqrt{5x-9}$.

Example 7.1.6 *Consider* $\int 7x\,\sin^5 x^2 \cos x^2\,dx$.

For such an antiderivative, our ability to guess the form depends upon our expertise with the original substitution method. Each of these were of a form $\int f(u)\,K\,du$*, where we could anticipate both* u *and* f*, with* du *accounting for remaining terms, and* $K \in \mathbb{R}$ *which we can initially ignore by taking our shortcut path described in this section. Looking ahead, the student well-versed in substitution will expect* $u = \sin x^2$*, and the integral being of the approximate form* $\int u^5\,du$ *(times a constant). Thus we will have an approximate antiderivative of* u^6 *(times a constant), i.e., the approximate form should be* $\sin^6 x^2$*. Now we differentiate this and see what compensating factors must be included to reconcile with the original integrand:*[2]

$$\frac{d}{dx}(\sin x^2)^6 = 6(\sin x^2)^5 \cdot \cos x^2 \cdot 2x = 12x\,\sin^5 x^2 \cos x^2.$$

Of course we want 7 *in the place of the* 12 *(or separately,* $2 \cdot 6$*), so we multiply by* $\frac{7}{12}$ *(or again,* $7 \cdot \frac{1}{6\cdot 2}$*). With this we have*

$$\int 7x\,\sin^5 x^2 \cos x^2\,dx = \frac{7}{12}\sin^6 x^2 + C.$$

It would be perfectly natural to forego this method of "guess and adjust" in favor of the old-fashioned substitution method for this problem. Indeed the full substitution method has some advantages (see the next subsection). For instance, it is more "constructive," and thus less error-prone; one is less tempted to skip steps while employing substitution, while one might attempt a purely mental derivative computation of the answer here and thus easily be off by a factor. It is important that each student find the comfortable level of brevity for himself or herself.[3]

But the the method of this section is still often worthwhile.

Example 7.1.7 *Compute* $\int x^3 \sin x^4\,dx$.

Solution: This is of the approximate form $\int \sin u\,du$*, with* $u = x^4$*. The approximate form of the solution is thus* $\cos x^4 + C$ *(or* $-\cos x^4 + C$*, but these differ by a multiplicative constant* -1*), which has derivative* $-\sin x^4 \cdot 4x^3$*. We introduce a factor of* $-\frac{1}{4}$ *to compensate for the extra factor of* -4*:*

$$\int x^3 \sin x^4\,dx = -\frac{1}{4}\cos x^4 + C,$$

which can be quickly verified by differentiation.

[2]Notice that we are assuming fluency in the chain rule as we compute the derivative of $\sin^6 x^2$, rather than writing out every step as we did in Chapter 4. Each student must gage personal ability to omit steps.

[3]It is the author's experience that students in engineering and physics programs are often more interested in arriving at the answer quickly, while mathematics and other science students usually prefer the presentation of the full substitution method. The latter are somewhat less likely to be wrong by a multiplicative constant, though the former tend to progress through the topics faster. There are, of course, spectacular exceptions, and each group benefits from camaraderie with the other.

Example 7.1.8 *Compute* $\displaystyle\int x\sqrt{9-x^2}\,dx$.

Solution: It is advantageous to read this integral as $\int x(9-x^2)^{1/2}\,dx$, *which is of approximate form* $\int u^{1/2}\,du$ *(where* $u = 9-x^2$*). These observations, and the approximate form* $(9-x^2)^{3/2}$ *of the integral, can be gotten by this mental observation we are developing in this section. The approximate antiderivative's derivative is* $\frac{3}{2}(9-x^2)^{1/2}\cdot(-2x)$, *which has an extra factor of* -3 *(after cancellation). Thus*

$$\int x\sqrt{9-x^2}\,dx = -\frac{1}{3}\left(9-x^2\right)^{3/2} + C.$$

7.1.2 Limitations of the Method

There are two very important points to be made about the limitations of the method. The first point is illustrated by an example, and the second by making several related points.

(I) It is imperative that the derivative of the approximate form differs from the original function to be integrated by at most a multiplicative constant.

In particular, an extra variable function cannot be compensated for. To illustrate this point, and simultaneously warn against a common mistake, consider

$$\int \frac{1}{x^2+1}\,dx.$$

The mistake to avoid here is to take erroneously the approximate solution to be $\ln(x^2+1)$, which we then notice has derivative

$$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot 2x.$$

Unfortunately we cannot compensate by dividing by the extra factor $2x$, because[4]

$$\frac{d}{dx}\left[\frac{\ln(x^2+1)}{2x}\right] = \frac{2x\cdot\frac{d\ln(x^2+1)}{dx} - \ln(x^2+1)\cdot\frac{d(2x)}{dx}}{(2x)^2} = \frac{2x\cdot\frac{2x}{x^2+1} - 2\ln(x^2+1)}{4x^2},$$

which is guaranteed (by the presence of the non-cancelling logarithm in the result) to be something other than our original function $\frac{1}{x^2+1}$. The method does not work because multiplicative *functions* do not "go along for the ride" in derivative (or antiderivative) problems the way multiplicative constants do. Of course we knew from before that

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C,$$

so this integral is not really suitable for a substitution argument, but is rather a special form in its own right.

[4]Alternatively, a product rule computation can be used:

$$\frac{d}{dx}\left[\frac{1}{2x}\ln(x^2+1)\right] = \frac{1}{2x}\cdot\frac{d\ln(x^2+1)}{dx} + \ln(x^2+1)\cdot\frac{d}{dx}\left[\frac{1}{2x}\right] = \frac{1}{2x}\cdot\frac{2x}{x^2+1} - \frac{1}{2x^2}\ln(x^2+1),$$

which eventually gives the original function for the first product, but the second part of the product rule is a complication we cannot rid ourselves of easily, though a partial solution to this problem of extra, function-type factors in the integrand is given in the next section.

(II) This method can not totally replace the earlier substitution method.

(a) The skills used in the substitution method will be needed for later methods. In particular, the idea which is crucial and recurring is that the entire integral in x is often replaced by one in u (for instance)—including the dx-term and possibly others replaced by du—and, if a definite integral, the interval of integration also represents u-values.

(b) If an integral is difficult enough, the more "constructive" substitution method is less error-prone than is this shortcut style developed here.

(c) Anyhow, the idea of the substitution method is embedded in this method; anticipating what to set equal to u is equivalent to guessing the approximate form of the integral in u, and thus the approximate form of the antiderivative.

(d) When using numerical and other methods with definite integrals, a substitution can sometimes make for a much simpler integral to be approximated or otherwise analyzed, even if the antiderivative is never computed. For instance, with $u = x^2$, giving then $du = 2x\,dx$, we can write

$$\int_{-1}^{2} xe^{x^4}\,dx = \frac{1}{2}\int_{1}^{4} e^{u^2}\,du.$$

None of our usual techniques will yield antiderivatives for either integrand (as the reader is invited to try), but numerical methods such Riemann sums, Trapezoidal and Simpson's Rules can find approximations for the definite integrals. The latter form of the integral (in u) will yield accurate numerical results more easily than the former (in x).

To summarize, the method here has us making an educated guess about the form of the antiderivative, perhaps writing down our guess as our tentative (or "candidate") answer, taking its derivative, and, assuming it is the same as the integrand except for some multiplicative constant(s), inserting other multiplicative constants into our answer to adjust for discrepancies. It will only work if the tentative antiderivative has derivative equal to some constant times the original integrand.

The method is not sophisticated, but will be useful for streamlining later, much longer integration techniques introduced in the rest of this chapter.

Exercises

For each of the following, first attempt to compute the antiderivative by finding an approximate form of the antiderivative, differentiating it, and inserting a constant factor to compensate for any extra or missing constants. If that method is too unwieldy, compute the integral by the substitution method, showing all details.

1. $\int (x^2+1)^7 \cdot 2x\,dx$
2. $\int \cos x^4 \cdot 4x^3\,dx$
3. $\int 15x^2 \sec^2 5x^3\,dx$
4. $\int \dfrac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}\,dx$
5. $\int \dfrac{\csc^2\left(\frac{1}{x}\right)}{x^2}\,dx$
6. $\int \tan^7 x\,\sec^2 x\,dx$
7. $\int \dfrac{x}{(x^2+1)^3}\,dx$
8. $\int (2x-11)^9\,dx$
9. $\int \cos 5x\,dx$
10. $\int \csc 9x \cot 9x\,dx$
11. $\int \cos x \sin x\,dx$ (See #13)
12. $\int \tan^3 5x \sec^2 5x\,dx$
13. $\int \sin x \cos x\,dx$ (See #11)
14. $\int \sin^3 x \cos x\,dx$
15. $\int \tan^5 x \sec^2 x\,dx$
16. $\int x \sin x^2\,dx$
17. $\int x^3 \cdot \sqrt{x^4-2}\,dx$
18. $\int (x^3+x^2)^4(3x^2+2x)\,dx$
19. $\int \sec^5 3x \cdot \sec 3x \tan 3x\,dx$
20. $\int \dfrac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
21. $\int x^2 \sin x^3 \cos^5 x^3\,dx$
22. $\int \dfrac{\sin^3\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)}{x^2}\,dx$
23. $\int e^x \cos e^x\,dx$
24. $\int xe^{x^2}\,dx$
25. $\int e^{2x} \sin e^{2x}\,dx$
26. $\int e^{-x} \sec^2 e^{-x}\,dx$
27. $\int e^{5x}\,dx$
28. $\int \dfrac{e^x}{e^{2x}+1}\,dx$
29. $\int \dfrac{e^{3x}}{\sqrt{1-e^{6x}}}\,dx$
30. $\int \dfrac{dx}{\sqrt{e^{2x}-1}}$ (Hint: multiply the integrand by e^x/e^x.)
31. $\int e^{4x}(9+e^{4x})^{10}\,dx$
32. $\int xe^{-2x^2}\,dx$
33. $\int \dfrac{e^{1/x}}{x^2}\,dx$

34. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

35. $\displaystyle\int e^{3\cos 2x}\sin 2x\,dx$

36. $\displaystyle\int \frac{\cos x}{\sin x+1}\,dx$

37. $\displaystyle\int \frac{\cos x}{\sin x}\,dx$

38. $\displaystyle\int \frac{\sin x}{\cos x}\,dx$

39. $\displaystyle\int \frac{2x+1}{x^2+x}\,dx$

40. $\displaystyle\int \frac{x}{x^2+1}\,dx$

41. $\displaystyle\int \frac{1}{x^2+1}\,dx$

42. $\displaystyle\int \frac{1}{x\ln x}\,dx$

43. $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx$

44. $\displaystyle\int \frac{e^{2x}}{1+e^{4x}}\,dx$

45. $\displaystyle\int \frac{\sec^2 x}{1+\tan x}\,dx$

46. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

47. $\displaystyle\int \frac{\ln x}{x}\,dx$

48. $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

49. $\displaystyle\int \frac{1}{x(1+(\ln x)^2)}\,dx$

50. $\displaystyle\int \frac{1}{x|\ln x|\sqrt{(\ln x)^2-1}}\,dx$

51. $\displaystyle\int \frac{\sec^2(\ln x)}{x}\,dx$

52. $\displaystyle\int \frac{(9+\ln x)^6}{x}\,dx$

53. $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$

54. $\displaystyle\int \cot 2x\,dx$

55. $\displaystyle\int \frac{\tan(\ln x)}{x}\,dx$

56. $\displaystyle\int \csc\frac{x}{9}\,dx$

57. $\displaystyle\int x^2\sec 5x^3\,dx$

7.2 Integration By Parts

While integration by substitution in its elementary form takes advantage of the chain rule, by contrast integration by parts exploits the product rule. In applications it is a bit more complicated than substitution, and there are perhaps more variations on the theme than with substitution, at least at the college calculus level. For these reasons, being fluent in this method usually requires seeing more steps ahead than substitution required. However it can be similarly mastered with practice.

7.2.1 The Idea by Examples

Suppose that we need to find an antiderivative of the function $f(x) = x\sec^2 x$. It is not hard to see that normal substitution in not going to easily yield a formula for our desired antiderivative $F(x)$:

$$\int x\sec^2 x\,dx = F(x) + C.$$

However, a clever student might notice that $x\sec^2 x$ contains terms that could have arisen from a product rule derivative such as:

$$\begin{aligned}\frac{d}{dx}\left[x\tan x\right] &= x\cdot\frac{d\tan x}{dx} + \tan x\cdot\frac{dx}{dx}\\ &= x\sec^2 x + \tan x.\end{aligned}$$

If we rearrange the terms above, we can rewrite this product rule derivative computation as follows:

$$x\sec^2 x = \frac{d}{dx}\left[x\tan x\right] - \tan x. \tag{7.9}$$

In fact (7.9) above is perhaps where the spirit of the method is most on display: that the given function to be integrated is indeed one *part* of a product rule derivative. If we are fortunate, the other *part* of the product rule formula is easier to integrate, because the derivative term, namely $\frac{d}{dx}\left[x\tan x\right]$ is trivial to integrate (see below). Indeed, if we take antiderivatives of both sides of (7.9), we then get the desired formula for the antiderivative $\int x\sec^2 x\,dx$:

$$\begin{aligned}\int x\sec^2 x\,dx &= \int\left[\left(\frac{d[x\tan x]}{dx}\right) - \tan x\right]dx\\ &= x\tan x - \int\tan x\,dx\\ &= x\tan x - \ln|\sec x| + C.\end{aligned}$$

From such as the above emerges a method whereby we identify our given function (here $x\sec^2 x$) as a *part* of a product rule computation ($\frac{d}{dx}[x\tan x]$), and integrate our original function by instead (trivially) integrating the product rule derivative term (again $\frac{d}{dx}[x\tan x]$), and then integrating the other *part* ($\tan x$) of the product rule output. Often the other, hidden part of the underlying product rule is easier to integrate than the original function, and therein lies much of the usefulness of the method.

While we will have a formal procedure to implement the method, one more example from first principles can further illustrate its spirit.

Example 7.2.1 *Compute* $\int x\cos x\,dx$.

Solution: The integrand is one part of a product rule computation for $\frac{d}{dx}[x\sin x]$, *namely* $\frac{d}{dx}[x\sin x] = x\cos x + \sin x$, *so we can write*

$$\begin{aligned}\int x\cos x\,dx &= \int\left[\left(\frac{d}{dx}[x\sin x]\right) - \sin x\right]dx\\ &= x\sin x + \cos x + C.\end{aligned}$$

It is interesting to compute the derivative of our answer, and see how the term we desire $(x\cos x)$ emerges, and how other terms which naturally emerge cancel each other. This is left to the reader.

7.2.2 The Technique in Its Simpler Applications

Recall that when we completely developed the substitution method, the underlying principle—the chain rule—was not written out in complete derivative form, but rather in the more compact *differential* form. Having supposed that F was an antiderivative of f, i.e., $F' = f$, we eventually settled on writing the argument below without the first two integrals:

$$\int f(u(x))u'(x)\,dx = \int f(u(x))\cdot\frac{du(x)}{dx}\,dx = \int f(u)\,du = F(u) + C = F(u(x)) + C.$$

At first we did write the first steps because the proof was in the chain rule: $\frac{d}{dx}F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$. However, we eventually opted for the differential form, though for most it takes some practice for it to seem natural.

We will adopt differential notation in integration by parts as well. For instance, recall that the product rule for derivatives,

$$\frac{d(uv)}{dx} = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}, \tag{7.10}$$

can be rewritten in differential form

$$d(uv) = u\,dv + v\,du. \tag{7.11}$$

This came from multiplying both sides of (7.10) by dx, all along assuming u and v are in fact functions of x. Now we rearrange (7.11) as follows:

$$u\,dv = d(uv) - v\,du. \tag{7.12}$$

Equation (7.12) is perhaps the best equation to visualize the principle behind the eventual integration formula, because it is an easy step from the product rule. The actual formula quoted in most textbooks is still two steps away. First we integrate both sides:

$$\int u\,dv = \int d(uv) - \int v\,du. \tag{7.13}$$

Next we notice that the first integral on the right hand side of (7.13) is simply $\int d(uv) = uv + \text{Constant}$. Since there is also a constant present in the second integral on the right hand side of (7.13), we can omit mentioning the first constant and arrive at our final working formula for our integration by parts technique:

$$\boxed{\int u\,dv = uv - \int v\,du.} \tag{7.14}$$

Most textbooks and instructors use the formula above in exactly this form (7.14). It should be memorized, though its derivation—particularly from (7.12)—should also not be forgotten. Furthermore, anytime it is used it is good practice to write the (boxed) formula (7.14) within the problem at the point at which it is used.

Next we look at an example of the actual application of (7.14). In the example below, the arrangement of terms is as one would work the problem with pencil and paper, except for the implication arrows which we will omit in subsequent problems, and possibly the under-braces which become less and less necessary with practice.

Example 7.2.2 *Compute* $\int xe^x\,dx$.

Solution: *First recall our boxed formula (7.14). For this problem, we write*

$$\begin{array}{cc} u = x & dv = e^x\,dx \\ \Downarrow & \Uparrow \\ du = dx & v = e^x \end{array}$$

$$\begin{aligned} \int \underbrace{x}_{u}\ \underbrace{e^x\,dx}_{dv} &= uv - \int v\,du \\ &= (x)(e^x) - \int (e^x)\,dx \\ &= xe^x - e^x + C. \end{aligned}$$

It is interesting to note that we *choose* u and dv, and then *compute* du and v, with one qualification. That is that v is not unique; the computation from dv to v is of an antidifferentiation nature and so we really only know v up to an additive constant. In fact *any* v so that $dv = e^x\,dx$ (we took $v = e^x \implies dv = e^x\,dx$) will work in (7.14). Any additive constant, while legitimate, will eventually cancel in the final computation, and so we usually omit it. For instance, if we had chosen $v = e^x + 100$, we would have had

$$\begin{aligned} uv - \int v\,du &= x(e^x + 100) - \int (e^x + 100)\,dx \\ &= xe^x + 100x - e^x - 100x + C \\ &= xe^x - e^x + C, \end{aligned}$$

with the same final answer as before. For most cases, we will just assume that the additive constant is zero and we will use the simplest antiderivative for v. We will also not continue to write the implication arrows as they are technical and perhaps confusing.

Now we will revisit the first example we gave in Subsection 7.2.1, using what will be our basic style for this method.

Example 7.2.3 *Compute* $\int x\sec^2 x\,dx$.

Solution: *First, as is standard for these, we complete a chart like in the previous example.*

$$\begin{array}{cc} u = x & dv = \sec^2 x\,dx \\ du = dx & v = \tan x \end{array}$$

$$\begin{aligned}\int x\sec^2 x\,dx &= \int u\,dv\\ &= uv - \int v\,du\\ &= x\tan x - \int \tan x\,dx\\ &= x\tan x - \ln|\sec x| + C.\end{aligned}$$

Since this method is more complicated than substitution, there are more complicated considerations in how to apply it. First of course, one should attempt an earlier, simpler method. But if those fail, and integration by parts is to be attempted,[5] the following guidelines for choosing u and dv should be considered for our formula $\int u\,dv = uv - \int v\,du$:

1. u and dv **must** account for all factors of the original integral, and no more.

   1.5. Of course, dv **must** contain the differential term (for example, dx) as a factor, but can contain more terms.

2. $v = \int dv$ should be computable with relative ease.

3. $du = u'(x)\,dx$ (assuming the original integral was in x) should not be overly complicated.

4. The integral $\int v\,du$ should be simpler than the original integral $\int u\,dv$.[6]

The next example illustrates the importance of consideration 2 above.

Example 7.2.4 *Compute* $\int x^3\sin x^2\,dx$.

Solution: We do not want to make $u = \sin x^2$, *because then* $dv = x^3\,dx$, *giving* $du = 2x\cos x^2$ *and* $v = \frac{1}{4}x^4$, *and our* $\int v\,du$ *will be* $\int \frac{1}{2}x^5\cos x^2\,dx$, *which is worse than our original integral.*

We will instead take u *to be some power of* x, *but not all of* x^3, *else the terms remaining for* dv *would be* $dv = \sin x^2\,dx$, *which we cannot integrate with ordinary methods.*[7]

What we will settle on is $dv = x\sin x^2\,dx$, *because its integral is an easy substitution we can short-cut as in Section 7.1. We leave the remaining terms, collectively* x^2, *for* u*:*

$$\begin{aligned} u &= x^2 & dv &= x\sin x^2\,dx\\ du &= 2x\,dx & v &= -\frac{1}{2}\cos x^2\end{aligned}$$

[5] Of course with practice one can see ahead whether or not integration by parts is likely to achieve an answer for a particular integral.

[6] Later, in a twist on the method, we will see that the we do not require $\int v\,du$ be easier than the original, $\int u\,dv$, in all cases, but it is desirable in most cases.

[7] In fact, we cannot compute $\int \sin x^2\,dx$ using any kind of substitution or parts, or any other method of this text for that matter, and arrive at an antiderivative in simple terms of the functions we know so far such as powers, exponentials, logarithms, trigonometric or hyperbolic functions or their inverses. However, when we study series we will find other expressions with which we can fashion an antiderivative of $\sin x^2$.

$$\begin{aligned}\int x^3 \sin x^2\,dx &= \int \underbrace{x^2}_{u} \cdot \underbrace{x \sin x^2\,dx}_{dv} \\ &= uv - \int v\,du \\ &= (x^2)\left(-\frac{1}{2}\cos x^2\right) - \int \left(-\frac{1}{2}\cos x^2\right) 2x\,dx \\ &= -\frac{x^2}{2}\cos x^2 + \int \cos x^2 \cdot x\,dx \\ &= -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2 + C.\end{aligned}$$

We omitted the details of computing v from dv, and computing the last integral, in the spirit of Section 7.1.

This last example shows how the requirement that $v = \int dv$ (up to an additive constant) be computable helps to guide us to the proper choice of u and dv. It was lucky that the second integral was easily computable (which would not have been the case if the original integral were, say, $\int x^2 \sin x^2\,dx$ or $\int x^4 \sin x^2\,dx$), but anyhow we can not even get to the second integral if we can not compute v.

Example 7.2.5 *Compute* $\displaystyle\int \frac{x^9}{\sqrt{1-x^5}}\,dx$.

Solution: This is similar to the previous example, in the sense that computability of $v = \int dv$ dictates our choices of u and dv.

$$\begin{aligned} u &= x^5 & dv &= x^4(1-x^5)^{-1/2}\,dx \\ du &= 5x^4\,dx & v &= -\frac{2}{5}(1-x^5)^{1/2}\end{aligned}$$

$$\begin{aligned}\int \frac{x^9}{\sqrt{1-x^5}}\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= -\frac{2}{5}x^5(1-x^5)^{1/2} + 2\int x^4(1-x^5)^{1/2}\,dx \\ &= -\frac{2}{5}x^5\sqrt{1-x^5} + 2\cdot\frac{-1}{5}\cdot\frac{2}{3}(1-x^5)^{3/2} + C \\ &= -\frac{2}{5}x^5\sqrt{1-x^5} - \frac{4}{15}(1-x^5)^{3/2} + C \\ &= -\frac{2}{15}\sqrt{1-x^5}\left[3x^5 + 2(1-x^5)\right] + C \\ &= -\frac{2}{15}(x^5+2)\sqrt{1-x^5} + C.\end{aligned}$$

As will often be the case in this section and subsequent sections, to check our answers it will usually be much easier to carefully check each step in our work of computing the integrals, than to compute the derivatives of our answers. (If we do wish to check answers by differentiation, it is often simpler to do so with a nonsimplified expression of the solution.)

7.2.3 Repeated Use of Integration by Parts

The next examples show a different lesson: that it is sometimes appropriate to integrate by parts more than once in a given problem. At each step, the hope is that the integral $\int v\,du$ is simpler than the integral it came from ($\int u\,dv$) in the integration by parts formula. Sometimes the new integral $\int v\,du$ is indeed simpler, but not so much so that it can be integrated readily. Indeed, at times that second integral also needs integration by parts, and so on, until we arrive at an integral that we can compute easily.

Example 7.2.6 *Compute* $\displaystyle\int x^2 \cos 3x\,dx$.

Solution: The x^2 term is complicating our integral, and so we reduce its effect somewhat by an integration by parts step.

$$\begin{aligned} u &= x^2 & dv &= \cos 3x\,dx \\ du &= 2x\,dx & v &= \frac{1}{3}\sin 3x \end{aligned}$$

$$\begin{aligned} \int x^2 \cos 3x\,dx &= uv - \int v\,du \\ &= \frac{1}{3}x^2 \sin 3x - \int \frac{2}{3} x \sin 3x\,dx. \end{aligned} \tag{7.15}$$

While we still cannot compute this last integral directly with old methods, it is better than the original in the sense that our trigonometric function is multiplied by a first-degree polynomial, where in the original the polynomial was second-degree. One more application of integration by parts and there will be no polynomial factor at all in the new integral.

A strict use of the language would force us to introduce two new variables other than u and v, but since they have "disappeared" in the present form of our answer, namely $\frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x\,dx$, it is not considered such bad form to "reset" (or "recycle") u and v for another integration by parts step, this time involving the integral $\int x \sin 3x\,dx$:

$$\begin{aligned} u &= x & dv &= \sin 3x\,dx \\ du &= dx & v &= -\frac{1}{3}\cos 3x \end{aligned}$$

$$\begin{aligned} \int x \sin 3x\,dx &= uv - \int v\,du \\ &= -\frac{x}{3}\cos 3x + \frac{1}{3}\int \cos 3x\,dx \\ &= -\frac{x}{3}\cos 3x + \frac{1}{3}\cdot\frac{1}{3}\sin 3x + C_1. \end{aligned}$$

Now we insert this last result into our original computation (7.15):

$$\begin{aligned} \int x^2 \cos 3x\,dx &= \frac{x^2}{3}\sin 3x - \frac{2}{3}\left[-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + C_1\right] \\ &= \frac{x^2}{3}\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C. \end{aligned}$$

Several lessons can be gleaned from the example above. (1) it is very important for proper "bookkeeping," as these problems can beget several "subproblems," and the proper placement of resulting terms is crucial for getting the correct final answer; (2) it is not unknown to use integration by parts more than once in a problem; (3) if we can take as many antiderivatives of a function $f(x)$ as we like (i.e., the antiderivative, the antiderivative of the antiderivative, etc.), then for an integral $\int x^n f(x)\,dx$ we can let $u = x^n$, and integration by parts will yield a second integral with a reduction in the power of x, namely

$$\int x^n f(x)\,dx = x^n F(x) - \int nx^{n-1} F(x)\,dx \tag{7.16}$$

where $F' = f$. If this process is repeated often enough, the complicating polynomial factors will have their degrees diminished until there is no polynomial factor, and we can finally finish the integration.

7.2.4 Integrals of Certain Other Functions

Here we look at cases where functions appear whose derivatives are known, but whose antiderivatives might not be standard knowledge of the average calculus student. In these cases we choose that function to be u, and the other terms to constitute dv, assuming of course we can compute $v = \int dv$.

Example 7.2.7 *Compute* $\displaystyle\int (x^2+1)\ln x\,dx$.

Solution: We cannot let $dv = \ln x\,dx$, since as yet we do not know the antiderivative of $\ln x$. (Even if we did, such a choice for dv would not be advantageous, as Example 7.2.8 will help to show later.) So we have little choice but to let $\ln x$ be u.

$$\begin{aligned} u &= \ln x & dv &= (x^2+1)\,dx \\ du &= \frac{1}{x}\,dx & v &= \frac{x^3}{3} + x \end{aligned}$$

$$\begin{aligned} \int (x^2+1)\ln x\,dx &= uv - \int v\,du \\ &= (\ln x)\left(\frac{x^3}{3}+x\right) - \int \left(\frac{x^3}{3}+x\right)\frac{1}{x}\,dx \\ &= \frac{1}{3}(\ln x)(x^3+3x) - \int \left(\frac{x^2}{3}+1\right)\,dx \\ &= \frac{1}{3}(x^3+3x)\ln x - \frac{1}{9}x^3 - x + C. \end{aligned}$$

Next we have an interesting case where we are forced to take $dv = dx$.

Example 7.2.8 *Compute* $\displaystyle\int \ln x\,dx$.

Solution: Here we can not let $dv = \ln x\,dx$, for computing $v = \int dv$ would be the same as computing the whole, original integral. As in Example 7.2.7, we also note that placing $\ln x$ in the u-term makes for a simple enough du term. Thus we write

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x}\,dx & v &= x \end{aligned}$$

so that

$$\begin{aligned}\int \ln x\,dx &= uv - \int v\,du \\ &= (\ln x)(x) - \int (x)\left(\frac{1}{x}\,dx\right) \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C.\end{aligned}$$

In fact the same type of computation will be used for finding antiderivatives of arctrigonometric functions (though arcsecant and arccosecant need later techniques for finding the resulting second integral $\int v\,du$).

Example 7.2.9 *Compute* $\displaystyle\int \sin^{-1} x\,dx$.

Solution: Again we have no choice but to let $u = \sin^{-1} x$, *and* $dv = dx$.

$$\begin{aligned} u &= \sin^{-1} x & dv &= dx \\ du &= \frac{1}{\sqrt{1-x^2}}\,dx & v &= x\end{aligned}$$

For brevity, we label the desired integral $(\boldsymbol{\mathcal{I}})$, *so here* $(\boldsymbol{\mathcal{I}}) = \int \sin^{-1} x\,dx$. *(The second integral below is computed "by inspection.")*

$$\begin{aligned}(\boldsymbol{\mathcal{I}}) = uv - \int v\,du &= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx \\ &= x\sin^{-1} x + \left(1-x^2\right)^{1/2} + C.\end{aligned}$$

7.2.5 An Indirect Method

The following method, summarized at the end, is useful in surprisingly many settings.

Example 7.2.10 *Compute* $\displaystyle\int e^{2x}\cos 3x\,dx = (\boldsymbol{\mathcal{I}})$.

Solution: Here we again name the desired integral $(\boldsymbol{\mathcal{I}})$ *for brevity in later steps. It should be obvious (especially after a few attempts) that simple substitution methods will not work. So we attempt an integration by parts.*

Step 1. *We will let the trigonometric function be part of* dv:

$$\begin{aligned} u &= e^{2x} & dv &= \cos 3x\,dx \\ du &= 2e^{2x}\,dx & v &= \frac{1}{3}\sin 3x\end{aligned}$$

So far, after some rearrangement and simplifying, we have

$$\begin{aligned}(\boldsymbol{\mathcal{I}}) = uv - \int v\,du &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\underbrace{\int e^{2x}\sin 3x\,dx}_{(\boldsymbol{\mathcal{II}})}. \end{aligned} \tag{7.17}$$

This does not seem any easier than the first integral, so perhaps we might continue, but this time let the trigonometric function be u and the exponential (along with dx) be contained in dv.

Step 2–First Attempt. *Compute* $(\boldsymbol{\mathcal{II}}) = \int e^{2x} \sin 3x\, dx$ *in light of the comments at the end of the first step.*

$$\begin{aligned} u &= \sin 3x & dv &= e^{2x}\, dx \\ du &= 3\cos 3x\, dx & v &= \frac{1}{2}e^{2x} \end{aligned}$$

$$(\boldsymbol{\mathcal{II}}) = uv - \int v\, du = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\, dx.$$

Combining this with the conclusion (7.17) of Step 1 gives us:

$$\begin{aligned} (\boldsymbol{\mathcal{I}}) &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\, dx\right] \\ &= \frac{1}{3}e^{2x}\sin 3x - \frac{1}{3}e^{2x}\sin 3x + \int e^{2x}\cos 3x\, dx \\ &= \int e^{2x}\cos 3x\, dx. \end{aligned}$$

Unfortunately that puts us right back where we started. However, a minor change in our effort above will eventually lead us to the solution. While keeping Step 1, our next step towards a solution is to replace Step 2 by the same strategy as used in Step 1, namely that we use the exponential function for u and the trigonometric function in the dv term.

Step 2—Second Attempt. *Again we attempt to compute* $(\boldsymbol{\mathcal{II}}) = \int e^{2x}\sin 3x\, dx$, *though with different choices of u and dv.*

$$\begin{aligned} u &= e^{2x} & dv &= \sin 3x\, dx \\ du &= 2e^{2x}\, dx & v &= -\frac{1}{3}\cos 3x \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\mathcal{II}}) &= uv - \int v\, du \\ &= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\, dx. \end{aligned} \tag{7.18}$$

It may seem that (7.18) is also a dead end, since it contains the original integral. But this attempt is different. In fact, when we combine (7.18) with (7.17) we get

$$\begin{aligned} (\boldsymbol{\mathcal{I}}) &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}(\boldsymbol{\mathcal{II}}) \\ &= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[-\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\, dx\right] \\ &= \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\underbrace{\int e^{2x}\cos 3x\, dx}_{(\boldsymbol{\mathcal{I}})}, \end{aligned}$$

which we can summarize by the following equation:

$$(\mathcal{I}) = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}(\mathcal{I}). \tag{7.19}$$

Now we are ready to derive $(\mathcal{I})$*, not by another calculus computation, but in fact by simple algebra: we* **solve** *for it.*

Step 3. *Solve (7.19) for* $(\mathcal{I})$*. First we add* $\frac{4}{9}(\mathcal{I})$ *to both sides of (7.19):*

$$\frac{13}{9}(\mathcal{I}) = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x + C_1. \tag{7.20}$$

Here we include C_1 *because in fact each* $(\mathcal{I})$ *in (7.19) represents all antiderivatives, which differ from each other by additive constants. Now (7.19) made sense because of the fact that there are (hidden) additive constants on both sides of that equation (though on the right side they are multiplied by* $-\frac{4}{9}$*, but that still yields additive constants).*[8] *Solving (7.20) for* $(\mathcal{I})$ *we now have*

$$\begin{aligned}(\mathcal{I}) &= \frac{9}{13}\left[\frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x + C_1\right] \\ &= \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + C,\end{aligned} \tag{7.21}$$

where $C = \frac{9}{13}C_1$*.*

Of course with this our original problem is solved:

$$\int e^{2x}\cos 3x\,dx = \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + C.$$

What is important to understand about the example above is that sometimes, though we cannot perhaps directly compute a particular integral, it may happen that an indirect method gives us the answer. Here we found an equation, namely (7.19), which our desired integral satisfies, and for which $(\mathcal{I})$ could be solved algebraically. We must be open to the possibility—indeed, the opportunity—of finding a desired quantity by such indirect methods, as well as direct computations.

It should be pointed out that we could have computed the integral in Example 7.2.10 by instead letting u be the trigonometric function, and $dv = e^{2x}\,dx$ in *both* Steps 1 and 2. In fact it is usually best to pick similar choices for u and dv when an integration by parts will take more than one step. (Recall the discussion for $\int x^n f(x)\,dx$.)

The method of Example 7.2.10, namely solving for $(\mathcal{I})$ after an integration by parts step, is available perhaps more often than one would think, though it is not a method of first resort.

The next example is also one in which we will eventually "solve" for the integral algebraically.

Example 7.2.11 *Compute* $\int \sin^2 x\,dx = (\mathcal{I})$.

[8] In fact, two simultaneous appearances of $(\mathcal{I})$ do not have to have the same additive constants, so $(\mathcal{I}) - (\mathcal{I}) = C_2$, not zero.

Solution: *The only reasonable choice here seems to be to let* $u = \sin x$ *and* $dv = \sin\, dx$*, if we are to integrate this by parts.*[9]

$$\begin{aligned} u &= \sin x & dv &= \sin x\, dx \\ du &= \cos x\, dx & v &= -\cos x \end{aligned}$$

$$(\boldsymbol{\mathcal{I}}) = \int uv - \int v\, du = -\sin x \cos x + \int \cos^2 x\, dx.$$

We could perform the same integration by parts with the second integral, which might or might not yield an equation we can solve for $(\boldsymbol{\mathcal{I}})$ *(as the reader is invited to explore), but instead we will use the fact that* $\cos^2 x = 1 - \sin^2 x$*:*

$$\begin{aligned} (\boldsymbol{\mathcal{I}}) &= -\sin x \cos x + \int (1 - \sin^2 x)\, dx \\ &= -\sin x \cos x + x - \int \sin^2 x\, dx \\ &= x - \sin x \cos x - (\boldsymbol{\mathcal{I}}). \end{aligned}$$

Adding $(\boldsymbol{\mathcal{I}})$ *to both sides we get*[10]

$$\begin{aligned} 2(\boldsymbol{\mathcal{I}}) &= -\sin x \cos x + x + C_1 \\ \Longrightarrow \qquad (\boldsymbol{\mathcal{I}}) &= \frac{1}{2}(x - \sin x \cos x) + C. \end{aligned}$$

7.2.6 Miscellaneous Considerations

First we look at a definite integral arising from integration by parts. It should be pointed out that the general formula will look like the following:[11]

$$\int_{x=a}^{x=b} u\, dv = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} v\, du. \tag{7.22}$$

[9]The integral in Example 7.2.11 can also be computed directly if we first use the trigonometric identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, and then the identity $\sin 2x = 2 \sin x \cos x$:

$$\begin{aligned} \int \sin^2 x\, dx &= \int \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C \\ &= \frac{1}{2}x - \frac{1}{4} \cdot 2 \sin x \cos x + C = \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C, \end{aligned}$$

which is the same as the answer in the text of Example 7.2.11. In Section 7.3 we will opt for this alternative method, and indeed will make quite an effort to exploit the algebraic properties of the trigonometric functions wherever possible, but some integrals there will still *require* integration by parts.

[10]In fact many textbooks do not bother writing the C_1 term, preferring to remind the student at the end that an indefinite integral problem necessitates a "$+C$."

[11]Some texts leave out the "$x =$" parts, assuming they are understood, but we will continue to use the convention that, unless otherwise stated, the "limits of integration" should correspond to values of the differential's variable. Another popular way to write (7.22) avoids the issue:

$$\int_a^b u(x)v'(x)\, dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\, dx.$$

Example 7.2.12 *Compute* $\displaystyle\int_{-\pi}^{\pi} x\sin x\,dx$.

Solution: The antiderivative is an easier case than many of our previous examples, but care has to be taken to keep track of all the signs (+/−) in computing the definite integral:

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= dx & v &= -\cos x \end{aligned}$$

$$\begin{aligned} \int_{-\pi}^{\pi} x\sin x\,dx &= \left.(-x\cos x)\right|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos x\,dx \\ &= [-\pi\cos\pi] - [-(-\pi)\cos(-\pi)] + \left.\sin x\right|_{-\pi}^{\pi} \\ &= (-\pi)(-1) - (\pi)(-1) + \sin\pi - \sin(-\pi) \\ &= \pi + \pi + 0 - 0 \\ &= 2\pi. \end{aligned}$$

In the example above, we could also have noticed that $\int_{-\pi}^{\pi}\cos x\,dx$ is zero because we are integrating over a whole period $[-\pi,\pi]$ of $\cos x$, and both $\sin x$ and $\cos x$ have definite integral zero over any full period $[a, a+2\pi]$. (Think of their graphs, or their definite integrals over any such period.)

It is typical to compute that part $u(x)v(x)\big|_a^b$ separately, but one could instead separately compute the entire antiderivative, and then evaluate at the two limits and take the difference:

$$\begin{aligned} \int_{-\pi}^{\pi} x\sin x\,dx &= \left.(-x\cos x + \sin x)\right|_{-\pi}^{\pi} \\ &= (\pi + 0) - (-\pi + 0) \\ &= 2\pi. \end{aligned}$$

The choice of method is a matter of bookkeeping preferences, and perhaps whether or not part of the right-hand side of (7.22) is particularly simple. If not, it is reasonable to solve the indefinite integral $\int x\sin x\,dx$ as a separate matter, and then write the definite integral with the formula for the antiderivative inserted, as in $\int f(x)\,dx = F(x)\big|_a^b$ and so on as above.

The next example gives us several options for computing the new integral $\int v\,du$ along the way, though in each the original choices of u and dv are the same.

Example 7.2.13 *Compute* $\displaystyle\int x\tan^{-1}x\,dx = (\boldsymbol{\mathcal{I}})$.

Solution: Again we have little choice on our selection of u and dv.

$$\begin{aligned} u &= \tan^{-1}x & dv &= x\,dx \\ du &= \frac{1}{x^2+1}\,dx & v &= \frac{1}{2}x^2 \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\mathcal{I}}) &= uv - \int v\,du \\ &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{x^2+1}\,dx. \end{aligned}$$

Now this last integral can be found by first rewriting the integrand using either polynomial long division, or by using a little cleverness:

$$\frac{x^2}{x^2+1} = \frac{x^2+1-1}{x^2+1} = \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} = 1 - \frac{1}{x^2+1}.$$

Polynomial long division would have yielded the same result, and is a bit more straightforward, but the technique we used here is good to have available. Either way we then have

$$\begin{aligned}(\boldsymbol{\mathcal{I}}) &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx \\ &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C.\end{aligned}$$

Though our choice of u and dv was limited, our choice of v was actually not as limited. Recall that we could have chosen any $v = \frac{1}{2}x^2 + C_1$. While previously we chose $C_1 = 0$ for its apparent simplicity, for this particular integral we could have actually saved ourselves some effort if we had chosen v more strategically, with a different C_1:

$$\begin{aligned} u &= \tan^{-1}x & dv &= x\,dx \\ du &= \frac{1}{x^2+1}\,dx & v &= \frac{1}{2}\left(x^2+1\right)\end{aligned}$$

In effect we chose $C_1 = \frac{1}{2}$. This gives us

$$\begin{aligned}(\boldsymbol{\mathcal{I}}) &= uv - \int v\,du \\ &= \frac{1}{2}\left(x^2+1\right)\tan^{-1}x - \int\frac{\frac{1}{2}(x^2+1)}{x^2+1}\,dx \\ &= \frac{1}{2}\left(x^2+1\right)\tan^{-1}x - \int\frac{1}{2}\,dx \\ &= \frac{1}{2}\left(x^2+1\right)\tan^{-1}x - \frac{1}{2}x + C,\end{aligned}$$

which is the same as before, though slightly rearranged.

Though rare, and not crucial, strategically adding a particular constant to the natural choice for v can on occasion make for easier computations.

Integration by parts is a very important technique to have at one's disposal when tackling integration problems. As has been demonstrated here, such problems can range from fairly simple computations which are perhaps just slightly more involved than more routine substitution-type integration problems, to some which utilize or indeed require much more clever applications of the technique. At the center of all these is a rearranged product rule,

$$u\cdot\frac{dv}{dx} = \frac{d(uv)}{dx} - v\cdot\frac{du}{dx},$$

which we can integrate to get

$$\int u\cdot\frac{dv}{dx}\,dx = \int\frac{d(uv)}{dx}\,dx - \int v\cdot\frac{du}{dx}\,dx,$$

or our working formula which then follows:

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

Of course the applications can be quite clever and detailed, but they all are based upon this relatively simple formula.

Example 7.2.14 *An amount of money P invested at an annual interest rate r (written as a decimal), compounded continuously for t years will then be worth $A(t) = Pe^{rt}$. Solving for P we would get the* **present value** *of A dollars t years from now to be*

$$P = Ae^{-rt}.$$

The idea is that a smaller amount of money P today (presently) could be invested to return the amount A in t years. Naturally so we expect $P < A$.

If instead we have revenue flowing into the account at a constant yearly rate of R dollars per year (but flowing at that yearly rate constantly)

for T years, then the present value P is instead given by[12]

$$P = \int_0^T Re^{-rt}\,dt.$$

This can also be seen by examining the general case, where the revenue flow rate is dependent upon time, so $R = R(t)$. Then then the present value is given by

$$P = \int_0^T R(t)e^{-rt}\,dt. \tag{7.23}$$

This is reasonable because it represents the accretion of the present values of each injected revenue $R(t)$ for $t \in [0, T]$, because:

- *$R(t)\,dt$ represents the revenue injected at time t at a rate of $R(t)$ for an infinitesimal time interval of length dt;*
- *$R(t)e^{-0.05t}\,dt$ therefore represents the present value of that revenue, each of these to be accumulated in the integral in* (7.23).

Use (7.23) *to calculate the present value P of the first three years of revenue when $R(t) = 2000 + 100t$ (in dollars per year) and the compounded interest rate is 5% (compounded continuously).*

Solution: Given $R(t) = 2000 + 100t$ and $r = 0.05$, then

$$P = \int_0^3 (2000 + 100t)e^{-0.05t}\,dt.$$

We can break this into two integrals—one of which requires integration by parts—or we can wrap both integrals into one integration by parts process. Here we choose the latter approach.

$$\begin{aligned} u &= 2000 + 100t & dv &= e^{-0.05t}\,dt \\ du &= 100\,dt & v &= \frac{1}{-0.05}e^{-0.05t} = -20e^{-0.05t}. \end{aligned}$$

[12] To be clear, note that we can have a velocity of 60 miles/hour for any period of time, including significantly less than an hour, or even as an *instantaneous rate*. Here we have a constant (and instantaneous) flow rate of R dollars per year. It does not mean that we invest R dollars in a lump sum at the end of each year, but that we have a constant flow which would amount to R dollars if it were allowed to proceed for an entire year, just as 60 miles/hour would accumulate to 60 miles after one hour. Next we let R vary, as in $R = R(t)$. The rate will still be in dollars per year, but it will be an instantaneous rate.

From this we get

$$\begin{aligned} P &= (2000+100t)\left(-20e^{-0.05t}\right)\Big|_0^3 + \int_0^3 20e^{-0.05t}\cdot 100\,dt \\ &= \left[-2000(20+t)e^{-0.05t} + \frac{2000}{-0.05}e^{-0.05t}\right]\Big|_0^3 \\ &= (-80,000-2000t)e^{-0.05t}\Big|_0^3 = -80,000e^{-0.15} + 80,000e^0 \approx -74,021 + 80,000 \\ \implies P &\approx 5,979. \end{aligned}$$

Ultimately, the computation in this example means that we would have to invest a lump sum of approximately \$5979 at the same 5% (compounded continuously) rate today to accumulate the total value of our more complicated investment strategy with a nonconstant revenue stream $R(t)$ in the same three years (at the same interest rate), with that total being approximately (in part because we have rounded to the nearest dollar for $P \approx \$5979$) $A = 5979e^{0.05(3)} \approx 6947$.

Another way to compute this total (future) value \$6947 of our investment scheme after three years is to use the following computation (shown in general and for our specific case):

$$A(T) = \int_0^T R(t)e^{r(T-t)}\,dt, \qquad A(3) = \int_0^3 (2000+100t)\,e^{0.05(3-t)}\,dt. \tag{7.24}$$

This is because we can look at the quantities inside the integral in the following way:

- $R(t) = (2000+100t)$ is the rate of revenue (investment) flowing into the account per unit time at time t.
- $R(t)\,dt = (2000+100t)\,dt$ represents the amount of new revenue invested during an infinitesimal time interval of length dt at time t.
- $T-t = 3-t$ is the total length in years that the investment made at time t earns interest.

Thus the integral (7.24) represents the accretion of values of each injected revenue $R(t)$ for all times $t \in [0,T]$. For our case of $R(t) = 2000+100t$, $r = 0.05$ and $[0,T] = [0,3]$ we integrate this as before:

$$\begin{aligned} u &= 2000+100t & dv &= e^{0.05(3-t)}\,dt \\ du &= 100\,dt & v &= -20e^{0.05(3-t)}. \end{aligned}$$

$$\begin{aligned} A(3) = \int_0^3 (2000+100t)\,e^{0.05(3-t)}\,dt &= -20(2000+100t)e^{0.05(3-t)}\Big|_0^3 + 2000\int_0^3 e^{0.05(3-t)}\,dt \\ &= -20(2000+100t)e^{0.05(3-t)}\Big|_0^3 - 40,000e^{0.05(3-t)}\Big|_0^3 \\ &= -80,000\left(e^0 - e^{\cdot 15}\right) - 6000 \\ &\approx 12,947 - 6000 = 6947, \end{aligned}$$

so indeed we see that the value \$6947 of the revenue stream in 3 years (calculated by (7.24) above) is the same as what our calculated present value $P \approx \$5979$ would be worth in 3 years if invested all at once now at the same 5% interest rate, compounded continuously.

While in this model revenue $R(t)$ is a continuous function approximating what in reality is a discrete depositing and interest phenomenon, we can nonetheless approximate (very well in fact) with an integral what would be the present and future values of a somewhat complicated revenue injection and investment scheme, running for several years. This is one of many ways in which calculus is applied in business and economic settings.

Exercises

Compute the following integrals, all of which can be computed "by parts."

1. $\int x \sin x \, dx$
2. $\int x^2 \sin x \, dx$
3. $\int x \cos 3x \, dx$
4. $\int x \sec x \tan x \, dx$
5. $\int x \sec^2 x \, dx$
6. $\int x \ln x \, dx$
7. $\int x \tan^{-1} x \, dx$
8. $\int x \sec^{-1} x \, dx$, $x > 1$
9. $\int x \sec^{-1} x \, dx$, $x < 1$
10. $\int x\sqrt{1-x}\, dx$. (Parts optional)
11. $\int \dfrac{x}{\sqrt{1-x}}\, dx$ (Parts optional)
12. $\int xe^x \, dx$
13. $\int xe^{x/2} \, dx$
14. $\int x^3 e^{x^2} \, dx$
15. $\int x^5 \sin x^3 \, dx$
16. $\int x^2 e^{3x} \, dx$
17. $\int \ln x \, dx$
18. $\int \tan^{-1} x \, dx$
19. $\int \sin^{-1} x \, dx$
20. $\int x\sqrt{1-x^2}\ \sin^{-1} x \ dx$
21. $\int x^3 \sin 2x \, dx$
22. $\int (\ln x)^2 \, dx$
23. $\int \sin^2 5x \, dx$ (Parts optional)
24. $\int \cos^2 x \, dx$ (Parts optional)
25. $\int e^{5x} \cos 2x \, dx$
26. $\int \sec^3 x \, dx$
27. The current flowing in a particular circuit as a function of time t is given by $i = e^{-3t \sin t}$. Determine the charge q which has passed through the circuit in $[0, t]$. Recall that $i = \frac{dq}{dt}$.
28. The slope of a curve is given by $dy/dx = x^3\sqrt{1+x^2}$. Find the equation of the curve if it passes through the point $(0, 1)$.

29. The root-mean-square value of a function $f(x)$ over an interval $[0, T]$ is given by

$$f(x)_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T [f(x)]^2\, dx}. \quad (7.25)$$

Find the root-mean-square value of the function $f(x) = \sqrt{\sin^{-1} x}$ over $[0, 1]$.

30. Find the root-mean-square value of the function $f(x) = \sqrt{e^x \cos x}$ over $[0, \pi/2]$.

31. Find the present value P of the first 2 years of revenue when $R(t) = (800 + 10 \sin t)$ dollars per year and the compounded interest rate (continuously compounded) is 4.5%. (See Example ?????)

7.3 Trigonometric Integrals

We have already looked at two basic types of trigonometric integrals: those arising from the derivatives of the trigonometric functions (Subsection 6.1.4, page 532), and those of the elementary substitution types in Section 6.5. In this section we are mainly interested in computing integrals where the integrands are combinations of powers of trigonometric functions. In such cases, the angles of each trigonometric function appearing are all the same. Another important topic considered here is how to deal with trigonometric combinations where the angles differ, and we will examine how to deal with several of those cases.

In the first examples where the angles agree, we rearrange the terms in the integrand and use the three basic trigonometric identities to write the entire integral as function of one trigonometric function, and its differential as the final factor. A substitution step then leads to one or more power rules. Unfortunately this only leads to a solution if the combinations of powers are of a few simple forms. Still, these combinations occur often enough to warrant study.

After we look at those simplest forms, we look at other combinations of powers where the angles agree. Techniques include other algebraic manipulations, as well as integration by parts.

In the final forms, where the angles do not agree, we look at several trigonometric identities which help us to rewrite the integrals into simpler forms.

7.3.1 Sample Problems

These first three examples illustrate an approach we develop in Subsections 7.3.2, 7.3.3 and 7.3.4.

Example 7.3.1 *Compute* $\displaystyle\int \tan^2 x\,dx$.

Solution: $\displaystyle\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C.$

The example above used the facts that $\tan^2 x = \sec^2 x - 1$, and that we know the antiderivative of $\sec^2 x$ (where we might not have known the antiderivative of $\tan^2 x$ immediately). The integral above does not in itself contain a general method. Indeed there is no general method, but there are ways to rewrite many trigonometric integrals to make their computations more elementary.

Example 7.3.2 *Compute* $\displaystyle\int \sin^6 x\,\cos^5 x\,dx$.

Solution: Here we will use the fact that $\cos^2 x = 1 - \sin^2 x$, *and so* $\cos^{2k} x = \left(1 - \sin^2 x\right)^k$. *Eventually we will take* $u = \sin x$, *implying* $du = \cos x\,dx$:

$$\begin{aligned}
\int \sin^6 x\,\cos^5 x\,dx &= \int \sin^6 x\,\cos^4 x\,\cos x\,dx \\
\left.\begin{array}{r} u = \sin x \\ du = \cos x\,dx \end{array}\right\} \qquad &= \int \sin^6 x\left(1-\sin^2 x\right)^2 \cos x\,dx \\
&= \int u^6\left(1-u^2\right)^2\,du = \int u^6\left(1-2u^2+u^4\right)\,du \\
&= \int \left[u^6 - 2u^8 + u^{10}\right]\,du \\
&= \frac{u^7}{7} - \frac{2u^9}{9} + \frac{u^{11}}{11} + C \\
&= \frac{1}{7}\sin^7 x - \frac{2}{9}\sin^9 x + \frac{1}{11}\sin^{11} x + C.
\end{aligned}$$

Example 7.3.3 *Compute* $\int \sec^5 x \, \tan^3 x \, dx$.

Solution: Here we will borrow a factor of secant, and another of tangent, to form the functional part of du, where $u = \sec x$*:*

$$\begin{aligned}
\int \sec^5 x \, \tan^3 x \, dx &= \int \sec^4 x \, \tan^2 x \, \sec x \tan x \, dx \\
&= \int \sec^4 x \left(\sec^2 x - 1\right) \sec x \tan x \, dx \\
&= \int u^4 \left(u^2 - 1\right) \, du \\
&= \int \left[u^6 - u^4\right] \, du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C.
\end{aligned}$$

Now we look at these three specific techniques more closely and generalize them.

7.3.2 Odd Powers of Sine or Cosine

Here we are interested in the cases of integrals

$$\int \sin^m \theta \, \cos^n \theta \, d\theta. \tag{7.26}$$

where either m or n is odd. Suppose, for example, that m is odd, so that we can write $m = 2k+1$ for some integer k. Then we rewrite the form (7.26) as

$$\int \sin^m \theta \, \cos^{2k+1} \theta \, d\theta = \int \sin^m \theta \, \cos^{2k} \theta \, \cos\theta \, d\theta.$$

The $\cos\theta$ term which we "peeled away" becomes the functional part of the du, where $u = \sin\theta$ (so $du = \cos\theta \, d\theta$). We then write the rest of the integral in terms of $u = \sin\theta$. To do so we use

$$\begin{aligned}
& \sin^2\theta + \cos^2\theta = 0 \\
\iff \quad & \cos^2\theta = 1 - \sin^2\theta \\
\implies \quad & \cos^{2k}\theta = \left(1 - \sin^2\theta\right)^k.
\end{aligned}$$

Using this fact in the integral above, and setting $u = \sin\theta$, we get

$$\begin{aligned}
\int \sin^m \theta \, \cos^{2k+1} \theta \, d\theta &= \int \sin^m \theta \, \cos^{2k} \theta \, \cos\theta \, d\theta \\
&= \int \sin^m \theta \left(1 - \sin^2\theta\right)^k \cos\theta \, d\theta \\
&= \int u^m \left(1 - u^2\right)^k \, du.
\end{aligned}$$

This yields a polynomial integrand, which we may then wish to expand before computing (with a sequence of power rules).

Similarly, if there is an odd power of the sine, we can use the fact that $\sin^2\theta = 1 - \cos^2\theta$, and eventually using $u = \cos\theta$, to rewrite such an integral

$$\begin{aligned}\int \sin^{2k+1}\theta\,\cos\theta\,d\theta &= \int \sin^{2k}\theta\,\cos\theta\sin\theta\,d\theta \\ &= \int \left(1-\cos^2\theta\right)^k \cos^n\theta\,\sin\theta\,d\theta \\ &= \int \left(1-u^2\right)^k u^n(-du) \\ &= -\int \left(1-u^2\right)^k u^n\,du.\end{aligned}$$

In both of these it was crucial that we had an odd number of factors of either the sine or cosine, since "peeling off" one factor then leaves an even number, which can be easily written in terms of the other trigonometric function. The peeled off factor is then the functional part of the differential after substitution.

Note that while any even power of a sine or cosine function can be written entirely in terms of the other, this is not the case with odd powers.[13] This technique works because removing a factor from an odd power of sine or cosine, both provides the functional part of du and leaves an even power, which we write in terms of the other function which is then u in the substitution.

Example 7.3.4 *Compute* $\int \sin^5 x\,\cos^4 x\,dx$.

Solution: Here we see an odd number of sine factors, as so we peel one away to be part of the differential term, and write the entire integral in terms of the cosine:

$$\begin{aligned}\int \sin^5 x\,\cos^4 x\,dx &= \int \sin^4 x\,\cos^4 x\,\sin x\,dx \\ &= \int \left(\sin^2 x\right)^2 \cos^4 x\,\sin x\,dx \\ &= \int \left(1-\cos^2 x\right)^2 \cos^4 x\,\sin x\,dx.\end{aligned}$$

(In most future computations we will skip the second line above.) Now we take

$$\begin{aligned} & u = \cos x \\ \implies \quad & du = -\sin x\,dx \\ \iff & -du = \sin x\,dx.\end{aligned}$$

With the substitution we will have a polynomial to integrate. To summarize and finish the

[13] Consider the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. When solved for either the sine or cosine function, we get one of the following:

$$\begin{aligned}\sin\theta &= \pm\sqrt{1-\cos^2\theta}, \\ \cos\theta &= \pm\sqrt{1-\sin^2\theta}.\end{aligned}$$

We see the ambiguity in the $\pm$, and the introduction of a radical which itself can very much complicate an integral. However, when we raise these to even powers the radicals *and* the $\pm$ both disappear, and we are left with sums of nonnegative, integer powers.

problem, we have:

$$\begin{aligned}\int \sin^5 x\, \cos^4 x\, dx &= \int \left(1-\cos^2 x\right)^2 \cos^4 x\, \sin x\, dx \\ &= \int \left(1-u^2\right)^2 u^4 (-du) \\ &= -\int \left(1-2u^2+u^4\right) u^4\, du \\ &= -\int \left(u^4-2u^6+u^8\right)\, du \\ &= -\frac{1}{5}\,u^5+\frac{2}{7}\,u^7-\frac{1}{9}\,u^9+C \\ &= -\frac{1}{5}\cos^5 x+\frac{2}{7}\cos^7 x-\frac{1}{9}\cos^9 x+C.\end{aligned}$$

It should be clear that one can not easily differentiate the final answer and immediately recognize the original integrand. This is because some trigonometric identities were used to get an integrand form which was computable using these methods. Indeed, it is best to check the validity of the steps from the beginning, rather than to differentiate a tentative answer. However, it is an interesting exercise—left to the interested reader—in trigonometric identities to perform the differentiation, and then validate that the answer there is the original integrand.

It is not necessary that the angle is always x. However, for this technique we do require the angles inside the trigonometric functions to always match, and for the approximate differential of the variable of substitution to be present.

Example 7.3.5 *Compute* $\int \sin^4 5x\, \cos^3 5x\, dx$.

Solution: Here there is an odd number of cosine terms, and we act accordingly.

$$\begin{aligned}\int \sin^4 5x\, \cos^3 5x\, dx &= \int \sin^4 5x\, \cos^2 5x\, \cos 5x\, dx \\ &= \int \sin^4 5x \left(1-\sin^2 5x\right) \cos 5x\, dx.\end{aligned}$$

Here we have

$$\begin{aligned} & u = \sin 5x \\ \Longrightarrow \quad & du = 5\cos 5x\, dx \\ \Longleftrightarrow \quad & \frac{1}{5}\, du = \cos 5x\, dx.\end{aligned}$$

Now we begin again, incorporating this new information into our computation:

$$\begin{aligned}\int \sin^4 5x\, \cos^3 5x\, dx &= \int \sin^4 5x \left(1-\sin^2 5x\right) \cos 5x\, dx \\ &= \int u^4 \left(1-u^2\right) \cdot \frac{1}{5}\, du \\ &= \frac{1}{5}\int \left(u^4-u^6\right)\, du \\ &= \frac{1}{5}\cdot\frac{1}{5}\,u^5-\frac{1}{5}\cdot\frac{1}{7}\,u^7+C \\ &= \frac{1}{25}\sin^5 5x-\frac{1}{35}\sin^7 5x+C.\end{aligned}$$

The technique works even if only one of the trigonometric functions sine or cosine appears, as long as it is to an odd power.

Example 7.3.6 *Compute* $\displaystyle\int \sin^3 7x\,dx$.

Solution: Here we can still peel off a sine factor to be the functional part of our differential, and then write the remaining factors in terms of the cosine.

$$\begin{aligned}\int \sin^3 7x\,dx &= \int \sin^2 7x\,\sin 7x\,dx \\ &= \int \left(1-\cos^2 7x\right)\sin 7x\,dx.\end{aligned}$$

Using the substitution $u=\cos 7x$, *so* $du=-7\sin 7x\,dx$, *implying* $-\frac{1}{7}\,du=\sin 7x\,dx$, *we get*

$$\begin{aligned}\int \sin^3 7x\,dx &= \int \left(1-\cos^2 7x\right)\sin 7x\,dx \\ &= \int \left(1-u^2\right)\cdot\frac{-1}{7}\,du \\ &= -\frac{1}{7}\left[u-\frac{1}{3}\,u^3\right]+C \\ &= -\frac{1}{7}\cos 7x+\frac{1}{21}\cos^3 7x+C.\end{aligned}$$

Furthermore, not all the powers need to be positive integer powers, as long as one is odd.

Example 7.3.7 *Compute* $\displaystyle\int \frac{\cos^7 x}{\sqrt{\sin x}}\,dx$.

Solution: Here we have an odd number of cosine terms, so we will peel one off to be the functional part of our differential. That is, we will have $u=\sin x$, *so* $du=\cos x\,dx$. *Thus*

$$\begin{aligned}\int \frac{\cos^7 x}{\sqrt{\sin x}}\,dx &= \int \frac{\cos^6 x}{\sqrt{\sin x}}\,\cos x\,dx \\ &= \int \frac{\left(1-\sin^2 x\right)^3}{\sqrt{\sin x}}\,\cos x\,dx \\ &= \int \frac{\left(1-u^2\right)^3}{\sqrt{u}}\,du \\ &= \int \frac{1-3u^2+3u^4-u^6}{u^{1/2}}\,du \\ &= \int \left[u^{-1/2}-3u^{3/2}+3u^{7/2}-u^{11/2}\right]\,du \\ &= 2u^{1/2}-3\cdot\frac{2}{5}\,u^{5/2}+3\cdot\frac{2}{9}\,u^{9/2}-\frac{2}{13}\,u^{13/2}+C \\ &= 2u^{1/2}\left[1-\frac{3}{5}u^2+\frac{1}{3}u^4-\frac{1}{13}u^6\right]+C \\ &= 2\sqrt{\sin x}\left[1-\frac{3}{5}\sin^2 x+\frac{1}{3}\sin^4 x-\frac{1}{13}\sin^6 x\right]+C.\end{aligned}$$

It is possible that both powers are odd, and either function can be peeled off, and the integral written in terms of the other. However, if one of these odd powers is greater than the other, it is more efficient to peel off a factor from the lower power, as the next example demonstrates.

Example 7.3.8 *Compute* $\int \sin^3 x \, \cos^7 x \, dx$.

Solution: We will consider both methods for computing this antiderivative. First we peel off a sine to be part of the differential, and let $u = \cos x$ *(so* $du = -\sin x \, dx$*).*

$$\begin{aligned}
\int \sin^3 x \, \cos^7 x \, dx &= \int \sin^2 x \, \cos^7 x \, \sin x \, dx \\
&= \int \left(1 - \cos^2 x\right) \cos^7 x \, \sin x \, dx \\
&= \int \left(1 - u^2\right) u^7 (-du) \\
&= -\int \left(u^7 - u^9\right) \, du \\
&= -\frac{1}{8} u^8 + \frac{1}{10} u^{10} + C \\
&= -\frac{1}{8} \cos^8 x + \frac{1}{10} \cos^{10} x + C.
\end{aligned}$$

Next we instead peel off a cosine factor, and let $w = \sin x$ *(so that* $dw = \cos x \, dx$*).*

$$\begin{aligned}
\int \sin^3 x \, \cos^7 x \, dx &= \int \sin^3 x \, \cos^6 x \, \cos x \, dx \\
&= \int \sin^3 x \left(1 - \sin^2 x\right)^3 \cos x \, dx \\
&= \int w^3 \left(1 - w^2\right)^3 \, dw \\
&= \int w^3 \left(1 - 3w^2 + 3w^4 - w^6\right) \, dw \\
&= \int \left(w^3 - 3w^5 + 3w^7 - w^9\right) \, dw \\
&= \frac{1}{4} w^4 - \frac{3}{6} w^6 + \frac{3}{8} w^8 - \frac{1}{10} w^{10} + C \\
&= \frac{1}{4} \sin^4 x - \frac{1}{2} \sin^6 x + \frac{3}{8} \sin^8 x - \frac{1}{10} \sin^{10} x + C.
\end{aligned}$$

As we see in the above example, there can be different valid choices for some integrals. The answers may look very different, but that is a reflection of the wealth of trigonometric identities available. In fact the antiderivatives, excluding the arbitrary constants, need not be equal but the difference should be accounted for in the constants.[14]

[14] Recall the integral $\int 2 \sin x \cos x \, dx$, for which one can let either $u = \sin x$ or $u = \cos x$, yielding

$$\int 2 \sin x \cos x \, dx = \sin^2 x + C_1, \qquad \text{or}$$
$$\int 2 \sin x \cos x \, dx = -\cos^2 x + C_2.$$

Since these differ by a constant, specifically $\sin^2 x = -\cos^2 x + 1$, both are valid. But clearly $\sin^2 x \neq -\cos^2 x$.

7.3.3 Even Powers of Secant or Odd Powers of Tangent

This technique of peeling off some factors of a trigonometric function to be part of the du (after substitution) has two workable versions for integrals of the type

$$\int \sec^m \theta \tan^n \theta \, d\theta. \tag{7.27}$$

These rely upon the following facts from trigonometry and calculus:

$$\tan^2\theta + 1 = \sec^2\theta, \qquad \frac{d}{d\theta}\tan\theta = \sec^2\theta,$$
$$\sec^2\theta - 1 = \tan^2\theta, \qquad \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta.$$

The techniques we will employ are as follow:

1. If an integral of the form (7.27) contains an odd power of tangent, we peel off a factor $\sec\theta\tan\theta$ to be the functional part of the differential. This leaves an even power of tangent, which can be written as a power of $\left(\sec^2\theta - 1\right)$.

2. If an integral of the form (7.27) contains an even power of secant, we peel off a factor $\sec^2\theta$ to be the functional part of the differential. The remaining, still even power of secant is then written as a power of $\left(\tan^2\theta + 1\right)$.

Example 7.3.9 *Compute* $\int \sec^6 x \, \tan^8 x \, dx$.

Solution *Here we have an even number of secant factors, and so we can peel off two. Eventually we will let* $u = \tan x$, *implying* $du = \sec^2 x \, dx$.

$$\begin{aligned}
\int \sec^6 x \, \tan^8 x \, dx &= \int \sec^4 x \, \tan^8 x \, \sec^2 x \, dx \\
&= \int \left(\tan^2 x + 1\right)^2 \tan^8 x \, \sec^2 x \, dx \\
&= \int \left(u^2 + 1\right)^2 u^8 \, du \\
&= \int \left(u^4 + 2u^2 + 1\right) u^8 \, du \\
&= \int \left(u^{12} + 2u^{10} + u^8\right) du \\
&= \frac{1}{13}u^{13} + \frac{2}{11}u^{11} + \frac{1}{9}u^9 + C \\
&= \frac{1}{13}\tan^{13} x + \frac{2}{11}\tan^{11} x + \frac{1}{9}\tan^9 x + C.
\end{aligned}$$

Example 7.3.10 *Compute* $\int \sec^7 2x \tan^5 2x \, dx$

Solution: *Here we have an odd number of tangent factors, so we peel off a* $\sec 2x \tan 2x$ *factor to be the functional part of the differential. Eventually we then have* $u = \sec 2x$, *giving*

$du = 2\sec 2x \tan 2x\,dx$ and thus $\frac{1}{2}\,du = \sec 2x \tan 2x\,dx$.

$$\begin{aligned}
\int \sec^7 2x\, \tan^5 2x\,dx &= \int \sec^6 2x\, \tan^4 2x\, \sec 2x \tan 2x\,dx \\
&= \int \sec^6 2x \left(\sec^2 2x - 1\right)^2 \sec 2x \tan 2x\,dx \\
&= \int u^6 \left(u^2 - 1\right)^2 \cdot \frac{1}{2}\,du \\
&= \frac{1}{2}\int u^6 \left(u^4 - 2u^2 + 1\right)\,du \\
&= \frac{1}{2}\int \left(u^{10} - 2u^8 + u^6\right)\,du \\
&= \frac{1}{2}\left[\frac{1}{11}\,u^{11} - \frac{2}{9}\,u^9 + \frac{1}{7}\,u^7\right] + C \\
&= \frac{1}{22}\sec^{11} 2x - \frac{1}{9}\sec^9 2x + \frac{1}{14}\sec^7 2x + C.
\end{aligned}$$

In fact this last example could be computed by first rewriting the integral in terms of cosines and sines:

$$\begin{aligned}
\int \frac{\sin^5 2x}{\cos^{12} 2x}\,dx &= \int \frac{\sin^4 2x}{\cos^{12} 2x} \sin 2x\,dx = \int \frac{\left(1 - \cos^2 2x\right)^2}{\cos^{12} 2x} \sin 2x\,dx \\
&= \int \frac{\left(1 - u^2\right)^2}{u^{12}} \cdot \frac{-1}{2}\,du = -\frac{1}{2}\int u^{-12}\left(1 - 2u^2 + u^4\right)\,du, \text{ etc.}
\end{aligned}$$

Thus, the relationships involving the secant and tangent are not required in this last example. However, rewriting the integral in the previous problem, Example 7.3.9, in terms of sines and cosines would not yield an odd power of either. Thus Example 7.3.9 illustrates an integral which does benefit from the extra structure (algebraic and calculus) of the secant-tangent relationship.

Example 7.3.11 *Compute* $\displaystyle\int \tan^4 x\,dx$.

Solution: Here we look at two solutions. In the first, instead of exploiting the fact that there are an even number of factors of secant (namely zero) present here, we will repeatedly use the fact that $\tan^2\theta + 1 = \sec^2\theta$. *(In the second line, we let* $u = \tan x$.*)*

$$\begin{aligned}
\int \tan^4 x\,dx &= \int \tan^2 x \left(\sec^2 x - 1\right)\,dx \\
&= \int \underbrace{\tan^2 x}_{u^2}\underbrace{\sec^2 x}_{du}\,dx - \int \tan^2 x\,dx \\
&= \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx \\
&= \frac{1}{3}\tan^3 x - \int \left(\sec^2 x - 1\right)\,dx \\
&= \frac{1}{3}\tan^3 x - \tan x + x + C.
\end{aligned}$$

Of course the other method is to "peel off" a factor of $\sec^2 x$, *which we do even though it does not really appear. To have it appear, we will multiply and divide the integrand by* $\sec^2 x$. *Then*

we will let $u = \tan x$. A long division will give us the sum of powers in our final integral below.[15]

$$\begin{aligned}
\int \tan^4 x\, dx &= \int \frac{\tan^4 x}{\sec^2 x} \sec^2 x\, dx \\
&= \int \frac{\tan^4 x}{\tan^2 x + 1} \sec^2 x\, dx \\
&= \int \frac{u^4}{u^2+1}\, du \\
&= \int \left(u^2 - 1 + \frac{1}{u^2+1} \right) du \\
&= \frac{1}{3} u^3 - u + \tan^{-1} u + C_1 \\
&= \frac{1}{3} \tan^3 x - \tan x + \tan^{-1}(\tan x) + C_1 \\
&= \frac{1}{3} \tan^3 x - \tan x + x + C.
\end{aligned}$$

Here we did have the extra complication of long division. Furthermore, to see that the two answers were the same we had to notice $\tan^{-1}(\tan x) = x + n\pi$, where $n \in \mathbb{Z}$, i.e., n is an integer. Thus the final constant C takes into account $C_1 - n\pi$, still a constant.

7.3.4 Even Powers of Cosecant or Odd Powers of Cotangent

Here we just point out that a similar relationship exists between the cosecant and cotangent, as exists between the secant and tangent. We briefly look at two examples to illustrate this. The integral type is

$$\int \csc^m \theta\, \cot^n \theta\, d\theta. \tag{7.28}$$

We begin with the following facts from trigonometry and calculus:

$$\begin{aligned}
\cot^2 \theta + 1 &= \csc^2 \theta, & \frac{d}{d\theta} \cot \theta &= -\csc^2 \theta, \\
\csc^2 \theta - 1 &= \cot^2 \theta, & \frac{d}{d\theta} \csc \theta &= -\csc \theta \cot \theta.
\end{aligned}$$

The techniques we will employ mirror those used for the secant-tangent integrals:

1. If an integral of the form (7.28) contains an odd power of cotangent, we peel off a factor $\csc\theta \cot\theta$ to be the functional part of the differential. This leaves an even power of cotangent, which can be written as a power of $\left(\csc^2\theta - 1\right)$.

2. If an integral of the form (7.28) contains an even power of cosecant, we peel off a factor $\csc^2\theta$ to be the functional part of the differential. The remaining even power of cosecant is then written as a power of $\left(\cot^2\theta + 1\right)$.

[15] A clever student might also note that

$$\frac{u^4}{u^2+1} = \frac{u^4-1+1}{u^2+1} = \frac{u^4-1}{u^2+1} + \frac{1}{u^2+1} = \frac{(u^2-1)(u^2+1)}{u^2+1} + \frac{1}{u^2+1} = u^2 - 1 + \frac{1}{u^2+1}.$$

However, such "clever" methods are difficult to generalize, and polynomial long division is more straightforward.

Example 7.3.12 *Compute* $\displaystyle\int \csc^8 x \, \cot^2 x \, dx$.

Solution: *We see an even number of cosecants, so we peel off two to be part of the differential.*

$$\begin{aligned}\int \csc^8 x \, \cot^2 x \, dx &= \int \csc^6 x \, \cot^2 x \, \csc^2 x \, dx \\ &= \int \left(\csc^2 x\right)^3 \cot^2 x \, \csc^2 x \, dx \\ &= \int \left(\cot^2 x + 1\right)^3 \cot^2 x \, \csc^2 x \, dx.\end{aligned}$$

Taking $u = \cot x$, *giving* $du = -\csc^2 x \, dx$, *so* $-du = \csc^2 x \, dx$, *we get*

$$\begin{aligned}\int \csc^8 x \, \cot^2 x \, dx & \int \left(\cot^2 x + 1\right)^3 \cot^2 x \, \csc^2 x \, dx \\ &= \int \left(u^2 + 1\right)^3 u^2 (-du) \\ &= -\int \left(u^6 + 3u^4 + 3u^2 + 1\right) u^2 \, du \\ &= -\int \left(u^8 + 3u^6 + 3u^4 + u^2\right) \, du \\ &= -\frac{1}{9} u^9 - \frac{3}{7} u^7 - \frac{3}{5} u^5 - \frac{1}{3} u^3 + C \\ &= -\frac{1}{9} \cot^9 x - \frac{3}{7} \cot^7 x - \frac{3}{5} \cot^5 x - \frac{1}{3} \cot^3 x + C.\end{aligned}$$

Example 7.3.13 *Compute* $\displaystyle\int \csc^3 \frac{x}{2} \, \cot^3 \frac{x}{2} \, dx$.

Solution: *Here the cotangent appears to an odd power, so we will peel of one cosecant and one cotangent.*

$$\begin{aligned}\int \csc^3 \frac{x}{2} \, \cot^3 \frac{x}{2} \, dx &= \int \csc^2 \frac{x}{2} \, \cot^2 \frac{x}{2} \, \csc \frac{x}{2} \, \cot \frac{x}{2} \, dx \\ &= \int \csc^2 \frac{x}{2} \left(\csc^2 \frac{x}{2} - 1\right) \csc \frac{x}{2} \cot \frac{x}{2} \, dx.\end{aligned}$$

Now we let $u = \csc \frac{x}{2}$, *implying* $du = -\csc \frac{x}{2} \cot \frac{x}{2} \cdot \frac{1}{2} \, dx$, *whence* $-2 \, du = \csc \frac{x}{2} \cot \frac{x}{2} \, dx$. *Our integral then becomes*

$$\begin{aligned}\int \csc^3 \frac{x}{2} \, \cot^3 \frac{x}{2} \, dx &= \int \csc^2 \frac{x}{2} \left(\csc^2 \frac{x}{2} - 1\right) \csc \frac{x}{2} \cot \frac{x}{2} \, dx \\ &= \int u^2 \left(u^2 - 1\right) (-2) \, du \\ &= -2 \int \left(u^4 - u^2\right) \, du \\ &= -\frac{2}{5} u^5 + \frac{2}{3} u^3 + C \\ &= -\frac{2}{5} \csc^5 \frac{x}{2} + \frac{2}{3} \csc^3 \frac{x}{2} + C.\end{aligned}$$

7.3.5 Even Powers of Sine and Cosine

Now we turn our attention to the question of integration when both powers of sine and cosine are even. There are two standard methods for handling this: integration by parts, and "half-angle formulas." The former is more useful when the powers are small than when they are large, and the latter is perhaps more general.[16]

Example 7.3.14 *Compute* $\int \sin^2 x\,dx$ *using integration by parts.*

Solution: This exact computation was performed in Example 7.2.11, page 609. So that it is in front of us here, we summarize that computation:

$$\begin{aligned}(\boldsymbol{\mathcal{I}}) &= \int \underbrace{\sin x}_{u}\underbrace{\sin x\,dx}_{dv} = \underbrace{(\sin x)}_{v}\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\underbrace{\cos x\,dx}_{du} = -\sin x\cos x + \int \cos^2 x\,dx \\ &= -\sin x\cos x + \int \left(1-\sin^2 x\right)\,dx = -\sin x\cos x + x - \int \sin^2 x\,dx \\ &= x - \sin x\cos x - (\boldsymbol{\mathcal{I}}).\end{aligned}$$

At this point we add $(\boldsymbol{\mathcal{I}}) = \int \sin^2 x\,dx$ *to both sides to get*

$$\begin{aligned}2\int \sin^2 x\,dx &= x - \sin x\cos x + C_1 \\ \implies \int \sin^2 x\,dx &= \frac{1}{2}\left(x - \sin x\cos x\right) + C.\end{aligned}$$

The method above works well for integrating $\sin^2 x$ or $\cos^2 x$, but higher, even powers become more cumbersome. For this reason it is common to opt for alternatives involving slightly more sophisticated trigonometric identities. There is some redundancy in the list below, as (7.29), (7.30), (7.31) and (7.33) together imply the others.

$$\sin(-\theta) = -\sin\theta, \tag{7.29}$$
$$\cos(-\theta) = \cos\theta, \tag{7.30}$$
$$\sin(A+B) = \sin A\cos B + \sin B\cos A \tag{7.31}$$
$$\sin(A-B) = \sin A\cos B - \sin B\cos A \tag{7.32}$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{7.33}$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{7.34}$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{7.35}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{7.36}$$
$$\cos 2\theta = 2\cos^2\theta - 1 \tag{7.37}$$
$$\cos 2\theta = 1 - 2\sin^2\theta. \tag{7.38}$$

It is left for the exercises to show that

1. Equation (7.32) follows from replacing B with $-B$ in (7.31),

[16] In today's calculus texts, integration by parts is less prominently presented for such integrals, while half-angle methods are more popular among authors. We present both here for the lower powers, as some of the phenomena found in the integration by parts for such integrals are found later in this section.

2. Similarly, (7.34) follows from (7.33).

3. Equation (7.35) follows from (7.31), if we let $A, B = \theta$.

4. Similarly (7.36) follows from (7.33).

5. (7.37) and (7.38) follow from (7.36) and the identity $\sin^2\theta + \cos^2\theta = 1$.

Now (7.37) and (7.38) can be rewritten as follow:

$$\cos 2\theta + 1 = 2\cos^2\theta,$$
$$2\sin^2\theta = 1 - \cos 2\theta.$$

Dividing each of these by 2 gives us the so-called half-angle formulas:[17]

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos 2\theta\right), \tag{7.39}$$
$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right). \tag{7.40}$$

Using (7.40), we see

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1 - \cos 2x)\,dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C.$$

For reasons which will be clear in the next section, it is often desirable that the angle in the final answer agree with the original angle, in this case x. For that we use the double-angle formula (7.35), to get

$$\begin{aligned}\int \sin^2 x\,dx &= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C = \frac{1}{2}\left[x - \frac{1}{2}\cdot 2\sin x\cos x\right] + C\\ &= \frac{1}{2}x - \frac{1}{2}\sin x\cos x + C.\end{aligned}$$

This agrees with the answer we obtained through integration by parts, in Example 7.3.14, page 626.

Example 7.3.15 *Compute* $\int \sin^4 x\,dx$.

[17] Equations (7.39) and (7.40) are called half-angle formulas because the angle θ on the left is half of the angle 2θ on the right. In fact, knowing the location of the terminal side of an angle does not tell us where its half is located. Indeed, 90° and 450° are coterminal, but their half angles, 45° and 225° are not. This is reflected in what are given as "half-angle" formulas in most trigonometry texts (compare to (7.39) and (7.40)):

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

However, knowing where an angle terminates does determine where twice the angle terminates, as is reflected in (7.35)–(7.38).

Solution: *Here we use the half-angle formulas repeatedly, until our integral has no positive, even powers of sine or cosine:*

$$\begin{aligned}\int \sin^4 x\,dx &= \int \left(\sin^2 x\right)^2\,dx = \int \left[\frac{1}{2}\left(1-\cos 2x\right)\right]^2\,dx \\ &= \frac{1}{4}\int \left(1-2\cos 2x+\cos^2 2x\right)\,dx \\ &= \frac{1}{4}\int \left[1-2\cos 2x+\frac{1}{2}(1+\cos 4x)\right]\,dx \\ &= \frac{1}{4}\int \left[\frac{3}{2}-2\cos 2x+\frac{1}{2}\cos 4x\right]\,dx \\ &= \frac{1}{4}\left[\frac{3}{2}x-\sin 2x+\frac{1}{2}\cdot\frac{1}{4}\sin 4x\right]+C \\ &= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.\end{aligned}$$

This answer is correct, but if we want to match the angles to the original (x), we can use some double-angle formulas (7.35) and (7.36):

$$\begin{aligned}\int \sin^4 x\,dx &= \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C \\ &= \frac{3}{8}x-\frac{1}{4}\cdot 2\sin x\cos x+\frac{1}{32}\cdot 2\sin 2x\cos 2x+C \\ &= \frac{3}{8}x-\frac{1}{2}\sin x\cos x+\frac{1}{16}(2\sin x\cos x)(\cos^2 x-\sin^2 x)+C,\end{aligned}$$

which can again be simplified and rewritten in several ways.

Example 7.3.16 *Compute* $\int \sin^2 3x\,\cos^2 3x\,dx$.

Solution:

$$\begin{aligned}\int \sin^2 3x\,\cos^2 3x\,dx &= \int \frac{1}{2}(1-\cos 6x)\cdot\frac{1}{2}(1+\cos 6x)\,dx = \frac{1}{4}\int \left(1-\cos^2 6x\right)\,dx \\ &= \frac{1}{4}\int \left[1-\frac{1}{2}(1+\cos 12x)\right]\,dx = \frac{1}{4}\int \left[\frac{1}{2}-\frac{1}{2}\cos 12x\right]\,dx \\ &= \frac{1}{4}\left[\frac{x}{2}-\frac{1}{2}\cdot\frac{1}{12}\sin 12x\right]+C = \frac{x}{8}-\frac{1}{96}\sin 12x+C.\end{aligned}$$

(We could have used $1-\cos^2 6x=\sin^2 6x$ after the first line.) As before, if we would like to have our answer in terms of the original angle, we need to utilize the double angle formulas (7.35) and (7.36):

$$\begin{aligned}\int \sin^2 3x\,\cos^2 3x\,dx &= \frac{x}{8}-\frac{1}{96}\sin 12x+C \\ &= \frac{x}{8}-\frac{1}{96}\cdot 2\sin 6x\cos 6x+C \\ &= \frac{x}{8}-\frac{1}{48}(2\sin 3x\cos 3x)(\cos^2 3x-\sin^2 3x)+C \\ &= \frac{x}{8}-\frac{1}{24}\sin 3x\cos 3x(\cos^2 3x-\sin^2 3x)+C.\end{aligned}$$

Another method would be to have used the identity

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta, \tag{7.41}$$

which follows quickly from the double-angle formula $\sin 2\theta = 2\sin\theta\cos\theta$, i.e., (7.35). From this we can compute

$$\begin{aligned}\int \sin^2 3x\,\cos^2 3x\,dx &= \int (\sin 3x\cos 3x)^2\,dx \\ &= \int \frac{1}{4}\sin^2 6x\,dx \\ &= \frac{1}{4}\int \frac{1}{2}(1-\cos 12x)\,dx \\ &= \frac{1}{8}\left(x - \frac{1}{12}\sin 12x\right) + C.\end{aligned}$$

While the trigonometric parts of these antiderivatives look very different, the polynomial parts are all $\frac{1}{8}x$. The trigonometric parts often take interesting manipulations via identities to verify that they are the same, or off by an additive constant. The conclusion is that two students seeking antiderivatives of trigonometric functions may well have very different answers, though the non-trigonometric parts are usually the same.

7.3.6 Miscellaneous Problems and Methods-I

There are many trigonometric integrals that either do not fit one of the above categories, or for which those methods are unwieldy. We will look at several such here. The reader should realize, however, that we cannot exhaust all possibilities here, and a particular problem may have a particularly clever solution which does not generalize well to other problems.

The methods of the earlier subsections are all standard and any successful calculus student is expected to know them. The first few examples here are of this class also, in that the better students should be able to handle these without resorting to references. We will, however, eventually have methods in this subsection which such a student should be aware of, but is understandably less likely to be able to recite from memory. All are derivable, but again, the latter are somewhat more obscure and even an excellent students might prefer to use a reference. It is important, however, that all students be aware of these latter classes of problems, and the available methods of solution, regardless of whether a reference is used ultimately.

Example 7.3.17 *Compute* $\displaystyle\int \sec^3 x\,dx$.

Solution: Here we have an odd number of secants, and an even (zero) number of tangents. Unfortunately our earlier methods called for an even number of secants or an odd number of tangents. We could notice that the integrand represents an odd number (−3) of cosines, and then with $u = \sin x$, *we could write*

$$\int \sec^3 x\,dx = \int \frac{1}{\cos^3 x}\,dx = \int \frac{1}{\cos^4 x}\,\cos x\,dx = \int \frac{1}{\left(1-\sin^2 x\right)^2}\cos x\,dx = \int \frac{1}{(1-u^2)^2}\,du,$$

but in fact we have yet to discuss how to integrate that final form. (We will in Section 7.5, and while it will be somewhat long, it will be a straightforward computation there). Instead we will

next try integration by parts. Since the integrand contains an easily integrated $\sec^2 x\,dx$ *factor, we will let that be* dv*:*

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x\,dx \\ du &= \sec x \tan x dx & v &= \tan x \end{aligned}$$

$$\int \sec^3 x\,dx = uv - \int v\,du = \sec x \tan x - \int \sec x \tan^2 x\,dx.$$

Now it is tempting to do another parts step, with $dv = \sec x \tan x\,dx$*, but—as happened in some previous examples—we would then have our original integral on the left, and the same on the right (with all other functions cancelling). What works here is to instead use one of the basic trigonometric identities at this step:*

$$\begin{aligned} \int \sec^3 x\,dx &= \sec x \tan x - \int \sec x \tan^2 x\,dx \\ &= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) + C \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\ &= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. \end{aligned}$$

Adding $\int \sec^3 x\,dx$ *to both sides gives*

$$\begin{aligned} 2\int \sec^3 x\,dx &= \sec x \tan x + \ln|\sec x + \tan x| + C_1 \\ \implies \int \sec^3 x\,dx &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C. \end{aligned}$$

When integrating arbitrary integer powers of the trigonometric functions, a common technique is to make use of so-called *reduction formulas*. These are derived using integration by parts, often incorporating the kind of computation above. For instance, let us consider the general problem of integrating $\sec^n x$, where $n \geq 3$. Such an integral contains within its integrand the factor $\sec^2 x$, which we use in the dv term. Integration by parts can proceed as follows:

$$\begin{aligned} u &= \sec^{n-2} x & dv &= \sec^2 x\,dx \\ du &= (n-2)\sec^{n-3} x\,\sec x \cdot \tan x\,dx & v &= \tan x \\ \text{i.e.,}\quad du &= (n-2)\sec^{n-2} x\,\tan x\,dx \end{aligned}$$

giving us

$$\begin{aligned} \int \sec^n x\,dx &= \int \underbrace{\sec^{n-2} x}_{u}\underbrace{\sec^2 x\,dx}_{dv} \\ &= \sec^{n-2} x\,\tan x - \int (n-2)\sec^{n-2} x\,\tan^2 x\,dx \\ &= \sec^{n-2} x\,\tan x - \int (n-2)\sec^{n-2} x\left(\sec^2 x - 1\right)\,dx \\ &= \sec^{n-2} x\,\tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx \\ \implies (n-1)\int \sec^n x\,dx &= \sec^{n-2} x\,\tan x + (n-2)\int \sec^{n-2} x\,dx. \end{aligned}$$

Now we can divide by $(n-1)$ to get a general reduction formula

$$\int \sec^n x\,dx = \frac{1}{n-1}\,\sec^{n-2} x\,\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx. \tag{7.42}$$

This is called a reduction formula because the resulting integral is of a lower power of secant. A quick inspection reveals that this formula is valid for $n=2$ as well, so it is if fact valid for $n \geq 2$.

Example 7.3.18 *Compute* $\int \sec^5 x\,dx$.

Solution: Here we will invoke the formula twice: once for $n=5$, and then again for $n=3$ to deal with the resulting integral. That will give an integral of secant to the first power, which is one which should be already memorized.

$$\begin{aligned}
\int \sec^5 x\,dx &= \frac{1}{4}\sec^3 x\,\tan x + \frac{3}{4}\int \sec^3 x\,dx && \text{(n=5 in (7.42))}\\
&= \frac{1}{4}\sec^3 x\,\tan x + \frac{3}{4}\left[\frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx\right] && \text{(n=3 in (7.42))}\\
&= \frac{1}{4}\sec^3 x\,\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x + \tan x| + C.
\end{aligned}$$

Other reduction formulas which can be arrived at similarly include

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x\,\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx, \tag{7.43}$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x\,\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx, \tag{7.44}$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx. \tag{7.45}$$

In fact (7.45) is simpler, not requiring one to "solve" for the integral, but just using that $\tan^n x = \tan^{n-2} x(\sec^2 x - 1)$. It is left as an exercise.

Example 7.3.19 *Compute* $\int \cos^6 5x\,dx$.

Solution: Here we cannot use the formula (7.44) directly, because our angle does not match our differential. To compensate, we will perform a substitution step first. Specifically, we will let $u = 5x$, so $du = 5dx$ and thus $\frac{1}{5}\,du = dx$, giving

$$\begin{aligned}
\int \cos^6 5x\,dx &= \int \cos^6 u \cdot \frac{1}{5}\,du\\
&= \frac{1}{5}\int \cos^6 u\,du\\
&= \frac{1}{5}\left[\frac{\cos^5 u\,\sin u}{6} + \frac{5}{6}\int \cos^4 u\,du\right] && \text{(n = 6 in (7.44))}\\
&= \frac{\cos^5 u\,\sin u}{30} + \frac{1}{6}\left[\frac{\cos^3 u\,\sin u}{4} + \frac{3}{4}\int \cos^2 u\,du\right] && \text{(n = 4 in (7.44))}\\
&= \frac{\cos^5 u\,\sin u}{30} + \frac{\cos^3 u\,\sin u}{24} + \frac{1}{8}\left[\frac{\cos u\sin u}{2} + \frac{1}{2}\int 1\,du\right] && \text{(n = 2 in (7.44))}\\
&= \frac{\cos^5 u\,\sin u}{30} + \frac{\cos^3 u\,\sin u}{24} + \frac{\cos u\sin u}{16} + \frac{1}{16}\,u + C\\
&= \frac{\cos^5 5x\,\sin 5x}{30} + \frac{\cos^3 5x\,\sin 5x}{24} + \frac{\cos 5x\sin 5x}{16} + \frac{5x}{16} + C.
\end{aligned}$$

Clearly reduction formulas can be very useful. Indeed they provide an iterative method for reducing certain integral computations, step by step, until—hopefully—easily manageable integrals appear. In fact in both examples above, we used the reduction formula for one more step than necessary, because it was easier than recomputing $\int \sec^3 x\,dx$ or $\int \cos^2 u\,du$ as before. Furthermore, many of the technical details for finding these integrals are built into the reduction formulas.

As useful as the reduction formulas are, they have a couple of minor drawbacks. First, when the angle does not match the differential, some substitution needs to be performed to compensate. Second—and more serious—is that any attempt to memorize these is likely to result in error. Thus the student of calculus needs to learn the earlier methods and be able to perform such calculations unaided, and also know that these reduction formulas (and others) are available and know how to use them.[18]

Now we consider integrals of following three forms, where $m \neq n$:

$$\int \sin mx\, \sin nx\, dx, \qquad \int \sin mx\, \cos nx\, dx, \qquad \int \cos mx\, \cos nx\, dx.$$

What distinguishes these is that the angles of the trigonometric functions do not agree. There are two methods for computing these: integration by parts, and utilizing the following trigonometric identities:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right], \tag{7.46}$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right], \tag{7.47}$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]. \tag{7.48}$$

For obvious reasons, these are called *product-sum formulas*. These follow from adding or subtracting (7.31), (7.32), (7.33), and (7.34), which we repeat here for reference:

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \sin B \cos A,\\
\sin(A-B) &= \sin A \cos B - \sin B \cos A,\\
\cos(A+B) &= \cos A \cos B - \sin A \sin B,\\
\cos(A-B) &= \cos A \cos B + \sin A \sin B.
\end{aligned}$$

For instance, (7.46) follows from adding the first two of these and solving for $\sin A \cos B$. All three, (7.46)–(7.48), are left to the exercises.

Example 7.3.20 *Compute* $\int \sin 2x \cos 5x\, dx$.

Solution: Here we use (7.46), with $A = 2x$ and $B = 5x$:

$$\begin{aligned}
\int \sin 2x \cos 5x\, dx &= \int \frac{1}{2}[\sin(2x-5x) + \sin(2x+5x)]\, dx\\
&= \int \frac{1}{2}[\sin(-3x) + \sin 7x]\, dx\\
&= \int \frac{1}{2}[-\sin 3x + \sin 7x]\, dx\\
&= \frac{1}{3}\cos 3x - \frac{1}{7}\cos 7x + C.
\end{aligned}$$

[18] Such formulas can be found in most engineering/science calculus texts, as well as books containing tables of integration formulas such as the *CRC Standard Mathematical Tables and Formulae*.

The method above has the drawback that the solution does not contain the same angles as the integrand. One can get back to the original angles using the formulas

$$\cos 3x = \cos(5x - 2x) = \cos 5x \cos 2x + \sin 5x \sin 2x,$$
$$\cos 7x = \cos(5x + 2x) = \cos 5x \cos 2x - \sin 5x \sin 2x.$$

Alternatively, an integration by parts argument leaves intact the angles. It requires two integration by parts steps, and we need to solve for the integral. Furthermore, we have to make the analogous substitution for u both times, and for dv both times. By analogous, here we mean using the same angle, $2x$ or $5x$, as the argument of the trigonometric function both times. If we always let the u-term have angle $2x$, and the dv-term have angle $5x$, eventually the solution there will be

$$\int \sin 2x \cos 5x \, dx = \frac{5}{21} \sin 2x \sin 5x + \frac{2}{21} \cos 2x \cos 5x + C.$$

While it is far from obvious that these results are the same, it is an interesting exercise to check this last computation by computing the derivative of our answer here, so see how it simplifies to the integrand.

7.3.7 Miscellaneous Problems and Methods-II

It is often the case that trigonometric integrals arise in the process of using another technique, and so the form of the integral may be more awkward than we have illustrated here so far. (This will be the case particularly in Section 7.4.) In such cases it is often necessary to experiment with rewriting the integral using whatever trigonometric identities apply. One also has to be aware that a substitution or integration-by-parts argument may be required eventually. Even then, if it is a multi-step application of integration by parts, it helps to recall when it helps to continue the "parts" step, and when it is better to use a trigonometric identity and solve for the integral algebraically, as when we integrated $\sec^3 x$ or $\csc^3 x$ (see Example 7.3.17, completed on page 630). In fact, once part of the process yields such an integral, it helps to know that such a complication was probably inevitable and we might as well deal with it from there, rather than attempt to bypass that problem.

With experience one learns to look ahead a few steps and anticipate which algebraic manipulation or identity will yield positive progress towards a form we can integrate (even if it requires some cleverness such as needed to integrate $\sec^3 x$), but it is not uncommon to require multiple attempts to integrate a trigonometric integral before finding a strategy that will ultimately succeed.

Example 7.3.21 *Compute* $\int \tan^2\theta \sec\theta \, d\theta$.

Solution: *We will attempt to compute this integral two different methods. First we will integrate by parts, noting that*

$$\int \tan^2\theta \sec\theta \, d\theta = \int \tan\theta \cdot \sec\theta \tan\theta \, d\theta.$$

$$\begin{aligned} u &= \tan\theta & dv &= \sec\theta \tan\theta \, d\theta \\ du &= \sec^2\theta \, d\theta & v &= \sec\theta \end{aligned}$$

$$\begin{aligned}
\int \tan^2\theta\sec\theta\,d\theta &= \int \underbrace{\tan\theta}_{u}\underbrace{\sec\theta\tan\theta\,d\theta}_{dv} \\
&= \underbrace{\tan\theta\sec\theta}_{uv} - \int \underbrace{\sec\theta}_{v}\underbrace{\sec^2\theta\,d\theta}_{du} \\
&= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta \\
&= \sec\theta\tan\theta - \int \sec^2\theta\sec\theta\,d\theta \\
&= \sec\theta\tan\theta - \int \left[\left(\tan^2\theta+1\right)\sec\theta\right]\,d\theta \\
&= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta - \int \sec\theta\,d\theta \\
&= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta - \ln|\sec\theta+\tan\theta| + C_1 \\
\implies 2\int \tan^2\theta\sec\theta\,d\theta &= sec\theta\tan\theta - \ln|\sec\theta+\tan\theta| + C_1 \\
\implies \int \tan^2\theta\sec\theta\,d\theta &= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C.
\end{aligned}$$

Because an intermediate step included the integral of $\sec^3\theta$*, which by itself would require an integration by parts technique that includes solving for the integral, it was likely that part of the solution would include our original integral on the left. Rather than beginning an integration by parts, the simple experiment of rewriting* $\sec^3\theta$ *as* $\sec^2\theta\sec\theta$*, and ultimately* $(\tan^2\theta+1)\sec\theta$*, was successful in producing a form in which we could solve for our original integral.*

The other method of attack is to write the entire integral in terms of $\sec x$ *in the first place:*

$$\begin{aligned}
\int \tan^2 x\sec x\,dx &= \int (\sec^2 x - 1)\sec x\,dx \\
&= \int (\sec^3 x - \sec x)\,dx.
\end{aligned}$$

While the second function in the integral has a standard and known antiderivative, as mentioned earlier there seems to be no way to avoid integrating $\sec^3 x$ *or similarly complicated terms, so for it we then either use a reduction formula or integrate by parts, using the indirect method where we solve for the integral. The latter approach was built into our first (successful) approach in this example.*

Alternatively we could have computed $\int \sec^3\theta\,d\theta$ elsewhere and inserted the computation here. That computation could have been carried out as before (Example 7.3.17), or through the appropriate reduction formula (7.42), page 631.

The next example can be computed in two ways, but the first that we show here in fact requires a technique from a later section, namely Section 7.5. We show it here anyways in anticipation of the time when the student can call upon the ideas of more than one of these sections for a single problem, as we did previously (in using integration by parts to solve a trigonometric integral, for instance).

Example 7.3.22 *Compute* $\displaystyle\int \frac{\cos^4 x}{\sin x}\,dx$.

Solution 1: *Note that this integral does in fact contain an odd number of sine factors, even though that number is* -1. *We can therefore make the integral one in terms of cosine only as the integrand, with the differential containing the sine:*

$$\begin{aligned}\int \frac{\cos^4 x}{\sin x}\,dx &= \int \frac{\cos^4 x}{\sin^2 x}\cdot \sin x\,dx \\ &= \int \frac{\cos^4 x}{1-\cos^2 x}\cdot \sin x\,dx.\end{aligned}$$

At this point, we have $u = \cos x \implies du = -\sin x\,dx \implies -du = \sin x\,dx$, *and then we eventually us some long division of polynomials to get*

$$\begin{aligned}\int \frac{\cos^4 x}{\sin x}\,dx &= \int \frac{\cos^4 x}{1-\cos^2 x}\cdot \sin x\,dx = -\int \frac{u^4}{1-u^2}\,du = \int \frac{u^4}{u^2-1}\,du \\ &= \int \left[u^2+1+\frac{1}{u^2-1}\right]\,du.\end{aligned}$$

At this point we need a technique from the Partial Fractions section (Section 7.5) to expand the fraction still inside the integral to get[19]

$$\frac{1}{u^2-1} = -\frac{1}{2}\cdot\frac{1}{u+1}+\frac{1}{2}\cdot\frac{1}{u-1}.$$

While we mention only the final result of that computation, it is easily enough verified by combining the fractions on the right-hand side. To finish our integral computation, we would then write

$$\begin{aligned}\int \frac{\cos^4 x}{\sin x}\,dx &= \int \left[u^2+1-\frac{1/2}{u+1}+\frac{1/2}{u-1}\right]\,du \\ &= \frac{1}{3}u^3+u-\frac{1}{2}\ln|u+1|+\frac{1}{2}|u-1|+C \\ &= \frac{1}{3}\cos^3 x+\cos x-\frac{1}{2}\ln|\cos x+1|+\frac{1}{2}\ln|\cos x-1|+C.\end{aligned}$$

Solution 2: *An alternative approach is to rewrite the entire integral in terms of the sine function:*

$$\begin{aligned}\int \frac{\cos^4 x}{\sin x}\,dx &= \int \frac{\left(\cos^2 x\right)^2}{\sin x}\,dx = \int \frac{\left(1-\sin^2 x\right)^2}{\sin x}\,dx \\ &= \int \frac{1-2\sin^2 x+\sin^4 x}{\sin x}\,dx = \int \left(\csc x-2\sin x+\sin^3 x\right)\,dx \\ &= -\ln|\csc x+\cot x|+2\cos x+\int \sin^3 x\,dx.\end{aligned}$$

This last integral is then one with an odd power of sine, so we use our previous techniques, with

[19] Here we could instead recall the derivative formula for $\tanh^{-1} x$ and use it for our antiderivative here, since $u = \cos x \in [-1,1]$:

$$\int \frac{1}{1-u^2}\,du = \tanh^{-1} u + C.$$

$u = \cos x \implies du = -\sin x\,dx$:

$$\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx = \int \underbrace{(1-\cos^2 x)}_{1-u^2}\underbrace{\sin x\,dx}_{-du}$$
$$= -\cos x + \frac{1}{3}\cos^3 x + C.$$

Thus

$$\int \frac{\cos^4 x}{\sin x}\,dx = -\ln|\csc x + \cot x| + 2\cos x - \cos x + \frac{1}{3}\cos^3 x + C$$
$$= -\ln|\csc x + \cot x| + \cos x + \frac{1}{3}\cos^3 x + C.$$

So far the major theme of this section is that we often benefit from rewriting trigonometric integrals using trigonometric identities, but with an eye towards the forms that make the calculus more agreeable. We developed some guidelines, but we should also be aware that there are simple cases which might not fit neatly into the guidelines, or might have an easier manipulation than the guidelines would direct.

Example 7.3.23 *Consider* $\displaystyle\int \frac{\cos^2 x}{\sin x}\,dx$.

This integral does have an odd number of sines, so we could multiply the numerator and denominator by $\sin x$*, with the numerator's to be used in the du term, but (as the reader is invited to verify) the resulting integral is not as simple as we get from the following.*

$$\int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{1-\sin^2 x}{\sin x}\,dx$$
$$= \int (\csc x - \sin x)\,dx$$
$$= \ln|\csc x - \cot x| + \cos x + C.$$

Trigonometry students know to consider replacing $\cos^2 x$ *with* $1-\sin^2 x$ *at any such opportunity, to see if it is advantageous. Calculus students should do the same.*

Example 7.3.24 *Compute* $\displaystyle\int \frac{1}{1+\sin x}\,dx$.

Solution: Here we consider multiplying the integrand by $(1-\sin x)/(1-\sin x)$ *to see if it is advantageous. (It is.)*

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\,dx$$
$$= \int \frac{1-\sin x}{\cos^2 x}\,dx$$
$$= \int (\sec^2 x - \sec x\tan x)\,dx$$
$$= \tan x - \sec x + C.$$

The second integral could have been done fairly easily using sines and cosines, but would have taken more steps.

Example 7.3.25 *Compute* $\displaystyle\int \frac{\tan x}{\sec x - 1}\,dx$.

Solution: Here we do a similar computation but using secants and tangents.

$$\begin{aligned}
\int \frac{\tan x}{\sec x - 1}\,dx &= \int \frac{\tan x}{\sec x - 1} \cdot \frac{\sec x + 1}{\sec x + 1}\,dx \\
&= \int \frac{\tan x \sec x + \tan x}{\sec^2 x - 1}\,dx \\
&= \int \frac{\tan x \sec x + \tan x}{\tan^2 x}\,dx \\
&= \int (\csc x + \cot x)\,dx \\
&= \ln|\csc x - \cot x| - \ln|\csc x| + C.
\end{aligned}$$

That there are so many techniques here is not surprising, since even if we keep the angles the same, there are six trigonometric functions and a great many identities relating them. The calculus facts guide us somewhat in where to look for identities which can aid in finding a particular antiderivative, but even with these twenty-five examples there are still unexplored possibilities. However these give a reasonable sample of the types of trigonometric integrals we are likely to encounter in the rest of the text.

Exercises

Evaluate the following integrals.

1. $\int \sin x \cos x \, dx$
2. $\int \sin^2 x \cos x \, dx$
3. $\int \sin x \cos^2 x \, dx$
4. $\int \sin^3 x \cos^2 x \, dx$
5. $\int \sin^4 x \cos^5 x \, dx$
6. $\int \dfrac{\sin^3 x}{\cos^2 x} \, dx$
7. $\int \dfrac{\sin^3 x}{\cos^2 x + 1} \, dx$
8. $\int \sin^4 x \cos^5 x \, dx$
9. $\int \sin^3 2x \cos^{15} 2x \, dx$
10. $\int \dfrac{\sin^2 x}{\cos x} \, dx$
11. $\int \sin x \ln |\sin x| \, dx$
12. $\int \cos x \ln |\sin x| \, dx$
13. Prove (7.45), page 631.

7.4 Trigonometric Substitution

In this section we explore how integrals can sometimes be solved by making some clever substitutions involving trigonometric functions, even though the original integrals themselves do not involve such functions.

7.4.1 Introduction

Before developing the general mechanics, we look at a few examples below for motivation.

Example 7.4.1 *Compute* $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$, *using a nontrivial substitution method.*

Solution: We already know the answer to this integral, because we know $\frac{d}{dx}\sin^{-1} x = 1/\sqrt{1-x^2}$, *so a "clever" but unnecessary substitution* $u = \sin^{-1} x$ *would yield the antiderivative quickly.*[20]

But let us imagine for a moment that we do not have this antiderivative at our disposal, and need to tackle this integral from first principles. Can we accomplish this without first developing a theory of derivatives of arc-trigonometric functions?

The answer is yes, if we are clever in a different way, which offers a more general method we can apply to a class of more complicated integrals, as we will see. This substitution method is to notice that $-1 < x < 1$ *is required in the integral, and that is nearly the same as the range of* $\sin\theta$, *namely* $-1 \le \sin\theta \le 1$, *and is in fact contained in it. So we make a substitution, albeit somewhat "backwards" from what we are used to, where* x *will be a function of* θ *rather than a function of* x *being explicitly some new variable:*

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \frac{\cos\theta}{\cos\theta}\,d\theta = \int d\theta = \theta + C = \sin^{-1} x + C$$

$$\left.\begin{array}{r} x = \sin\theta \\ \implies dx = \cos\theta\,d\theta \end{array}\right|$$

The above computation is completely correct, but there are a few technicalities to check.

1. *Why are we allowed to take the nonnegative case above, when we know* $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$*? In other words, usually* $\cos\theta = \pm\sqrt{1-\sin^2\theta}$, *but we took the "+" case.*

2. *Why can we automatically say* $x = \sin\theta \implies \theta = \sin^{-1} x$*? After all, there are infinitely many angles with the same sine, and they need not necessarily even be coterminal when we graph them in standard position. (Example:* $\sin\frac{\pi}{4} = \sin\frac{3\pi}{4} = \sin\frac{9\pi}{4} = \sin 11\pi 4$, *etc.)*

The answer to both questions lies in the values for θ *that we can choose when we make the substitution. In fact we were negligent by not fixing the range of* θ *from the outset, but we*

[20] Anytime we know the antiderivative, we can perform a substitution where we define u to be that antiderivative, and the resulting integral will be $\int du = u + C$, but this has relatively little use for discovery. For the integral in Example 7.4.1, we could have written

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int du = u + C = \sin^{-1} x + C.$$

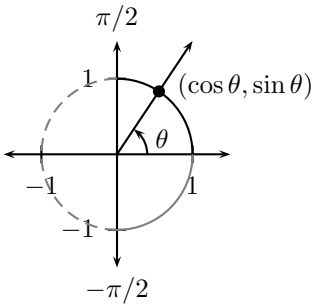
$$\left.\begin{array}{r} u = \sin^{-1} x \\ \implies du = \dfrac{1}{1-x^2}\,dx \end{array}\right|$$

will see there is a standard practice in which we will take exactly those angles θ (or a subset of them) which are the same as the range of the relevant arc-trigonometric function. For the problem above, it is assumed $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, since $\sin\theta$ is one-to-one for these values, and $\sin\theta$ ranges over the whole range of the sine function for these values, i.e., for the same reason the arcsine is defined to output these values.

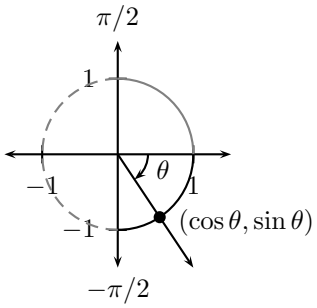
$$\underbrace{[-1,1]}_{x \in} \xrightarrow{\sin^{-1} x} \underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}_{\theta \in} \xrightarrow{\sin\theta} \underbrace{[-1,1]}_{x \in}$$

Thus we have mappings from x to θ and back. (When we actually compute the integral, we concentrate on the second mapping more than the first, until we compute the antiderivative, in which case we may need the first mapping to "get back" to x.)

When we look at the values of θ and their terminal points on the unit circle, all doubt about our casual computation $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ is laid to rest, because cosine is nonnegative in our range of angles θ:



$x = \sin\theta \geq 0$
$\theta \in [0, \pi/2]$
$\cos\theta \geq 0$



$x = \sin\theta \leq 0$
$\theta \in [-\pi/2, 0]$
$\cos\theta \geq 0$

Now "x" here does not signify the coordinate on the horizontal axis. Indeed, for this particular case "x" is the same as $\sin\theta$, which is the vertical ("y") coordinate where our angle in standard position (initial side being the positive horizontal axis) pierces the unit circle.

Note that having the angles θ ranging from $-\pi/2$ to $\pi/2$ means that $-1 \leq \sin\theta \leq 1$ is equivalent to $\theta = \sin^{-1} x$, and also implies $\cos\theta \geq 0$ and so $\cos\theta = \sqrt{1-\sin^2\theta}$, where normally we only have $\cos\theta = \pm\sqrt{1-\sin^2\theta}$:

1. $\left(\forall\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right)\left[x = \sin\theta \longleftrightarrow \theta = \sin^{-1} x\right]$, *and*
2. $\left(\forall\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right)\left[\cos\theta = \sqrt{1-\sin^2\theta}\right]$.

The example above illustrates another useful approach for some integrals:

1. Make a substitution to rewrite the integral in terms of an angle θ (with appropriate range of θ matching the range of values for x).
2. Compute the resultant trigonometric integral.
3. Rewrite the antiderivative in terms of x.

Drawing diagrams on an appropriate circle as above will be quite useful in subsequent problems. The process can not only clarify somewhat our substitution process, but it can also allow us to check that we have correct signs for all the various cases for θ. Moreover, there are times when we need to read actual values of *other* trigonometric functions of θ, but in terms of x.

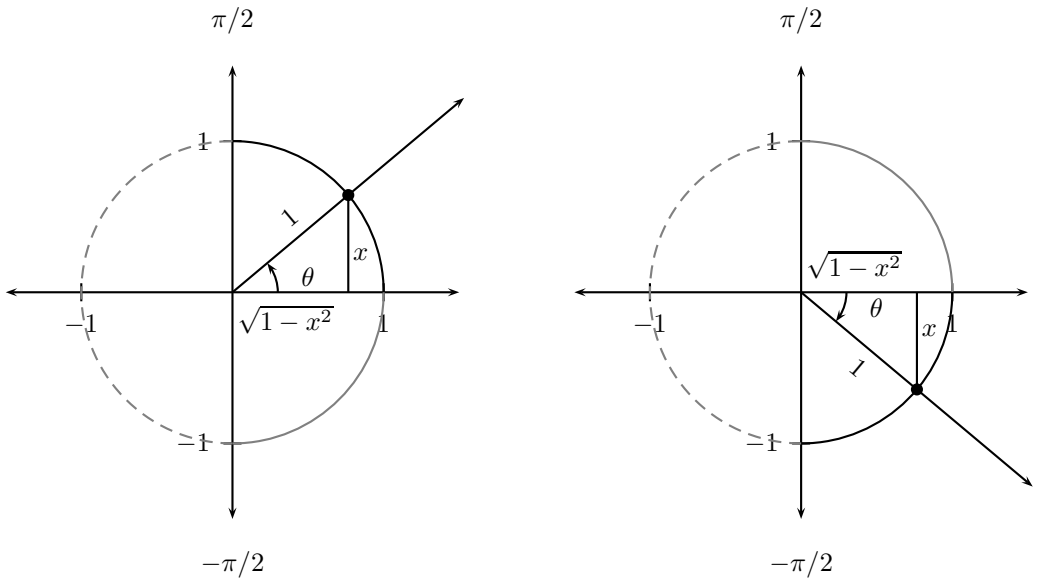
The process above may seem unnecessarily complicated, especially for an integral for which we knew the answer from the beginning. However, this advanced technique generalizes to integrals which do not succumb to previous methods (though they should always be considered first!). For instance we would be hard-pressed to compute the antiderivative in the next example using earlier techniques.

Example 7.4.2 *Compute* $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\,dx$.

Solution: Note first that this integral will not simply yield to earlier techniques. (The reader is welcome to try, to see where those methods eventually fall short.)

Note also that, due to the square root, we require $-1 \le x \le 1$. *In fact we also cannot have* $x = 0$, *but that constraint will be consistent with our new integral upon substitution. Ultimately it is the radical which is giving us the most difficulty here.*

The key to solving this problem is to again realize that $[-1, 1]$ *is exactly the range of the function* $\sin\theta$, *so we will again use the substitution* $x = \sin\theta$ *in the integral above, with the understanding that* $\theta \in [-\pi/2, \pi/2]$ *(excluding zero due to the denominator). This time we will again draw the diagram, but will label the various parts of the resulting triangles for future reference. Note that in the second drawing,* $\sin\theta = x < 0$ *while* $\cos\theta = \sqrt{1-x^2}$ *is still a positive quantity, and again, where the angle's terminal ray pierces the unit circle is the point* $(\cos\theta, \sin\theta)$.



Note that we can now read all trigonometric functions of θ *from the diagrams. For instance,* $\tan\theta = x/\sqrt{1-x^2}$, *regardless of which of the two quadrants contains the terminal ray of* θ.

Now we perform the substitution, noting as usual that the differential must also be accounted for:

$$\begin{aligned} & x = \sin\theta \\ \implies & dx = \cos\theta\, d\theta. \end{aligned}$$

Thus

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x}\, dx &= \int \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\cos\theta\, d\theta = \int \frac{\cos\theta}{\sin\theta}\cos\theta\, d\theta \\ &= \int \frac{\cos^2\theta}{\sin\theta}\, d\theta = \int \frac{1-\sin^2\theta}{\sin\theta}\, d\theta = \int \left[\csc\theta - \sin\theta\right]\, d\theta \\ &= \ln|\csc\theta - \cot\theta| + \cos\theta + C. \end{aligned}$$

Now we have a trigonometric form of the antiderivative, but of course the original integral was in x and not θ. Our diagram allows us to read the other trigonometric functions of θ in terms of x. We note that it does not matter, for this example, which of the two quadrants contains θ:

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x}\, dx &= \ln|\csc\theta - \cot\theta| + \cos\theta + C = \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C \\ &= \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C \\ &= \ln\left(1-\sqrt{1-x^2}\right) - \ln|x| + \sqrt{1-x^2} + C.^{21} \end{aligned}$$

Note that the calculus was finished by the end of the first line in the above equations, and the rest were algebraic rewritings. That absolute values were not needed in the first expression on the last line was due to the fact that $1 \geq \sqrt{1-x^2} \implies 1-\sqrt{1-x^2} \geq 0$.

Unlike the first example, we are not likely to be anxious to check our work by taking the derivative of our solution (though it would be an interesting exercise, particularly to see how terms cancel), so instead we strive to be careful and clear about our derivation, so we can minimize errors and easily re-read our computations to verify our result.

It should be noted at the outset that the trigonometric integrals which arise here may require some re-writing before they succumb to our trigonometric integral methods of Section 7.3; a problem which naturally gives rise to trigonometric substitution (as in the previous example) may or may not yield a simple trigonometric integral. However, all trigonometric integrals we

[21] Another student might instead use $\int \csc = -\ln|\csc\theta + \cot\theta + C_1$, but the final answer would be the same:

$$\begin{aligned} \int \frac{\sqrt{1-x^2}}{x}\, dx &= -\ln|\csc\theta + \cot\theta| + \cos\theta + C = -\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C \\ &= -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\cdot\frac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| + \sqrt{1-x^2} + C = -\ln\left|\frac{1-(1-x^2)}{x\left(1-\sqrt{1-x^2}\right)}\right| + \sqrt{1-x^2} + C \\ &= -\ln\left|\frac{x}{1-\sqrt{1-x^2}}\right| + \sqrt{1-x^2} + C = -\left(\ln|x| - \ln\left(1-\sqrt{1-x^2}\right)\right) + \sqrt{1-x^2} + C \\ &= \ln\left(1-\sqrt{1-x^2}\right) - \ln|x| + \sqrt{1-x^2} + C, \end{aligned}$$

which is the same as the answer written in the example.

will encounter here are of classes we considered in Section 7.3, so ultimately those techniques equipped us for our work here.

Before we delve into other trigonometric substitutions, we will perform one more involving the sine.

Example 7.4.3 *Compute* $\displaystyle\int \sqrt{9-25x^2}\,dx$.

Solution: As always, we should look to see if previous methods apply. They do not, without extraordinary cleverness, though it is interesting to note that simple substitution would work if we had an extra factor of x in the integrand. (We do not.)

Now in previous examples, we wanted to exploit the trigonometric identity $1-\sin^2\theta = \cos^2\theta$. Here we will do the same, except we will multiply this equation by 9, giving us $9 - 9\sin^2\theta = 9\cos^2\theta$.

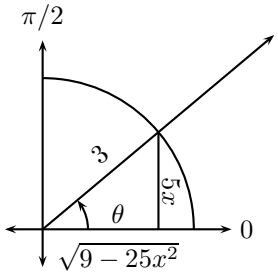
So we wish to have $25x^2 = 9\sin^2\theta$, i.e., $x^2 = \frac{9}{25}\sin^2\theta$. We get this if we let $x = \frac{3}{5}\sin\theta$. Below is what a student (or professor) might hand-write to compute this integral:

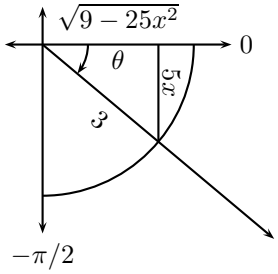
$$\begin{aligned}
\int \sqrt{9-25x^2}\,dx &= \int \sqrt{9-25\cdot\frac{9}{25}\sin^2\theta}\cdot\frac{3}{5}\cos\theta\,d\theta \\
25x^2 = 9\sin^2\theta \;\Big|\; &= \int \sqrt{9-9\sin^2\theta}\cdot\frac{3}{5}\cos\theta\,d\theta \\
x^2 = \frac{9}{25}\sin^2\theta \;\Big|\; &= \int \sqrt{9(1-\sin^2\theta)}\cdot\frac{3}{5}\cos\theta\,d\theta \\
\therefore\ \text{Take } x = \frac{3}{5}\sin\theta \;\Big|\; &= \int 3\sqrt{1-\sin^2\theta}\cdot\frac{3}{5}\cos\theta\,d\theta \\
\implies dx = \frac{3}{5}\cos\theta\,d\theta \;\Big|\; &= \int \frac{9}{5}\cos^2\theta\,d\theta \\
&= \int \frac{9}{5}\cdot\frac{1}{2}\left(1+\cos 2\theta\right)\,d\theta \\
&= \frac{9}{10}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C \\
&= \frac{9}{10}\theta+\frac{9}{10}\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta+C \\
&= \frac{9}{10}\theta+\frac{9}{10}\sin\theta\cos\theta+C.
\end{aligned}$$

We require the final trigonometric antiderivative to be in terms of the angle θ (and not 2θ for example), so that we can read the trigonometric functions of θ from the diagram. In constructing the diagram, we need to solve our substitution for $\sin\theta$:

$$x = \frac{3}{5}\sin\theta \implies \sin\theta = \frac{5x}{3}.$$

Note that our diagram below is not constructed on the unit circle, but on a more general circle of positive radius (in this case 3). As before, we have to consider both quadrants in which θ may fall.



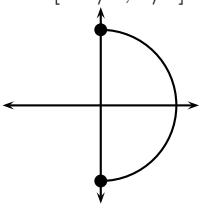
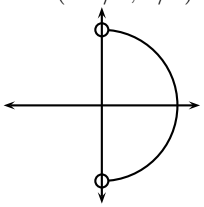
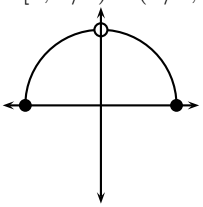


From these and our expression for $\sin\theta$ above we can complete our integral computation:

$$\begin{aligned}\int \sqrt{9-25x^2}\,dx &= \frac{9}{10}\theta + \frac{9}{10}\sin\theta\cos\theta + C \\ &= \frac{9}{10}\sin^{-1}\left(\frac{5x}{3}\right) + \frac{9}{10}\cdot\frac{5x}{3}\cdot\frac{\sqrt{9-25x^2}}{3} + C \\ &= \frac{9}{10}\sin^{-1}\left(\frac{5x}{3}\right) + \frac{x}{2}\sqrt{9-25x^2} + C.\end{aligned}$$

7.4.2 The General Approach

There are cases where a different trigonometric substitution is appropriate and useful. In fact the choices are mutually exclusive, and the form to be used can be deduced from the range of x-values in the domain of the original integrand, though one instead usually sees how problematic terms would simplify. In the chart below, we use x, though "x" can be x or $5x$, or similar, with the necessary algebra and calculus to compensate. Also for simplicity, we assume $a > 0$:[22]

Integral contains:	$\sqrt{a^2-x^2}$	$\sqrt{a^2+x^2}$	$\sqrt{x^2-a^2}$
Substitute:	$x = a\sin\theta$	$x = a\tan\theta$	$x = a\sec\theta$
Motivation:	$\sqrt{a^2-x^2} = a\cos\theta$	$\sqrt{a^2+u^2} = a\sec\theta$	$\sqrt{x^2-a^2} = a\lvert\tan\theta\rvert$
Range of x:	$-a \le x \le a$	$x \in \mathbb{R}$	$x \in (-\infty, -a] \cup [a, \infty)$
Range of θ:	$\theta \in [-\pi/2, \pi/2]$	$\theta \in (-\pi/2, \pi/2)$	$\theta \in [0, \pi/2) \cup (\pi/2, \pi]$
			

The graphs of the positions of θ are important for verification purposes, as well as for filling in with x-expressions the trigonometric functions of θ that usually arise from the trigonometric form of the antiderivative.

[22] In fact, sometimes these substitutions are useful even when the radicals are not present, particularly for the tangent case, such as in computing $\int \frac{1}{(x^2+a^2)^n}\,dx$. So we can perhaps make a general statement that these substitutions should be considered if we have powers of (a^2-x^2), (a^2+x^2) or (a^2-x^2), respectively, when previous methods will not work, and the range of values for x and θ are compatible.

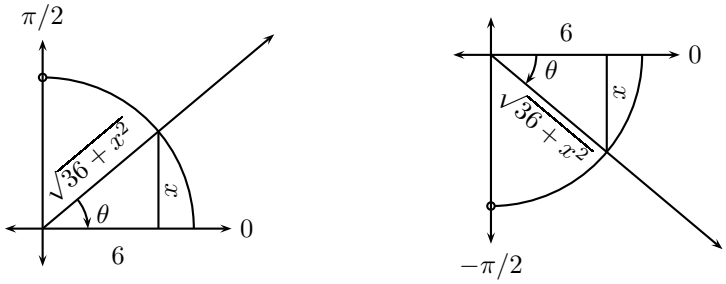
Example 7.4.4 *Compute* $\displaystyle\int \frac{1}{\sqrt{36+x^2}}\,dx$.

Solution: Here we will let $x = 6\tan\theta$. *Note that as part of the method (but usually left unstated),* $-\pi/2 < \theta < \pi/2 \iff -\infty < x < \infty$.

$$\int \frac{1}{\sqrt{36+x^2}}\,dx = \int \frac{1}{\sqrt{36+36\tan^2\theta}}\cdot 6\sec^2\theta\,d\theta = \int \frac{6\sec^2\theta}{\sqrt{36\sec^2\theta}}\,d\theta = \int \frac{6\sec^2\theta}{6\sec\theta}\,d\theta$$

$$\left.\begin{array}{r} x = 6\tan\theta \\ \implies dx = 6\sec^2\theta\,d\theta \end{array}\right\} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1.$$

At this point we have the integral solved in terms of θ. *To get back to* x, *we draw the relevant triangles in which* $x = \tan\theta$, *i.e.,* $\tan\theta = x/6$. *Note* $\sec\theta = \left(\sqrt{36+x^2}\right)/6$ *in both quadrants.*



Thus

$$\begin{aligned}\int \frac{1}{\sqrt{36+x^2}}\,dx &= \ln|\sec\theta+\tan\theta| + C_1 \\ &= \ln\left|\frac{\sqrt{36+x^2}}{6} + \frac{x}{6}\right| + C_1 \\ &= \ln\left|\sqrt{36+x^2}+x\right| - \ln 6 + C_1 \\ &= \ln\left|\sqrt{36+x^2}+x\right| + C.\end{aligned}$$

It is customary to have the solution in simplest possible form, which is why we expanded the logarithm and absorbed the $-\ln 6$ *term into the constant. (One usually only puts the subscript on the* C_1 *term after further lines reveal it is useful.)*

As we see in the drawings of the relevant angles, when we use a tangent-type substitution we get the same, simple expressions for the derived sides regardless of which of the two quadrants holds the terminal ray of θ. It is not always the case with the secant substitutions.

Example 7.4.5 *Compute* $\displaystyle\int \sqrt{x^2-9}\,dx$.

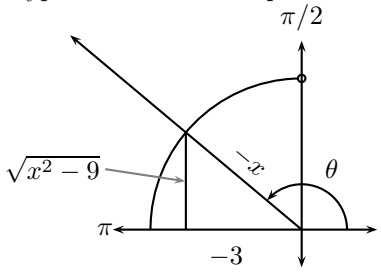
Solution: Here we let $x = 3\sec\theta$.

$$\int \sqrt{x^2-9}\,dx = \int \sqrt{9\sec^2\theta-9}\cdot 3\sec\theta\tan\theta\,d\theta$$

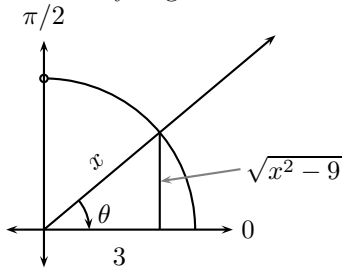
$$\left.\begin{array}{r} x = 3\sec\theta \\ \implies dx = 3\sec\theta\tan\theta\,d\theta \end{array}\right\} = \int \sqrt{9\tan^2\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int 3|\tan\theta|\cdot 3\sec\theta\tan\theta\,d\theta$$

Recall that our choices for θ terminate in either the first or second quadrant when we use a secant-type substitution, but while $\tan\theta \geq 0$ in Quadrant I, we have $\tan\theta \leq 0$ in Quadrant II. These coincide with the cases x positive and x negative (or more precisely $x \geq 3$ and $x \leq -3$), respectively. Below we graph the two cases, noting that $\sec\theta = x/3$, i.e., $\cos\theta = 3/x$, but also that the "hypotenuse" must be positive in each case, as dictated by trigonometric theory.



case $x \leq -3$



case $x \geq 3$

The signs of the two "legs" of the representative triangles are also useful in checking the expression for the hypotenuse. Note $\sec\theta = x/3 = (-x)/(-3)$, and also that $-x > 0$ for the case θ in QII. (The hypotenuse is a radius, and therefore always positive.)

For this particular example, the antiderivatives for the two cases differ by a factor of -1, so we do most of the work by finding the antiderivative for one case, and changing sign for the other. For simplicity we will compute the antiderivative for the case θ in Quadrant I first.

1. *Case $x \geq 3$:* $\displaystyle\int \sqrt{x^2-9}\,dx = 9\int \tan^2\theta\sec\theta\,d\theta.$

   This requires integration by parts, of the type where we "solve" for the integral. We will use

$$\begin{aligned} u &= \tan\theta & dv &= \sec\theta\tan\theta\,d\theta \\ du &= \sec^2\theta\,d\theta & v &= \sec\theta. \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\mathcal{I}}) &= uv - \int v\,du = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta \\ &= \sec\theta\tan\theta - \int (\tan^2\theta + 1)\sec\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\,d\theta - \int \tan^2\theta\sec\theta\,d\theta \\ &= \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| - (\boldsymbol{\mathcal{I}}). \end{aligned}$$

   Solving for $(\boldsymbol{\mathcal{I}})$, we have

$$\begin{aligned} 2(\boldsymbol{\mathcal{I}}) &= \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C_1 \\ \Longrightarrow\ (\boldsymbol{\mathcal{I}}) &= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C. \end{aligned}$$

   Using our definition of $\sec\theta$ and the previous diagram (for QI), we have

$$\begin{aligned} (\boldsymbol{\mathcal{I}}) &= \frac{1}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} - \frac{1}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C_2 \\ &= \frac{x\sqrt{x^2-9}}{18} - \frac{1}{2}\left[\ln\left|x + \sqrt{x^2-9}\right| - \ln 3\right] + C_1 \\ &= \frac{x\sqrt{x^2-9}}{18} - \frac{1}{2}\ln\left|x + \sqrt{x^2-9}\right| + C. \end{aligned}$$

2. *Case $x \le -3$: That is, θ in QII, where we will have the same antiderivative in θ except for a sign. Here $\tan\theta \le 0$, so $|\tan\theta| = -\tan\theta$, so the trigonometric form of the antiderivative is the same as above except for an extra factor of (-1):*

$$\begin{aligned}(\mathcal{I}) &= \int 3(-\tan\theta)\cdot 3\sec\theta\tan\theta\,d\theta \\ &= \frac{-1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\ &= \frac{-1}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{-3} + \frac{1}{2}\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{-3}\right| + C_3 \\ &= \frac{x\sqrt{x^2-9}}{18} + \frac{1}{2}\left[\ln\left|x-\sqrt{x^2-9}\right| - \ln 3\right] + C_3 \\ &= \frac{x\sqrt{x^2-9}}{18} + \frac{1}{2}\ln\left|x-\sqrt{x^2-9}\right| + C.\end{aligned}$$

Summarizing,

$$\int\sqrt{x^2-9}\,dx = \begin{cases} \dfrac{1}{18}x\sqrt{x^2-9} - \dfrac{1}{2}\ln\left|x+\sqrt{x^2-9}\right| + C, & x \ge 3, \\ \dfrac{1}{18}x\sqrt{x^2-9} + \dfrac{1}{2}\ln\left|x-\sqrt{x^2-9}\right| + C, & x \le -3. \end{cases}$$

Often, in a problem like this latest example we will know from the start what will be the range of interest of values of x. For instance, if we know $x \ge 0$ (more precisely, $x \ge 3$) we can finish the problem by drawing one diagram only, namely that in the first quadrant. If this were a definite integral, for instance, we would know which range of x we need by looking at the endpoints of our integral.

It should be emphasized that only the secant-type trigonometric substitutions require us to check both quadrants, because $\sqrt{\sec^2\theta - 1} = \pm\tan\theta$ for the range of θ we use.[23]

Sometimes, particularly for tangent substitutions, no radical is present in the original integral yet the trigonometric substitution is the method of choice.

Example 7.4.6 *Compute* $\displaystyle\int\frac{1}{(x^2+9)^2}\,dx$.

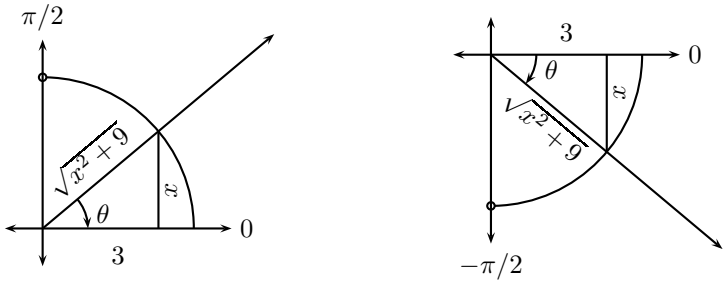
Solution: Note that this is very different from the case where we have $(x^2+9)^1$ in the denominator, which would eventually yield a arctangent, namely $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$. But it gives us

[23] There is an approach where one assumes $\theta \in [0,\pi/2)\cup[\pi,3\pi/2)$ or similar QI and QIII angles, and then $\sqrt{\sec^2\theta-1} = \tan\theta$ since $\tan\theta \ge 0$ in those quadrants. However we avoid this because sometimes the antiderivative includes the angle θ itself, which for our case would be $\theta = \sec^{-1}\frac{x}{3} = \cos^{-1}\frac{3}{x}$. This other approach simply redefines the arcsecant as well, so $\sec^{-1} z \in [0,\pi/2)\cup[\pi,3\pi/2)$. The approach has many advantages (for instance the derivative of the arcsecant does not contain an absolute value), but at a cost of such conveniences as $\sec^{-1} z = \cos^{-1}\frac{1}{z}$. One also redefines the arccosecant in that approach to arc-trigonometric functions.

a hint of what to do, namely let $x = 3\tan\theta$.

$$\begin{aligned}
\int \frac{1}{(x^2+9)^2}\,dx &= \int \frac{1}{\left(9\tan^2\theta+9\right)^2}\cdot 3\sec^2\theta\,d\theta \\
\left.\begin{aligned} x &= 3\tan\theta \\ dx &= 3\sec^2\theta\,d\theta \end{aligned}\right\} &= \int \frac{3\sec^2\theta\,d\theta}{\left(9\sec^2\theta\right)^2} \\
&= \int \frac{3\sec^2\theta\,d\theta}{81\sec^4\theta} \\
&= \frac{1}{27}\int \cos^2\theta\,d\theta \\
&= \frac{1}{27}\cdot\frac{1}{2}\int (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{27}\left[\frac{1}{2}\cdot\theta+\frac{1}{2}\sin 2\theta\right]+C \\
&= \frac{1}{54}\theta+\frac{1}{27}\cdot\frac{1}{2}\cdot 2\sin\theta\cos\theta+C.
\end{aligned}$$

Now the integral is solved in terms of θ*, so we need to get back to* x*, which we again do by looking at diagrams of the relation between* x *and* θ*, namely that* $\tan\theta = x/3$*, and* $\theta \in (-\pi/2, \pi/2)$.



Finishing off our integral is now fairly easy:

$$\begin{aligned}
\int \frac{1}{(x^2+9)^2}\,dx &= \frac{1}{54}\theta+\frac{1}{27}\cdot\sin\theta\cos\theta+C \\
&= \frac{1}{54}\tan^{-1}x+\frac{1}{27}\cdot\frac{x}{\sqrt{x^2+9}}\cdot\frac{3}{\sqrt{x^2+9}}+C \\
&= \frac{1}{54}\tan^{-1}x+\frac{1}{9}\cdot\frac{x}{x^2+9}+C.
\end{aligned}$$

While this is a powerful approach, it is not always the technique of choice. For example,

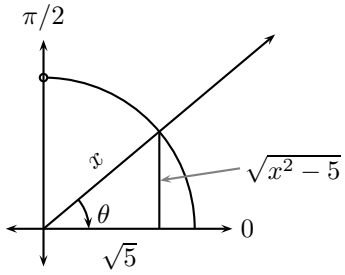
$$\int \frac{x}{\sqrt{x^2-9}}\,dx = \sqrt{x^2-9}+C,$$

from a simple substitution, or perhaps even an anticipation of the general form followed by a check of multiplicative constants. Similarly, if we replaced x with x^2 in the numerator we could integrate by parts. If we found ourselves integrating by parts twice, for instance (perhaps it is x^3 in the numerator) it may well be more efficient to use trigonometric substitution, or an integration by parts step may ultimately require it! But trigonometric substitution is usually not the method of choice if previous methods apply.

Example 7.4.7 *Compute* $\displaystyle\int \frac{\sqrt{x^2-5}}{x}\,dx$ *assuming that* $x \geq \sqrt{5}$.

Solution: Here we let $x^2 = 5\sec^2\theta$, *i.e.,* $x = \sqrt{5}\sec\theta$, *giving* $dx = \sqrt{5}\sec\theta\tan\theta\,d\theta$. *Note that we are assuming positive* x, *so* $\theta \in [0, \pi/2)$ *so* θ *terminates in QI, where* $\tan\theta \geq 0$. *Thus,*

$$\begin{aligned}
\int \frac{\sqrt{x^2-5}}{x}\,dx &= \int \frac{\sqrt{5\sec^2\theta - 5}}{\sqrt{5}\sec\theta} \cdot \sqrt{5}\sec\theta\tan\theta\,d\theta \\
&= \int \frac{\sqrt{5}\tan\theta}{\sqrt{5}\sec\theta} \cdot \sqrt{5}\sec\theta\tan\theta\,d\theta \\
&= \sqrt{5}\int \tan^2\theta\,d\theta \\
&= \sqrt{5}\int \left(\sec^2\theta - 1\right)\,d\theta \\
&= \sqrt{5}\tan\theta - \sqrt{5}\theta + C \\
&= \sqrt{5}\cdot\frac{\sqrt{x^2-5}}{\sqrt{5}} - \sqrt{5}\sec^{-1}\frac{x}{\sqrt{5}} + C \\
&= \sqrt{x^2-5} + \sec^{-1}\frac{x}{\sqrt{5}} + C.
\end{aligned}$$



Exercises

1. Use trigonometric substitution to derive the general formula ($a > 0$)
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C.$$

2. Use trigonometric substitution to derive the general formula ($a > 0$)
$$\int \frac{dx}{a^2+x^2} = \frac{x}{a}\tan^{-1}\frac{x}{a} + C.$$

3. Use trigonometric substitution to derive the general formula for $x > a > 0$:
$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{x}{a}\sec^{-1}\frac{x}{a} + C.$$

4. Compute $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$.

5. Compute $\displaystyle\int \left(9-x^2\right)^{3/2}\,dx$.

6. Compute $\displaystyle\int \frac{1}{(25+9x^2)^{5/2}}\,dx$.

7. Compute $\displaystyle\int \sqrt{x^2+2x}\,dx$. (Complete the square.)

7.5 Partial Fractions and Integration

In this section we are interested in techniques for computing integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx, \tag{7.49}$$

where $P(x)$ and $Q(x)$ are polynomials. This is not in general a simple problem, unless the integral in (7.49) is from very particular classes. However, with the techniques we explore here, we can break $\frac{P(x)}{Q(x)}$ into simpler fractions whose integrals are relatively easy. To see an advantage of such an approach, consider the following example.

Example 7.5.1 *Compute* $\displaystyle\int \frac{5x-1}{x^2-x-2}\,dx$.

Solution: Note that the numerator is not a simple (constant number) multiple of the derivative of the denominator, so substitution will not give a simple $\int \frac{1}{u}\,du$*-form.*

However, it happens that

$$\frac{5x-1}{x^2-x-2} = \frac{5x-1}{(x-2)(x+1)} = \frac{2}{x+1} + \frac{3}{x-2}. \tag{7.50}$$

We well see later a technique for deriving the right-hand side of (7.50). This gives us

$$\begin{aligned}\int \frac{5x-1}{x^2-x-2}\,dx &= \int \left(\frac{2}{x+1} + \frac{3}{x-2}\right) dx \\ &= 2\ln|x+1| + 3\ln|x-2| + C \\ &= \ln\left|(x+1)^2(x-2)^3\right| + C.\end{aligned}$$

In this section we will develop the methods needed to find expansions of fractions such as in (7.50). The idea is to reverse the high school algebra exercises, which would have us *combine* sums or differences of fractions into a single fraction. For purposes of integral calculus, it is almost always better to instead deal with several, simpler fractions than their combination into a single, more complicated fraction.

A significant amount of our work in this section will be algebraic, specifically, developing the method of decomposing a fraction $P(x)/Q(x)$ into simpler, "partial fractions." In order for the method to work, we will require $P(x)$ to have lower degree $Q(x)$. (If the degree of P is at least that of Q, we can use long division to write the function as a polynomial plus $r(x)/Q(x)$, where the degree of r is less than that of Q.)

Though not crucial for the calculus, we will spend the next subsection looking roughly at the theory behind the general form of these *partial fraction decompositions* (PFD's), in hopes it will help reinforce the rules themselves. In Subsection 7.5.2 we will definitively write the rules for PFD's without reference to integrals. Finally, we will see how to solve for the coefficients of a particular PFD, in the context of computing antiderivatives of these.

7.5.1 Theory Behind the Forms of PFD's (Optional)

The argument here is usually omitted from calculus texts, and instead left to linear algebra courses (if discussed at all!). However, the basic intuition is not difficult so we include it here, though the real work is in later subsections. In all of these we are looking at functions

$$\frac{P(x)}{Q(x)}, \qquad P \text{ and } Q \text{ polynomials, degree } P < \text{degree } Q. \tag{7.51}$$

Before stating the general rules for PFD's, we look at several examples illustrating the underlying theory. (Note that a *linear combination* of two functions $f(x)$ and $g(x)$ is a function of the form $a \cdot f(x) + b \cdot g(x)$, where $a, b \in \mathbb{R}$ are constants.)

Example 7.5.2 *For this example, we will argue in steps.*

1. *Consider all functions of the form*
$$\frac{ax+b}{(x+1)(x-2)}. \tag{7.52}$$

2. *Now there are two linearly independent*[24] *functions, specifically* $\frac{x}{(x+1)(x-2)}$ *and* $\frac{1}{(x+1)(x-2)}$ *which—with linear combinations—can give us any such function of form (7.52). Indeed,*
$$\frac{ax+b}{(x+1)(x-2)} = a \cdot \left[\frac{1}{(x+1)(x-2)}\right] + b \cdot \left[\frac{x}{(x+1)(x-2)}\right].$$
In a linear-algebraic sense, we would say the functions of the form (7.52) *form a 2-dimensional space (or 2-dimensional vector space), because to specify such a function requires two constants, a and b.*

3. *Now instead consider another 2-dimensional space of functions given by linear combinations of the form*
$$\frac{A}{x+1} + \frac{B}{x-2} = A \cdot \left[\frac{1}{x+1}\right] + B \cdot \left[\frac{1}{x-2}\right]. \tag{7.53}$$

4. *The functions* $\frac{1}{x+1}$ *and* $\frac{1}{x-2}$ *are indeed also linearly independent, so the set of all functions of the form* (7.53) *also forms a 2-dimensional vector space; to specify any such function requires specifying two constants, A and B.*

5. *Now notice that*
$$\frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)} = \frac{(A+B)x+(-2A+B)}{(x+1)(x-2)},$$
which is of form (7.52) *with $a = A+B$ and $b = -2A+B$. In other words, any function of the form* (7.53) *can also be written in the from* (7.52).

6. *This tells us that the two-dimensional space of functions* $\frac{A}{x+1} + \frac{B}{x-2}$ *is contained in the two-dimensional space of functions* $\frac{ax+b}{(x+1)(x-2)}$. *It is a fact of linear algebra that the only way for a two-dimensional space to be contained in another two-dimensional space is for them to be the same spaces. (Think about a plane being contained in another plane, and realize that they must then be the same plane.)*

7. *Finally, since (by 6 above) the space of all functions of the form* $\frac{ax+b}{(x+1)(x-2)}$ *is the* **same** *as the space of all functions of the form* $\frac{A}{x+1} + \frac{B}{x-2}$, *it follows that any function of the form* $\frac{ax+b}{(x+1)(x-2)}$ *can also be written in the form* $\frac{A}{x+1} + \frac{B}{x-2}$.

[24] We call functions $f_1, f_2, \cdots, f_n$ *linearly independent* if and only if it is impossible to write any of these as linear combinations of the others. In other words, we *exclude* cases where there exist *constants* $a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n \in \mathbb{R}$ such that
$$f_k = a_1 f_1 + a_2 f_2 + \cdots + a_{k-1} f_{k-1} + a_{k+1} f_{k+1} + \cdots + a_n f_n, \qquad \textit{i.e.},$$
$$(\forall x)[f_k(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_{k-1} f_{k-1}(x) + a_{k+1} f_{k+1}(x) + \cdots + a_n f_n(x)].$$

Notice that functions of the form (7.52) are indeed also of the form $P(x)/Q(x)$ where P is of degree less than Q, since the degree of P is at most 1 (zero if $a = 0$) and the degree of Q is 2.

The argument above guarantees that a PFD such as (7.50) exists. It is more desirable for integration purposes to have form (7.53) than (7.52).

Example 7.5.3 *An argument similar to that of the previous example shows that the following forms give exactly the same* set *of functions:*

$$\frac{ax^2+bx+c}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}. \tag{7.54}$$

Of course a, b, c are likely to differ from A, B, C. Here the underlying sets of linearly independent functions are, respectively,

$$U = \left\{ \frac{x^2}{(x+1)(x+2)(x+3)}, \frac{x}{(x+1)(x+2)(x+3)}, \frac{1}{(x+1)(x+2)(x+3)} \right\},$$
$$V = \left\{ \frac{1}{x+1}, \frac{1}{x+2}, \frac{1}{x+3} \right\}.$$

Both sets of vectors span[25] *3-dimensional spaces. It is not hard to see that functions on the right-hand side of* (7.54) *can also be in the form on the left. Indeed, if we combine the fractions on the right, we get*

$$\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} = \frac{\overbrace{A(x+2)(x+3)}^{\text{degree} \le 2} + \overbrace{B(x+1)(x+3)}^{\text{degree} \le 2} + \overbrace{C(x+1)(x+2)}^{\text{degree} \le 2}}{(x+1)(x+2)(x+3)},$$

which gives us a polynomial in the numerator with degree at most 2, as on the left-hand side of (7.54). *Here we have the span of V contained in the span of U, though they are both 3-dimensional spaces. Thus they must be the same spaces (we have one 3-dimensional space inside of another, so they must be the same!), so in fact, anything written like the left-hand side of* (7.54) *can be written like the right-hand side. (In a later subsection we will show how to find A, B, C if we are given a, b, c.)*

It should be clear that integrating a function written like the right-hand side of (7.54) *is likely much simpler than integrating one in the form on the left.*

Example 7.5.4 *Next we argue that the following forms describe the same (space of) functions:*

$$\frac{ax^2+bx+c}{(x+7)^3} = \frac{A}{x+7} + \frac{B}{(x+7)^2} + \frac{C}{(x+7)^3}. \tag{7.55}$$

The underlying sets of linearly independent functions are, respectively, $\left\{ \frac{x^2}{(x+7)^3}, \frac{x}{(x+7)^3}, \frac{1}{(x+7)^3} \right\}$ *and* $\left\{ \frac{1}{x+7}, \frac{1}{(x+7)^2}, \frac{1}{(x+7)^3} \right\}$, *both spanning 3-dimensional spaces. To show they are the same spaces, we note that*

$$\frac{A}{x+7} + \frac{B}{(x+7)^2} + \frac{C}{(x+7)^3} = \frac{A(x+7)^2 + B(x+7) + C}{(x+7)^3} = \frac{Ax^2 + (14A+B)x + (49A+C)}{(x+7)^3},$$

[25] The noun form of **span** has a precise technical meaning. The *span* of "vectors" $v_1, v_2, \cdots, v_n$ is the set of all possible linear combinations of those vectors. Thus for example

$$\text{Span}\left\{ \frac{1}{x+1}, \frac{1}{x+2}, \frac{1}{x+3} \right\} = \left\{ a \cdot \left[\frac{1}{x+1}\right] + b \cdot \left[\frac{1}{x+2}\right] + c \cdot \left[\frac{1}{x+3}\right] \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

We would then say that the functions (vectors, in the linear algebra sense) $\frac{1}{x+1}, \frac{1}{x+2}, \frac{1}{x+3}$, taken together, *span* the set described above.

thus simplifying to the form on the left-hand side of (7.55)*, with* $a = A$*,* $b = 14A + B$*, and* $c = 49A + C$*, all constants. Arguing as before, the underlying sets of linearly independent functions must span the same 3-dimensional spaces, so anything of the form on the left-hand side of* (7.55) can be written also in the form on the right-hand side.

Example 7.5.5 *For our last example, we claim the following forms give the same functions:*

$$\frac{ax^4 + bx^3 + cx^2 + dx + e}{x^3(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}. \tag{7.56}$$

Here the spanning sets of linearly independent functions are respectively[26]

$$U = \left\{ \frac{x^4}{x^3(x^2+1)}, \frac{x^3}{x^3(x^2+1)}, \frac{x^2}{x^3(x^2+1)}, \frac{x}{x^3(x^2+1)}, \frac{1}{x^3(x^2+1)} \right\},$$
$$V = \left\{ \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{x}{x^2+1}, \frac{1}{x^2+1} \right\}.$$

Again, the form on the right of (7.56) *can be rewritten as below, simplifying into the form on the left-hand side of* (7.56) *as*

$$\frac{Ax^2(x^2+1) + Bx(x^2+1) + C(x^2+1) + (Dx+E)(x^2+1)}{x^3(x^2+1)}$$
$$= \frac{Ax^4 + (B+D)x^3 + (A+C+E)x^2 + (B+D)x + (C+E)}{x^3(x^2+1)}. \tag{7.57}$$

(Note that this has numerator of degree at most 4.) So anything that is a linear combination of functions of V *can also be written as a linear combination of functions of* U*. Because both sides of (7.56) must therefore describe exactly the same functions in a 5-dimensional vector space, it follows that anything written in the form on the left-hand side of (7.56) can also be written in the form on the right-hand side.*

Note that it is a little tricky to find the new coefficients of $1, x, x^2, x^3$, etc., in the new numerator of (7.57). They all follow from gathering what we would get if we multiplied out the original numerator, but it would be easy to miss a term. (The reader should verify the computation (7.57).)

In the next subsection we generalize the logic of the examples above to write exact rules for the form of a PFD based upon the original fraction's denominator. Then in Subsection 7.5.3 we look at three methods of finding the coefficients, A, B, C, etc., of the PFD expansion, and immediately apply the methods to problems of computing integrals of such functions.

[26] Of course we can also simplify the functions in U as the five linearly independent functions

$$U = \left\{ \frac{x}{x^2+1}, \frac{1}{x^2+1}, \frac{1}{x(x^2+1)}, \frac{1}{x^2(x^2+1)}, \frac{1}{x^3(x^2+1)} \right\}.$$

7.5.2 Partial Fraction Decompositions: The Rules

It is a fact of algebra (corollary to the Fundamental Theorem of Algebra) that any polynomial with real coefficients can be factored uniquely—up to rearrangement of multiplicative constants—into powers of linear terms $(ax+b)^n$ and powers of irreducible quadratic[27] terms $(ax^2+bx+c)^m$ with real coefficients. So for instance,

$$x^3 - x^2 + x - 1 = (x-1)(x^2+1),$$

and there is no other way to factor it, except for instance $2(x-1)(\frac{1}{2}x^2+\frac{1}{2})$, etc. With that in mind, the rules for partial fraction decompositions follow.

First, we are given a rational function $\frac{P(x)}{Q(x)}$, where P and Q are polynomials.

0. In 1 and 2 below we assume deg $P <$ deg Q. If the deg $P \geq$ deg Q, then first we apply polynomial long division to achieve

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)},$$

where p, r are polynomials and deg $r <$ deg Q. Then the following rules apply to $\frac{r(x)}{Q(x)}$.

1. If $(ax+b)^n$, where $a \neq 0$ occurs as a factor in $Q(x)$, then the partial fraction decomposition (PFD) of $\frac{P(x)}{Q(x)}$ will contain terms

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

2. If ax^2+bx+c is an irreducible quadratic, and $(ax^2+bx+c)^m$ occurs as a factor in $Q(x)$, then the PFD of $\frac{P(x)}{Q(x)}$ will contain terms

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}.$$

The application of these rules can be somewhat confusing at first, so we will look at several examples before proceeding to solve for the coefficients A_1, B_1, etc. For the second case, we will mostly be interested in irreducible quadratics of the form x^2+k^2, where $k>0$. Note that we will usually use letters without subscripts, such as A, B, C, and so on for our PFD coefficients (to be found later).

Example 7.5.6 *Write the partial fraction decompositions for the given rational functions:*

- $\dfrac{3x^5-11x^3+15x-2}{(x+1)^2(x-3)^4} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2} + \dfrac{E}{(x-3)^3} + \dfrac{F}{(x-3)^4}.$
- $\dfrac{1}{(x+1)^2(x-3)^4} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-3} + \dfrac{D}{(x-3)^2} + \dfrac{E}{(x-3)^3} + \dfrac{F}{(x-3)^4}.$

[27] It is easy to see when a quadratic term is "irreducible over the real numbers," meaning we cannot write it as $(ex+f)(gx+h)$, where $e, f, g, h \in \mathbb{R}$, the latter being equivalent to there being real numbers α, β such that the polynomial is zero there (i.e., at $\alpha = -f/e$, $\beta = -h/g$). Using the quadratic formula, it is plain that no such real solutions to the quadratic being zero occur if and only if $b^2 - 4ac < 0$ (i.e., when the term under the radical in the quadratic formula is negative).

In both cases, we had a polynomial of degree less than 6 divided by a polynomial of degree 6, so the exception to "Rule 0" is not invoked. We also had two factors of $x+1$ in the denominator, so we needed a constant over the first power, plus another constant over the second power of $x+1$. With $(x-3)$ appearing to the third power in the denominator, we needed a constant over each of the first, second, and third powers of $x-3$. (Of course the choice of constants A, B, C, D, E will be different for these two functions above, but the abstract **form** *of their PFD's is the same.)*

We do not want to be redundant in our PFD's, so if $Q(x)$ contains the factor $(x-3)^4$ but does not contain $(x-3)^5$, for instance, we require constants divided by $(x-3)$, $(x-3)^2$, $(x-3)^3$ and $(x-3)^4$ (but not $(x-3)^5$). Now one could say that such a $Q(x)$ also contains $(x-3)^2$, but we do not then require in our PFD constants divided by $(x-3)$ and $(x-3)^2$ *again*, since these are already taken care of by those required by the factor $(x-3)^4$ in $Q(x)$.

To rephrase the rules in light of the last paragraph, if exactly n factors of $(ax+b)$ appear in $Q(x)$, then the PFD contains terms $\frac{A_1}{ax+b} + \cdots + \frac{A_n}{(ax+b)^n}$. If exactly m factors of (ax^2+bx+c) appear, with $b^2-4ac<0$, then the PFD contains terms $\frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$.

Example 7.5.7 *Here are more PFD expansion forms. (We do not solve for the coefficients yet.)*

- $\dfrac{x^4+x+1}{x^3(x^2+9)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+9} + \dfrac{Fx+G}{(x^2+9)^2}.$
- $\dfrac{1}{(x^2+1)(x^2+4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+4}.$
- $\dfrac{x^5-8}{x(2x+1)^2(9-x)^3} = \dfrac{A}{x} + \dfrac{B}{2x+1} + \dfrac{C}{(2x+1)^2} + \dfrac{D}{9-x} + \dfrac{E}{(9-x)^2} + \dfrac{F}{(9-x)^3}.$
- $\dfrac{2}{x^2-5} = \dfrac{2}{\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)} = \dfrac{A}{x-\sqrt{5}} + \dfrac{B}{x+\sqrt{5}}.$
- $\dfrac{1}{x^4-1} = \dfrac{1}{(x^2-1)(x^2+1)} = \dfrac{1}{(x+1)(x-1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1}.$

The last two PFD's above required us to factor the denominators before we started to implement the rules. Note that we must be careful to identify factors which are truly distinct. Consider the following:

$$\frac{1}{x(x-3)(3x-9)} = \frac{1}{3x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}.$$

The factors $(x-3)$ and $(3x-9)$ were not really distinct factors, but were constant multiples of each other. If we do not notice this we will find ourselves attempting a PFD with $\frac{A}{x} + \frac{B}{x-3} + \frac{C}{3x-9}$, but the "$B$" and "$C$" terms are not independent, so we will miss one dimension of possibilities for our PFD. Note also that the factor $\frac{1}{3}$ can be included in the first PFD term (i.e., we could replace $\frac{A}{x}$ with $\frac{A}{3x}$, of course giving a different "A"), or its influence absorbed into the A, B and C terms. We will usually opt for the latter approach (as we did above).

7.5.3 Finding the Coefficients for PFD's

There are two main methods, and one auxiliary method, for finding the coefficients A, B, etc., for a PFD. The most efficient method for a particular PFD is usually a mixture of the two main

methods; perhaps the first method can be used to find A and C, and the second to find B, for instance. Efficiently computing the coefficients is thus somewhat of an art.[28]

The methods are based upon some properties of polynomials. Consider two polynomials

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$
$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_2 x^2 + b_1 x + b_0.$$

The statement that $f(x) = g(x)$ is an "equality of polynomials," i.e., that $f(x)$ and $g(x)$ are the *same* polynomial is equivalent to each of the following two conditions (separately):[29]

1. $(\forall x \in \mathbb{R})[f(x) = g(x)]$. In other words, f and g are the same functions.
2. $(\forall i \in \{1, 2, \cdots, \max\{m, n\}\})[a_i = b_i]$, that is, all the coefficients are the same. (Note that it is possible, for instance, that $m < n$, in which case we just take $b_{m+1}, \cdots, b_n = 0$.)

Furthermore, if $f(x)$ and $g(x)$ are the same polynomials, then $f'(x) = g'(x)$, $f''(x) = g''(x)$, $f'''(x) = g'''(x) \cdots$, in the sense of being the same polynomials, and so 1 and 2 from above apply to these derivatives as well.

Our first method will exploit 1, the second 2, and the auxiliary method will make use of final observation about derivatives. The first method essentially "probes" the two polynomials at different points, usually chosen strategically, to get some quick information out of a polynomial equality, and is often called an "evaluation method." The second method is often referred to as "comparing coefficients," and can also be useful for finding quick information. The auxiliary method exploits the fact that the first methods can be applied to the derivatives (of any order) of f and g to get further information quickly.

Example 7.5.8 *Compute the integral* $\displaystyle\int \frac{1}{x^2 - 5x + 6}\,dx$.

Solution: Here we have a degree-0 polynomial divided by a degree-2 polynomial, so the PFD rules apply. Now one usually writes the PFD form of the integrand, complete with the unknown coefficients, before proceeding to the methods of computing the coefficients. In other words, our first step would be to write:

$$\int \frac{1}{x^2 - 5x + 6}\,dx = \int \frac{1}{(x-2)(x-3)}\,dx = \int \left[\frac{A}{x-2} + \frac{B}{x-3}\right] dx.$$

The next two lines can be skipped with practice, though the first time one works this section they are worth writing so the mechanics of the method can be understood and reinforced. First we write the algebraic step (PFD) which was contained in the rewriting of the integrands above:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

This came from the fact that we have $x - 2$ as a factor in the denominator, but only once, and the same for $x - 3$. Next we multiply both sides by **the denominator on the left***:*

$$(x-2)(x-3)\left[\frac{1}{(x-2)(x-3)}\right] = (x-2)(x-3)\left[\frac{A}{x-2} + \frac{B}{x-3}\right].$$

[28] In fact either method is—strictly speaking—sufficient, and indeed there are textbooks which teach only one or the other method. However, trying to fit a particularly complicated PFD into any single method will make for much more difficult computations than are necessary. That said, for computer programming one would likely choose one method and let the computer calculate the coefficients by "brute force."

[29] Some texts use the notation $f(x) \equiv g(x)$, read "$f(x)$ is identically equal to $g(x)$." In other words, $f(x)$ and $g(x)$ are the same functions. The word *identically* is in the spirit of, for instance, trigonometric identities, so one could write for example $\sin^2\theta + \cos^2\theta \equiv 1$.

On the left, the whole denominator cancels and we have the numerator of the original fraction. (This always happens.) On the right we have to distribute the $(x-2)(x-3)$ across the sum in the brackets. For the "A" term the $(x-2)$ cancels, while for the "B" term the $(x-3)$ cancels, giving us an equality of polynomials:

$$1 = A(x-3) + B(x-2). \tag{7.58}$$

Because this is an equality of polynomials ($f(x) = g(x)$ where $f(x) = 1$ and $g(x) = A(x-3) + B(x-2)$), it must be true for any $x \in \mathbb{R}$. Now we choose two values of x strategically.

$$\begin{aligned} \underline{x=3}: \qquad & 1 = A(3-3) + B(3-2) \implies \boxed{1 = B} \\ \underline{x=2}: \qquad & 1 = A(2-3) + B(2-2) \\ \implies & 1 = -A \implies \boxed{A = -1}. \end{aligned}$$

Now we summarize what we have so far, and compute the desired integral:

$$\begin{aligned} \int \frac{1}{x^2-5x+6}\,dx &= \int \left[\frac{-1}{x-2} + \frac{1}{x-3}\right] dx \\ &= -\ln|x-2| + \ln|x-3| + C \\ &= \ln\left|\frac{x-3}{x-2}\right| + C. \end{aligned}$$

(The last step is not necessary, but for reasons of style many textbooks combine logarithmic terms into a single logarithm.)

Because (7.58) was an equality of polynomials (meaning the polynomial on the left is *the same polynomial* as that on the right[30]), we could substitute any number for x in (7.58) and still have a true statement. Fortunately, there were choices which could eliminate an unknown, leaving an equation in the other unknown which is easily solved.

The second method for finding A and B (not preferred here but not terribly difficult here either) is to look at the coefficients of the polynomials on the left-hand side and right-hand side of (7.58), and realize that the coefficients of the various powers of x must agree for these to be the *same* polynomials. Though perhaps not necessary for this simple case, one sometimes expands the right-hand side and collects like terms

$$1 = (A+B)x + (-3A-2B).$$

From this or just reading from (7.58), we can in turn set equal the coefficients of the x^1 terms and the constant (some say x^0) terms to get the following system of two equations in two unknowns:

$$\begin{cases} 0 &= A &+ B \\ 1 &= -3A &- 2B. \end{cases} \tag{7.59}$$

The first equation came from the fact that there is no x^1-term on the left-hand side of (7.59), or alternatively, the x^1-term is $0x^1$ on the left. To solve such a system one might add three

[30] It should be pointed out that when we write a PFD, for instance

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3},$$

we mean that these are the same functions as well, so once we find A and B, the right-hand side would simplify to become the left-hand side. To find A and B we actually solve the *polynomial* equality (7.58) for A and B.

Note also the distinction between "equations" such as $2x-1=5$, which is true only for $x=3$, and "equalities" such as $(x+1)^2 = x^2+2x+1$, true for all x, meaning the function $(x+1)^2$ is the same as the function x^2+2x+1.

times the first equation to the second, to get $B = 1$, and use that information in the first to get $A = -1$, as before.

Whenever the denominator of our function $P(x)/Q(x)$ has a linear factor $(ax+b)$, evaluating the associated *polynomial* equality—such as (7.58)—at that x-value which makes this linear factor zero (namely $x = -b/a$) will quickly yield one of the coefficients, since all but one term in the polynomial equation will have $(ax + b)$ as a factor, and therefore vanish at $x = -b/a$. Thus this first method should always be employed to find that coefficient if the denominator Q has a linear factor. If the denominator has all linear terms to the first power, then this "evaluation" method will quickly yield all coefficients.

Example 7.5.9 *Compute* $\displaystyle\int \frac{2x^2 - 3x + 2}{x(x+5)(2x+1)}\,dx$.

Solution: *It is important to notice that the numerator is degree 2, and the denominator degree 3, so the PFD rules do apply.*

$$\int \frac{2x^2 - 3x + 2}{x(x+5)(2x+1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x+5} + \frac{C}{2x+1}\right]\,dx.$$

Eventually we will cease writing the next two lines, but to be sure we will include them here so that the logic is clear:

$$\frac{2x^2 - 3x + 2}{x(x+5)(2x+1)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{2x+1}$$

$$\implies x(x+5)(2x+1)\left[\frac{2x^2 - 3x + 2}{x(x+5)(2x+1)}\right] = x(x+5)(2x+1)\left[\frac{A}{x} + \frac{B}{x+5} + \frac{C}{2x+1}\right]$$

$$\implies \qquad 2x^2 - 3x + 2 = A(x+5)(2x+1) + Bx(2x+1) + Cx(x+5).$$

Into this last line we can now enter values for x which will quickly yield the coefficients.

$$\underline{x = 0}: \qquad 2 = A(5)(1) \implies \boxed{A = \frac{2}{5}}$$

$$\underline{x = -5}: \qquad 50 + 15 + 2 = B(-5)(-9)$$

$$\implies 67 = 45B \implies \boxed{B = \frac{67}{45}}.$$

$$\underline{x = -\frac{1}{2}}: \qquad 2\cdot\frac{1}{4} - 3\cdot\left(-\frac{1}{2}\right) + 2 = C\left(-\frac{1}{2}\right)\left(-\frac{1}{2} + 5\right)$$

$$\implies \frac{1}{2} + \frac{3}{2} + 2 = C\left(-\frac{1}{2}\right)\left(\frac{9}{2}\right)$$

$$\implies 4 = -\frac{9}{4}C \implies \boxed{C = -\frac{16}{9}}.$$

Putting this together with our original integral, we get

$$\begin{aligned}\int \frac{2x^2 - 3x + 2}{x(x+5)(2x+1)}\,dx &= \int \left[\frac{2/5}{x} + \frac{67/45}{x+5} + \frac{-16/9}{2x+1}\right]\,dx \\ &= \frac{2}{5}\ln|x| + \frac{67}{45}\ln|x+5| - \frac{16}{9}\cdot\frac{1}{2}\ln|2x+1| + C \\ &= \frac{2}{5}\ln|x| + \frac{67}{45}\ln|x+5| - \frac{8}{9}\ln|2x+1| + C.\end{aligned}$$

Next we look at an example where all factors of $Q(x)$ are linear, but one of these linear factors appears to the second power.

Example 7.5.10 *Compute* $\displaystyle\int \frac{x+1}{x^2(x-5)(x+4)}\,dx$.

Solution: *This time we will describe but omit the explicit multiplication step in the PFD.*[31]

$$\int \frac{x+1}{x^2(x-5)(x+4)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5} + \frac{D}{x+4}\right] dx,$$

where

$$\frac{x+1}{x^2(x-5)(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-5} + \frac{D}{x+4}.$$

Multiplying by $x^2(x-5)(x+4)$ *then gives us*

$$x+1 = Ax(x-5)(x+4) + B(x-5)(x+4) + Cx^2(x+4) + Dx^2(x-5). \tag{7.60}$$

With this equation, choosing $x = 0, 5, -4$ *will yield three of the four coefficients.*

$$\begin{aligned}
\underline{x=0}: \qquad & 1 = B(-5)(4) \implies \boxed{B = -\frac{1}{20}} \\
\underline{x=5}: \qquad & 6 = C(5^2)(9) \\
\implies & \frac{2\cdot 3}{5^2\cdot 3\cdot 3} = C \implies \boxed{C = \frac{2}{75}} \\
\underline{x=-4}: \qquad & -3 = D((-4)^2)(-9) \\
\implies & -3 = D[-16\cdot 9] \\
\implies & 3 = D\cdot 16\cdot 3\cdot 3 \implies \boxed{D = \frac{1}{48}}
\end{aligned}$$

This exhausts the evaluations which give equations in one coefficient. Next we have several methods for finding A.

Method 1. *Compare coefficients. In particular, we look at the highest-order x-terms which appear—at least initially—in the polynomial equality, which for* (7.60) *means the* x^3*-terms. Here we have no* x^3*-terms on the left, and on the right, even without a complete expansion, we can see that the* x^3*-terms will be* $A + C + D$*. (The middle-order terms are more difficult to read from* (7.60)*.) Fortunately we already know the values of* C *and* D*, so we have enough information to find A:*

$$\begin{aligned}
\underline{x^3\text{-term}}: \qquad & 0 = A + C + D \\
\implies & 0 = A + \frac{2}{75} + \frac{1}{48} = A + \frac{2}{3\cdot 5^2} + \frac{1}{2^4\cdot 3} \\
\iff & 0 = A + \frac{2\cdot 2^4 + 1\cdot 5^2}{3\cdot 5^2\cdot 2^4} = A + \frac{32+25}{3\cdot 5^2\cdot 2^4} \\
\iff & -\frac{57}{1200} = A \implies \boxed{A = -\frac{19}{400}}
\end{aligned}$$

[31] The pattern of cancellation, when we multiply the PFD by the original denominator $Q(x)$, should become second nature with a small amount of practice. That said, it is important to remember what we are doing (multiplying by $Q(x)$) to get from the PFD to the polynomial equality, and how the various factors cancel (or do not cancel) in that multiplication.

From this we can complete the integration:

$$\int \frac{x+1}{x^2(x-5)(x+4)}\,dx = \int \left[\frac{-\frac{19}{400}}{x} + \frac{-\frac{1}{20}}{x^2} + \frac{\frac{2}{75}}{x-5} + \frac{\frac{1}{48}}{x+4}\right] dx$$
$$= -\frac{19}{400}\ln|x| - \frac{1}{20}\cdot\frac{-1}{x} + \frac{2}{75}\ln|x-5| + \frac{1}{48}\ln|x+4| + C.$$

Method 2. *One can instead evaluate the polynomial equality (7.60) at still another x-value, though no such value will produce A alone:*

$$\underline{x=1}: \qquad 2 = A(1)(-4)(5) + B(-4)(5) + C(1)^2(5) + D(1)^2(-4).$$

Since we already know B, C and D, we can insert that information and solve for A.

Method 3. *This will be more useful later, but this method (referred to earlier as the* auxiliary *method) certainly applies. The idea is that we apply $\frac{d}{dx}$ to both sides of* (7.60)*, which is valid because the left-hand side and right-hand side of* (7.60) *are the same* functions.

In order to use this method, it is useful to recall the generalized product rule. For three functions $u(x)$, $v(x)$ and $w(x)$, for instance, we have

$$(uvw)' = u'vw + uv'w + uvw'.$$

For reference we recall (7.60)*, from which we then compute the derivatives. Equation* (7.60) *reads:*

$$x+1 = Ax(x-5)(x+4) + B(x-5)(x+4) + Cx^2(x+4) + Dx^2(x-5).$$

$$\underline{\frac{d}{dx}}: \qquad 1 = A[(1)(x-5)(x+4) + x(1)(x+4) + x(x-5)(1)]$$
$$+ B[(1)(x+4) + (x-5)(1)]$$
$$+ C[(2x)(x+4) + x^2(1)]$$
$$+ D[(2x)(x-5) + x^2(1)].$$

Now when we evaluate this at $x = 0$, we see that we get

$$1 = A[(1)(-5)(4)] + B[(1)(4) + (-5)(1)] + 0 + 0.$$

This gives us $1 = -20A - B$, so then $A = (1+B)/(-20) = (19/20)/(-20) = -19/400$, as before.

A simple principle buried in the third method is the following:

Theorem 7.5.1 *If $(x-a)^m$, where $m > 1$ is a factor of a polynomial $f(x)$, then $(x-a)^{m-1}$ is a factor of $f'(x)$.*

For a proof, we note that $(x-a)^m$ being a factor of $f(x)$ is equivalent to $f(x) = (x-a)^m g(x)$, where $g(x)$ is another polynomial. Thus

$$f'(x) = (x-a)^m g'(x) + m(x-a)^{m-1}(1)g(x) = (x-a)^{m-1}\underbrace{[(x-a)g'(x) + mg(x)]}_{\text{polynomial}},$$

so indeed $(x-a)^{m-1}$ is a factor of $f'(x)$.

Now evaluating both sides of (7.60) at $x=0$ caused those terms with x and x^2 factors to vanish, leaving an equation with B only. When we differentiate (7.60), those terms with x^2 factors *still vanish* at $x=0$—because one power of x remains—leaving only the B-term (as before) and the A-term (which had a factor x but not x^2). Already knowing B, we could solve for A.

A few guidelines for efficiently finding the PFD coefficients should be made at this point.

1. When linear factors are present in $Q(x)$, it is best to exhaust this method for finding some of the coefficients easily. This means evaluating the relevant polynomial equality at each value for which $Q(x)=0$.
2. When those values are exhausted, we should next compare coefficients of the powers of x, particularly the highest power which occurs on the right-hand side.
3. If $(ax+b)^m$ is a factor of $Q(x)$, where $m>1$, and the first two methods fail to get all coefficients, then differentiation of the polynomial equality may yield more coefficients.
4. If there are still coefficients to be found, then further evaluations, differentiations, or coefficient comparisons should be implemented.

Example 7.5.11 *Compute* $\displaystyle\int \frac{5x^3-17x^2+19x-13}{(x+1)(x-2)^3}\,dx$.

Solution: *As usual we start with the PFD, since the denominator has higher degree than the numerator.*

$$\frac{5x^3-17x^2+19x-13}{(x+1)(x-2)^3}=\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}$$

$$5x^3-17x^2+19x-13=A(x-2)^3+B(x+1)(x-2)^2+C(x+1)(x-2)+D(x+1) \quad (7.61)$$

$$\begin{aligned}
\underline{x=-1}: \qquad -5-17-19-13&=A(-27)\\
\implies -54&=-27A \implies \boxed{A=2}\\
\underline{x=2}: \qquad 5(8)-17(4)+19(2)-13&=D(3)\\
\implies -3&=3D \implies \boxed{D=-1}\\
\underline{x^3\text{-term}}: \qquad 5&=A+B\\
\implies 5&=2+B \implies \boxed{B=3}.
\end{aligned}$$

While we could perform another evaluation ($x=0$ comes to mind), or look at another coefficient (prone to error), instead we will differentiate (7.61):

$$\begin{aligned}
15x^2-34x+19=&A[3(x-2)^2(1)]\\
&+B[(1)(x-2)^2+(x+1)\cdot 2(x-2)(1)]\\
&+C[(1)(x-2)+(x+1)(1)]\\
&+D(1).
\end{aligned}$$

Now we evaluated at $x=2$:

$$\begin{aligned}
15(4)-34(2)+19&=C(3)+D\\
\implies 11&=3C-1 \implies \boxed{C=4}.
\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{5x^3-17x^2+19x-13}{(x+1)(x-2)^3}\,dx &= \int \left[\frac{2}{x+1}+\frac{3}{x-2}+\frac{4}{(x-2)^2}-\frac{1}{(x-2)^3}\right]dx\\ &= 2\ln|x+1|+3\ln|x-2|-\frac{4}{x-2}-\frac{1}{-2}\cdot\frac{1}{(x-2)^2}+C\\ &= \ln\left|(x+1)^2(x-2)^3\right|-\frac{4}{x-2}+\frac{1}{2(x-2)^2}+C.\end{aligned}$$

A quick corollary to our Theorem 7.5.1 is that if $(x-a)^m$ is a factor of a polynomial $f(x)$, then for $k<m$ we have $(x-a)^{m-k}$ is a factor of $f^{(k)}(x)=\frac{d^k}{dx^k}f(x)$. This follows from repeated applications of the theorem, which can be paraphrased as saying that we lose at most one factor of $(x-a)$ for each derivative we take, until we run out of factors of $(x-a)$. If our latest example had $(x-2)^4$ in the denominator, we could have taken a second derivative of the corresponding polynomial equation, and then those terms with $(x-2)^3$ or $(x-2)^4$ will still be zero at $x=2$, but the other terms would likely be nonzero.[32]

Now we turn our attention to PFD's where the denominators contain irreducible quadratic factors.[33] One problem with such factors is that they are nonzero for any $x\in\mathbb{R}$,[34] so the evaluation method's usefulness is limited in these cases. For such PFD's, we will need to rely more upon the coefficient comparison method to find our coefficients.

Example 7.5.12 *Compute* $\displaystyle\int \frac{4x^3-7x^2+31x-38}{x^4+13x^2+36}\,dx$.

Solution: PFD rules apply since the degree of the numerator is less than that of the denominator. We need to begin by factoring the denominator of the integrand, after which we can write the general form of the PFD.

$$\int \frac{4x^3-7x^2+31x-38}{x^4+13x^2+36}\,dx = \int \frac{4x^3-7x^2+31x-38}{(x^2+4)(x^2+9)}\,dx = \int\left[\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+9}\right]dx.$$

Now taking the second equation, we underlying PFD becomes the polynomial equality

$$4x^3-7x^2+31x-38=(Ax+B)(x^2+9)+(Cx+D)(x^2+4). \tag{7.62}$$

Now we look at the coefficients.[35]

$$\begin{aligned}\underline{x^3\text{-term}}:&\quad 4=A+C\\ \underline{x^2\text{-term}}:&\quad -7=B+D\\ \underline{x^1\text{-term}}:&\quad 31=9A+4C\\ \underline{x^0\text{-term}}:&\quad -38=9B+4D.\end{aligned}$$

[32] Note that it is quite possible that $x-a$ is not a factor of a polynomial $f(x)$, but *is* a factor of $f'(x)$. That is the case when $x=a$ is a *critical point* of $f(x)$. For example, $x-1$ is not a factor of $f(x)=x^2-2x+21$, but is a factor of $f'(x)=2x-2=2(x-1)$.

Note also that the theorem applies to any linear factor $ax+b$, where $a\neq 0$, since $ax+b=a\left(x+\frac{b}{a}\right)$. Thus if x^m is a factor of a polynomial $f(x)$, the x^{m-1} is a factor of $f'(x)$, etc., as is the case if we replace x^m with $(ax+b)^m=a^m\left(x+\frac{b}{a}\right)$.

[33] Until the next section, we will not be able to integrate the general case where we have an irreducible quadratic factor to a power greater than 1, with some exceptional cases.

[34] Recall that if $f(x)$ is a polynomial of degree ≥ 1, then $f(a)=0 \iff (x-a)$ is a factor of $f(x)$.

[35] Note that the constant ("x^0") term equation is what we would get if we evaluated (7.62) at $x=0$. It is easy to see that this is always the case.

Though this looks like (and is) four equations in four unknowns, in fact it "decouples" into two systems, each with two unknowns, since the first and third equations have only A and C, and the second and fourth have B and D only. We solve these in turn.

$$\begin{array}{rcrcr} 4 &=& A &+& C \\ 31 &=& 9A &+& 4C \end{array} \qquad \begin{array}{rcrcr} -7 &=& B &+& D \\ -38 &=& 9B &+& 4D \end{array}$$

For the first system, we multiply the first equation by -9 and add to the second, to get $-5 = 0A - 5C \implies C = 1$. From that we have the original first equation giving $A = 4 - C = 4 - 1 = 3$.

For the second system, we do the same, that is, multiply the first equation by -9 and add to the second, giving $63 - 38 = -5D \implies 25 = -5D \implies -5 = D$. From the original first equation in that system, we then get $B = -7 - D = -7 + 5 = -2$.

Now we compute the integral, noting that it is easier if we break the PFD into four distinct terms:

$$\begin{aligned} \int \frac{4x^3 - 7x^2 + 31x - 38}{(x^2+4)(x^2+9)}\,dx &= \int \left[\frac{3x}{x^2+4} - \frac{2}{x^2+4} + \frac{x}{x^2+9} - \frac{5}{x^2+9}\right] dx \\ &= \frac{3}{2}\ln(x^2+4) - \frac{2}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln(x^2+9) - \frac{5}{3}\tan^{-1}\frac{x}{3} + C \\ &= \ln\sqrt{(x^2+4)^3(x^2+9)} - \tan^{-1}\frac{x}{2} - \frac{5}{3}\tan^{-1}\frac{x}{3} + C. \end{aligned}$$

In the example above, we used the following common integration formula, which is particularly useful in problems encountered in this section. It is derivable with the usual substitution methods, and not too difficult to verify by differentiation. The formula is the following:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C. \tag{7.63}$$

We also used

$$\int \frac{x}{x^2+k^2}\,dx = \frac{1}{2}\ln(x^2+k^2) + C,$$

assuming $k \neq 0$. Note that we do not need absolute values inside the logarithm since $x^2 + k^2 \geq k^2 > 0$.

When we have irreducible quadratic factors in the denominator $Q(x)$, it is likely that we will need to compare coefficients.[36] After all, there are no real numbers which will make all but one of those coefficients vanish. (We can make two vanish with $x = 0$, but that still leaves two.) If linear terms are also present, however, the evaluation method will yield one or more of the coefficients quickly.

Example 7.5.13 *Compute* $\displaystyle\int \frac{12x^4 + 190x^2 + 13x - 6}{(2x-1)(x^2+16)}\,dx.$

Solution: First we note that the numerator has degree which is not less than the denominator, so we must use long division. To do so we need to expand the denominator: $(2x-1)(x^2+16) = 2x^3 - x^2 + 32x - 16$.

[36] Or something equivalent to comparing coefficients. For instance, $x = 0$ gives $-38 = 9B + 4D$, and one derivative of (7.62) gives

$$12x^2 - 14x + 31 = (A)(x^2+9) + (Ax+B)(2x) + (C)(x^2+4) + (Cx+D)(2x),$$

which, when we consider the datum $x = 0$ gives $31 = 9A + 4C$. Both of these we had before. More derivatives, evaluated at $x = 0$, give multiples of the other two equations in our system (four equations in four unknowns).

Now through polynomial long division we get

$$\frac{12x^4+190x^2+13x-6}{(2x-1)(x^2+16)} = \frac{12x^4+190x^2+13x-6}{2x^3-x^2+32x-16} = 6x+3+\frac{x^2+13x+42}{2x^3-x^2+32x-16}. \quad (7.64)$$

Refactoring our denominator, our integral now becomes

$$\int \left[6x+3+\frac{x^2+13x+42}{(2x-1)(x^2+16)}\right]\,dx = \int \left[6x+3+\frac{A}{2x-1}+\frac{Bx+C}{x^2+16}\right]\,dx.$$

The first two terms are easy enough. For our PFD, we need only concern ourselves with the remaining fraction:

$$\frac{x^2+13x+42}{(2x-1)(x^2+16)} = \frac{A}{2x-1}+\frac{Bx+C}{x^2+16}.$$

The corresponding polynomial equation is then

$$x^2+13x+42 = A(x^2+16)+(Bx+C)(2x-1). \quad (7.65)$$

We begin with an evaluation, followed by a coefficient comparison.

$$\begin{aligned}
\underline{x=\frac{1}{2}}: \qquad & \frac{1}{4}+\frac{13}{2}+42 = A\left(\frac{1}{4}+16\right) \\
\implies & \frac{1+26+168}{4} = \frac{65}{4}A \\
\implies & 195 = 65A \implies \boxed{A=3} \\
\underline{x^2\text{-term}}: \qquad & 1 = A+2B \\
\implies & 1 = 3+2B \implies \boxed{B=-1}.
\end{aligned}$$

Perhaps the simplest next step is to find C by evaluation of (7.65) at, say, $x=0$:

$$\begin{aligned}
\underline{x=0}: \qquad & 42 = 16A-C \\
\implies & 42 = 16(3)-C \\
\implies & C = 16(3)-33 = 48-16 \implies \boxed{C=6}.
\end{aligned}$$

Thus our original integral, including the polynomial terms, becomes

$$\begin{aligned}
\int \frac{12x^4+190x^2+13x-6}{(2x-1)(x^2+16)}\,dx &= \int \left[6x+3+\frac{3}{2x-1}-\frac{x}{x^2+16}+\frac{6}{x^2+16}\right]\,dx \\
&= 3x^2+3x+\frac{3}{2}\ln|2x-1|-\frac{1}{2}\ln(x^2+16)+\frac{6}{4}\tan^{-1}\frac{x}{4}+C \\
&= 3x(x+1)+3\ln\sqrt{|2x-1|}-\ln\sqrt{x^2+16}+\frac{3}{2}\tan^{-1}\frac{x}{4}+C.
\end{aligned}$$

The second from the last line was complete; the last line just gives some alternative styles for the particular terms.

Of course with any new technique, we have to be sure that we do not neglect the earlier methods.

7.6 Miscellaneous Methods

In this section we will use completing the square, and other methods to rewrite several types of integrals into forms where we can more easily use either partial fractions or trigonometric substitution. We will also look at examples where a substitution will bring us to such forms. Finally, we will consider the use of integration tables, which can be found in numerous publications, but which require some sophistication to be used properly.